SLICING AND FINE PROPERTIES OF FUNCTIONS WITH BOUNDED $\mathcal{A}$-VARIATION

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Abstract. We study the slicing and fine properties of functions in $\mathrm{BV}^{\mathcal{A}}$, the space of functions of bounded $\mathcal{A}$-variation. Here, $\mathcal{A}$ is a homogeneous linear differential operator with constant coefficients (of arbitrary order). Our main result is the characterization of all $\mathcal{A}$ satifying the following one-dimensional structure theorem: every $u \in \mathrm{BV}^{\mathcal{A}}$ can be sliced into one-dimensional BV-sections. Moreover, decomposing $\mathcal{A}u$ into an absolutely continuous part $\mathcal{A}^a u$, a Cantor part $\mathcal{A}^c u$ and a jump part $\mathcal{A}^j u$, each of these measures can be recovered from the corresponding classical D^a, D^c and D^j BV-derivatives of its one-dimensional sections. By means of this result, we are able to analyze the set of Lebesgue points as well as the set of jump points where these functions have approximate one-sided limits. Thus, proving a structure and fine properties theorem in $\mathrm{BV}^{\mathcal{A}}$. Our results extend most of the classical fine properties of BV (and all of those known for BD) to $\mathrm{BV}^{\mathcal{A}}$. Applications of our results are discussed for operators that are not covered by the existing theory.

Contents

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1. Introduction

In this article we consider the space

$$\mathrm{BV}^{\mathcal{A}}(\Omega) = \left\{ \mathrm{L}^1(\Omega;\mathbb{R}^N) : \mathcal{A}u \in \mathcal{M}(\Omega;\mathbb{R}^M) \right\}$$

of *functions of bounded $\mathcal{A}$-variation* over an open set $\Omega \subset \mathbb{R}^n$. Here, $\mathcal{M}(\Omega;\mathbb{R}^M)$ denotes the space of $\mathbb{R}^M$-valued Radon measures and $\mathcal{A}$ is a homogeneous system of linear partial differential operators with constant coefficients. More precisely, the operator $\mathcal{A}$ acts on functions $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ as

$$\mathcal{A}u := \sum_{|\alpha|=k} A_\alpha \partial^\alpha u, \tag{1}$$

where the coefficients A_α are tensors (matrices) in $\mathbb{R}^M \otimes \mathbb{R}^N \cong \mathbb{R}^{M\times N}$, ∂^α denotes the distributional partial derivative $\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$ of order $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha = (\alpha_1,\ldots,\alpha_n)$ is a multi-index in $\mathbb{N}_0^n$.

The purpose of this work is to give a comprehensive determination of the structural and fine properties of functions in $\mathrm{BV}^{\mathcal{A}}(\Omega)$, very much in the fashion of what is known for $\mathrm{BV}(\Omega;\mathbb{R}^N)$. A fundamental part of this endeavor is to construct a unified theory that circumvents the use of tools that are exclusive to the theory of gradients (such as the co-area formula and the theory of sets of finite perimeter). In this regard, we have found that some of the core ideas used to establish the *same* fine properties in BD remain valid in the $\mathrm{BV}^{\mathcal{A}}$-framework; particularly, the one-dimensional slicing theory (see [2, 9]). In its current form, the implementation of slicing techniques appeals directly to the unique structure that the gradient and the symmetric gradient possess. As such, little is known about the overall structural properties of operators admitting *slicing* into lower dimensional elements. Our main contribution in this vein is the following: First, we introduce the notion of « $\mathrm{rank}_{\mathcal{A}}(w)$ » for a vector w in the target space « $\mathbb{R}^M$ » of the operator $\mathcal{A}$. This concept extends the classical notion of rank when $\mathbb{R}^M \cong \mathbb{R}^{m\times d}$ is a space of matrices, but is also sensible to $\mathcal{A}$ in a suitable algebraic way. For first-order operators (when $k = 1$), $\mathrm{rank}_{\mathcal{A}}$-one vectors w can be formally defined as those vectors satisfying

$$w \cdot \mathcal{A}u = \partial_\xi(u \cdot e) \quad \text{for all } u \in \mathrm{C}^\infty(\mathbb{R}^n;\mathbb{R}^N),$$

for some direction $\xi \in \mathbb{R}^n$ and some $e \in \mathbb{R}^N$. In this case, one may think of (ξ, e) as coordinates on which the operator $\mathcal{A}$ controls the partial derivative operator $\partial_\xi(\ )^e$. Naturally, the more $\mathrm{rank}_{\mathcal{A}}$-one tensors exist, the more individual partial derivatives are controlled by the operator $\mathcal{A}$. In this regard, we show that the *algebraic* mixing property

$$\bigcap_{\substack{\pi \leq \mathbb{R}^n \\ \dim(\pi)=n-1}} \mathrm{span}\left\{ \mathrm{Im}\, \mathbb{A}^k(\xi) : \xi \in \pi \right\} = \{0\}, \qquad \mathbb{A}^k(\xi) := \sum_{i=1}^n \xi^\alpha A_\alpha,$$

is equivalent to the existence of a family $\{w_1, \dots, w_M\} \subset \mathbb{R}^M$ such that

$$\mathrm{span}\{w_1, \dots, w_M\} = \mathbb{R}^M, \qquad \mathrm{rank}_{\mathcal{A}}(w_j) \leq 1.$$

Then, we show that this spanning property is equivalent to the following *functional* property: the space $\mathrm{BV}^{\mathcal{A}}(\Omega)$ admits a definition by slicing into one-dimensional BV-sections. We proceed to develop a slicing theory in $\mathrm{BV}^{\mathcal{A}}$, which is based on the notion of $\mathrm{rank}_{\mathcal{A}}$ and the understanding of the functional properties that stem from the mixing condition. These slicing methods are crucial for carrying the analysis of Lebesgue point properties, which we subsequently use to establish the structure and fine properties for $\mathrm{BV}^{\mathcal{A}}$-spaces.

Before embarking on a formal discussion of the main slicing and fine properties theorems, let us bring some perspective to our results by briefly recalling the slicing and fine properties of the classical BV-theory. The space of functions of bounded variation $\mathrm{BV}(\Omega; \mathbb{R}^N)$ consists of all functions $u \in \mathrm{L}^1(\Omega; \mathbb{R}^N)$ whose distributional gradient can be represented by an $\mathbb{R}^N \otimes \mathbb{R}^n$ tensor-valued Radon measure $Du \in \mathcal{M}(\Omega; \mathbb{R}^N \otimes \mathbb{R}^n)$. The theory surrounding this space of functions originated by the work of Caccioppoli [11,12], De Giorgi [13–16] and Federer [19,20], who studied a particular class of BV functions that consists of characteristic functions (sets of finite perimeter). Independently, Fleming & Rishel[22] proved the co-area formula, which cast into the context of sets of finite perimeter evolved into the following well-known identity for gradient measures:

$$|Du|(B) = \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(\partial^*\{u > t\} \cap B)\, \mathrm{d}t.$$

The existence of such a decomposition into a family of $(n-1)$-dimensional rectifiable sections is an example of the *structural properties* for BV-functions. It implies, among other things, that the total variation of a gradient measure vanishes on sets of zero $\mathcal{H}^{n-1}$-measure. In fact, it implies the stronger bound $|Du| \ll \mathcal{I}^{n-1} \ll \mathcal{H}^{n-1}$. Later on, Federer [20, §3.2.14] and Vol'pert [32] showed that, for every $u \in \mathrm{BV}(\Omega; \mathbb{R}^M)$, the set S_u of Lebesgue discontinuous points is $\mathcal{H}^{n-1}$-countably rectifiable. They also showed that the measure Du can be decomposed, with respect to the n-dimensional Lebesgue measure $\mathscr{L}^n$, into a singular part $D^s u$ and an absolutely continuous part $D^a u = \nabla u \mathscr{L}^n$ with density given by the approximate differential $\nabla u : \Omega \to \mathbb{R}^N \otimes \mathbb{R}^n$ of u. Furthermore, the singular $D^s u$ may be split into a *Cantor part* and a *jump part* as

$$\begin{aligned} D^s u &= D^c u + D^j u \\ &= D^s u \llcorner (\Omega \setminus S_u) + (u^+ - u^-) \otimes \nu_u \mathcal{H}^{n-1} \llcorner J_u, \end{aligned}$$

where the Cantor part $D^c u$ is the restriction of $D^s u$ to the set $\Omega \setminus S_u$ of Lebesgue continuity points of u, and the jump set $J_u \subset S_u$ is the set of approximate discontinuity points $x \in \Omega$ where u has one-sided limits $u^+(x) \neq u^-(x)$ with respect to a suitable direction $\nu_u(x)$ normal to S_u in a measure theoretical sense. Thus, $|Du|$ and each of the terms $D^a u, D^c u, D^j u$ possess particular geometrical properties that correlate directly with the Lebesgue continuity properties of u. These properties ultimately conform the so-called *fine properties* of functions of bounded variation.

Another remarkable fact is that BV-functions can be characterized by their decomposition into one-directional sections (see, for example [3, Remark 3.104]). Namely, an integrable function u lies in $\mathrm{BV}(\Omega;\mathbb{R}^N)$ if and only if, for any direction $\xi \in \mathbf{S}^{n-1} = \{\, \zeta \in \mathbb{R}^n : |\zeta| = 1 \,\}$ and every coordinate $e \in \mathbb{R}^N$, its one-dimensional sections $u^e_{y,\xi}$ belong to $\mathrm{BV}(\Omega^\xi_y;\mathbb{R}^N)$ for $\mathcal{H}^{n-1}$-almost every $y \in \pi_\xi$, and

$$\int_{\pi_\xi} |Du^e_{y,\xi}|(\Omega^\xi_y) \,\mathrm{d}\mathcal{H}^{n-1}(y) < \infty. \tag{2}$$

Here, $\Omega^\xi_y = \{\, s \in \mathbb{R} : y + s\xi \in \Omega \,\}$, where π_ξ is the plane orthogonal to ξ, passing through the origin and $u^e_{y,\xi}(t) = u(y + t\xi) \cdot e$. As a matter of fact, the structure theorem extends to each of these one-dimensional sections:

$$(D^\sigma u : e \otimes \xi) = \int_{\pi_\xi} D^\sigma u^e_{y,\xi} \,\mathrm{d}\mathcal{H}^{n-1} \qquad \text{for all } \sigma = a, c, j. \tag{3}$$

Related decompositions and slicing techniques hold for

$$\mathrm{BD}(\Omega) = \left\{\, u \in \mathrm{L}^1(\Omega;\mathbb{R}^n) : Eu = Du + Du^T \in \mathcal{M}(\Omega;\mathbb{R}^n \odot \mathbb{R}^n) \,\right\},$$

the space of functions $u : \mathbb{R}^n \to \mathbb{R}^n$ of *bounded deformation* over Ω. In analogy with (2), the slicing properties of BD present a significant reduction of coordinates « $e \in \mathbb{R}^N$ » that are required to *control* the symmetric gradient. This was observed by Bellettini & Coscia, who showed (see [9]) that function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ has bounded deformation if and only if $u^\xi_{y,\xi} \in \mathrm{BV}(\Omega^\xi_\xi)$ for $\mathcal{H}^{n-1}$-almost every $y \in \pi_\xi$ and

$$\int_{\pi_\xi} |Du^\xi_{y,\xi}| \,\mathrm{d}\mathcal{H}^{n-1}(y) < \infty.$$

The full range of fine properties for BD is contained in the celebrated work of Ambrosio, Coscia & Dal Maso [2], where the authors establish a crucial one-dimensional structure theorem of the flavor of (3). More precisely, they showed that if $u \in \mathrm{BD}(\Omega)$, then

$$(E^\sigma u : \xi \otimes \xi) = \int_{\pi_\xi} D^\sigma u^\xi_{y,\xi} \,\mathrm{d}\mathcal{H}^{n-1}(y) \qquad \text{for all } \sigma = a, c, j. \tag{4}$$

In analogy with BV, here, $Eu = E^a u + E^s u$ is the Radon–Nykodým decomposition of Eu with respect to to $\mathscr{L}^d$, $E^c u = E^s u \llcorner (S_u \setminus J_u)$ is the Cantor part of Eu, and $E^j u = E^s u \llcorner J_u$ is the jump part where $a \odot b := \frac{1}{2}(a \otimes b + b \otimes a)$ for vectors $a, b \in \mathbb{R}^n$. Just as for the BV-theory, it is shown that $|Eu|$-almost every point is either an approximate continuity point or an approximate jump point, i.e.,

$$|Eu|(S_u \setminus J_u) = 0.$$

Other interesting properties such as approximate differentiability are also discussed in [2] (though this was formerly established by Hajłasz [24] in a more general framework). Gathering these results into a single statement results in the following structure theorem for functions of bounded deformation: if $u \in \mathrm{BD}(\Omega)$, then one may split Eu into the mutually singular measures

$$Eu = \mathrm{sym}(\nabla u)\,\mathscr{L}^n \,+\, E^c u \,+\, (u^+ - u^-) \odot \nu_u \,\mathcal{H}^{n-1} \llcorner J_u,$$

where $\nabla u : \Omega \to \mathbb{R}^n \otimes \mathbb{R}^n$ is the approximate gradient map of u and J_u is indeed an $\mathcal{H}^{n-1}$-countably rectifiable set.

Having revised the elements behind the slicing and fine properties of BV and BD, we are now in position to give a brief account of the most important results presented in this work. In order to keep this preliminary exposition accurate and simple, we shall for now focus on the case when $\mathcal{A}$ is a first-order operator ($k = 1$). Our main assumption on $\mathcal{A} : \mathrm{C}^\infty(\mathbb{R}^n;\mathbb{R}^N) \to \mathrm{C}^\infty(\mathbb{R}^n;\mathbb{R}^M)$ is that it is an *elliptic* operator, i.e., there exists a positive constant c such that

$$|\mathbb{A}(\xi)[v]| \geq c|\xi||v|, \qquad \mathbb{A}(\xi) := \sum_{j=1}^n \xi_j A_j. \tag{5}$$

A vector $w \in \mathbb{R}^M$ with $\mathrm{rank}_{\mathcal{A}}$-one satisfies the following identity in Fourier space: there exists a pair $(\xi, e) \in \mathbb{R}^n \times \mathbb{R}^N$ such that

$$w \cdot \mathbb{A}(\eta)[v] = (\xi \cdot \eta)(e \cdot v) \qquad \text{for all } \eta \in \mathbb{R}^n,\ v \in \mathbb{R}^N.$$

Any such pair (ξ, e) associated to a $\mathrm{rank}_{\mathcal{A}}$-one tensor is said to belong to « $\partial\sigma(\mathcal{A})$ », the *directional spectrum* of $\mathcal{A}$. Notice that, for the gradient and symmetric gradient, we have $\partial\sigma(D) = \mathbb{R}^n \times \mathbb{R}^N$ and $\partial\sigma(E) = \{\, (\xi,\xi) : \xi \in \mathbb{R}^n \,\}$ respectively. These are precisely the pairs of directions and coordinates where the one-dimensional slicing holds for each of this operators; compare this to the identities (3), (4) and the fact that both $e \otimes \xi$, $\xi \otimes \xi$ are *rank-one* tensors.

In Theorem 2.1 we show that (5) and the mixing algebraic condition

$$\bigcap_{\substack{\pi \leq \mathbb{R}^n \\ \dim(\pi) = n-1}} \mathrm{span}\,\{\, \mathrm{Im}\,\mathbb{A}(\xi) : \xi \in \pi \,\} = \{0\} \tag{6}$$

are equivalent to the following slicing representation of $\mathrm{BV}^{\mathcal{A}}(\Omega)$: a function u belongs to $\mathrm{BV}^{\mathcal{A}}(\Omega)$ if and only if, for every pair $(\xi, e) \in \partial\sigma(\mathcal{A})$, the one-dimensional sections

$$\begin{aligned} u^e_{y,\xi} \colon \Omega^\xi_y &\longrightarrow \mathbb{R} \\ t &\longmapsto u(y + t\xi) \cdot e \end{aligned}$$

belong to $\mathrm{BV}(\Omega^\xi_y)$ for $\mathcal{H}^{n-1}$-almost every $y \in \pi_y$ and

$$\int_{\pi_\xi} |Du^e_{y,\xi}|(\Omega^y_\xi)\, \mathrm{d}\mathcal{H}^{n-1}(y) < \infty.$$

Furthermore, in Theorem 2.2 we show that $\mathcal{A}u$ satisfies a one-dimensional *structure theorem* in the following sense: if w is a $\mathrm{rank}_{\mathcal{A}}$-one vector with an associated spectral pair $(\xi, e) \in \partial\sigma(\mathcal{A})$, then

$$(w \cdot \mathcal{A}^\sigma u) = \int_{\pi_\xi} \left(\int D^\sigma u^e_{y,\xi} \right) \mathrm{d}\mathcal{H}^{n-1}(y), \quad \text{for all } \sigma \in \{a, c, j\}.$$

By exploiting this structural property, we are able to give a purely geometric proof that $|\mathcal{A}u|$ vanishes when projected on purely unrectifiable σ-finite $(n-1)$-dimensional sets (see Corollary 2.1): if $u \in \mathrm{BV}^{\mathcal{A}}(\Omega)$ and $B \subset \Omega$ is a Borel set satisfying

$$\mathcal{H}^{n-1}(\mathbf{p}_\xi(B)) = 0 \quad \text{for } \mathcal{H}^{n-1}\text{-almost every } \xi \in \mathbf{S}^{n-1},$$

where $\mathbf{p}_\xi : \mathbb{R}^n \to \pi_\xi$ is the canonical orthogonal projection, then $|\mathcal{A}u|(B) = 0$ and in particular the Besicovitch–Federer Theorem implies the measure theoretical estimate

$$|\mathcal{A}u| \ll \mathcal{I}^{n-1} \ll \mathcal{H}^{n-1}.$$

This accounts for rectifiability estimates of $\mathcal{A}$-gradient measures (see also [6] for a proof of this result that relies on harmonic analysis techniques for elliptic operators).

We also show that, for all $\mathcal{A}$ satisfying (5)-(6), most of the well-known fine properties of the BV-theory extend to $u \in \mathrm{BV}^{\mathcal{A}}$ and its parts $\mathcal{A}^a u, \mathcal{A}^c u, \mathcal{A}^j u$. Let us give a brief account of the most relevant ones, all of which are contained in Theorem 2.3 (for $k = 1$) and Theorem 2.6 (for arbitrary order k); see also Theorem 2.4. Following the same principles of the BV-splitting, one can decompose $\mathcal{A}u$, with respect to the n-dimensional Lebesgue, into a singular part $\mathcal{A}^s u$ and an absolutely continuous part $\mathcal{A}^a u = A(\nabla u)\mathscr{L}^n$ where A is a linear map acting on the approximate differential ∇u of u (this follows from a result for elliptic operators of Alberti, Bianchini and Crippa [1, Thm. 3.4] and an observation of Raita [27]). A priori, one can trivially split $\mathcal{A}^s u$ into a Cantor part, a *diffuse discontinuous* part, and a jump part as

$$\begin{aligned} \mathcal{A}^s u &= \mathcal{A}^c u + \mathcal{A}^d u + \mathcal{A}^j u \\ &:= \mathcal{A}^s u \llcorner (\Omega \setminus S_u) \,+\, \mathcal{A}^s u \llcorner (S_u \setminus J_u) \,+\, \mathcal{A}^s u \llcorner J_u. \end{aligned} \tag{7}$$

In [8] the author and Skorobogatova showed that J_u is $\mathcal{H}^{n-1}$-countably recitifiable with a measure-theoretic orientation normal $\nu_u : J_u \to \mathbf{S}^{n-1}$ (see also [26]), and that the density of the jump part is characterized as

$$\mathcal{A}^j u = (u^+ - u^-) \otimes_{\mathbb{A}} \nu_u \, \mathcal{H}^{n-1} \llcorner J_u, \qquad v \otimes_{\mathbb{A}} \xi := \mathbb{A}(\xi)[v].$$

Hence, in the $\mathrm{BV}^{\mathcal{A}}$-setting the representation of the jump part differs only in that the density $(u^+ - u^-) \otimes_{\mathbb{A}} \nu_u$ replaces the classical BV-jump density $(u^+ - u^-) \otimes \nu_u$. This and other *soft* fine properties have been also discussed in [8] in a slightly more general framework.

In the realm of continuity properties and the classical structure theorem, we show that there is *essentially* only one type of discontinuities by proving the estimate

$$|\mathcal{A}u|(S_u \setminus J_u) = 0. \tag{8}$$

This implies that only jump-type discontinuities are recorded by the total variation measure $|\mathcal{A}u|$. Notice that, then, the diffuse discontinuous part $\mathcal{A}^d u$ becomes a superfluous term of (7). Therefore, from (8) we also conclude that $\mathcal{A}u$ satisfies the classical structure theorem decomposition

$$\mathcal{A}u \,=\, A(\nabla u)\,\mathscr{L}^n \,+\, \mathcal{A}^s u \llcorner (\Omega \setminus S_u) \,+\, (u^+ - u^-) \otimes_{\mathbb{A}} \nu_u \, \mathcal{H}^{n-1} \llcorner J_u, \tag{9}$$

where $\nabla u : \Omega \to \mathbb{R}^M \otimes \mathbb{R}^n$ is the approximate differential of u and A is a linear map that is expressed solely in terms of the principal symbol $\mathbb{A}$. Parting from (8) and following by verbatim the proof of [2, Thm. 6.1], one may upgrade (8) to the following statement: the set of Lebesgue discontinuity points of u that are not jump points is negligible for all $\mathcal{A}$-gradient measures, i.e.,

$$|\mathcal{A}v|(S_u \setminus J_u) = 0 \quad \text{for all } u, v \in \mathrm{BV}^{\mathcal{A}}(\Omega).$$

In the last section, we address the concepts and main results for operators of arbitrary order under the additional assumption of complex-ellipticity. We introduce a linearization principle that allows us to transform the PDE measure-constraint $\mathcal{A}u$ into a first-order PDE constraint $d\mathcal{A}$ over $\nabla^{k-1}u$. This method seems to be new (see also [8]) in the study of general elliptic operators and might be interesting in its own right.

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2. Statement of the main results

In its most general setting, we shall consider a k^{th} order homogeneous linear partial differential operator with constant coefficients operator acting on spaces of smooth functions as

$$\mathcal{A} : \mathrm{C}^\infty(\mathbb{R}^n; V) \to \mathrm{C}^\infty(\mathbb{R}^n; W).$$

Here, V and W are finite-dimensional euclidean spaces, of respective dimensions N and M (up to a linear isomorphism the reader may think of V and W as $\mathbb{R}^N$ and $\mathbb{R}^M$ respectively). More precisely, we shall consider operators $\mathcal{A} : \mathrm{C}^\infty(\Omega; V) \to \mathrm{C}^\infty(\Omega; W)$ acting on smooth maps $u \in \mathrm{C}^\infty(\Omega; V)$ as

$$\mathcal{A}u = \sum_{|\alpha|=k} A_\alpha \partial^\alpha u \in \mathrm{C}^\infty(\Omega; W),$$

where the coefficients $A_\alpha \in W \otimes V^* \cong \mathrm{Lin}(V; W)$ are constant tensors.

2.1. **Slicing of first-order operators.** We commence by discussing the so-called *slicing* theory, which extends the known theory for gradients and symmetric gradients [2, 9].

In all that follows we consider $\Omega \subset \mathbb{R}^n$ to be an open set with Lipschitz boundary. For a non-zero vector $\xi \in \mathbb{R}^n$, we write

$$\pi_\xi = \{\, \eta \in \mathbb{R}^n : \xi \cdot \eta = 0 \,\}$$

to denote the orthogonal hyper-plane to ξ passing through the origin. For a Borel set $B \subset \Omega$ and $y \in \pi_\xi$, we define the one-dimensional slice of B in the ξ-direction and passing through $y \in \pi_\xi$ as

$$B_y^\xi = \{\, s \in \mathbb{R} : y + s\xi \in B \,\}.$$

For a given covector $e \in V^*$ and a function $u : \Omega \to V$, we define a function of one variable $u_{y,\xi}^e : \Omega_y^\xi \to \mathbb{R}$ by setting

$$u_{y,\xi}^e(t) := \langle e, u(y + t\xi) \rangle,$$

the one-dimensional section of u^e restricted to the one-dimensional slice Ω_y^ξ.

For stating our results it will be crucial to give a name to those partial derivative operators $\partial_\xi(\ )^e$ that are controlled by $\mathcal{A}$.

Definition 2.1 (Directional spectrum)**.** Let $\mathcal{A} : \mathrm{C}^\infty(\mathbb{R}^n; V) \to \mathrm{C}^\infty(\mathbb{R}^n; W)$ be a first-order homogeneous partial differential operator. We define the directional spectrum of $\mathcal{A}$ as the set of pairs $(\xi, e) \in \mathbb{R}^n \times V^*$ with the following property: there exists a covector $w \in W^*$ such that

$$\langle w, \mathbb{A}(\eta)v \rangle = \langle \xi, \eta \rangle \langle e, \eta \rangle \quad \text{for all } \eta \in \mathbb{R}^n \text{ and all } v \in V. \tag{10}$$

We write $\partial\sigma(\mathcal{A})$ to denote the directional spectrum of $\mathcal{A}$.

The elements of $\partial\sigma(\mathcal{A})$ have the following slicing property:

Proposition 2.1. *Let $\mathcal{A} : \mathrm{C}^\infty(\mathbb{R}^n; V) \to \mathrm{C}^\infty(\mathbb{R}^n; W)$ be a first-order partial differential operator and let u be a function in $\mathrm{BV}^{\mathcal{A}}(\Omega)$. Then, for every $(\xi, e) \in \partial\sigma(\mathcal{A})$ and $w \in W^*$ satisfying*

$$\langle w, \mathbb{A}(\eta)v \rangle = \langle \xi, \eta \rangle \langle e, \eta \rangle \quad \textit{for all } (\eta, v) \in \mathbb{R}^n \times V,$$

it holds that $u_{y,\xi}^e \in \mathrm{BV}(\Omega_y^\xi)$ for $\mathcal{H}^{n-1}$-almost every $y \in \pi_\xi$ and

$$\int_{\pi_\xi} |Du_{y,\xi}^e|(\Omega_y^\xi) \,\mathrm{d}\mathcal{H}^{n-1}(y) \le |\mathcal{A}u|(\mathbb{R}^n) \cdot |w|.$$

Moreover,

$$\langle w, \mathcal{A}u\rangle(B) = \int_{\pi_\xi} Du^e_{y,\xi}(B^\xi_y)\,\mathrm{d}\mathcal{H}^{n-1}(y),$$

$$|\langle w, \mathcal{A}u\rangle|(B) = \int_{\pi_\xi} |Du^e_{y,\xi}|(B^\xi_y)\,\mathrm{d}\mathcal{H}^{n-1}(y),$$

for all Borel sets $B \subset \Omega$.

Notice that the algebraic identity (10) is equivalent to requiring that $\langle w, \mathbb{A}\rangle = \xi \otimes v$, when considering $\mathbb{A}$ as a bi-linear form from $\mathbb{R}^n \times V$ to W. This motivates us to think of such covectors w as some sort of "rank-one vectors" with respect to $\mathcal{A}$. Consider the linear map

$$\begin{aligned} \bar{f}_{\mathcal{A}} \colon \mathbb{R}^n \otimes V &\longrightarrow W \\ \eta \otimes v &\longmapsto \mathbb{A}(\eta)[v] \end{aligned}.$$

For a covector w in W^*, the composition of linear maps $w \circ f_{\mathcal{A}}$ belongs to the tensor space $(\mathbb{R}^n \otimes V)^*$. We introduce the following concept of rank:

Definition 2.2 ($\mathrm{rank}_{\mathcal{A}}$)**.** Let $w \in W^*$, we define

$$\mathrm{rank}_{\mathcal{A}}(w) := \mathrm{rank}(w \circ f_{\mathcal{A}}),$$

where the latter is the canonical rank acting on $(\mathbb{R}^n \otimes V)^*$.

Our first main result is the following BV one-dimensional representation $\mathrm{BV}^{\mathcal{A}}(\Omega)$, which we show to be equivalent to an algebraic mixing condition imposed on the principal symbol (this condition was introduced by Spector and Van Schaftingen [30] as a sufficient condition to establish endpoint $\mathrm{L}^{\frac{n}{n-1},1}$ Lorentz-type estimates for first-order elliptic systems).

Theorem 2.1 (Slicing representation)**.** *Let $\mathcal{A} : \mathrm{C}^\infty(\mathbb{R}^n; V) \to \mathrm{C}^\infty(\mathbb{R}^n; W)$ be a first-order homogeneous linear elliptic differential operator. The following are equivalent:*

(*1*) *The principal symbol of $\mathcal{A}$ satisfies the algebraic mixing property*

$$\bigcap_{\substack{\pi \leq \mathbb{R}^n \\ \dim(\pi)=n-1}} \mathrm{span}\,\{\,\mathbb{A}(\eta)v : \eta \in \pi, v \in V\,\} = \{0\}. \qquad (\mathfrak{m})$$

(*2*) *There exist covectors $w_1, \dots, w_M$ in W^* such that*

$$\mathrm{span}\{w_1, \dots, w_M\} = W^*, \qquad \mathrm{rank}_{\mathcal{A}}(w_i) \leq 1.$$

In particular, there exist directions $\xi_1, \dots, \xi_M$ in $\mathbb{R}^n$ and covectors $e_1, \dots, e_M$ in V^* with the following property: a function u belongs to $\mathrm{BV}^{\mathcal{A}}(\Omega)$ if and only if, for every $i = 1, \dots, M$, the one-dimensional sections $u^{e_i}_{y,\xi_i}$ satisfy

$$u^{e_i}_{y,\xi_i} \in \mathrm{BV}(\Omega^{\xi_i}_y) \text{ for } \mathcal{H}^{n-1}\text{-almost every } y \in \pi_{\xi_i}$$

and

$$\int_{\pi_{\xi_i}} |Du^{e_i}_{y,\xi_i}|(\Omega^{\xi_i}_y)\,\mathrm{d}\mathcal{H}^{n-1}(y) < \infty.$$

Remark 2.1. For $\Omega = \mathbb{R}^n$, the slicing representation theorem also holds if one dispenses with the ellipticity assumption. Effectively, the ellipticity of $\mathcal{A}$ is only used to find an extension operator $E : \mathrm{BV}^{\mathcal{A}}(\Omega) \to \mathrm{BV}^{\mathcal{A}}(\mathbb{R}^n)$, which allows for localization arguments up to the boundary.

Remark 2.2. If $\mathcal{A}$ is a first-order elliptic operator satisfying the mixing property $(\mathfrak{m})$, then one can show that every coordinate in $\mathbb{R}^n$ (or V^*) is the first (or second component) of a spectral pair, i.e.,

$$\mathrm{Proj}_{\mathbb{R}^n} \partial\sigma(\mathcal{A}) = \mathbb{R}^n, \qquad \mathrm{Proj}_{V^*} \partial\sigma(\mathcal{A}) = V^*.$$

A related statement holds for k^{th} order elliptic operators satisfying the generalized mixing property $(\mathfrak{m}_k)$ defined below.

In analogy with BV, we decompose $\mathcal{A}u = \mathcal{A}^a u + \mathcal{A}^s u$, where $\mathcal{A}^a u$ is the absolutely continuous part of $\mathcal{A}u$ with respect to $\mathscr{L}^n$ and $\mathcal{A}^s u$ is the singular part of $\mathcal{A}u$ with respect to $\mathscr{L}^n$. We also define the Cantor and jump parts of $\mathcal{A}^s u$ as follows:

$$\mathcal{A}^c u := \mathcal{A}^s u \llcorner (\Omega \setminus S_u), \qquad \mathcal{A}^j u := \mathcal{A}^s u \llcorner J_u.$$

Our second main result is a one-dimensional structure theorem for first-order elliptic operators that satisfy the mixing property $(\mathfrak{m})$:

Theorem 2.2 (One-dimensional structure theorem). *Let $\mathcal{A} : \mathrm{C}^\infty(\mathbb{R}^n; V) \to \mathrm{C}^\infty(\mathbb{R}^n; W)$ be a first-order linear elliptic partial differential operator satisfying the mixing condition $(\mathfrak{m})$. Let $(\xi, e) \in \partial\sigma(\mathcal{A})$ and let $w \in W^*$ satisfy*

$$\langle w, \mathbb{A}(\eta)v\rangle = \langle \xi, \eta\rangle\langle e, v\rangle \quad \text{for all } \eta \in \mathbb{R}^n \text{ and all } v \in V.$$

Then, for every $u \in \mathrm{BV}^{\mathcal{A}}(\Omega)$, we have

$$\left.\begin{aligned} \langle w, \mathcal{A}^\sigma u\rangle &= \int_{\pi_\xi} D^\sigma u^e_{y,\xi} \,\mathrm{d}\mathcal{H}^{n-1}(y) \\ |\langle w, \mathcal{A}^\sigma u\rangle| &= \int_{\pi_\xi} |D^\sigma u^e_{y,\xi}| \,\mathrm{d}\mathcal{H}^{n-1}(y) \end{aligned}\right. \qquad \text{for all } \sigma \in \{a, c, j\},$$

as measures over Ω.

2.2. **Rectifiability of $\mathcal{A}$-gradient measures.** A closely related (weaker) mixing condition was introduced in [6] (see also [4])—in the context of $\mathcal{A}$-free measures— as a sufficient condition to establish $\mathcal{H}^{n-1}$-rectifiability of PDE-constrained vector measures. In our context, the results contained in [6] imply that $|\mathcal{A}u| \ll \mathcal{I}^{n-1} \ll \mathcal{H}^{n-1}$ as measures.[1] Theorem 2.1 and Remark 2.2 allow us to give a purely geometric proof of the following dimensional and rectifiability results (which we state for operators of arbitrary order):

Corollary 2.1 (Rectifiability of $\mathcal{A}$-gradients). *Let $\mathcal{A} : \mathrm{C}^\infty(\mathbb{R}^n; V) \to \mathrm{C}^\infty(\mathbb{R}^n; W)$ be a homogeneous linear elliptic differential operator satisfying the the mixing condition.*

$$\bigcap_{\substack{\pi \leq \mathbb{R}^n \\ \dim(\pi) = n-1}} \mathrm{span}\left\{ \mathbb{A}^k(\eta)[v] : \eta \in \pi, v \in V \right\} = \{0\}. \tag{$\mathfrak{m}_k$}$$

If u is a function in $\mathrm{BV}^{\mathcal{A}}(\Omega)$ and $B \subset \Omega$ is a Borel set satisfying

$$\mathcal{H}^{n-1}(\mathbf{p}_\xi(B)) = 0 \quad \text{for } \mathcal{H}^{n-1}\text{-almost every } \xi \in \mathbf{S}^{n-1},$$

where $\mathbf{p}_\xi : \mathbb{R}^n \to \pi_\xi$ is the canonical linear orthogonal projection onto π_ξ, then

$$|\mathcal{A}u|(B) = 0.$$

In particular, $|\mathcal{A}u| \ll \mathcal{I}^{n-1} \ll \mathcal{H}^{n-1}$ as measures and

$$\dim_{\mathcal{H}}(|\mathcal{A}u|) \geq n - 1.$$

[1]Here, $\mathcal{I}^{n-1}$ is the $(n-1)$-dimensional Integral-geometric-measure.

2.3. **Structural and fine properties for first-order elliptic operators.** We are now in position to state the main fine properties result, which is key towards establishing a reliable variational theory for functions of bounded $\mathcal{A}$-variation. It establishes that $\mathcal{A}$-gradients vanish when projected on purely unrectifiable σ-finite $(n-1)$-dimensional sets. It also says that, essentially all approximate discontinuities are jump-type discontinuities and that all Cantor points are Lebesgue continuity points.

Theorem 2.3 (Structural and fine properties)**.** *Let $\mathcal{A} : \mathrm{C}^\infty(\mathbb{R}^n;V) \to \mathrm{C}^\infty(\mathbb{R}^n;W)$ be a first-order homogeneous linear elliptic operator satisfying*

$$\bigcap_{\substack{\pi \leq \mathbb{R}^n \\ \dim(\pi)=n-1}} \operatorname{span}\{\, \mathbb{A}(\eta)[v] : \eta \in \pi_\xi, v \in V \,\} = \{0\}.$$

If $u \in \mathrm{BV}^{\mathcal{A}}(\Omega)$, then $\mathcal{A}u$ decomposes into mutually singular measures as

$$\begin{aligned} \mathcal{A}u &= \mathcal{A}^a u + \mathcal{A}^c u + \mathcal{A}^j u \\ &= \mathcal{A}^a u + \mathcal{A}^s u \llcorner (\Omega \setminus S_u) + \mathcal{A}^s u \llcorner J_u\,, \end{aligned}$$

and the following properties hold:

(*i*) *$\mathcal{A}^a u = A(\nabla u)\,\mathscr{L}^n$, where $\nabla u : \Omega \to V \otimes \mathbb{R}^n$ is the approximate gradient of u and*

$$A(P) \coloneqq \sum_{i=1}^n A_i P[\mathbf{e}_i] \quad P \in V \otimes \mathbb{R}^n.$$

(*ii*) *The jump set J_u is $\mathcal{H}^{n-1}$-countably rectifiable and the jump part is characterized by the identity of measures*

$$\mathcal{A}^j u = \mathbb{A}(\nu_u)[u^+ - u^-]\,\mathcal{H}^{n-1} \llcorner J_u,$$

where $(u^+, u^-, \nu_u) : J_u \to V \times V \times \mathbf{S}^{n-1}$ is the triplet Borel map associated to the jump discontinuities on J_u (see Definition 3.1).

(*iii*) *The set $S_u \setminus J_u$ is purely $\mathcal{H}^{n-1}$-unrectifiable, $J_u \subset \Theta_u$, and*

$$\mathcal{H}^{n-1}(\Theta_u \setminus J_u) = 0.$$

(*iv*) *The Cantor part vanishes on sets that are σ-finite with respect to $\mathcal{H}^{n-1}$:*

$$E \subset \Omega \text{ Borel with } \mathcal{H}^{n-1}(E) < \infty \quad \Longrightarrow \quad |\mathcal{A}^c u|(E) = 0.$$

In particular,

$$|\mathcal{A}^c u|(\Theta_u) = 0.$$

(*v*) *The set of Lebesgue discontinuity points that are not jump points is negligible for all $\mathcal{A}$-gradient measures, that is,*

$$|\mathcal{A}v|(S_u \setminus J_u) = 0 \quad \text{for all } v \in \mathrm{BV}^{\mathcal{A}}(\Omega).$$

In particular,

$$|\mathcal{A}u|(S_u \setminus J_u) = 0.$$

Remark 2.3. Property (i) holds for elliptic operators (this follows from a result of Alberti, Bianchini and Crippa [1, Thm. 3.4] and an observation of Raita [27]). Property (ii) and that $(S_u \setminus J_u)$ is $\mathcal{H}^{n-1}$-purely unrectifiable also hold for complex-elliptic operators (see [8, Thm. 1.2]).

If we restrict ourselves to operators $\mathcal{A}$ acting on scalar functions $u : \Omega \to \mathbb{R}$, we show that $\mathcal{A}$ is equivalent to a rather simple elliptic operator (see Lemma 3.1) This allows us to state a version of the Structure Theorem for first-order elliptic operators acting on scalar maps, where we can dispense with any of the mixing assumptions:

Theorem 2.4. *Let $\mathcal{A} : \mathrm{C}^\infty(\mathbb{R}^n) \to \mathrm{C}^\infty(\mathbb{R}^n; V)$ be a first-order elliptic partial differential operator and let u be a function in $\mathrm{BV}^{\mathcal{A}}(\Omega)$. Then, $\mathcal{A}u$ decomposes into mutually singular measures as*

$$\mathcal{A}u = A(\nabla u)\,\mathscr{L}^n + \mathcal{A}^c u + \mathbb{A}(\nu_u)[u^+ - u^-]\,\mathcal{H}^{n-1} \llcorner J_u,$$

and u satisfies the properties (i)-(v) in Theorem 2.3.

2.4. **Statements for higher-order operators.** For a k^{th} order elliptic operator $\mathcal{A}$, the classical Calderón–Zygmund theory implies the local embedding

$$\mathrm{BV}^{\mathcal{A}}_{\mathrm{loc}}(\Omega) \hookrightarrow \mathrm{W}^{k-1,p}_{\mathrm{loc}}(\Omega; V), \qquad 1 \le p < \frac{n}{n-1}. \tag{11}$$

Since the fine properties of Sobolev spaces are already well-understood (see for instance [18, Sec. 4.8]), we shall only focus on the fine properties of the $(k-1)^{\text{th}}$-order derivative map $\nabla^{k-1}u \in \mathrm{L}^1(\Omega; V \otimes E_{k-1}(\mathbb{R}^n))$. Here, $E_m(\mathbb{R}^n)$ is the space of m^{th} order symmetric tensors on $\mathbb{R}^n$. In analogy with the first-order case, we shall now give a name to those pairs $(\xi, E) \in \mathbb{R}^n \times (V^* \otimes E_{k-1}(\mathbb{R}^n))$ such that

$$\langle w, \mathcal{A}u\rangle = \partial_\xi(\langle E, D^{k-1}u\rangle) \quad \text{for some } w \in W^*.$$

Definition 2.3 (Tensor spectrum)**.** Let $\mathcal{A} : \mathrm{C}^\infty(\mathbb{R}^n; V) \to \mathrm{C}^\infty(\mathbb{R}^n; W)$ be a k^{th} order homogeneous linear differential operator. The tensor spectrum of $\mathcal{A}$ is defined as the set of pairs $(\xi, E) \in \mathbb{R}^n \times (V^* \otimes E_{k-1}(\mathbb{R}^n))$ with the following property: there exists $w^* \in W$ such that

$$\langle w, \mathbb{A}^k(\eta)v\rangle = \langle \xi, \eta\rangle\langle E, v \otimes^{k-1} \eta\rangle \quad \text{for all } \eta \in \mathbb{R}^n,\ v \in V. \tag{12}$$

We write $(\xi, E) \in \partial\sigma(\mathcal{A})$.

For a higher-order operator, the $\mathrm{rank}_{\mathcal{A}}$ that we shall consider does not extending directly from the one for first-order operators. As already hinted above, this stems from the fact that we want to understand the slicing properties of $\nabla^{k-1}u$ as opposed to that ones of u. The rigorous definition of $\mathrm{rank}_{\mathcal{A}}$ requires of a linearization of the operator $\mathcal{A}$, but that will be postponed to Section 8. To make sense of the next statements, we say that $\mathrm{rank}_{\mathcal{A}}(w) = 1$ if and only if $w \in W^*$ satisfies (12) for a non-trivial pair $(\xi, E) \in \partial\sigma(\mathcal{A})$.

In order to establish the one-dimensional structure theorem and the fine properties for higher order operators we require $\mathcal{A}$ to be complex-elliptic: there exists a positive constant $c > 0$ such that

$$|\mathbb{A}^k(\xi)[v]| \ge c|\xi|^k|v| \quad \text{for all } \xi \in \mathbb{C}^n \text{ and all } v \in \mathbb{C} \otimes V.$$

We are now in position to state the general version of the slicing theorem:

Theorem 2.5. *Let $\mathcal{A} : \mathrm{C}^\infty(\mathbb{R}^n; V) \to \mathrm{C}^\infty(\mathbb{R}^n; W)$ be a k^{th} order homogeneous linear differential complex-elliptic operator. The following are equivalent:*

(*1*) *$\mathcal{A}$ satisfies the mixing condition*

$$\bigcap_{\substack{\pi \le \mathbb{R}^n \\ \dim(\pi) = n-1}} \operatorname{span}\left\{ \mathbb{A}^k(\eta)v : \eta \in \pi, v \in V \right\} = \{0\}.$$

(2) *There exist covectors $w_1, \dots, w_M \in W^*$ such that*

$$\operatorname{span}\{w_1, \dots, w_M\} = W^*, \qquad \operatorname{rank}_{\mathcal{A}}(w_i) \le 1.$$

In particular, there exist directions $\xi_1, \dots, \xi_M$ in $\mathbb{R}^n$ and tensors $E_1, \dots, E_M$ in $V^* \otimes E_{k-1}(\mathbb{R}^n)$ with the following property: A function u belongs to $\mathrm{BV}^{\mathcal{A}}(\Omega)$ if and only if $U := \nabla^{k-1}u \in \mathrm{W}^{1,1}(\Omega; V)$ and, for every $i = 1, \dots, M$,

$$U^{E_i}_{y,\xi_i} \in \mathrm{BV}(\Omega^{\xi_i}_y) \text{ for } \mathcal{H}^{n-1}\text{-almost every } y \in \pi_{\xi_i}$$

and

$$\int_{\pi_{\xi_i}} |DU^{E_i}_{y,\xi_i}|(\Omega^{\xi_i}_y) \, \mathrm{d}\mathcal{H}^{n-1}(y) < \infty.$$

Lastly, we state the higher-order version of the fine properties Theorem:

Theorem 2.6. *Let $\mathcal{A} : \mathrm{C}^\infty(\mathbb{R}^n; V) \to \mathrm{C}^\infty(\mathbb{R}^n; W)$ be a homogeneous k^{th} order linear complex-elliptic operator satisfying*

$$\bigcap_{\substack{\pi \le \mathbb{R}^n \\ \dim(\pi) = n-1}} \operatorname{span}\Big\{ \mathbb{A}^k(\eta)[v] : \eta \in \pi, v \in V \Big\} = \{0\}.$$

If $u \in \mathrm{BV}^{\mathcal{A}}(\Omega)$ and $U := \nabla^{k-1}u$, then $\mathcal{A}u$ decomposes into mutually singular measures as

$$\begin{aligned} \mathcal{A}u &= \mathcal{A}^a u + \mathcal{A}^c u + \mathcal{A}^j u \\ &:= \mathcal{A}^a u + \mathcal{A}^s u \llcorner (\Omega \setminus S_U) + \mathcal{A}^s u \llcorner J_U \end{aligned}$$

and the following properties hold

(i) *$\mathcal{A}^a u = A(\nabla^k u) \mathscr{L}^n$, where $\nabla^k u : \Omega \to V \otimes E_k(\mathbb{R}^n)$ is the approximate gradient of U and*

$$A(F) := \sum_{|\alpha| = k} \frac{1}{\alpha!} A_\alpha\big[\langle F, \mathbf{e}_1^{\alpha_1} \odot \cdots \odot \mathbf{e}_n^{\alpha_n} \rangle\big], \qquad F \in V \otimes E_k(\mathbb{R}^n).$$

(ii) *The jump set $J_U \subset \Theta_u$ is countably $\mathcal{H}^{n-1}$-rectifiable and the jump part is characterized by the identity of measures*

$$\mathcal{A}^j u = \mathbb{A}^k(\nu_u)[\![\nabla^{k-1}u]\!] \, \mathcal{H}^{n-1} \llcorner J_U,$$

where

$$[\![F]\!] := \langle F^+ - F^+, \otimes^{k-1}\nu_u \rangle.$$

(iii) *The Cantor part $\mathcal{A}^c u$ vanishes on sets that are σ-finite with respect to $\mathcal{H}^{n-1}$, that is,*

$$E \subset \Omega \text{ Borel with } \mathcal{H}^{n-1}(E) < \infty \quad \Longrightarrow \quad |\mathcal{A}^c u|(E) = 0.$$

(iv) *The set $S_U \setminus \Theta_u$ is purely $\mathcal{H}^{n-1}$-unrectifiable and*

$$\mathcal{H}^{n-1}(\Theta_u \setminus J_U) = 0.$$

(v) *The set of Lebesgue discontinuity points that are not jump points is negligible for all $\mathcal{A}$-gradient measures, that is,*

$$|\mathcal{A}v|(S_U \setminus J_U) = 0 \quad \text{for all } v \in \mathrm{BV}^{\mathcal{A}}(\Omega).$$

3. Preliminaries

3.1. **Disintegration into one-dimensional sections.** We begin by recalling a basic concept of the slicing theory for Radon measures. Let $\Omega \subset \mathbb{R}^n$ be an open set and let $\xi \in \mathbb{R}^n$ be a non-zero vector and. Suppose that for $\mathcal{H}^{n-1}$-almost every

$$y \in \Omega^\xi := \left\{ z \in \pi_\xi : \Omega^\xi_z \neq \emptyset \right\},$$

we are given a measure μ_y in $\mathcal{M}(\Omega^\xi_y)$. Further assume that, for every $\varphi \in \mathrm{C}(\mathbb{R})$, the assignment

$$y \mapsto \int_{\Omega^\xi_y} \varphi(t) \,\mathrm{d}\mu_y(t)$$

is well-defined and Borel measurable $\mathcal{H}^{n-1}$-almost everywhere on Ω^ξ. Lastly, assume that

$$\int_{\Omega^\xi} |\mu_y|(\Omega^\xi_y) \,\mathrm{d}\mathcal{H}^{n-1}(y) < \infty.$$

Then, for every Borel set $B \subset \Omega$, the assignment $y \mapsto \mu_y(B^\xi_y)$ is Borel $\mathcal{H}^{n-1}$-measurable on Ω^ξ_y and the set function $\lambda : \mathfrak{B}(\Omega) \to \mathbb{R}$ defined as

$$\lambda(B) := \int_{\Omega^y} \mu_y(\Omega^\xi_y) \,\mathrm{d}\mathcal{H}^{n-1} \quad \text{for all Borel sets } B \subset \Omega,$$

is a bounded Radon measure on Ω, which we shall denote by

$$\int_{\Omega^\xi} \mu_y \,\mathrm{d}\mathcal{H}^{n-1}(y).$$

In this case we say that λ is disintegrated by $\mathcal{H}^{n-1}$ into one-dimensional sections μ_y. By a density argument, it is easy to show that the total variation measure $|\lambda|$ coincides with the measure

$$\int_{\Omega^\xi} |\mu_y| \,\mathrm{d}\mathcal{H}^{n-1}.$$

3.2. **The jump set and approximate continuity.** Next, we give rigorous definitions and introduce notation belonging to the classical *fine properties* theory. We begin by recalling the formal definitions of the approximate jump set and points of approximate continuity.

Definition 3.1 (Approximate jump)**.** Let $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega;\mathbb{R}^M)$. We say that a point x is an *approximate jump point* of u ($x \in J_u$) if there exist distinct vectors $a, b \in \mathbb{R}^M$ and a direction $\nu \in \mathbf{S}^{n-1}$ satisfying

$$\begin{cases} \lim\limits_{r\downarrow 0} \fint_{B^+_r(x,\nu)} |u(y) - a| \,\mathrm{d}y = 0, \\ \lim\limits_{r\downarrow 0} \fint_{B^-_r(x,\nu)} |u(y) - b| \,\mathrm{d}y = 0, \end{cases} \tag{13}$$

where $B^\pm_r(x,\nu) := \{ y \in B_r(x) : \pm \langle \nu, y \rangle > 0 \}$ are the ν-oriented half-balls centered at x, where $B_r(x)$ is the open unit ball of radius $r > 0$ and centered at x.

We refer to a, b as the one-sided limits of u at x with respect to the orientation ν. Since the jump triplet (a, b, ν) is well-defined up to a sign in ν and a permutation of (a, b), we shall write $(u^+, u^-, \nu_u) : J_u \to \mathbb{R}^M \times \mathbb{R}^M \times \mathbf{S}^{n-1}$ to denote the triplet Borel map associated to the jump discontinuities on J_u, i.e., $x \in J_u$ if and only if (13) holds with $(a, b, \nu) = (u^+(x), u^-(x), \nu_u(x))$.

We now define what it means for a locally integrable function to be approximately continuous at a given point:

Definition 3.2 (Approximate continuity)**.** Let $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega;\mathbb{R}^M)$ and let $x \in \Omega$. We say that u has an *approximate limit* $z \in \mathbb{R}^M$ at x if

$$\lim_{r\downarrow 0} \fint_{B_r(x)} |u(y) - z| \,\mathrm{d}y = 0.$$

The set of points $S_u \subset \Omega$ is called the *approximate discontinuity set*.

3.3. **Symbolic calculus.** In stating our results it will be fundamental to recall a couple of basic definitions for partial differential operators. For a Schwartz function $u \in \mathcal{S}(\mathbb{R}^n;V)$, the Fourier transform applied to $\mathcal{A}u$ gives

$$\widehat{\mathcal{A}u}(\xi) = (2\pi\mathrm{i})^k \mathbb{A}^k(\xi)[\widehat{u}(\xi)], \qquad \mathbb{A}^k(\xi) := \sum_{i=1} \xi^\alpha A_\alpha, \quad \xi^\alpha := \xi_1^{\alpha_1} \cdot \dots \cdot \xi_n^{\alpha_n}.$$

The homogeneous polynomial $\mathbb{A}^k : \mathbb{R}^n \to W \otimes V^*$ is called the *principal symbol* associated to $\mathcal{A}$ (when $k = 1$ we shall simply write $\mathbb{A} = \mathbb{A}^1$). The *image cone* of $\mathcal{A}$ (which contains all $\mathcal{A}$-gradients in Fourier space) and the *essential range* of $\mathcal{A}$ are respectively defined as

$$\mathrm{I}_{\mathbb{A}} := \bigcup_{\xi\in\mathbb{R}^n} \operatorname{Im}\mathbb{A}(\xi), \qquad W_{\mathcal{A}} := \operatorname{span}\mathrm{I}_{\mathbb{A}} \subset W.$$

A simple Fourier transform argument (e.g., Sec. 2.5 in [7]) shows that

$$\mathcal{A}u(x) \in W_{\mathcal{A}} \quad \text{for all } u \in \mathrm{C}_c^\infty(\Omega;V).$$

Thus, by a standard approximation argument (e.g., Theorem 1.3 in [5]) we find that $\mathrm{BV}^{\mathcal{A}}(\Omega) \subset \mathrm{L}^1(\Omega;W_{\mathcal{A}})$. One of the main structural assumptions on $\mathcal{A}$ will be to assume that it is an elliptic operator in the following sense:

Definition 3.3 (Ellipticity)**.** We say that an operator $\mathcal{A}$ as in (2) is elliptic if there exists a positive constant c such that

$$|\mathbb{A}^k(\xi)[v]| \geq c|\xi|^k|v| \quad \text{for all } (\xi,v) \in \mathbb{R}^n \times V.$$

3.4. **Traces of complex elliptic operators.** The concept of complex-ellipticity defined below was introduced by Smith [28,29] and has recently been shown (see [10]) to be a necessary and sufficient condition for the existence of trace operators on $\mathrm{BV}^{\mathcal{A}}$-spaces when $\mathcal{A}$ is a first-order operators.

Definition 3.4. We say that an operator $\mathcal{A}$ as in (2) is complex-elliptic when the complexification of the principal symbol map $\mathbb{A}^k$ is injective, i.e., if there exists a positive constant c such that

$$|\mathbb{A}^k(\xi)v| \geq c|\xi||v| \qquad \text{for all } \xi \in \mathbb{C}^n \text{ and all } v \in \mathbb{C} \otimes V.$$

Remark 3.1. For first-order elliptic operators, the mixing property is a sufficient condition for complex-ellipticity, i.e.,

$$(\mathfrak{m}) + \text{elliptic} \quad \Longrightarrow \quad \text{complex-elliptic.} \tag{14}$$

However, complex-ellipticity is *not* a sufficient condition for an operator to satisfy the mixing condition (see Examples 10.1 and 10.2).

We shall may make use of the following trace properties for $u \in \mathrm{BV}^{\mathcal{A}}(\Omega)$ when $\mathcal{A}$ is complex-elliptic: If Ω is a Lipschitz domain, then there exists a continuous linear trace operator $\operatorname{tr} : \mathrm{BV}^{\mathcal{A}}(\Omega) \to \mathrm{L}^1(\partial\Omega;\mathcal{H}^{n-1})$, satisfying $u = \operatorname{tr} u$ for all $u \in \mathrm{C}(\overline{\Omega};V) \cap \mathrm{BV}^{\mathcal{A}}(\Omega)$. In particular, there the extension by zero

$$\bar{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega \end{cases}$$

belongs to $\mathrm{BV}^{\mathcal{A}}(\mathbb{R}^n)$ and satisfies $\mathcal{A}u \llcorner \partial\Omega = \mathbb{A}(-\nu_\Omega)[\mathrm{tr}(u)]\,\mathcal{H}^{n-1} \llcorner \partial\Omega$, where ν_Ω is the outer unit normal of Ω.

3.4.1. *Characterization of elliptic equations.* For first-order equations in divergence form, there are no difference among the concepts of ellipticity, complex-ellipticity and the mixing property:

Proposition 3.1 (First-order equations)**.** *Let $n \geq 2$ and let $R \in \mathbb{R}^M \otimes \mathbb{R}^n$. Consider the operator*

$$\mathcal{A}_R u := \mathrm{div}(Ru) = \left(\sum_{j=1}^n R_{ij}\partial_j u\right)_i \qquad i = 1, \ldots, N\,,$$

defined on scalar maps $u : \mathbb{R}^n \to \mathbb{R}$. The following are equivalent:

(1) $\mathrm{rank}(R) \geq n$*,*
(2) $\mathcal{A}_R$ is elliptic,
(3) $\mathcal{A}_R$ is complex-elliptic,
(4) $\mathcal{A}_R$ satisfies the the mixing condition $(\mathfrak{m})$*.*

Proof. We shall see that (a) $\Leftrightarrow$ (b),(c) and $\neg$(a) $\Leftrightarrow$ $\neg$(d). Ellipticity is equivalent to the principal symbol $\mathbb{A}_R(\xi)$ being injective for all $\xi \in \mathbb{R}^n$, which is equivalent to $|M \cdot \xi| > 0$ for all non-zero $\xi \in \mathbb{R}^n$; this shows that (a) $\Leftrightarrow$ (b). However, since R is a tensor with real coefficients the same holds for all $\xi \in \mathbb{C}^n$; this shows (a) $\Leftrightarrow$ (c). Lastly, in this particular case (d) fails if and only if there exists a non-zero $\xi \in \mathbb{R}^n$ such that $M\xi \in R[\eta^\perp]$ for all $\eta \in \mathbb{R}^n$. However, this is equivalent to M not being one-to-one, or equivalently, that $\mathrm{rank}(R) < n$. ☐

We can now use the above result to show that, for first-order equations, ellipticity, complex-ellipticity, and the mixing condition are all equivalent:

Lemma 3.1. *Let $\mathcal{A} : \mathrm{C}^\infty(\mathbb{R}^n) \to \mathrm{C}^\infty(\mathbb{R}^n;\mathbb{R}^M)$ be a first-order elliptic operator. There exists an full-rank tensor $R \in \mathbb{R}^n \otimes \mathbb{R}^n$ such that the operator*

$$\mathcal{A}_R u := \mathrm{div}(Ru), \quad u : \mathbb{R}^n \to \mathbb{R},$$

is complex-elliptic, satisfies the mixing property, and

$$\|\mathcal{A}_R u\|_{\mathrm{L}^1} \leq \|\mathcal{A}u\|_{\mathrm{L}^1} \leq c\|\mathcal{A}_R u\|_{\mathrm{L}^1}$$

for all $u \in \mathrm{C}_c^\infty(\mathbb{R}^n)$.

Proof. Since $\mathcal{A}$ is of of first order, the characteristic polynomial $p_j(\xi)$ corresponding to each scalar operator P_j must be of the form $p_j(\xi) = e_j \cdot \xi$ for some $e_j \in \mathbb{R}^M$. Therefore, the $\mathbb{C}$-ellipticity of $\mathcal{A}$ is equivalent to the family $\{e_1, \ldots, e_M\}$ possessing a basis of an n-dimensional subspace of $\mathbb{R}^M$. Hence, we may find a permutation $\sigma \in S_M$ such that

$$E_\sigma := \{e_{\sigma(1)}, \ldots, e_{\sigma(n)}\} \qquad \text{spans an } n\text{-dimensional space.}$$

The matrix $R_{\ell i} := [e_{\sigma(\ell)}]_i$ has rank n and therefore the operator defined by

$$[\mathcal{A}_R u]_\ell := \sum_{i=1}^n R_{\ell i}\partial_i u = [\mathrm{div}(Ru)]_j, \qquad \ell = 1, ..., n, \quad u \in \mathrm{C}^\infty(\Omega),$$

is complex-elliptic. The L^1-bound $\|\mathcal{A}_R u\|_{\mathrm{L}^1(\Omega)} \leq \|\mathcal{A}u\|_{\mathrm{L}^1(\Omega)}$ follows immediately from the fact that $\mathcal{C} = p \circ \mathcal{A}$ where $p : \mathbb{R}^M \to \mathrm{span}\, E_\sigma$ is the orthogonal projection onto span E_σ. It remains to show that we can again estimate $\mathcal{A}u$ by $\mathcal{C}u$ in L^1. Since E_σ forms a basis of $\mathbb{R}^n$, for each scalar operator P_k we can write

$$P_k u = \sum_{\ell=1}^n c_\ell^k e_{\sigma(\ell)} \cdot \nabla u = \sum_{i,\ell=1}^n c_\ell^k R_{\ell i}\partial_i u = \sum_\ell c_\ell^k [\mathcal{A}_R u]_\ell$$

for some choice of constants c_ℓ^k. From this, we immediately obtain the desired estimate $\|\mathcal{A}u\|_{\mathrm{L}^1(\Omega)} \leq c\|\mathcal{A}_R u\|_{\mathrm{L}^1(\Omega)}$. □

4. Slicing theory for first-order operators

The purpose of this section is to study first-order elliptic operators $\mathcal{A}$ that allow for the measure $\mathcal{A}u \in \mathcal{M}(\Omega; W)$ to be disintegrated into one-dimensional sectional derivatives of its potential u. We aim to show that if a function u belongs to $\mathrm{BV}^{\mathcal{A}}(\Omega)$, then there exist directions $\{\xi_1, \ldots, \xi_r\} \subset \mathbf{S}^{n-1}$ and covectors $\{e_1, \ldots, e_r\} \subset V^*$ such that

$$u_{y,\xi_i}^{e_i} \in \mathrm{BV}(\Omega_y^{\xi_i}) \text{ for } \mathcal{H}^{n-1}\text{-almost every } y \in \Omega^{\xi_i}$$

for every $i = 1, \ldots, r$, and

$$\mathcal{A}u = \sum_{i=1}^r \bigg(\int_{\pi_{\xi_i}} Du_{y,\xi_i}^{e_i} \,\mathrm{d}\mathcal{H}^{n-1}(y) \bigg) P_i, \tag{15}$$

for some family of vectors $\{P_1, ..., P_r\} \in W$. Let us begin by making the following observation about the inherent relationship between the principal symbol of $\mathcal{A}$ and the slicing of $\mathcal{A}u$: Recall that the principal symbol $\mathbb{A}$ induces the linear map

$$\begin{aligned} \bar{f}_{\mathcal{A}} &: \mathbb{R}^n \otimes V \to W, \\ &(\eta \otimes v) \mapsto \mathbb{A}(\eta)[v]. \end{aligned}$$

Let us assume for a moment that $\Omega = \mathbb{R}^n$ and that, for some covector $w \in W^*$, the composition map $w \circ \bar{f}_{\mathcal{A}}$ may be represented by a rank-one tensor (viewed as an element of $(\mathbb{R}^n \otimes V)^*$):

$$w \circ \bar{f}_{\mathcal{A}} = \xi^* \otimes e^* \quad \text{for some } \xi \in \mathbb{R}^n \text{ and } e \in V^*. \tag{16}$$

Here we have identified $\mathbb{R}^n$ with $(\mathbb{R}^n)^*$ by the canonical isomorphism. It follows that

$$\langle w, \bar{f}_{\mathcal{A}}(\eta \otimes v)\rangle = \langle \xi^*, \eta\rangle\langle e^*, v\rangle \qquad \text{for all } (\eta, v) \in \mathbb{R}^n \times V.$$

Now, let $u \in \mathcal{S}(\mathbb{R}^n; V)$. We recall the following idea contained in the proof of Theorem 5 in [30]: applying the Fourier transform to the identity above we deduce the point-wise identity

$$\langle w, \mathbb{A}(\eta)[\widehat{u}(\eta)]\rangle = \langle \xi, \widehat{u}(\eta)\rangle\langle e, \widehat{u}(\eta)\rangle \quad \text{for all } \eta \in \mathbb{R}^n.$$

Inverting the Fourier transform we discover that $\langle w, \mathcal{A}u\rangle = \partial_\xi u^e$. Then, by using the differential π_ξ-independence of the right hand it is relatively simple to show that

$$\langle w, \mathcal{A}u\rangle = \int_{\pi_\xi} Du_{y,\eta}^e \,\mathrm{d}\mathcal{H}^{n-1}(y) \qquad \text{as measures in } \mathcal{M}(\mathbb{R}^n). \tag{17}$$

The key feature of this this equality of measures is that the left hand side is absolutely continuous with respect to $|\mathcal{A}u|$. This, in turn, allows one to apply standard smooth approximation methods to extend it to all functions $u \in \mathrm{BV}^{\mathcal{A}}(\mathbb{R}^n)$. In fact, it is easy to verify that (16) is not only a sufficient, but a necessary condition for (17) to hold on arbitrary functions $u \in \mathrm{BV}^{\mathcal{A}}(\mathbb{R}^n)$.

4.1. **The rank-one property.** Motivated by the previous discussion, we next study those elements $w \in W^*$ for which (17) holds —the rank$_{\mathcal{A}}$-one vectors. It will also be convenient to give a name to the cone of rank$_{\mathcal{A}}$-m covectors: for an integer $0 \le m \le \min\{n, \dim(V)\}$, we write

$$\mathcal{A}_m^{\otimes} := \{\, w \in W^* : \operatorname{rank}_{\mathcal{A}}(w) = m \,\}$$

to denote the cone of rank$_{\mathcal{A}}$-m vectors.

Definition 4.1. We say that $\mathcal{A}$ has the rank-one property if and only if

$$\operatorname{span} \mathcal{A}_1^{\otimes} = (W_{\mathcal{A}})^*,$$

or equivalently,

$$\operatorname{span}\{\mathcal{A}_0^{\otimes} \cup \mathcal{A}_1^{\otimes}\} = W^*.$$

The following lemma shows that the rank-one property is equivalent to the mixing condition.

Lemma 4.1. *The following are equivalent:*

(*1*) *$\mathcal{A}$ satisfies the rank-one property,*

(*2*) *$\mathcal{A}$ satisfies the mixing condition*

$$\bigcap_{\xi \in \mathbf{S}^{n-1}} \operatorname{span}\{\, \operatorname{Im} \mathbb{A}(\eta) : \eta \in \pi_\xi \,\} = \{0_W\}.$$

Proof. First, we show that (1)⇒(2). Let us first fix a $P \in (W_{\mathcal{A}})^*$ with $\operatorname{rank}_{\mathcal{A}}(P) = 1$. By definition there exist vectors $\xi \in \mathbb{R}^n$ and $e \in V$ such that

$$\langle P, \mathbb{A}(\eta)v\rangle = \langle \xi, \eta\rangle\langle e, v\rangle \quad \forall\, (\eta, v) \in \mathbb{R}^n \times V.$$

In particular,

$$P \in \left(\sum_{\eta \in \pi_\xi} \operatorname{Im} \mathbb{A}(\eta) \right)^{\perp}. \tag{18}$$

Now, by assumption, we may find a family $\{P_j\}_{j=1}^r \subset W \cap \{\operatorname{rank}_{\mathcal{A}} = 1\}$ spanning $(W_{\mathcal{A}})^*$. Write $\pi_j = \pi_{\xi_j} \in \operatorname{Gr}(n-1, n)$ to denote the hyper-plane for which (18) holds with $P = P_j$. Next, consider

$$Q \in \bigcap_{\zeta \in \mathbf{S}^{n-1}} \operatorname{span}\{\, \operatorname{Im} \mathbb{A}(\eta) : \eta \in \pi_\zeta \,\} \subset W_{\mathcal{A}}.$$

Since $Q \in \operatorname{span}_{\eta \in \pi_j}\{\operatorname{Im} \mathbb{A}(\eta)\}$ for all $j = \{1, \dots, r\}$, we conclude from (18) with $P = P_j$ that $\langle P_j, Q\rangle = 0$ for all $j = \{1, \dots, r\}$. Recalling that $\{P_j\}_{j=1}^r$ spans ${W_{\mathcal{A}}}^*$, we conclude that Q must be the zero vector and (2) follows.

We now show (2)⇒(1). Fix a direction $\xi \in \mathbf{S}^{n-1}$ and notice that

$$P \in \bigcap_{\eta \in \pi_\xi} \ker \mathbb{A}(\eta)^* \quad \Rightarrow \quad \operatorname{rank}_{\mathcal{A}}(P) \le 1. \tag{19}$$

Here, $\mathbb{A}(\xi)^*$ is the adjoint operator of $\mathbb{A}(\xi)$. Indeed, $\langle Q, \mathbb{A}(\eta)v\rangle = \langle \mathbb{A}(\eta)^*Q, v\rangle = 0$ for all $\eta \in \pi_\xi$ and all $v \in V$. Equivalently, $\langle Q, \mathbb{A}(\eta)v\rangle = \langle \xi, \eta\rangle\langle \mathbb{A}(\eta)^*Q, v\rangle$ for all $\eta \in \pi_\xi$ and $v \in V$. The claim then follows by taking $e^* = \mathbb{A}(\eta)^*Q$ and observing that $Q \circ f_{\mathcal{A}} = \xi^* \otimes e^*$. Now, by an application of De Morgan's laws (for orthogonal complements) we obtain

$$W^* = \operatorname{span}\left\{ \bigcap_{\eta \in \pi_\xi} \ker \mathbb{A}(\eta)^* : \xi \in \mathbf{S}^{n-1} \right\}.$$

The sufficiency then follows from (19). □

4.2. **Proof of the sectional representation theorem.** We begin this section with the proof of Proposition 2.1:

Proof of Proposition 2.1. By assumption we can find $w^* \in W^*$ such that

$$\langle w^*, \mathbb{A}(\eta)\widehat{\varphi}(\eta)\rangle = \langle \xi, \eta\rangle\langle e, \widehat{\varphi}(\eta)\rangle \qquad \text{for every } \eta \in \mathbb{R}^n, \tag{20}$$

and all $\varphi \in \mathrm{C}_c^\infty(\mathbb{R}^n; V)$. Inverting the Fourier transform, we find that this is equivalent to the pointwise identity $\langle w^*, \mathcal{A}\varphi(x)\rangle = \partial_\xi \varphi^e(x)$ for all $x \in \mathbb{R}$. This identity implies that $w^* \circ \mathcal{A} = \partial_\xi(\ \ )^e$ as distributional differential operators; we shall recall this identity as the proof develops.

By Fubini's Theorem and a change of variables, we have

$$\begin{aligned}\int \langle w^*, \mathcal{A}u_\varepsilon(x)\rangle \,\mathrm{d}x &= \int \partial_\xi (u_\varepsilon)^e(x)\,\mathrm{d}x \\ &= \int_{\pi_\xi} \left(\int_{\mathbb{R}} D(u_\varepsilon)^e_{y,\xi}(t)\,\mathrm{d}t\right) \mathrm{d}\mathcal{H}^{n-1}(y).\end{aligned}$$

Clearly we have

$$\int \langle w^*, \mathcal{A}u_\varepsilon(x)\rangle \,\mathrm{d}x \longrightarrow \langle w^*, \mathcal{A}u\rangle(\mathbb{R}^n) \qquad \text{as } \varepsilon \to 0^+.$$

From Young's inequality and the identity above we get the uniform bound

$$|w^*||\mathcal{A}u|(\mathbb{R}^N) \geq |\langle w^*, \mathcal{A}u_\varepsilon \mathscr{L}^n\rangle|(\mathbb{R}^n) \geq \int_{\pi_\xi} V[(u_\varepsilon)^a_{\xi,y}]\,\mathrm{d}\mathcal{H}^{n-1}(y),$$

where, for $h : \mathbb{R} \to \mathbb{R}$, Vh is the pointwise variation of h.

By Arguing as in the proof of [2, Proposition 3.2], passing to the limit $\varepsilon \to 0^+$ we deuce (from the lower semicontinuity properties of the pointwise variation, Fatou's lemma, and standard measure theoretic arguments) that

$$|w^*||\mathcal{A}u|(\mathbb{R}^n) \geq \int_{\pi_\xi} V[\tilde{u}^e_{\xi,y}](\mathbb{R})\,\mathrm{d}\mathcal{H}^{n-1}(y) = \int_{\pi_\xi} |Du^e_{\xi,y}|(\mathbb{R})\,\mathrm{d}\mathcal{H}^{n-1}(y),$$

where « $\tilde{\ }$ » denotes the Lebesgue representative of a locally integrable function. This shows that if $u \in \mathrm{BV}^{\mathcal{A}}(\mathbb{R}^n)$, then $u^e_{\xi,y} \in \mathrm{BV}(\mathbb{R})$ for $\mathcal{H}^{n-1}$-almost every $y \in \pi_\xi$. Now, let $\varphi \in \mathrm{C}_c^\infty(\mathbb{R}^n)$ be an arbitrary test function. Using that the identity $\langle w^*, \mathcal{A}u\rangle = \partial_\xi u^e$ in the sense of distributions, we get

$$\begin{aligned}\int \varphi(x)\,\mathrm{d}[\langle w^*, \mathcal{A}u\rangle](x) &= \int \varphi\,\mathrm{d}[\partial_\xi u^e](x) \\ &= -\int \partial_\xi \varphi_{\xi,y}(x)\, u^e(x)\,\mathrm{d}x.\end{aligned}$$

Notice that $\partial_\xi \varphi(y + t\xi) = D(\varphi_{\xi,y})(t)$. Hence, by Fubini's theorem and a change of variables we further deduce

$$\begin{aligned}\langle w^*, \mathcal{A}u\rangle(\varphi) &= -\int_{\pi_\xi} \left(\int_{\mathbb{R}} D(\varphi_{\xi,y})(t)\, u^e_{\xi,y}(t)\,\mathrm{d}t\right) \mathrm{d}\mathcal{H}^{n-1}(y) \\ &= \int_{\pi_\xi} \left(\int_{\mathbb{R}} \varphi_{\xi,y}(t)\,\mathrm{d}[Du^e_{y,\xi}](t)\right) \mathrm{d}\mathcal{H}^{n-1}(y),\end{aligned}$$

where in the last equality we have used that $u^e_{y,\xi} \in \mathrm{BV}(\mathbb{R})$ for $\mathcal{H}^{n-1}$-almost every $y \in \pi_\xi$. Herewith, a standard density argument implies the equalities of measures

$$\langle w^*, \mathcal{A}u\rangle(B) = \int_{\pi_\xi} Du^e_{y,\xi}(B^\xi_y)\,\mathrm{d}\mathcal{H}^{n-1}(y),$$

$$|\langle w^*, \mathcal{A}u\rangle|(B) = \int_{\pi_\xi} |Du^e_{y,\xi}|(B^\xi_y)\,\mathrm{d}\mathcal{H}^{n-1}(y)$$

for all Borel sets $B \in \mathbb{R}^n$. □

We are now ready to give the proof of slicing theorem:

Proof of Theorem 2.1. First, observe that due to the nature of the boundary $\partial\Omega$, one can extend u by zero to a function in $\mathrm{BV}^{\mathcal{A}}(\mathbb{R}^n)$ (see Section 3.4). Therefore, there is no loss of generality in assuming that $\Omega = \mathbb{R}^n$. Also, by a localization argument we may assume that the support of u is compact. Now, let us mollify u at scale $\varepsilon > 0$ —denote this regularization by $u_\varepsilon \in \mathrm{C}^\infty_c(\mathbb{R}^n; V)$.

Let us prove that (1) ⇒ (2). By Lemma 4.1 we may suppose that $\mathcal{A}$ satisfies the rank-one property. This means that we can find a family of $\mathrm{rank}_{\mathcal{A}}$-one covectors $\{P_1, \dots, P_r\}$ spanning $(W_{\mathcal{A}})^*$ and such that

$$\langle P_i, \mathbb{A}(\eta)v\rangle = \langle \xi_i, \eta\rangle\langle e_i, v\rangle \quad \text{for some } (\xi_i, e_i) \in \partial\sigma(\mathcal{A}),$$

for all $i = 1, \dots, r$. We may complete this to a basis $\{P_1, \dots, P_r, P_{r+1}, \dots, P_M\}$ of W^* with $\mathrm{rank}_{\mathcal{A}}(P_i) = 0$ for all $i = r+1, \dots, M$. In a natural way, this basis induces a canonical isomorphism $W \to W^*$. Let $\{w_1, \dots, w_M\} \subset W$ be the pre-image of $\{P_1, \dots, P_M\}$ under this isomorphism and let $u \in \mathrm{BV}^{\mathcal{A}}(\Omega)$ so that

$$\mathcal{A}u = \sum_{i=1}^{r} \langle P_i, \mathcal{A}u\rangle w_i.$$

Now, invoking Proposition 2.1 for each individual term $\langle P_i, \mathcal{A}u\rangle$ —each (ξ_i, e_i) lies in the directional spectrum— we conclude that

$$\mathcal{A}u = \sum_{i=1}^{M} \left(\int_{\pi_{\xi_i}} Du^{e_i}_y \,\mathrm{d}\mathcal{H}^{n-1}(y) \right) w_i. \tag{21}$$

The sought assertion then follows from the triangle inequality.

The implication (2) ⇒ (1) follows directly from Lemma 4.1. □

4.3. **Algebraic constructions.** We shall see how, given a spectral pair $(\xi, e) \in \partial\sigma(\mathcal{A})$, one can algebraically set the foundations of what the slice $\mathcal{A}^e_\xi$ of $\mathcal{A}$ should be: an operator that is invariant with respect to ∂_ξ and with respect to the e-coordinate. In all that follows and for the rest of this section let us be given a non-trivial pair (so that $\xi \otimes e$ is a non-zero tensor)

$$(\xi, e) \in \partial\sigma(\mathcal{A}).$$

Let us recall the notation $\bar{f}_{\mathcal{A}}(\xi \otimes v) = f_{\mathcal{A}}(\xi, v) := \mathbb{A}(\xi)[v]$, defined in the introduction, and consider the pullback map

$$\begin{aligned} g_{\mathcal{A}} \colon W^* &\longrightarrow (\mathbb{R}^n \otimes V)^* \\ w^* &\longmapsto w^* \circ \bar{f}_{\mathcal{A}} \end{aligned},$$

Notice that by the definition of essential image, the map $g_{\mathcal{A}}$ is injective when restricted to $(W_{\mathcal{A}})^*$. We are now in position to start the construction of $\mathcal{A}^e_\xi$. The first step is to remove the coordinates of V spanned by e:

Definition 4.2. We define the subspace V_e of V as

$$V_e := \begin{cases} V & \text{if } \dim(V) = 1, \\ e^\perp & \text{if } \dim(V) \geq 2. \end{cases}$$

We write $\mathbf{p}^e : V \to V_e$ to denote the canonical orthogonal projection onto V_e.

The second step is to remove all coordinates of $\mathcal{A}u$ (by separation with W^*) containing partial derivatives on directions that interact with ξ, as well as all the coordinates related to u^e. To this end, let us consider the subspace of $(\mathbb{R}^n \otimes V)^*$ defined by

$$Y_\xi^e := \begin{cases} \operatorname{span}\{\xi \otimes e\} & \text{if } \dim(V) = 1, \\ \mathbb{R}^n \otimes e + \xi \otimes V^* & \text{if } \dim(V) \geq 2. \end{cases}$$

Since we are targeting elements in W^*, it is natural to work with the pre-image of the isometry $g_\mathcal{A} : (W_\mathcal{A})^* \to (\mathbb{R}^n \otimes V)^*$. We thus consider the subspace $X_\xi^e := g_\mathcal{A}^{-1}[Y_\xi^e]$, which leads us to the following definition:

Definition 4.3. We write W_ξ^e to denote the subspace of W defined by the property

$$W_\xi^e := (X_\xi^e)^\perp.$$

In all that follows we shall write $\mathbf{p}_\xi^e : W \to W_\xi^e$ to denote the linear canonical orthogonal projection from W onto W_ξ^e.

The following result is fundamental for key posterior arguments. It guarantees that the results contained in the forthcoming Lemma 4.2 and Corollary 4.1 are non-trivial under our main assumptions:

Proposition 4.1. *Let $n \geq 2$ and let $\mathcal{A} : \mathrm{C}^\infty(\mathbb{R}^n; V) \to \mathrm{C}^\infty(\mathbb{R}^n; W)$ be a first-order elliptic operator. If $(\xi, e) \in \partial\sigma(\mathcal{A})$, then W_ξ^e is non-trivial.*

Proof. Let us argue by a contradiction argument. If indeed W_ξ^e was trivial, then $\dim(X_\xi^e) = \dim(W_\mathcal{A})$ —here we are using that $g_\mathcal{A}$ is one-to-one when restricted to $(W_\mathcal{A})^*$. If $\dim(V) = 1$, then $\dim(X_\xi^e) = 1$ and by ellipticity we also have $\dim(W_\mathcal{A}) \geq n \geq 2$ —thus, reaching a contradiction. Else, we may find non-zero vectors $\eta \in \pi_\xi$ and $a \in e^\perp$. Therefore, using that $\mathbf{p}_\xi^e \equiv 0$, we get $\mathbb{A}(\eta)[a] = (\mathrm{id}_W - \mathbf{p}_\xi^e) \circ \mathbb{A}(\eta)[a] = 0$. Here, in reaching the last equality we have used the representation of linear maps and the fact that $X_\xi^e = g_\mathcal{A}^{-1}\{\, \xi^* \otimes v + \zeta^* \otimes e^* : \zeta \in \pi_\xi, v \in V \,\}$. This poses a contradiction with the assumption that $\mathcal{A}$ is elliptic. $\square$

Notation 1. Let a be a vector in a euclidean vector-space X. We write ℓ_a to denote the span of a in X.

Next, we show that both the ξ- and e-coordinates are algebraically invariant for the PDE when restricted to W_ξ^e.

Proposition 4.2. *Let $\mathcal{A} : \mathrm{C}^\infty(\mathbb{R}^n; V) \to \mathrm{C}^\infty(\mathbb{R}^n; W)$ be a first-order operator and let $(\xi, e) \in \partial\sigma(\mathcal{A})$. Then,*

$$\mathbf{p}_\xi^e \circ f_\mathcal{A} \equiv 0 \quad \text{on } (\mathbb{R}^n \times \ell_e) + (\ell_\xi \times V). \tag{22}$$

In particular, we obtain the equivalence of bi-linear forms

$$(\mathbf{p}_\xi^e \circ f_\mathcal{A})|_{\pi_\xi \times V_e} \equiv f_\mathcal{A}|_{\pi_\xi \times V_e}. \tag{23}$$

Proof. By definition $\mathbf{p}_\xi^e \circ f_\mathcal{A} \in (X_\xi^e)^\perp$. The sought assertion then follows directly from the definition of X_ξ^e. $\square$

We have the following direct consequence:

Proposition 4.3. *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and let $\mathcal{A} : \mathrm{C}^\infty(\mathbb{R}^n; V) \to \mathrm{C}^\infty(\mathbb{R}^n; W)$ be a first-order $\mathbb{K}$-elliptic operator. Assume that $(\xi, e) \in \partial\sigma(\mathcal{A})$, then*

$$\mathbf{p}_\xi^e \circ f_\mathcal{A} : \mathbb{K}\pi_\xi \times (\mathbb{K} \otimes V_e) \to (\mathbb{K} \otimes W_\xi^e)$$

defines a non-singular $\mathbb{K}$ bi-linear form.

Proof. The result follows form (23) and the fact that $f_{\mathcal{A}}$ is itself a non-singular $\mathbb{K}$ bi-linear form on the state domain of definition. □

4.4. **Stability properties of co-dimension one slicing.** With the constructions we have done so far we have now a toolbox to guarantee the existence of non-trivial slices « $\mathcal{A}^e_\xi \llcorner \pi_\xi$ » of $\mathcal{A}$, which will ultimately be the pivotal tool towards the slicing on codimension-one planes and the proof of the structure and fine properties theorems.

Let us introduce the restriction operator for differential operators:

Definition 4.4. Let $\mathcal{A} : \mathrm{C}^\infty(\mathbb{R}^n; V) \to \mathrm{C}^\infty(\mathbb{R}^n; W)$ be a differential operator (of arbitrary order k) and let $\pi \in \mathrm{Gr}(n)$. We define the operator

$$\mathcal{A} \llcorner \pi : \mathrm{C}^\infty(\pi; V) \to \mathrm{C}^\infty(\pi; W),$$

which is associated to the symbol

$$\mathbb{A} \llcorner \pi := \mathbb{A}^k|_\pi.$$

This operator has the intrinsic property that

$$\mathcal{A} \llcorner \pi(\varphi) = \mathcal{A}(\varphi \circ \mathbf{p}) \quad \text{for all } \mathrm{C}^\infty_c(\pi; V).$$

Let us recall the notation $\mathbf{p}_\xi : \mathbb{R}^n \to \pi_\xi$ for the canonical linear orthogonal projection from $\mathbb{R}^n$ onto π_ξ. The next Lemma studies the stability properties of the slices $\mathcal{A}^e_\xi$ with respect to its restriction on π_ξ.

Lemma 4.2 (Sub-operators). *Let $n \geq 2$ and let $\mathcal{A} : \mathrm{C}^\infty(\mathbb{R}^n; V) \to \mathrm{C}^\infty(\mathbb{R}^n; W)$ be a first-order operator. Suppose that $(\xi, e) \in \partial\sigma(\mathcal{A})$ is a non-trivial pair and define the first-order operator*

$$\mathcal{A}^e_\xi : \mathrm{C}^\infty(\mathbb{R}^n; V_e) \to \mathrm{C}^\infty(\mathbb{R}^n; W^e_\xi),$$

which is associated to the principal symbol

$$\mathbb{A}^e_\xi(\eta)[v] := \mathbf{p}^e_\xi \circ \mathbb{A}(\eta)[v] \quad \textit{for all } \eta \in \mathbb{R}^n \textit{ and } v \in V_e.$$

Then,

$$(\mathcal{A}^e_\xi \llcorner \pi_\xi)(\varphi) = \mathcal{A}(\mathbf{p}^e \circ \varphi \circ \mathbf{p}_\xi) \quad \textit{for all } \varphi \in \mathrm{C}^\infty(\pi_\xi; V)$$

and the following implications hold:

1. *$\mathcal{A}^e_\xi$ is ∂_ξ-invariant;*
2. *$\mathcal{A}$ is $\mathbb{K}$-elliptic $\Longrightarrow$ $\mathcal{A}^e_\xi \llcorner \pi_\xi$ is $\mathbb{K}$-elliptic ($\mathbb{K} = \mathbb{R}, \mathbb{C}$);*
3. *$\mathcal{A}$ satisfies $(\mathfrak{m})$ $\Longrightarrow$ $\mathcal{A}^e_\xi \llcorner \pi_\xi$ satisfies $(\mathfrak{m})$;*
4. *The rank-one cones of these operators satisfy the set contention*

$$(\mathcal{A}^e_\xi \llcorner \pi_\xi)^{\otimes}_1 \subset \mathcal{A}^{\otimes}_1.$$

Proof. The proof (i) follows directly from (22), whilst (ii) follows from the previous proposition. To show (iii) we first notice that the case $n = 2$ is trivial: in this case $\mathcal{A}^e_\xi \llcorner \pi_\xi$ is morally an elliptic operator acting on functions of one variable and hence every non-zero element in W^e_ξ is $\mathrm{rank}_{\mathcal{A}}$-one. This shows that $\mathcal{A}^e_\xi$ satisfies the rank-one property and therefore also $(\mathfrak{m})$. Now, assume that $n \geq 3$ so that $\pi \in \mathrm{Gr}(n-1, n)$ if and only if there exists $\gamma \in \mathbb{R}^n$ and a non-trivial $\tilde{\pi} \in \mathrm{Gr}(n-2, \pi_\xi)$ such that $\pi = \mathrm{span}\{\gamma, \tilde{\pi}\}$. Using the identity $\mathbf{p}^e_\xi[f_{\mathcal{A}}] \equiv f_{\mathcal{A}}$ (as bi-linear forms on $\pi_\xi \times V_e$) we deduce that $\mathrm{span}\{\, \mathbb{A}^e_\xi(\eta)[v] : \eta \in \tilde{\pi}, v \in V_e \,\} \subset \mathrm{span}\,\{\, \mathbb{A}(\eta)[v] : \eta \in \pi, v \in V \,\}$. Moreover, if P is in the intersection of the first linear space for all $\tilde{\pi} \in \mathrm{Gr}(n-2, \pi_\xi)$, then so it is in the intersections of the second space for all $\pi \in \mathrm{Gr}(n-1)$. Indeed, if there exists π for which P does not belong to it, then P would also not belong to the first sum for any $\tilde{\pi} \in \mathrm{Gr}(n-2, \pi \cap \pi_\xi) \subset \mathrm{Gr}(n-2, \pi_\xi)$. This shows that

$$\bigcap_{\tilde{\pi} \in \mathrm{Gr}(n-2,\pi_\xi)} \operatorname{span}\left\{ \mathbb{A}^e_\xi(\eta)[v] : \eta \in \tilde{\pi}, v \in V_e \right\} \subset \bigcap_{\pi \in \mathrm{Gr}(n-1)} \operatorname{span}\{ \mathbb{A}(\eta) : \eta \in \pi \}.$$

We are thus in position to use that $\mathcal{A}$ satisfies $(\mathfrak{m})$, from where it directly follows that $\mathcal{A}^e_\xi \llcorner \pi_\xi$ must also satisfy the mixing property $(\mathfrak{m})$ with π_ξ in place of $\mathbb{R}^n$. Lastly, we show (iv). Let (ω, h) be an arbitrary pair in $\partial\sigma(\mathcal{A}^e_\xi \llcorner \pi_\xi)$. We must check that indeed $(\omega, h) \in \partial\sigma(\mathcal{A})$. To this end let $w \in (W^e_\xi)^*$ be the vector satisfying $g_{\mathcal{A}^e_\xi \llcorner \pi_\xi}(w) = \omega^* \otimes h^*$. Invoking (23) we find that

$$w \circ f_{\mathcal{A}}(\eta, v) = w \circ f_{\mathcal{A}^e_\xi}(\eta, v) = \langle \omega^*, \eta \rangle \langle h^*, v \rangle \quad \text{for all } \eta \in \pi_\xi, v \in V_e.$$

Thus, in order to show that $(\omega, h) \in \partial\sigma(\mathcal{A})$, the standard representation of linear maps tells us that it suffices to show that $w \circ f_{\mathcal{A}}$ vanishes on $(\ell_\xi \times V) \cup (\mathbb{R}^n \times \ell_e)$. This however follows from (22) and the fact that $w \in (W^e_\xi)^*$. ☐

We are now in position to state the slicing on co-dimension one planes:

Corollary 4.1 (Slicing on hyper-planes)**.** *Let $\mathcal{A} : \mathrm{C}^\infty(\mathbb{R}^n; V) \to \mathrm{C}^\infty(\mathbb{R}^n; W)$ be a first-order elliptic operator and let (ξ, e) be a non-trivial pair in $\partial\sigma(\mathcal{A})$. Consider the restriction operator $\mathcal{B}^e_\xi : \mathrm{C}^\infty(\pi_\xi; V_e) \to \mathrm{C}^\infty(\pi_\xi; W^e_\xi)$ defined by*

$$\mathcal{B}^e_\xi \coloneqq \mathcal{A}^e_\xi \llcorner \pi_\xi,$$

where $\mathcal{A}^e_\xi$ is the sub-operator defined in Lemma 4.2. Given $u : \mathbb{R}^n \to V$ and a vector $z \in \ell_\xi$, we define a function $v_z : \pi_\xi \to V_e$ as

$$v_z(y) = \mathbf{p}^e u(z + y)$$

where as usual $\mathbf{p}^e : V \to V_e$ is the linear orthogonal projection onto V_e.

Then, for every $u \in \mathrm{BV}^{\mathcal{A}}(\mathbb{R}^n)$, the following holds:

(1) The functions v_z are well-defined and belong to $\mathrm{BV}^{\mathcal{B}^e_\xi}(\pi_\xi)$ for $\mathcal{H}^1$-almost every $z \in \ell_\xi$. Moreover,

$$\int_{\ell_\xi} |\mathcal{B}^e_\xi v_z|(\pi_\xi) \, \mathrm{d}\mathcal{H}^1(z) < \infty.$$

(2) For every Borel set $B \subset \mathbb{R}^n$ we have

$$(\mathbf{p}^e_\xi[\mathcal{A}u])(B) = \int_{\ell_\xi} \mathcal{B}^e_\xi v_z(B_z) \, \mathrm{d}\mathcal{H}^1(z),$$

$$|\mathbf{p}^e_\xi[\mathcal{A}u]|(B) = \int_{\ell_\xi} |\mathcal{B}^e_\xi v_z|(B_z) \, \mathrm{d}\mathcal{H}^1(z),$$

where $B_z \coloneqq \{ y \in \pi_\xi : y + z \in B \}$ is the π_ξ-slice of B at $z \in \ell_\xi$.

Proof. Let $\varphi \in \mathrm{C}^\infty_c(\mathbb{R}^n; V)$ and let us first assume that $u \in \mathrm{C}^\infty(\mathbb{R}^n; V)$. We write $\varphi_z(y) = \varphi(z + y)$. By Fubini's Theorem and integration by parts we obtain

$$\begin{aligned}
\langle \mathbf{p}^e_\xi[\mathcal{A}u], \varphi \rangle &= \langle \mathcal{A}^e_\xi u, \varphi \rangle \\
&= \int_{\ell_\xi} \left(\int_{\pi_\xi} ([\mathcal{A}^e_\xi \llcorner \pi_\xi] u)(z + y) \varphi(z + y) \, \mathrm{d}\mathcal{H}^{n-1}(y) \right) \mathrm{d}\mathcal{H}^1(z) \\
&= \int_{\ell_\xi} \left(\int_{\pi_\xi} \mathcal{B}^e_\xi v_z(y) \varphi_z(y) \, \mathrm{d}\mathcal{H}^{n-1}(y) \right) \mathrm{d}\mathcal{H}^1(z).
\end{aligned}$$

Here, in the second equality we have used that $\mathcal{A}^e_\xi$ is ∂_ξ-invariant; in passing to the last equality we have used that $(\mathcal{A}^e_\xi \llcorner \pi_\xi)u(z+\bullet) = \mathcal{A}(\mathbf{p}_e u(z+\bullet)\mathbf{p}_\xi) = \mathcal{A}^e_\xi \llcorner \pi_\xi(v_z)$ (cf. Lemma 4.2). Taking the supremum over all φ gives the inequality

$$|\mathcal{A}u|(\mathbb{R}^n) \geq \int_{\ell_\xi} \left(\int_{\pi_\xi} |\mathcal{B}^e_\xi v_z| \,\mathrm{d}\mathcal{H}^{n-1} \right) \mathrm{d}\mathcal{H}^1(z).$$

This estimate is stable under mollification: Let ρ_ε be a standard mollifier at ε-scale. Using the standard notation $u_\varepsilon := u \star \rho_\varepsilon$ and Young's inequality we obtain the uniform bound

$$|\mathcal{A}u|(\mathbb{R}^n) \geq |\mathbf{p}^e_\xi[\mathcal{A}u_\varepsilon]|(\mathbb{R}^n) \geq \int_{\ell_\xi} |\mathcal{B}^e_\xi \mathbf{p}^e[u_\varepsilon(y+z)]|(\pi_\xi)\,\mathrm{d}\mathcal{H}^1(z).$$

Fubini's theorem guarantees that, for $\mathcal{H}^1$-almost every $z \in \ell_\xi$, we have the convergence $w_\varepsilon := u_\varepsilon(z+\bullet) \to v_z$ in $\mathrm{L}^1(\pi_\xi)$. This also means that $\mathcal{B}^e_\xi w_\varepsilon \to \mathcal{B}^e_\xi v_z$ in the sense of distributions on π_ξ. Therefore, the map $w \mapsto |\mathcal{B}^e_\xi w|(\pi_\xi)$ is lower semicontinuous with respect to $\mathrm{L}^1(\pi_\xi)$ convergence. Then, the estimate above and Fatou's lemma further imply that

$$|\mathcal{A}u|(\mathbb{R}^n) \geq \int_{\ell_\xi} |\mathcal{B}^e_\xi v_z|(\pi_\xi)\,\mathrm{d}\mathcal{H}^1(z).$$

This shows that if $u \in \mathrm{BV}^{\mathcal{A}}(\mathbb{R}^n)$, then (1) holds.

Similarly to the identity for smooth functions, using Fubini's theorem one shows that the identity

$$\begin{aligned}\langle \mathbf{p}^e_\xi[\mathcal{A}u], \varphi\rangle_{\mathbb{R}^n} &= \int_{\ell_\xi} \left(\int_{\pi_\xi} v_z\,[\mathcal{B}^e_\xi]^* \varphi_z \,\mathrm{d}\mathcal{H}^{n-1} \right) \mathrm{d}\mathcal{H}^1(z) \\ &= \int_{\ell_\xi} \langle \mathcal{B}^e_\xi v_z, \varphi_z\rangle_{\pi_\xi}\,\mathrm{d}\mathcal{H}^1(z).\end{aligned}$$

holds, in the sense of distributions for, for all $u \in \mathrm{L}^1(\mathbb{R}^n;V)$. Here, the suffix $\langle \bullet, \bullet\rangle_X$ indicates that the pairing is to be taken in $\mathcal{D}'(X) \times \mathcal{D}(X)$. The first statement in (2) then follows from a classical approximation argument for sets and (1), which ensures that $\langle \mathcal{B}^e_\xi v_z, \varphi_z\rangle_{\pi_\xi} = \int_{\pi_\xi} \varphi_z \,\mathrm{d}\mathcal{B}^e_\xi v_z$ for $\mathcal{H}^1$-almost every $z \in \ell_\xi$. The second statement in (2) follows from characterization of the total variation of generalized product measures.

This finishes the proof. □

4.5. **Slices of arbitrary co-dimension.** The previous proposition extends to the following more general context: given a subspace $\mathscr{V} \in \mathrm{Gr}(\ell, n)$, there exists an operator $\mathcal{B}_\mathscr{V} : \mathrm{C}^\infty(\mathscr{V};V) \to \mathrm{C}^\infty(\mathscr{V};W)$ and a linear projection $p_\mathscr{V} : W \to W_\mathscr{V}$ such that

$$p_\mathscr{V}[\mathcal{A}u] = \int_{\mathscr{V}^\perp} \mathcal{B}_\mathscr{V} v_z \,\mathrm{d}\mathcal{H}^{n-\ell}(z).$$

In fact, the proof of this statement only requires the rank-ℓ property

$$\operatorname{span} \mathcal{A}^\otimes_\ell = W_\mathcal{A},$$

or equivalently (the proof is left to the reader),

$$\bigcap_{\substack{\pi \leq \mathbb{R}^d \\ \dim(\pi) = n-\ell}} \operatorname{span}\{\,\mathrm{Im}\,\mathbb{A}(\eta) : \eta \in \pi\,\} = \{0\}.$$

However, since $\mathcal{A}$ may not have a self-similar algebraic design (as the gradient or the symmetric gradient have), in general there is no straightforward formula for $\mathcal{B}_{\mathscr{V}}$ in terms of $\mathcal{A}$, other than hefty one given by iteration of slicing:

$$\mathcal{B}_{\mathscr{V}} = (((\mathcal{A}^{e_1}_{\xi_1})^{e_2}_{\xi_2})\dots)^{e_{n-\ell}}_{\xi_{n-\ell}} \llcorner \mathscr{V},$$

where $\{\xi_i, e_i\}_{i=1}^{n-\ell} \subset \partial\sigma(\mathcal{A})$ and $\{\xi_1, \dots, \xi_{n-\ell}\}$ is a basis of $\mathscr{V}^\perp$ (the last step requires the projection result contained in Corollary 4.2 below).

Remark 4.1. As we shall see later (in Section 10), the deviatoric operator

$$Lu := Eu - \frac{\mathrm{id}_{\mathbb{R}^n}}{n}\operatorname{div}(u), \quad u : \mathbb{R}^n \to \mathbb{R}^n,$$

not only does not satisfy the rank-one property, but its set of rank_L-one tensors is empty. In light of the slicing theorem, it is clear that no coordinate of Eu cannot be sliced into one-dimensional sections. However, it is easy to check that L satisfies the rank-two property and therefore it can be sliced into two-dimensional slices.

4.6. **Polarization properties.** The purpose of this section is to verify that, under ellipticity and the rank-one condition, there exist of sufficient transversal spectral pairs. The next lemma is inspired in a key transversality result of [2], in which the one-dimensional structure theorem there hinges on.

Proposition 4.4 (Polarization)**.** *Let $n \geq 2$ and let $\mathcal{A} : \mathrm{C}^\infty(\mathbb{R}^n; V) \to \mathrm{C}^\infty(\mathbb{R}^n; W)$ be a first-order elliptic operator satisfying the rank-one property. Assume that (ξ, e) is a non-trivial pair in the spectrum $\partial\sigma(\mathcal{A})$. Then there exists a non-trivial pair*

$$(\eta, f) \in \partial\sigma(\mathcal{A}) \textit{ for some } \eta \in \pi_\xi \textit{ and } f \in (V_e)^*.$$

Moreover, there exists a direction $v \in \ell_f$ with

$$(\xi + \eta, e + v), (\xi - \eta, e - v) \in \partial\sigma(\mathcal{A})$$

and $\operatorname{span}\{\xi, \eta\} \subset \operatorname{Proj}_{\mathbb{R}^n} \partial\sigma(\mathcal{A})$.

Proof. The existence of (η, f) is a direct consequence of Lemma 4.2 and Proposition 4.1. We shall therefore focus on the second statement:

1. Reduction to the case $n = 2$. Let us assume that $n \geq 3$ for otherwise there is nothing to show. By construction we get

$$(\xi, e) \in \partial\sigma(\mathcal{B}), \quad \mathcal{B} := \mathcal{A}^f_\eta \llcorner \pi_\eta,$$

where $\mathcal{B}$ is elliptic and satisfies the rank-one property. Since $\dim(\pi_\eta) \geq 2$, then the slice $\mathcal{B}^e_\xi \llcorner (\pi_\xi \cap \pi_\eta)$ is a non-trivial elliptic operator satisfying the rank-one property. This conveys the existence of a pair $(\omega, h) \in \partial\sigma(\mathcal{A}) \cap ((\pi_\xi \cap \pi_\eta) \times (V_e \cap V_\eta)^*)$ and, in particular,

$$(\xi, a), (\eta, f) \in \partial\sigma(\mathcal{A}^h_\omega \llcorner \pi_\omega) \subset \partial\sigma(\mathcal{A}).$$

Once again, we observe that $\mathcal{A}^h_\omega \llcorner \pi_\omega$ is elliptic and satisfies the rank property. Therefore, the statement of the proposition holds if and only if an analogous statement holds for the operator $\mathcal{A}^h_\omega \llcorner \pi_\omega$. An iteration of this argument tells us there is no loss of generality in assuming that $n = 2$.

2. The case when $\dim(V) = N > 2$. Let us assume that $N > 2$ (and recall from the previous step that $n = 2$). As before, consider the slice $\mathcal{B} = \mathcal{A}^e_\xi \llcorner \pi_\xi$, which in this case is an elliptic differential operator in one variable (the η-variable). More precisely $\mathcal{B} : \mathrm{C}^\infty(\ell_\eta; V_e) \to \mathrm{C}^\infty(\ell_\eta; W^e_\xi)$. It follows that $\mathcal{B}$ contains a gradient operator and therefore $\{\eta\} \times (V_e)^* \in \partial\sigma(\mathcal{B}) \subset \partial\sigma(\mathcal{A})$. Similarly, the slice $\mathcal{A}^f_\eta \llcorner \ell_\xi$ acts on $\mathrm{C}^\infty(\ell_\xi; V_f)$ and by the same reasoning above we get $\{\xi\} \times (V_f)^* \subset \partial\sigma(\mathcal{A})$. Since $\dim(V) \geq 3$, we also have $\dim(V_v \cap V_e) \geq N-2 \geq 1$. In particular, using the bi-linearity of $f_{\mathcal{A}}$ we find that

$$\operatorname{span}\{\xi, \eta\} \times H^* \subset \partial\sigma(\mathcal{A}), \quad H \coloneqq V_e \cap V_f. \tag{24}$$

Up to a change of variables we may assume that $\xi + \eta$ and $\xi - \eta$ are orthogonal vectors. Up to constant multiplication, we may also assume that $|\xi| = |\eta| = 1$. Now, working with the slice $\mathcal{A}^h_{\xi+\eta} \llcorner \ell_\omega$ for some non-zero $h \in H^*$, we find (this slice must be a gradient) that $\ell_{\xi-\eta} \times \operatorname{span}\{e, f\} \subset \partial\sigma(\mathcal{A})$. Thus, again by the bi-linearity of $f_{\mathcal{A}}$ and the fact that $(\xi, e) \cup (\eta, f) \in \partial\sigma(\mathcal{A})$, we conclude that $\operatorname{span}\{\xi, \eta\} \times \operatorname{span}\{e, f\} \subset \partial\sigma(\mathcal{A})$, from where the sought assertion trivially follows.

3. The case when $\dim(V) \leq 2$. The proof when $\dim(V) = 1$ is trivial since then (24) holds trivially for the generating vector f of V^*. We shall hence focus in the case when $n = \dim(V) = 2$. The first observation is that

$$\operatorname{Proj}_{\mathbb{R}^2}[\partial\sigma(\mathcal{A})] = \mathbb{R}^2, \quad \operatorname{Proj}_{V^*}[\partial\sigma(\mathcal{A})] = V^*. \tag{25}$$

This means that every $\eta \in \mathbb{R}^2$ and every $e \in V^*$ are the first and second coordinates (respectively) of some element in the directional spectrum. The proof of this follows directly from the mixing property in two-dimensions: for a given $\eta \in \mathbf{S}^1$, the image $\operatorname{Im} \mathbb{A}(\eta_\perp)$ cannot be the whole of W and therefore there exists a non-zero $w^* \in \mathbb{A}(\eta_\perp)^\perp$ such that $\langle w^*, \mathbb{A}(\eta_\perp)[e]\rangle = 0$ for all $e \in V$. By the representation of linear maps, this implies that

$$\langle w^*, \mathbb{A}(\omega)[v]\rangle = \langle \eta, \omega\rangle \langle \mathbb{A}(\eta)^*[w^*], v\rangle \quad \text{for all } (\omega, v) \in \mathbb{R}^n \times V.$$

Therefore we obtain $(\eta, \mathbb{A}(\eta)^*[w^*]) \in \partial\sigma(\mathcal{A})$. Since $\eta \in \mathbf{S}^1$ was arbitrarily chosen, this proves the first claim. The second claim follows from a symmetric argument on the V-variable.

The second observation is that, qualitatively speaking, there are only two possible cases (recall that $g_{\mathbb{A}} : (W_{\mathcal{A}})^* \to (\mathbb{R}^2)^* \otimes V^*$ is one-to-one): $\dim(W_{\mathcal{A}}) \in \{3, 4\}$. Let us first understand the case when $\dim(W_{\mathcal{A}}) = 4$, which is easier. Clearly, this is the case when $W^*_{\mathcal{A}} \cong (\mathbb{R}^2)^* \otimes V^*$. In particular, $\mathbb{R}^2 \times V^* = \partial\sigma(\mathcal{A})$. Thus, the conclusion of the Proposition holds trivially. Let us now address the case when $\dim(W_{\mathcal{A}}) = 3$. Firstly, we claim that to each $[\zeta] \in \mathbb{PR}^1$ there corresponds one (and only one) $[v_\zeta] \in \mathbb{P}V^*$ such that $(\zeta, v_\zeta) \in \partial\sigma(\mathcal{A})$; moreover, this assignment is injective. The fact that to each line in $\mathbb{PR}^1$ corresponds at least one representative non-zero vector $v_\zeta \in V^*$ follows from (25). We are left to check that there cannot be more than one spectral V^*-coordinate attached to any direction $\zeta \in \mathbf{S}^1$. If this was the case for some $\zeta \in \mathbf{S}^1$, then a linearity argument would give $\{\zeta_0\} \times V^* \subset \partial\sigma(\mathcal{A})$ (here we are using that $\dim(V) = 2$). However, by a similar linearity argument (now on the ζ-variable), this would also imply that $\mathbf{S}^1 \times \{v_0\} \subset \partial\sigma(\mathcal{A})$ for some non-zero $v_0 \in V^*$. Hence, all four pairs $(\zeta_0, v_0), (\zeta_0, (v_0)_\perp), ((\zeta_0)_\perp, v_0)((\zeta_0)_\perp, (v_0)_\perp)$ would belong to $\partial\sigma(\mathcal{A})$. This however implies that $\dim(W_{\mathcal{A}}) \geq 4 > 3$; therefore reaching a contradiction. This proves that the assignment $\zeta \mapsto [v_\zeta]$ is well defined. That the map is one-to-one follows by inverting the roles of ζ and v.

From this observation, we can give a basis for $g_{\mathcal{A}}(W^*_{\mathcal{A}}) \leq (\mathbb{R}^2 \otimes V)^*$ conformed by the rank-one tensors $\{\xi \otimes e, \eta \otimes f, (\xi + \eta) \otimes (\alpha e + \beta f)\}$. Here, $\alpha, \beta \in \mathbb{R}$ are chosen so that $(\xi + \eta) \otimes (\alpha e + \beta f)$ belongs to $\partial\sigma(\mathcal{A})$. Observe that since $[\zeta] \mapsto [v_\zeta]$ is one-to-one, then both α, β are non-zero reals. We are now in position to determine precisely the map $\mathbb{PR}^1 \to \mathbb{P}V^* : [\eta] \mapsto [v_\eta]$. An equivalent basis to the one given above, corresponds to the elements $\{\xi \otimes e, \eta \otimes f, \alpha(\xi \otimes e) + \beta(\eta \otimes f)\}$. Now, let $\zeta \in \mathbf{S}^1$ be an arbitrary direction, which we may write as $\zeta = c\xi + d\eta$ for some reals $c, d \in \mathbb{R}$. Likewise we may write $v_\zeta = he + gf$. Developing the tensorial product we discover that $\zeta \otimes v_\zeta = ch(\xi \otimes e) + dg(\eta \otimes f) + cg(\xi \otimes f) + dh(\eta \otimes e)$. Since the left hand side is rank-one tensor, it must hold that $[(\alpha, \beta)] = [(cg, dh)] \in \mathbb{PR}^1$. Herewith, we deduce that $[(h, g)] = [(\alpha c, \beta d)]$ and thus the map $\nu \mapsto [v_\nu]$ is explicitly given by the assignment $[c\xi + d\eta] \mapsto [(\alpha c)e + (\beta d)f]$. The sought polarization property follows by taking $c = \alpha^{-1}$, $d = \pm 1$ and $v = \pm\beta f$. This finishes the proof of the last possible case. □

Corollary 4.2. *Let $\mathcal{A} : \mathrm{C}^\infty(\mathbb{R}^n; V) \to \mathrm{C}^\infty(\mathbb{R}^n; W)$ be a first-order operator satisfying the the rank-one property. Then,*

$$\mathbb{R}^n = \mathrm{Proj}_{\mathbb{R}^n} \partial\sigma(\mathcal{A}) \quad \text{and} \quad V^* = \mathrm{Proj}_{V^*} \partial\sigma(\mathcal{A}).$$

Proof. If $n = 1$, then the proof is trivial. For $n = 2$, the proof follows from the last statement in the previous proposition. The case for $n \geq 3$ follows by induction (using Lemma 4.2) and a simple geometric argument. The assertion for the projection onto V follows by a symmetric argument. □

Corollary 4.2 allows us to make an improvement in the dimensional estimates for the total variation measure of $\mathcal{A}u$. Namely, we pass from absolute continuity with $\mathcal{H}^{n-1}$ to absolute continuity with respect to the $(n-1)$-dimensional integral-geometric-measure:

Corollary 4.3. *Let $\mathcal{A} : \mathrm{C}^\infty(\mathbb{R}^n; V) \to \mathrm{C}^\infty(\mathbb{R}^n; W)$ be a first-order operator satisfying the the mixing condition $(\mathfrak{m})$. Let u be a function in $\mathrm{BV}^\mathcal{A}(\Omega)$ and let $B \subset \Omega$ be a Borel set satisfying*

$$\mathcal{H}^{n-1}(\mathbf{p}_\xi(B)) = 0 \quad \text{for } \mathcal{H}^{n-1}\text{-almost every } \xi \in \mathbf{S}^{n-1}.$$

Then,

$$|\mathcal{A}u|(B) = 0.$$

Proof. Let $B \subset \Omega$ satisfy the assumptions of the Corollary for all $\xi \in \mathbf{S}$, where $\mathbf{S} \subset \mathbf{S}^{n-1}$ is a full $\mathcal{H}^{n-1}$-measure subset. By virtue of the previous result and a continuity argument (the manifold of rank-one tensors is closed in $\mathbb{R}^n \otimes V$), we may find a family of coordinates $\{P_1, \dots, P_M\}$ spanning $(W_\mathcal{A})^*$, and, for each $j \in \{1, \dots, M\}$, we may find covectors $\xi_j \in \mathbf{S}$ and $e_j \in V^*$ such that

$$\langle P_j, \mathbb{A}(\xi)[v] \rangle = \langle \xi_j, \xi \rangle \langle e_j, v \rangle, \quad \text{for all } \xi \in \mathbb{R}^n, v \in V.$$

Then, the Structure Theorem 2.1 gives

$$\begin{aligned} |\langle P_j, \mathcal{A}u \rangle|(B) &= \int_{\pi_{\xi_j}} |Du^{e_j}_{y,\xi_j}|(B^y_{\xi_j}) \, \mathrm{d}\mathcal{H}^{n-1}(y) \\ &= \int_{\mathbf{p}_{\xi_j}(B)} |Du^{e_j}_{y,\xi_j}|(B^y_{\xi_j}) \, \mathrm{d}\mathcal{H}^{n-1}(y) = 0 \qquad (\xi_j \in \mathbf{S}). \end{aligned}$$

Since $\mathrm{span}\{P_1, \dots, P_M\} = (W_\mathcal{A})^*$, we conclude that $|\mathcal{A}u|(B) = 0$. □

Remark 4.2. For future reference let us recall from [8, Thm, 1.1] that $J_u \subset \Theta_u$ whenever $\mathcal{A}$ is an elliptic operator satisfying the rank-one property.

Corollary 4.4. *Let $u \in \mathrm{BV}^\mathcal{A}(\Omega)$. Then*

(1) $|\mathcal{A}u|(\Theta_u \setminus J_u) = \mathcal{H}^{n-1}(\Theta_u \setminus J_u) = 0$,
(2) Θ_u *is $\mathcal{H}^{n-1}$-countably rectifiable,*
(3) $\mathcal{A}^c u$ *vanishes on σ-finite sets with respect to the $\mathcal{H}^{n-1}$-measure.*

Proof. A standard covering argument implies that Θ_u is a σ-finite set with respect to $\mathcal{H}^{n-1}$. We may split $S := \Theta_u \setminus J_u$ into two disjoint Borel sets

$$G \cup F := \{\theta^{*(n-1)}(|\mathcal{A}u|) \in (0,\infty)\} \cup \{\theta^{*(n-1)}(|\mathcal{A}u|) = \infty\}.$$

Following a standard procedure, we proceed to split G as $R \cup U$ where R is countably $\mathcal{H}^{n-1}$-rectifiable, U is $\mathcal{H}^{n-1}$-purely unrectifiable, and $\mathcal{H}^{n-1}(R \cap U) = 0$. Once again, a standard covering argument gives $\mathcal{H}^{n-1}(F \cup (R \cap U)) = 0$. On the other hand, the Besicovitch–Federer Theorem implies that $\mathcal{I}^{n-1}(U) = 0$. By the dimensional estimates of the previous corollary we get $|\mathcal{A}u|(S) = 0$. By the definition of Θ_u, it also holds $\mathcal{H}^{n-1} \llcorner \Theta_u \ll |\mathcal{A}u| \llcorner \Theta_u$, whence we conclude that $\mathcal{H}^{n-1}(S) = 0$. This proves (1).

Since J_u itself is countably $\mathcal{H}^{n-1}$-rectifiable and $\mathcal{H}^{n-1}(\Theta_u \setminus J_u) = 0$, then Θ_u is also countably $\mathcal{H}^{n-1}$-rectifiable and (2) follows.

Let $B \subset \Omega$ be a σ-finite Borel set with respect to $\mathcal{H}^{n-1}$ so that $B = \bigcup_{i=1}^{\infty} B_i$ where $\mathcal{H}^{n-1}(B_i) < \infty$. From (1) we know that

$$|\mathcal{A}^c u|(B) = |\mathcal{A}^c u|(B \setminus \Theta_u).$$

Since at every point in $\Omega \setminus \Theta_u$, the $(n-1)$-dimensional upper density of $|\mathcal{A}u|$ vanishes, a standard covering argument implies that, for any $\varepsilon > 0$ it holds

$$|\mathcal{A}^c u|(B_i) \leq \varepsilon \mathcal{H}^{n-1}(B_i) \quad \text{for all } i = 1, 2, \ldots.$$

Letting $\varepsilon \to 0^+$ and using the finiteness of $\mathcal{H}^{n-1}(B_i)$ for all $i \in \mathbb{N}$, we find that $|\mathcal{A}^c u|(B) = 0$. This proves (3), which finishes the proof. □

5. Analysis of Lebesgue points

Now we focus on the Lebesgue continuity properties. We shall see, by the end of this section that the discontinuous diffusion part $\mathcal{A}^d u := \mathcal{A}^s u \llcorner (S_u \setminus J_u)$ vanishes for elliptic operators satisfying the mixing property. Moreover, we establish that the only one form of approximate discontinuity for $\mathrm{BV}^{\mathcal{A}}$-functions is the jump-type discontinuity.

The results of this section hinge on the algebraic robustness of the rank-one property. Often, the main difficulty of the proofs will reside in finding the correct way to cast the algebraic structure of $\mathbb{A}$ into the well-established techniques developed for the symmetric gradient from [2].

Lemma 5.1. *Let $\mathcal{A} : \mathrm{C}^\infty(\mathbb{R}^n; V) \to \mathrm{C}^\infty(\mathbb{R}^n; W)$ be a first-order elliptic operator satisfying the mixing condition* $(\mathfrak{m})$.

Let $u \in \mathrm{BV}^{\mathcal{A}}(\mathbb{R}^n)$, $x \notin \Theta_u$, and $\xi \in \mathbf{S}^{n-1}$. Assume that $\mathcal{H}^{n-1}$-almost every point in the slice $x + \pi_\xi$ is a Lebesgue point of u. If $(\xi, e) \in \partial\sigma(\mathcal{A})$ and the function $v : \pi_\xi \to V_e$ defined by $v(y) = \mathbf{p}^e u(x+y)$ has one-sided Lebesgue point limits at $0 \in \pi_\xi$ with respect to a suitable direction $\nu \in \pi_\xi$, then

(1) 0 is a Lebesgue point of v in π_ξ,
(2) x is a Lebesgue point of $\mathbf{p}^e u$ in $\mathbb{R}^n$.

Remark 5.1. In proving property (2), we use that if $\mathcal{A}$ is complex-elliptic, then every $u \in \mathrm{BV}^{\mathcal{A}}(\mathbb{R}^n)$ is quasi-continuous on the set Θ_u (see [8, Propisition 1.2]).

Proof. Step 1. Preparations. We may assume without loss of generality that $x = 0$. Let (e^+, e^-, ν) be the triple describing the one-sided limits of v at $0 \in \pi_\xi$. Clearly, we may assume that $|e^+ - e^-| > 0$ for otherwise the first statement is trivial (and we may pass to the next step). By construction we have $e := e^+ - e^- \in V_e$.

We claim that there exists a pair $(\eta, f) \in \partial\sigma(\mathcal{A})$ satisfying

$$(\eta, f) \in \pi_\xi \times V_e, \qquad \eta \cdot \nu \neq 0, \qquad f \cdot e \neq 0. \tag{26}$$

First, we use that $\mathrm{Proj}_{\pi_\xi} \partial\sigma(\mathcal{A}^e_\xi \llcorner \pi_\xi) = \pi_\xi$ to find a non-zero coordinate $g \in (V_e)^*$ with $(\nu, g) \in \partial\sigma(\mathcal{A})$. If $|\langle g, e\rangle| > 0$, then we simply set $(\eta, f) = (\nu, g)$. If however $g \cdot e = 0$, then we have to solve two further sub-cases: This time we use the identity $\mathrm{Proj}_{(V_e)^*} \partial\sigma(\mathcal{A}^e_\xi \llcorner \pi_\xi) = (V_e)^*$ to find a direction $\omega \in \mathbf{S}^{n-1}$ with $(\omega, e) \in \partial\sigma(\mathcal{A})$. If $|\langle \omega, \nu\rangle| > 0$, then we set $(\eta, f) = (\omega, e)$. Else, we use Lemma 4.4 to find a non-zero coordinate $v \in \ell_e$ such that

$$(\nu + \omega, g + v) \in \partial\sigma(\mathcal{A}).$$

This pair satisfies the sought properties. Since this is the only other possible case, this proves the claim.

Observation: Slicing the operator $\mathcal{A}^e_\xi \llcorner \pi_\xi$ with respect to the pair (η, f), we may find another pair $(\eta_2, f_2) \in \partial\sigma(\mathcal{A})$ satisfying (26). Repeating the same argument yields a family $\{\eta_j, f_j\}_{j=1}^\ell \subset \partial\sigma(\mathcal{A})$ satisfying (26) where the V-coordinates also satisfy

$$\mathrm{span}\{f_j\}_{j=1}^\ell = (V_e)^*. \tag{27}$$

This will be used in Step 3.

Step 2. Approximate continuity on π_ξ. The idea is to show first that $0 \in \pi_\xi$ is a Lebesgue point of v. This part of the proof mimics the proof of Theorem 5.1 in [2] into our context. Let Q_r be the open cube of radius r that is centered at y and with two of its axes oriented by the ξ- and η-directions respectively. We write $C_r = Q_r \cap \pi_\xi$ and $C_r^\pm = \{\, y \in C_r : \pm y \cdot \nu \geq 0 \,\}$. Lastly, we write A_r to denote the $(n-1)$-dimensional open ball in π_ξ with radius $\rho(\nu \cdot \eta)$. With these conventions we have $A_r \pm r\eta \in C_{2r}^\pm$.

Let Y be the set of all real numbers $\rho \geq 0$ such that both $y + \rho\eta$ are Lebesgue points of u in $\mathbb{R}^n$ for $\mathcal{H}^{n-1}$-almost every $y \in \pi_\xi$. We know that $\mathcal{H}^1(\mathbb{R}^+ \setminus Y) = 0$ and that, by assumption, also $0 \in Y$.

Let us record for later use that the one-sided limits assumption implies

$$\lim_{r \to 0^+} \frac{1}{r^{n-1}} \int_{C_{2r}^\pm} |v(y) - e^\pm| = 0. \tag{28}$$

The triangle inequality, (26), and a change of variables yield the estimate

$$\begin{aligned} |a^+ - a^-| \lesssim \frac{1}{(2r)^{n-1}} \bigg(&\int_{C_{2r}^+} |v(y) - e^+| \, \mathrm{d}\mathcal{H}^{n-1}(y) \\ &+ \int_{C_{2r}^-} |v(y) - e^-| \, \mathrm{d}\mathcal{H}^{n-1}(y) \\ &+ \int_{A_r} |u^f(y + r\eta) - u^f(y - r\eta)| \, \mathrm{d}\mathcal{H}^{n-1}(y) \bigg). \end{aligned} \tag{29}$$

The change of variables $\tilde{y} = y \pm r\eta$ and the one-sided continuity (28) give that the first two terms of the right-hand side above are of order $\mathrm{O}(r)$. Therefore, we only need to show the last term vanishes as $r \to 0^+$. In all that follows we write $\tilde{u}$ to denote the Lebesgue representative of u. Let $\rho \in Y$. Using the polarization from Lemma 4.4 we may decompose, for $\mathcal{H}^{n-1}$-a.e. $y \in \pi_\xi$, the difference $\tilde{u}^f(y-\rho\eta) - \tilde{u}^f(y+\rho\eta)$ as

$$\begin{aligned}2[\tilde{u}^f(y+\rho\eta) - \tilde{u}^f(y-\rho\eta)] = \tilde{u}^{f+e}(y+\rho\xi) - \tilde{u}^{f+e}(t-\rho\eta)\\ + \tilde{u}^{f-e}(y+\rho\eta) - \tilde{u}^{f-e}(y+\rho\xi)\\ + \tilde{u}^{f-e}(y-\rho\xi) - \tilde{u}^{f-e}(y-\rho\eta)\\ + \tilde{u}^{f+e}(y+\rho\eta) - \tilde{u}^{f+e}(y-\rho\xi)\\ - 2[\tilde{u}^e(y+\rho\xi) - \tilde{u}^e(y-\rho\xi)].\end{aligned} \tag{30}$$

E ach of this terms may be estimated by a total variation term in a transversal direction to ξ. Following a measure theoretic argument as in the proof of Theorem 5.1 in [2], we may (for $\mathcal{H}^{n-1}$-almost every $y \in \pi_\xi$ and $\rho \in Y$) estimate $|\tilde{u}^f(t-\rho\eta) - \tilde{u}^f(t+\rho\eta)|$, up to a multiplicative constant, by the sum

$$V\tilde{u}^{f+e}_{\eta+\xi,y}([-\rho,\rho]) + V\tilde{u}^{f-e}_{\eta-\xi,y}([-\rho,\rho]) + V^e_{\xi,y}\tilde{u}([-\rho,\rho]),$$

where the latter is the one-dimensional total variation in terms of difference quotients (see, e.g., (2.8) in [2]). Returning to the first estimate (29), we then deduce from the bound $V\tilde{h} \le |Dh|$ for functions h of one variable that

$$\begin{aligned}|e^+ - e^-| \lesssim \frac{c}{r^{n-1}}\bigg(\int_{A_r} |Du^{f+e}_{\eta+\xi,y}|([-r,r])\,\mathrm{d}\mathcal{H}^{n-1}(y)\\ + \int_{A_r} |Du^{f-e}_{\eta-\xi,y}|([-r,r])\,\mathrm{d}\mathcal{H}^{n-1}(y)\\ + \int_{A_r} |Du^{e}_{\xi,y}|([-r,r])\,\mathrm{d}\mathcal{H}^{n-1}(y)\bigg) + \mathrm{O}(r).\end{aligned}$$

Since all pairs $(\eta+\xi, f+e), (\eta-\xi, f-e), (\xi, e)$ all belong to $\partial\sigma(\mathcal{A})$, we may control each of the terms on the right hand side by $|\mathcal{A}u|(Q_{2r})$. The first part of the proof and the fact that $0 \notin \Theta_u$ give

$$|e^+ - e^-| \lesssim \limsup_{r\to 0^+}\left(\frac{C}{r^{n-1}}|\mathcal{A}u|(Q_{2r} + \mathrm{O}(r)\right) = 0.$$

This shows that $0 \in \pi_\xi$ is a Lebesgue point of v.

Step 3. Proof of continuity by transversality. Now, we use the continuity proved in Step 2, and another suitable transversality argument to show that $0 \in \mathbb{R}^n$ is a Lebesgue point of $\mathbf{p}^e u$ in $\mathbb{R}^n$. This part of the proof follows by verbatim the arguments contained in the proof of Theorem 6.4 in [2]. Since $0 \notin \Theta_u$, the quasi-continuity established in [8, Prop. 1.2] implies that there exist a family of reals $\{d_r\}_{r\in\mathbb{R}} \subset V$ such that

$$\frac{1}{r^n}\int_{Q_r} |u^{\mathbf{v}}(y) - d_r^{\mathbf{v}}|\,\mathrm{d}y \le \frac{1}{r^n}\int_{Q_r}|u - d_r| = \mathrm{O}(r) \quad \forall\, \mathbf{v} \in (V_e)^*, \tag{31}$$

where we have used the short-hand notation $d_r^{\mathbf{v}} := \langle \mathbf{v}, d_r\rangle$. Notice that a priori this is not enough to ensure Lebesgue continuity since the sequence $\{d_r\}$ may not be convergent as $t \to 0^+$. Applying Fubini's Theorem we further obtain

$$\lim_{r\to 0^+} \frac{1}{r^{n-1}}\int_{C_r}\left(\frac{1}{r}\int_{-r}^{r}|u^{\mathbf{v}}(y+t\xi) - d_r^{\mathbf{v}}|\,\mathrm{d}t\right)\mathrm{d}\mathcal{H}^{n-1}(y) = 0,$$

for all $\mathbf{v} \in (V_e)^*$. Now, fix $r > 0$ an arbitrary radius. A standard measure theoretic argument applied to the previous estimate yields the existence of $\rho = \rho(r) \in (r/2, r)$

such that

$$\frac{1}{r^{n-1}}\int_{C_r}|\tilde{u}^{\mathbf{v}}(y+\rho\xi)-d_r^{\mathbf{v}}|+|\tilde{u}^{\mathbf{v}}(y-\rho\xi)-d_r^{\mathbf{v}}|\,\mathrm{d}\mathcal{H}^{n-1}(y)\leq 2\mathrm{O}(r). \tag{32}$$

We know that v is approximately continuous at 0 when restricted to π_ξ (cf. Step 2). Let us write $d \coloneqq \tilde{v}(0)$ to denote its Lebesgue point at $0 \in \pi_\xi$. Let $j \in \{1,\dots,\ell\}$ where ℓ is the integer in (27). Similarly to the previous step, we write the difference $[\tilde{u}^{f_j}(y+\rho\xi)-\tilde{u}^{f_j}(y-\rho\xi)-2d^{f_j}]$ as a sum of good transversal differences by setting:

$$\begin{aligned}\tilde{u}^{f_j}(y+\rho\xi)-&\tilde{u}^{f_j}(y-\rho\xi)-2d^{f_j}\\ &=\tilde{u}^{f_j+e}(y+\rho\xi)-\tilde{u}^{f_j+e}(y-\rho\eta_j)\\ &\quad+\tilde{u}^{f_j-e}(y-\rho\xi)-\tilde{u}^{f_j-e}(y-\rho\eta_j)\\ &\qquad+\tilde{u}^{e}(y-\rho\xi)-\tilde{u}^{e}(y+\rho\xi)+\underbrace{2[\tilde{u}^{f_j}(y-\rho\eta_j)-d^{f_j}]}_{\text{app. cont. difference over }\pi_\xi},\end{aligned}$$

Integrating both sides over C_r, we are now in position to use a suitable transversality reasoning as in the previous step (see also (6.1) in [2]) where we exploit that $(\eta_j,f_j)\in\partial\sigma(\mathcal{A})$:

$$\begin{aligned}\frac{1}{r^{n-1}}\int_{C_r}&|\tilde{u}^{f_j}(\rho+t\xi)-\tilde{u}^{f_j}(y-\rho\xi)-2d^{f_j}|\,\mathrm{d}\mathcal{H}^{n-1}(y)\\ &\lesssim\frac{1}{r^{n-1}}|\mathcal{A}(Q_{2r})|+\frac{1}{r^{n-1}}\int_{C_r}|\tilde{u}^{f_j}(y-\rho\eta_j)-d^{f_j}|\,\mathrm{d}\mathcal{H}^{n-1}(y).\end{aligned}$$

Then, the continuity of $p_e u$ at $0\in\pi$ and the fact that $0\notin\Theta_u$ imply

$$\lim_{r\to0^+}\frac{1}{r^{n-1}}\int_{C_r}|\tilde{u}^{f_j}(\rho+t\xi)-\tilde{u}^{f_j}(y-\rho\xi)-2d^{f_j}|\,\mathrm{d}\mathcal{H}^{n-1}(y)=0.$$

This estimate and (32) imply that $d_r^{f_j}$ converges to d^{f_j} as $r\to0^+$. Since $\{f_j\}_{j=1}^{\ell}$ spans $(V_e)^*$, we deduce that $\mathbf{p}^e[d_r]\to\mathbf{p}^e[d]$ as $r\to0^+$. We conclude from (31) that

$$\lim_{r\to0^+}\frac{1}{r^{n-1}}\int_{C_r}|\mathbf{p}^e u(y)-\mathbf{p}^e[d]|\,\mathrm{d}y=0,$$

which proves that 0 is indeed a Lebesgue point of $\mathbf{p}^e u$ at $0\in\mathbb{R}^n$. This finishes the proof. ☐

Corollary 5.1. $|\mathcal{A}u|$*-almost every* $x\in\Omega\setminus\Theta_u$ *is a Lebesgue point of* u*. In particular,*

$$|\mathcal{A}u|(S_u\setminus J_u)=0.$$

Proof. With all the algebraic structure we have developed now, the proof should follow the same lines as the one given in [2, Proposition 6.8]. For the convenience of the reader we include it here.

We may assume that $n\geq2$ for otherwise the proof follows from the one-dimensional classical BV-theory. Since the claim is local, we may as well assume that $u\in\mathrm{BV}^{\mathcal{A}}(\mathbb{R}^n)$. Our proof is based on an induction argument over the dimension n. We assume that continuity assertion holds for all elliptic operators satisfying the rank-one property that act on spaces of functions of $(n-1)$-variables, and we prove the result for operators acting on functions spaces of n-variables. The step of induction is clear since the case $n-1=1$ is covered by the BV-theory.

Claim 1. Let us write $\mu=\mathcal{A}u$ and $\mu_\xi^e\coloneqq\langle w,\mu\rangle$ where $g_{\mathbb{A}}(w)=\xi\otimes e$ so that $\mu_\xi^e=\partial_\xi u^e$. The first task will be to show that $|\mu_\xi^e|$-a.e. point in Ω must either be contained in Θ_u or be a Lebesgue point of u^e. To this end, let us choose a pair $(\eta, f) \in \partial\sigma(\mathcal{A})$ satisfying $\eta \in \pi_\xi$ and $f \in (V_e)^*$. In particular, this ensures that

$$e \in (\mathbf{p}^f[V])^* = (V_f)^*, \quad |\mu^a_\xi| \lesssim |\mathbf{p}^f_\eta[\mu]|. \tag{33}$$

The codimension-one slicing tells us that $|\mathbf{p}^f_\eta[\mu]|$ is concentrated on points of the form $y + z\eta$, where $y \in \pi_\eta$ and $z \in S$ for some $S \subset \ell_\eta$ with $\mathcal{H}^1(\ell_\eta \setminus S) = 0$. Moreover, the slices $\pi_\eta \ni y \mapsto v_z(y) := \mathbf{p}^f u(y + z)$ are well-defined and all belong to $\mathrm{BV}^{\mathcal{B}}(\pi_\xi)$ for every $z \in S$, where $\mathcal{B} = \mathcal{A}^f_\eta \llcorner \pi_\xi$ is elliptic and satisfies the rank-one property. The hypothesis of induction then ensures that, at every $z \in S$, $|\mathcal{B}v_z|$-almost every point in π_η has one-sided Lebesgue points (with respect to some direction). The slicing identity

$$|\mathbf{p}^f_\eta[\mu]|(B) = \int_S |\mathcal{B}v_z|(B_z)\, \mathrm{d}\mathcal{H}^1(z).$$

and Lemma 5.1 further imply that every such point either belongs to Θ_u or it must be a Lebesgue point of $\mathbf{p}^f \circ u$ in $\mathbb{R}^n$. In sight of (33) this shows that

$$|\mu^e_\xi|\text{-almost every } x \in \Omega \setminus \Theta_u \text{ is a Lesbegue point of } u^e.$$

This proves the first claim.

The next step is to use this property of u and $\Theta_u^\complement$ to actually lift the Lebesgue continuity to all the coordinates of u, at $|\mu|$-almost every $x \in \Omega \setminus \Theta_u$. In light of the rank-one property, it shall be enough to show that u^e is Lebesgue continuous at $|\mu^f_\eta|$-almost every point in $\Omega \setminus \Theta_u$, where (η, f) is an arbitrary pair in $\partial\sigma(\mathcal{A})$. Clearly, we may assume that $f \notin \ell_e$ for otherwise the claim above implies the desired result. By a change of variables we may further assume that $f \in (V_e)^* = 0$. Let $\xi \in \pi_\eta$ be such that $(\xi, e) \in \partial\sigma(\mathcal{A})$. In view of the polarization result in Lemma 4.4 we may find a direction $\omega \in \ell_\xi$ and a coordinate $v \in \ell_e$ for which

$$(\eta + \omega, f + v), (\eta - \omega, f - v) \in \partial\sigma(\mathcal{A}).$$

On the other hand, the linearity of (distributional) differentiation ensures that

$$\begin{aligned} \mu^{f\pm v}_{\eta\pm\omega} &= \partial_{\eta\pm\omega} u^{f\pm v} \\ &= \partial_\xi u^f \pm (\partial_\eta u^v + \partial_\omega u^f) + \partial_\omega u^v \\ &= \mu^f_\eta \pm (\partial_\eta u^v + \partial_\omega u^f) + \mu^v_\omega \\ &=: \mu^f_\eta \pm \sigma + \mu^v_\omega. \end{aligned}$$

In particular the distribution σ is a measure, which is absolutely continuous with respect to Λ. let $\mathcal{O} \subset \Omega$ be a full $|\mu|$-measure set where the density $\mu/|\mu|$ exists and consider the measure $\Lambda := |\mu^f_\eta| + |\sigma| + |\mu^v_\omega| \ll |\mu|$. There exist Borel functions $\lambda_1, \lambda_2, \lambda_3 : \mathcal{O} \to \mathbb{R}$ such that

$$\mu^f_\eta = \lambda_1\Lambda, \quad \mu^v_\omega = \lambda_2\Lambda, \quad \sigma = \lambda_3\Lambda.$$

We split $(\mathbb{R}^n - \Theta_u)$ into the disjoint Borel sets

$$\begin{aligned} A_1 &:= \{\, x \in \mathbb{R}^n - \Theta_u : \lambda_1(x) + \lambda_2(x) = 0 \,\}, \\ A_2 &:= \{\, x \in \mathbb{R}^n - \Theta_u : \lambda_1(x) + \lambda_2(x) \neq 0 \,\}. \end{aligned}$$

On A_1 the total variation measures of μ^f_η and μ^v_ξ are identical and therefore (recall that $v \in \ell_e$) u^e is Lebesgue continuous $|\mu^f_\eta|$-almost everywhere on A_1. On the other hand, the measure $|\mu^{f+v}_{\eta+\omega}| + |\mu^{f-v}_{\eta-\omega}|$ is strictly positive in A_2. This implies that $|\mu|$-almost every point in $x \in A_2$ is either a Lebesgue point of u^{f+v} or a Lebesgue point of u^{f-v}. By linearity, we conclude that u^e is Lebesgue continuous $|\mu^f_\eta|$-almost everywhere in A_2. This shows that u^e is Lebesgue continuous $|\mu^f_\eta|$-almost everywhere on $\mathbb{R}^n \setminus \Theta_u$, as desired. Lastly, since both $e \in V^*$ and $(\eta, f) \in \partial\sigma(\mathcal{A})$ were chosen in an arbitrary manner, the rank-one property implies that (recall that $J_u \subset \Theta_u$ and $|\mathcal{A}u|(\Theta_u \setminus J_u) = 0$)

$$|\mathcal{A}u|(S_u \setminus J_u) = 0.$$

This finishes the proof. □

Arguing as in [2, Thm. 6.1] we obtain the following more general statement:

Corollary 5.2. *Let $\mathcal{A}$ be as in the previous Corollary and let u, v be functions in $\mathrm{BV}^{\mathcal{A}}(\Omega)$. Then*

$$|\mathcal{A}u|(S_v \setminus J_v) = 0.$$

Corollary 5.3. *Let $u \in \mathrm{BV}^{\mathcal{A}}(\mathbb{R}^n)$ and let $(\xi, e) \in \partial\sigma(\mathcal{A})$ be a non-trivial pair. Set $\mathcal{B} := \mathcal{A}^e_\xi \llcorner \pi_\xi$, and, for a vector $z \in \ell_\xi$, write $v_z : \pi_\xi \to V_e$ to denote the function defined by*

$$v_z(y) := \mathbf{p}^e u(z + y).$$

There exists a full $\mathscr{L}^1$-measure set $Z \subset \ell_\xi$ with the following property: if $z \in Z$, then

1. *$\mathcal{H}^{n-1}$-almost every point in $z + \pi_\xi$ is a Lebesgue point of u,*
2. *$v_z \in \mathrm{BV}^{\mathcal{B}}(\pi_\xi)$,*
3. *$|\mathcal{B}^j v_z|(\pi_\xi \setminus (\Theta_u)_z) = 0$.*

Proof. By Fubini's theorem, there exists a full $\mathscr{L}^1$-measure set $Z_1 \subset \ell_\xi$ such that $\mathcal{H}^{n-1}$-almost every point of $z + \pi_\xi$ is a Lebesgue point of u. On the other hand, the slicing on co-dimension one planes yields another full $\mathscr{L}^1$-measure set $Z_2 \subset \ell_\xi$ where $v_z \in \mathrm{BV}^{\mathcal{B}}(\pi_\xi)$. Set $Z := Z_1 \cap Z_2$. Then Z is a full $\mathscr{L}^1$-measure set of ℓ_ξ. Moreover, since $\mathcal{B}$ is elliptic and satisfies the rank-one property, v_z has one-sided Lebesgue limits at $|\mathcal{B}^j v_z|$-almost every $y \in \pi_\xi$ (for all $z \in Z$). Therefore, the first assertion of the previous lemma implies that, for every $z \in Z$, the function v_z is Lebesgue continuous $|\mathcal{B}v_z|$-almost everywhere in $\pi_\xi \setminus (\Theta_u)_z$. □

6. Proof of the one-dimensional structure theorem

We have now all the necessary tools to give a proof of the one-dimensional structure theorem (Theorem 2.2).

Proof of Theorem 2.2. It suffices to show that

$$\left[\int_{\pi_\xi} Du^e_{y,\xi} \,\mathrm{d}\mathcal{H}^{n-1}(y)\right]^\sigma = \int_{\pi_\xi} D^\sigma u^e_{y,\xi} \,\mathrm{d}\mathcal{H}^{n-1}(y), \qquad \sigma = a, c, j, \tag{34}$$

For ease of notation, let

$$\nu = \int_{\pi_{\xi_i}} Du^e_{y,\xi} \,\mathrm{d}\mathcal{H}^{n-1}(y).$$

Then, we have

$$\nu^a - \int_{\pi_\xi} D^a u^e_{y,\xi_i} \,\mathrm{d}\mathcal{H}^{n-1}(y) = \int_{\pi_\xi} D^s u^e_{y,\xi} \,\mathrm{d}\mathcal{H}^{n-1}(y) - \nu^s.$$

Fubini's Theorem yields that the left-hand side is absolutely continuous with respect to $\mathscr{L}^n$, meanwhile the right-hand side is concentrated on a set of $\mathscr{L}^n$-measure zero. Thus, both sides must vanish, and so we conclude that (34) indeed holds for $\sigma = a$.

Now, let us further decompose the singular part into the Cantor and jump parts:

$$\nu^j - \int_{\pi_\xi} D^j u^e_{y,\xi} \,\mathrm{d}\mathcal{H}^{n-1}(y) = \nu^c - \int_{\pi_\xi} D^c u^e_{y,\xi} \,\mathrm{d}\mathcal{H}^{n-1}(y) \\ (\nu^j - \mu^j \triangleq \nu^c - \mu^c). \tag{35}$$

Once again, we aim to show that both sides vanish. For a start, consider the restriction of both sides to the jump set J_u. We know that J_u is countably $\mathcal{H}^{n-1}$ rectifiable (see [8, 26]), and hence an elementary measure-theoretic argument shows that for $\mathcal{H}^{n-1}$-almost every point $y \in \pi_\xi$, the line $\{ y + t\xi : t \in \mathbb{R} \}$ and J_u intersect at most on a countable set. However, $D^c u^e_{y,\xi}$ is non-atomic, so it vanishes on this intersection. In conclusion, we have $\mu^c \llcorner J_u = 0$. Let us recall from Corollary 4.4 that $\mathcal{H}^{n-1}(\Theta_u \setminus J_u) = |\mathcal{A}u|(\Theta_u \setminus J_u)$. Therefore,

$$\nu^c \llcorner (\Theta_u \setminus J_u) = \mu^c \llcorner (\Theta_u \setminus J_u) = 0.$$

Moreover, $\nu^c \llcorner \Theta_u = 0$, since this is a component of $\mathcal{A}^c u \llcorner J_u \equiv 0$ in some direction. We conclude that the left-hand side of (35) vanishes when restricted to Θ_u. We also know that the essential support of ν^j is contained in J_u, so it remains to show that the other term μ^j is concentrated purely on Θ_u. Indeed, if this was the case, then both sides must vanish identically, whence we obtain $\nu^j \equiv \mu^j$ and $\nu^c \equiv \mu^c$ as desired.

Let us prove that μ^j vanishes in $\Omega \setminus \Theta_u$. The proof we give here follows closely the n-dimensional induction-based proof in [2]. The main difficultly, however, lies in circumventing the lack of a well-defined structure of elliptic operators satisfying the slicing property.

Step 1. The step of induction $n = 2$. The measure μ^j is concentrated on slices $\ell_\xi + y$ where $y \in Y$ and $\mathscr{L}^1(\pi_\xi \setminus Y) = 0$. By Fubini's theorem, we may assume without loss of generality that, for all $y \in Y$, $\mathcal{H}^1$-almost every point of $\mathbb{R}$ is a Lebesgue point of $v_y := u^e_{y,\xi}$. Now, let $(\eta, f) \in \partial\sigma(\mathcal{A}^e_\xi \llcorner \pi_\xi)$ so that

$$(\xi, e) \in \partial\sigma(\mathcal{A}^f_\eta \llcorner \pi_\eta).$$

Set $\mathcal{B} := \mathcal{A}^f_\eta \llcorner \pi_\eta$ and consider the isometry $\psi : \ell_\xi \to \mathbb{R} : t\xi \mapsto t$. By construction we have $\ell_\xi = \pi_\eta$ and we can find a positive constant $c > 0$ such that (since $n = 2$, we have $\ell_\eta = \pi_\xi$)

$$|Du^e_{y,\xi}| \le c\psi_\# |\mathcal{B}(\mathbf{p}^f u)_y| \quad \text{as measures over } \mathbb{R}, \text{ for } \mathscr{L}^1\text{-a.e. } y \in \pi_\xi. \tag{36}$$

Notice also that $(\mathbf{p}^f u)_y(\psi(t)) := \mathbf{p}^f u(y + \psi(t)) = \mathbf{p}^f u_{y,\xi}(t)$ for all $t \in \mathbb{R}$. Thus, using Corollary 4.1 we verify that

$$\int_{\pi_\xi} |Du^e_{y,\xi}| \,\mathrm{d}\mathcal{H}^1(y) \le c \int_{\ell_\eta} \psi_\# |\mathcal{B}(\mathbf{p}^f u)_y| \,\mathrm{d}\mathcal{H}^1(y),$$

where the inequality holds in the sense of (bounded) measures on $\mathbb{R}^2$. Therefore, up to adding a set of $\mathscr{L}^1$-measure zero to Y, we may assume that the measure on the right hand side is concentrated on slices of the form $\ell_\xi + y$, with $y \in Y$. Moreover, since $\mathcal{B}$ is an elliptic on functions of one variable, it must be that $(\mathbf{p}^f u)_y \in \mathrm{BV}(\mathbb{R})$ for all $y \in Y$. Therefore, the one-dimensional BV-theory implies that, for all $y \in Y$, the map $(\mathbf{p}^f u)_y$ has one-sided Lebesgue points for $\psi_\# |\mathcal{B}(\mathbf{p}^f u)_y|$-a.e. $t \in \mathbb{R}$ and all $y \in Y$. Applying Lemma 5.1 to u and (η, f) we thus infer from (36) that $(\mathbf{p}^f u)_z \circ \psi$ is Lebesgue continuous at $|Du^e_{y,\xi}|$-a.e. $t \in \mathbb{R}$ such that $y + t\xi \notin \Theta_u$. Since also $e \in (V_f)^*$, we conclude that

$$\left(\int_{\pi_\xi} |D^j u^e_{y,\xi}| \,\mathrm{d}\mathcal{H}^{n-1} \right) \llcorner \Theta_u^{\complement} \equiv 0.$$

This proves that μ^j vanishes on $\Omega \setminus \Theta_u$ as desired.

Step 3. The induction argument. Let $n \geq 3$ and assume that the the conclusion of the Theorem holds for all operators satisfying the mixing condition in dimensions $m \leq n-1$. Let $(\eta, f) \in \partial\sigma(\mathcal{A})$ be a pair satisfying

$$(\xi, e) \in \partial\sigma(\mathcal{A}^f_\eta \llcorner \pi_\eta).$$

Let $\mathcal{B} := \mathcal{A}^f_\eta \llcorner \pi_\eta$ and write $v_z(y) := \mathbf{p}^f u(z+y)$. The slicing on co-dimension one planes implies the existence of a set $Z \subset \ell_\eta$ with $\mathscr{L}^1(\ell_\eta \setminus Z) = 0$, and such that the slices v_z belong to $\mathrm{BV}^{\mathcal{B}}(\pi_\eta)$ for all $z \in \ell_\eta$. Moreover, the hypothesis of induction applied to the operator $\mathcal{B}$ and v_z for all $z \in Z$ yield the identity

$$I^\sigma := \int_{\ell_\eta} \bigg(\int_{\pi_\xi \cap \pi_\eta} |D^\sigma (v_z)^e_{\xi,\tilde{y}}| \, \mathrm{d}\mathcal{H}^{n-2}(\tilde{y}) \bigg) \, \mathrm{d}z \lesssim \int_{\ell_\eta} |\mathcal{B}^\sigma v_z| \, \mathrm{d}\mathcal{H}^1(z) \qquad (37)$$

for all $\sigma = a, c, j$. Observe that I^σ is non-trivial since we have assumed that $n > 2$.

Next, recall that by construction $e \in (V_f)^*$. This observation and Fubini's theorem imply the equality of functions

$$(v_z)^e_{\xi,\tilde{y}} \equiv (v_z)^e(\tilde{y} + \bullet\, \xi) \equiv (\mathbf{p}^f u)^e(z + \tilde{y} + \bullet\, \xi) \equiv u^e_{\xi, z+\tilde{y}} \quad \text{in } \mathrm{BV}(\mathbb{R})$$

for $\mathcal{H}^{n-2}$-almost every $\tilde{y} \in (\pi_\xi \cap \pi_\eta)$ and every $z \in Z$. Therefore, applying Fubini's theorem and appealing to the classical structure theorem for functions of bounded variation we further deduce that

$$I^\sigma = \int_{\pi_\xi} |D^\sigma u^e_{y,\xi}| \, \mathrm{d}\mathcal{H}^{n-1}(y) = |\mu^j|, \qquad \sigma \in \{a, c, j\}. \qquad (38)$$

On the other hand, from Corollary 5.3 we find that $|\mathcal{B}^j v_z|((\pi_\xi \setminus \Theta_u)_z) = 0$ for $\mathscr{L}^1$-almost every $z \in \ell_\eta$. This proves that

$$\bigg(\int_{\ell_\eta} |\mathcal{B}^j v_z| \, \mathrm{d}\mathcal{H}^1(z) \bigg) \llcorner \Theta_u^{\complement} \equiv 0 \quad \text{as measures over } \mathbb{R}^n.$$

To conclude we recall the estimate (37), which in light of the previous equality of measure implies that $I^j \llcorner (\mathbb{R}^n \setminus \Theta_u)$ must be the zero measure. The sought assertion $\mu^j \llcorner (\mathbb{R}^n \setminus \Theta_u) \equiv 0$ follows from (38). $\square$

7. Proof of the fine properties statements

With all the analysis developed so far, we are now able to collect the proofs of the statements for the fine properties:

Proof of Theorem 2.3. In light of Remark 2.3, we shall only discuss the structure theorem decomposition and statements (iii)-(v). The assertions contained in (iii)-(iv) follow directly from the results of Corollary 4.4, while assertion (v) follows from Corollary 5.2. As discussed in the introduction, one can trivially decompose $\mathcal{A}u$ into mutually singular measures

$$\mathcal{A}u = \mathcal{A}^a u + \mathcal{A}^c u + \mathcal{A}^d u + \mathcal{A}^j,$$

where $\mathcal{A}^d u = \mathcal{A}^s u \llcorner (S_u \setminus J_u)$. By virtue of (v) with $v = u$, we deduce that $|\mathcal{A}^d u| \equiv 0$ whenever $\mathcal{A}$ is elliptic and satisfies the mixing property. This finishes the proof. $\square$

Proof of Theorem 2.4. The proof of this result follows directly from Theorem 2.3 and the results contained in Section 3.4.1. $\square$

8. Notions and proofs for operators of arbitrary order

The task of the following lines is to extend the slicing theory that we have already introduced for first order operators. As it has already been discussed in the introduction, due to the ellipticity assumption, we shall focus on the fine properties of the $(k-1)^{\text{th}}$ gradients of functions in $\mathrm{BV}^{\mathcal{A}}$-spaces. Therefore, we also look for a slicing theory in terms of $\nabla^{k-1}u$ rather than u itself. This leads us to consider a notion of $\operatorname{rank}_{\mathcal{A}}$ that extends the one for first-order operators and whose $\operatorname{rank}_{\mathcal{A}}$-one elements satisfy

$$\langle w, \mathcal{A}u\rangle = \partial_\xi(\langle E, \nabla^{k-1}u\rangle) =: \partial_\xi U^E$$

for some $\xi \in \mathbb{R}^n$ and some $E \in V^{k-1} := V^* \otimes E_{k-1}(\mathbb{R}^n)$. We are naturally led to define, at least formally, the $\operatorname{rank}_{\mathcal{A}}$-one vectors as those $w \in W^*$ such that

$$\langle w, \mathbb{A}^k(\eta)v\rangle = \langle \xi, \eta\rangle\langle E, v \otimes^{k-1} \eta\rangle \qquad \text{for all } \xi \in \mathbb{R}^n,\ M \in V \otimes E_{k-1}(\mathbb{R}^n) \tag{39}$$

for some $\xi \in \mathbb{R}^n$ and $E \in V^{k-1}$.

Definition 8.1. Let $w \in W^*$. We say that $w \in \mathcal{A}_1^{\otimes}$ if and only if w satisfies (39) for some $(\xi, E) \in \partial\sigma(\mathcal{A})$.

Notice that, such covectors w, are precisely those with the property that

$$\langle w, \mathbb{A}^k(\eta)\widehat{u}(\eta)\rangle = \langle \xi \cdot \eta\rangle\langle E, \mathcal{F}(\nabla^{k-1}u)(\eta)\rangle,$$

so that inverting the Fourier transform we obtain the desired identity $\langle w, \mathcal{A}u\rangle = \partial_\xi U^E$ for all $u \in \mathcal{S}(\mathbb{R}^n; V)$. If slicing is to be useful, we need that $\mathcal{A}$ controls a sufficient number of partial derivatives $\partial_\xi(\ \ )^E$. Now, observe that if $\operatorname{rank}_{\mathcal{A}}(w) = 1$, then

$$w \in \bigcap_{\eta \in \pi_\xi} \ker \mathbb{A}^k(\eta)^*. \tag{40}$$

Hereby we deduce that the mixing property $(\mathfrak{m}_k)$ is a necessary condition for the slicing of $\mathcal{A}u$ and $\nabla^{k-1}u$. The remaining question is whether this is also a sufficient condition. The first issue at hand is that $\mathbb{A}^k$ is no longer a linear map, but a k-homogeneous map. Our first step will be to deal with the k-homogeneity of $\mathcal{A}$:

Definition 8.2 (linearized symbol)**.** Let $\mathcal{A} : \mathrm{C}^\infty(\mathbb{R}^n; V) \to \mathrm{C}^\infty(\mathbb{R}^n; W)$ be a k^{th} order partial differential operator. Let us recall that the principal symbol $\mathbb{A}^k$ is a k-homogeneous map $\mathbb{A}^k : \mathbb{R}^n \to W \otimes V^*$, and therefore there exists a (uniquely determined) linear map

$$\overline{\mathbb{A}^k} : V \otimes E_k(\mathbb{R}^n) \to W$$

satisfying

$$\overline{\mathbb{A}^k}[v \otimes^k \xi] = \mathbb{A}^k(\xi)[v] \quad \text{for all } \xi \in \mathbb{R}^n,\ v \in V.$$

The main difference is that we are considering a linear map. Of course, the cost to pay is that we are adding a considerable amount of k-order tensors. In analogy with the first order case, we consider the rank induced by the pullback of $\overline{\mathbb{A}^k}$:

Now, we lift the algebraic linearization to a linearization of $\mathcal{A}$, which will allow us to make use the better part of the theory we have developed for first-order operators. In order to preserve ellipticity, the idea is to add a curl-operator to compensate for the addition of $(k-1)^{\text{th}}$ order tensors occurring in the linearization process of $\mathbb{A}^k$:

Definition 8.3 (Linearized operator)**.** We define the linearization of $\mathcal{A}$ to be the first-order operator given by

$$d\mathcal{A} := \overline{\mathbb{A}^k}(DU) \times \operatorname{curl}_{k-1} U, \qquad U : \mathbb{R}^n \to V^k,$$

where, for a positive integer m,

$$\operatorname{curl}_m U := (\partial_i U^\ell_{\mathbf{e}_j+\beta} - \partial_j U^\ell_{\mathbf{e}_i+\beta})_{i,j,\beta}, \qquad |\beta| = m-1,\ \ell = 1,\dots,\dim(V),$$

is the generalized curl operator on V^{m+1}-valued tensor fields.

It is well-known (see [23, Example 10.3(d)]) that $\ker \operatorname{curl}_m(\xi) = \{\, v \otimes^m \xi : v \in V \,\}$. Therefore, by definition, we have

$$d\mathbb{A}(\xi)[v \otimes^{k-1} \xi] = (\mathbb{A}^k(\xi)[v], 0) \qquad \text{for all } \xi \in \mathbb{R}^n,\ v \in V. \tag{41}$$

This shows that $d\mathcal{A}$ indeed behaves a derivative, in the sense that it is a linear operator that acts on $(k-1)$-order gradients as $\mathcal{A}$, i.e.,

$$d\mathcal{A}(\nabla^{k-1}u) = (\mathcal{A}u, 0), \qquad u : \mathbb{R}^n \to V. \tag{42}$$

The linearization satisfies the following crucial properties:

Lemma 8.1. *Let $\mathcal{A} : \mathrm{C}^\infty(\mathbb{R}^n; V) \to \mathrm{C}^\infty(\mathbb{R}^n; W)$ be a k^{th} order partial differential operator. The following holds:*

(1) $d\mathcal{A}$ is $\mathbb{K}$-elliptic if and only if $\mathcal{A}$ is $\mathbb{K}$-elliptic for $\mathbb{K} = \mathbb{C}, \mathbb{R}$,
(2) if $\mathcal{A}$ is elliptic, then $u \in \mathrm{BV}^{\mathcal{A}}(\Omega)$ if and only if $\nabla^{k-1}u \in \mathrm{BV}^{d\mathcal{A}}(\Omega)$,
(3) $\{\, w \in W^* : w$ satisfies (40)$\,\} = \operatorname{Proj}_{W^*}(d\mathcal{A})^{\otimes}_1 \subset \mathcal{A}^{\otimes}_1$,
(4) $\partial\sigma(\mathcal{A}) = \partial\sigma(d\mathcal{A})$.

Remark 8.1 (Non-stability of the mixing property under linearization)**.** In general it is not true that if $\mathcal{A}$ is elliptic and satisfies $(\mathfrak{m}_k)$, then its linearization $d\mathcal{A}$ satisfies $(\mathfrak{m})$. In particular, we cannot expect the k^{th} order theory to follow trivially from linearization.

Proof of Proposition 8.1. The proof of (1) and (2) is contained in [8, Sec. 5].

Let us prove (3). Let $(w, h) \in (d\mathcal{A})^{\otimes}_1$. By definition, we may find (w, h)

$$\langle w, \overline{\mathbb{A}^k}[M \odot \eta]\rangle + \langle h, \operatorname{curl}(\eta)[M]\rangle = \langle \xi, \eta\rangle\langle E, M\rangle.$$

Taking $M = v \otimes^{k-1} \eta$ above, we deduce that

$$\langle w, \mathbb{A}^k(\eta)[v]\rangle = \langle \xi, \eta\rangle\langle E, v \otimes^{k-1} \eta\rangle.$$

This shows that $w \in \mathcal{A}^{\otimes}_1$ and therefore w satisfies (40). The same argument shows that $\partial\sigma(d\mathcal{A}) \subset \partial\sigma(\mathcal{A})$. The challenging part is to show the other contention of the equality. Let us assume that w and (ξ, E) satisfy (40). Using the linearity of $\overline{\mathbb{A}^k}$ and a polarization argument, we find that

$$\langle w, \overline{\mathbb{A}^k}[Q]\rangle = 0 \qquad \text{for all } Q \in V \otimes E_k(\pi_\xi).$$

Let $\{\xi, \zeta_2, \dots, \zeta_n\}$ be an orthonormal basis of $\mathbb{R}^n$ so that

$$\Big\{\, e^\alpha := (\alpha!)^{-1}(\otimes^{\alpha_1}\xi) \odot (\otimes^{\alpha_2}\zeta_2) \odot \cdots \odot (\otimes^{\alpha_n}\zeta_n) : |\alpha| = k \,\Big\}$$

is an orthogonal basis of $E_k(\mathbb{R}^n)$. Notice that the orthogonal complement of $E_k(\pi_\xi)$ in V_k is given by $\operatorname{span}\{\, e^\alpha : \alpha_1 \geq 1, |\alpha| = k \,\}$. Then, by the representation of linear maps, we may find reals $\gamma^\ell_\alpha \in \mathbb{R}$ such that

$$w \cdot \overline{\mathbb{A}^k}[M \odot \eta] = \sum_{\substack{\ell=1,\dots,N. \\ \alpha^\ell_1 \geq 1, \\ |\alpha^\ell| = k}} \gamma_\alpha \langle \mathbf{v}^\ell \otimes e^\alpha, M \odot \eta\rangle, \qquad M \in V_{k-1},$$

where $\{\mathbf{v}^1, \dots, \mathbf{v}^N\}$ is any orthonormal basis of V^*. Now, writing $\eta = \eta_1\xi + \eta_2\zeta_2 + \cdots + \zeta_n\eta_n$, and $(M)_{\ell,\beta} = m^\ell_\beta$, we find that

$$M \odot \eta = \sum_{\substack{j=1,\dots,n,\\ \ell=1,\dots,N,\\ |\beta|=k-1}} \eta_j m^\ell_\beta (\mathbf{v}_\ell \otimes e^{\beta+\mathbf{e}_j}).$$

This, in turn, yields the identity

$$\langle \mathbf{v}^\ell \otimes e^\alpha, M \odot \eta \rangle = \eta_j m^\ell_\beta \delta_{(\beta+\mathbf{e}_j)\alpha}, \qquad |\alpha| = k.$$

Next, we exploit the representation of linear maps by elements in the complement of their kernel to deduce that

$$\langle w, \overline{\mathbb{A}^k}[M \odot \eta] \rangle = \sum_{\substack{j=1,\dots,n\\ \ell=1,\dots,N}} \sum_{\substack{\beta+\mathbf{e}_j=\alpha,\\ |\alpha|=k,\\ \alpha_1\geq 1}} \eta_j m^\ell_\beta \gamma^\ell_\alpha = I_1 + \cdots + I_n,$$

where

$$I_m(\eta)[M] \coloneqq \sum_{\substack{j=1,\dots,n\\ \ell=1,\dots,N}} \sum_{\substack{\beta+\mathbf{e}_j=\alpha,\\ |\alpha|=k,\\ \alpha_1=m}} \eta_j m^\ell_\beta \gamma^\ell_\alpha.$$

The goal now is to prove that $I_m(\eta)[M] = \langle \xi, \eta \rangle \langle E_m, M \rangle + \langle h_m, \mathrm{curl}_{k-1}(\eta)[M] \rangle$ for some constant tensors E_m, h_m. Since the argument for arbitrary m is analogous to the one for $m = 1$, we shall focus on proving the sought representation for I_1. We may write

$$I_1 = II_1 + III_1 \coloneqq \sum_{\substack{\ell=1,\dots,N,\\ |\beta|=k-1}} \eta_1 m^\ell_\beta \gamma^\ell_{\beta+\xi} + \sum_{\substack{j=2,\dots,n\\ \ell=1,\dots,N}} \Bigg(\sum_{\substack{\xi+\omega+\zeta_j=\alpha,\\ |\omega|=k-2,}} \eta_j m^\ell_{\omega+\xi} \gamma^\ell_\alpha \Bigg).$$

Now, the first term II_1 can be written as $\langle \xi, \eta \rangle \langle E_1, M \rangle$ where $E_1 \in E_{k-1}(\mathbb{R}^n)$ is expressed as in coordinates as $E_1 = (\gamma^\ell_{\beta+\mathbf{e}_1})_{\ell,\beta}$. Thus, we only have to check that the second term on the right-hand side can be expressed as a linear combination of terms of $\mathrm{curl}_{k-1}(\eta)[M]$. First, let us calculate the $(k-1)$-curl operator on simple tensors for $j \geq 2$:

$$\begin{aligned} \mathrm{curl}_{k-1}(\zeta_j)[\mathbf{v}_\ell \otimes e^{\omega+\xi}] &= \mathbf{v}_\ell (\delta_{jp}\delta_{(\omega+\xi)(\beta+\zeta_q)} - \delta_{jq}\delta_{(\omega+\xi)(\beta+\zeta_p)})_{p,q,\beta} \\ &= \mathbf{v}_\ell \sum_{\substack{p,q=1,\dots,n\\ |\beta|=k-2}} \zeta_j \wedge (\zeta_q \odot e^{\omega+\xi-\zeta_q}) + \zeta_j \wedge (\mathbf{e}_p \odot e^{\omega+\xi-\zeta_p}) \\ &\eqqcolon \mathbf{v}_\ell \otimes \mathbf{m}_{j,\omega}. \end{aligned}$$

Notice that $|\mathbf{m}_{j,\omega}| \geq 1$ for all $j = 2, \dots, n$ and all $|\omega| = k-2$. This follows since $\zeta_j \wedge \xi \odot e^\omega$ is a non-zero tensor. Now, consider the tensor

$$h_1 \coloneqq - \sum_{\substack{j=2,\dots,n,\\ \ell=1,\dots,N,\\ |\omega|=k-2,\, \omega_q\geq 1}} \gamma^\ell_{\omega+\xi+\zeta_j} \left(\frac{\mathbf{v}^\ell \otimes \mathbf{m}_{j,\omega}}{|\mathbf{m}_{j,\omega}|^2} \right).$$

By construction, we discover that (here we use that $|\mathbf{m}_{j,\omega}| > 0$)

$$h_1 \cdot \mathrm{curl}(\eta)[M] = -III_1.$$

We have thus have found h_1 such that

$$I_1(\eta)[M] + \langle h_1, \mathrm{curl}_{k-1}[M] \rangle = \langle \xi, \eta \rangle \langle E_1, M \rangle.$$

As mentioned beforehand, the process of showing that

$$I_m(\eta)[M] + \langle h_m, \cdot \operatorname{curl}_{k-1}[M]\rangle = \langle \xi, \eta\rangle\langle E_m, M\rangle$$

is analogous and therefore the details are left to reader for its verification. Summing over $m = 1, \dots, k$ we find that

$$\langle (w,h), d\mathbb{A}(\eta)[M]\rangle = \langle \xi, \eta\rangle\langle E, M\rangle$$

where we have set $h := h_1 + \cdots + h_k$ and $\tilde{E} := E_1 + \cdots + E_k$. This proves that $(w,h) \in (d\mathcal{A})_1^{\otimes}$. It also implies that $\tilde{E} = E$, which shows that $\partial\sigma(\mathcal{A}) \subset \partial\sigma(d\mathcal{A})$.

This proves properties (3) and (4). □

Proposition 8.1. *Let $\mathcal{A} : \mathrm{C}^\infty(\mathbb{R}^n; V) \to \mathrm{C}^\infty(\mathbb{R}^n; W)$ be a k^{th} order homogeneous linear elliptic differential operator satisfying $(\mathfrak{m}_k)$. Then*

$$\operatorname{Proj}_{\mathbb{R}^n} \partial\sigma(d\mathcal{A}) = \mathbb{R}^n, \qquad \operatorname{span}\Big\{ \operatorname{Proj}_{V^k} \partial\sigma(d\mathcal{A})\Big\} = V^k.$$

Moreover, $d\mathcal{A}$ satisfies the polarization property contained in Proposition 4.4.

Remark 8.2. In general and in the context of the previous assumptions, it does not hold that

$$\operatorname{Proj}_{V^k} \partial\sigma(d\mathcal{A}) = V^k.$$

In particular, one *cannot* expect the linearization $d\mathcal{A}$ to satisfy the rank-one property. As an example, consider the operator

$$\mathcal{D}^k u = (\partial_1^k u, \dots, \partial_n^k u), \quad k \geq 3.$$

It is easy to verify that the only pure derivatives $\partial_\eta \partial_\xi^{k-1}$ controlled by $\mathscr{D}^k$ are the ones where $\eta = \xi$ are elements of the canonical axis. In particular, if $\xi \in \mathbf{S}^{n-1}$ does not belong to the canonical axis, then the tensor $\otimes^{k-1}\xi$ cannot be the second coordinate of a spectral pair of $d\mathcal{A}$.

Proof. The mixing property $(\mathfrak{m}_k)$ and the previous lemma imply that there exists a non-trivial pair $(w,h) \in \partial\sigma(d\mathcal{A})$. Let $(\xi, E) \in \mathbb{R}^n \times V^{k-1}$ be a spectral direction associated to this pair and consider the slice $\mathcal{B} := (d\mathcal{A})_\xi^E \llcorner \pi_\xi$.

Invoking Proposition 4.1 and Lemma 4.2 we find that $\mathcal{B}$ is a non-trivial elliptic operator. Moreover, by a similar argument to the one used to prove point (4) in the same lemma, one can show that $\mathcal{B}$ also satisfies the mixing property $(\mathfrak{m}_k)$ as an operator with variables in π_ξ. We claim that $\mathcal{B}$ is the linearization of a k^{th} order operator from V to W with variables in π_ξ. First, notice that the projection $\mathbf{p}_\xi^E$ induces a projection $\mathbf{p} : W \to X$, where $X = \mathbf{p}_\xi^E[W \times \{0\}]$. Now, let us consider the operator $\mathcal{L}$ that is associated to the principal symbol

$$\mathbb{L}^k(\zeta)[v] = \mathbf{p} \circ \mathbb{A}^k(\zeta)[v] \quad \text{for all } \zeta \in \pi_\xi \text{ and } v \in V.$$

This construction conveys the identity $\overline{\mathbb{L}^k} = \mathbf{p} \circ \overline{\mathbb{A}^k}$, whereby we obtain

$$d\mathbb{L} = (\mathbf{p} \circ \overline{\mathbb{A}^k \llcorner \pi_\xi}, \operatorname{curl}_{k-1}).$$

Testing this identity with $(\operatorname{curl}_{k-1})$-free tensors yields

$$\begin{aligned} d\mathbb{L}(\zeta)[v \otimes^{k-1} \zeta] &= (\mathbf{p} \circ \mathbb{A}^k(\zeta)[v], 0) \\ &= \mathbf{p}_\xi^E(d\mathbb{A}(\zeta)[v \otimes^{k-1} \zeta], 0) \\ &= \mathbb{B}(\xi)[v \otimes^{k-1} \zeta]. \end{aligned}$$

This proves that $\mathbb{B}$ is indeed the linearization of $\mathcal{L}$ (that $\mathcal{L}$ is elliptic follows from the fact that $\mathcal{B}$ is elliptic). This proves the claim.

Thus, $\mathcal{B}$ satisfies the very same assumptions that $\mathcal{A}$ in the hypotheses of this lemma. In other words, slicing is stable under the given properties and therefore we can iterate the slicing until we find a one-dimensional (elliptic) slice. This yields (notice that $n \leq \dim(V_{k-1})$)

(1) an orthonormal basis $\{\xi_1 \coloneqq \xi, \xi_2, \ldots, \xi_n\}$ of $\mathbb{R}^n$,
(2) a basis $\{E_1, \ldots, E_{n-1}, E_n, \ldots, E_r\}$ of V_{k-1}

such that (ξ_i, E_i) is a spectral pair of the i^{th} slice of $d\mathcal{A}$ for all $i = 1, \ldots, n-1$, and $\ell_{\xi_n} \times \operatorname{span}\{E_n, \ldots, E_{n+r}\}$ is a subset of the directional spectrum of the last (one-dimensional and elliptic) slice (and therefore containing a gradient). This, and point (4) in the previous lemma yield

$$\operatorname{span}\Big\{\operatorname{Proj}_{\mathbb{R}^n} \partial\sigma(d\mathcal{A})\Big\} = \mathbb{R}^n, \qquad \operatorname{span}\Big\{\operatorname{Proj}_{V^k} \partial\sigma(d\mathcal{A})\Big\} = V^k.$$

The iteration can be re-engineered in a way that, for any distinct $p, q \in \{1, \ldots, n\}$, we slice with respect to (ξ_i, E_i) for all $i \notin \{p, q\}$. This gives a slice of $d\mathcal{A}$ is then defined on the 2-plane $\{\xi_p, \xi_q\}$. At this point of the proof, one may follow by verbatim the arguments in the proof of Proposition 4.4 to show that $\operatorname{span}\{\xi_p, \xi_q\} \subset \operatorname{Proj}_{\mathbb{R}^n} \partial\sigma(d\mathcal{A})$. Moreover, since the argument is independent of the initial pair (ξ, E), we may also follow the same ideas of that proof to show the polarization property. □

8.1. **Proofs of the main results.** The idea will be to discuss, in chronological order, suitable versions of the main propositions, lemmas and theorems that are valid for first-order operators. Since most of the ideas remain largely similar, we shall mainly focus on those details and adaptations which are non-trivial.

The slicing theorem. If $(w, h) \in (d\mathcal{A})^1_{\otimes}$ is a $\operatorname{rank}_{d\mathcal{A}}$-one tensor with an associated spectral pair $(\xi, E) \in \partial\sigma(d\mathcal{A})$, then $\langle w, \mathcal{A}u\rangle = \langle (w, h), d\mathcal{A}(\nabla^{k-1}u)\rangle = \partial_\xi(\langle E, \nabla^{k-1}u\rangle)$. Invoking Proposition 2.1 and (42) we get

$$\langle w, \mathcal{A}u\rangle = \int_{\pi_\xi} D(\nabla^{k-1}u)^E_{y,\xi} \, \mathrm{d}\mathcal{H}^{n-1}(y),$$
$$|\langle w, \mathcal{A}u\rangle| = \int_{\pi_\xi} |D(\nabla^{k-1}u)^E_{y,\xi}| \, \mathrm{d}\mathcal{H}^{n-1}(y),$$

as long as $u \in \mathrm{BV}^{\mathcal{A}}(\mathbb{R}^n)$. The same holds for Ω instead of $\mathbb{R}^n$ when $d\mathcal{A}$ is complex-elliptic, or equivalently, that $\mathcal{A}$ is complex-elliptic (see Proposition 8.1). Applying the rules for orthogonal complements we find that the mixing condition $(\mathfrak{m}_k)$ is equivalent to the existence of a family $\{w_1, \ldots, w_M\}$ spanning $(W_{\mathcal{A}})^*$ and satisfying (40). Invoking Lemma 8.1, we find that these covectors are, in fact, elements of $\mathcal{A}_1^{\otimes}$ that can be extended to elements of $d\mathcal{A}_1^{\otimes}$ by adding an appropriate coordinate. Hence, the same concluding argument in the proof of Theorem 2.1 serves just as well as a proof for Theorem 2.5.

Dimensional estimates and fine properties I. The proof of Corollary 2.1 follows from the first conclusion of the previous lemma and the same geometric argument used in the proof of Corollary 4.3. As a consequence, we also obtain a suitable version of Corollary 4.4 for higher order operators —thus proving that $|\mathcal{A}u| \ll \mathcal{I}^{n-1}$, that $\mathcal{H}^{n-1}(\Theta_u \setminus J_U) = 0$, and that $\mathcal{A}^c u$ vanishes on $\mathcal{H}^{n-1}$ σ-finite sets.

Analysis of Lebesgue points on Θ_u (Lemma 5.1). The basis of the one-dimensional structure theorem and also of the fact that $|\mathcal{A}u|(S_u \setminus J_u)$, in the first-order case, hinges on on the first and second statements of Lemma 5.1 respectively. The first statement relies purely on slicing and the polarization properties, which we have now established in Proposition 8.1 in the general case. The second statement requires slightly more: firstly, we need that $d\mathcal{A}$ is complex-elliptic in order to be able to use the quasi-continuity property at Θ_u (see [8, Prop. 1.2]); this is covered by the properties of the linearization and the assumption that $\mathcal{A}$ is complex-elliptic. The other necessary tool is that there are sufficient spectral V^k-coordinates to span V^k, which we have also established in Proposition 8.1 for the general case —despite that $d\mathcal{A}$ may not satisfy the rank-one property.

One-dimensional structure theorem and fine properties II. The one-dimensional structure theorem on one-dimensional BV-sections follows by using $d\mathcal{A}$ and the identity $d\mathcal{A}U = (\mathcal{A}u, 0)$ from the analysis of Lebesgue points and the aforementioned version of Corollary 4.4. The latter also conveys a suitable version of Corollary 5.1, which then proves that $|\mathcal{A}u|(S_U \setminus \Theta_u) = 0$, or equivalently, that $\mathcal{A}^d u := \mathcal{A}^s u \llcorner (S_U \setminus J_U) \equiv 0$. This covers the fine properties of the Cantor part $\mathcal{A}^c u$. The characterization of $\mathcal{A}^a u$ follows from the use of the linearization properties and the existing theory (see Remark 2.3). Lastly, the characterization of $\mathcal{A}^j u$ follows from the results contained in [8]. This proves Theorem 2.6. □

9. Applications

In this section we review some well-known operators that satisfy our main assumptions. In particular, we revise the details of some interesting cases, which are not covered by the BV-theory.

9.1. **Gradients.** The gradient operator

$$Du = (\partial_1 u, \dots, \partial_n u), \qquad u : \mathbb{R}^n \to \mathbb{R}^N,$$

is complex-elliptic operator. Indeed, the symbol associated to D is simply $D(\xi)[a] = a \otimes \xi$, which has no complex non-trivial zeros. Clearly, D also satisfies the mixing property since $D_1^{\otimes} = \mathrm{I}_D = \{\, a \otimes \xi : \xi \in \mathbb{R}^n, a \in \mathbb{R}^N \,\}$, and, in particular, $\partial\sigma(D) = \mathbb{R}^n \times \mathbb{R}^N$.

9.2. **Higher Gradients.** The k^{th} gradient operator

$$D^k u = \left(\frac{\partial^k u^j}{\partial x_{i_1} \cdots \partial x_{i_k}} \right)^j_{i_1,\dots,i_k}, \qquad u : \mathbb{R}^n \to \mathbb{R}^N,$$

is complex-elliptic since its symbol is given by $D^k(\xi)[a] = a \otimes^k \xi$. It also satisfies the rank-one property or the mixing condition since every element in the image cone is clearly rank-one.

If u belongs to the space

$$\mathrm{BV}^k(\mathbb{R}^n; \mathbb{R}^N) := \{\, u \in \mathrm{L}^1(\mathbb{R}^n) : D^k u \in \mathcal{M}(\mathbb{R}^n; \mathbb{R}^N \otimes E_k(\mathbb{R}^n)) \,\},$$

then $\nabla^{k-1} u \in \mathrm{BV}(\mathbb{R}^n; \mathbb{R}^N \otimes E_{k-1}(\mathbb{R}^n))$. The classical BV-theory implies that $\nabla^{k-1} u$ and $D^k u$ satisfy the fine properties. Moreover, the structure theorem takes the form

$$D^k u = \nabla^k u \, \mathscr{L}^n \, + \, D^c(\nabla^{k-1} u) \, + \, [\![\nabla^{k-1} u]\!] \otimes^k \nu_u \, \mathcal{H}^{n-1} \llcorner J_{\nabla^{k-1} u},$$

where

$$[\![\nabla^{k-1} u]\!] := \langle (\nabla^{k-1} u)^+ - (\nabla^{k-1} u)^-, \underbrace{\nu_u, \dots, \nu_u}_{(k-1)\text{-times}} \rangle \in \mathbb{R}^N.$$

The following example is particularly interesting, since it does not follow from the BV-theory. It says that it suffices to control the pure derivatives $\partial_1^k, \dots, \partial_n^k$ in order to deduce a structure theorem and fine properties for all the lower order derivatives:

9.3. **Fine properties of $\mathscr{BV}^k$-functions.** Consider the diagonal of the k^{th} gradient

$$u \mapsto \mathscr{D}^k := \operatorname{diag}(D^k u) = (\partial_1^k u, \ldots, \partial_n^k u), \quad u : \mathbb{R}^n \to \mathbb{R}.$$

The principal symbol of $\mathscr{D}^k$ given by the map $\xi \mapsto (\xi_1^k, \ldots, \xi_n^k)$, whence we verify that $\mathscr{D}^k$ is complex-elliptic. Moreover, $\mathscr{D}^k$ satisfies the rank-one property. Indeed, every element of the canonical basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of $\mathbb{R}^n$ is a $\text{rank}_{\mathscr{D}^k}$-one tensor.

We conclude that if u belongs to the space

$$\mathscr{BV}^k(\mathbb{R}^n) := \{\, u \in \mathrm{L}^1(\mathbb{R}^n) : \mathscr{D}^k u \in \mathcal{M}(\mathbb{R}^n; \mathbb{R}^n) \,\},$$

then $\nabla^{k-1}u$ is integrable and approximately differentiable, The jump set of $\nabla^{k-1}u$ is countably $\mathcal{H}^{n-1}$-rectifiable, $|\mathscr{D}^k u| \ll \mathcal{I}^{n-1} \ll \mathcal{H}^{n-1}$, and $\mathscr{D}^k u$ decomposes into its absolutely continuous, Cantor, and jump parts as

$$\begin{aligned}\mathscr{D}^k u = \operatorname{diag}(\nabla^k u)\, \mathscr{L}^n \,+\, (\mathscr{D}^k)^s u \llcorner (\mathbb{R}^n \setminus S_{\nabla^{k-1}u}) \\ +\, [\![\nabla^{k-1}u]\!]\, \big((\nu_1)_1^k, \ldots, (\nu_u)_n^k\big)\, \mathcal{H}^{n-1} \llcorner J_{\nabla^{k-1}u}.\end{aligned}$$

Moreover, $\nabla^{k-1}u$ satisfies the fine properties (i)-(v) contained in Theorem 2.6.

9.4. **Fine properties of BD-functions.** For vector-valued map $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ we define its symmetric gradient

$$Eu := \frac{1}{2}(Du + Du^t),$$

which takes values on the space $E_2(\mathbb{R}^n)$ of symmetric bilinear forms of $\mathbb{R}^n$. One readily verifies that E is elliptic since $\mathrm{I}_E = \{\, a \odot \xi : a, \xi \in \mathbb{R}^n \,\}$. We have $E_1^{\otimes} = \{\, \xi \otimes \xi : \xi \in \mathbb{R}^n \,\}$ and $\partial\sigma(E) = \{\, (\xi, \xi) : \xi \in \mathbb{R}^n \,\}$. A standard polarization argument shows that the family $\{\, \xi \otimes \xi : \xi \in \mathbb{R}^n \,\}$ is a spanning set of $E_2(\mathbb{R}^n)$, which further implies that that E has the rank-one property. The structure theorem in $\mathrm{BD}(\mathbb{R}^n)$, which is well-known, reads

$$Eu = \operatorname{sym}(\nabla u)\, \mathscr{L}^n \,+\, E^c u \,+\, [\![u]\!] \odot \nu_u\, \mathcal{H}^{n-1} \llcorner J_u.$$

More generally, one may consider the symmetrization of the gradient of a symmetric k-tensor field:

9.5. **Fine properties of BD^k-functions.** Let $k \in \mathbb{N}$. For a symmetric k-tensor $v \in E_k(\mathbb{R}^n)$, we define the operator with symbol $E^k(\xi)[v]$ satisfying

$$E^k(\xi)[v]a = \operatorname{sym}^{k+1}(\xi)[v](a_1, ..., a_{k+1}) := \frac{1}{(k+1)!} \sum_{\sigma \in S_{k+1}} (\xi \cdot a_{\sigma(k+1)}) v(a_{\sigma(1)}, ..., a_{\sigma(k)}),$$

for all $(k+1)$-tuples $\mathbb{R}^n$-vector-fields $(a_1, \ldots, a_{k+1})$. Thus, for a k-tensor-field $u : \mathbb{R}^n \to E_k(\mathbb{R}^n)$, we have $E^k u = \operatorname{sym}^{k+1}(Du)$. We verify that this operator is complex-elliptic by the pointwise definition: let $\xi \in \mathbb{C}^n$ be a non-zero vector and let $v \in E_k(\mathbb{C}^n)$. Then $\operatorname{sym}^k(\xi)v = 0$ implies $\operatorname{sym}^{k+1}(\xi)[v](a, ..., a) = 0$ for each $a \in \mathbb{C}^n$. This however implies $v(a, \ldots, a) = 0$ for all $a \in \mathbb{C}^n$ with $a \cdot \xi \neq 0$. Now, consider any $b \in \pi_\xi$, and take $a = b + \xi$. Applying v to this choice of a, we see by a polarization argument that v vanishes on $E_{k+1}(\mathbb{C}^n)$ and hence $v = 0$. This shows that E^k is complex-elliptic. Moreover,

$$(E^k)_1^{\otimes} = \left\{ \otimes^{k+1}\xi : \xi \in \mathbb{R}^n \right\}, \qquad \partial\sigma(E^k) = \left\{ (\xi, \otimes^k \xi) : \xi \in \mathbb{R}^n \right\}.$$

This family of k-tensors can be seen to generate $E_{k+1}(\mathbb{R}^n)$ and therefore E^k satisfies the rank-one property. Thus, for a function in the space

$$\mathrm{BD}^k(\mathbb{R}^n) = \{\, u \in \mathrm{L}^1(\mathbb{R}^n; E_k(\mathbb{R}^n)) : E^k u \in \mathcal{M}(\mathbb{R}^n; E_{k+1}(\mathbb{R}^n)) \,\},$$

the structure theorem takes the form

$$Eu = \mathrm{sym}^k(\nabla u)\,\mathscr{L}^n + (E^k)^c u + \mathrm{sym}^k([\![u]\!] \otimes \xi)\,\mathcal{H}^{n-1} \llcorner J_u$$

and furthermore u satisfies the fine properties established in Theorem 2.3.

10. Counterexamples

We review a number of well-known differential operators, which fail to satisfy the main mixing property.

10.1. **Insufficiency of complex-ellipticity.** The following two examples show that complex-ellipticity is not a sufficient condition to ensure the rank-one the validity of the mixing property:

Example 10.1 (Deviatoric operator)**.** Consider the operator that measures the shear part of the symmetric gradient:

$$Lu = Eu - \frac{\mathrm{div}(u)}{n}\,\mathrm{id}_{\mathbb{R}^n}, \qquad u : \mathbb{R}^n \to \mathbb{R}^n.$$

This operator satisfies the following structural properties:

(1) L is elliptic,
(2) L is complex-elliptic for $n \geq 3$,
(3) L does not satisfy the rank-one property for all $n \geq 2$ and its $\mathrm{rank}_{\mathbb{L}}$-one cone is trivial, i.e., $\mathbb{L}_1^{\otimes} = \{0\}$. In particular, L does not satisfy $(\mathfrak{m})$.

Proof. The principal symbol is given by $\mathbb{L}(\xi)[a] = a \odot \xi - n^{-1}(a \cdot \xi)\,\mathrm{id}_{\mathbb{R}^n}$. Therefore, $\mathbb{L}(\xi)[a] = 0$ if and only if $n(a \odot \xi) = (a \cdot \xi)\,\mathrm{id}_{\mathbb{R}^n}$. When $(a \odot \xi)$ is non-trivial, it is either a rank-one matrix or a rank-two matrix with eigenvalues of discordant sign. This shows that L is elliptic. That L is complex-elliptic follows from the fact that it has a finite dimensional kernel if and only if $n \geq 3$.

Notice that Lu takes values on the space $\mathrm{sym}_0(n)$ of trace-free symmetric $(n \times n)$-matrices. The spectral theorem ensures that $\mathrm{sym}_0(n)$ contains no rank-one elements, whereby we conclude that $\mathbb{L}_1^{\otimes} = \{0\}$. □

Example 10.2. Let $\mathcal{B} : \mathrm{C}^\infty(\mathbb{R}^{n+1}; \mathbb{R}^{N+1}) \to \mathrm{C}^\infty(\mathbb{R}^n; \mathbb{R}^{N+n+1})$ be the operator associated ti the principal symbol

$$\mathbb{B}(\xi_0, \dots, \xi_n)[v_0, \dots, v_N] := \bigg(\sum_{r+s=m} \xi_r v^s\bigg)_m, \qquad m = 0, 1, \dots, n+N.$$

Then,

(1) $\mathcal{B}$ is complex-elliptic
(2) $\mathcal{B}$ does not satisfy the rank-one property for all $n + N \geq 3$.

Proof. A symmetry argument shows that there is no loss of generality in assuming that $n \geq N$ (otherwise, we simply reverse the roles of $\mathbb{R}^n$ and V). We only show the failure of the mixing property since the complex-ellipticity follows directly from a simple induction argument. The associated principal symbol of the operator satisfies

$$(\mathbb{B}(\xi)[v])_m = z_m(\xi, v) := \sum_{r+s=m} \xi_r v_s = \sum_{r+s=m} (\mathbf{e}_r \cdot \xi)(\mathbf{v}_s \cdot v),$$

where $\{\mathbf{e}_0,\dots,\mathbf{e}_n\}$ are $\{\mathbf{v}_0,\dots\mathbf{v}_n\}$ are orthonormal basis of $\mathbb{R}^{n+1}$ and V. Therefore, z_m may be identified (under a suitable isometry) with a matrix of the form

$$\tilde{z}_m = \begin{bmatrix} & & & & & 1 & \\ & & & & 1 & & \\ & & & 1 & & & \\ & & 1 & & & & \\ & 1 & & & & & \\ 1 & & & & & & \\ & & & & & & \end{bmatrix} \in \mathbb{R}^{N+1}\otimes\mathbb{R}^{n+1},$$

where the vector in the first row only has a non-zero coordinate in its $(m+1)^{\text{th}}$ entry. Notice that $\operatorname{rank}(\tilde{z}_m)\geq m+1$. More generally, for a vector $P=(P_0,\dots,P_{p+q})$, we discover that the bi-linear form $P\cdot f_{\mathcal{B}}$ may be identified (under the same isometry) with a matrix of the form

$$\tilde{P} = \begin{bmatrix} a & b & c & d & e & f & g \\ b & c & d & e & f & g & h \\ c & d & e & f & g & h & i \\ d & e & f & g & h & i & j \\ e & f & g & h & i & j & k \\ f & g & h & i & j & k & l \\ g & h & i & j & k & l & m \end{bmatrix},$$

and accordingly $\operatorname{rank}(\tilde{P})=1$ only when considering multiples of $(1,0,\dots,0)$, $(0,\dots,0,1)$, or $(1,1,\dots,1)$. Since clearly these three vectors do not span $\mathbb{R}^{n+N+1}$ (when $n+N\geq 3$), this shows that $\mathcal{A}$ does not satisfy the rank-one property in the conjectured range. ☐

10.2. **Non-canceling operators.** The following are some relevant examples of elliptic operators that *fail* to satisfy the cancellation property

$$\bigcap_{\xi\in\pi_\xi}\operatorname{Im}\mathbb{A}^k(\xi)=\{0\},$$

introduced by Van Schaftingen in [31] to establish limiting Sobolev inequalities on $\mathrm{BV}^{\mathcal{A}}(\mathbb{R}^n)$. Every operator satisfying the mixing property $(\mathfrak{m}_k)$ is clearly canceling Therefore, the operators discussed next, all fail to satisfy the rank-one property (for more details we refer the reader to [8, 31] and references therein):

Example 10.3. Let $n\geq 2$. The div-curl operator

$$\mathcal{F}_n u := (\operatorname{div}\times\operatorname{curl})u.$$

defined for vector-fields $u:\mathbb{R}^n\to\mathbb{R}^n$.

An equivalent formulation of the previous example comes from the *del-bar* operator or the Cauchy-Riemann equations, as well as conformal gradients:

Example 10.4 (Cauchy-Riemann equations)**.** A function $u:\mathbb{R}^2\to\mathbb{R}^2$ satisfies

$$\bar{\partial}(u^1+\mathrm{i}u^2) := (\partial_1-\mathrm{i}\partial_2)(u^1+\mathrm{i}u^2)=0$$

if and only if $w(x+\mathrm{i}y):=u^1(x,y)+\mathrm{i}u^2(x,y)$ is holomorphic. The $\bar{\partial}$-operator in two dimensions is equivalent to the div-curl operator: if we set $\psi=(u^1,-u^2)$, then

$$(\operatorname{div}\times\operatorname{curl})\psi=0.$$

Example 10.5 (Conformal maps)**.** The same conclusions apply to the differential inclusion

$$Du(x) \in K := \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R} \right\},$$

where K the is the set of conformal (2×2) matrices.

Example 10.6 (Compensated compactness)**.** For $n \geq 3$ and let $m \in \{1, ..., n-1\}$, consider the first order operator (d, d^*), whose symbol is given by

$$[(d, d^*)(\xi)]v := (\xi \wedge v, *(\xi \wedge *v)), \qquad v \in \Lambda^m(\mathbb{R}^n).$$

10.3. **Scalar-valued elliptic operators.** All elliptic and scalar-valued operators, in dimensions $n \geq 2$, are non-canceling In particular the following well-known examples do not satisfy our assumptions:

Example 10.7 (Laplacian)**.** The easiest example of a second-order operator that is elliptic but fails to the mixing condition is the Laplacian

$$\Delta u = \sum_{i=1}^{n} \partial_i^2 u, \qquad n \geq 2.$$

Observe also that there is no hope for either dimensional to hold for Laplace's operator for all $n \geq 2$: There exists a fundamental solution $\Phi \in \mathrm{C}^\infty(\mathbb{R}^n \setminus \{0\})$ satisfying $\Delta\Phi = \delta_0$ on $\mathbb{R}^n$. Accordingly, $\dim_{\mathcal{H}}(\Delta\Phi) = 0$.

Example 10.8 (The $\mathcal{A}$-Laplacian operator)**.** For any homogeneous elliptic partial differential operator $\mathcal{A}$ from V to W, we may consider the operator

$$\Delta_{\mathcal{A}} u := (\mathcal{A}^* \circ \mathcal{A})u.$$

Such operators are elliptic and possess a fundamental solution $\Delta_{\mathcal{A}}\Phi_{\mathcal{A}} = \delta_0$.

Example 10.9 (Laplace-Beltrami operator)**.** The Laplace-Beltrami operator

$$\Delta := dd^* + d^*d$$

is elliptic and scalar-valued and fails to satisfy the mixing property.

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