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EXPLICIT SOLUTIONS OF SOME LINEAR-QUADRATIC MEAN FIELD GAMES

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Abstract. We consider N-person differential games involving linear systems affected by white noise, running cost quadratic in the control and in the displacement of the state from a reference position, and with long-time-average integral cost functional. We solve an associated system of Hamilton-Jacobi-Bellman and Kolmogorov-Fokker-Planck equations and find explicit Nash equilibria in the form of linear feedbacks. Next we compute the limit as the number N of players goes to infinity, assuming they are almost identical and with suitable scalings of the parameters. This provides a quadratic-Gaussian solution to a system of two differential equations of the kind introduced by Lasry and Lions in the theory of Mean Field Games [22]. Under a natural normalization the uniqueness of this solution depends on the sign of a single parameter. We also discuss some singular limits, such as vanishing noise, cheap control, vanishing discount. Finally, we compare the L-Q model with other Mean Field models of population distribution.

1. Introduction.

Consider a system of linear stochastic differential equations

$$dX_t^i = (A^i X_t^i - \alpha_t^i)dt + \sigma^i dW_t^i, \quad X_0^i = x^i, \quad i = 1, \dots, N, \tag{1}$$

where W_t^i is a Brownian motion and α_t^i is the control of the i-th player of the differential game that we now describe. For each initial vector of positions $X = (x^1, \dots, x^N)$ we consider for the i-th player the long-time-average cost functional with quadratic running cost

$$J^i(X, \alpha^1, \dots, \alpha^N) := \liminf_{T\to+\infty} \frac{1}{T} E\left[\int_0^T \frac{R_i}{2}(\alpha_t^i)^2 + \left(X_t - \overline{X}_i\right)^T Q^i \left(X_t - \overline{X}_i\right)\, dt\right],$$

where E denotes the expected value, $R_i > 0$, Q^i is a symmetric matrix, and $\overline{X}_i$ is a given reference position. We wish to study the Nash equilibrium strategies of this N-person game and understand the limit behavior as $N \to +\infty$ within the theory of Mean Field Games initiated by Lasry and Lions [22, 23, 24]. We recall that this theory is intimately connected to the modeling of economic equilibrium with rational anticipations, following the fundamental contribution of Aumann [5].

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We refer to the recent survey [15] for several applications of Mean Field Games to economics and mathematical finance.

In this paper we limit ourselves to the case of 1-dimensional state space for each player, because our goal is to give solutions as explicit as possible to the systems of Hamilton-Jacobi-Bellman and Kolmogorov-Fokker-Planck (briefly, HJB and KFP) equations arising in the theory. Thus A^i is a given scalar here. However the case of general d-dimensional X^i_t can be studied similarly via the solution of suitable matrix Riccati equations and will be treated in a forthcoming paper.

In Section 2 we define the admissible strategies and introduce the system of $2N$ HJB and KFP equations associated to the N-person game, as in [22, 24]. Under a generic condition we find explicit quadratic solutions v^i for the HJB equations and Gaussian solutions m_i for the KFP equations, and affine feedback strategies that give a Nash equilibrium of the game.

In Section 3 we introduce the assumption that the running cost of the i-th player is symmetric with respect to the positions of any two other players. It leads to reducing the $(N+1)N/2$ coefficients of Q^i to just four parameters: the primary costs of self-displacement, $q_i > 0$, and cross-displacement, β_i, and the secondary costs of self and cross-displacement, η_i and γ_i; also the N entries of the reference position $\overline{X}_i$ reduce to two, the preferred value h_i for the i-th player and his reference value r_i for the other agents. Next we assume the players are almost identical, i.e., they have the same parameters in the dynamical system and cost functional, except possibly the secondary costs of displacement. Then there is a unique identically distributed quadratic-Gaussian solution of the $2N$ HJB-KFP equations, i.e., such that all v^i are equal and so are all m_i.

Section 4 is devoted to the limit as $N \to +\infty$ for parameters such that

$$q^N \to \overline{q}, \quad \beta^N \sim \overline{\beta}/N, \quad \eta^N_i \sim \overline{\eta}/N, \quad \gamma^N_i \sim \overline{\gamma}/N^2.$$

Then the identically distributed solution (v^N, m^N, λ^N) of the preceding $2N$ system converges to a solution (v, m, λ) of the Mean Field system of two equations

$$\begin{cases} -\nu v_{xx} + \frac{(v_x)^2}{2R} - Axv_x + \lambda = \overline{V}[m](x) \quad \text{in } I\!R, \\ -\nu m_{xx} - \left(\frac{v_x}{R} m - Axm\right)_x = 0 \quad \text{in } I\!R, \\ \min\left[v(x) - \frac{RAx^2}{2}\right] = 0, \quad \int_{I\!R} m(x)dx = 1, \quad m > 0 \text{ in } I\!R, \end{cases} \tag{2}$$

where $\nu = \sigma^2/2$ and $\overline{V}[m]$ is the non-local operator

$$\overline{V}[m](x) := \overline{q}(x-h)^2 + \overline{\beta}(x-h)\int_{I\!R}(y-r)\,dm(y) + \overline{\gamma}\left(\int_{I\!R}(y-r)\,dm(y)\right)^2 + \overline{\eta}\int_{I\!R}(y-r)^2\,dm(y). \tag{3}$$

Such solution is explicit and unique among quadratic-Gaussian ones, except for one critical value of $\overline{\beta}$. Moreover it is the unique solution of (2) if $\overline{\beta} \geq 0$, by a monotonicity argument of Lasry and Lions [22, 24]. Note, however, that the normalization condition on v in (2) is different from the null-average condition of the periodic case [22, 24].

The results of these three sections parallel those of the seminal papers [22, 24] on games with ergodic cost criterion with the following main differences. Lasry and

Lions consider system (1) in dimension $d \geq 1$ with $A^i = 0$, running costs of the form $L^i(X_t^i, \alpha_t^i) + F^i(X_t^1, \ldots, X_t^N)$ with L^i superlinear in α and F^i $\mathbb{Z}^d$-periodic in each entry, so their state space for each agent is a torus. No explicit formulas can be expected for these general costs and the proofs rely on some hard estimates for the HJB equations. In our Linear-Quadratic (briefly, L-Q) case the explicit quadratic-Gaussian formulas for solutions allow rather elementary calculations; on the other hand the unboundedness of data and solutions requires some additional care in the proof of the verification theorem.

In Section 5 we exploit the formulas for solutions to study several singular limits. For vanishing noise $\sigma^i \to 0$ we show that the distributions m_i become Dirac masses, the Nash equilibrium feedback remains the same for the limit deterministic game, a fact known for finite horizon problems [7, 10], and the vanishing viscosity limit commutes with $N \to +\infty$. For the cheap control limit, that is, $R_i \to 0$, the distributions m_i become again Dirac masses and the limit commutes with $N \to +\infty$. After solving in quadratic-Gaussian form the HJB-KFP equations of the discounted infinite horizon problem, we show that for vanishing discount there is convergence to the long-time-average cost problem, and also this limit commutes with $N \to +\infty$. Finally we study the scaling $1/N = o(\beta^N)$ that is related to a singular perturbation of (2).

Section 6 discusses an interpretation of the L-Q Mean Field Game as a model of the distribution of a population. We compare it to the Mean Field model with local log utility studied by Guéant [14, 13] and reported in [15], where explicit quadratic-Gaussian solutions are also found. The important parameters of $\overline{V}$ (3) in this discussion are $\overline{q}$ and $\overline{\beta}$, because $\overline{\beta} > 0$ means that it is costly for an individual to imitate his/her peers, whereas for $\overline{\beta} < 0$ resembling the others is rewarding as in the log model of [14, 13].

We conclude this introduction with some bibliographical remarks. Huang, Caines, and Malhamé studied L-Q stochastic games with discounted cost and large number of players motivated by several engineering applications [17, 19]. They also developed their approach to encompass nonlinear systems and more general costs [18, 20], independently of the Lasry-Lions theory. Discrete Mean Field Games were studied by Gomes et al. [12], numerical methods by Achdou and Capuzzo-Dolcetta [2], see also [21] and [1]. For the background on N-person differential games we refer to the books [7, 10] and for the ergodic stochastic case to [8], see also the references therein.

2. Games with N players and ergodic payoff.

The notations of the paper are chosen to be consistent with those of [22], [24]. We assume that each player controls a 1-dimensional state variable with linear dynamics, that is,

$$dX_t^i = (A^i X_t^i - \alpha_t^i)dt + \sigma^i dW_t^i, \quad X_0^i = x^i \in \mathbb{R}, \quad i = 1, \ldots, N \tag{4}$$

where $A^i, \sigma^i \in \mathbb{R}$ are given, $\sigma^i \neq 0$, $(W_t^1, \ldots, W_t^N)$ are N independent Brownian motions, the control $\alpha^i : [0, +\infty) \to \mathbb{R}$ of the i-th player is a bounded process adapted to W_t^i. For each initial vector of positions $X = (x^1, \ldots, x^N) \in \mathbb{R}^N$ we consider the long-time-average cost functional

$$J^i(X, \alpha^1, \ldots, \alpha^N) := \liminf_{T \to +\infty} \frac{1}{T} E\left[\int_0^T \frac{R_i}{2}(\alpha_t^i)^2 + F^i(X_t^1, \ldots, X_t^N) dt\right], \quad R_i > 0,$$

and we assume that F^i is quadratic in the following sense. For each player i there is a reference position of the whole state vector $\bar{X}_i$, and F^i is a quadratic form in $X - \bar{X}_i$, i.e., for a symmetric matrix Q^i,

$$F^i(x^1, \dots, x^N) := \left(X - \overline{X}_i\right)^T Q^i \left(X - \overline{X}_i\right) = \sum_{j,k=1}^{N} q^i_{jk}(x^j - \overline{x}^j_i)(x^k - \overline{x}^k_i), \quad q^i_{ii} > 0. \tag{5}$$

The condition $q^i_{ii} > 0$ means that $\overline{x}^i_i$ is a preferred position for the i-th player, but the matrix Q^i is not assumed positive semidefinite.

We are interested in *Nash equilibrium points*, that is, vectors of admissible control strategies $\overline{\alpha} = (\overline{\alpha}^1, \dots, \overline{\alpha}^N)$ such that

$$J^i(X, \overline{\alpha}) = \min_{\alpha^i} J^i(X, \overline{\alpha}^1, \dots, \overline{\alpha}^{i-1}, \alpha^i, \overline{\alpha}^{i+1}, \dots, \overline{\alpha}^N) \quad \forall\, i = 1, \dots, N. \tag{6}$$

For the current cost functional it is natural to choose as *admissible strategy*, or admissible control function, for the i-th player any bounded process α^i adapted to W^i_t such that the corresponding solution X^i_t of (4) satisfies, for some $C > 0$,

$$E[X^i_t] \le C, \quad E[(X^i_t)^2] \le C, \quad \forall\, t > 0, \tag{7}$$

and is *ergodic* in the following sense: there exists a probability measure m_{α^i} such that $\int_{\mathbb{R}} x dm_{\alpha^i}(x), \int_{\mathbb{R}} x^2 dm_{\alpha^i}(x) < +\infty$, and for any polynomial g of degree at most two

$$\lim_{T\to+\infty} \frac{1}{T} E\left[\int_0^T g\left(X^i_t\right)\, dt\right] = \int_{\mathbb{R}} g(x) dm_{\alpha^i}(x), \tag{8}$$

locally uniformly with respect to the initial position x^i of X^i_t.

The last condition is a standard property of an ergodic process with invariant measure m_{α^i}, for bounded and continuous functions g (see, e.g., [16], [4], and the references therein). Here we assume it for second-degree polynomials because our running cost is quadratic, so the cost functional $J^i(X, \alpha)$ does not depend on the initial position X for a N-vector α of such controls.

Important examples of admissible strategies are the *affine feedbacks* whose trajectory is ergodic, as made precise by the next Lemma.

Lemma 2.1. *For the feedback*

$$\alpha^i(x) = K^i x + c_i, \quad x \in \mathbb{R}, \qquad K^i > A^i, \tag{9}$$

consider the process $\alpha^i_t := \alpha^i(X^i_t)$ *where* X^i_t *solves*

$$dX^i_t = [(A^i - K^i)X^i_t - c_i]dt + \sigma^i dW^i_t.$$

Then α^i *is admissible.*

Proof. The explicit solution of the linear equation for X^i_t satisfies (7) and it is also known to be ergodic with Gaussian invariant measure m_{α^i} of mean $-c_i/(K^i - A^i)$ and variance $\nu^i/(K^i - A^i)$ (see, e.g., [16]). Then (8) holds for any bounded and continuous g. To get the conclusion it's enough to check it for $g(x) = x$ and $g(x) = x^2$ and this is easily done by integrating on $[0, T]$ the explicit expressions for $E[X^i_t]$ and $E[(X^i_t)^2]$. ∎

In order to write the system of HJB-KFP equations associated to the game as in [22, 24] we observe that the i-th Hamiltonian is

$$H^i(x,p) = \frac{p^2}{2R_i} - A^i xp, \quad x, p \in \mathbb{R},$$

and for a N-vector of probability measures on $\mathbb{R}$ $(m_1, \dots, m_N)$ we denote

$$f^i(x; m_1, \dots, m_N) := \int_{\mathbb{R}^{N-1}} F^i(x^1, \dots, x^{i-1}, x, x^{i+1}, \dots, x^N) \prod_{j \neq i} dm_j(x^j), \tag{10}$$

$$\nu^i := \frac{(\sigma^i)^2}{2}.$$

We want to solve the system

$$\begin{cases} -\nu^i v^i_{xx} + \frac{(v^i_x)^2}{2R_i} - A^i x v^i_x + \lambda_i = f^i(x; m_1, \dots, m_N) \ \text{ in } \mathbb{R}, \ i = 1, \dots, N \\ -\nu^i (m_i)_{xx} - \left(\frac{v^i_x}{R_i} m_i - A^i x m_i \right)_x = 0 \quad \text{in } \mathbb{R}, \quad i = 1, \dots, N \\ \int_{\mathbb{R}} m_i(x) dx = 1, \quad m_i > 0 \ \text{ in } \mathbb{R}, \end{cases} \tag{11}$$

where with a slight abuse of notation we are denoting with m_i a measure as well as its density. Since we are not in the periodic setting of [22, 24] the solutions v^i are expected to be unbounded and cannot be normalized by prescribing the value of their average. In the next result we produce solutions with v^i a quadratic polynomial and m_i Gaussian, namely,

$$v^i(x) = \frac{(x-\mu_i)^2}{2s_i} + \frac{R_i A^i x^2}{2}, \quad m_i(x) = \frac{1}{\sqrt{2\pi s_i \nu^i R_i}} \exp\left(-\frac{(x-\mu_i)^2}{2 s_i \nu^i R_i} \right), \tag{12}$$

for two vectors $\mu = (\mu_1, \dots, \mu_N)$ and $s = (s_1, \dots, s_N)$ with $s_i > 0$ that we will compute explicitly. We define the $N \times N$ matrix B by

$$B_{ii} := 2q^i_{ii} + R_i (A^i)^2, \quad B_{ij} := 2q^i_{ij} \quad i \neq j.$$

Theorem 2.2. *Under the above conditions, if* $\det B \neq 0$ *then*
i) there exists a unique triple $(\lambda_1, \dots, \lambda_N) \in \mathbb{R}^N$, $\mu \in \mathbb{R}^N$, $s \in (0, +\infty)^N$, *such that* (12) *solves* (11), *and*

$$s_i = \left(2q^i_{ii} R_i + (R_i A^i)^2 \right)^{-1/2}, \tag{13}$$

$$\mu = B^{-1} p, \quad p_i := 2 \sum_{j=1}^N q^i_{ij} \overline{x}^j_i, \tag{14}$$

ii) the affine feedback

$$\overline{\alpha}^i(x) = \frac{x - \mu_i}{s_i R_i} + A^i x, \quad x \in \mathbb{R}, \quad i = 1, \dots, N, \tag{15}$$

is a Nash equilibrium point for all initial positions $X \in \mathbb{R}^N$ *among the admissible strategies and* $J^i(X, \overline{\alpha}) = \lambda_i$ *for all* X *and* i.

Proof. For any $v^i \in C^1(\mathbb{R})$, the i-th equation of the second group on N equations in (11) can be integrated to get

$$m_i(x) = c_i \exp\left(\frac{1}{\nu^i R_i} \left(\frac{R_i A^i x^2}{2} - v^i(x) \right) \right).$$

Therefore we are left with the first group of N equations and we plug into them v^i of the form (12) to get

$$-\nu^i\left(\frac{1}{s_i}+R_iA^i\right)+\frac{1}{2R_i}\left(\frac{(x-\mu_i)^2}{s_i^2}+(R_iA^ix)^2\right)-R_i(A^ix)^2+\lambda_i=f^i. \quad (16)$$

Next we compute $f^i=f^i(x;m_1,\dots,m_N)$ using that m_i is the distribution of a Gaussian random variable $\mathcal{N}(\mu_i,s_i\nu^iR_i)$:

$$f^i(x;m_1,\dots,m_N)=q^i_{ii}(x-\overline{x}^i_i)^2+2(x-\overline{x}^i_i)\sum_{j\neq i}q^i_{ij}(\mu_j-\overline{x}^j_i)+b_i$$

$$b_i:=\sum_{j,k\neq i,\,j\neq k}q^i_{jk}(\mu_j-\overline{x}^j_i)(\mu_k-\overline{x}^k_i)+\sum_{j\neq i}q^i_{jj}(s_j\nu^jR_j+(\mu_j-\overline{x}^j_i)^2).$$

Then (16) is an equality between two quadratic polynomials. By equating the coefficients of x^2 we get

$$\frac{1}{2R_is_i^2}-\frac{R_i(A^i)^2}{2}=q^i_{ii} \quad (17)$$

that gives (13). By equating the coefficients of x we get

$$-\frac{\mu_i}{R_is_i^2}=-2q^i_{ii}\overline{x}^i_i+2\sum_{j\neq i}q^i_{ij}(\mu_j-\overline{x}^j_i),\quad i=1,\dots,N, \quad (18)$$

and using (17) we get the matrix equation $B\mu=p$ with p given by (14). Finally, by equating the remaining terms we obtain

$$\lambda_i=\nu^i\left(\frac{1}{s_i}+R_iA^i\right)-\frac{\mu_i^2}{2R_is_i^2}+q^i_{ii}(\overline{x}^i_i)^2-2\overline{x}^i_i\sum_{j\neq i}q^i_{ij}(\mu_j-\overline{x}^j_i)+b_i. \quad (19)$$

This completes the proof of *i)*.

Consider the feedback $\overline{\alpha}=(\overline{\alpha}^1,\dots,\overline{\alpha}^N)$ given by (15) and note it is admissible by Lemma 2.1. Let α^i be an admissible strategy for the i-th player. By (7) we can use Dynkin's formula (see, e.g., [11]) and the first equation in (11) to get

$$\begin{aligned}E\left[v^i(X^i_T)-v^i(x^i)\right]&=E\left[\int_0^T\left(\nu^iv^i_{xx}+A^ixv_x-\alpha^i_tv^i_x\right)(X^i_t)\,dt\right]\\&\geq E\left[\int_0^T\left(\nu^iv^i_{xx}+A^ixv_x-\frac{\left(v^i_x\right)^2}{2R_i}\right)(X^i_t)-\frac{R^i}{2}(\alpha^i_t)^2\,dt\right]\\&=\lambda_iT-E\left[\int_0^T\left(f^i(X^i_t)+\frac{R^i}{2}(\alpha^i_t)^2\right)\,dt\right],\end{aligned}$$

where the inequality is an equality if $\alpha^i=\overline{\alpha}^i$. We divide both sides by T and let $T\to+\infty$. The left hand side vanishes because v^i is a quadratic polynomial and the estimates (7) hold. Then

$$\lambda_i\leq\liminf_{T\to+\infty}\frac{1}{T}E\left[\int_0^T\left(f^i(X^i_t;m_1,\dots,m_N)+\frac{R^i}{2}(\alpha^i_t)^2\right)\,dt\right]$$

with equality if $\alpha^i=\overline{\alpha}^i$. We claim that the right hand side is

$$J^i(X,\overline{\alpha}^1,\dots,\overline{\alpha}^{i-1},\alpha^i,\overline{\alpha}^{i+1},\dots,\overline{\alpha}^N).$$

Then $\lambda_i=J^i(X,\overline{\alpha})$ and (6) holds.

To prove the claim we consider each term of the running cost F^i. We begin with the terms with $j = k$, $j \neq i$. Then

$$\lim_{T\to+\infty} \frac{1}{T} E\left[\int_0^T (X_t^j - \overline{x}_i^j)^2\, dt\right] = \int_{I\!R} (x^j - \overline{x}_i^j)^2 dm_j(x^j) = s_j \nu^j R_j + (\mu_j - \overline{x}_i^j)^2$$

by (8) and the fact that the invariant measure m_j of the process X_t^j corresponding to the control $\overline{\alpha}^j$ is a Gaussian $\mathcal{N}(\mu_j, s_j\nu^j R_j)$.

Next, we consider the terms with X_t^i and X_t^j with $j \neq i$. By the definition of the admissible controls, the process X_t^i corresponding to α^i is ergodic with invariant measure m_{α^i}. Then, for $\tilde{\mu} := \int_{I\!R} x\, dm_{\alpha^i}(x)$,

$$\lim_{T\to+\infty} \frac{1}{T} E\left[\int_0^T (X_t^i - \overline{x}_i^i)(\mu_j - \overline{x}_i^j)\, dt\right] = (\tilde{\mu} - \overline{x}_i^i)(\mu_j - \overline{x}_i^j).$$

To compute the corresponding term in J^i we observe that X_t^j solves the linear Langevin equation

$$dX_t^j = b(\mu_j - X_t^j)dt + \sigma^j dW_t^j, \quad X_0^j = x^j, \quad b := \frac{1}{s_j R_j} > 0.$$

A standard computation gives $E[X_t^j] = \mu_j + (x^j - \mu_j)e^{-bt}$. Then

$$E\left[\int_0^T (X_t^j X_t^i - \mu_j\tilde{\mu})\, dt\right] = \mu_j \int_0^T (E[X_t^i] - \tilde{\mu})\, dt + (x^j - \mu_j)\int_0^T e^{-bt} E[X_t^i]\, dt.$$

If we divide by T and let $T \to +\infty$ the first integral on the right hand side vanishes by (8) and the second tends to 0 by (7). Therefore, using (8) again,

$$\lim_{T\to+\infty} \frac{1}{T} E\left[\int_0^T (X_t^j - \overline{x}_i^j)(X_t^i - \overline{x}_i^i)\, dt\right] =$$
$$\int_{I\!R^2} (x^j - \overline{x}_i^j)(x^i - \overline{x}_i^i) dm_j(x^j) dm_{\alpha^i}(x^i) = (\mu_j - \overline{x}_i^j)(\tilde{\mu} - \overline{x}_i^i).$$

The remaining terms are those involving $j \neq k$, $j, k \neq i$. By the same argument as above we get

$$\lim_{T\to+\infty} \frac{1}{T} E\left[\int_0^T (X_t^j - \overline{x}_i^j)(X_t^k - \overline{x}_i^k)\, dt\right] =$$
$$\int_{I\!R^2} (x^j - \overline{x}_i^j)(x^k - \overline{x}_i^k) dm_j(x^j) dm_k(x^k) = (\mu_j - \overline{x}_i^j)(\mu_k - \overline{x}_i^k).$$

This completes the proof of the claim and of the theorem. □

Remark 1. The solution associated to the optimal feedback $\overline{\alpha}^i$ is the Ornstein-Uhlenbeck process

$$dX_t^i = -\frac{X_t^i - \mu_i}{s_i R_i} dt + \sigma^i dW_t^i$$

and therefore it is mean-reverting with an explicit Gaussian density for X_t^i. Note also that the Nash equilibrium $\overline{\alpha}$ does not depend on the noise intensities σ^i.

Remark 2. The minimal assumption on q_{ii}^i for the validity of the preceding theorem is $q_{ii}^i > -R_i(A^i)^2/2$ because s_i remains well-defined by (13).

Remark 3. From the proof of Theorem 2.2 it is easy to see that the condition $\det B \neq 0$ is necessary for the existence of a unique solution of (11) of the quadratic-Gaussian form (12). The next example shows that, if it fails, there may be either infinitely many solutions or none.

Example 1. For $N = 2$ we consider almost identical players, as we will do for all N later on, namely, $q^1_{11} = q^2_{22} =: q$, $R_1 = R_2 =: R$, $A^1 = A^2 =: A$, $q^1_{12} = q^2_{12} =: \beta/2$, $x^1_1 = x^2_2 =: h$, $x^1_2 = x^2_1 =: r$. Then $\det B = (2q + RA^2)^2 - \beta^2$. For $\beta = 2q + RA^2$ all vectors $\mu = (\mu_1, \mu_2)$ satisfying $\mu_1 + \mu_2 = r + 2qh/\beta$ solve the equation $B\mu = p$ with p defined by (14). Then there are infinitely many Gaussian solutions. In the case $\beta = -2q - RA^2$, instead, there are no solutions of $B\mu = p$ (and therefore no Gaussian solutions) unless $2qh = -\beta r$, and in this last case there are again infinitely many (all μ such that $\mu_1 = \mu_2$).

Remark 4. Note that the matrix B depends only on the drift terms $A^1, \dots, A^N$, the control costs $R^1, \dots, R^N$, and the i-th line of each matrix Q^i. The $i-$th diagonal term $B_{ii} = 2q^i_{ii} + R_i(A^i)^2$ is twice the cost for the $i-$th player to keep his state X^i_t at distance 1 from his reference position $\bar{X}_i$, whereas the off-diagonal term $B_{ij} = 2q^i_{ij}$ is twice his cost if also the j-th player stays at distance 1 from his reference position $\bar{X}_j$ and on the same side (i.e., both at left or both at right). The meaning of condition $\det B \neq 0$ is that one of these two kinds of costs prevails on the other.

3. **Symmetric and almost identical players.** A natural assumption that we will use in the rest of the paper is the following condition saying that the i-th player is influenced in the same way by any two other players.
Symmetry Assumption: the cost F^i of the i-player is symmetric with respect to the position of any two other players, i.e.,

$$F^i(x^1, \dots, x^j, \dots, x^k, \dots, x^N) = F^i(x^1, \dots, x^k, \dots, x^j, \dots, x^N) \quad \forall\, j, k \neq i. \tag{20}$$

Recall that F^i is the quadratic form (5) with coefficients $q^i_{jk},\ j, k = 1, \dots, N$.

Lemma 3.1. *The Symmetry Assumption holds if and only if*

$$q^i_{ij} = q^i_{ik} =: \frac{\beta_i}{2}, \quad q^i_{jj} = q^i_{kk} =: \eta_i, \quad \overline{x}^j_i = \overline{x}^k_i =: r_i, \quad \forall\, j, k \neq i,$$
$$q^i_{lj} = q^i_{kl} = q^i_{km} =: \gamma_i \quad \forall\, l, j, k, m \neq i, \quad l \neq j, k \neq l, k \neq m.$$

Proof. The sufficiency is trivial. For the necessity note that the Symmetry Assumption is an identity between two second degree polynomials. By equating their coefficients one easily gets the conclusions. □

If we also set

$$q_i := q^i_{ii}, \qquad h_i := \overline{x}^i_i \qquad i = 1, \dots, N,$$

the positional cost F^i takes the simpler form

$$F^i(x^1, \dots, x^N) = q_i(y^i)^2 + \beta_i y^i \sum_{j \neq i} y^j + \gamma_i \sum_{j,k \neq i,\, j \neq k} y^j y^k + \eta_i \sum_{j \neq i} (y^j)^2, \tag{21}$$

where $y^i = x^i - h_i$ and $y^j = x^j - r_i$ for $j \neq i$. The parameters involved in the running cost of the i-th player are only six, besides the control cost R_i, and they can be called

- h_i = preferred own position (happy state),
- r_i = reference position of the other players,
- q_i = primary cost of self-displacement,

- η_i = secondary cost of self-displacement,
- β_i = primary cost of cross-displacement,
- γ_i = secondary cost of cross-displacement.

The only sign condition on these parameters is $q_i > 0$, that can be relaxed if $A^i \neq 0$, see Remark 2. Under the Symmetry Assumption the formulas for the Gaussian solution of the system (11) simplify a bit. For instance the cost corresponding to the Nash equilibrium becomes

$$\lambda_i = \frac{\nu^i}{s_i} + \nu^i R_i A^i - \frac{\mu_i^2}{2}\left(2q_i + R_i(A^i)^2\right) + q_i h_i^2 - h_i\beta_i \sum_{j\neq i}(\mu_j - r_i) + \gamma_i \sum_{j,k\neq i,\, j\neq k}(\mu_j - r_i)(\mu_k - r_i) + \eta_i \sum_{j\neq i}(s_j \nu^j R_j + (\mu_j - r_i)^2). \quad (22)$$

A more important consequence of the Symmetry Assumption is that the positional cost F^i can be written in the form arising in Mean Field Games, that is,

$$F^i(x^1, \dots, x^N) = V_i\left[\frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}\right](x^i) \quad \forall\, i = 1, \dots, N, \quad (23)$$

where δ_{x_j} is the Dirac measure on $I\!R$ concentrated at x_j, and the operator V maps probability measures on $I\!R$ to quadratic polynomials and is given by

$$V_i[m](x) := q_i(x-h_i)^2 + \beta_i(x-h_i)(N-1)\int_{I\!R}(y-r_i)\,dm(y) + \gamma_i\left((N-1)\int_{I\!R}(y-r_i)\,dm(y)\right)^2 + (\eta_i - \gamma_i)(N-1)\int_{I\!R}(y-r_i)^2\,dm(y). \quad (24)$$

This is easy to check using the identity $\sum_{j,k\neq i,\, j\neq k} y^j y^k = \left(\sum_{j\neq i} y^j\right)^2 - \sum_{j\neq i}(y^j)^2$.

Remark 5. The Symmetry Assumption is essentially necessary for representing F^i as in (23) with $V_i[m](x)$ depending only on x and on $\int_{I\!R} K_l(x,y)\,dm(y)$ for a finite number of smooth kernels K_l. In fact, imposing such a form to each of the four terms of F^i leads to the conditions of Lemma 3.1.

Definition 3.2. We say that *the players are almost identical* if F^i satisfies the Symmetry Assumption (20) and the players have the same
- control system, i.e., $A^i = A$ and $\sigma^i = \sigma$ (and hence $\nu^i = \nu > 0$) for all i,
- cost of the control, i.e., $R_i = R > 0$ for all i,
- reference positions, i.e., $h_i = h$ and $r_i = r$ for all i,
- primary costs of displacement, i.e., $q_i = q > 0$ and $\beta_i = \beta$ for all i.

The term almost identical is motivated by the independence on i of four of the parameters appearing in the operator V_i, whereas the two secondary costs of displacement γ_i, η_i are still allowed to depend on i. Note also that the reference state vectors $\overline{X}_i$ are all different if $h \neq r$.

For almost identical players we produce solutions of (11) that are Gaussian and also identically distributed.

Theorem 3.3. *Assume the players are almost identical and*

$$2q + RA^2 \neq \beta(1-N). \quad (25)$$

Then

i) there exist unique $\mu \in I\!R$, $s > 0$, such that

$$v^i(x) = v(x) := \frac{(x-\mu)^2}{2s} + \frac{RAx^2}{2},$$
$$m_i(x) = m(x) := \frac{1}{\sqrt{2\pi s\nu R}} \exp\left(-\frac{(x-\mu)^2}{2s\nu R}\right) \quad (26)$$

solve (11) for some $(\lambda_1, \dots, \lambda_N) \in I\!R^N$, moreover

$$s = \left(2qR + R^2A^2\right)^{-1/2}, \quad \mu = \frac{2qh + r\beta(N-1)}{2q + \beta(N-1) + RA^2}, \quad (27)$$

$$\lambda_i = \frac{\nu}{s} + \nu RA - \frac{\mu^2}{2Rs^2} + qh^2 - h\beta(N-1)(\mu - r) + \gamma_i(N-1)(N-2)(\mu-r)^2 + \eta_i(N-1)(s\nu R + (\mu-r)^2), \quad i = 1, \dots, N; \quad (28)$$

ii) the affine feedback

$$\overline{\alpha}^i(x) = \frac{x-\mu}{sR} + Ax, \quad x \in I\!R, \ i = 1, \dots, N,$$

is a Nash equilibrium point for all initial positions $X \in I\!R^N$ among the admissible strategies, and $J^i(X, \overline{\alpha}) = \lambda_i$ for all X and i.

Proof. We plug solutions of the form (26) into (11) and we arrive, as in the proof of Theorem 2.2, at the equations (16), where now all terms on the left hand side are independent of i, but λ_i, and the right hand side f^i is given by

$$f^i(x; m_1, \dots, m_N) = q(x-h)^2 + (x-h)\beta(N-1)(\mu-r) + b_i$$
$$b_i := \gamma_i(N-1)(N-2)(\mu-r)^2 + \eta_i(s\nu R + (\mu-r)^2).$$

The equality of the coefficients of x^2 gives the expression for s in (27), as in the proof of Theorem 2.2. Next, by equating the coefficients of x the system (18) decouples and reduces to

$$-\frac{\mu}{Rs^2} = -2qh + \beta(N-1)(\mu - r),$$

that is solvable by (25) and gives

$$\mu = Rs^2 \frac{2qh + r\beta(N-1)}{Rs^2\beta(N-1) + 1}$$

as well as (27). Finally, by equating the remaining terms we obtain (28). The proof of *ii)* is the same as in Theorem 2.2. □

Remark 6. The assumption (25) of the last theorem is weaker than the one of Theorem 2.2, namely $\det B \neq 0$. In fact, $2q + RA^2 = \beta(1-N)$ implies $\det B = 0$. On the other hand, if $N = 2$ and $\beta = 2q + RA^2$, (25) is satisfied and $\det B = 0$. In this case we saw in Example 1 that there are infinitely many Gaussian solutions to (11) and only one is identically distributed.

Remark 7. If the drift $A = 0$ and $\beta \geq 0$ the expected value μ of the distribution m is a weighted average of the two reference states h and r, and it coincides with the preferred state h if the cost of cross-displacement β vanishes.

Remark 8. If the drift $A \neq 0$ we can allow any $q > -RA^2/2$ in the Theorem, instead of $q > 0$.

4. **The limit as $N \to +\infty$.** In this section we study the limit as the number of players N goes to $+\infty$. For simplicity we assume that the control system, the cost per unit control, and the reference positions remain the same, i.e., A, ν, R, h, r are independent of N. To underline the dependence of all the other quantities on N we add a superscript N to them. We assume the following scaling of the coefficients $q^N, \beta^N, \gamma_i^N, \eta_i^N$ involved in the running cost:

$$\lim_N q^N = \overline{q}, \quad \lim_N \beta^N(N-1) = \overline{\beta}, \\ \lim_N \gamma_i^N (N-1)^2 = \overline{\gamma}, \quad \lim_N \eta_i^N (N-1) = \overline{\eta} \quad \forall\, i. \tag{29}$$

This is natural because in each running cost F^i (21) there are $N-1$ terms multiplied by β^N, $(N-1)^2$ multiplied by γ_i^N, and $N-1$ by η_i^N. In fact, if we denote with V_i^N the operator defined by (24), for any probability measure m on $\mathbb{R}$

$$V_i^N[m](x) \to \overline{V}[m](x) \quad \text{as } N \to +\infty, \text{ locally uniformly in } x,$$

where

$$\overline{V}[m](x) := \overline{q}(x-h)^2 + \overline{\beta}(x-h)\int_{\mathbb{R}} (y-r)\, dm(y) \\ + \overline{\gamma}\left(\int_{\mathbb{R}} (y-r)\, dm(y)\right)^2 + \overline{\eta}\int_{\mathbb{R}} (y-r)^2\, dm(y). \tag{30}$$

Finally we denote with v^N, m^N, λ_i^N the Gaussian identically distributed solution of (11) produced in Theorem 3.3. As in [22, 24] we expect the limit of these solution to satisfy the system of two Mean Field equations

$$\begin{cases} -\nu v_{xx} + \frac{(v_x)^2}{2R} - Axv_x + \lambda = \overline{V}[m](x) \quad \text{in } \mathbb{R}, \\ -\nu m_{xx} - \left(\frac{v_x}{R}m - Axm\right)_x = 0 \quad \text{in } \mathbb{R}, \\ \min\left[v(x) - \frac{RAx^2}{2}\right] = 0, \quad \int_{\mathbb{R}} m(x)dx = 1, \quad m > 0 \text{ in } \mathbb{R}. \end{cases} \tag{31}$$

Note the normalization condition on v that replaces the null-average condition of the periodic case in [22, 24].

Theorem 4.1. *Assume the players are almost identical,* (29) *holds, and*

$$2\overline{q} + RA^2 > 0, \qquad 2\overline{q} + RA^2 \neq -\overline{\beta}. \tag{32}$$

Then
i) system (31) *has exactly one solution* (v, m, λ) *of the quadratic-Gaussian form*

$$v(x) := \frac{(x-\overline{\mu})^2}{2\overline{s}} + \frac{RAx^2}{2}, \quad m(x) := \frac{1}{\sqrt{2\pi\overline{s}\nu R}} \exp\left(-\frac{(x-\overline{\mu})^2}{2\overline{s}\nu R}\right), \tag{33}$$

given by

$$\overline{s} = \left(2\overline{q}R + R^2A^2\right)^{-1/2}, \quad \overline{\mu} = \frac{2\overline{q}h + r\overline{\beta}}{\overline{\beta} + 2\overline{q} + RA^2}, \tag{34}$$

$$\lambda = \frac{\nu}{\overline{s}} + \nu RA - \frac{\overline{\mu}^2}{2R\overline{s}^2} + \overline{q}h^2 - h\overline{\beta}(\overline{\mu} - r) + (\overline{\gamma} + \overline{\eta})(\overline{\mu} - r)^2 + \overline{\eta}\overline{s}\nu R; \tag{35}$$

ii) as $N \to +\infty$, $v^N \to v$ in $C^1_{loc}(\mathbb{R})$ with second derivative converging uniformly in $\mathbb{R}$, $m^N \to m$ in $C^k(\mathbb{R})$ for all k, and $\lambda_i^N \to \lambda$ for all i;
iii) if in addition $\overline{\beta} \geq 0$, then (v, m, λ) given in i) is the unique solution of (31).

Proof. i) We plug a solution of the form (33) into (31) and get

$$-\frac{\nu}{\overline{s}}-\nu RA+\frac{1}{2R}\left(\frac{(x-\overline{\mu})^2}{\overline{s}^2}+(RAx)^2\right)-RA^2x^2+\lambda=$$
$$\overline{q}(x-h)^2+\overline{\beta}(x-h)(\overline{\mu}-r)+(\overline{\gamma}+\overline{\eta})(\overline{\mu}-r)^2+\overline{\eta}\overline{s}\nu R.$$

By equating the coefficients of x^2 and x on both sides we get (34), whereas the remaining terms give (35).

ii) Note that (32) and (29) imply $q^N > -RA^2/2$ and (25) for N large enough. Then Remark 8, the explicit formulas (26), (27), (28), and the assumption (29) give immediately the stated convergence.

iii) To prove uniqueness let us first check the monotonicity of $\overline{V}$ with respect to the scalar product of functions in the Lebesgue space L^2, if $\overline{\beta}\geq 0$. For two probability measures with densities m,n

$$\int_{I\!R}\left(\overline{V}[m]-\overline{V}[n]\right)(x)\,(m-n)(x)\,dx=$$
$$\overline{\beta}\int_{I\!R}(x-h)\int_{I\!R}(y-r)(m-n)(y)\,dy\,(m-n)(x)\,dx=$$
$$\overline{\beta}\int_{I\!R}x\int_{I\!R}y(m-n)(y)\,dy\,(m-n)(x)\,dx=$$
$$\overline{\beta}\left(\int_{I\!R}x(m-n)(x)\,dx\right)^2\geq 0.$$

Now we follow the method of [22, 24]. Let u,n,λ_1 be another solution of (31). Multiply the first equation in (31) by $m-n$ and subtract the same expression computed on u,n,λ_1; next multiply the second equation in (31) by $v-u$ and subtract the same expression computed on u,n,λ_1; subtract the second identity from the first and integrate on $I\!R$. By using $\int_{I\!R}m(x)dx=\int_{I\!R}n(x)dx$ we arrive at

$$\int_{I\!R}\left(\overline{V}[m]-\overline{V}[n]\right)(x)\,(m-n)(x)\,dx+$$
$$\int_{I\!R}\frac{m(x)}{R}\left(\frac{(u_x)^2}{2}-\frac{(v_x)^2}{2}-v_x(u_x-v_x)\right)dx+$$
$$\int_{I\!R}\frac{n(x)}{R}\left(\frac{(v_x)^2}{2}-\frac{(u_x)^2}{2}-u_x(v_x-u_x)\right)dx=0.$$

Since each of the three terms is non-negative, it must vanish. Then $m>0$ and $p^2/2$ strictly convex imply $u_x\equiv v_x$. Next, the condition $\min\left[v(x)-RAx^2/2\right]=\min\left[u(x)-RAx^2/2\right]$ gives $u\equiv u$. Therefore $w(x):=m(x)-n(x)$ solves

$$-\nu w_{xx}-\left(\frac{v_x}{R}w-Axw\right)_x=0\quad\text{in } I\!R,\qquad\int_{I\!R}w(x)dx=0$$

and by direct integration it is easy to see that $w\equiv 0$. Finally $\lambda_1=\lambda_2$ by the first equation in (31). □

Remark 9. In the proof we showed the operator $\overline{V}$ defined by (30) is monotone with respect to the scalar product in L^2, as defined in [22, 24], if and only if $\overline{\beta}\geq 0$, and strictly monotone if and only if $\overline{\beta}>0$. The parameter $\overline{\beta}$ is the signed cost per unit time and per unit of displacement of the single player from h and of the average player from r. If $\overline{\beta} > 0$ there is a positive cost if both displacements are in the same direction, i.e., both to the right or both to the left, and a negative cost if they are in opposite directions. If $\overline{\beta} < 0$ the reverse situation occurs. Therefore we can say that imitation among players is costly if $\overline{\beta} > 0$ and rewarding if $\overline{\beta} < 0$. The statement *iii)* of Theorem 4.1 says that there cannot be multiple solutions to (31) unless imitation is rewarding.

Remark 10. The conditions (32) have a simple meaning. The quantity $2\overline{q} + RA^2$ is twice the cost per unit time of staying at distance 1 from the preferred state h, and it is positive if either $\overline{q} > 0$ or $A \neq 0$. The second inequality in (32) is always satisfied if $\overline{\beta} \geq 0$, i.e., imitation is costly or indifferent. If, instead, $\overline{\beta} < 0$ the inequality is satisfied unless the reward $-\overline{\beta}$ for imitating a unitary displacement of the other players exactly balances twice the cost $\overline{q} + RA^2/2$ of staying at distance 1 from h.

Remark 11. The case left out of the theorem above is $\overline{\beta} = -2\overline{q} - RA^2$. If $2\overline{q}h \neq -\overline{\beta}r$ then there is no solution of the Gaussian form (33). If instead $2\overline{q}h = -\overline{\beta}r$ there is a continuum of solutions, because for every $\overline{\mu} \in I\!R$ the functions (33) with $\overline{s} = \left(2\overline{q}R + R^2A^2\right)^{-1/2}$ and the constant (35) solve (31).

This is also an example that statement *iii)* of the last theorem concerning uniqueness may not hold if $\overline{\beta} < 0$. A different example of Mean Field system with infinitely many Gaussian solutions was given by Guéant [14], see Section 6 for a discussion.

Remark 12. The assumption $\beta = \beta^N \geq 0$ does not imply uniqueness for the system (11) associated to N players, different from the case (31) describing infinitely many players. In fact , for $N = 2$ and $\beta = 2q + RA^2$ there are infinitely many Gaussian solutions, see Remark 6.

Remark 13. If $A \neq 0$ negative values of $\overline{q}$ are allowed. In this case h is not a preferred positions as it is rewarding to stay far from it.

Remark 14. If we have expansions of the parameters in powers of $1/N$, such as $q^N = \overline{q} + \frac{q_1}{N} + \frac{q_2}{N^2} + \dots$, $\beta^N = \frac{\overline{\beta}}{N} + \frac{\beta_2}{N^2} + \dots$, etc., we can easily get expansions of the solution v^N, m^N, λ_i^N in powers of $1/N$. Note also that we can assume the parameters A, ν, R, h, r depend on N, provided they converge as $N \to \infty$.

Example 2. In [17, 19] the authors considered infinite horizon discounted functionals, as in Section 5.4 below, with positional cost

$$F^i(x^1, \dots, x^N) = \left[x^i - b\left(\frac{1}{N}\sum_{j\neq i} x^j + c\right)\right]^2.$$

The case $c \neq 0$ requires some minor modifications to our previous calculations, but for $c = 0$ F^i is a quadratic form in X with symmetric and almost identical players and the parameters are

$$q_i^N = 1, \quad \beta_i^N = -\frac{2b}{N}, \quad \gamma_i^N = \eta_i^N = \frac{b^2}{N^2}, \quad \forall\, i, N.$$

Then the scaling assumption (29) holds, with

$$\overline{q} = 1, \quad \overline{\beta} = -2b, \quad \overline{\gamma} = b^2, \quad \overline{\eta} = 0,$$

and the limit positional cost is

$$\overline{V}[m](x) = \left(x - b\int_{I\!R} y\,dm(y)\right)^2 .$$

Therefore the quadratic-Gaussian solution of Theorem 4.1 has mean $\overline{\mu} = 0$, $\overline{s} = \left(2\overline{R} + R^2A^2\right)^{-1/2}$, and cost $\lambda = \nu\left(2\overline{R} + R^2A^2\right)^{1/2} + \nu RA$.

5. **Other limiting cases.**

5.1. **The small noise or vanishing viscosity limit.** We consider the limit as the noise coefficient σ^j tends to 0 in the dynamics (4) of the j-th player. This is a vanishing viscosity limit $\nu^j \to 0$ for the j-th HJB equation and the j-th KFP equation in (11). The limit of the Gaussian solutions (12) found in Theorem 2.2 is easy: the function v^j in (12) does not change because s_j and μ_j do not depend on ν^j, whereas m_j converge to the Dirac mass δ_{μ_j} in the sense of distributions. This limit satisfies the system (11) with $\nu^j = 0$, although the j-th KFP equation is verified only in the sense of distributions by the measure δ_{μ_j}. The affine feedbacks α^i (15) still define a Nash equilibrium point, since the proof of Theorem 2.2 *ii)* holds unchanged. If $\sigma_i = 0$ for all $i = 1, \dots, N$ we have therefore found a Nash equilbrium for a deterministic N-person differential game. The fact that the equilibrium feedback is the same for the limit deterministic game as for all positive noise intensities is remarkable, and it was known for finite horizon L-Q problems [7, 10].

Next we perform the vanishing viscosity limit $\nu \to 0$ in the Mean Field system of equations (31). As before v is unchanged and $m \to \delta_{\overline{\mu}}$ in the sense of distributions. Then

$$v(x) = \frac{(x-\overline{\mu})^2}{2\overline{s}} + \frac{RAx^2}{2}, \quad m(x) = \delta_{\overline{\mu}},$$
$$\lambda = \overline{q}h^2 - \frac{\overline{\mu}^2}{2R\overline{s}^2} - h\overline{\beta}(\overline{\mu} - r) + (\overline{\gamma} + \overline{\eta})(\overline{\mu} - r)^2, \tag{36}$$

with $\overline{s}, \overline{\mu}$ given by (34), solve the first order Mean Field system

$$\begin{cases} \frac{(v_x)^2}{2R} - Axv_x + \lambda = \overline{V}[m](x) \quad \text{in } I\!R, \\ -\left(\frac{v_x}{R}m - Axm\right)_x = 0 \quad \text{in } I\!R, \\ \min\left[v(x) - \frac{RAx^2}{2}\right] = 0, \quad \int_{I\!R} m(x)dx = 1, \quad m > 0 \text{ in } I\!R. \end{cases} \tag{37}$$

with $\overline{V}[m]$ defined by (30) and the second equation verified in the sense of distributions. It is easy to see that $\overline{s}$ and $\overline{\mu}$ are uniquely determined by the form of v and m in (36). Moreover (36) is the unique solution of (37) if $\overline{\beta} \geq 0$, by the same proof as part *iii)* of Theorem 4.1.

Finally note that the limits $N \to +\infty$ and $\nu \to 0$ commute: if we assume the N players be almost identical, v^N, m^N, λ_i^N be the limit as $\nu^i = \nu \to 0$ of the Gaussian identically distributed solution of (11), and (29), then as $N \to +\infty$ $v^N \to v$ in $C^2_{loc}(I\!R)$, $m^N \to m$ in the sense of distributions, and $\lambda_i^N \to \lambda$ for all i, where v, m, λ are given by (36). We refer to [24] for an example where the small noise limit does not commute with $N \to +\infty$.

5.2. **Cheap control.** We investigate the limit as $R_i \to 0$ and for simplicity we limit ourselves to the case of almost identical players. Note that the equations in (11) become very degenerate for $R_i = 0$, but we can use the explicit formulas of Theorem 3.3. We see that $s \to +\infty$ but $sR \to 0$ as $R \to 0$, whereas

$$\mu \to \frac{2qh + r\beta(N-1)}{2q + \beta(N-1)} =: \mu^N.$$

Then $v(x) \to 0$ in $C^2_{loc}(\mathbb{R})$ and $m \to \delta_{\mu^N}$ in the sense of distributions. Therefore the Gaussian solution converges to a Dirac mass as in the small noise limit, but here v vanishes. Moreover

$$\lambda_i \to -\frac{(\mu^N)^2}{4q} + qh^2 - h\beta(N-1)(\mu^N - r) \\ + \gamma_i(N-1)(N-2)(\mu^N - r)^2 + \eta_i(N-1)(\mu^N - r)^2 =: \lambda^N \quad (38)$$

for all $i = 1, \ldots, N$.

A very similar result holds for the Gaussian solutions of the Mean Field equations (31) of Theorem 4.1: $v(x) \to 0$ in $C^2_{loc}(\mathbb{R})$, $m \to \delta_{\tilde{\mu}}$ in the sense of distributions, where

$$\overline{\mu} \to \frac{2\overline{q}h + r\overline{\beta}}{2\overline{q} + \overline{\beta}} =: \tilde{\mu},$$

and

$$\lambda \to -\frac{\tilde{\mu}^2}{4\overline{q}} + \overline{q}h^2 - h\overline{\beta}(\tilde{\mu} - r) + \overline{\gamma}(\tilde{\mu} - r)^2 + \overline{\eta}(\tilde{\mu} - r)^2 =: \tilde{\lambda}.$$

Note that both μ^N and $\tilde{\mu}$ are weighted averages of the two reference states h and r, if $\overline{\beta} \geq 0$.

Finally we remark that also the cheap control limit $R \to 0$ commutes with $N \to +\infty$ (of course under the assumption (29)). In fact $\mu^N \to \tilde{\mu}$ and therefore $\delta_{\mu^N} \to \delta_{\tilde{\mu}}$ in the sense of distributions and $\lambda^N \to \tilde{\lambda}$.

5.3. **Large cost of cross-displacement.** If the parameters of the cost functional scale in a different way from (29) the convergence of the Gaussian identically distributed solution (v^N, m^N, λ_i^N) is much harder. A case that we find interesting is a decay of β^N slower than $1/N$ as $N \to \infty$, or even no decay at all. Therefore we assume

$$\lim_{N\to\infty} |\beta^N|(N-1) = +\infty \quad (39)$$

instead of the second condition in (29) and keep the other three assumptions on the behavior at infinity of $q^N, \gamma_i^N, \eta_i^N$. Then the second term in $V_i^N[m^N](x)$ diverges for $x \neq h$, unless $\int_{\mathbb{R}} y\, dm^N(y) \to r$, and we wonder about the validity of a Mean Field system of the form (31) for some new limit operator $\overline{V}$.

We pass to the limit in the formulas (26) (27) (28) for v^N, m^N, λ_i^N and get (33) with $\overline{s}$ as in (34) and $\overline{\mu} = r$, and

$$\lambda = \frac{\nu}{\overline{s}} + \nu RA - \frac{r^2}{2R\overline{s}^2} + \overline{q}h^2 - h(2qh - 2qr - rRA^2) + \overline{\eta}\overline{s}\nu R.$$

Therefore the mean value $\overline{\mu}$ of the limit distribution is just the reference state r instead of a linear combination of h and r as in all cases studied before. Moreover the limits v, μ, λ solve the Mean Field system (31) with the limit operator $\overline{V}$ given by (30) with the second term $\overline{\beta}\int_{\mathbb{R}}(y - r)\, dm(y)(x - h)$ replaced by $\left(2\overline{q}(h - r) - rRA^2\right)(x - h)$.

The same limit is obtained if we let the cost of cross-displacement $\overline{\beta}$ tend to $+\infty$ or $-\infty$. This is a *singular perturbation* or penalization problem for the Mean Field system (30) (31).

5.4. **Discounted problems and the vanishing discount limit.** Consider cost functionals with infinite horizon and discounted running cost, that is, for some $\rho > 0$,

$$J^i = E\left[\int_0^{+\infty}\left(\frac{R_i}{2}(\alpha_t^i)^2 + f^i(X_t^i)\right)e^{-\rho t}dt\right],$$

where $f^i(x^i) = \int_{I\!R^{N-1}} F^i \prod_{j\neq i} dm_j(x^j)$ as in (10) and F^i is quadratic as in (5). The associated system of $2N$ HJB and KFP equations is the same as (11) with λ_i replaced by ρv^i. By the same calculations as in Theorem 2.2 one finds Gaussian solutions with m_i as in (12) and

$$v_\rho^i(x) = \frac{(x-\mu_i)^2}{2s_i} + \frac{R_i A^i x^2}{2} + c_i,$$

under the condition that $\det(B - \rho D) \neq 0$, where D is the diagonal matrix with $D_{ii} = R_i A^i$. It is easy to write explicit formulas for the $3N$ unknown parameters $s_i^\rho, \mu_i^\rho, c_i^\rho$; in particular,

$$\rho c_i^\rho = \lambda_i - \frac{\rho(\mu_i^\rho)^2}{2s_i^\rho},$$

where λ_i is given by (19).

The vanishing discount limit is the limit as $\rho \to 0$. We easily get that $s_i^\rho \to s_i$, $\mu_i^\rho \to \mu_i$, $c_i^\rho \to \infty$, where the limits are given in Theorem 2.2, so v_ρ^i diverges but $\rho v_\rho^i \to \lambda_i$ and $v_\rho^i(x) - v_\rho^i(0) + (\mu_i^\rho)^2/2s_i^\rho \to v^i(x)$ given by (12). Thus in the limit $\rho \to 0$ we recover the unique Gaussian solution of (11). This link between discounted infinite horizon problems and ergodic control is well known for problems with a single player [6] or two-person zero-sum games [3], see also the references therein.

Next for $\rho > 0$ fixed we let $N \to +\infty$ under the assumption of almost identical players, see Definition 3.2, and with the scaling (29) of the parameters. Denote with v_ρ^N, m_ρ^N the identically distributed Gaussian solutions of the discounted N-player problem. By the usual method we see that if

$$2\overline{q} + RA^2 + \rho RA \neq -\overline{\beta}$$

there is exactly one quadratic v_ρ and Gaussian m_ρ solving the Mean Field system for the discounted problem

$$\begin{cases} -\nu v_{xx} + \frac{(v_x)^2}{2R} - Axv_x + \rho v = \overline{V}[m](x) \quad \text{in } I\!R, \\ -\nu m_{xx} - \left(\frac{v_x}{R}m - Axm\right)_x = 0 \quad \text{in } I\!R, \\ \min\left[v(x) - \frac{RAx^2}{2}\right] = 0, \quad \int_{I\!R} m(x)dx = 1, \quad m > 0 \text{ in } I\!R. \end{cases} \tag{40}$$

Moreover one checks on the explicit formulas that $v_\rho^N \to v_\rho$ and $m_\rho^N \to m_\rho$ as $N \to +\infty$. If we let now $\rho \to 0$ in (40) we get the Gaussian solution of the Mean Field system (31) for the ergodic problem found in Theorem 4.1, and therefore the limits $N \to +\infty$ and $\rho \to 0$ commute.

6. **Models of population distribution.** The Mean Field equations (31) can be interpreted as modeling the stationary states of a population formed by a continuum of identical individuals who move around seeking to minimize their own cost functional depending on the distribution function m of the entire population. The form and the parameters of the cost functional describe the preferences of the individuals. The reference model of this kind has the discounted cost functional

$$E\left[\int_0^{+\infty}\left(\frac{|\alpha_t^i|^2}{2}+\overline{q}|X_t^i-h|^2-\log m(X_t^i)\right)e^{-\rho t}dt\right], \qquad \overline{q}\geq 0, \tag{41}$$

and was studied in depth by Guéant [14, 13] in many space dimensions and with drift $A=0$, see also the survey paper [15]. This model is not derived as the limit as $N\to+\infty$ of a N-person game; in fact the right hand side of the H-J-B equation in (40) is $\overline{q}|x-h|^2-\log m(x)$ instead of a non-local integral operator $V[m](x)$, although it can be recovered as a limit of such operators, as argued in [24].

We want to compare some of the results by Guéant [14, 13] with Theorem 4.1 on the L-Q model (30), (31) (and the same can be done for the discounted problem (40) with similar results). For simplicity we restrict to the case $A=0$, as in [14, 13], and $h=r$, so there is only one reference position in the state space, as in (41).

The choice of the strictly decreasing running cost $-\log m$ in (41) aims at modeling a population whose agents wish to resemble their peers as much as possible. On the other hand, a consequence of this monotonicity assumption is that no uniqueness is guaranteed for the corresponding Mean Field equations for any value of the parameters. For this reason an interesting stability analysis is performed in [14, 13, 15].

In our model, the term of $\overline{V}$ that describes the willingness to resemble the other individuals is $\overline{\beta}(x-h)\int_{\mathbb{R}}(y-h)\,dm(y)$. The term $\overline{q}(x-h)^2$ is the same as in the log model (41) and the other two terms are irrelevant for the present discussion. The parameter $\overline{\beta}$ is the signed cost per unit time and per unit of displacement of the single player and of the average player from h. If $\overline{\beta}>0$ there is a positive cost if both displacements are in the same direction, i.e., both to the right or both to the left, and a negative cost if they are in opposite directions. If $\overline{\beta}<0$ the reverse situation occurs. Therefore we can say that imitation among players is costly if $\overline{\beta}>0$ and rewarding if $\overline{\beta}<0$. The statement *iii)* of Theorem 4.1 says that there is uniqueness of solution to (31) unless imitation is rewarding.

We continue the comparison in the range $\overline{\beta}<0$, so that both models describe a preference for imitation, although in different forms. One of the results of [14, 13, 15] for the log model is the existence of a unique Gaussian solution for any $\overline{q}>0$, with mean h, whereas for $\overline{q}=0$ there is a continuum of Gaussian solutions, with arbitrary mean μ. In the L-Q model (with $A=0$ and $h=r$) there is existence and uniqueness of a Gaussian solution for any $\overline{q}>0$ and $\overline{q}\neq-\beta/2$, with mean h, and for $\overline{q}=-\beta/2$ there is a continuum of Gaussian solutions with arbitrary mean, cfr. Remark 11. So there is the same bifurcation phenomenon, but for a different value of $\overline{q}$. On the other hand, in our model as $\overline{q}\to 0$ the variance of m goes to $+\infty$: the distribution tries to become uniform on $\mathbb{R}$, so this limit is singular.

Finally we recall that two other models of population distribution are proposed in Section 5 of the survey [15] besides (41). They involve the following operators

$$V^{(1)}[m](x)=b\left(x-\int y\,dm(y)\right)^2, \qquad V^{(2)}[m](x)=b\int (x-y)^2\;dm(y).$$

Note that both are special cases of $\overline{V}$ given by (30): it is enough to take $h = r = 0$ and $\overline{q} = -\overline{\beta}/2 = b$ in both, $\overline{\gamma} = b$ and $\overline{\eta} = 0$ for $V^{(1)}$, $\overline{\gamma} = 0$ and $\overline{\eta} = b$ for $V^{(2)}$.

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