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On the quasistatic limit of some dynamical problems with dissipative terms

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Abstract

This thesis is devoted to the analysis of the asymptotic behaviour of two damped dynamical problems as inertia vanishes. Thanks to the presence of dissipative terms, we prove that in both cases the limit evolution is quasistatic and rate-independent; the role of the damping is crucial for the validity of the result, since counterexamples in the dissipation-free setting are known.

The first problem, covering several mechanical models, deals with the abstract differential inclusion

$$\varepsilon^2\ddot{x}^\varepsilon(t)+\partial_v\mathcal{R}(t,\dot{x}^\varepsilon(t))+D_x\mathcal{E}(t,x^\varepsilon(t))\ni 0,\quad t>0,$$

in finite dimensional spaces. Here the damping term is given by the dissipation potential $\mathcal{R}$, which represents a dry friction possibly depending on time. After the establishment of an existence (and uniqueness) result of dynamic solutions under suitable assumptions, we show that the limit function obtained sending $\varepsilon\to 0^+$ solves (in various suitable senses) the rate-independent inclusion $\partial_v\mathcal{R}(t,\dot{x}(t))+D_x\mathcal{E}(t,x(t))\ni 0$.

The second model describes the debonding of a one-dimensional object (a bar or a tape) from an adhesive brittle substrate. To illustrate the process, the viscous-damped wave equation

$$\varepsilon^2u^\varepsilon_{tt}(t,x)-u^\varepsilon_{xx}(t,x)+\varepsilon u^\varepsilon_t(t,x)=0,\quad t>0\,,\,0<x<\ell^\varepsilon(t),$$

is considered in the time-dependent domain $(0,\ell^\varepsilon(t))$ representing the unbonded part of the object, and is coupled with a dynamic Griffith's criterion governing the evolution of the debonding front ℓ^ε. We first prove existence, uniqueness and continuous dependence results for this coupled problem. Then, exploiting the presence of the viscous term $\varepsilon u^\varepsilon_t$, we are able to deduce that the dynamic solution $(u^\varepsilon,\ell^\varepsilon)$ converges as $\varepsilon\to 0^+$ to a pair (u,ℓ) solving the equilibrium equation $u_{xx}(t,x)=0$ in $(0,\ell(t))$ together with a quasistatic Griffith's criterion for ℓ.

Our main contribution is thus the confirmation for the two considered models of the tendency of dynamical systems to be close to their quasistatic counterpart (when inertia is small) only if suitable dissipation mechanisms are taken into account. Their presence is indeed necessary to erase in the limit all the kinetic effects, which otherwise survive and can not be detected by a pure quasistatic analysis.

Introduction

In most of the models arising in the framework of mechanical systems the involved physical process is assumed to be quasistatic if the external forces act slowly. In this setting all the rate-dependent effects, such as viscosity or inertia, are neglected and the evolution is usually driven by two criteria: the body is at equilibrium at every time (stability), and the total energy of the system is balanced by the work of the external forces (energy balance). We refer for instance to the monograph [65] for a wide and complete presentation of quasistatic and rate-independent evolutions.

The common choice of adopting a quasistatic viewpoint is twofold. On one hand quasistatic models are much easier to treat than the viscous or dynamic ones which naturally emerge in mechanics. On the other hand they are believed to provide a reasonable approximation of the aformentioned more natural and richer models, still keeping all the peculiar features and properties. Of course, this approximation makes sense and can be possible only when external forces (and initial velocity) are slow enough with respect to a suitable time-scale, otherwise inertia is triggered and internal vibrations must be taken into account.

Due to the above reasons, in the last twenty years a deep mathematical comprehension of quasistatic models has been obtained, see for instance [14, 18, 21, 27, 33, 34, 71] in the context of Fracture Mechanics, or [64, 67] for damage models. Among the main consequences of this extensive understanding, we may mention the development of simple and efficient algorithms leading to proficuous numerical investigations of the involved processes, see [5, 6, 44, 70].

However, although the quasistatic approximation is often adopted and accepted, a rigorous mathematical proof of its validity is really far from being achieved in a general framework, due to the high complexity and diversity of the phenomena under consideration. We may roughly recognise two different kinds of approach: the first one, which takes only viscosity into account, consists in the asymptotic analysis of a first order singular perturbation of the quasistatic model. The procedure, called for clear reasons vanishing viscosity method, has been widely analysed and understood under different points of view thanks to the contribution of several authors. We quote for instance [4, 61, 62, 63, 86] for an abstract investigation and the concepts of Balanced Viscosity and parametrised solutions, or [32, 47, 79] for applications to mechanics and elasticity. Exploiting the characteristic parabolic structure of the viscous problem, the main techniques adopted in this framework come from the by now well consolidated realm of gradient flows, even treated in metric spaces (see [9] for a general discussion on the topic).

The second approach, which we follow in this thesis, deals instead with the quasistatic limit of dynamical problems, an approximation of second order due to the presence of inertia; because of several mathematical difficulties brought by the underlying hyperbolic structure, it still offers a huge variety of open questions and hard challenges. To illustrate its formulation we consider as an example the following system of ordinary differential equations:

$$\begin{cases} \ddot{x}(t) = f(t, x(t), \dot{x}(t)), & t > 0, \\ x(0) = x_0, \quad \dot{x}(0) = x_1. \end{cases}$$

Usually in many applications the function f is the (spatial) gradient of a suitable potential energy functional emerging from the mechanical model one is interested in. In order to consider slow forces and slow initial velocity, a small parameter $\varepsilon > 0$ is included in the system, which thus becomes:

$$\begin{cases} \ddot{x}_\varepsilon(t) = f(\varepsilon t, x_\varepsilon(t), \dot{x}_\varepsilon(t)), & t > 0, \\ x_\varepsilon(0) = x_0, \quad \dot{x}_\varepsilon(0) = \varepsilon x_1. \end{cases} \tag{0.0.1}$$

This is still not yet the correct system of which to study the limit as $\varepsilon \to 0^+$ (indeed the second derivative, i.e. the inertia, is not vanishing); we first need to recast it in the proper time-scale of internal oscillations. With this aim, it is convenient to introduce the change of variable $t \mapsto t/\varepsilon$ and the rescaled function $x^\varepsilon(t) = x_\varepsilon(t/\varepsilon)$; hence (0.0.1) takes finally the form:

$$\begin{cases} \varepsilon^2 \ddot{x}^\varepsilon(t) = f(t, x^\varepsilon(t), \varepsilon \dot{x}^\varepsilon(t)), & t > 0, \\ x^\varepsilon(0) = x_0, \quad \dot{x}^\varepsilon(0) = x_1. \end{cases} \tag{0.0.2}$$

The problem thus consists in understanding whether or not solutions of (0.0.2) converge as $\varepsilon \to 0^+$ to functions x solving the quasistatic problem

$$\begin{cases} 0 = f(t, x(t), 0), & t > 0, \\ x(0) = x_0, \end{cases} \tag{0.0.3}$$

which is exactly the formal limit of (0.0.2). Other nontrivial issues often emerging in this asymptotic analysis are the regularity in time of the limit function and the characterisation of possible jumps, which are expected to occur in a nonconvex setting.

Because of the aforementioned difficulties, nowadays only partial results on the theme are available; we refer for instance to [26] in a case of perfect plasticity, to [54, 79] for damage models, to [82] for a model of delamination, and to [60, 80] for systems with hardening in viscoelastic solids. The issue of the quasistatic limit of dynamic evolutions has also been studied in a finite-dimensional setting where, starting from [3] and with the contribution of [69], an almost complete understanding on the topic has been reached in [83]. A common feature appearing both in finite and in infinite dimension is the validation of the quasistatic approximation via dynamic problems only in presence of a damping term in the dynamical model. A simple explanation could be the following: roughly speaking, if no dissipations are considered, kinetic energy persists in the limit precluding the resulting evolution to be rate-independent.

The content of this thesis work goes in the same direction; we indeed present two different dynamical problems with different dissipative terms and we analyse their asymptotic behaviour when inertia vanishes, as explained in the previous illustrative example. Thanks to the presence of dissipation, we prove that in both cases the limit evolution is quasistatic and rate-independent; thus the quasistatic approximation is justified. We want to highlight that the role of the damping is crucial for the validity of the result, since counterexamples in the undamped situation of both models are known, see [52] and [69]. We hence confirm the tendency possessed by dynamical systems to be related (when inertia is small) to their quasistatic counterpart only if suitable dissipative mechanisms are taken into account. Their presence is indeed necessary to delete in the limit all the dynamic effects, which otherwise survive and can not be detected by a pure quasistatic analysis.

Part I
Finite-dimensional systems with non-autonomous dry friction

The subject of the first part of the thesis regards the investigation of the quasistatic approximation for abstract dynamical systems in finite dimension, in which the role of

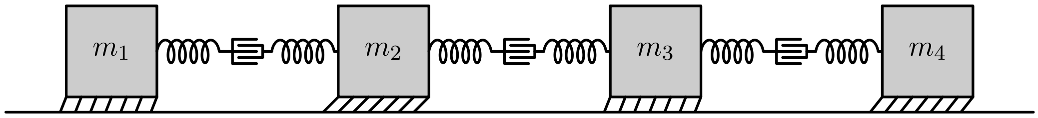

Figure 1: A model of soft crawler, discussed in Subsection 1.2.2.

principal dissipation is played by a (possibly) time-dependent dry friction. Although several mechanisms appearing in nature show the occurence of a non-autonomous damping, from a mathematical point of view this feature has been barely taken into account: we refer for instance to [39], where this aspect is studied, but only in a quasistatic framework.

Among the multitude of standard mechanical models which can be included in the abstract formulation we consider, a particular focus is reserved to an application to a discrete model of soft crawler, analysed in [36]. To explain it, let us consider the example illustrated in Figure 1. The system is composed by $N = 4$ masses on a line. Each couple of adjacent masses is joined by a soft actuated link and subjected to dry friction; this means that the actual shape of the locomotor is not explicitly controlled, but undergoes to hysteresis. Soft actuation is standard in nature, where soft bodies and soft body parts, compliant joints and soft shells are the norm. This feature is even more evident for worm-like locomotion: for instance earthworms and leeches are entirely soft-bodied, while no lever action on the skeleton is employed by snakes during rectilinear locomotion.

In coordination with the soft actuation on the links, a second active control is sometimes available to crawlers: the ability to change the friction coefficients in time. The most remarkable example is inching, i.e. the locomotion strategy of leeches and inchworms, which has been also reproduced in soft robotic devices [85]. In inching locomotion the crawler can be modelled as a single link, periodically elongating and contracting, with the two extremities alternately increasing the friction coefficient (anchoring): during elongation the backward extremity has more grip, so it remains steady while the forward extremity advances, and vice versa during contraction. Other examples of active control of the friction coefficients can be observed in crawlers using anisotropic friction: changing the tilt angle of bristles – such as *setae* and *chaetae* in anellids [1, 72] – or scales – such as in snakes [43] and in robotic replicas [57, 73] – produces a change in the friction coefficients [37], that is used to facilitate sliding or gripping.

The description of the dynamic evolution for the above system can be covered by the following general formulation, involving an abstract differential inclusion:

$$\begin{cases} \varepsilon^2\mathbb{M}\ddot{x}^\varepsilon(t) + \varepsilon\mathbb{V}\dot{x}^\varepsilon(t) + \partial_v\mathcal{R}(t,\dot{x}^\varepsilon(t)) + D_x\mathcal{E}(t,x^\varepsilon(t)) \ni 0, \quad t>0, \\ x^\varepsilon(0) = x_0^\varepsilon, \quad \dot{x}^\varepsilon(0) = x_1^\varepsilon, \end{cases} \tag{0.0.4}$$

where the small parameter $\varepsilon > 0$ models a slow loading regime, and the reparametrisation $t \mapsto t/\varepsilon$ has already been applied. We stress that, differently from the illustrative example (0.0.2), in the argument of the subdifferential of $\mathcal{R}(t,\cdot)$ there is no ε in front of the velocity $\dot{x}^\varepsilon(t)$. This is motivated by the fact that $\mathcal{R}(t,\cdot)$ is usually, and in particular here, positively one-homogeneous; thus its subdifferential turns out to be homogeneous of degree zero, and so the term ε can be neglected. Moreover, we also allow the initial position and velocity to depend on ε.

Likewise [83], we confine ourselves to a finite dimensional framework due to mathematical issues arising in infinite dimension, where only specific and concrete models have been treated (see [26, 54, 60, 79, 82, 80] or also the second part of the thesis). Thus, the ambient space where the variable $x^\varepsilon(t)$ lives will be a Banach space X with finite dimension. In the abstract setting, $\mathbb{M}\colon X \to X^*$ is a symmetric positive-definite linear operator representing masses, and $\mathbb{V}\colon X \to X^*$ a positive-semidefinite (hence, possibly $\mathbb{V} = 0$) linear operator including the possible presence of viscosity in the model. The function $\mathcal{E}\colon [0,+\infty) \times X \to [0,+\infty)$ instead represents a driving potential energy, whereas $\mathcal{R}\colon [0,+\infty) \times X \to [0,+\infty]$ is a time-dependent dissipation potential, modelling for instance dry friction. While the dependence on time of the function $\mathcal{E}$ is customary in these kind of problems, in order to include external forces in the system, we point out that non-autonomous dissipation potentials $\mathcal{R} = \mathcal{R}(t,v)$ are very rare in the mathematical literature. The general assumptions on $\mathcal{E}$ and $\mathcal{R}$ will be discussed later on in the Introduction, see also Chapter 1.

In the specific example of the locomotion model of Figure 1, the ambient space X is $\mathbb{R}^N$ and the components $x_i^\varepsilon \in \mathbb{R}$ of the solution represent the position of the i-th block. The matrix $\mathbb{M} := \mathrm{Diag}\{m_1,\dots,m_N\}$ denotes the mass distribution, while the matrix $\mathbb{V}$ could describe for instance viscous resistances to length changes in the links, or the linear component of a Bingham type friction on the blocks, caused by lubrication with a non-Newtonian fluid, see [29]. The functional $\mathcal{E}$ represents the elastic energy of the system. We emphasize that, since we are dealing with a locomotion problem, rigid translations must be included in the space of admissible configuration. This implies that the elastic energy $\mathcal{E}$ takes the form

$$\mathcal{E}(t,x) = \mathcal{E}_{sh}(t,\pi_Z(x)),$$

where the restricted functional $\mathcal{E}_{sh}$ is defined on a smaller subspace $Z \subseteq X$. The linear operator $\pi_Z\colon X \to Z$ assigns to each configuration $x \in X$ the corresponding shape of the locomotor; in the example of Figure 1 with $N = 4$, a natural choice could be $Z = \mathbb{R}^3$ and $\pi_Z(x) = (x_2 - x_1, x_3 - x_2, x_4 - x_3)$. Furthermore, to model the soft actuated links joining the masses, it is common to consider a quadratic restricted energy, namely

$$\mathcal{E}_{sh}(t,z) = \sum_{i=1}^{N-1} \frac{k_i}{2}(z_i - \ell_i(t))^2,$$

where ℓ_i is the actuator of the i-th link, usually a Lipschitz function, while $k_i > 0$ is its elastic constant. We however remark that the abstract formulation also includes other kinds of standard model, not related to locomotion, where $Z = X$ and π_Z can be considered as the identity over X.

Finally, the functional $\mathcal{R}$ will have the form

$$\mathcal{R}(t,v) = \chi_K(v) + \mathcal{R}_{\mathrm{finite}}(t,v), \tag{0.0.5}$$

where χ_K is the characteristic function of a closed convex cone K and $\mathcal{R}_{\mathrm{finite}}$ has finite values. In the considered example of the crawler, one could take

$$K = \bigcap_{i=1}^{N} \{x \in \mathbb{R}^N \mid \nu_i\, x_i \geq 0\}, \qquad \text{for suitable } \nu_i \in \{-1,0,1\},$$

and

$$\mathcal{R}_{\mathrm{finite}}(t,v) = \sum_{i=1}^{N} \alpha_i(t)|v_i|,$$

with $\alpha_i\colon [0,+\infty) \to (0,+\infty)$ Lipschitz and bounded.

The dissipation potential $\mathcal{R}_{\mathrm{finite}}$ accounts for dry friction forces which, as explained before, may change in time. The term χ_K instead represents a constraint on velocities and may be used to describe situations in which hooks or hard scales [58] are used to create an extreme anisotropy in the interaction with the surface, so that motion “against the hair” may be considered impossible.

Chapter 1. Setting of the problem and applications

The first chapter of the thesis, included in the submitted work [38] in collaboration with P. Gidoni, contains a detailed presentation on the (rescaled) dynamic problem (0.0.4), and on its quasistatic counterpart:

$$\begin{cases}\partial_v\mathcal{R}(t,\dot{x}(t))+D_x\mathcal{E}(t,x(t))\ni 0, \quad t>0,\\ x(0)=x_0.\end{cases} \tag{0.0.6}$$

Differently from (0.0.3), we notice that the time derivative $\dot{x}(t)$ survives in the quasistatic formulation since, as explained before, the subdifferential $\partial_v\mathcal{R}(t,\cdot)$ is positively homogeneous of degree zero: this in particular ensures rate-independence.

First of all we introduce all the main assumptions we require on $\mathcal{E}$ and $\mathcal{R}$. We list them also here for the sake of completeness. We recall that the elastic energy has the form $\mathcal{E}(t,x)=\mathcal{E}_{sh}(t,\pi_Z(x))$, where Z is a linear subspace of X and $\pi_Z\colon X\to Z$ is a linear and surjective operator; we thus suppose that $\mathcal{E}_{sh}\colon[0,+\infty)\times Z\to[0,+\infty)$ satisfies:

(E1) $\mathcal{E}_{sh}(\cdot,z)$ is absolutely continuous in $[0,T]$ for every $z\in Z$ and for every $T>0$;

(E2) $\mathcal{E}_{sh}(t,\cdot)$ is μ–uniformly convex for some $\mu>0$ for every $t\in[0,+\infty)$, namely for every $\theta\in[0,1]$ and $z_1,z_2\in Z$:

$$\mathcal{E}_{sh}(t,\theta z_1+(1-\theta)z_2)\le\theta\mathcal{E}_{sh}(t,z_1)+(1-\theta)\mathcal{E}_{sh}(t,z_2)-\frac{\mu}{2}\theta(1-\theta)|z_1-z_2|^2_Z;$$

(E3) $\mathcal{E}_{sh}(t,\cdot)$ is differentiable for every $t\in[0,+\infty)$ and the differential $D_z\mathcal{E}_{sh}$ is continuous in $[0,+\infty)\times Z$;

(E4) for a.e. $t\in[0,+\infty)$ and for every $z\in Z$ it holds

$$\left|\frac{\partial}{\partial t}\mathcal{E}_{sh}(t,z)\right|\le\omega(\mathcal{E}_{sh}(t,z))\gamma(t),$$

where $\omega\colon[0,+\infty)\to[0,+\infty)$ is nondecreasing and continuous, while the nonnegative function γ is in $L^1(0,T)$ for every $T>0$;

(E5) for every $R>0$ and $T>0$ there exists a nonnegative function $\eta_R\in L^1(0,T)$ such that for a.e. $t\in[0,T]$ and for every $z_1,z_2\in\overline{\mathcal{B}^Z_R}$ it holds

$$\left|\frac{\partial}{\partial t}\mathcal{E}_{sh}(t,z_2)-\frac{\partial}{\partial t}\mathcal{E}_{sh}(t,z_1)\right|\le\eta_R(t)|z_2-z_1|_Z.$$

We observe that condition (E2) yields convexity of the whole functional $\mathcal{E}(t,\cdot)$; we however point out that this property will not be necessary when dealing with the dynamic problem (0.0.4) and for the first part of the subsequent vanishing inertia analysis performed in Chapter 2, where also non convex energies are allowed.

As regards $\mathcal{R}$ we require that it can be written as in (0.0.5), assuming $K\subseteq X$ is a nonempty closed convex cone, and $\mathcal{R}_{\text{finite}}\colon[0,+\infty)\times X\to[0,+\infty)$ fulfils:

(R1) for every $t\in[0,+\infty)$, the function $\mathcal{R}_{\text{finite}}(t,\cdot)$ is convex, positively homogeneous of degree one, and satisfies $\mathcal{R}_{\text{finite}}(t,0)=0$;

(R2) for every $T > 0$ there exist two positive constants $\alpha^* \geq \alpha_* > 0$, possibly depending on T, for which

$$\alpha_* |v| \leq \mathcal{R}_{\text{finite}}(t, v) \leq \alpha^* |v|, \quad \text{for every } (t, v) \in [0, T] \times X;$$

(R3) for every $T > 0$ there exists a non-negative function $\rho = \rho_T \in L^1(0, T)$ for which

$$|\mathcal{R}_{\text{finite}}(t, v) - \mathcal{R}_{\text{finite}}(s, v)| \leq |v| \int_s^t \rho(\tau) \, \mathrm{d}\tau, \text{ for every } 0 \leq s \leq t \leq T \text{ and } v \in X.$$

We notice that the first inequality in (R2) ensures coercivity of the dissipation potential. We point out that such a strong request, crucial for our analysis, is absolutely natural in the finite dimensional setting we are considering, as we will see in the examples of Section 1.2. On the contrary, it becomes very restrictive in infinite dimension: indeed, in standard models of elasticity where the simplest ambient space is $H^1_0(\Omega)$, a common choice of dissipation potential is $\int_\Omega |v(x)| \, \mathrm{d}x$, which of course is not coercive.

We then introduce different notions of solution for the dynamic and the quasistatic problem. In both cases, we say that a function is a *differential solution* if initial data are attained and the differential inclusion holds true in the dual space X^* for almost every time. A weaker concept is instead given only for the quasistatic formulation (0.0.6); to proper present it, we need to consider a suitable generalisation of functions of bounded variation from $[0, +\infty)$ to X, already used in [39], in which the norm of the target space is replaced by the time-dependent functional $\mathcal{R}$. For this reason we call them functions of bounded $\mathcal{R}$–variation. In Section 1.3 we discuss in detail the properties possessed by this generalised version of BV functions, comparing them with the standard case.

Thus, a function x of bounded $\mathcal{R}$–variation is said to be an *energetic solution* for (0.0.6) if the following global stability condition and weak energy balance hold true:

(GS) $\mathcal{E}(t, x(t)) \leq \mathcal{E}(t, v) + \mathcal{R}(t, v - x(t))$, for every $v \in X$ and for every $t \in [0, +\infty)$;

(WEB) $\mathcal{E}(t, x(t)) + V_{\mathcal{R}}(x; 0, t) = \mathcal{E}(0, x_0) + \int_0^t \frac{\partial}{\partial t} \mathcal{E}(\tau, x(\tau)) \, \mathrm{d}\tau$, for every $t \in [0, +\infty)$,

where $V_{\mathcal{R}}(x; 0, t)$ denotes the $\mathcal{R}$–variation of x in the time interval $[0, t]$. The motivations behind such a definition are well known, and a deep overview can be found for instance in [65]. Briefly speaking, if the potential energy is convex, any differential solution of (0.0.6) satisfies (GS) and (WEB), replacing the $\mathcal{R}$–variation $V_{\mathcal{R}}(x; 0, t)$ with the more common term $\int_0^t \mathcal{R}(\tau, \dot{x}(\tau)) \, \mathrm{d}\tau$.

Once that the setting of the problem and the main assumptions have been clarified, we then propose several applications that can be covered by systems of this form. Besides the model of discrete soft crawlers already shortly discussed here in the Introduction, we present the formulation of scalar and vectorial play operator and an example of rheological model.

Chapter 2. Vanishing inertia analysis

In Chapter 2 we discuss existence, uniqueness, and the main properties of the solutions of both the dynamical system (0.0.4) and the quasistatic one (0.0.6). We finally perform the asymptotic analysis as $\varepsilon \to 0^+$ for a dynamic solution x^ε of (0.0.4). Also the results of this chapter are contained in the work [38] in collaboration with Paolo Gidoni.

We first deal with the dynamical problem; we prove existence of differential solutions under assumptions (E1), (E3)–(E5) and (R1)–(R3). As previously remarked, convexity here is not needed. The strategy is based on an equivalent formulation of (0.0.4): by setting $\eta^\varepsilon := \varepsilon^2\mathbb{M}x^\varepsilon$ we can transform the system in the following second order perturbed sweeping process:

$$\begin{cases}\ddot\eta^\varepsilon(t) \in -\mathcal{N}_{\mathbb{M}K}(\dot\eta^\varepsilon(t)) - F(t,\eta^\varepsilon(t),\dot\eta^\varepsilon(t)),\\ \eta^\varepsilon(0) = \varepsilon^2\mathbb{M}x_0^\varepsilon, \quad \dot\eta^\varepsilon(0) = \varepsilon^2\mathbb{M}x_1^\varepsilon,\end{cases} \tag{0.0.7}$$

where $\mathcal{N}_{\mathbb{M}K}$ is the normal cone to the convex set $\mathbb{M}K$, and the multivalued map F has the form

$$F(t,u,v) = \varepsilon\mathbb{V}\,\mathbb{Q}^\varepsilon v + \partial_v\mathcal{R}_{\text{finite}}(t,\mathbb{Q}^\varepsilon v) + D_x\mathcal{E}(t,\mathbb{Q}^\varepsilon u),$$

where $\mathbb{Q}^\varepsilon$ is the inverse operator of $\varepsilon^2\mathbb{M}$. We then rely on a more general result of [2], which in particular ensures existence of solutions to (0.0.7). Uniqueness is also obtained under a slightly stronger assumption of $\mathcal{E}$, namely requiring its gradient to be, roughly speaking, Lipschitz in space uniformly with respect to time. The precise result is stated in Theorem 2.1.8, see also Proposition 2.1.7.

We also remark that previous argument exploits another equivalent formulation of (0.0.4), which will be useful for the vanishing inertia analysis, described by the following pair of conditions:

(LS^ε) for a.e. time $t \in [0,+\infty)$ and for every $v \in X$

$$\mathcal{R}(t,v) + \langle D_x\mathcal{E}(t,x^\varepsilon(t)) + \varepsilon^2\mathbb{M}\ddot x^\varepsilon(t) + \varepsilon\mathbb{V}\dot x^\varepsilon(t), v\rangle \geq 0;$$

(EB^ε) for every $t \in [0,+\infty)$

$$\begin{aligned}&\frac{\varepsilon^2}{2}|\dot x^\varepsilon(t)|^2_{\mathbb{M}} + \mathcal{E}(t,x^\varepsilon(t)) + \int_0^t \mathcal{R}(\tau,\dot x^\varepsilon(\tau))\,\mathrm{d}\tau + \varepsilon\int_0^t |\dot x^\varepsilon(\tau)|^2_{\mathbb{V}}\,\mathrm{d}\tau\\ =\;& \frac{\varepsilon^2}{2}|x_1^\varepsilon|^2_{\mathbb{M}} + \mathcal{E}(0,x_0^\varepsilon) + \int_0^t \frac{\partial}{\partial t}\mathcal{E}(\tau,x^\varepsilon(\tau))\,\mathrm{d}\tau.\end{aligned}$$

Here (LS^ε) stands for local stability, while (EB^ε) for energy(-dissipation) balance. In particular in the energy balance we recognise, in order of appearance, kinetic energy, potential energy, energy dissipated by friction and by viscosity; the last term in the second line represents instead the work done by external forces.

Next, we turn our attention to the quasistatic problem, in particular on the notion of energetic solution. The main outcome is a temporal regularity result, under the assumptions (E1)–(E5) and (R1)–(R3). We indeed prove that actually energetic solutions are absolutely continuous, thus recovering their equivalence with differential solutions. For this fact, convexity– i.e. assumption (E2)– is crucial, and the argument adapts the ideas of [67] to our framework in which uniform convexity is available only on the subspace Z and the dissipation potential $\mathcal{R}$ depends on time. The first step employs uniform convexity to deduce from (GS) the following improved version of stability:

$$\mathcal{E}(t,x(t)) + \frac{\mu}{2}|\pi_Z(x(t)) - \pi_Z(v)|^2_Z \leq \mathcal{E}(t,v) + \mathcal{R}_{sh}(t,\pi_Z(v) - \pi_Z(x(t))), \quad \text{for every } v \in X,$$

where $\mathcal{R}_{sh}$ is a suitable restricted version of $\mathcal{R}$, defined on Z; see (2.2.1). By considering as competitor in the above estimate the solution at a different time $x(s)$, and then exploiting the weak energy balance (WEB) together with suitable properties of the $\mathcal{R}$–variation, we are able to get absolute continuity of x. The rigorous and detailed result is stated in Proposition 2.2.8. We then present some known cases in which uniquess is granted, see Lemmas 2.2.9 and 2.2.10.

Finally, we focus on the limit behaviour of a dynamic solution x^ε to (0.0.4) as $\varepsilon \to 0^+$. By combining the energy balance (EB^ε) together with Grönwall Lemma, for every $T > 0$ we first deduce the following uniform bound:

$$\frac{\varepsilon^2}{2}|\dot{x}^\varepsilon(t)|^2_{\mathbb{M}} + \mathcal{E}(t, x^\varepsilon(t)) + \int_0^t \mathcal{R}(\tau, \dot{x}^\varepsilon(\tau))\, \mathrm{d}\tau + \varepsilon \int_0^t |\dot{x}^\varepsilon(\tau)|^2_{\mathbb{V}}\, \mathrm{d}\tau \le C_T, \qquad \text{for every } t \in [0, T],$$

where C_T is a constant independent of ε and possibly dependent on T. As a simple byproduct, exploiting the coercivity given by condition (R2) and by means of Helly's Selection Theorem, we thus obtain the existence of a convergent subsequence, here not relabelled, and of a limit function x such that:

- $\lim\limits_{\varepsilon \to 0^+} x^\varepsilon(t) = x(t)$, for every $t \in [0, +\infty)$;
- $\lim\limits_{\varepsilon \to 0^+} \varepsilon|\dot{x}^\varepsilon(t)|_{\mathbb{M}} = 0$, for every $t \in (0, +\infty) \setminus J_x$, where J_x is the jump set of x.

Notice that the second limit is saying that kinetic energy vanishes outside the (possible) discontinuity points of x. We then characterise the limit function x as an energetic solution to the quasistatic problem (0.0.6) by means of an asymptotic analysis of the stability condition (LS^ε) and the energy balance (EB^ε). Passing to the limit in (LS^ε), we first manage to deduce that the right and left limit of x are locally stable, meaning that:

(LS^+) $\mathcal{R}(t, v) + \langle D_x\mathcal{E}(t, x^+(t)), v\rangle \ge 0$, for every $v \in X$ and for every $t \in [0, +\infty)$;

(LS^-) $\mathcal{R}(t, v) + \langle D_x\mathcal{E}(t, x^-(t)), v\rangle \ge 0$, for every $v \in X$ and for every $t \in (0, +\infty)$.

Letting $\varepsilon \to 0^+$ in the energy balance, and by using lower semicontinuity of the $\mathcal{R}$–variation, we instead get the following inequality:

$$\mathcal{E}(t, x^+(t)) + V_{\mathcal{R}}(x; s-, t+) \le \mathcal{E}(s, x^-(s)) + \int_s^t \frac{\partial}{\partial t}\mathcal{E}(\tau, x(\tau))\, \mathrm{d}\tau, \qquad \text{for every } 0 < s \le t. \tag{0.0.8}$$

We want to highlight that, up to this point, convexity assumption (E2) on the potential energy $\mathcal{E}$ is not needed. But from now on we require it to complete the argument; indeed convexity improves the local stability conditions $(LS^\pm)$ to their global counterparts:

(GS^+) $\mathcal{E}(t, x^+(t)) \le \mathcal{E}(t, v) + \mathcal{R}(t, v - x^+(t))$, for every $v \in X$ and for every $t \in [0, +\infty)$;

(GS^-) $\mathcal{E}(t, x^-(t)) \le \mathcal{E}(t, v) + \mathcal{R}(t, v - x^-(t))$, for every $v \in X$ and for every $t \in (0, +\infty)$.

By standard techniques, the above conditions allow us to show also the other inequality in (0.0.8), obtaining as a consequence the fact that the right limit x^+ is an energetic solution of (0.0.6). Thanks to the regularity properties previously proved (based on convexity), we finally deduce that x coincides with x^+ and is actually absolutely continuous; hence x itself is an energetic (actually a differential) solution to the quasistatic problem. The rigorous result is contained in Theorem 2.3.9, see also Theorem 2.3.8.

Part II
A mechanical model of debonding with viscous damping

The second part of the thesis deals with the analysis of the quasistatic limit for a particular dynamic debonding model involving one spatial dimension (in some context also called peeling test) when also viscosity is taken into account. The interest of the physical and engineering community on this kind of models originates in the '70s from the works of Hellan [40, 41, 42], Burridge & Keller [17] and carries on in the '90s with the ones of Freund and Slepyan collected in [35] and [84], respectively. Their importance relies on the fact that they possess deep similarities to the theory of dynamic crack growth based on Griffith's criterion, but at the same time they are much easier to treat, allowing an exhaustive comprehension of the involved physical processes. Among the features and difficulties shared with general Fracture Dynamics we can list the time dependence of the domain of the wave equation and the presence of an energy criterion governing the evolution of the system; the obvious simplification of these debonding models relies instead in the monodimensionality of the domain. This one-dimensional spatial structure indeed allows to study and inspect more easily the growth of the debonding front, in contrast to the very hard task of detecting the path and the evolution of a crack in a multidimensional body. For the sake of completeness, among the multitude of recent works about dynamic crack propagation we quote for instance [22, 23, 25, 49, 78].

In the last few years the model of a tape peeled away from a substrate has been studied from different points of view by several authors, see for instance [24, 31, 30, 50, 51, 52, 53]. In particular, a complete mathematical analysis has been given in [24, 52], where the authors firstly prove well-posedness of the problem and then show how the quasistatic limit question has a negative answer if no dampings are taken into account. Here we consider the same model, with the addition of a viscosity term thanks to which we will be able to validate the quasistatic approximation.

The model can be interpreted in two different ways. The first one, following [30, 50], describes a dynamic peeling test for a one-dimensional tape, which is assumed to be perfectly flexible and inextensible, initially attached to a flat rigid substrate and placed in some environment which causes a viscous damping on its surface. We assume the deformation of the tape takes place in a vertical plane with orthogonal coordinates (x, y), where the positive x-axis represents the substrate as well as the reference configuration of the tape. For the sake of simplicity we neglect incompenetration between the tape and the substrate. During the evolution the tape is described by $x \mapsto (x + h(t, x), u(t, x))$, namely the pair $(h(t, x), u(t, x))$ is the displacement at time $t \geq 0$ of the point $(x, 0)$, and it is glued to the substrate on the half line $\{x \geq \ell(t), y = 0\}$, where ℓ is a nondecreasing function satisfying $\ell_0 := \ell(0) > 0$ which represents the debonding front. This implies that both the horizontal displacement $h(t, x)$ and the vertical one $u(t, x)$ are equal to 0 for $x \geq \ell(t)$. As in [24] we make the crucial assumption that $\ell_0 > 0$, namely at the initial time $t = 0$ the tape is already debonded in the segment $\{(x, 0) \mid x \in [0, \ell_0)\}$; see instead [53] for the analysis of the singular case in which initially the tape is completely glued to the substrate. At the endpoint $x = 0$ we prescribe a boundary condition $u(t, 0) = w(t)$, where w is the time-dependent vertical loading. See Figure 2.

Linear approximation and inextensibility of the tape lead to the following formula for the horizontal displacement h:

$$h(t, x) = \frac{1}{2} \int_x^{+\infty} u_x(t, \sigma)^2 \, \mathrm{d}\sigma,$$

and thus the only unknowns of the problem are the vertical displacement u and the debonding front ℓ. Introducing a parameter $\nu \geq 0$ which tunes viscosity, it turns out that the vertical displacement u solves the problem

$$\begin{cases} u_{tt}(t, x) - u_{xx}(t, x) + \nu u_t(t, x) = 0, & t > 0\,,\ 0 < x < \ell(t), \\ u(t, 0) = w(t), & t > 0, \\ u(t, \ell(t)) = 0, & t > 0, \\ u(0, x) = u_0(x), & 0 < x < \ell_0, \\ u_t(0, x) = u_1(x), & 0 < x < \ell_0, \end{cases} \tag{0.0.9}$$

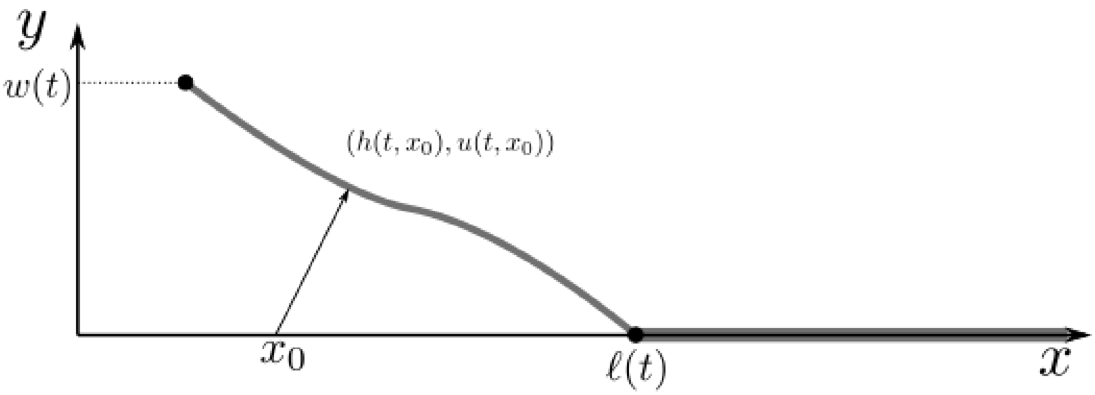

Figure 2: The deformation of the film at time t is represented by the displacement $(x_0, 0) \mapsto (x_0 + h(t, x_0), u(t, x_0))$. The function $w(t)$ is the vertical loading, while $\ell(t)$ is the debonding front.

where the initial conditions u_0 and u_1 are given functions.

The second and, in our opinion, much proper and simpler interpretation of the model is the one of a bar, initially glued to a flat rigid support, loaded horizontally and thus exhibiting only horizontal displacement. In this setting the function $u(t, x)$ represents the horizontal displacement of the bar, while $w(t)$ is the horizontal loading acting in $x = 0$; as before, the nondecreasing function $\ell(t)$ denotes the debonding front, and system (0.0.9) governs the evolution of u. To comply with both the interpretations we will always omit the adjective vertical or horizontal when we refer to the displacement u.

To establish the rules governing the evolution of the debonding front ℓ we need first to introduce for $t \in [0, +\infty)$ the internal energy of the displacement u, namely the sum of kinetic energy:

$$\mathcal{K}(t) := \frac{1}{2} \int_0^{\ell(t)} u_t(t, \sigma)^2 \, \mathrm{d}\sigma,$$

and potential energy:

$$\mathcal{E}(t) := \frac{1}{2} \int_0^{\ell(t)} u_x(t, \sigma)^2 \, \mathrm{d}\sigma.$$

We then consider the energy dissipated by viscosity:

$$\mathcal{V}(t) := \nu \int_0^t \int_0^{\ell(\tau)} u_t(\tau, \sigma)^2 \, \mathrm{d}\sigma \, \mathrm{d}\tau,$$

and the work done by the external loading:

$$\mathcal{W}(t) := -\int_0^t \dot{w}(\tau) u_x(\tau, 0) \, \mathrm{d}\tau.$$

Remark 0.0.1. Differently from Part I we now prefer considering the potential energy $\mathcal{E}$ to depend only on time. This is mainly due to two reasons: first of all the term $-u_{xx}$ in the wave equation (of course meant in a weak sense) is not exactly the (Frechèt) differential of the energy due to the non homogeneous boundary conditions. Furthermore, the fact that the external loading w is time-dependent and that the debonding front ℓ grows during the evolution makes also the space where the displacement u lives to change in time.

In this framework, the work of the external loading is thus no more represented by the integral of $\frac{\partial}{\partial t}\mathcal{E}$ along the evolution, but it takes the form introduced just above.

Moreover, we assume that the glue between the substrate and the tape (or the bar) behaves in a brittle fashion, thus the energy dissipated during the debonding process in the time interval $[0,t]$ is given by the formula

$$\int_{\ell_0}^{\ell(t)} \kappa(\sigma)\,\mathrm{d}\sigma,$$

where $\kappa\colon [\ell_0,+\infty) \to (0,+\infty)$ is a measurable function representing the local toughness of the glue.

In the context of Griffith's theory, we postulate that the debonding front ℓ has to evolve following two principles: the first one, called energy(-dissipation) balance, simply states that during the evolution the following equality between internal energy, dissipated energy and work of the external loading has to be satisfied:

$$\mathcal{K}(t)+\mathcal{E}(t)+\mathcal{V}(t)+\int_{\ell_0}^{\ell(t)} \kappa(\sigma)\,\mathrm{d}x = \mathcal{K}(0)+\mathcal{E}(0)+\mathcal{W}(t), \qquad \text{for every } t\in[0,+\infty). \tag{0.0.10}$$

The second one, called maximum dissipation principle, states that ℓ has to grow at the maximum speed which is consistent with the energy(-dissipation) balance (see also [48]):

$$\dot\ell(t) = \max\{\alpha\in[0,1) \mid \kappa(\ell(t))\alpha = G_\alpha(t)\alpha\}, \qquad \text{for a.e. } t\in[0,+\infty), \tag{0.0.11}$$

where $G_\alpha(t)$ is the so-called dynamic energy release rate at speed α, a quantity which measures the amount of energy spent by the debonding process. It is obtained as the opposite of a sort of partial derivative of the energy with respect to the elongation of the debonding front; we refer to Subsection 3.4.2 for the rigorous definition and all the details.

We only want to anticipate that the two principles (0.0.10) and (0.0.11) together are equivalent to the following system, called dynamic Griffith's criterion:

$$\begin{cases} 0\le \dot\ell(t) < 1, \\ G_{\dot\ell(t)}(t) \le \kappa(\ell(t)), \\ \left[G_{\dot\ell(t)}(t) - \kappa(\ell(t))\right]\dot\ell(t) = 0, \end{cases} \qquad \text{for a.e. } t\in[0,+\infty), \tag{0.0.12}$$

The first row is an irreversibility condition, which ensures that the debonding front can only increase; the second one is a stability condition, and says that the dynamic energy release rate cannot exceed the threshold given by the toughness; the third one is simply the energy(-dissipation) balance (0.0.10).

The addition of the damping term to the wave equation, harmless at a first sight, makes instead the coupled problem (0.0.9)&(0.0.12) much more difficult to treat than the undamped case $\nu=0$ previously analysed in [24, 52]. Indeed, the arguments they adopted do not work anymore because of a real coupling between the two unknowns u and ℓ which appears if ν is positive. The role of our contribution is thus to develop an original approach which allows us to overcome the technical difficulties related to the damping term and to get and improve the results obtained in [24, 52].

Chapter 3. Existence and uniqueness

Chapter 3, whose contents are contained in [76] in collaboration with L. Nardini, is devoted to the proof of existence and uniqueness of dynamic evolutions for the debonding model we previously described, namely of a pair (u,ℓ) satisfying the coupled problem (0.0.9)&(0.0.12). The strategy relies in the introduction of the auxiliary function

$v(t,x) := e^{\nu t/2}u(t,x)$ which solves the equivalent problem

$$\begin{cases} v_{tt}(t,x) - v_{xx}(t,x) - \dfrac{\nu^2}{4}v(t,x) = 0, & t > 0\,,\, 0 < x < \ell(t), \\ v(t,0) = z(t), & t > 0, \\ v(t,\ell(t)) = 0, & t > 0, \\ v(0,x) = v_0(x), & 0 < x < \ell_0, \\ v_t(0,x) = v_1(x), & 0 < x < \ell_0, \end{cases} \tag{0.0.13}$$

where z, v_0 and v_1 are suitable transformations of the data w, u_0 and u_1. The main advantage of (0.0.13) with respect to (0.0.9) is that we get rid of the first derivative (in time) in the equation.

We first assume that the debonding front ℓ is prescribed and we prove the validity of the classical Duhamel's principle in our context in which the domain of the wave equation increases in time. Namely we show that any solution to (0.0.13) satisfies the representation formula

$$v(t,x) = A(t,x) + \frac{\nu^2}{8} \iint_{R(t,x)} v(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau, \tag{0.0.14}$$

where A is the d'Alembert's solution of the undamped wave equation and R is a suitable space-time domain which encodes the reflection of the travelling waves in the two extrema of the tape (or the bar). Thanks to (0.0.14) we are able to exploit a contraction argument to deduce existence and uniqueness of solutions to (0.0.13) (and hence to (0.0.9)), see Theorem 3.2.12.

We then introduce in a rigorous way the dynamic energy release rate $G_\alpha(t)$ and dynamic Griffith's criterion (0.0.12) and we prove its equivalence with the following ordinary differential equation for the debonding front ℓ:

$$\dot\ell(t) = \max\left\{\frac{G_0(t) - \kappa(\ell(t))}{G_0(t) + \kappa(\ell(t))}, 0\right\}, \qquad \text{for a.e. } t \in [0,+\infty). \tag{0.0.15}$$

By exploiting (0.0.14) together with (0.0.15) we employ again the contraction fixed point theorem to finally get existence and uniqueness of solutions to the coupled problem (0.0.9) together with (0.0.12), see Theorem 3.5.6.

Chapter 4. Continuous dependence

In Chapter 4 we deal with the continuous dependence problem for the model under consideration. The related results have been published in [74]. Surprisingly we are not aware of the presence in literature of any kind of continuous dependence results for debonding models, despite the importance of the issue and despite partial achievements in this direction have already been obtained in the more complicated framework of Fracture Dynamics, see for instance [19, 25]. Therefore the significance of our contribution is filling this gap, giving a positive answer to the question of continuous dependence for the one-dimensional dynamic debonding model we are analysing. We indeed consider the following convergence assumptions on all the involved data:

$$\ell_0^n \to \ell_0 \quad \text{and} \quad \nu^n \to \nu; \tag{0.0.16a}$$

$$u_0^n \to u_0 \text{ in } H^1(0,+\infty),\ u_1^n \to u_1 \text{ in } L^2(0,+\infty) \text{ and } w^n \to w \text{ in } \widetilde H^1(0,+\infty); \tag{0.0.16b}$$

$$\kappa^n \to \kappa \text{ uniformly in } [0,X] \text{ for every } X > 0. \tag{0.0.16c}$$

and we show that the related solutions (u^n, ℓ^n) and (u, ℓ) fulfils for every $T > 0$:

$$
\begin{aligned}
&\bullet\ \dot\ell^n \to \dot\ell \text{ in } L^1(0,T), \text{ and thus } \ell^n \to \ell \text{ uniformly in } [0,T];\\
&\bullet\ u^n \to u \text{ uniformly in } [0,T] \times [0,+\infty);\\
&\bullet\ u^n \to u \text{ in } H^1((0,T) \times (0,+\infty));\\
&\bullet\ u^n \to u \text{ in } C^0([0,T]; H^1(0,+\infty)) \text{ and in } C^1([0,T]; L^2(0,+\infty));\\
&\bullet\ u^n_x(\cdot,0) \to u_x(\cdot,0) \text{ and}\\
&\quad \sqrt{1-\dot\ell^n(\cdot)^2}\,u^n_x(\cdot,\ell^n(\cdot)) \to \sqrt{1-\dot\ell(\cdot)^2}\,u_x(\cdot,\ell(\cdot)) \text{ in } L^2(0,T).
\end{aligned}
\tag{0.0.17}
$$

As we did in Chapter 3, our analysis is based on the auxiliary problem (0.0.13) and on Duhamel's principle (0.0.14) and the ordinary differential equation (0.0.15) for the debonding front. In particular, we exploit the same estimates we previously obtained while performing the contraction argument.

We first assume a priori that the sequence of debonding fronts ℓ^n converges to ℓ as in the first line of (0.0.17); in this case we show that (0.0.16a) and (0.0.16b) imply all the above convergences for the displacements u^n towards u. This is a continuous dependence result for problem (0.0.9) still not coupled with (0.0.12). We then show that, assuming in addition (0.0.16c) and exploiting equation (0.0.15), the convergence of the sequence of debonding fronts we postulated before actually happens; thus the continuous dependence result (0.0.17) for the coupled problem is proved. See Theorem 4.3.6.

Chapter 5. Vanishing inertia and viscosity analysis

Chapter 5, whose contents have been published in [75], finally treats the issue of the quasistatic limit for the damped debonding model, namely the asymptotic analysis of the behaviour of the system in the case of slow loading and slow initial velocity. We thus introduce a small parameter $\varepsilon > 0$ and after the time-reparametrisation $t \mapsto t/\varepsilon$ we are led to consider the rescaled dynamic problem:

$$
\begin{cases}
\varepsilon^2 u^\varepsilon_{tt}(t,x) - u^\varepsilon_{xx}(t,x) + \nu\varepsilon u^\varepsilon_t(t,x) = 0, & t > 0\,,\ 0 < x < \ell^\varepsilon(t),\\
u^\varepsilon(t,0) = w^\varepsilon(t), & t > 0,\\
u^\varepsilon(t,\ell^\varepsilon(t)) = 0, & t > 0,\\
u^\varepsilon(0,x) = u^\varepsilon_0(x), & 0 < x < \ell_0,\\
u^\varepsilon_t(0,x) = u^\varepsilon_1(x), & 0 < x < \ell_0,
\end{cases}
\tag{0.0.18}
$$

coupled with the rescaled dynamic Griffith's criterion:

$$
\begin{cases}
0 \le \dot\ell^\varepsilon(t) < 1/\varepsilon,\\
G^\varepsilon_{\varepsilon\dot\ell^\varepsilon(t)}(t) \le \kappa(\ell^\varepsilon(t)),\\
\left[G^\varepsilon_{\varepsilon\dot\ell^\varepsilon(t)}(t) - \kappa(\ell^\varepsilon(t))\right]\dot\ell^\varepsilon(t) = 0,
\end{cases}
\qquad \text{for a.e. } t \in [0,+\infty).
\tag{0.0.19}
$$

As we did in Part I, in (0.0.18) we also allow the external loading and the initial data to depend on the small parameter ε.

The final aim is thus to investigate the limit as ε goes to 0^+ of the pair $(u^\varepsilon, \ell^\varepsilon)$ solution of the rescaled coupled problem (0.0.18)&(0.0.19) and to understand if such a limit pair behaves like a quasistatic evolution. To develop the analysis several assumptions on the toughness κ will be crucial; for the sake of clarity we list them here:

(K0) the function κ is not integrable in $[\ell_0, +\infty)$;

(K1) the function $x \mapsto x^2\kappa(x)$ is nondecreasing on $[\ell_0, +\infty)$;

(K2) the function $x \mapsto x^2\kappa(x)$ is strictly increasing on $[\ell_0, +\infty)$;

(K3) the function $x \mapsto x^2\kappa(x)$ is strictly increasing on $[\ell_0, +\infty)$ and its derivative is strictly positive almost everywhere.

Condition (K0) states that an infinite amount of energy is needed to debond all the tape (or the bar), while conditions (K1), (K2), (K3) prevent the toughness from being too oscillating.

Of course, before studying the limit of the pair $(u^\varepsilon, \ell^\varepsilon)$ we need to introduce and analyse the concept of quasistatic evolution for the debonding model: we consider different definitions based on global and local minima of the energy. The first one is the classical *energetic evolution* à la Mielke and Roubíček (see [65]), already adopted in Part I, namely the pair (u, ℓ) has to comply a suitable global stability condition and energy(-dissipation) balance for every time $t \in [0 + \infty)$:

$$\text{(GS)}\quad \frac{1}{2}\int_0^{\ell(t)} u_x(t,\sigma)^2 \,\mathrm{d}\sigma + \int_{\ell_0}^{\ell(t)} \kappa(\sigma)\,\mathrm{d}\sigma \le \frac{1}{2}\int_0^{\hat\ell} \dot{\hat u}(\sigma)^2\,\mathrm{d}\sigma + \int_{\ell_0}^{\hat\ell}\kappa(\sigma)\,\mathrm{d}\sigma,$$

for every $\hat\ell \ge \ell(t)$ and for every $\hat u \in H^1(0, \hat\ell)$ satisfying $\hat u(0) = w(t)$ and $\hat u(\hat\ell) = 0$;

$$\text{(EB)}\quad \frac{1}{2}\int_0^{\ell(t)} u_x(t,\sigma)^2 \,\mathrm{d}\sigma + \int_{\ell_0}^{\ell(t)} \kappa(\sigma)\,\mathrm{d}\sigma = \frac{1}{2}\int_0^{\ell_0} u_x(0,\sigma)^2\,\mathrm{d}\sigma - \int_0^t \dot w(\tau)u_x(\tau,0)\,\mathrm{d}\tau.$$

In the energy balance (EB) we recognise the potential energy, the energy dissipated in the debonding process and, as last term, the work of the external loading. A second definition based on local minima of the energy and requiring some temporal regularity is the following one, which we call *absolutely continuous quasistatic evolution*:

(i) ℓ is absolutely continuous on $[0, T]$ for every $T > 0$ and $\ell(0) = \ell_0$;

(ii) $u(t,x) = w(t)\left(1 - \frac{x}{\ell(t)}\right)\mathbb{1}_{[0,\ell(t)]}(x)$, for every $(t, x) \in [0, +\infty) \times [0, +\infty)$;

(iii) the quasistatic version of Griffith's criterion holds true, namely:

$$\begin{cases} \dot\ell(t) \ge 0, \\ \frac{1}{2}\frac{w(t)^2}{\ell(t)^2} \le \kappa(\ell(t)), \\ \left[\frac{1}{2}\frac{w(t)^2}{\ell(t)^2} - \kappa(\ell(t))\right]\dot\ell(t) = 0, \end{cases} \qquad \text{for a.e. } t \in [0, +\infty).$$

Similarities with dynamic Griffith's criterion (0.0.12) are evident, with the exception of the term $\frac{1}{2}\frac{w(t)^2}{\ell(t)^2}$ which requires some explanations. Likewise the dynamic framework we can introduce the notion of quasistatic energy release rate as the opposite of the partial derivative of the internal energy with respect to the elongation of the debonding front. Namely $G_{\mathrm{qs}}(t) = -\frac{\partial}{\partial\ell}\mathcal{E}(t)$, since kinetic energy is negligible in a quasistatic setting. By means of (ii) we can compute $\mathcal{E}(t) = \frac{1}{2}\int_0^{\ell(t)} u_x(t,\sigma)^2\,\mathrm{d}\sigma = \frac{1}{2}\frac{w(t)^2}{\ell(t)}$, from which we recover $G_{\mathrm{qs}}(t) = \frac{1}{2}\frac{w(t)^2}{\ell(t)^2}$.

In Section 5.5 we compare the different notions of quasistatic evolutions proving their equivalence under the strongest assumption (K3). We then provide an existence and uniqueness result by writing down explicitely the solution:

- $u(t,x) = w(t)\left(1 - \frac{x}{\ell(t)}\right)\mathbb{1}_{[0,\ell(t)]}(x), \quad$ for every $(t, x) \in [0, +\infty) \times [0, +\infty)$,
- $\ell(t) = \phi_\kappa^{-1}\left(\max\left\{\frac{1}{2}(w^2)_*(t), \phi_\kappa(\ell_0)\right\}\right), \quad$ for every $t \in [0, +\infty)$.

In the above formulas we introduced the functions $\phi_\kappa(x) = x^2\kappa(x)$ (this first one appearing in the conditions (K1)–(K3)) and $(w^2)_*(t) = \sup\limits_{\tau\in[0,t]} w(\tau)^2$.

Once that the concept of quasistatic evolution in this setting has been clarified, we start analysing the asymptotic behaviour of the dynamic evolution $(u^\varepsilon, \ell^\varepsilon)$ as ε vanishes. The argument is similar to the one used in Part I, with the main difference and difficulty given of course by the infinite dimension of the problem.

The first step consists indeed in exploiting the energy balance to get uniform bounds and estimates for the displacement u^ε and the debonding front ℓ^ε in order to gain compactness in suitable spaces. Of course we need to consider the rescaled energies, namely:

$$\mathcal{K}^\varepsilon(t) = \frac{1}{2}\int_0^{\ell^\varepsilon(t)} \varepsilon^2 u_t^\varepsilon(t,\sigma)^2 \,\mathrm{d}\sigma;$$

$$\mathcal{E}^\varepsilon(t) = \frac{1}{2}\int_0^{\ell^\varepsilon(t)} u_x^\varepsilon(t,\sigma)^2 \,\mathrm{d}\sigma;$$

$$\mathcal{V}^\varepsilon(t) = \nu\int_0^t\int_0^{\ell^\varepsilon(\tau)} \varepsilon u_t^\varepsilon(\tau,\sigma)^2 \,\mathrm{d}\sigma\,\mathrm{d}\tau;$$

$$\mathcal{W}^\varepsilon(t) = -\int_0^t \dot{w}^\varepsilon(\tau) u_x^\varepsilon(\tau,0)\,\mathrm{d}\tau.$$

The first estimate we obtain is

$$\mathcal{K}^\varepsilon(t) + \mathcal{E}^\varepsilon(t) + \mathcal{V}^\varepsilon(t) + \int_{\ell_0}^{\ell^\varepsilon(t)} \kappa(\sigma)\,\mathrm{d}\sigma \le C_T, \qquad \text{for every } t\in[0,T], \tag{0.0.21}$$

where C_T is a positive constant depending on $T > 0$ and not on ε. Exploiting (0.0.21), (K0) and Helly's Selection Theorem, we deduce that, up to subsequences, the sequence of rescaled debonding front ℓ^ε pointwise converges to a nondecreasing function ℓ. We then take advantage of the presence of viscosity, managing to adapt the classical estimate used to show exponential stability of the weakly damped wave equation, see for instance [68], to our time-dependent domain setting. We thus obtain for every $t\in[0,T]$:

$$\mathcal{K}^\varepsilon(t)+\widetilde{\mathcal{E}}^\varepsilon(t) \le 4\left(\mathcal{K}^\varepsilon(0)+\widetilde{\mathcal{E}}^\varepsilon(0)\right)e^{-m\frac{t}{\varepsilon}} + C_T\int_0^t \left(\dot{\ell}^\varepsilon(\tau)+\dot{w}^\varepsilon(\tau)^2+u_x^\varepsilon(\tau,0)^2+1\right)e^{-m\frac{t-\tau}{\varepsilon}}\,\mathrm{d}\tau, \tag{0.0.22}$$

where $m\in(0,\nu)$ and the modified (shifted) potential energy is defined as

$$\widetilde{\mathcal{E}}^\varepsilon(t) := \frac{1}{2}\int_0^{\ell^\varepsilon(t)}\left(u_x^\varepsilon(t,\sigma)+\frac{w^\varepsilon(\tau)}{\ell^\varepsilon(\tau)}\right)^2\mathrm{d}\sigma, \qquad \text{for } t\in[0,+\infty).$$

Of course assumption $\nu > 0$ is crucial for the validity of (0.0.22), while for the toughness only the minimal assumption (K0) is needed.

Differently to the undamped case studied in [52], where only weak convergence is achieved from the bound (0.0.21), thanks to (0.0.22) we are able to infer the following strong convergence of the displacements u^ε:

- $\varepsilon u_t^\varepsilon(t,\cdot)\to 0$ strongly in $L^2(0,+\infty)$, for every $t\in(0,+\infty)\setminus J_\ell$,
- $u^\varepsilon(t,\cdot)\to u(t,\cdot)$ strongly in $H^1(0,+\infty)$, for every $t\in(0,+\infty)\setminus J_\ell$,

where u is the proper affine function $u(t,x) := w(t)\left(1-\frac{x}{\ell(t)}\right)\mathbb{1}_{[0,\ell(t)]}(x)$ and J_ℓ is the jump set of ℓ. In particular the first limit ensures that kinetic energy vanishes as $\varepsilon\to 0^+$. This feature is missing in the undamped setting [52], where in turn the persistence of kinetic energy is the main reason why the limit evolution is not rate-independent.

After the extraction of convergent subsequences we characterise the limit debonding front ℓ passing to the limit in the rescaled dynamic Griffith's criterion (0.0.19). By means of a generalisation of Duhamel's representation formula (0.0.14) valid for every time $t \in [0,+\infty)$, and exploiting the dynamic stability condition $G^{\varepsilon}_{\varepsilon\dot\ell^\varepsilon(t)}(t) \le \kappa(\ell^\varepsilon(t))$ we are able to prove that the limit function ℓ fulfils:

$$\frac{1}{2}\frac{w(t)^2}{\ell^+(t)^2} \le \kappa(\ell^+(t)), \quad \text{for every } t \in [0,+\infty), \tag{0.0.23a}$$

$$\frac{1}{2}\frac{w(t)^2}{\ell^-(t)^2} \le \kappa(\ell^-(t)), \quad \text{for every } t \in (0,+\infty), \tag{0.0.23b}$$

where ℓ^+, ℓ^- are the right and left limit of ℓ, respectively. Moreover, letting $\varepsilon \to 0^+$ in the rescaled energy(-dissipation) balance we show that the following weak version of condition (EB) holds true in the limit for every $0 < s \le t$:

$$\frac{1}{2}\frac{w(t)^2}{\ell^+(t)} + \int_{\ell_0}^{\ell^+(t)} \kappa(\sigma)\,\mathrm{d}\sigma + \mu_D([s,t]) = \frac{1}{2}\frac{w(s)^2}{\ell^-(s)} + \int_{\ell_0}^{\ell^-(s)} \kappa(\sigma)\,\mathrm{d}\sigma + \int_s^t \dot w(\tau)\frac{w(\tau)}{\ell(\tau)}\,\mathrm{d}\tau, \tag{0.0.24}$$

where μ_D is a so–called defect measure which takes into account the loss of energy in the limit due to viscosity. The above equality is exactly the counterpart of inequality (0.0.8) in this context, since the measure μ_D is nonnegative.

Somehow replacing the role of convexity, assumption (K2) allows to deduce from (0.0.23) and (0.0.24) that the limit debonding front ℓ is absolutely continuous and satisfies quasistatic Griffith's criterion (iii), but it might have a jump at the initial time time $t = 0$. Thus the limit pair (u, ℓ) is an absolutely continuous quasistatic evolution starting from $\ell^+(0)$.

We are finally able to characterise this first jump by proving that $\ell^+(0) = \lim\limits_{t\to+\infty} \tilde\ell(t)$, where $\tilde\ell$ is the debonding front related to the unscaled dynamical coupled problem:

$$\begin{cases} \tilde u_{tt}(t,x) - \tilde u_{xx}(t,x) + \nu\tilde u_t(t,x) = 0, & t > 0\,,\ 0 < x < \tilde\ell(t), \\ \tilde u(t,0) = w(0), & t > 0, \\ \tilde u(t,\tilde\ell(t)) = 0, & t > 0, \\ \tilde u(0,x) = u_0(x), & 0 < x < \ell_0, \\ \tilde u_t(0,x) = 0, & 0 < x < \ell_0, \end{cases}$$

$$\begin{cases} 0 \le \dot{\tilde\ell}(t) < 1, \\ G_{\dot{\tilde\ell}(t)}(t) \le \kappa(\tilde\ell(t)), \\ \left[G_{\dot{\tilde\ell}(t)}(t) - \kappa(\tilde\ell(t))\right]\dot{\tilde\ell}(t) = 0, \end{cases} \qquad \text{for a.e. } t \in [0,+\infty),$$

where the external loading is freezed at the initial value $w(0)$ and there is no initial velocity. The complete result is stated in Theorem 5.5.21.

Appendix

At the end of the thesis we attach two Appendixes. In Appendix A we gather all the technical proofs regarding the notions of $\mathcal{R}$–absolutely continuous functions and functions of bounded $\mathcal{R}$–variation we used throughout the first part of the thesis.

Appendix B instead contains some useful results, employed in the second part of the thesis, about the Chain rule and the Leibniz differentiation rule under low regularity assumptions, namely valid for absolutely continuous functions.

Notation

Basic notation.

$\alpha \wedge \beta$, $\min\{\alpha, \beta\}$,	minimum between α and β.
$\alpha \vee \beta$, $\max\{\alpha, \beta\}$,	maximum between α and β.
$\lvert \cdot \rvert$,	modulus, norm of finite dimensional normed spaces.
$\lVert \cdot \rVert_X$,	norm of the normed space X.
$\langle \cdot, \cdot \rangle_X$,	duality product between X and X^*.
$\lvert A \rvert$,	Lebesgue measure of the set $A \subseteq \mathbb{R}^n$.
$\overline{A}$,	closure of the set A.
$A \setminus B$,	set difference between the sets A and B.
$A \triangle B$,	symmetric difference between the sets A and B.
$\mathbb{1}_A$,	indicator function of the set A, namely $\mathbb{1}_A(x) = 1$ if $x \in A$ and $\mathbb{1}_A(x) = 0$ otherwise.
χ_A,	characteristic function of the set A, namely $\chi_A(x) = 0$ if $x \in A$ and $\chi_A(x) = +\infty$ otherwise.
$\mathcal{N}_{\mathcal{K}}^{\mathbb{A}}(x)$,	normal cone to the convex set $\mathcal{K}$ at the point x, with respect to the scalar product $\langle \mathbb{A}\cdot, \cdot \rangle$.
$\mathcal{B}_R^X$,	open ball of radius R with center 0 in the normed space X.

Derivatives.

$\dot{f}$,	first derivative of the function of only one variable f.
$\ddot{f}$,	second derivative of the function of only one variable f.
u_t, u_x, $\dfrac{\partial u}{\partial t}$, $\dfrac{\partial u}{\partial x}$,	time and space partial first derivatives of $u = u(t, x)$.
u_{tt}, u_{tx}, u_{xx},	time and space partial second derivatives of $u = u(t, x)$.
$D_x \mathcal{E}(t, x)$,	differential with respect to space of the function $\mathcal{E} = \mathcal{E}(t, x)$.
$\partial \phi$,	subdifferential of the convex function ϕ.
$\partial_v \mathcal{R}(t, v)$,	subdifferential with respect to v of the function $\mathcal{R} = \mathcal{R}(t, v)$.

Functional spaces.

Let X be a Banach space, Ω an open set in $\mathbb{R}^N$, and $b > a$ two real numbers.

$L^p(a, b; X)$,	Lebesgue space of (Bochner) p–integrable functions from (a, b) to X.
$W^{k,p}(a, b; X)$,	Sobolev space of functions from (a, b) to X with p–integrable kth derivatives.
$\widetilde{W}^{k,p}(a, +\infty; X)$,	functions from $(a, +\infty)$ to X belonging to $W^{k,p}(a, b; X)$ for every $b > a$.
$H^k(a, b; X)$,	Sobolev space $W^{k,2}(a, b; X)$.
$\widetilde{H}^k(a, +\infty; X)$,	functions from $(a, +\infty)$ to X belonging to $H^k(a, b; X)$ for every $b > a$.

$C^k([a,b];X)$,	functions from $[a,b]$ to X with continuous kth derivative.
$C^{k,1}([a,b];X)$,	functions from $[a,b]$ to X with Lipschitz kth derivative.
$\widetilde{C}^{k,1}([a,+\infty);X)$,	functions from $[a,+\infty)$ to X belonging to $C^{k,1}([a,b];X)$ for every $b > a$.
$AC([a,b];X)$,	absolutely continuous functions from $[a,b]$ to X.
$AC_{\mathcal{R}}([a,b];X)$,	$\mathcal{R}$–absolutely continuous functions from $[a,b]$ to X.
$\widetilde{AC}_{\mathcal{R}}([a,+\infty);X)$,	functions from $[a,+\infty)$ to X belonging to $AC_{\mathcal{R}}([a,b];X)$ for every $b > a$.
$BV_{\mathcal{R}}([a,b];X)$,	functions of bounded $\mathcal{R}$–variation from $[a,b]$ to X.
$\widetilde{BV}_{\mathcal{R}}([a,+\infty);X)$,	functions from $[a,+\infty)$ to X belonging to $BV_{\mathcal{R}}([a,b];X)$ for every $b > a$.

In the previous spaces, X is omitted if it is $\mathbb{R}$.

$L^p(\Omega)$,	Lebesgue space of p–integrable scalar functions on Ω.
$W^{k,p}(\Omega)$,	Sobolev space of scalar functions on Ω with p–integrable kth derivative.
$C^k(\overline{\Omega})$,	scalar functions on $\overline{\Omega}$ with continuous kth derivatives up to the boundary.
$H^1_0(a,b)$,	functions in $H^1(a,b)$ vanishing at the boundary.
$H^{-1}(a,b)$,	dual of $H^1_0(a,b)$.

Remark 0.0.2. Every function in $W^{k,p}(a,b;X)$ is always identified with its continuous representative on $[a,b]$.

Part I

Finite-dimensional systems with non-autonomous dry friction

Chapter 1

Setting of the problem and applications

Contents

In this chapter we present in details the (rescaled) abstract dynamic problem:

$$\begin{cases} \varepsilon^2\mathbb{M}\ddot{x}^\varepsilon(t) + \varepsilon\mathbb{V}\dot{x}^\varepsilon(t) + \partial_v\mathcal{R}(t, \dot{x}^\varepsilon(t)) + D_x\mathcal{E}(t, x^\varepsilon(t)) \ni 0, \quad t > 0, \\ x^\varepsilon(0) = x_0^\varepsilon, \quad \dot{x}^\varepsilon(0) = x_1^\varepsilon, \end{cases} \tag{1.0.1}$$

whose mechanical interpretation has been widely explained in the Introduction, and its quasistatic counterpart:

$$\begin{cases} \partial_v\mathcal{R}(t, \dot{x}(t)) + D_x\mathcal{E}(t, x(t)) \ni 0, \quad t > 0 \\ x(0) = x_0. \end{cases} \tag{1.0.2}$$

In particular, we discuss the hypotheses we require on the involved energies, as well as we propose several applications that can be covered by problems of this form.

The chapter is organised as follows: in Section 1.1 we list the properties of the elastic energy $\mathcal{E}$ and of the time-dependent dissipation potential $\mathcal{R}$ we need to develop the whole analysis of this first part of the thesis. We then introduce the notions of solution we want to investigate, see Definitions 1.1.7, 1.1.8 and 1.1.10.

Section 1.2 contains some example of the main mechanical models which can be described by systems (1.0.1) and (1.0.2), under the assumptions introduced in the previous section.

Finally, Section 1.3 is devoted to the presentation of a mathematical tool used to deal with the time-dependent dissipation potential $\mathcal{R}$. It consists in a generalisation of the concepts of absolutely continuous functions and functions of bounded variation, where the norm of the target space is replaced by the functional $\mathcal{R}$. We first introduce the two concepts, see Definitions 1.3.3 and 1.3.6; we then present their main properties, which will be used throughout the whole part, leaving all the technical proofs in Appendix A.

The contents of this chapter are contained in the submitted paper [38] in collaboration with P. Gidoni.

1.1 Main assumptions

Let X be a finite dimensional vector space endowed with the norm $|\cdot|$. The same symbol will be also adopted for the modulus in $\mathbb{R}$; however, its meaning will be always clear from the context. We denote by X^* the topological dual of X, and by $\langle x^*, x\rangle$ the duality product between $x^* \in X^*$ and $x \in X$. The operator norm in X^* will be denoted by $|\cdot|_*$. Given $R > 0$, by $\mathcal{B}^X_R$ we denote the open ball in X of radius R and centered at the origin, and with $\overline{\mathcal{B}^X_R}$ its closure.

Let us also recall some basic notions on set-valued maps; for more details, we refer to the monographs [10, 77]. Given two topological spaces A_1, A_2, we denote with $F\colon A_1 \rightrightarrows A_2$ a map from A_1 having as values subsets of A_2. We say that such a set-valued map is upper continuous in a point $a \in A_1$ if for every neighbourhood $U \subseteq A_2$ of $F(a)$ there exists a neighbourhood $V \subseteq A_1$ of a such that $F(\tilde{a}) \subset U$ for every $\tilde{a} \in V$. We say that a map is upper semicontinuous if it is so for every point of its domain. We recall that if a set-valued map has compact values, then it is upper semicontinuous if and only if its graph is closed.

Given a convex, lower semicontinuous map $\phi\colon X \to [0, +\infty]$, we define its subdifferential $\partial\phi(x_0) \subseteq X^*$ at each point $x_0 \in X$ as

$$\partial\phi(x_0) = \{\xi \in X^* \mid \phi(x_0) + \langle \xi, x - x_0\rangle \leq \phi(x) \quad \text{for every } x \in X\}.$$

Notice that $\partial\phi$ has closed convex values. Moreover, if $\phi(x_0) = +\infty$ and ϕ is finite in at least one point, then $\partial\phi(x_0) = \emptyset$. Given a subset $\mathcal{K} \subset X$, we denote with $\chi_{\mathcal{K}}\colon X \to [0, +\infty]$ its characteristic function:

$$\chi_{\mathcal{K}}(x) := \begin{cases} 0, & \text{if } x \in \mathcal{K}, \\ +\infty, & \text{if } x \notin \mathcal{K}. \end{cases}$$

Let us now present in detail our assumptions on the mechanical problems which will be the subject of our investigation.

1.1.1 Mass and viscosity

Let $\mathbb{M}\colon X \to X^*$ be a symmetric positive-definite linear operator, which will represent mass distribution. Since X has finite dimension, we observe that there exist two constants $M \geq m > 0$ such that

$$m|x|^2 \leq |x|^2_{\mathbb{M}} := \langle \mathbb{M}x, x\rangle \leq M|x|^2, \qquad \text{for every } x \in X. \tag{1.1.1}$$

We want to stress that the requirement on $\mathbb{M}$ of being positive definite, crucial for our analysis, fits well with the finite dimensional setting in which we are working; in particular, all the applications we have in mind, see Section 1.2, fulfil this assumption. On the contrary, in infinite dimensional models usually the mass operator is null on a subspace (see for instance [60]), thus $\mathbb{M}$ turns out to be only positive-semidefinite.

We consider also the (possible) presence of viscous dissipation, by introducing the positive-semidefinite linear operator $\mathbb{V}\colon X \to X^*$ (symmetry is not needed here). As before, we notice that there exists a non-negative constant $V \geq 0$ such that

$$0 \leq |x|^2_{\mathbb{V}} := \langle \mathbb{V}x, x\rangle \leq V|x|^2, \qquad \text{for every } x \in X. \tag{1.1.2}$$

We point out that we include also the case $\mathbb{V} \equiv 0$, corresponding to the absence of viscous friction forces in the dynamic problem (1.1.3). Indeed in this first part we are mostly interested in the presence of a stronger notion of dissipation, which will be introduced in the following, and which actually overwhelmes the effects of viscosity for the purposes of the vanishing inertia analysis.

1.1.2 Elastic energy

Before introducing our assumptions on the elastic energy $\mathcal{E}$, we recall that our main application concerns a locomotion problem. This implies that the space of admissible states X must include translations, for which the elastic energy is invariant. Hence the elastic energy will be coercive only on a subspace, intuitively corresponding to the shape of the locomotor.

Let us therefore consider a linear subspace $Z \subseteq X$, which is often convenient to endow with its own norm $|\cdot|_Z$, cf. the examples in [36]. We assume that the elastic energy $\mathcal{E}\colon [0,+\infty) \times X \to [0,+\infty)$ has the form $\mathcal{E}(t,x) = \mathcal{E}_{sh}(t, \pi_Z(x))$, where $\pi_Z\colon X \to Z$ is a linear and surjective operator and $\mathcal{E}_{sh}\colon [0,+\infty) \times Z \to [0,+\infty)$ satisfies:

(E1) $\mathcal{E}_{sh}(\cdot, z)$ is absolutely continuous in $[0,T]$ for every $z \in Z$ and for every $T > 0$;

(E2) $\mathcal{E}_{sh}(t,\cdot)$ is μ–uniformly convex for some $\mu > 0$ for every $t \in [0,+\infty)$, namely for every $\theta \in [0,1]$ and $z_1, z_2 \in Z$:

$$\mathcal{E}_{sh}(t, \theta z_1 + (1-\theta)z_2) \leq \theta\mathcal{E}_{sh}(t,z_1) + (1-\theta)\mathcal{E}_{sh}(t,z_2) - \frac{\mu}{2}\theta(1-\theta)|z_1 - z_2|^2_Z;$$

(E3) $\mathcal{E}_{sh}(t,\cdot)$ is differentiable for every $t \in [0,+\infty)$ and the differential $D_z\mathcal{E}_{sh}$ is continuous in $[0,+\infty) \times Z$;

(E4) for a.e. $t \in [0,+\infty)$ and for every $z \in Z$ it holds

$$\left|\frac{\partial}{\partial t}\mathcal{E}_{sh}(t,z)\right| \leq \omega(\mathcal{E}_{sh}(t,z))\gamma(t),$$

where $\omega\colon [0,+\infty) \to [0,+\infty)$ is nondecreasing and continuous, while the nonnegative function γ is in $L^1(0,T)$ for every $T > 0$;

(E5) for every $R > 0$ and $T > 0$ there exists a nonnegative function $\eta_R \in L^1(0,T)$ such that for a.e. $t \in [0,T]$ and for every $z_1, z_2 \in \overline{\mathcal{B}^Z_R}$ it holds

$$\left|\frac{\partial}{\partial t}\mathcal{E}_{sh}(t,z_2) - \frac{\partial}{\partial t}\mathcal{E}_{sh}(t,z_1)\right| \leq \eta_R(t)|z_2 - z_1|_Z.$$

Let us also introduce some additional assumptions on the energy $\mathcal{E}$, which are in general not required, but provide sharper results.

(E6) for every $\lambda > 0$ and $R > 0$ there exists $\delta = \delta(\lambda, R) > 0$ such that if $|t - s| \leq \delta$ and $z \in \mathcal{B}^Z_R$, then

$$\left|\frac{\partial}{\partial t}\mathcal{E}_{sh}(t,z) - \frac{\partial}{\partial t}\mathcal{E}_{sh}(s,z)\right| \leq \lambda;$$

(E7) for every $R > 0$ and $T > 0$ there exists a nonnegative function $\varsigma_R \in L^1(0,T)$ such that for a.e. $t \in [0,T]$ and for every $z_1, z_2 \in \overline{\mathcal{B}_R^Z}$ it holds

$$|D_z\mathcal{E}_{sh}(t,z_2) - D_z\mathcal{E}_{sh}(t,z_1)|_* \leq \varsigma_R(t)|z_2 - z_1|_Z.$$

We finally present the classical case of a quadratic energy:

(QE) $\mathcal{E}_{sh}(t,z) = \frac{1}{2}\langle \mathbb{A}_{sh}(z-\ell_{sh}(t)), z-\ell_{sh}(t)\rangle_Z$, where $\mathbb{A}_{sh}\colon Z \to Z^*$ is a symmetric, positive-definite linear operator and $\ell_{sh} \in \widetilde{AC}([0,+\infty); Z)$, i.e. it is in $AC([0,T]; Z)$ for every $T > 0$.

It can be easily verified that (QE) implies conditions (E1)–(E5) and (E7), whereas it satisfies (E6) if and only if ℓ_{sh} has continuous derivative. However, for our purposes, the additional structure of (QE) will alone provide a suitable alternative to (E6).

Remark 1.1.1. We point out that the more common case $Z \equiv X$ is also included in our formulation. In such a case all the assumptions above on $\mathcal{E}_{sh}$ are taken directly on $\mathcal{E}$.

Remark 1.1.2. Let us notice that, since π_Z is linear, if any of (E1), (E3)–(E7) holds, the same property enunciated for $\mathcal{E}_{sh}$ is satisfied also "directly" by the entire function $\mathcal{E}$ on $[0,+\infty) \times X$, with the only change of the addition of the multiplicative term $|\pi_Z|_*$ in the bounds of (E5), (E7). The only caveat is with (E2), which implies that $\mathcal{E}(t,\cdot)$ is convex, but in general not uniformly convex in the whole X.

Thanks to the above remark, we observe that by (E1) and (E3) we deduce that $\mathcal{E}$ is continuous in $[0,+\infty) \times X$, while from (E1) and (E5) we get that $\frac{\partial}{\partial t}\mathcal{E}$ is a Caratheodory function. Thus for every $x\colon [0,+\infty) \to X$ measurable, the function $t \mapsto \frac{\partial}{\partial t}\mathcal{E}(t,x(t))$ is measurable too. Moreover if x is also bounded, namely $\sup\limits_{t\in[0,T]} |x(t)| \leq R_T$ for every $T > 0$, then (E4) implies that $\frac{\partial}{\partial t}\mathcal{E}(\cdot,x(\cdot))$ is summable in $[0,T]$ for every $T > 0$, indeed:

$$\int_0^T \left|\frac{\partial}{\partial t}\mathcal{E}(\tau,x(\tau))\right| \mathrm{d}\tau \leq \int_0^T \omega(\mathcal{E}(\tau,x(\tau)))\gamma(\tau)\,\mathrm{d}\tau \leq \omega(M_{R_T})\int_0^T \gamma(\tau)\,\mathrm{d}\tau < +\infty,$$

where M_{R_T} denotes the maximum of $\mathcal{E}$ on the compact set $[0,T] \times \overline{\mathcal{B}^X_{R_T}}$. If in addition x is absolutely continuous from $[0,T]$ to X, by (E1), (E3) and (E4) we also deduce that $t \mapsto \mathcal{E}(t,x(t))$ is absolutely continuous in $[0,T]$ too, indeed for every $0 \leq s \leq t \leq T$ it holds:

$$\begin{aligned}|\mathcal{E}(t,x(t)) - \mathcal{E}(s,x(s))| &\leq |\mathcal{E}(t,x(t)) - \mathcal{E}(t,x(s))| + |\mathcal{E}(t,x(s)) - \mathcal{E}(s,x(s))|\\ &\leq C_{R_T}|x(t)-x(s)| + \int_s^t \left|\frac{\partial}{\partial t}\mathcal{E}(\tau,x(s))\right| \mathrm{d}\tau\\ &\leq C_{R_T}|x(t)-x(s)| + \omega(M_{R_T})\int_s^t \gamma(\tau)\,\mathrm{d}\tau,\end{aligned}$$

where C_{R_T} is the maximum of $|D_x\mathcal{E}|_*$ on $[0,T] \times \overline{\mathcal{B}^X_{R_T}}$.

1.1.3 Dissipation potential

Let us now consider the main dissipative forces involved in the system, described by a time-dependent dissipation potential $\mathcal{R}\colon [0,+\infty) \times X \to [0,+\infty]$ which takes into account both possible constraints on the velocity and the presence of dry friction. It originates from a function $\mathcal{R}_{\text{finite}}\colon [0,+\infty) \times X \to [0,+\infty)$ with finite values on which we make the following assumptions:

(R1) for every $t \in [0,+\infty)$, the function $\mathcal{R}_{\text{finite}}(t,\cdot)$ is convex, positively homogeneous of degree one, and satisfies $\mathcal{R}_{\text{finite}}(t,0) = 0$;

(R2) for every $T > 0$ there exist two positive constants $\alpha^* \geq \alpha_* > 0$, possibly depending on T, for which

$$\alpha_* |v| \leq \mathcal{R}_{\text{finite}}(t,v) \leq \alpha^* |v| , \qquad \text{for every } (t,v) \in [0,T] \times X;$$

(R3) for every $T > 0$ there exists a non-negative function $\rho = \rho_T \in L^1(0,T)$ for which

$$|\mathcal{R}_{\text{finite}}(t,v) - \mathcal{R}_{\text{finite}}(s,v)| \leq |v| \int_s^t \rho(\tau)\,\mathrm{d}\tau, \text{ for every } 0 \leq s \leq t \leq T \text{ and } v \in X.$$

Remark 1.1.3. We observe that the second inequality in (R2) actually follows from (R1) and (R3). Indeed, since we are in finite dimension, the convex function $\mathcal{R}_{\text{finite}}(t,\cdot)$ is automatically continuous on X; by (R3) this easily implies $\mathcal{R}_{\text{finite}}$ is continuous on the whole $[0,T]\times X$, and hence by one-homogeneity we get $\mathcal{R}_{\text{finite}}(t,v) \leq C\,|v|$ for some constant $C > 0$ and every $(t,v) \in [0,T] \times X$.

As regards $\mathcal{R}$ we finally assume that:

(R4) there exists a nonempty closed convex cone $K \subseteq X$, independent of time, and there exists a function $\mathcal{R}_{\text{finite}} \colon [0,+\infty) \times X \to [0,+\infty)$ satisfying (R1)–(R3) such that for every $(t,v) \in [0,+\infty) \times X$ it holds

$$\mathcal{R}(t,v) = \chi_K(v) + \mathcal{R}_{\text{finite}}(t,v).$$

We will denote with $\partial_v\mathcal{R}$ the subdifferential of $\mathcal{R}$ with respect to its second variable. The choice of the letter v when dealing with the dissipation potential reminds the fact that the second argument of $\mathcal{R}$ is usually a velocity.

As an immediate consequence of condition (R4) we can rephrase conditions (R1)–(R3) directly on $\mathcal{R}$:

Corollary 1.1.4. *Let $\mathcal{R}$ be as in (R4). Then it holds:*

(I) for every $t \in [0,+\infty)$, the function $\mathcal{R}(t,\cdot)$ is convex, positively homogeneous of degree one, lower semicontinuous, and satisfies $\mathcal{R}(t,0) = 0$;

(II) for every $T > 0$ and for every $(t,v) \in [0,T] \times K$ one has

$$\alpha_* |v| \leq \mathcal{R}(t,v) \leq \alpha^* |v| ,$$

with the same constants α^* and α_* of (R2);

(III) for every $T > 0$, for every $0 \leq s \leq t \leq T$ and $v \in K$ one has

$$|\mathcal{R}(t,v) - \mathcal{R}(s,v)| \leq |v| \int_s^t \rho(\tau)\,\mathrm{d}\tau,$$

with the same function ρ of (R3).

Moreover the following properties hold true:

(IV) the function $\mathcal{R}$ is continuous on $[0,+\infty) \times K$;

(V) the multivalued map $\partial_v\mathcal{R}_{\text{finite}}$ is upper semicontinuous on $[0,+\infty) \times X$;

(VI) for every $T > 0$, and $(t, v) \in [0, T] \times X$ one has

$$\partial_v \mathcal{R}_{\text{finite}}(t, v) \subseteq \overline{\mathcal{B}^{X^*}_{\alpha^*}},$$

with α^* as in (R2). In particular $\partial_v \mathcal{R}_{\text{finite}}$ has compact, convex, non-empty values.

Proof. The first three points are a trivial byproduct of (R1)–(R3), respectively, due to the form of $\mathcal{R}$ given by (R4). We indeed notice that, since K is a nonempty closed convex cone, its characteristic function χ_K is convex, positively homogeneous of degree one, lower semicontinuous, and vanishes at $v = 0$. Moreover, the restriction of $\mathcal{R}$ on $[0, +\infty) \times K$ is $\mathcal{R}_{\text{finite}}$, thus with finite values; this means that by convexity each functional $\mathcal{R}(t, \cdot)$ is continuous on K. By an easy application of (R3) one deduces (IV).

To prove (V), since $\partial_v \mathcal{R}_{\text{finite}}$ has compact values (see point (VI)), it is sufficient to show that for every sequence (t_k, v_k, ξ_k) in $[0, +\infty) \times X \times X^*$ such that $\xi_k \in \partial_v \mathcal{R}_{\text{finite}}(t_k, v_k)$, if $(t_k, v_k, \xi_k) \to (\bar{t}, \bar{v}, \bar{\xi}) \in [0, +\infty) \times X \times X^*$ then $\bar{\xi} \in \partial_v \mathcal{R}_{\text{finite}}(\bar{t}, \bar{v})$. By definition of subdifferential, for every $k \in \mathbb{N}$ we have

$$\mathcal{R}_{\text{finite}}(t_k, v_k) + \langle \xi_k, v - v_k \rangle \leq \mathcal{R}_{\text{finite}}(t_k, v), \quad \text{for every } v \in X.$$

By the continuity of $\mathcal{R}_{\text{finite}}$ on $[0, +\infty) \times X$ and of the dual coupling, passing to the limit in the above estimate gives

$$\mathcal{R}_{\text{finite}}(\bar{t}, \bar{v}) + \langle \bar{\xi}, v - \bar{v} \rangle \leq \mathcal{R}_{\text{finite}}(\bar{t}, v), \quad \text{for every } v \in X,$$

namely $\bar{\xi} \in \partial_v \mathcal{R}_{\text{finite}}(\bar{t}, \bar{v})$.

We finally prove (VI). Since $\mathcal{R}_{\text{finite}}$ has finite values, we deduce that $\xi \in \partial_v \mathcal{R}_{\text{finite}}(t, v)$ if and only if

$$\langle \xi, \tilde{v} \rangle \leq \mathcal{R}_{\text{finite}}(t, \tilde{v} + v) - \mathcal{R}_{\text{finite}}(t, v), \quad \text{for every } \tilde{v} \in X.$$

We now recall that convexity plus one-homogeneity easily yield subadditivity, thus we can continue the above inequality getting

$$\langle \xi, \tilde{v} \rangle \leq \mathcal{R}_{\text{finite}}(t, \tilde{v}), \quad \text{for every } \tilde{v} \in X.$$

By means of (R2) we thus deduce that $|\xi|_* \leq \alpha^*$ and so we conclude. □

Remark 1.1.5 (Comparison with ψ-regularity [39]). Let us remark that our assumptions on $\mathcal{R}$ are very close to the notion of ψ-regularity introduced in [39] (see also Definition 1.3.1). Most of the differences between the two frameworks are due to the fact that [39] deals with functionals $\mathcal{R}$ defined on a general Banach space X, but with finite values. For instance, if the functional $\mathcal{R}$ has finite values, we observe that assumption (R4) is automatically satisfied with $K = X$.

The are only two points in which our assumptions are actually slightly stricter than [39], and both are motivated. The first one is the left inequality in (R2), corresponding in the framework of [39] to the additional assumption $c\,|v| \leq \psi(v)$. This is related to the fact that we have renounced to coercivity in the energy $\mathcal{E}$, and such loss has to be compensated with a coercivity in the dissipation potential $\mathcal{R}$, in order to recover some a priori estimates, such as (i) in Corollary 2.1.4. We however point out that such a request is absolutely natural in the finite dimensional setting we are considering, as we will see in the examples of Section 1.2. On the contrary, it becomes very restrictive in infinite dimension: indeed, in standard models of elasticity where the simplest ambient space is $H^1_0(\Omega)$, a common choice of dissipation potential is $\int_\Omega |v(x)| \,\mathrm{d}x$, which of course lacks of coercivity.

The second stronger assumption is that the modulus of continuity appearing in (R3) is of integral type. This is because we are interested in absolutely continuous solutions of the quasistatic problem (1.1.5), not just continuous ones, cf. Proposition 5.2.4. However, a general modulus of continuity (as the one used in [39]) would be enough to get all the results presented in Section 1.3.

Let us also introduce an optional assumption on $\mathcal{R}$ (actually on the set K), which will be used to improve the regularity of the quasistatic solutions:

(R5) there exists a constant $C_K > 0$ such that, for every $z \in Z$

- either $\pi_Z(x) \neq z$ for every $x \in K$;
- or there exists $x \in K$ such that $\pi_Z(x) = z$ and $|x| \leq C_K |z|_Z$.

We remark that, by a physical point of view, assumption (R5) is usually satisfied. Indeed, violating (R5) would mean that the constraints allow a locomotor to achieve an arbitrarily large displacement with an arbitrarily small change in shape. All the models we consider in Section 1.2 satisfy (R5). By a mathematical point of view, let us highlight some common situations where (R5) is true.

Proposition 1.1.6. *Each of the following is a sufficient condition for (R5):*

1. $K = X$ *or* $K = \{0\}$*;*
2. $\dim Z = \dim X$*;*
3. $\dim X = 1 + \dim Z$ *and* K *is a polyhedral closed cone, i.e. there exist* J *covectors* $f_1^K, \dots f_J^K \in X^*$ *such that*

$$K = \{x \in X \mid \langle f_j^K, x \rangle \geq 0 \quad \text{for every } j = 1, \dots, J\}.$$

Proof. The first two points are trivial. Let us therefore prove the third point. First of all we observe that for $z = 0_Z$ the second alternative of (R5) is satisfied by $x = 0_X$. For $z \neq 0_Z$, by homogeneity, it is sufficient to consider the case $|z|_Z = 1$. Moreover, without loss of generality we can assume $\left| f_j^K \right|_* = 1$.

Let $i\colon Z \times \ker \pi_Z \to X$ be the canonical identification. For every $z \in Z$ we write $\hat{z} := i(z, 0) \in X$; moreover, given any non-zero vector $y \in \ker \pi_Z$, we set $\eta := i(0, y)/|i(0, y)|$. Since $\dim X = 1 + \dim Z$, we deduce that $\pi_Z(x) = z$ if and only if $x = \hat{z} + \lambda\eta$ for some $\lambda \in \mathbb{R}$.

Let us denote with $\mathcal{S} = \{\hat{z} = i(z, 0) \in X \mid |z|_Z = 1\}$ and set

$$\begin{aligned}
C_1 &:= \max_{j=1,\dots,J} \max_{\hat{z} \in \mathcal{S}} \left| \langle f_j^K, \hat{z} \rangle \right|, \\
C_2 &:= \min_{j=1,\dots,J} \{ \left| \langle f_j^K, \eta \rangle \right| \mid \langle f_j^K, \eta \rangle \neq 0 \}, \\
C_3 &:= \max_{\hat{z} \in \mathcal{S}} |\hat{z}| .
\end{aligned}$$

We claim that we can take $C_K = C_3 + (C_1/C_2)$. Fix z with norm 1, and consider the corresponding $\hat{z} \in \mathcal{S}$. Since K is closed, we have two alternative possibilities:

- either $\pi_Z(x) \neq z$ for every $x \in K$;
- or there exists $\bar{\lambda} \in R$ such that $\hat{z} + \bar{\lambda}\eta \in K$ and

$$\left| \hat{z} + \bar{\lambda}\eta \right| \leq |\hat{z} + \lambda\eta| , \qquad \text{for every } \lambda \in \mathbb{R} \text{ such that } \hat{z} + \lambda\eta \in K.$$

To prove (R5) it is sufficient to show, if the second option holds, that $|\bar{\lambda}| \leq C_1/C_2$, so that $|\hat{z} + \bar{\lambda}\eta| \leq |\hat{z}| + |\bar{\lambda}| \leq C_3 + (C_1/C_2)$. To show this estimate on $|\bar{\lambda}|$, let us observe that, in order to minimize the absolute value, either $\bar{\lambda} = 0$ or there exists an index j such that

$$\langle f_j^K, \hat{z} \rangle + \bar{\lambda} \langle f_j^K, \eta \rangle = 0, \qquad \text{and} \qquad \langle f_j^K, \eta \rangle \neq 0,$$

which implies $|\lambda| \leq C_1/C_2$. □

An example of a closed convex cone which does not satisfy (R5) is the following: let us set $X = \mathbb{R}^3$, $Z = \mathbb{R}^2$, $\pi_Z(x) = (x_2, x_3)$ and

$$K := \{(\lambda, \lambda a, \lambda b) \mid \lambda \geq 0,\ a^2 + (b-1)^2 \leq 1\}.$$

Let us pick $z = (\cos\theta, \sin\theta)$, with $\theta \in (0, \pi/2)$, so that $|z|_Z = 1$. A simple computation shows that

$$(\lambda, \cos\theta, \sin\theta) \in K, \quad \iff \quad \cos^2\theta + (\sin\theta - \lambda)^2 \leq \lambda^2, \quad \text{and } \lambda > 0, \iff \quad \lambda \sin\theta \geq \frac{1}{2}.$$

Hence (R5) is violated by any sequence $\theta_n \to 0^+$. We point out however that this counterexample is purely mathematical: we are not aware of any reasonable mechanical model described by such a choice of K.

We now present the dynamic and quasistatic problems we will study.

1.1.4 Dynamic problem

Let $\mathbb{M}, \mathbb{V}$ be as above, and assume that (E1), (E3)–(E5) and (R4) are satisfied. For $\varepsilon > 0$ we refer as *dynamic problem* to the differential inclusion

$$\begin{cases} \varepsilon^2 \mathbb{M} \ddot{x}^\varepsilon(t) + \varepsilon \mathbb{V} \dot{x}^\varepsilon(t) + \partial_v \mathcal{R}(t, \dot{x}^\varepsilon(t)) + D_x \mathcal{E}(t, x^\varepsilon(t)) \ni 0, \quad t > 0, \\ x^\varepsilon(0) = x_0^\varepsilon, \quad \dot{x}^\varepsilon(0) = x_1^\varepsilon, \end{cases} \tag{1.1.3}$$

where the initial velocity satisfy the admissibility condition

$$x_1^\varepsilon \in K, \tag{1.1.4}$$

for K as in (R4). To give the definition of solution we recall that by $\widetilde{W}^{2,1}(0, +\infty; X)$ we mean the space of functions from $(0, +\infty)$ to X belonging to $W^{2,1}(0, T; X)$ for every $T > 0$.

Definition 1.1.7. *We say that a function $x^\varepsilon \in \widetilde{W}^{2,1}(0, +\infty; X)$ is a* differential solution *of (1.1.3) if the differential inclusion holds true in X^* for a.e. $t \in [0, +\infty)$ and initial position and velocity are attained.*

We discuss existence and uniqueness of differential solution for (1.1.3) in Section 2.1, see Theorem 2.1.8.

1.1.5 Quasistatic problem

Assume that (E1)–(E5) and (R4) are satisfied. We refer as *quasistatic problem* to the differential inclusion

$$\begin{cases} \partial_v \mathcal{R}(t, \dot{x}(t)) + D_x \mathcal{E}(t, x(t)) \ni 0, \quad t > 0 \\ x(0) = x_0. \end{cases} \tag{1.1.5}$$

For the quasistatic problem we introduce two notions of solution. Conditions for existence of each type of solution are a direct consequence of the vanishing inertia analysis, although they could be derived separately (see for instance [39, 65] for a general argument based on time-discretisation).

Definition 1.1.8. *We say that a function $x \in \widetilde{AC}([0,+\infty);X)$ is a* differential solution *of* (1.1.5) *if the differential inclusion holds true in X^* for a.e. $t \in [0,+\infty)$ and the initial position is attained.*

We observe that the existence of differential solutions for (1.1.5) requires the admissibility condition on the initial datum

$$-D_x\mathcal{E}(0,x_0) \in \partial_v\mathcal{R}(0,0). \tag{1.1.6}$$

In order to introduce the second (weaker) notion of solution, let us first present a suitable generalisation of functions of bounded variation, which we will discuss in detail in section 1.3.

Definition 1.1.9. *Given a function $f\colon [a,b] \to X$, we define its $\mathcal{R}$–variation in $[s,t]$, with $a \le s < t \le b$, as:*

$$V_\mathcal{R}(f;s,t) := \lim_{n\to+\infty}\sum_{k=1}^n \mathcal{R}(t_{k-1}, f(t_k)-f(t_{k-1})), \tag{1.1.7}$$

where $\{t_k\}_{k=1}^n$ is a fine *sequence of partitions of $[s,t]$, namely it is of the form $s=t_0<t_1<\dots<t_n=t$ and satisfies*

$$\lim_{n\to+\infty}\sup_{k=1,\dots,n}(t_k-t_{k-1})=0. \tag{1.1.8}$$

We also set $V_\mathcal{R}(f;t,t):=0$, for every $t\in[a,b]$.

We say that f is a function of bounded $\mathcal{R}$–variation in $[a,b]$ if its $\mathcal{R}$–variation in $[a,b]$ is finite, i.e. $V_\mathcal{R}(f;a,b)<+\infty$. In this case we write $f\in BV_\mathcal{R}([a,b];X)$,

As before, since the time interval is $[0,+\infty)$, we denote by $\widetilde{BV}_\mathcal{R}([0,+\infty);X)$ the space of functions belonging to $BV_\mathcal{R}([0,T];X)$ for every $T>0$.

Definition 1.1.10. *We say that $x\in\widetilde{BV}_\mathcal{R}([0,+\infty);X)$ is an* energetic solution *for the quasistatic problem* (1.1.5) *if the initial position is attained and the following global stability condition and weak energy balance hold true:*

(GS) $\mathcal{E}(t,x(t)) \le \mathcal{E}(t,v)+\mathcal{R}(t,v-x(t))$, *for every $v\in X$ and for every $t\in[0,+\infty)$;*

(WEB) $\mathcal{E}(t,x(t))+V_\mathcal{R}(x;0,t)=\mathcal{E}(0,x_0)+\int_0^t \frac{\partial}{\partial t}\mathcal{E}(\tau,x(\tau))\,\mathrm{d}\tau$, *for every $t\in[0,+\infty)$.*

The justification of this definition, together with the main properties of energetic solutions, will be given in Section 2.2; see in particular Proposition 2.2.2.

We remark that the notion of energetic solution is more flexible than the one of differential solution, since it does not involve derivatives and in general allows for discontinuous solutions. We refer to [65] for a wide and complete presentation on the topic.

1.2 Applications and examples

In this section we illustrate several examples which can be described by our abstract formulation; in particular they explain and motivate our framework. Since the applications we present here are all set in $X=\mathbb{R}^N$, endowed with the euclidean norm, for simplicity we will always identify canonically the dual space X^* with $\mathbb{R}^N$, so that the dual coupling $\langle\cdot,\cdot\rangle$ coincides with the scalar product.

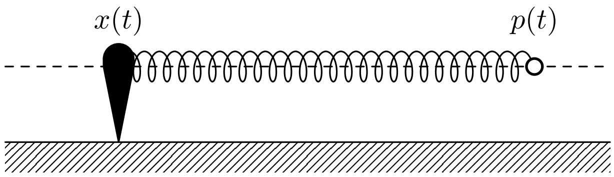

Figure 1.1: A mechanical model of the scalar play operator, discussed in Subsection 1.2.1.

1.2.1 The minimal example: the play operator

Let us begin by presenting a very simple model, illustrated in Figure 1.1, to which our results may be applied. We have a mass $m > 0$ with position $x(t)$ on a line, and subject to (isotropic) dry friction. The mass is connected to a (linear) spring, whose other end is moved according to the function $p(t) \in W^{1,1}(0,T)$. Thus the dynamic evolution of the system is described by the inclusion (1.1.3), where:

$$X = Z = K = \mathbb{R}, \quad \mathcal{R}(t,v) = \mathcal{R}(v) = \alpha\,|v|\,, \quad \mathcal{E}(t,x) = \mathcal{E}_{sh}(t,x) = \frac{k}{2}(x - p(t) + L^{\mathrm{rest}})^2,$$

and π_Z is the identity. Notice that (QE) holds. Clearly $\mathbb{M} = m > 0$, while we may assume either $\mathbb{V} = 0$, or add an additional viscous resistance to $\dot{x}$, so that the resulting friction force-velocity law for the mass is of Bingham type.

The relevance of this model is due to the fact that its quasistatic evolution corresponds to the *(scalar) play operator* [46]; indeed a straightforward computation shows that (1.1.5) in this case reads as

$$\begin{cases} p(t) - L^{\mathrm{rest}} - x(t) \in \frac{\alpha}{k}\,\partial\,|\dot{x}(t)|\,, \\ x(0) = x_0, \end{cases} \tag{1.2.1}$$

and hence, setting $u(t) = p(t) - L^{\mathrm{rest}}$, we notice that (1.2.1) is equivalent to

$$\begin{cases} |u(t) - x(t)| \leq \frac{\alpha}{k}, \\ (u(t) - x(t) - v)\,\dot{x}(t) \geq 0, \quad \text{for every } v \in \left[-\frac{\alpha}{k}, \frac{\alpha}{k}\right], \\ x(0) = x_0. \end{cases}$$

Of course, more advanced models may be built by considering analogously a mass on a plane (or abstractly in an N-dimensional space), or considering nonautonomous friction coefficients.

1.2.2 Soft crawlers

We now illustrate minutely how the family of models represented in Figure 1 and described in the Introduction fits in our mathematical framework. Their quasistatic version has been extensively discussed in [36], to which we refer for more details. We also mention [12], where similar models have been studied in the dynamic case.

We are considering a model with $N \geq 2$ blocks on a line, with adjacent blocks joined by an actuated soft link. We describe with x_i the position of the i-th block. The elastic energy of the system will not depend directly on any of the positions of the block, but only on the distances $x_i - x_{i-1}$ between two consecutive blocks. Hence we set

$$X = \mathbb{R}^N, \qquad Z = \mathbb{R}^{N-1}, \qquad \pi_Z(x_1, \dots, x_N) = (x_2 - x_1, \dots, x_N - x_{N-1}).$$

We now discuss separately each of the elements of the dynamics.

Mass distribution

Denoting with $m_i > 0$ the mass of the i-th block, the linear operator $\mathbb{M}$ is

$$\mathbb{M} = \mathrm{Diag}(m_1, \dots, m_N).$$

Viscous friction

There are two main situation in which we may consider viscous friction. The first one is to assume an additional viscous friction resistance when the blocks slide, in addition to dry friction we discuss below. Such forces are described by a diagonal matrix

$$\mathbb{V}_{\mathrm{ext}} = \mathrm{Diag}(\nu_1^{\mathrm{ext}}, \dots, \nu_N^{\mathrm{ext}}),$$

for some non-negative coefficients $\nu_i^{\mathrm{ext}} \geq 0$. This also means that the total friction force acting on each block is of Bingham type, and may be justified by lubrication with a non-Newtonian fluid [29].

The second possible way to introduce viscosity in the model is to assume a viscous resistance to deformation of the links. This is represented by the matrix

$$\mathbb{V}_{\mathrm{link}} = \begin{pmatrix} \nu_1^{\mathrm{link}} & -\nu_1^{\mathrm{link}} & 0 & \cdots & 0 & 0 \\ -\nu_1^{\mathrm{link}} & \nu_1^{\mathrm{link}} + \nu_2^{\mathrm{link}} & -\nu_2^{\mathrm{link}} & \cdots & 0 & 0 \\ 0 & -\nu_2^{\mathrm{link}} & \nu_2^{\mathrm{link}} + \nu_3^{\mathrm{link}} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \nu_{N-2}^{\mathrm{link}} + \nu_{N-1}^{\mathrm{link}} & -\nu_{N-1}^{\mathrm{link}} \\ 0 & 0 & 0 & \cdots & -\nu_{N-1}^{\mathrm{link}} & \nu_{N-1}^{\mathrm{link}} \end{pmatrix}$$

for some non-negative coefficients $\nu_i^{\mathrm{link}} \geq 0$.

Accounting for these two effects, a general viscosity matrix $\mathbb{V}$ takes the form $\mathbb{V} = \mathbb{V}_{\mathrm{link}} + \mathbb{V}_{\mathrm{ext}}$.

Dry friction

Since each block is affected independently by dry friction, the rate-independent dissipation potential can be represented as the sum

$$\mathcal{R}_{\mathrm{finite}}(t, v) = \sum_{i=1}^{N} \mathcal{R}_i(t, v_i),$$

of N dissipation potentials $\mathcal{R}_i\colon [0, T] \times \mathbb{R} \to [0, +\infty)$, each of the form

$$\mathcal{R}_i(t, v) = \begin{cases} \alpha_i^+(t) v, & \text{if } v \geq 0, \\ \alpha_i^-(t) v, & \text{if } v \leq 0, \end{cases} \tag{1.2.2}$$

where the functions $\alpha_i^{\pm}\colon [0, T] \to (0, +\infty)$ are strictly positive and absolutely continuous. Concretely, it means that each block has two dry friction coefficients, one for forward and one for backward movements, possibly varying in time. By compactness, we observe that in this framework the assumptions (R1)–(R3) are satisfied. As argued in [36, Lemma 3.2], the uniqueness condition $(*)$ of Lemma 2.2.10 for the quasistatic problem is satisfied if, for every subset of indices $J \subseteq \{1, 2, \dots, N\}$ we have

$$\sum_{i \in J} \alpha_i^+(t) \neq \sum_{i \in J^C} \alpha_i^-(t), \qquad \text{for a.e. } t \in [0, T], \tag{1.2.3}$$

where $J^C = \{1, 2, \dots, N\} \setminus J$.

Velocity constraint

Most of the models of crawlers usually fit in the $K = X$ case: indeed, the possibility to move the body both backwards and forwards is often appreciable in locomotion. In some situations, however, backward friction is extremely higher than forward friction, so that in fact no backwards movement occurs. For this reason, sometimes it is convenient to assume an infinite friction coefficient, namely a constraint on velocities. With our notation, this corresponds to set

$$K = \bigcap_{i=1}^{N} K_i^+, \qquad \text{where} \quad K_i^+ = \{v \in \mathbb{R}^N \mid v_i \geq 0\}.$$

We observe that the set K is a polyhedral cone, satisfying condition 3 of Proposition 1.1.6. Notice also that, in this case, the coefficients α_i^- in (1.2.2) can be freely chosen, for instance equal to a positive constant, since they are not involved in the dynamics. More generally, we can introduce analogously the halfplanes $K_i^- = \{v \in \mathbb{R}^N \mid v_i \leq 0\}$, and set K as the intersection of an arbitrary selection of sets $K_i^\pm$, although this would result often in something less pragmatical in terms of locomotion. In particular, if $K \subseteq K_i^+ \cap K_i^-$, the i-th block would be completely anchored on the surface.

Elastic energy

The total elastic energy will be the sum of the elastic energies of each link. Hence we have

$$\mathcal{E}(t,x) = \sum_{i=1}^{N-1} \mathcal{E}_i^{\text{link}}(t, x_{i+1} - x_i), \qquad \text{or equivalently} \qquad \mathcal{E}_{sh}(t,z) = \sum_{i=1}^{N-1} \mathcal{E}_i^{\text{link}}(t, z_i).$$

In order for $\mathcal{E}_{sh}$ to satisfy any of the properties (E1)–(E7), it is sufficient to ask each of the energies $\mathcal{E}_i^{\text{link}} \colon [0,T] \times \mathbb{R} \to [0,+\infty)$ of the links to satisfy the same condition being required on $\mathcal{E}_{sh}$. The quadratic case (QE) corresponds to the case in which each of the link energies is quadratic, namely it follows Hooke's law

$$\mathcal{E}_i^{\text{link}}(t, z_i) = \frac{k_i}{2}\,(z_i - \ell_i(t))^2\,,$$

for a positive elastic constant $k_i > 0$ and an absolutely continuous $\ell_i \colon [0,T] \to \mathbb{R}$. Notice that our results hold also for nonlinear models of elasticity. For instance, the soft link may behave like a Duffing-type nonlinear spring, i.e.

$$\mathcal{E}_i^{\text{link}}(t, z_i) = \frac{k_i}{2}\,(z_i - \ell_i(t))^2 + \frac{\beta_i}{4}\,(z_i - \ell_i(t))^4\,,$$

where the quartic term produces a hardening of the spring. In such a case the assumptions (E1)–(E5) and (E7) are all satisfied. Pay attention that (E6) holds only if ℓ_i are continuosly differentiable; however in this specific example one can argue as in Lemma 2.3.6, thus (E6) is not really necessary.

1.2.3 A rheological model

In order to illustrate a second example with multiple material points, we propose here, with our notation, a rheological model presented in [11, Sec. 2.2.6], and illustrated in Figure 1.2.

The model consists on N material points and N P_i-elements connected in series. A P_i element is composed of a St-Venant element with threshold $\alpha_i > 0$ and a linear spring with

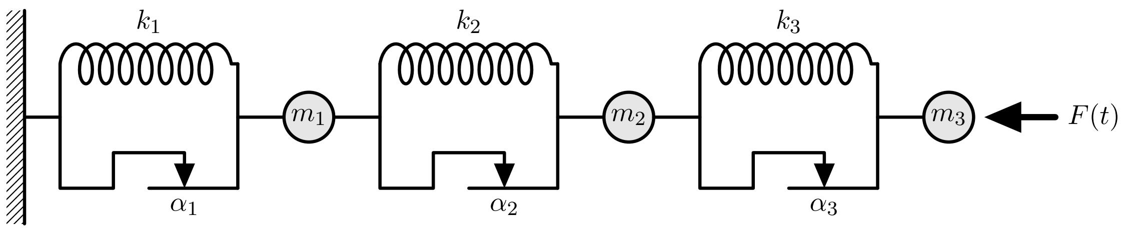

Figure 1.2: A rheological model discussed in Subsection 1.2.3, cf. also [11, Sec. 2.2.6]

constant $k_i > 0$ connected in parallel. As before, we denote with x_i the position on the line of the i-th material point, having mass $m_i > 0$. The first P_i-element is connected to the first material point at one end, whereas the other end is fixed in the origin. Moreover, the N-th material point is subject to an external force $F(t)$, absolutely continuous in time. Hence

$$X = Z = K = \mathbb{R}^N, \qquad \pi_Z = \mathrm{I}, \qquad \mathbb{M} = \mathrm{Diag}(m_1, \ldots, m_N).$$

The energy $\mathcal{E}$ will be the sum of a potential energy $F(t)x_N$ used to describe the external force, plus the elastic energies of the P_i-elements, namely:

$$\mathcal{E}(t,x) = \mathcal{E}_{sh}(t,x) = F(t)x_N + \frac{k_1}{2}\, x_1^2 + \sum_{i=2}^{N} \frac{k_i}{2}(x_i - x_{i-1})^2.$$

Similarly, the dissipation potential $\mathcal{R}$ will be the sum of the dissipation potentials associated to each St-Venant element, namely

$$\mathcal{R}(t,v) = \mathcal{R}(v) = \alpha_1 \left|v_1\right| + \sum_{i=2}^{N} \alpha_i \left|v_i - v_{i-1}\right|,$$

where we recall that in the first P_i-element one end is fixed. The assumptions (E1)-(E5), (E7), (R4) are easily verified, as also (E6) if in addition F is continuously differentiable. As before, however, (E6) could be avoided by arguing as in Lemma 2.3.6.

1.2.4 A planar model

Let us now consider the two-dimensional analogous of the simple model discussed in Subsection 1.2.1 and illustrated in Figure 1.1. Setting for simplicity the rest length of the spring to zero, we have

$$X = Z = \mathbb{R}^2, \qquad \pi_Z = \mathrm{I}, \qquad \mathcal{E}(t,x) = \mathcal{E}_{sh}(t,x) = \frac{k}{2}\left|p(t) - x\right|^2,$$

and (QE) again holds. A point mass at x can be therefore considered as a test particle (or more concretely, the point of a cantilever), probing the frictional properties of the surface. For simplicity, here we limit ourselves to autonomous dissipation. Until now we have presented only models lying on a line, so that the friction forces possibly acting on each mass are described by two parameters α^+ and α^-. If instead the test mass lies on a plane, dry friction is described by a function on the unit circle. Whereas the isotropic case $\mathcal{R}(v) = \alpha\left|v\right|$ is simple, the nature of friction when the surface is anisotropic is a complicated matter.

Experimentally, friction of scaly surfaces, for instance snakes or sharks skins, is usually measured only in four orthogonal directions: forwards, backwards, and the in two transversal directions (usually showing a symmetric behaviour), cf. e.g. [13, 56]. We are not aware of experimental characterizations of the friction coefficients with respect to all the other intermediate directions. There is however a mathematical restriction on the scenarios that can be effectively described by the subdifferential of a function $\mathcal{R}$. What we aim to show here is that, by introducing the constraint K, we allow to study a qualitatively different class of models, non included in the case $\mathcal{R} < +\infty$.

If $X = K$, namely there is no velocity constraint, then the functional $\mathcal{R}$ is continuous by convexity, and so the friction coefficient changes continuously with respect to the direction of the velocity. Moreover, we notice that convexity affects ulteriorly the structure of the friction coefficient: for instance, oscillations arbitrarily both ample and frequent of the friction coefficient as the direction varies are not allowed.

When hooks or scales introduce anisotropic friction on a plane, a scenario that can be expected, or at least desirable, is as follows:

- friction is extremely high for all velocities with a non-zero backward component (i.e. for all $v = (v_1, v_2)$ with $v_1 < 0$);
- friction is low for all the remaining velocities ($v_1 \geq 0$), in particular also for purely lateral velocities ($v_1 = 0$).

If $X = K$, such a case can be portrayed only approximatively, since a smooth transition is compulsory from low to high friction. The scenario can instead be better described by setting

$$K = \{v \in \mathbb{R}^2 \mid v_1 \geq 0\}.$$

Indeed, we emphasize that $\mathcal{R}$ is in general lower semicontinuous, but not continuous, on the boundary of K.

A situation even more radical is usually considered in the modelling of slithering locomotion, with "snake in a tube" models [20]. While slithering on a plane, snakes experience a very large resistance to transversal sliding, compared to the longitudinal one, so that the whole body of the snake follows the same path covered by its head. Hence, according to the description in such models, a test particle on a snake skin would experience:

- extremely high friction for all velocities with a non-zero lateral component ($v_2 \neq 0$);
- high friction for a purely backward velocity ($v_1 < 0$ and $v_2 = 0$);
- low friction for a purely forward velocity ($v_1 > 0$ and $v_2 = 0$).

Again, the situation can be portrayed only approximatively by a finite dissipation functional $\mathcal{R}$, while it is effectively described by introducing the constraint K as

$$K = \{v \in \mathbb{R}^2 \mid v_2 = 0\}, \qquad \text{or} \qquad K = \{v \in \mathbb{R}^2 \mid v_1 \geq 0, v_2 = 0\}.$$

Notice that all the three examples of cones K in this subsection satisfy condition 3 of Proposition 1.1.6.

1.3 $AC_{\mathcal{R}}$ and $BV_{\mathcal{R}}$ functions

In this section we introduce and present the main properties of the analogue of absolutely continuous (vector-valued) functions and of functions of bounded variation when the norm $|\cdot|$ is replaced by a general time-dependent functional $\mathcal{R}$. These two notions will be useful to deal with both problems (1.1.3) and (1.1.5). Here we consider the case of a reflexive Banach space X and instead of limiting ourselves to potentials $\mathcal{R}$ satisfying (R4) we consider the larger class of ψ–regular functionals used in [39] (but still with the additional coercivity assumption, see (ψ4) below). This choice is motivated by two reasons: first of all we provide new results which are not investigated in [39] and thus we prefer to state them in the broadest possible setting; furthermore all the proofs, which for the sake of clarity we collect in Appendix A, do not become easier in our simpler framework, neither exploiting the finite dimension of the space nor using the explicit form of $\mathcal{R}$ given by (R4). We want also to recall that a more general theory can be developed even in a metric setting, see for instance [9], Chapter 1.

We follow the presentation given in [39] for the definition and the main features of functions of bounded $\mathcal{R}$–variation when $\mathcal{R}$ depends on time, and we provide some more properties we will need during the thesis. We also refer to the Appendix of [15] for a very well detailed presentation of the classical case in which $\mathcal{R}$ is the norm of the Banach space X.

We thus consider a reflexive Banach space X and a ψ–regular function $\mathcal{R}\colon [a,b]\times X \to [0,+\infty]$ in the sense of the following definition, see also [39]:

Definition 1.3.1. *Given an* admissible *function* $\psi\colon X\to[0,+\infty]$*, namely satisfying*

(ψ0) $\psi(0)=0$*;*

(ψ1) ψ *is convex;*

(ψ2) ψ *is positively homogeneous of degree one;*

(ψ3) ψ *is lower semicontinuous;*

(ψ4) *there exists a positive constant* $c>0$ *such that* $c|\cdot|\le\psi(\cdot)$*,*

we say that $\mathcal{R}\colon[a,b]\times X\to[0,+\infty]$ *is* ψ–regular *if:*

- *for every* $t\in[a,b]$*,* $\mathcal{R}(t,\cdot)$ *is convex, positively homogeneous of degree one, lower semicontinuous, and satisfies* $\mathcal{R}(t,0)=0$*;*
- *there exist two positive constants* $\alpha^*\ge\alpha_*>0$ *for which*

$$\alpha_*\psi(v)\le\mathcal{R}(t,v)\le\alpha^*\psi(v),\qquad \textit{for every } (t,v)\in[a,b]\times X; \tag{1.3.1}$$

- *there exists a nonnegative and nondecreasing function* $\sigma\in C^0([0,b-a])$ *satisfying* $\sigma(0)=0$ *and for which*

$$|\mathcal{R}(t,v)-\mathcal{R}(s,v)|\le\psi(v)\sigma(t-s),\ \textit{for every } a\le s\le t\le b \textit{ and } v\in\{\psi<+\infty\}. \tag{1.3.2}$$

Remark 1.3.2. We again notice that this definition actually differs from the one considered in [39] due to the additional assumption (ψ4), which gives coercivity. Most of the results of this section are however valid without (ψ4), as the reader can check from the proofs (see Appendix A). Indeed we always stress the points where it is really necessary.

We want to point out that if $\mathcal{R}$ satisfies (R4), then it is ψ^K–regular (with an absolutely continuous σ) with respect to the admissible function

$$\psi^K(v)=\chi_K(v)+|v|\,, \tag{1.3.3}$$

where K is given by (R4). On the other hand, any ψ–regular functional $\mathcal{R}$ can be written as

$$\mathcal{R}(t,v)=\chi_{\{\psi<+\infty\}}(v)+\mathcal{R}_{|_{\{\psi<+\infty\}}}(t,v),$$

where $\mathcal{R}_{|\{\psi<+\infty\}}$ has finite values due to (1.3.1) and the set $\{\psi < +\infty\}$ is a nonempty convex cone thanks to $(\psi 0)$–$(\psi 2)$. However, in general, this set is not closed and moreover the second inequality in (1.3.1) cannot be improved to (R2), since no bounds from above for ψ are available. These are the main differences between ψ–regular functionals and functionals satisfying (R4).

We first deal with the notion of $\mathcal{R}$–absolutely continuous functions:

Definition 1.3.3. *We say that a function $f\colon [a,b] \to X$ is $\mathcal{R}$–absolutely continuous, and we write $f \in AC_{\mathcal{R}}([a,b];X)$ if f is absolutely continuous and $\int_a^b \mathcal{R}(\tau, \dot f(\tau))\,\mathrm{d}\tau < +\infty$.*

Next proposition provides a natural link between $\mathcal{R}$–absolutely continuous and classical absolutely continuous functions. We recall that we gather all the proofs of this section in Appendix A.

Proposition 1.3.4. *Given a function $f\colon [a,b] \to X$, the following are equivalent:*

(1) f is $\mathcal{R}$–absolutely continuous;

(2) f is absolutely continuous and $\int_a^b \psi(\dot f(\tau))\,\mathrm{d}\tau < +\infty$;

(3) there exists a nonnegative function $m \in L^1(a,b)$ such that:

$$\psi(f(t) - f(s)) \le \int_s^t m(\tau)\,\mathrm{d}\tau, \qquad \textit{for every } a \le s \le t \le b.$$

Remark 1.3.5. In the special case of a potential $\mathcal{R}$ satisfying (R4), namely when ψ has the form (1.3.3), from (2) we deduce that $f \in AC_{\mathcal{R}}([a,b];X)$ if and only if f is absolutely continuous and $\dot f(t) \in K$ for almost every time $t \in [a,b]$.

We now recall the notion of functions of bounded $\mathcal{R}$–variation:

Definition 1.3.6. *Given a function $f\colon [a,b] \to X$, we define its $\mathcal{R}$–variation in $[s,t]$, with $a \le s < t \le b$, as:*

$$V_{\mathcal{R}}(f;s,t) := \lim_{n\to+\infty} \sum_{k=1}^{n} \mathcal{R}(t_{k-1}, f(t_k) - f(t_{k-1})), \tag{1.3.4}$$

where $\{t_k\}_{k=1}^n$ is a fine *sequence of partitions of $[s,t]$, namely it is of the form $s = t_0 < t_1 < \dots < t_n = t$ and satisfies*

$$\lim_{n\to+\infty} \sup_{k=1,\dots,n} (t_k - t_{k-1}) = 0. \tag{1.3.5}$$

We also set $V_{\mathcal{R}}(f;t,t) := 0$, for every $t \in [a,b]$.

We say that f is a function of bounded $\mathcal{R}$–variation in $[a,b]$ if its $\mathcal{R}$–variation in $[a,b]$ is finite, i.e. $V_{\mathcal{R}}(f;a,b) < +\infty$. In this case we write $f \in BV_{\mathcal{R}}([a,b];X)$,

Remark 1.3.7. We want to say that the limit in (1.3.4) exists and it does not depend on the fine sequence of partitions chosen, thus the definition is well-posed. If $\mathcal{R}$ does not depend on time, the limit in (1.3.4) can be replaced by a supremum. For a proof of these facts we refer to [39], Appendix A.

Remark 1.3.8 (Notation). During the section will be useful to consider the variation of a function with respect to the time-independent function $\mathcal{R}(\bar{t},\cdot)$, namely when the time $t=\bar{t}$ is frozen. In this case we denote the variation by $V_{\mathcal{R}(\bar{t})}(f;s,t)$. We notice that $V_{\mathcal{R}(\bar{t})}(f;s,t)$ can be obtained by replacing $\mathcal{R}(t_k, f(t_k)-f(t_{k-1}))$ with $\mathcal{R}(\bar{t}, f(t_k)-f(t_{k-1}))$ in (1.3.4), or by taking the supremum over finite partitions since the frozen potential does not depend on time.

From the Definition 1.3.6 we easily notice that (1.3.1) allows us to deduce that a function f belongs to $BV_{\mathcal{R}}([a,b];X)$ if and only it it is a function of bounded ψ–variation, i.e. $V_\psi(f;a,b)<+\infty$; moreover by (ψ4) we deduce that f is a function of bounded variation in the classical sense. As a byproduct, see for instance the Appendix in [15], we obtain that any $f\in BV_{\mathcal{R}}([a,b];X)$ has at most a countable number of discontinuity points, and at every $t\in[a,b]$ there exist right and left (strong) limits of f, namely:

$$f^+(t) := \lim_{t_k\searrow t} f(t_k), \qquad \text{and} \qquad f^-(t) := \lim_{t_k\nearrow t} f(t_k). \tag{1.3.6}$$

Remark 1.3.9. Given a function $f\colon [a,b]\to X$, with a little abuse of notation we will always consider and still denote by f its constant extension to a slightly larger interval $(a-\delta, b+\delta)$, for some $\delta>0$; namely $f(t)=f(a)$ if $t\in(a-\delta,a]$ and $f(t)=f(b)$ if $t\in[b,b+\delta)$. This ensures that the limits in (1.3.6) are well defined also in $t=a,b$ and in particular it holds $f^-(a)=f(a)$ and $f^+(b)=f(b)$.

Remark 1.3.10. In the particular case in which $\mathcal{R}$ satisfies (R4), namely when ψ is given by (1.3.3), it is easy to see that $f\in BV_{\mathcal{R}}([a,b];X)$ if and only if f has bounded variation (in the classical sense) and $f(t)-f(s)\in K$ for every $a\le s\le t\le b$.

Trivially the $\mathcal{R}$–variation of f is monotone in both entries (see (a) in the next Proposition), thus for every $a\le s\le t\le b$ they are well defined:

$$V_{\mathcal{R}}(f;s,t+) := \lim_{t_k\searrow t} V_{\mathcal{R}}(f;s,t_k), \qquad V_{\mathcal{R}}(f;s,t-) := \lim_{s\le t_k, t_k\nearrow t} V_{\mathcal{R}}(f;s,t_k),$$
$$V_{\mathcal{R}}(f;s-,t) := \lim_{s_k\nearrow s} V_{\mathcal{R}}(f;s_k,t), \qquad V_{\mathcal{R}}(f;s+,t) := \lim_{s_k\le t, s_k\searrow s} V_{\mathcal{R}}(f;s_k,t),$$
$$V_{\mathcal{R}}(f;s-,t+) := \lim_{s_k\nearrow s, t_k\searrow t} V_{\mathcal{R}}(f;s_k,t_k),$$
$$V_{\mathcal{R}}(f;s-,t-) := \lim_{s_k\le t_k, s_k\nearrow s, t_k\nearrow t} V_{\mathcal{R}}(f;s_k,t_k),$$
$$V_{\mathcal{R}}(f;s+,t+) := \lim_{s_k\le t_k, s_k\searrow s, t_k\searrow t} V_{\mathcal{R}}(f;s_k,t_k).$$

Next proposition gathers all the properties of the $\mathcal{R}$–variation we will need throughout the thesis.

Proposition 1.3.11. *Given a function $f\colon[a,b]\to X$, the following properties hold true:*

(a) for every $a\le r\le s\le t\le b$ it holds:

$$V_{\mathcal{R}}(f;r,t)=V_{\mathcal{R}}(f;r,s)+V_{\mathcal{R}}(f;s,t);$$

(b) for every $a\le s\le t\le b$ it holds:

$$V_{\mathcal{R}}(f;s-,t+)=V_{\mathcal{R}}(f;s-,s)+V_{\mathcal{R}}(f;s,t)+V_{\mathcal{R}}(f;t,t+);$$

(c) if $f\in BV_{\mathcal{R}}([a,b];X)$, then for every $t\in[a,b]$ the following equalities hold true:

$$V_{\mathcal{R}}(f;t,t+)=V_{\mathcal{R}(t)}(f;t,t+)=\lim_{t_k\searrow t}\mathcal{R}(t,f(t_k)-f(t)), \qquad V_{\mathcal{R}}(f;t,t-)=0,$$
$$V_{\mathcal{R}}(f;t-,t)=V_{\mathcal{R}(t)}(f;t-,t)=\lim_{t_k\nearrow t}\mathcal{R}(t,f(t)-f(t_k)), \qquad V_{\mathcal{R}}(f;t+,t)=0$$
$$V_{\mathcal{R}}(f;t-,t-)=0, \qquad V_{\mathcal{R}}(f;t+,t+)=0;$$

(d) *if* $f \in BV_{\mathcal{R}}([a,b];X)$*, then* f^+, f^- *belong to* $BV_{\mathcal{R}}([a,b],X)$ *and for every* $a \le s \le t \le b$ *the following inequalities hold true:*

$$\begin{aligned} V_{\mathcal{R}}(f;s-,t+) &\ge \max\left\{V_{\mathcal{R}}(f^+;s-,t+), V_{\mathcal{R}}(f^-;s-,t+)\right\}, \\ V_{\mathcal{R}}(f;s+,t+) &\ge V_{\mathcal{R}}(f^+;s,t+), \\ V_{\mathcal{R}}(f;s-,t-) &\ge V_{\mathcal{R}}(f^-;s-,t). \end{aligned}$$

As in the classical case, the inclusion $AC_{\mathcal{R}}([a,b];X) \subseteq BV_{\mathcal{R}}([a,b];X)$ holds true, as stated in the next proposition:

Proposition 1.3.12. *A function* $f\colon [a,b] \to X$ *is* $\mathcal{R}$*–absolutely continuous if and only if it is of bounded* $\mathcal{R}$*–variation and the function* $t \mapsto V_{\mathcal{R}}(f;a,t)$ *is absolutely continuous. In this case it holds*

$$V_{\mathcal{R}}(f;s,t) = \int_s^t \mathcal{R}(\tau, \dot{f}(\tau))\,\mathrm{d}\tau, \qquad \text{for every } a \le s \le t \le b.$$

Like in the classical case, the $\mathcal{R}$–variation is pointwise weakly lower semicontinuous, as stated in the following lemma:

Lemma 1.3.13. *Let* $\{f_n\}_{n\in\mathbb{N}}$ *be a sequence of functions from* $[a,b]$ *to* X *such that* $f_n(t) \rightharpoonup f(t)$ *weakly for every* $t \in [a,b]$*. Then one has*

$$V_{\mathcal{R}}(f;s,t) \le \liminf_{n\to+\infty} V_{\mathcal{R}}(f_n;s,t), \qquad \text{for every } a \le s \le t \le b.$$

We finally state and prove a useful generalisation in $BV_{\mathcal{R}}([a,b];X)$ of the following classical result: a sequence of nondecreasing and continuous scalar functions pointwise converging to a continuous function (in a compact interval) actually converges uniformly.

Lemma 1.3.14. *Let* $\{f_n\}_{n\in\mathbb{N}} \subseteq BV_{\mathcal{R}}([a,b];X)$ *be a sequence of functions pointwise strongly converging to* $f \in BV_{\mathcal{R}}([a,b];X)$*. Assume that:*

- $V_{\mathcal{R}}(f_n;a,\cdot)$ *are continuous in* $[a,b]$ *for every* $n \in \mathbb{N}$ *and* $V_{\mathcal{R}}(f;a,\cdot)$ *is continuous in* $[a,b]$*;*
- $\lim\limits_{n\to+\infty} V_{\mathcal{R}}(f_n;a,t) = V_{\mathcal{R}}(f;a,t)$*, for every* $t \in [a,b]$*.*

Then the (strong) convergence of f_n *to* f *is actually uniform in* $[a,b]$*.*

Chapter 2

Vanishing inertia analysis

Contents

In this chapter we discuss existence, uniqueness, and the main properties of the solutions both of the dynamical system (1.0.1) and the quasistatic one (1.0.2). We finally perform the asymptotic analysis as $\varepsilon \to 0^+$ for a dynamic solution x^ε of (1.0.1).

The chapter is organised as follows: Section 2.1 is focused on the dynamic problem (1.0.1). We first present useful energy bounds on x^ε, which will be also exploited in Section 2.3 to deal with the quasistatic limit. We then prove an existence (and uniqueness) result, see Proposition 2.1.7 and Theorem 2.1.8.

In Section 2.2 we turn our attention to the quasistatic problem (1.0.2). In particular we introduce and analyse the notion of energetic solution, providing a temporal regularity result, see Proposition 2.2.8, and presenting some known cases in which uniquess is granted, see Lemmas 2.2.9 and 2.2.10.

Section 2.3 finally contains the main result of this first part of the thesis, regarding the vanishing inertia limit of dynamic solutions x^ε, see Theorem 2.3.9. To obtain it, we first employ the energy bounds we previously gained in order to deduce the existence of convergent subsequences (Theorem 2.3.1). We then characterise the limit function as an energetic solution to the quasistatic problem (1.0.2) by means of an asymptotic analysis of a suitable stabilty condition and of an energy balance fulfilled by the dynamic solution x^ε.

Also the contents of this chapter are contained in the work [38] in collaboration with Paolo Gidoni.

2.1 Existence of solutions for the dynamic problem

This section is devoted to the analysis of the dynamic problem (1.1.3) and to the proof of an existence result under the main assumptions (E1), (E3)–(E5) and (R4). Convexity, i.e. (E2), here is not needed. Condition (E7) will be also added to obtain uniqueness of differential solutions, see Theorem 2.1.8. Of course in this section the parameter $\varepsilon > 0$ is fixed; however, since some results we obtain here will be useful also in the rest of the chapter where ε is sent to 0, for the sake of brevity we prefer to assume that the initial data are uniformly bounded in ε. Namely we require there exists a positive constant $\Lambda > 0$

for which

$$|x_0^\varepsilon| \leq \Lambda, \qquad \text{and} \qquad |\varepsilon x_1^\varepsilon| \leq \Lambda, \qquad \text{for every } \varepsilon > 0. \tag{2.1.1}$$

Before starting the analysis we recall the following Grönwall-type estimate:

Lemma 2.1.1 (Grönwall inequality). *Let $f\colon [a,b] \to [0,+\infty)$ be a bounded measurable function such that*

$$f(t) \leq C + \int_a^t \omega(f(\tau))g(\tau)\,\mathrm{d}\tau, \qquad \textit{for every } t \in [a,b], \tag{2.1.2}$$

where $C > 0$ is a positive constant, $\omega\colon [0,+\infty) \to [0,+\infty)$ is a nondecreasing continuous function such that $\omega(x) > 0$ if $x > 0$, and $g \in L^1(a,b)$ is nonnegative.

Then it holds:

$$f(t) \leq \varphi^{-1}\left(\varphi(C) + \int_a^t g(\tau)\,\mathrm{d}\tau\right), \qquad \textit{for every } t \in [a,b],$$

where $\varphi(t) := \displaystyle\int_1^t \frac{1}{\omega(\tau)}\,\mathrm{d}\tau.$

Proof. We consider the auxiliary function $F(t) := \int_a^t \omega(f(\tau))g(\tau)\,\mathrm{d}\tau$. Since f is bounded, F is absolutely continuous in $[a,b]$ and $F(a) = 0$. Moreover by (2.1.2) we deduce:

$$\dot F(\tau) = \omega(f(\tau))g(\tau) \leq \omega(C + F(\tau))g(\tau), \qquad \text{for a.e. } \tau \in [a,b].$$

From the above inequality we thus infer for every $t \in [a,b]$:

$$\begin{aligned}\int_a^t g(\tau)\,\mathrm{d}\tau &\geq \int_a^t \frac{\dot F(\tau)}{\omega(C+F(\tau))}\,\mathrm{d}\tau = \int_C^{C+F(t)} \frac{1}{\omega(\tau)}\,\mathrm{d}\tau = \varphi(C+F(t)) - \varphi(C)\\ &\geq \varphi(f(t)) - \varphi(C),\end{aligned}$$

where in the last inequality we used again (2.1.2) and exploited the monotonicity of φ. Hence we conclude. □

For a reason which will be clear later, to develop all the arguments of this section we need to introduce a truncated version of the elastic energy $\mathcal{E}$. We argue as follows: for every $\rho \in (0,+\infty)$, let $\lambda^\rho\colon [0,+\infty) \to [0,\rho+1]$ be a C^∞, monotone increasing, concave function such that $\lambda^\rho(r) = r$ for $r \leq \rho$ and let us consider the truncated energies

$$\mathcal{E}^\rho(t,x) = \mathcal{E}\left(t, \sigma_\rho(x)\right), \quad \text{where } \sigma_\rho(x) := \frac{\lambda^\rho(|x|)x}{|x|}, \tag{2.1.3}$$

setting in the limit case $\mathcal{E}^{+\infty} \equiv \mathcal{E}$. Notice that σ_ρ is the identity on $\overline{\mathcal{B}_\rho^X}$ and that the Jacobian of σ_ρ at each point has (operator) norm less or equal than one.

We observe that the new functions $\mathcal{E}^\rho$ cannot be expressed any longer as function of $(t, \pi_Z(x))$. Yet they inherit many of the regularity properties of $\mathcal{E}$ and $\mathcal{E}_{sh}$. Indeed we observe that, by (E1) and (E3), the functions $\mathcal{E}^\rho$ and $D_x\mathcal{E}^\rho$ are continuous in $[0,+\infty) \times X$, while from (E1) and (E5) we get that $\frac{\partial}{\partial t}\mathcal{E}^\rho$ is a Caratheodory function. Moreover, by (E4) it holds

$$\left|\frac{\partial}{\partial t}\mathcal{E}^\rho(t,x)\right| \leq \omega(\mathcal{E}^\rho(t,x))\gamma(t), \qquad \text{for a.e. } t \in [0,+\infty) \text{ and for every } x \in X, \tag{2.1.4}$$

where ω and γ are the same of (E4) and in particular do not depend on ρ. Furthermore, by compactness and the properties of σ_ρ, if $\rho \in (0, +\infty)$ then for every $T > 0$ we get that $D_x\mathcal{E}^\rho$ is bounded on the whole $[0, T] \times X$, namely there exists a constant $C^T_\rho > 0$ such that

$$\sup_{(t,x)\in[0,T]\times X} |D_x\mathcal{E}^\rho(t, x)|_* \le C^T_\rho. \tag{2.1.5}$$

If in addition also (E7) holds, we deduce that for every $T > 0$ there exists a function $\widetilde{\varsigma}_\rho \in L^1(0, T)$ such that

$$|D_x\mathcal{E}^\rho(t, x_1) - D_x\mathcal{E}^\rho(t, x_2)|_* \le \widetilde{\varsigma}_\rho(t)\, |x_1 - x_2|\,, \tag{2.1.6}$$

for a.e. $t \in [0, T]$, and every $x_1, x_2 \in \overline{\mathcal{B}^X_\rho}$.

Let us thus introduce the approximated problems

$$\begin{cases} \varepsilon^2\mathbb{M}\ddot{x}^\varepsilon(t) + \varepsilon\mathbb{V}\dot{x}^\varepsilon(t) + \partial_v\mathcal{R}(t, \dot{x}^\varepsilon(t)) + D_x\mathcal{E}^\rho(t, x^\varepsilon(t)) \ni 0, \quad t > 0, \\ x^\varepsilon(0) = x^\varepsilon_0, \quad \dot{x}^\varepsilon(0) = x^\varepsilon_1, \end{cases} \tag{2.1.7}$$

where for the sake of clarity we do not stress the dependence on ρ. We recall that we are always assuming (E1), (E3)–(E5), (R4) and considering $\mathbb{M}, \mathbb{V}$ as in Section 1.1, in particular satisfying (1.1.1) and (1.1.2).

As a first step we present an alternative formulation of (2.1.7), based on the definition of subdifferential. We emphasize that the following results, where not otherwise explicitly stated, hold also for the original dynamic problem (1.1.3), corresponding to $\rho = +\infty$. In particular, the uniform estimates with respect to the initial data of Corollary 2.1.4 for the original dynamic problem will be employed later on.

Proposition 2.1.2. *For every $\varepsilon > 0$ and $\rho \in (0, +\infty]$, a function $x^\varepsilon \in \widetilde{W}^{2,1}(0, +\infty; X)$ is a differential solution of (2.1.7) if and only if initial data are attained and the following dynamic local stability condition and dynamic energy balance hold true:*

(LS^ε) *for a.e. time $t \in [0, +\infty)$ and for every $v \in X$*

$$\mathcal{R}(t, v) + \langle D_x\mathcal{E}^\rho(t, x^\varepsilon(t)) + \varepsilon^2\mathbb{M}\ddot{x}^\varepsilon(t) + \varepsilon\mathbb{V}\dot{x}^\varepsilon(t), v\rangle \ge 0;$$

(EB^ε) *for every $t \in [0, +\infty)$*

$$\begin{aligned} &\frac{\varepsilon^2}{2}|\dot{x}^\varepsilon(t)|^2_{\mathbb{M}} + \mathcal{E}^\rho(t, x^\varepsilon(t)) + \int_0^t \mathcal{R}(\tau, \dot{x}^\varepsilon(\tau))\,\mathrm{d}\tau + \varepsilon\int_0^t |\dot{x}^\varepsilon(\tau)|^2_{\mathbb{V}}\,\mathrm{d}\tau \\ &= \frac{\varepsilon^2}{2}|x^\varepsilon_1|^2_{\mathbb{M}} + \mathcal{E}^\rho(0, x^\varepsilon_0) + \int_0^t \frac{\partial}{\partial t}\mathcal{E}^\rho(\tau, x^\varepsilon(\tau))\,\mathrm{d}\tau. \end{aligned}$$

Proof. By definition of subdifferential we deduce that $x^\varepsilon \in \widetilde{W}^{2,1}(0, +\infty; X)$ is a differential solution of (2.1.7) if and only if initial data are attained and for a.e. $t \in [0, +\infty)$ and for every $\tilde{v} \in X$ it holds:

$$\begin{aligned} &\mathcal{R}(t, \tilde{v}) + \langle D_x\mathcal{E}^\rho(t, x^\varepsilon(t)) + \varepsilon^2\mathbb{M}\ddot{x}^\varepsilon(t) + \varepsilon\mathbb{V}\dot{x}^\varepsilon(t), \tilde{v}\rangle \\ &\ge \mathcal{R}(t, \dot{x}^\varepsilon(t)) + \langle D_x\mathcal{E}^\rho(t, x^\varepsilon(t)) + \varepsilon^2\mathbb{M}\ddot{x}^\varepsilon(t) + \varepsilon\mathbb{V}\dot{x}^\varepsilon(t), \dot{x}^\varepsilon(t)\rangle. \end{aligned} \tag{2.1.8}$$

We thus conclude if we show that (2.1.8) is equivalent to (LS^ε) and (EB^ε).

We first assume that (2.1.8) holds true. We fix $v \in X$ and we choose $\tilde{v} = nv$, with $n \in \mathbb{N}$; by means of the one homogeneity of $\mathcal{R}(t, \cdot)$ and letting $n \to +\infty$ we deduce the validity of (LS^ε). Choosing $\tilde{v} = 0$ and exploiting (LS^ε), we instead get the following local energy balance (also called power balance):

(LEB^ε) for a.e. time $t \in [0,+\infty)$ it holds

$$\mathcal{R}(t,\dot{x}^\varepsilon(t)) + \langle D_x\mathcal{E}^\rho(t,x^\varepsilon(t)) + \varepsilon^2\mathbb{M}\ddot{x}^\varepsilon(t) + \varepsilon\mathbb{V}\dot{x}^\varepsilon(t), \dot{x}^\varepsilon(t)\rangle = 0.$$

Integrating (LEB^ε) between 0 and t we finally get (EB^ε). Indeed we recall that, since x^ε is absolutely continuous, the map $\mathcal{E}^\rho(\cdot,x^\varepsilon(\cdot))$ is absolutely continuous too and $\frac{\partial}{\partial t}\mathcal{E}^\rho(\cdot,x^\varepsilon(\cdot))$ is summable in $[0,t]$.

We now assume that (LS^ε) and (EB^ε) hold true. By differentiating (EB^ε) we easily get (LEB^ε); combining it with (LS^ε) we thus obtain (2.1.8) and we conclude. □

Thanks to the energy balance (EB^ε) we are able to infer the following uniform bound of the involved energy along a differential solution. As we said before we assume that the initial data are uniformly bounded with respect to ε since this result will be useful also for the next sections.

Proposition 2.1.3. *Assume that the initial data satisfy (2.1.1) and let x^ε be a differential solution of (2.1.7). Then for every $T>0$ there exists a positive constant $C_T^\Lambda>0$, independent of $\varepsilon>0$ and of $\rho\in(0,+\infty]$, such that:*

$$\frac{\varepsilon^2}{2}|\dot{x}^\varepsilon(t)|^2_{\mathbb{M}} + \mathcal{E}^\rho(t,x^\varepsilon(t)) + \int_0^t \mathcal{R}(\tau,\dot{x}^\varepsilon(\tau))\,\mathrm{d}\tau + \varepsilon\int_0^t |\dot{x}^\varepsilon(\tau)|^2_{\mathbb{V}}\,\mathrm{d}\tau \le C_T^\Lambda, \quad \textit{for every } t\in[0,T]. \tag{2.1.9}$$

Proof. We denote by $\mathcal{F}^\varepsilon(t)$ the left-hand side of (2.1.9). By means of the energy balance (EB^ε), together with the estimates (1.1.1) and (2.1.4), we deduce that the following inequality holds true for every $t\in[0,+\infty)$:

$$\begin{aligned}\mathcal{F}^\varepsilon(t) &= \frac{\varepsilon^2}{2}|x_1^\varepsilon|^2_{\mathbb{M}} + \mathcal{E}^\rho(0,x_0^\varepsilon) + \int_0^t \frac{\partial}{\partial t}\mathcal{E}^\rho(\tau,x^\varepsilon(\tau))\,\mathrm{d}\tau \le \widetilde{C}^\Lambda + \int_0^t \omega(\mathcal{E}^\rho(\tau,x^\varepsilon(\tau)))\gamma(\tau)\,\mathrm{d}\tau\\ &\le \widetilde{C}^\Lambda + \int_0^t \omega(\mathcal{F}^\varepsilon(\tau))\gamma(\tau)\,\mathrm{d}\tau.\end{aligned}$$

We now conclude by means of Lemma 2.1.1. □

As a simple corollary we deduce:

Corollary 2.1.4. *Assume that the initial data satisfy (2.1.1) and let x^ε be a differential solution of (2.1.7). Then for every $T>0$ there exists a positive constant $C_T^\Lambda>0$, independent of $\varepsilon>0$ and of $\rho\in(0,+\infty]$, such that:*

(i) $\max\limits_{t\in[0,T]}|x^\varepsilon(t)| < C_T^\Lambda$;

(ii) $\displaystyle\int_0^T \mathcal{R}(\tau,\dot{x}^\varepsilon(\tau))\,\mathrm{d}\tau < C_T^\Lambda$;

(iii) $\max\limits_{t\in[0,T]}\varepsilon|\dot{x}^\varepsilon(t)|_{\mathbb{M}} < C_T^\Lambda$.

Proof. The bounds in (ii) and (iii) simply follow from (2.1.9). To get (i) we recall that x^ε belongs to $W^{2,1}([0,T];X)$, and hence by using (R2) we obtain:

$$|x^\varepsilon(t)| \le |x_0^\varepsilon| + |x^\varepsilon(t)-x_0^\varepsilon| \le \Lambda + \int_0^t |\dot{x}^\varepsilon(\tau)|\,\mathrm{d}\tau \le \Lambda + \frac{1}{\alpha_*}\int_0^t \mathcal{R}(\tau,\dot{x}^\varepsilon(\tau))\,\mathrm{d}\tau.$$

We indeed notice that $\dot{x}^\varepsilon(t)$ is forced to live in K for almost every time $t\in[0,T]$, otherwise $\partial_v\mathcal{R}(t,\dot{x}^\varepsilon(t))$ would be empty or alternatively (ii) could not be valid. Thus we conclude by exploiting (ii). □

We now introduce a slightly generalisation of the well known notion of normal cone adopted in convex analysis. For a convex subset $\mathcal{K} \subset X$ and a positive definite, symmetric linear operator $\mathbb{A}\colon X \to X^*$, we denote with $\mathcal{N}^{\mathbb{A}}_{\mathcal{K}}(x)$ the normal cone to the set $\mathcal{K}$ in the point $x \in \mathcal{K}$ with respect to the scalar product $\langle \mathbb{A}\cdot, \cdot\rangle : X \times X \to \mathbb{R}$, namely

$$\mathcal{N}^{\mathbb{A}}_{\mathcal{K}}(x) := \{v \in X \mid \langle \mathbb{A}v, \tilde{x} - x\rangle \le 0 \quad \text{for every } \tilde{x} \in \mathcal{K}\}. \tag{2.1.10}$$

If x instead does not belong to $\mathcal{K}$, for convention we set $\mathcal{N}^{\mathbb{A}}_{\mathcal{K}}(x) := \emptyset$. If finally the scalar product is the one endowed to the space, we simply write $\mathcal{N}_{\mathcal{K}}(x)$.

We also recall an existence and uniqueness result for the second order perturbed sweeping process, see [2].

Theorem 2.1.5. *Let E be an Euclidean space, $\mathcal{K} \subseteq E$ a non-empty closed convex subset, $F\colon [0,T] \times E \times \mathcal{K} \rightrightarrows E$ an upper semicontinuous set-valued map with non-empty compact convex values and satisfying for every $(t,\eta,\mu) \in [0,T] \times E \times \mathcal{K}$ the bound*

$$F(t,\eta,\mu) \subseteq \beta(1 + |\eta|_E + |\mu|_E)\mathcal{B}^E_1, \tag{2.1.11}$$

where $\mathcal{B}^E_1$ is the open unitary ball in E centered at the origin. Then, for every $(\eta_0,\eta_1) \in E \times \mathcal{K}$, the problem

$$\begin{cases} \ddot{\eta}(t) \in -\mathcal{N}_{\mathcal{K}}(\eta(t)) - F(t,\eta(t),\dot{\eta}(t)), \\ \eta(0) = \eta_0, \qquad \dot{\eta}(0) = \eta_1, \end{cases} \tag{2.1.12}$$

admits at least one differential solution, namely a function $\eta \in W^{2,1}(0,T;E)$ such that the differential inclusion holds true for a.e. $t \in [0,T]$ and the initial data are attained. Moreover we actually have $\eta \in W^{2,\infty}(0,T;E)$.

Theorem 2.1.6. *Under the assumptions of Theorem 2.1.5, suppose in addition that there exists an open set $\mathcal{U} \subseteq E$ such that*

- *(j) every solution η of (2.1.12) satisfies $\eta(t) \in \mathcal{U}$ for every $t \in [0,T]$;*
- *(jj) there exists a function $k \in L^1(0,T)$ such that*

$$\langle f_1 - f_2, \mu_1 - \mu_2\rangle_E \ge -k(t)(|\eta_1 - \eta_2|^2_E + |\mu_1 - \mu_2|^2_E), \tag{2.1.13}$$

  for a.e. $t \in [0,T]$ and for every $\eta_1,\eta_2 \in \mathcal{U}$, $\mu_1,\mu_2 \in \mathcal{K}$, $f_1 \in F(t,\eta_1,\mu_1)$, $f_2 \in F(t,\eta_2,\mu_2)$.

Then the solution of (2.1.12) provided by Theorem 2.1.5 is unique.

The existence Theorem 2.1.5 is a special case of [2, Theorem 3.1]. The uniqueness Theorem 2.1.5 is instead a straightforward corollary of [2, Theorem 3.3], noticing that once a uniform bound (j) on the solutions is available, it is sufficient to require (jj) in a region $\mathcal{U}$ where the solutions are contained.

In the next proposition we translate these results in our framework, obtaining existence (and uniqueness) of solutions to (2.1.7), but only for $\rho \in (0,+\infty)$.

Proposition 2.1.7. *Fix $\varepsilon > 0$ and $T > 0$. For every initial values $x^\varepsilon_0 \in X$ and $x^\varepsilon_1 \in K$, and for every $\rho \in (0,+\infty)$, there exists at least a differential solution $x^\varepsilon \in W^{2,\infty}(0,T;X)$ to problem (2.1.7).*

Moreover, let us assume that also (E7) holds. We take $\Lambda := \max\{|x^\varepsilon_0|, |\varepsilon x^\varepsilon_1|\}$ and consider C^Λ_T to be as in Corollary 2.1.4. Then for every $\rho \in (C^\Lambda_T,+\infty)$ the solution of (2.1.7) is unique.

Proof. We fix $T > 0$. Let us recall that by (R4) and the linearity of the subdifferential with respect to the sum of two convex functions, we can write

$$\partial_v \mathcal{R}(t,v) = \partial\chi_K(v) + \partial_v \mathcal{R}_{\text{finite}}(t,v), \quad \text{for every } (t,v) \in [0,T] \times X.$$

Hence we can rewrite problem (2.1.7) as

$$\begin{cases} \varepsilon^2 \mathbb{M}\ddot{x}^\varepsilon(t) \in -\partial\chi_K(\dot{x}^\varepsilon(t)) - \widetilde{F}(t, x^\varepsilon(t), \dot{x}^\varepsilon(t)), \\ x^\varepsilon(0) = x_0^\varepsilon, \quad \dot{x}^\varepsilon(0) = x_1^\varepsilon, \end{cases} \tag{2.1.14}$$

where

$$\widetilde{F}(t,u,v) := \varepsilon \mathbb{V} v + \partial_v \mathcal{R}_{\text{finite}}(t,v) + D_x \mathcal{E}^\rho(t,u). \tag{2.1.15}$$

We now observe that, by (V), (VI) in Corollary 1.1.4, the map $\partial_v \mathcal{R}_{\text{finite}} \colon [0,T] \times K \rightrightarrows X^*$ has compact, convex, non-empty values and it is upper semicontinuous. Thus trivially also the map $\widetilde{F} \colon [0,T] \times X \times K \rightrightarrows X^*$ has compact, convex, non-empty values and it is upper semicontinuous on the whole domain. Moreover, by (1.1.2), (VI) in Corollary 1.1.4 and (2.1.5), for every $\rho \in (0, +\infty)$ there exists a constant $\tilde{\beta}_\rho > 0$ such that

$$\widetilde{F}(t,u,v) \subseteq \tilde{\beta}_\rho (1 + |v|)\, \mathcal{B}_1^{X^*}, \qquad \text{for every } (t,u,v) \in [0,T] \times X \times K, \tag{2.1.16}$$

where $\mathcal{B}_1^{X^*}$ is the open unitary ball in X^* centered at the origin.

Let us now set $\mathbb{Q}^\varepsilon := \varepsilon^{-2}\mathbb{M}^{-1} \colon X^* \to X$, so that $\mathbb{Q}^\varepsilon$ is a positive definite, symmetric linear operator. Using also that K is a closed, convex cone, for every $\eta \in X^*$ we have

$$\begin{aligned} \partial\chi_K(\mathbb{Q}^\varepsilon \eta) &= \{\xi \in X^* \mid \chi_K(\mathbb{Q}^\varepsilon\eta) + \langle \xi, x\rangle \le \chi_K(\mathbb{Q}^\varepsilon \eta + x) \quad \text{for every } x \in X\} \\ &= \{\xi \in X^* \mid \chi_K(\mathbb{Q}^\varepsilon\eta) + \langle \xi, \mathbb{Q}^\varepsilon\zeta\rangle \le \chi_K(\mathbb{Q}^\varepsilon (\eta + \zeta)) \quad \text{for every } \zeta \in X^*\} \\ &= \{\xi \in X^* \mid \chi_{\mathbb{M}K}(\eta) + \langle \xi, \mathbb{Q}^\varepsilon\zeta\rangle \le \chi_{\mathbb{M}K}(\eta + \zeta) \quad \text{for every } \zeta \in X^*\} \\ &= \{\xi \in X^* \mid \chi_{\mathbb{M}K}(\eta) + \langle \xi, \mathbb{Q}^\varepsilon(\tilde\eta - \eta)\rangle \le \chi_{\mathbb{M}K}(\tilde\eta) \quad \text{for every } \tilde\eta \in X^*\} \\ &= \mathcal{N}^{\mathbb{Q}^\varepsilon}_{\mathbb{M}K}(\eta). \end{aligned}$$

In the third step we have used the fact that K is a cone to neglect the factor ε^2. The last step follows by observing that both sets are empty if $\eta \notin \mathbb{M}K$, since the inequality would fail for $\tilde\eta \in \mathbb{M}K$. On the other hand, if $\eta \in \mathbb{M}K$, the inequality is always true for $\tilde\eta \notin \mathbb{M}K$, while it is equivalent to $\langle \xi, \mathbb{Q}^\varepsilon(\tilde\eta - \eta)\rangle \le 0$ for $\tilde\eta \in \mathbb{M}K$.

Let us now introduce the Euclidean space E as the vector space X^* endowed with the scalar product $\langle \mathbb{Q}^\varepsilon \cdot, \cdot \rangle$ with $\mathbb{Q}^\varepsilon$ as above. By (1.1.1) we observe that

$$\frac{1}{\varepsilon\sqrt{M}} |\eta|_* \le |\eta|_E \le \frac{1}{\varepsilon\sqrt{m}} |\eta|_*, \quad \text{for every } \eta \in E. \tag{2.1.17}$$

Then, x^ε is a differential solution of (2.1.7) if and only if $\eta^\varepsilon := \varepsilon^2 \mathbb{M} x^\varepsilon$ is a differential solution of the following second order perturbed sweeping process on E:

$$\begin{cases} \ddot\eta^\varepsilon(t) \in -\mathcal{N}_{\mathbb{M}K}(\dot\eta^\varepsilon(t)) - F(t, \eta^\varepsilon(t), \dot\eta^\varepsilon(t)), \\ \eta^\varepsilon(0) = \varepsilon^2 \mathbb{M} x_0^\varepsilon, \quad \dot\eta^\varepsilon(0) = \varepsilon^2 \mathbb{M} x_1^\varepsilon, \end{cases} \tag{2.1.18}$$

where the function $F \colon [0,T] \times E \times \mathbb{M}K \rightrightarrows E$ is defined by

$$F(t,u,v) := \widetilde{F}(t, \mathbb{Q}^\varepsilon u, \mathbb{Q}^\varepsilon v).$$

We observe that, by (2.1.17) and the linearity of $\mathbb{Q}^\varepsilon$, we have that the map F has compact, convex, non-empty values and is upper semicontinuous on the whole domain with respect to the norm of E. Moreover, by (2.1.16) and (2.1.17), for every $\rho \in (0, +\infty)$ there exists a constant $\beta_\rho > 0$ such that

$$F(t,u,v) \subseteq \beta_\rho(1 + |v|_E)\mathcal{B}_1^E, \qquad \text{for every } (t,u,v) \in [0,T] \times E \times \mathbb{M}K, \tag{2.1.19}$$

where $\mathcal{B}_1^E$ is the unitary ball in E centered at the origin. We have therefore verified all the hypotheses of Theorem 2.1.5, hence proving the existence of a solution $\eta^\varepsilon \in W^{2,\infty}(0,T;E)$ of (2.1.18). Noticing that $x^\varepsilon = \mathbb{Q}^\varepsilon \eta^\varepsilon \in W^{2,\infty}(0,T;X)$, we complete the first part of the proof.

It remains to show that such a solution is unique. Therefore, let us now consider $\rho \in (C_T^\Lambda, +\infty)$ and assume (E7), with the consequence that also (2.1.6) holds.

Since to every solution η^ε of (2.1.18) corresponds a solution $x^\varepsilon = \mathbb{Q}^\varepsilon \eta^\varepsilon$ of (2.1.7), which by Corollary 2.1.4 is contained in the open ball $\mathcal{B}^X_{C_T^\Lambda}$, we deduce that every solution η^ε of (2.1.18) is contained in the set $\mathcal{U} := \varepsilon^2\mathbb{M}\mathcal{B}^X_{C_T^\Lambda}$, which is open also in the topology of E. Hence condition (j) of Theorem 2.1.6 is satisfied.

We then observe that the function $\widetilde{F}$ can be decomposed in two parts. The first part $\widetilde{F}^a(t,v) := \varepsilon\mathbb{V}v + \partial_v\mathcal{R}_{\text{finite}}(t,v)$, at each time t, is included in the subdifferential with respect to v of a convex function, namely $\widetilde{F}^a(t,v) \subseteq \partial_v[\varepsilon \langle \mathbb{V}v, v\rangle + \mathcal{R}_{\text{finite}}(t,v)]$. Hence by monotonicity of the subdifferential it holds:

$$\left\langle \widetilde{f}_1^a - \widetilde{f}_2^a, v_1 - v_2 \right\rangle \geq 0,$$

for every $t \in [0,T]$, $v_1, v_2 \in K$, $\widetilde{f}_1^a \in \widetilde{F}^a(t,v_1)$, $\widetilde{f}_2^a \in \widetilde{F}^a(t,v_2)$. Therefore, taking $\mu_1 = \varepsilon^2\mathbb{M}v_1$ and $\mu_2 = \varepsilon^2\mathbb{M}v_2$, we infer that

$$\left\langle \widetilde{f}_1^a - \widetilde{f}_2^a, \mu_1 - \mu_2 \right\rangle_E = \left\langle \widetilde{f}_1^a - \widetilde{f}_2^a, \mathbb{Q}^\varepsilon\mu_1 - \mathbb{Q}^\varepsilon\mu_2 \right\rangle = \left\langle \widetilde{f}_1^a - \widetilde{f}_2^a, v_1 - v_2 \right\rangle \geq 0, \tag{2.1.20}$$

for every $t \in [0,T]$, $\mu_1, \mu_2 \in \mathbb{M}K$, $\widetilde{f}_1^a \in \widetilde{F}^a(t,\mathbb{Q}^\varepsilon\mu_1)$, $\widetilde{f}_2^a \in \widetilde{F}^a(t,\mathbb{Q}^\varepsilon\mu_2)$.

Let us now consider the second part $\widetilde{F}^b(t,u) := D_x\mathcal{E}^\rho(t,u)$ of $\widetilde{F}$. By (2.1.6) there exists a function $\widetilde{\varsigma}_\rho \in L^1(0,T)$ such that

$$|\widetilde{F}^b(t,u_1) - \widetilde{F}^b(t,u_2)|_* \leq \widetilde{\varsigma}_\rho(t)\,|u_1 - u_2|\,,$$

for a.e. $t \in [0,T]$, and for every $u_1, u_2 \in \mathcal{B}^X_{C_T^\Lambda}$. As before, taking $\eta_1 = \varepsilon^2\mathbb{M}u_1$ and $\eta_2 = \varepsilon^2\mathbb{M}u_2$, we deduce that

$$\begin{aligned} |\widetilde{F}^b(t,\mathbb{Q}^\varepsilon\eta_1) - \widetilde{F}^b(t,\mathbb{Q}^\varepsilon\eta_2)|_E &\leq \frac{1}{\varepsilon\sqrt{m}}|\widetilde{F}^b(t,\mathbb{Q}^\varepsilon\eta_1) - \widetilde{F}^b(t,\mathbb{Q}^\varepsilon\eta_2)|_* \\ &\leq \frac{\widetilde{\varsigma}_\rho(t)}{\varepsilon\sqrt{m}}|\mathbb{Q}^\varepsilon\eta_1 - \mathbb{Q}^\varepsilon\eta_2| \leq \frac{\widetilde{\varsigma}_\rho(t)}{\varepsilon^2\sqrt{mM}}|\eta_1 - \eta_2|_E, \end{aligned} \tag{2.1.21}$$

which therefore holds for a.e. $t \in [0,T]$, and every $\eta_1, \eta_2 \in \mathcal{U}$.

Hence, by combining (2.1.20) and (2.1.21) we obtain

$$\begin{aligned} \langle f_1 - f_2, \mu_1 - \mu_2\rangle_E &\geq \left\langle \widetilde{F}^b(t,\mathbb{Q}^\varepsilon\eta_1) - \widetilde{F}^b(t,\mathbb{Q}^\varepsilon\eta_2), \mu_1 - \mu_2 \right\rangle_E \\ &\geq -|\widetilde{F}^b(t,\mathbb{Q}^\varepsilon\eta_1) - \widetilde{F}^b(t,\mathbb{Q}^\varepsilon\eta_2)|_E\,|\mu_1 - \mu_2|_E \\ &\geq -\frac{\widetilde{\varsigma}_\rho(t)}{2\varepsilon^2\sqrt{mM}}(|\eta_1 - \eta_2|_E^2 + |\mu_1 - \mu_2|_E^2), \end{aligned}$$

for a.e. $t \in [0,T]$, and for every $\eta_1, \eta_2 \in \mathcal{U}$, $\mu_1, \mu_2 \in \mathbb{M}K$, $f_1 \in F(t,\eta_1,\mu_1)$, $f_2 \in F(t,\eta_2,\mu_2)$.

Hence also condition (jj) of Theorem 2.1.6 is satisfied, yielding the uniqueness result of the proposition. □

The main result of this section, concerning the original problem (1.1.3), is a straightforward corollary of Proposition 2.1.7.

Theorem 2.1.8. *Fix $\varepsilon > 0$, let $\mathbb{M}, \mathbb{V}$ be as in Section 1.1, and assume that $\mathcal{R}$ satisfies (R4) and $\mathcal{E}(t,x) = \mathcal{E}_{sh}(t, \pi_Z(x))$ satisfies (E1), (E3)–(E5). Then for every initial values $x_0^\varepsilon \in X$ and $x_1^\varepsilon \in K$ there exists at least a differential solution $x^\varepsilon \in \widetilde{W}^{2,\infty}(0,+\infty;X)$ to problem (1.1.3).*

If in addition (E7) holds, then such a solution is unique.

Proof. Let us set $\Lambda := \max\{|x_0^\varepsilon|, |\varepsilon x_1^\varepsilon|\}$ and fix $T > 0$. Taken $C_T^\Lambda > 0$ given by Corollary 2.1.4, we fix $\rho \in (C_T^\Lambda, +\infty)$.

We observe that by definition of the truncated energy $\mathcal{E}^\rho$ the two problems (1.1.3) and (2.1.7) coincide in the region $(t, x^\varepsilon, \dot{x}^\varepsilon) \in [0,T] \times \mathcal{B}_\rho^X \times K$; moreover, by Corollary 2.1.4, the solutions of both the initial value problems are contained in that region. Hence, the solutions of (1.1.3) and (2.1.7) coincide. Since by Proposition 2.1.7 problem (2.1.7) admits at least one differential solution x^ε, which additionally satisfies $x^\varepsilon \in W^{2,\infty}(0,T;X)$ and which is unique if also (E7) is satisfied, so does the original dynamic problem (1.1.3). □

2.2 Properties of energetic solutions

In this section we discuss the quasistatic problem (1.1.5) and in particular the notion of *energetic solution*, which we recall here

Definition 2.2.1. *We say that $x \in \widetilde{BV}_{\mathcal{R}}([0,+\infty);X)$ is an* energetic solution *for the quasistatic problem (1.1.5) if the initial position is attained and the following global stability condition and weak energy balance hold true:*

$$\text{(GS)}\ \ \mathcal{E}(t,x(t)) \le \mathcal{E}(t,v) + \mathcal{R}(t, v - x(t)), \quad \text{for every } v \in X \text{ and for every } t \in [0,+\infty);$$

$$\text{(WEB)}\ \ \mathcal{E}(t,x(t)) + V_{\mathcal{R}}(x;0,t) = \mathcal{E}(0,x_0) + \int_0^t \frac{\partial}{\partial t}\mathcal{E}(\tau, x(\tau))\,\mathrm{d}\tau, \quad \text{for every } t \in [0,+\infty).$$

Hence all the assumption of the quasistatic problem (1.1.5), namely (E1)–(E5) and (R4), hold here. The main purpose of this section is to prove temporal regularity of the energetic solutions to (1.1.5), which we obtain in Proposition 2.2.8. Such regularity will allow us to deduce the equivalence between the two notions of energetic and differential solutions. We also present some well known cases in which uniqueness for energetic (and differential) solutions holds; we point out that for a general elastic energy, as the one we consider here, the question of uniqueness is still open.

To start, we notice that, in the quasistatic setting, it is possible to provide a characterisation of differential solutions analogous to that of Proposition 2.1.2 for the dynamic problem. In fact, convexity leads to a better result, which also clarifies Definition 2.2.1 of energetic solutions.

Proposition 2.2.2. *A function $x \in \widetilde{AC}([0,+\infty);X)$ is a differential solution of the quasistatic problem (1.1.5) if and only if initial position is attained and one of the following two equivalent conditions is satisfied:*

$$\text{(1)} \begin{cases} \text{(LS)} & \mathcal{R}(t,v) + \langle D_x\mathcal{E}(t,x(t)), v\rangle \ge 0, \quad \text{for every } (t,v) \in [0,+\infty) \times X; \\ \text{(LEB)} & \mathcal{R}(t,\dot{x}(t)) + \langle D_x\mathcal{E}(t,x(t)), \dot{x}(t)\rangle = 0, \quad \text{for a.e. } t \in [0,+\infty); \end{cases}$$

$$\text{(2)} \begin{cases} \text{(GS)} & \mathcal{E}(t,x(t)) \le \mathcal{E}(t,v) + \mathcal{R}(t, v - x(t)), \quad \text{for every } (t,v) \in [0,+\infty) \times X; \\ \text{(EB)} & \mathcal{E}(t,x(t)) + \int_0^t R(\tau, \dot{x}(\tau))\mathrm{d}\tau = \mathcal{E}(0,x_0) + \int_0^t \frac{\partial}{\partial t}\mathcal{E}(\tau, x(\tau))\mathrm{d}\tau, \quad \text{for all } t \ge 0. \end{cases}$$

Proof. The fact that $x \in \widetilde{AC}([0,+\infty);X)$ is a differential solution of (1.1.5) if and only if initial position is attained and (1) is fulfilled follows by arguing as in the proof of Proposition 2.1.2. Notice that the passage from a.e. to every time is granted by continuity. We only need to show that (1) and (2) are equivalent; first of all we notice that (LEB) is equivalent to (EB) since we can obtain the first one by differentiating the second one. The fact that (GS) implies (LS) follows since $\mathcal{R}(t,\cdot)$ is one homogeneous, while the contrary follows since the function $v \mapsto \mathcal{E}(t,x(t)+v)$ is convex by (E2). □

Remark 2.2.3. As the reader can check from the proof, convexity assumption (E2) is needed only to deduce the global stability (GS) from the local one (LS).

Remark 2.2.4. We point out that, by (EB), any differential solution of (1.1.5) is actually $\mathcal{R}$–absolutely continuous. In particular, due to Proposition 1.3.12, it is an energetic solution.

2.2.1 Temporal regularity

We now pass to the main object of this section, namely the temporal regularity of energetic solutions. The argument follows the by now consolidated ideas of [59], [65], and [67]; the first step exploits uniform convexity to improve the estimate furnished by the global stability condition (GS). However, since in our setting uniform convexity holds only for the restricted energy $\mathcal{E}_{sh}$, we need to introduce also the notion of restricted dissipation potential from [36].

Given any functional $\Phi\colon X \to [0,+\infty]$ we define its (shape-)restricted version $\Phi_{sh}\colon Z \to [0,+\infty]$ in the following way:

$$\Phi_{sh}(z) := \inf_{x \in \pi_Z^{-1}(\{z\})} \Phi(x). \tag{2.2.1}$$

The following properties are a trivial byproduct of the definition of Φ_{sh}:

- if $\Phi^1 \le \Phi^2$ on X, then $\Phi^1_{sh} \le \Phi^2_{sh}$ on Z;
- $\Phi_{sh}(\pi_Z(x)) \le \Phi(x)$ for every $x \in X$;
- if Φ is positively homogeneous of degree one, then Φ_{sh} is positively homogeneous of degree one.

We also observe that condition (R5) gives an upper bound on the restricted dissipation potential:

Lemma 2.2.5. *Suppose in addition that $\mathcal{R}$ satisfies (R5). If $(t,z) \in [0,T]\times Z$ is such that $\mathcal{R}_{sh}(t,z) < +\infty$, then*

$$\mathcal{R}_{sh}(t,z) \le \alpha^* C_K \left|z\right|_Z, \tag{2.2.2}$$

with α^* and C_K as in (R2) and (R5), respectively.

Proof. Since $\mathcal{R}_{sh}(t,z) < +\infty$, there exists $\tilde{x} \in K$ such that $\pi_Z(\tilde{x}) = z$. Thus, by (R5) it is possible to select this $\tilde{x}$ such that $|\tilde{x}| \le C_K \left|z\right|_Z$. Hence, recalling Corollary 1.1.4, we have

$$\mathcal{R}_{sh}(t,z) \le \mathcal{R}(t,\tilde{x}) \le \alpha^* \left|\tilde{x}\right| \le \alpha^* C_K \left|z\right|_Z,$$

and we conclude. □

We now prove that the global stability condition (GS) is actually equivalent to an enhanced version of stability.

Lemma 2.2.6 (**Improved Stability**). *Fix* $t \in [0, +\infty)$. *If* $x^* \in X$ *satisfies*

$$\mathcal{E}(t, x^*) \le \mathcal{E}(t, x) + \mathcal{R}(t, x - x^*), \quad \textit{for every } x \in X, \tag{2.2.3}$$

then also the following stronger version of stability holds true:

$$\mathcal{E}(t, x^*) + \frac{\mu}{2}|\pi_Z(x^*) - \pi_Z(x)|_Z^2 \le \mathcal{E}(t, x) + \mathcal{R}_{sh}(t, \pi_Z(x) - \pi_Z(x^*)), \quad \textit{for every } x \in X. \tag{2.2.4}$$

Proof. From the definition of restricted dissipation potential (2.2.1) and recalling that $\mathcal{E}(t, \cdot) = \mathcal{E}_{sh}(t, \pi_Z(\cdot))$, we deduce that (2.2.3) implies:

$$\mathcal{E}(t, x^*) \le \mathcal{E}(t, x) + \mathcal{R}_{sh}(t, \pi_Z(x) - \pi_Z(x^*)), \quad \text{for every } x \in X. \tag{2.2.5}$$

Furthermore, by means of (E2) we know that for every $x_1, x_2 \in X$ and for every $\theta \in (0, 1)$ it holds:

$$\mathcal{E}(t, \theta x_1 + (1-\theta)x_2) \le \theta\mathcal{E}(t, x_1) + (1-\theta)\mathcal{E}(t, x_2) - \frac{\mu}{2}\theta(1-\theta)|\pi_Z(x_1) - \pi_Z(x_2)|_Z^2. \tag{2.2.6}$$

We now fix $x \in X$ and we choose $\theta x + (1-\theta)x^*$ as competitor for x^* in (2.2.5); by using the one-homogeneity of $\mathcal{R}_{sh}(t, \cdot)$, the linearity of π_Z, and (2.2.6), we get:

$$\begin{aligned}\mathcal{E}(t, x^*) &\le \mathcal{E}(t, \theta x + (1-\theta)x^*) + \mathcal{R}_{sh}(t, \theta(\pi_Z(x) - \pi_Z(x^*))) \\ &\le \theta\mathcal{E}(t, x) + (1-\theta)\mathcal{E}(t, x^*) - \frac{\mu}{2}\theta(1-\theta)\,|\pi_Z(x) - \pi_Z(x^*)|_Z^2 \\ &\quad + \theta\mathcal{R}_{sh}(t, \pi_Z(x) - \pi_Z(x^*)).\end{aligned}$$

By subtracting $\mathcal{E}(t, x^*)$ from both sides and dividing by θ we hence obtain:

$$0 \le \mathcal{E}(t, x) - \mathcal{E}(t, x^*) - \frac{\mu}{2}(1-\theta)\,|\pi_Z(x) - \pi_Z(x^*)|_Z^2 + \mathcal{R}_{sh}(t, \pi_Z(x) - \pi_Z(x^*)).$$

We conclude letting $\theta \searrow 0$. □

Next lemma will be used in the proof of Proposition 2.2.8.

Lemma 2.2.7. *Let* $(V, \|\cdot\|)$ *be a normed space and let* $f\colon [a, b] \to V$ *be a bounded measurable function such that:*

$$\|f(t) - f(s)\|^2 \le \int_s^t \|f(t) - f(\tau)\|g(\tau)d\tau + \|f(t) - f(s)\| \int_s^t h(\tau)d\tau, \quad \textit{for every } a \le s \le t \le b, \tag{2.2.7}$$

for some nonnegative $g, h \in L^1(a, b)$. *Then it holds:*

$$\|f(t) - f(s)\| \le \int_s^t \big(g(\tau) + h(\tau)\big)d\tau, \quad \textit{for every } a \le s \le t \le b.$$

Proof. Fix $t \in [a, b]$. For $s \in [a, t]$ we define the functions $\beta_t(s) := \|f(t) - f(s)\|$ and $\overline{\beta_t}(s) := \sup\limits_{\theta \in [s,t]} \beta_t(\theta)$, where the latter is finite since f is bounded.

We now fix $s \in [a, t]$ and, by using (2.2.7), for every $\theta \in [s, t]$ we hence obtain:

$$\begin{aligned}\beta_t(\theta)^2 &\le \int_\theta^t \beta_t(\tau)g(\tau)\,\mathrm{d}\tau + \beta_t(\theta)\int_\theta^t h(\tau)\,\mathrm{d}\tau \\ &\le \overline{\beta_t}(s)\int_s^t \big(g(\tau) + h(\tau)\big)d\tau,\end{aligned}$$

which implies

$$\overline{\beta_t}(s)^2 \le \overline{\beta_t}(s)\int_s^t \big(g(\tau) + h(\tau)\big)d\tau, \quad \text{for every } a \le s \le t \le b.$$

Since $\beta_t(s) \le \overline{\beta_t}(s)$, we conclude. □

We are now in a position to state and prove the main result of this section:

Proposition 2.2.8. *Assume that $\mathcal{R}$ satisfies (R4) and $\mathcal{E}(t,x) = \mathcal{E}_{sh}(t,\pi_Z(x))$ satisfies (E1)–(E5). Then any energetic solution x for (1.1.5) is continuous.*

Suppose in addition that (R5) holds or, alternatively, that $\mathcal{R}$ does not depend on time. Then x is $\mathcal{R}$–absolutely continuous and, therefore, a differential solution of (1.1.5).

Proof. We fix $T > 0$ and $0 \le s \le t \le T$; since x satisfies (GS) we can pick $x(t)$ as a competitor for $x(s)$ in (2.2.4), getting:

$$\begin{aligned}
&\frac{\mu}{2}|\pi_Z(x(t)) - \pi_Z(x(s))|_Z^2\\
&\le \mathcal{E}(s,x(t)) + \mathcal{R}_{sh}(s,\pi_Z(x(t)) - \pi_Z(x(s))) - \mathcal{E}(s,x(s))\\
&= \mathcal{E}(s,x(t)) - \mathcal{E}(t,x(t)) + \mathcal{E}(t,x(t)) - \mathcal{E}(s,x(s)) + \mathcal{R}_{sh}(s,\pi_Z(x(t)) - \pi_Z(x(s)))\\
&= \int_s^t \Big(\frac{\partial}{\partial t}\mathcal{E}(\tau,x(\tau)) - \frac{\partial}{\partial t}\mathcal{E}(\tau,x(t))\Big)\,\mathrm{d}\tau + \mathcal{R}_{sh}(s,\pi_Z(x(t)) - \pi_Z(x(s))) - V_{\mathcal{R}}(x;s,t),
\end{aligned}$$

where for the last equality we exploited (WEB).

We recall that x is bounded in $[0,T]$ since it belongs to $BV_{\mathcal{R}}([0,T];X)$; thus there exists $R > 0$ such that $|x(t)| \le R$ for every $t \in [0,T]$. Hence we can use (E5) and continue the above inequality:

$$\begin{aligned}
&\frac{\mu}{2}|\pi_Z(x(t)) - \pi_Z(x(s))|_Z^2\\
&\le \int_s^t |\pi_Z(x(t)) - \pi_Z(x(\tau))|_Z \eta_R(\tau)\,\mathrm{d}\tau + \mathcal{R}_{sh}(s,\pi_Z(x(t)) - \pi_Z(x(s))) - V_{\mathcal{R}}(x;s,t).
\end{aligned} \tag{2.2.8}$$

To estimate the term outside the integral we exploit (R2) and (R3), getting:

$$\begin{aligned}
V_{\mathcal{R}}(x;s,t) &\ge V_{\mathcal{R}(s)}(x;s,t) - V(x;s,t)\int_s^t \rho(\tau)\,\mathrm{d}\tau\\
&\ge \left(1 - \frac{1}{\alpha_*}\int_s^t \rho(\tau)\,\mathrm{d}\tau\right) V_{\mathcal{R}(s)}(x;s,t).
\end{aligned}$$

The above inequality finally implies:

$$V_{\mathcal{R}}(x;s,t) \ge \left(1 - \frac{1}{\alpha_*}\int_s^t \rho(\tau)\,\mathrm{d}\tau\right) \mathcal{R}_{sh}(s,\pi_Z(x(t)) - \pi_Z(x(s))). \tag{2.2.9}$$

Indeed, if the term within parentheses is negative the inequality is trivial; otherwise we observe that $V_{\mathcal{R}(s)}(x;s,t) \ge \mathcal{R}(s,x(t) - x(s)) \ge \mathcal{R}_{sh}(s,\pi_Z(x(t)) - \pi_Z(x(s)))$.

By plugging (2.2.9) into (2.2.8) we thus obtain

$$\begin{aligned}
&\frac{\mu}{2}|\pi_Z(x(t)) - \pi_Z(x(s))|_Z^2\\
&\le \int_s^t |\pi_Z(x(t)) - \pi_Z(x(\tau))|_Z \eta_R(\tau)\,\mathrm{d}\tau + \frac{1}{\alpha_*}\Big(\int_s^t \rho(\tau)\,\mathrm{d}\tau\Big)\mathcal{R}_{sh}(s,\pi_Z(x(t)) - \pi_Z(x(s))).
\end{aligned} \tag{2.2.10}$$

Since x is bounded in $[0,T]$, we deduce that $|\pi_Z(x(t)) - \pi_Z(x(\tau))|_Z$ is also bounded by a constant independent of t and τ. Moreover, by (II) in Corollary 1.1.4, we have

$$\begin{aligned}
\mathcal{R}_{sh}(s,\pi_Z(x(t)) - \pi_Z(x(s))) &\le \mathcal{R}(s,x(t) - x(s)) \le V_{\mathcal{R}(s)}(x;s,t)\\
&\le \frac{\alpha^*}{\alpha_*}V_{\mathcal{R}}(x;s,t) \le \frac{\alpha^*}{\alpha_*}V_{\mathcal{R}}(x;0,T).
\end{aligned}$$

Hence, from estimate (2.2.10) we infer:

$$|\pi_Z(x(t))-\pi_Z(x(s))|_Z\le C\left(\int_s^t (\eta_R(\tau)+\rho(\tau))\,\mathrm{d}\tau\right)^{\frac12},$$

for some constant $C>0$, and thus $\pi_Z\circ x$ is continuous from $[0,T]$ to Z. Since $\mathcal{E}(t,x(t))=\mathcal{E}_{sh}(t,\pi_Z(x(t)))$ and $\mathcal{E}_{sh}$ is continuous in $[0,T]\times Z$ by (E1) and (E3), we easily deduce that $t\mapsto\mathcal{E}(t,x(t))$ is continuous too. Thus by (WEB) we obtain that the $\mathcal{R}$–variation of x is continuous as a function of $t\in[0,T]$; by employing (c) in Proposition 1.3.11 together with (R2), we finally obtain that x itself is continuous too.

Let us now prove the $\mathcal{R}$–absolute continuity of x under the stronger assumptions (R5) or $\mathcal{R}$ autonomous. The first step is to show that both the alternative assumptions imply

$$|\pi_Z(x(t))-\pi_Z(x(s))|_Z\le C\int_s^t \big(\eta_R(\tau)+\rho(\tau)\big)\,\mathrm{d}\tau,\qquad\text{for every } 0\le s\le t\le T,\tag{2.2.11}$$

for some constant $C>0$. With this aim we notice that, in the case where $\mathcal{R}$ does not depend on time, the term outside the integral in (2.2.8) is less or equal than zero, since in this case trivially it holds

$$\mathcal{R}_{sh}(\pi_Z(x(t))-\pi_Z(x(s)))\le\mathcal{R}(x(t)-x(s))\le V_{\mathcal{R}}(x;s,t).$$

Thus (2.2.11) follows, actually with only η_R inside the integral, from Lemma 2.2.7 applied to this improved version of (2.2.8).

If instead $\mathcal{R}$ depends on time, but satisfies (R5), we can apply Lemma 2.2.5 to the rightmost term of (2.2.10) and then apply directly Lemma 2.2.7 to obtain (2.2.11).

Now the we have obtained (2.2.11) in both the alternative cases, the second step is to deduce $\mathcal{R}$–absolute continuity. Firstly, we deduce from (2.2.11) that the function $\pi_Z\circ x$ is absolutely continuous from $[0,T]$ into Z. We now prove that $t\mapsto\mathcal{E}(t,x(t))$ is an absolutely continuous function. With this aim we fix $0\le s\le t\le T$ and we estimate:

$$\begin{aligned}|\mathcal{E}(t,x(t))-\mathcal{E}(s,x(s))|&\le|\mathcal{E}(t,x(t))-\mathcal{E}(t,x(s))|+|\mathcal{E}(t,x(s))-\mathcal{E}(s,x(s))|\\&\le C_R|\pi_Z(x(t))-\pi_Z(x(s))|_Z+\int_s^t\left|\frac{\partial}{\partial t}\mathcal{E}(\tau,x(s))\right|\mathrm{d}\tau\\&\le C_R|\pi_Z(x(t))-\pi_Z(x(s))|_Z+\int_s^t\omega(\mathcal{E}(\tau,x(s)))\gamma(\tau)\,\mathrm{d}\tau.\end{aligned}$$

The second term on the right-hand side have been estimated using (E4); instead for the first term we have used the fact that x is bounded by some $R>0$ and, by (E3) and compactness, $\mathcal{E}_{sh}(t,\cdot)$ is Lipschitz continuous on $\overline{\mathcal{B}^Z_R}$ with some constant C_R, which can be taken uniformly in $t\in[0,T]$. Moreover, since $\mathcal{E}$ is bounded on $[0,T]\times\overline{\mathcal{B}^X_R}$ by continuity, from the above inequality we deduce that for every $0\le s\le t\le T$:

$$|\mathcal{E}(t,x(t))-\mathcal{E}(s,x(s))|\le C_R|\pi_Z(x(t))-\pi_Z(x(s))|_Z+\omega(M_R)\int_s^t\gamma(\tau)\,\mathrm{d}\tau.$$

Thus we proved that $t\mapsto\mathcal{E}(t,x(t))$ is absolutely continuous. We now conclude since by using (WEB) we have:

$$V_{\mathcal{R}}(x;s,t)=\mathcal{E}(s,x(s))-\mathcal{E}(t,x(t))+\int_s^t\frac{\partial}{\partial t}\mathcal{E}(\tau,x(\tau))\,\mathrm{d}\tau,\qquad\text{for every } 0\le s\le t\le T,$$

and thus, by using Proposition 1.3.12, x is $\mathcal{R}$–absolutely continuous since $\frac{\partial}{\partial t}\mathcal{E}(\cdot,x(\cdot))\in L^1(0,T)$ thanks to (E4). □

2.2.2 Uniqueness

We conclude this section by listing some of the known important cases in which the quasistatic problem (1.1.5) admits at most one solution. In the general framework the issue of uniqueness is not completely clear yet. We first discuss the case $\dim Z = \dim X$, corresponding to a coercive energy $\mathcal{E}$.

Lemma 2.2.9. *Assume that* $\dim Z = \dim X$*,* $\mathcal{R}$ *satisfies* (R4) *and* $\mathcal{E}(t,x) = \mathcal{E}_{sh}(t,\pi_Z(x))$ *satisfies* (E1)–(E5)*. Then each of the following additional assumptions is a sufficient condition for uniqueness of energetic solutions to* (1.1.5)*:*

(U1) *$\mathcal{R}$ does not depend on time and $\mathcal{E}_{sh}$ belongs to $\mathcal{C}^3([0,+\infty)\times Z)$;*

(U2) *$\mathcal{R}$ does not depend on time, $\mathcal{E}_{sh}(t,z) = \mathcal{V}(z) - \langle g(t), z\rangle$ with $\mathcal{V}$ strictly convex, $g \in \widetilde{AC}([0,+\infty);Z^*)$, and the stable sets*

$$\mathcal{S}(t) = \{z \in Z \mid \mathcal{E}_{sh}(t,z) \le \mathcal{E}_{sh}(t,w) + \mathcal{R}(w-z) \text{ for every } w \in Z\},$$

are convex for every $t \in [0,+\infty)$;

(U3) *$K = X$ and $\mathcal{E}_{sh}$ satisfies (QE) with $\ell_{sh} \in \widetilde{W}^{1,\infty}(0,+\infty;Z)$.*

Proof. The case when $\mathcal{R}$ does not depend on time is well studied; the proof of uniqueness under (U1) or (U2), and several discussions on their applicability, can be found for instance in [59, Theorems 4.1 and 4.2], or [65, Section 3.4.4], or [66, Theorems 6.5 and 7.4]. Case (U3) has been proved in [39, Theorem 4.7]. □

The locomotion case $\dim Z < \dim X$ has been deeply analysed in [36] in the case of quadratic energies; in particular we mention Theorem 4.3 for the uniqueness result, and Example 3.2 to illustrate the necessity of condition $(*)$ below. We present here a generalized result applying the very same argument.

Lemma 2.2.10. *Assume that* $\mathcal{R}$ *satisfies* (R4) *and* $\mathcal{E}(t,x) = \mathcal{E}_{sh}(t,\pi_Z(x))$ *satisfies* (E1)–(E5)*. Suppose in addition that at least one of* (U1)*,* (U2) *or* (U3) *holds, and that for almost every* $t \in [0,+\infty)$ *we have*

$(*)$ *for every $z \in Z$ with $\mathcal{R}_{sh}(t,z) < +\infty$, there exists a unique $x \in X$ such that $\pi_Z(x) = z$ and*

$$\mathcal{R}_{sh}(t,z) = \mathcal{R}(t,x) < \mathcal{R}(t,v), \qquad \text{for every } v \neq x \text{ such that } \pi_Z(v) = z.$$

Then the differential solution to (1.1.5) *is unique. In particular, since in each case we can apply Proposition 2.2.8, uniqueness holds true also for energetic solutions.*

Proof. It is well known that $x(t)$ is a differential solution of (1.1.5) if and only if it satisfies the initial condition and the *variational inequality*

$$\langle D_x\mathcal{E}(t,x(t)), v - \dot{x}(t)\rangle + \mathcal{R}(t,v) - \mathcal{R}(t,\dot{x}(t)) \ge 0, \quad \text{for every } v \in X \text{ and a.e. } t \in [0,T]. \tag{2.2.12}$$

Writing $z(t) := \pi_Z(x(t))$, inequality (2.2.12) can be equivalently split in the two following conditions, which must hold for almost every $t \in [0,T]$:

$$\mathcal{R}_{sh}(t,\dot{z}(t)) = \mathcal{R}(t,\dot{x}(t)) \le \mathcal{R}(t,v), \text{ for every } v \in X \text{ such that } \pi_Z(v) = \dot{z}(t); \tag{2.2.13}$$

$$\langle D_z\mathcal{E}_{sh}(t,z(t)), w - \dot{z}(t)\rangle_Z + \mathcal{R}_{sh}(t,w) - \mathcal{R}_{sh}(t,\dot{z}(t)) \ge 0, \text{ for every } w \in Z. \tag{2.2.14}$$

Adopting the same argument of [36, Lemmata 2.1 and 4.1], it can be observed that the functional $\mathcal{R}_{sh}$, defined according to (2.2.1), inherits the regularity properties (I) and (III)

of Corollary 1.1.4, with also (II) if $K = X$. These, combined with the one of (U1), (U2) or (U3) which is holding, allows to apply the results mentioned in the proof of the previous lemma, to obtain the uniqueness of a solution $z(t)$ of (2.2.14). Hence, if two differential solutions x_1, x_2 of (1.1.5) exist, they must satisfy $\pi_Z(\dot{x}_1(t)) = \pi_Z(\dot{x}_2(t)) = \dot{z}(t)$ almost everwhere. This, combined with (2.2.13), implies that $\mathcal{R}(t, \dot{x}_1(t)) = \mathcal{R}(t, \dot{x}_2(t))$ a.e., in contradiction with $(*)$, since $\mathcal{R}(t, \dot{x}(t)) < +\infty$ a.e. along solutions. Therefore the differential solution of (1.1.5) is unique. □

2.3 Quasistatic limit

This last section is devoted to the proof of the main result of Part I, namely we discuss the convergence as ε goes to 0 of a differential solutions x^ε of the dynamic problems (1.1.3), given by Theorem 2.1.8, to a (energetic or differential) solution of the quasistatic problem (1.1.5).

Hence in this section we are assuming all the basic hypotheses of the dynamic and quasistatic problems: X is a finite dimensional Banach space, $\mathbb{M}$ and $\mathbb{V}$ are as in Section 1.1, $\mathcal{E}(t,x) = \mathcal{E}_{sh}(t, \pi_Z(x))$ satisfies (E1)–(E5) and $\mathcal{R}$ satisfies (R4). We however point out that (E2), i.e. convexity, will not be necessary for the first part of the vanishing inertia analysis, as stressed in Remark 2.3.4. Moreover we assume that the initial velocity x_1^ε satisfy the admissibility condition (1.1.4).

We proceed as follows. Firstly, we use the uniform bound on the energy of x^ε, obtained in Proposition 2.1.3, to deduce the existence of a convergent subsequence by means of a compactness argument involving Helly's Selection Theorem. Then, we prove that the limit obtained from the subsequence is actually an energetic (and thus, from Proposition 2.2.8, a differential) solution of the quasistatic problem (1.1.5). The main results are collected in Theorems 2.3.8 and 2.3.9.

Theorem 2.3.1. *Assume that x_0^ε and $\varepsilon x_1^\varepsilon$ are uniformly bounded, namely* (2.1.1) *is satisfied. Then there exists a subsequence $\varepsilon_n \searrow 0$ and a function $x \in \widetilde{BV}_{\mathcal{R}}([0,+\infty);X)$ such that:*

(a) $\lim\limits_{n\to+\infty} x^{\varepsilon_n}(t) = x(t)$, *for every* $t \in [0,+\infty)$*;*

(b) $V_{\mathcal{R}}(x; s, t) \le \liminf\limits_{n\to+\infty} \int_s^t \mathcal{R}(\tau, \dot{x}^{\varepsilon_n}(\tau))\,\mathrm{d}\tau$, *for every* $0 \le s \le t$*;*

(c) $\lim\limits_{n\to+\infty} \varepsilon_n |\dot{x}^{\varepsilon_n}(t)|_{\mathbb{M}} = 0$, *for every* $t \in (0,+\infty) \setminus J_x$*, where J_x is the jump set of the limit function x.*

Proof. It is enough to prove the result for $T > 0$ fixed, using then a diagonal argument. By the uniform bounds (i) and (ii) of Corollary 2.1.4 together with (R2), the family $\{x^\varepsilon\}_{\varepsilon>0}$ is uniformly equibounded with uniformly equibounded variation in $[0,T]$. By means of the classical Helly's Selection Theorem we get the existence of a subsequence $\varepsilon_n \searrow 0$ and a function $x \in BV([0,T];X)$ for which (a) holds true (in $[0,T]$). Thanks to Proposition 1.3.12 and Lemma 1.3.13, we also infer that actually x belongs to $BV_{\mathcal{R}}([0,T];X)$ and that property (b) holds.

To get (c) we first notice that, by (ii) of Corollary 2.1.4 and (R2), we deduce that

$$\lim_{\varepsilon\to 0} \varepsilon \int_0^T |\dot{x}^\varepsilon(\tau)|\,\mathrm{d}\tau = 0,$$

from which we can assume without loss of generality that

$$\lim_{n\to+\infty} \varepsilon_n \dot{x}^{\varepsilon_n}(t) = 0, \qquad \text{for a.e. } t \in [0,T], \tag{2.3.1}$$

which implies the validity of (c) almost everywhere thanks to (1.1.1).

Let us now fix $t \in (0,T] \setminus J_x$ and consider two sequences $s_k \nearrow t$ and $t_k \searrow t$ at which (2.3.1) holds true. By means of the energy balance (EB^{ε_n}) and exploiting the nonnegativity of $\mathcal{R}$ and $|\cdot|^2_{\mathbb{V}}$ we deduce:

$$\begin{aligned} &\frac{\varepsilon_n^2}{2}|\dot{x}^{\varepsilon_n}(t_k)|^2_{\mathbb{M}} + \mathcal{E}(t_k, x^{\varepsilon_n}(t_k)) - \mathcal{E}(t, x^{\varepsilon_n}(t)) - \int_t^{t_k} \frac{\partial}{\partial t}\mathcal{E}(\tau, x^{\varepsilon_n}(\tau))\,\mathrm{d}\tau \\ \le\, & \frac{\varepsilon_n^2}{2}|\dot{x}^{\varepsilon_n}(t)|^2_{\mathbb{M}} \\ \le\, & \frac{\varepsilon_n^2}{2}|\dot{x}^{\varepsilon_n}(s_k)|^2_{\mathbb{M}} + \mathcal{E}(s_k, x^{\varepsilon_n}(s_k)) - \mathcal{E}(t, x^{\varepsilon_n}(t)) + \int_{s_k}^{t} \frac{\partial}{\partial t}\mathcal{E}(\tau, x^{\varepsilon_n}(\tau))\,\mathrm{d}\tau. \end{aligned}$$

Letting first $n \to +\infty$ we obtain:

$$\begin{aligned} &\mathcal{E}(t_k, x(t_k)) - \mathcal{E}(t, x(t)) - \int_t^{t_k} \frac{\partial}{\partial t}\mathcal{E}(\tau, x(\tau))\,\mathrm{d}\tau \\ \le\, & \liminf_{n\to+\infty} \frac{\varepsilon_n^2}{2}|\dot{x}^{\varepsilon_n}(t)|^2_{\mathbb{M}} \le \limsup_{n\to+\infty} \frac{\varepsilon_n^2}{2}|\dot{x}^{\varepsilon_n}(t)|^2_{\mathbb{M}} \\ \le\, & \mathcal{E}(s_k, x(s_k)) - \mathcal{E}(t, x(t)) + \int_{s_k}^{t} \frac{\partial}{\partial t}\mathcal{E}(\tau, x(\tau))\,\mathrm{d}\tau. \end{aligned}$$

Here we used the continuity of $\mathcal{E}$ and the dominated convergence theorem on the integral terms, exploiting assumption (E5).

Since $t \notin J_x$, letting now $k \to +\infty$ we prove (c). □

Our aim now is to prove that such a limit function x is an energetic solution of problem (1.1.5); we thus need to show the validity of the global stability condition (GS) and the weak energy balance (WEB). The strategy consists in passing to the limit the dynamic local stability condition (LS^ε) and the dynamic energy balance (EB^ε). This first proposition deals with stability conditions:

Proposition 2.3.2. *Assume that x_0^ε and $\varepsilon x_1^\varepsilon$ are uniformly bounded. Then the limit function x obtained in Theorem 2.3.1 fulfils the following inequality:*

$$\int_s^t \Big(\mathcal{R}(\tau, v) + \langle D_x\mathcal{E}(\tau, x(\tau)), v\rangle\Big)\,\mathrm{d}\tau \ge 0, \quad \textit{for every } v \in X \textit{ and for every } 0 \le s \le t. \tag{2.3.2}$$

In particular the right and the left limit of x are locally stable, meaning that:

(LS^+) $\mathcal{R}(t, v) + \langle D_x\mathcal{E}(t, x^+(t)), v\rangle \ge 0,$ *for every $v \in X$ and for every $t \in [0, +\infty)$;*

(LS^-) $\mathcal{R}(t, v) + \langle D_x\mathcal{E}(t, x^-(t)), v\rangle \ge 0,$ *for every $v \in X$ and for every $t \in (0, +\infty)$.*

Proof. Let ε_n be the subsequence obtained in Theorem 2.3.1. We now fix $v \in K$, being (2.3.2) trivial if $v \notin K$, and by integrating the local stability condition (LS^{ε_n}) between arbitrary $0 \le s \le t$ we deduce:

$$\begin{aligned} 0 &\le \int_s^t \Big(\mathcal{R}(\tau, v) + \langle D_x\mathcal{E}(\tau, x^{\varepsilon_n}(\tau)) + \varepsilon_n^2\mathbb{M}\ddot{x}^{\varepsilon_n}(\tau) + \varepsilon_n\mathbb{V}\dot{x}^{\varepsilon_n}(\tau), v\rangle\Big)\,\mathrm{d}\tau \\ &= \int_s^t\Big(\mathcal{R}(\tau, v) + \langle D_x\mathcal{E}(\tau, x^{\varepsilon_n}(\tau)), v\rangle\Big)\,\mathrm{d}\tau + \varepsilon_n^2\langle\mathbb{M}(\dot{x}^{\varepsilon_n}(t) - \dot{x}^{\varepsilon_n}(s)), v\rangle + \varepsilon_n\int_s^t\langle\mathbb{V}\dot{x}^{\varepsilon_n}(\tau), v\rangle\,\mathrm{d}\tau. \end{aligned}$$

Letting $n \to +\infty$ we obtain (2.3.2) by dominated convergence on the first term (using (E3)), while the second and the third term vanish by means of (ii) and (iii) of Corollary 2.1.4 together with (1.1.1), (1.1.2), and (R2).

The validity of $(LS^\pm)$ easily follows from (2.3.2) since by (E3) and (R3) the map $t \mapsto \mathcal{R}(t, v) + \langle D_x\mathcal{E}(t, x^\pm(t)), v\rangle$ is right continuous with x^+ and left continuous with x^-. □

Next proposition exploits the lower semicontinuity of the $\mathcal{R}$–variation (Lemma 1.3.13) to obtain an estimate from above of the quasistatic energy:

Proposition 2.3.3 (**Lower Energy Estimates**). *Assume that x_0^ε and $\varepsilon x_1^\varepsilon$ are uniformly bounded. Then the limit function x obtained in Theorem 2.3.1 fulfils the following energy inequalities:*

$$\mathcal{E}(t,x^+(t))+V_\mathcal{R}(x;s-,t+)\le\mathcal{E}(s,x^-(s))+\int_s^t\frac{\partial}{\partial t}\mathcal{E}(\tau,x(\tau))\,\mathrm{d}\tau,\quad \textit{for every } 0<s\le t. \tag{2.3.3a}$$

$$\mathcal{E}(t,x^+(t))+V_\mathcal{R}(x;s+,t+)\le\mathcal{E}(s,x^+(s))+\int_s^t\frac{\partial}{\partial t}\mathcal{E}(\tau,x(\tau))\,\mathrm{d}\tau,\quad \textit{for every } 0\le s\le t. \tag{2.3.3b}$$

$$\mathcal{E}(t,x^-(t))+V_\mathcal{R}(x;s-,t-)\le\mathcal{E}(s,x^-(s))+\int_s^t\frac{\partial}{\partial t}\mathcal{E}(\tau,x(\tau))\,\mathrm{d}\tau,\quad \textit{for every } 0<s\le t. \tag{2.3.3c}$$

If in addition $\lim\limits_{\varepsilon\to0}\varepsilon x_1^\varepsilon=0$, then (2.3.3a) and (2.3.3c) hold true also for $s=0$.

Proof. We prove only (2.3.3a), being the other inequalities analogous. We fix $0<s\le t$ and we consider two sequences $s_k\nearrow s$ and $t_k\searrow t$ such that $s_k,t_k\notin J_x$. By means of Theorem 2.3.1 and by using the nonnegativity of $|\cdot|^2_\mathbb{V}$ together with the energy balance (EB^{ε_n}) we get:

$$\begin{aligned}
&\mathcal{E}(t_k,x(t_k))+V_\mathcal{R}(x;s_k,t_k)\\
&\le\liminf_{n\to+\infty}\left(\frac{\varepsilon_n^2}{2}|\dot{x}^{\varepsilon_n}(t_k)|^2_\mathbb{M}+\mathcal{E}(t_k,x^{\varepsilon_n}(t_k))+\int_{s_k}^{t_k}\mathcal{R}(\tau,\dot{x}^{\varepsilon_n}(\tau))\,\mathrm{d}\tau+\varepsilon_n\int_{s_k}^{t_k}|\dot{x}^{\varepsilon_n}(\tau)|^2_\mathbb{V}\,\mathrm{d}\tau\right)\\
&=\liminf_{n\to+\infty}\left(\frac{\varepsilon_n^2}{2}|\dot{x}^{\varepsilon_n}(s_k)|^2_\mathbb{M}+\mathcal{E}(s_k,x^{\varepsilon_n}(s_k))+\int_{s_k}^{t_k}\frac{\partial}{\partial t}\mathcal{E}(\tau,x^{\varepsilon_n}(\tau))\,\mathrm{d}\tau\right)\\
&=\mathcal{E}(s_k,x(s_k))+\int_{s_k}^{t_k}\frac{\partial}{\partial t}\mathcal{E}(\tau,x(\tau))\,\mathrm{d}\tau,
\end{aligned}$$

where in the last equality we employed once again the continuity of $\mathcal{E}$ and (E5). Letting now $k\to+\infty$ we obtain (2.3.3a).

If in addition $\lim\limits_{\varepsilon\to0}\varepsilon x_1^\varepsilon=0$, the same argument works choosing $s_k\equiv0$; thus we conclude. □

Remark 2.3.4. We want to highlight that up to this point the convexity assumption (E2) was not needed. Thus even without convexity the limit function x satisfies the right and left local stability conditions $(LS^\pm)$ plus the energy inequality (2.3.3a). Usually a function satisfying these properties is called local solution to the quasistatic problem (1.1.5), see [65] Chapter 3. Inequality (2.3.3a) can be also reformulated as an energy equality in a very implicit way by introducing a so called defect measure μ_D such that:

$$\mathcal{E}(t,x^+(t))+V_\mathcal{R}(x;s-,t+)+\mu_D([s,t])=\mathcal{E}(s,x^-(s))+\int_s^t\frac{\partial}{\partial t}\mathcal{E}(\tau,x(\tau))\,\mathrm{d}\tau,\ \text{for every } 0\le s\le t.$$

The positive measure μ_D is no other than the opposite of the distributional derivative of the function $t\mapsto\mathcal{E}(t,x(t))+V_\mathcal{R}(x;0,t)-\int_0^t\frac{\partial}{\partial t}\mathcal{E}(\tau,x(\tau))\,\mathrm{d}\tau$. The presence of such a defect measure, which somehow takes into account the possible losses of energy in the system, appears in many asymptotical studies of mechanical models: we refer for instance to [4, 32, 61, 62, 63, 79] for a vanishing viscosity analysis and the notion of Balanced

Viscosity solutions in both finite and infinite dimension, or to [83] for a vanishing inertia and viscosity analysis (without a rate-independent dissipation) in finite dimension.

The fine properties of μ_D in our context where a rate-independent dissipation is also present are beyond the scopes of the present investigation, thus we leave this analysis open for future research. We simply notice that, as we will see in Theorem 2.3.8, the (uniform) convexity assumption (E2) will ensure that μ_D is the null measure.

From now on we will exploit the convexity assumption (E2). This allows us to deduce that the local conditions (LS^+) and (LS^-) are equivalent to their global counterpart:

$$(GS^+)\quad \mathcal{E}(t,x^+(t))\le\mathcal{E}(t,v)+\mathcal{R}(t,v-x^+(t)),\quad \text{for every } v\in X \text{ and for every } t\in[0,+\infty);$$

$$(GS^-)\quad \mathcal{E}(t,x^-(t))\le\mathcal{E}(t,v)+\mathcal{R}(t,v-x^-(t)),\quad \text{for every } v\in X \text{ and for every } t\in(0,+\infty).$$

These global conditions permit to get also a bound from below of the energy, see Lemma 2.3.5 and Proposition 2.3.7. We warn the reader that for the proof of next lemma in the case of a general elastic energy $\mathcal{E}$ we need to add the assumption (E6), which we rewrite here for the sake of clarity:

(E6) for every $\lambda>0$ and $R>0$ there exists $\delta=\delta(\lambda,R)>0$ such that if $|t-s|\le\delta$ and $z\in\mathcal{B}^Z_R$, then
$$\left|\frac{\partial}{\partial t}\mathcal{E}_{sh}(t,z)-\frac{\partial}{\partial t}\mathcal{E}_{sh}(s,z)\right|\le\lambda$$
.

Lemma 2.3.5. *Assume (E6). Assume that x_0^ε and $\varepsilon x_1^\varepsilon$ are uniformly bounded. Then the right and left limit of the function x obtained in Theorem 2.3.1 fulfil the following inequalities:*

$$\mathcal{E}(t,x^+(t))+V_\mathcal{R}(x^+;s,t)\ge\mathcal{E}(s,x^+(s))+\int_s^t\frac{\partial}{\partial t}\mathcal{E}(\tau,x(\tau))\,\mathrm{d}\tau,\qquad \textit{for every } 0\le s\le t; \tag{2.3.4a}$$

$$\mathcal{E}(t,x^-(t))+V_\mathcal{R}(x^-;s,t)\ge\mathcal{E}(s,x^-(s))+\int_s^t\frac{\partial}{\partial t}\mathcal{E}(\tau,x(\tau))\,\mathrm{d}\tau,\qquad \textit{for every } 0< s\le t. \tag{2.3.4b}$$

If in addition $x_0:=x(0)$ satisfies (1.1.6), namely $\mathcal{E}(0,x_0)\le\mathcal{E}(0,v)+\mathcal{R}(0,v-x_0)$ for every $v\in X$, then (2.3.4b) holds true also for $s=0$.

Proof. Inequality (2.3.4a) is trivially satisfied for $s=t$, so let us fix $0\le s<t$ and consider a fine sequence of partitions of $[s,t]$ such that:

$$\lim_{n\to+\infty}\sum_{k=1}^n\left|(t_k-t_{k-1})\frac{\partial}{\partial t}\mathcal{E}(t_k,x^+(t_k))-\int_{t_{k-1}}^{t_k}\frac{\partial}{\partial t}\mathcal{E}(\tau,x(\tau))d\tau\right|=0. \tag{2.3.5}$$

Such a sequence of partitions exists since $\frac{\partial}{\partial t}\mathcal{E}(\cdot,x(\cdot))\in L^1(s,t)$, see for instance [34], Lemma 4.5.

So let us fix one of these partitions and by means of (GS^+) we deduce that for every $k=1,\dots,n$ we have:

$$\mathcal{E}(t_{k-1},x^+(t_{k-1}))\le\mathcal{E}(t_{k-1},x^+(t_k))+\mathcal{R}(t_{k-1},x^+(t_k)-x^+(t_{k-1})),$$

and thus we obtain:

$$\begin{aligned}&\mathcal{E}(t_k,x^+(t_k))-\mathcal{E}(t_{k-1},x^+(t_{k-1}))+\mathcal{R}(t_{k-1},x^+(t_k)-x^+(t_{k-1}))\\ &\ge\mathcal{E}(t_k,x^+(t_k))-\mathcal{E}(t_{k-1},x^+(t_k))=\int_{t_{k-1}}^{t_k}\frac{\partial}{\partial t}\mathcal{E}(\tau,x^+(t_k))d\tau.\end{aligned}$$

By summing the above inequality from $k = 1$ to $k = n$ we get:

$$\begin{aligned}&\mathcal{E}(t, x^+(t)) - \mathcal{E}(s, x^+(s)) + \sum_{k=1}^{n} \mathcal{R}(t_{k-1}, x^+(t_k) - x^+(t_{k-1})) \\ &\geq \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \frac{\partial}{\partial t}\mathcal{E}(\tau, x^+(t_k))d\tau =: I_n.\end{aligned} \tag{2.3.6}$$

By letting $n \to +\infty$, we get (2.3.4a) if we show that $\lim\limits_{n\to+\infty} I_n = \int_s^t \frac{\partial}{\partial t}\mathcal{E}(\tau, x(\tau))d\tau$. To prove it we argue as follows:

$$\begin{aligned}&\left| I_n - \int_s^t \frac{\partial}{\partial t}\mathcal{E}(\tau, x(\tau))d\tau \right| = \left| \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \Big(\frac{\partial}{\partial t}\mathcal{E}(\tau, x^+(t_k)) - \frac{\partial}{\partial t}\mathcal{E}(\tau, x(\tau)) \Big) d\tau \right| \\ &\leq \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left| \frac{\partial}{\partial t}\mathcal{E}(\tau, x^+(t_k)) - \frac{\partial}{\partial t}\mathcal{E}(t_k, x^+(t_k)) \right| \mathrm{d}\tau \\ &+ \sum_{k=1}^{n} \left| (t_k - t_{k-1}) \frac{\partial}{\partial t}\mathcal{E}(t_k, x^+(t_k)) - \int_{t_{k-1}}^{t_k} \frac{\partial}{\partial t}\mathcal{E}(\tau, x(\tau))d\tau \right|.\end{aligned}$$

The second term vanishes as $n \to +\infty$ thanks to (2.3.5), while to deal with the first one we exploit (E6): we first fix $\lambda > 0$ and we pick $R = C_\Lambda^t |\pi_Z|_*$, where C_Λ^t is the constant appearing in Corollary 2.1.4. Then let δ be given accordingly by (E6). By means of (1.3.5) we know that $\max\limits_{k=1,\dots,n} |t_k - t_{k-1}| \leq \delta$ for n large enough, thus (E6) implies:

$$\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left| \frac{\partial}{\partial t}\mathcal{E}(\tau, x^+(t_k)) - \frac{\partial}{\partial t}\mathcal{E}(t_k, x^+(t_k)) \right| \mathrm{d}\tau \leq \lambda(t - s),$$

and hence (2.3.4a) is proved.

Inequality (2.3.4b) can be obtained arguing in the same way replacing x^+ with x^-, and recalling that (GS^-) holds true only if $t > 0$. If in addition x_0 satisfies (1.1.6), then (GS^-) holds true also in $t = 0$ and the whole argument can be performed also in $s = 0$. □

We want to point out that condition (E6) is not necessary for the validity of Lemma 2.3.5, but it is useful to treat the case of a general elastic energy. Indeed, if we restrict for instance our attention to the concrete case of a quadratic energy $\mathcal{E}_{sh}(t, z) = \frac{1}{2}\langle \mathbb{A}_{sh}(z - \ell_{sh}(t)), z - \ell_{sh}(t)\rangle_Z$ as in (QE), it is easy to verify that conditions (E1)–(E5) are satisfied, but (E6) does not hold true if $\dot{\ell}_{sh}$ is not continuous. However, Lemma 2.3.5 is still valid.

Lemma 2.3.6. *If in Lemma 2.3.5 assumption (E6) is replaced by (QE), the same conclusions hold.*

Proof. The proof follows the same strategy used for Lemma 2.3.5, with some adaptation. Firstly, we need to choose fine partitions satisfying instead:

$$\begin{aligned}&\lim_{n\to+\infty} \sum_{k=1}^{n} (t_k - t_{k-1}) \langle \mathbb{A}_{sh}(\pi_Z(x^+(t_k)) - \ell_{sh}(t_k)), \dot{\ell}_{sh}(t_k)\rangle_Z \\ &= \int_s^t \langle \mathbb{A}_{sh}(\pi_Z(x(\tau)) - \ell_{sh}(\tau)), \dot{\ell}_{sh}(\tau)\rangle_Z d\tau;\end{aligned} \tag{2.3.7a}$$

$$\lim_{n\to+\infty} \sum_{k=1}^{n} \left| (t_k - t_{k-1})\dot{\ell}_{sh}(t_k) - \int_{t_{k-1}}^{t_k} \dot{\ell}_{sh}(\tau)\,\mathrm{d}\tau \right|_Z = 0. \tag{2.3.7b}$$

As before, the existence of such a sequence of partitions is ensured by [34], Lemma 4.5. In this case the integral term I_n defined in (2.3.6) takes the form:

$$I_n = -\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \langle \mathbb{A}_{sh}(\pi_Z(x^+(t_k)) - \ell_{sh}(\tau)), \dot{\ell}_{sh}(\tau) \rangle_Z d\tau,$$

and we conclude if we prove that $\lim_{n\to+\infty} I_n = -\int_s^t \langle \mathbb{A}_{sh}(\pi_Z(x(\tau)) - \ell_{sh}(\tau)), \dot{\ell}_{sh}(\tau) \rangle_Z d\tau$. With this aim we rewrite I_n as:

$$\begin{aligned} I_n =& -\sum_{k=1}^{n} (t_k - t_{k-1}) \langle \mathbb{A}_{sh}(\pi_Z(x^+(t_k)) - \ell_{sh}(t_k)), \dot{\ell}_{sh}(t_k) \rangle_Z \\ &+ \sum_{k=1}^{n} \left\langle \mathbb{A}_{sh}(\pi_Z(x^+(t_k)) - \ell_{sh}(t_k))\,,\,(t_k - t_{k-1})\dot{\ell}_{sh}(t_k) - \int_{t_{k-1}}^{t_k} \dot{\ell}_{sh}(\tau)\,\mathrm{d}\tau \right\rangle_Z \\ &+ \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \langle \mathbb{A}_{sh}(\ell_{sh}(t_k) - \ell_{sh}(\tau)), \dot{\ell}_{sh}(\tau) \rangle_Z \,\mathrm{d}\tau =: J_n^1 + J_n^2 + J_n^3. \end{aligned}$$

By means of (2.3.7b) it is easy to see that $\lim_{n\to+\infty} J_n^2 = 0$, while exploiting the absolute continuity of ℓ_{sh} together with (1.3.5) we also deduce that $\lim_{n\to+\infty} J_n^3 = 0$. By using (2.3.7a) we conclude. □

As a simple corollary we get:

Proposition 2.3.7 (**Upper Energy Estimate**). *Assume (E6) or (QE), and assume that x_0^ε and $\varepsilon x_1^\varepsilon$ are uniformly bounded. Then the limit function x obtained in Theorem 2.3.1 fulfils the following inequality for every $0 < s \le t$:*

$$\mathcal{E}(t, x^+(t)) + \min\left\{V_{\mathcal{R}}(x^+; s-, t), V_{\mathcal{R}}(x^-; s, t+)\right\} \ge \mathcal{E}(s, x^-(s)) + \int_s^t \frac{\partial}{\partial t}\mathcal{E}(\tau, x(\tau))\,\mathrm{d}\tau. \tag{2.3.8}$$

If in addition $x_0 = x(0)$ satisfies (1.1.6), then it also holds:

$$\mathcal{E}(t, x^+(t)) + V_{\mathcal{R}}(x^-; 0, t+) \ge \mathcal{E}(0, x_0) + \int_0^t \frac{\partial}{\partial t}\mathcal{E}(\tau, x(\tau))\,\mathrm{d}\tau, \qquad \textit{for every } t \in [0, +\infty). \tag{2.3.9}$$

Proof. We fix $0 < s \le t$ and we consider two sequences $s_k \nearrow s$ and $t_k \searrow t$. By means of (2.3.4a) and (2.3.4b) we thus deduce:

$$\mathcal{E}(t, x^+(t)) + V_{\mathcal{R}}(x^+; s_k, t) \ge \mathcal{E}(s_k, x^+(s_k)) + \int_{s_k}^t \frac{\partial}{\partial t}\mathcal{E}(\tau, x(\tau))\,\mathrm{d}\tau,$$

$$\mathcal{E}(t_k, x^-(t_k)) + V_{\mathcal{R}}(x^-; s, t_k) \ge \mathcal{E}(s, x^-(s)) + \int_s^{t_k} \frac{\partial}{\partial t}\mathcal{E}(\tau, x(\tau))\,\mathrm{d}\tau. \tag{2.3.10}$$

Letting $k \to +\infty$ and since $\mathcal{E}$ is continuous in $[0, +\infty) \times X$ we obtain (2.3.8).

If in addition x_0 satisfies (1.1.6) we can set $s = 0$ in (2.3.10), thus also (2.3.9) follows by letting $k \to +\infty$. □

Combining all the results of this section we are finally able to prove that the limit function x is actually an energetic solution of the quasistatic problem (1.1.5). The rigorous statement is the following:

Theorem 2.3.8. *Assume (E6) or (QE), and assume that x_0^ε and $\varepsilon x_1^\varepsilon$ are uniformly bounded. Then the limit function x obtained in Theorem 2.3.1 is continuous in $(0,+\infty)$ and its right limit x^+ is an energetic solution for (1.1.5) with initial position $x^+(0)$ in the sense of Definition 2.2.1.*

If in addition $x_0 = x(0)$ satisfies (1.1.6) and $\lim\limits_{\varepsilon\to 0} \varepsilon x_1^\varepsilon = 0$, then x is continuous also in $t = 0$ and it is an energetic solution for (1.1.5) with initial position x_0.

Proof. We first prove that the right limit x^+ is an energetic solution for (1.1.5) with initial position $x^+(0)$. We only need to prove the weak energy balance (WEB), since we already know x^+ is globally stable, see (GS^+). With this aim we first fix $t \in [0,+\infty)$ and by combining (2.3.3b) and (2.3.4a) we get:

$$\begin{aligned}\mathcal{E}(t,x^+(t)) + V_\mathcal{R}(x;0+,t+) \le \mathcal{E}(0,x^+(0)) + \int_0^t \frac{\partial}{\partial t}\mathcal{E}(\tau,x(\tau))\,\mathrm{d}\tau \le \mathcal{E}(t,x^+(t)) + V_\mathcal{R}(x^+;0,t)\\ \le \mathcal{E}(t,x^+(t)) + V_\mathcal{R}(x^+;0,t+).\end{aligned}$$

By means of (d) in Proposition 1.3.11 we hence deduce that

$$V_\mathcal{R}(x;0+,t+) = V_\mathcal{R}(x^+;0,t+) = V_\mathcal{R}(x^+;0,t),$$

and also the validity of (WEB):

$$\mathcal{E}(t,x^+(t)) + V_\mathcal{R}(x^+;0,t) = \mathcal{E}(0,x^+(0)) + \int_0^t \frac{\partial}{\partial t}\mathcal{E}(\tau,x(\tau))\,\mathrm{d}\tau, \qquad \text{for every } t \in [0,+\infty).$$

Thus x^+ is an energetic solution starting from $x^+(0)$ and in particular, by means of Proposition 2.2.8, it is continuous in $[0,+\infty)$ with continuous $\mathcal{R}$–variation $V_\mathcal{R}(x^+;0,\cdot)$.

We now show that $x(t) = x^+(t)$ for every $t \in (0,+\infty)$. By means of (2.3.3a) and (2.3.8) and reasoning as before we get:

$$V_\mathcal{R}(x;t-,t+) = V_\mathcal{R}(x^+;t-,t), \qquad \text{for every } t \in (0,+\infty). \tag{2.3.11}$$

Since x^+ has continuous $\mathcal{R}$–variation, we deduce that $V_\mathcal{R}(x;t-,t+) = V_\mathcal{R}(x^+;t-,t) = 0$ if $t \in (0,+\infty)]$; this implies that the $\mathcal{R}$–variation of x is continuous in $(0,+\infty)$, and thus in particular x itself is continuous in $(0,+\infty)$ (see (c) in Proposition 1.3.11). This means in particular that $x(t) = x^+(t)$ for every $t \in (0,+\infty)$.

If in addition x_0 satisfies (1.1.6) and $\lim\limits_{\varepsilon\to 0} \varepsilon x_1^\varepsilon = 0$, then we can exploit (2.3.3a) in $s = 0$ and (2.3.9); since we now know that both x and $V_\mathcal{R}(x;0,\cdot)$ are continuous in $(0,+\infty)$, arguing as before we obtain:

$$\mathcal{E}(t,x(t)) + V_\mathcal{R}(x;0,t) = \mathcal{E}(0,x_0) + \int_0^t \frac{\partial}{\partial t}\mathcal{E}(\tau,x(\tau))\,\mathrm{d}\tau, \qquad \text{for every } t \in (0,+\infty).$$

Since the above equality is trivially satisfied in $t = 0$, we deduce that x satisfies (WEB); since (1.1.6) holds, from (GS^-) we also deduce that x satisfies (GS), and thus it is an energetic solution for (1.1.5) with initial position x_0. Thus we conclude. □

We conclude this section by stating the main theorem of the first part, which gathers and summarises what we have proved up to now about the convergence of dynamic solutions of problem (1.1.3) to quasistatic solutions of (1.1.5) when inertia vanishes.

Theorem 2.3.9. *Let $\mathbb{M}, \mathbb{V}$ be as in Section 1.1, and assume that $\mathcal{R}$ satisfies (R4), and that $\mathcal{E}(t,x) = \mathcal{E}_{sh}(t,\pi_Z(x))$ satisfies (E1)–(E6) or (QE). For every $\varepsilon > 0$, let x^ε be a differential solution of the dynamic problem (1.1.3) related to the initial position $x_0^\varepsilon \in X$ and the initial velocity $x_1^\varepsilon \in K$, and assume that x_0^ε and $\varepsilon x_1^\varepsilon$ are uniformly bounded. Then there exist a subsequence $\varepsilon_n \searrow 0$ and a function $x \in \widetilde{BV}_\mathcal{R}([0,+\infty);X) \cap C^0((0,+\infty);X)$ such that its right limit x^+ is an energetic solution for (1.1.5) in the sense of Definition 2.2.1 with initial position $x^+(0)$ and:*

(a') $\lim_{n\to+\infty} x^{\varepsilon_n}(t) = x(t)$ *for every* $t \in [0, +\infty)$*, and the convergence is uniform in any compact interval contained in* $(0, +\infty)$*;*

(b') $\lim_{n\to+\infty} \int_s^t \mathcal{R}(\tau, \dot{x}^{\varepsilon_n}(\tau))\,\mathrm{d}\tau = V_{\mathcal{R}}(x; s, t)$ *for every* $0 < s \leq t$*, and the convergence is uniform in any compact interval contained in* $[s, +\infty)$*;*

(c') $\lim_{n\to+\infty} \varepsilon_n |\dot{x}^{\varepsilon_n}(t)|_{\mathbb{M}} = 0$ *for every* $t \in (0, +\infty)$*, and the convergence is uniform in any compact interval contained in* $(0, +\infty)$*;*

(d') $\lim_{n\to+\infty} \varepsilon_n \int_s^t |\dot{x}^{\varepsilon_n}(\tau)|^2_{\mathbb{V}}\,\mathrm{d}\tau = 0$ *for every* $0 < s \leq t$*.*

If in addition $x_0 := x(0)$ *satisfies* (1.1.6)*, namely* $\mathcal{E}(0, x_0) \leq \mathcal{E}(0, v) + \mathcal{R}(0, v - x_0)$ *for every* $v \in X$*, and* $\lim_{\varepsilon\to 0} \varepsilon x_1^\varepsilon = 0$*, then the limit function* x *is continuous in the whole* $[0, +\infty)$*, is itself an energetic solution of* (1.1.5) *with initial position* x_0 *and the convergence in* (a') *and* (c') *is uniform in compact intervals contained in* $[0, +\infty)$*; moreover* (b') *and* (d') *hold true also in* $s = 0$*.*

Finally, if also (R5) *holds or if* $\mathcal{R}$ *does not depend on time, then* x *is actually* $\mathcal{R}$*–absolutely continuous, and thus a differential solution of* (1.1.5)*.*

Remark 2.3.10 (Uniqueness). If in particular one of the conditions of Lemma 2.2.9 or Lemma 2.2.10 is satisfied, and if $\lim_{\varepsilon\to 0} \varepsilon x_1^\varepsilon = 0$ and $\lim_{\varepsilon\to 0} x_0^\varepsilon = x_0$, for some x_0 satisfying (1.1.6), then there is no need to pass to a subsequence in the previous theorem. Indeed in this case the whole sequence x^ε converges in the sense of (a')–(d') (even in $t = 0$) towards the unique differential solution x to (1.1.5).

Proof of Theorem 2.3.9. Combining Theorems 2.3.1, 2.3.8 and exploiting Proposition 2.2.8 we get the existence of a subsequence $\varepsilon_n \searrow 0$ and of a function $x \in \widetilde{BV}_{\mathcal{R}}([0, +\infty); X) \cap C^0((0, +\infty); X)$ with the property that the right limit x^+ is an energetic solution for (1.1.5) with initial position $x^+(0)$ and for which the pointwise convergence in (a') and (c') hold. We now observe that by the energy balances (EB^{ε_n}) and (WEB) for every $0 < s \leq t$ we have:

$$\begin{aligned}
&\varepsilon_n \int_s^t |\dot{x}^{\varepsilon_n}(\tau)|^2_{\mathbb{V}}\,\mathrm{d}\tau + \int_s^t \mathcal{R}(\tau, \dot{x}^{\varepsilon_n}(\tau))\,\mathrm{d}\tau - V_{\mathcal{R}}(x; s, t)\\
&= \frac{\varepsilon_n^2}{2}|\dot{x}^{\varepsilon_n}(s)|^2_{\mathbb{M}} - \frac{\varepsilon_n^2}{2}|\dot{x}^{\varepsilon_n}(t)|^2_{\mathbb{M}} + \mathcal{E}(s, x^{\varepsilon_n}(s)) - \mathcal{E}(s, x(s)) + \mathcal{E}(t, x(t)) - \mathcal{E}(t, x^{\varepsilon_n}(t))\\
&\quad + \int_s^t \Big(\frac{\partial}{\partial t}\mathcal{E}(\tau, x^{\varepsilon_n}(\tau)) - \frac{\partial}{\partial t}\mathcal{E}(\tau, x(\tau))\Big)\,\mathrm{d}\tau.
\end{aligned} \tag{2.3.12}$$

By means of the pointwise convergence in (a') and (c') and recalling (E5) we deduce that the right-hand side of the above inequality vanishes as $n \to +\infty$. Thus the pointwise convergence in (b') and (d') easily follows, since by (b) in Theorem 2.3.1 we already know that

$$\liminf_{n\to+\infty} \left(\int_s^t \mathcal{R}(\tau, \dot{x}^{\varepsilon_n}(\tau))\,\mathrm{d}\tau - V_{\mathcal{R}}(x; s, t)\right) \geq 0.$$

By means of Lemma 1.3.14 we now deduce that the convergence in (a') is uniform in any compact interval contained in $(0, +\infty)$, while the uniform convergence in (b') is due to the standard result that a sequence of nondecreasing and continuous scalar functions pointwise converging to a continuous function on a compact interval actually converges uniformly. The uniform convergence in (c') now follows by rearranging equality (2.3.12) and by exploiting (E3), (E5) and the just obtained uniform convergence in (a'), (b') and (d').

If in addition x_0 satisfy (1.1.6) and $\lim_{\varepsilon\to 0} \varepsilon x_1^\varepsilon = 0$, we know by Proposition 2.2.8 and Theorem 2.3.8 that x is continuous in $[0, +\infty)$ and it is an energetic solution with initial position x_0. Arguing as before we obtain the uniform convergence in $[0, T]$ for (a') and (c') and the validity of (b') and (d') also in $s = 0$.

To conclude, if (R5) holds or if $\mathcal{R}$ does not depend on time, always by means of Proposition 2.2.8 we deduce that x is $\mathcal{R}$–absolutely continuous. □

We want to point out that our result is sharp, in the sense that no better kind of convergences (for instance in $W^{1,1}$) can be achieved in the quasistatic limit. It is enough to consider the simplest case $X = \mathbb{R}$, with $\mathbb{M} = \mathrm{I}$, $\mathbb{V} = 0$, dissipation potential $\mathcal{R}(t, v) = |v|$ and a quadratic elastic energy $\mathcal{E}(t, x) = \frac{1}{2}(x - t - 1)^2$. Indeed it is easy to verify that in this setting the unique differential solution of the dynamic problem (1.1.3), with initial position $x_0^\varepsilon = 0$ and initial velocity $x_1^\varepsilon = 0$, is the function

$$x^\varepsilon(t) = t - \varepsilon \sin\left(\frac{t}{\varepsilon}\right),$$

which of course converges as $\varepsilon \to 0^+$ towards $x(t) = t$, namely the unique differential solution of the quasistatic problem (1.1.5) with initial position $x_0 = 0$, in the sense of previous theorem.

However x^ε does not converge to x in $W^{1,1}(0, T)$ for fixed $T > 0$, indeed

$$\int_0^T |\dot{x}^\varepsilon(\tau) - \dot{x}(\tau)| \, \mathrm{d}\tau = \int_0^T \left|\cos\left(\frac{\tau}{\varepsilon}\right)\right| \mathrm{d}\tau,$$

which does not vanish as $\varepsilon \to 0^+$.

Part II

A mechanical model of debonding with viscous damping

Chapter 3

Existence and uniqueness

Contents

In this chapter we present and analyse the dynamic debonding model depicted in the Introduction and described by the system

$$\begin{cases} u_{tt}(t,x) - u_{xx}(t,x) + \nu u_t(t,x) = 0, & t > 0\,,\ 0 < x < \ell(t), \\ u(t,0) = w(t), & t > 0, \\ u(t,\ell(t)) = 0, & t > 0, \\ u(0,x) = u_0(x), & 0 < x < \ell_0, \\ u_t(0,x) = u_1(x), & 0 < x < \ell_0, \end{cases} \tag{3.0.1}$$

coupled with dynamic Griffith's criterion

$$\begin{cases} 0 \le \dot\ell(t) < 1, \\ G_{\dot\ell(t)}(t) \le \kappa(\ell(t)), \\ \Big[G_{\dot\ell(t)}(t) - \kappa(\ell(t))\Big]\,\dot\ell(t) = 0, \end{cases} \qquad \text{for a.e. } t \in [0,+\infty). \tag{3.0.2}$$

In particular we focus on the existence and uniqueness of solutions to the coupled problem (3.0.1)&(3.0.2).

The chapter is organised as follows. Section 3.1 collects some notation and some definition on the geometric aspects of the debonding model which will be used several times during the whole second part of the thesis.

In Section 3.2 we prove there exists a unique solution u to problem (3.0.1) when the evolution of the debonding front ℓ is known a priori; the idea is to introduce an equivalent problem solved by the function $v(t,x) = e^{\nu t/2}u(t,x)$ (see (3.2.3)) and then, exploiting a suitable representation formula (Duhamel's principle), to perform a contraction argument (see Proposition 3.2.11 and Theorem 3.2.12).

In Section 3.3 we study the sum $\mathcal{S}$ of the internal energy of the solution u to problem (3.0.1) and the energy dissipated by viscosity. We prove that $\mathcal{S}$ is an absolutely continuous function and we provide an explicit formula (for small time) for its derivative (see Proposition 3.3.1).

In the rest of the chapter we take care of problem (3.0.1) when the evolution of the debonding front ℓ is unknown, but is governed by dynamic Griffith's criterion (3.0.2). In the first part of Section 3.4 we introduce in a rigorous way the dynamic energy release rate $G_\alpha(t)$ at time t corresponding to a speed $\alpha \in (0,1)$ of the debonding front (see Definition 3.4.4); in the second one we formulate Griffith's criterion under the assumption that the energy dissipated during the debonding process in the time interval $[0,t]$ is expressed by the formula

$$\int_{\ell_0}^{\ell(t)} \kappa(\sigma)\,\mathrm{d}\sigma,$$

where $\kappa\colon [\ell_0,+\infty) \to (0,+\infty)$ is the local toughness of the glue between the tape and the substrate. With this aim, as in [24] and [48], we formulate the evolution in terms of an energy-dissipation balance and of a maximum dissipation principle, and then we show their equivalence with Griffith's criterion (3.0.2).

In Section 3.5 we present the main result of the chapter: we solve the coupled problem showing existence and uniqueness of a pair (u,ℓ) satisfying (3.0.1)&(3.0.2) (see Theorem 3.5.6). Our result generalises Theorem 3.5 in [24] both for the presence of the damping term as well as for the weaker regularity we require on the data. The strategy for the proof is, like in Section 3.2, to rewrite (3.0.1)&(3.0.2) as a fixed point problem and then to use a contraction argument (see Proposition 3.5.5). Furthermore, our approach even allows us to consider the presence of an external force f in the model (see Remark 3.5.12), namely when the equation for the displacement u becomes

$$u_{tt}(t,x) - u_{xx}(t,x) + \nu u_t(t,x) = f(t,x), \quad t>0\,,\ 0<x<\ell(t).$$

The results contained in this chapter have been published in [76], in collaboration with L. Nardini.

3.1 Preliminaries

In this section we collect some notation and some definition that we will use several times throughout this second part of the thesis. Some of them have already been introduced and used in [24].

3.1.1 Geometric considerations

Fix $\ell_0 > 0$ and consider a function $\ell\colon [0,+\infty) \to [\ell_0,+\infty)$, which will play the role of the debonding front, satisfying:

$$\ell \in C^{0,1}([0,+\infty)) \text{ and } \ell(0)=\ell_0, \tag{3.1.1a}$$

$$0 \le \dot\ell(t) \le 1 \text{ for a.e. } t \in [0,+\infty). \tag{3.1.1b}$$

Given such a function, and given a time $T>0$, we define the sets (see Figure 3.1):

$$\begin{aligned}
\Omega &:= \{(t,x) \mid t>0\,,\ 0<x<\ell(t)\},\\
\Omega'_1 &:= \{(t,x)\in\Omega \mid t \le x \text{ and } t+x \le \ell_0\},\\
\Omega'_2 &:= \{(t,x)\in\Omega \mid t > x \text{ and } t+x < \ell_0\},\\
\Omega'_3 &:= \{(t,x)\in\Omega \mid t < x \text{ and } t+x > \ell_0\},
\end{aligned}$$

$$
\begin{aligned}
\Omega' &:= \Omega'_1 \cup \Omega'_2 \cup \Omega'_3,\\
\Omega_T &:= \{(t,x)\in\Omega \mid t<T\},\\
\Omega'_T &:= \{(t,x)\in\Omega' \mid t<T\},\\
(\Omega'_i)_T &:= \{(t,x)\in\Omega'_i \mid t<T\},\ \text{for } i=1,2,3,
\end{aligned}
$$

Moreover, for $t \in [0,+\infty)$, we introduce the functions:

$$\varphi(t) := t-\ell(t)\,, \quad \psi(t) := t+\ell(t), \tag{3.1.2}$$

and since ψ is strictly increasing we can define:

$$\omega\colon [\ell_0,+\infty) \to [-\ell_0,+\infty), \quad \omega(t) := \varphi\circ\psi^{-1}(t).$$

By (3.1.1b) ψ turns out to be a bilipschitz function, while φ turns out to be Lipschitz since

$$1 \le \dot\psi(t) \le 2, \quad \text{and} \quad 0 \le \dot\varphi(t) \le 1, \quad \text{for a.e. } t\in[0,+\infty).$$

As a byproduct we get that ω is Lipschitz too and for a.e. $t\in[\ell_0,+\infty)$ it holds true:

$$0 \le \dot\omega(t) = \frac{1-\dot\ell(\psi^{-1}(t))}{1+\dot\ell(\psi^{-1}(t))} \le 1. \tag{3.1.3}$$

If ℓ in addition satisfies the slightly stronger condition

$$0 \le \dot\ell(t) < 1 \text{ for a.e. } t\in[0,+\infty). \tag{3.1.4}$$

we obtain

$$0 < \dot\varphi(t) \le 1, \quad \text{and} \quad 0 < \dot\omega(t) = \frac{1-\dot\ell(\psi^{-1}(t))}{1+\dot\ell(\psi^{-1}(t))} \le 1, \quad \text{for a.e. } t\in[0,+\infty).$$

Hence in this case φ and ω are also invertible with absolutely continuous inverse (on compact sets).

For $(t,x)\in\Omega'$ we also introduce the set which will encode the reflection of the travelling waves in the two extrema of the tape (or the bar):

$$R(t,x) = \{(\tau,\sigma)\in\Omega' \mid 0<\tau<t,\ \gamma_1(\tau;t,x)<\sigma<\gamma_2(\tau;t,x)\}, \tag{3.1.5}$$

where

$$
\begin{aligned}
\gamma_1(\tau;t,x) &= \begin{cases} x-t+\tau, & \text{if } (t,x)\in\Omega'_1,\\ |x-t+\tau|, & \text{if } (t,x)\in\Omega'_2,\\ x-t+\tau, & \text{if } (t,x)\in\Omega'_3,\end{cases}\\
\gamma_2(\tau;t,x) &= \begin{cases} x+t-\tau, & \text{if } (t,x)\in\Omega'_1,\\ x+t-\tau, & \text{if } (t,x)\in\Omega'_2,\\ \tau-\omega(t+x), & \text{if } (t,x)\in\Omega'_3 \text{ and } \tau\le\psi^{-1}(t+x),\\ x+t-\tau, & \text{if } (t,x)\in\Omega'_3 \text{ and } \tau>\psi^{-1}(t+x),\end{cases}
\end{aligned} \tag{3.1.6}
$$

are the left and the right boundary of $R(t,x)$, respectively. See Figure 3.1.

Remark 3.1.1. We warn the reader that, for the sake of clarity, during the whole thesis we shall not write Ω_ℓ, Ω'_ℓ, $R_\ell(t,x)$, φ_ℓ or ω_ℓ, even if all of the sets and the functions introduced in this section depend explicitely on the function ℓ.

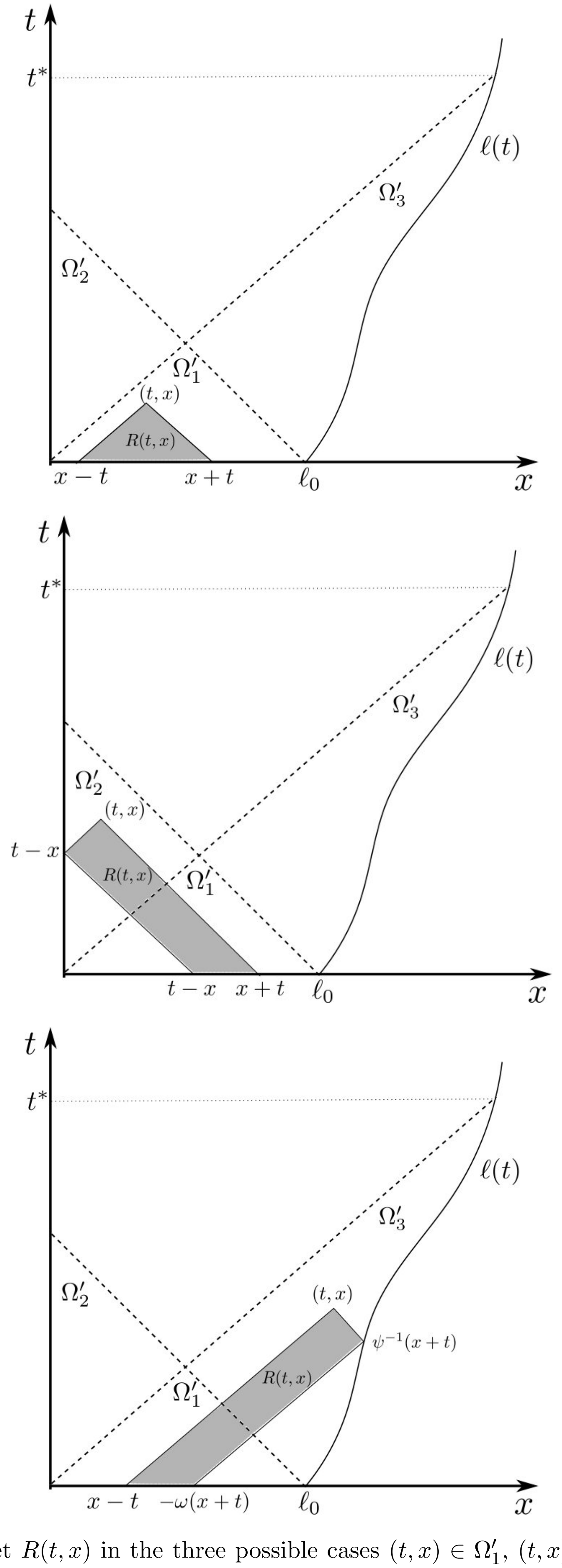

Figure 3.1: The set $R(t,x)$ in the three possible cases $(t,x) \in \Omega_1'$, $(t,x) \in \Omega_2'$, $(t,x) \in \Omega_3'$ and the time t^*.

3.1.2 Mathematical objects

For $a \in \mathbb{R}$ and for $k \geq 0$ integer we introduce the spaces:

$$\widetilde{H}^1(a,+\infty) := \{u \in H^1_{\text{loc}}(a,+\infty) \mid u \in H^1(a,b) \text{ for every } b > a\},$$
$$\widetilde{C}^{k,1}([a,+\infty)) := \{u \in C^k([a,+\infty)) \mid u \in C^{k,1}([a,b]) \text{ for every } b > a\}.$$

Finally let us define:

$$\widetilde{L}^2(\Omega') := \{u \in L^2_{\text{loc}}(\Omega') \mid u \in L^2(\Omega'_T) \text{ for every } T > 0\},$$
$$\widetilde{H}^1(\Omega') := \{u \in H^1_{\text{loc}}(\Omega') \mid u \in H^1(\Omega'_T) \text{ for every } T > 0\},$$
$$\widetilde{H}^1(\Omega) := \{u \in H^1_{\text{loc}}(\Omega) \mid u \in H^1(\Omega_T) \text{ for every } T > 0\}.$$

3.2 Prescribed debonding front

In this section we show existence and uniqueness of solutions to problem (3.0.1) when the evolution of the debonding front is prescribed. We first consider an auxiliary and equivalent problem, see (3.2.3), which will be easier to handle than the original one; then we provide a representation formula, given by (3.2.7), for a solution of this new problem. The result of existence and uniqueness will be finally obtained by means of a fixed point argument, as stated in Proposition 3.2.11 and Theorem 3.2.12.

We fix $\nu \geq 0$, $\ell_0 > 0$ and a function $\ell\colon [0,+\infty) \to [\ell_0,+\infty)$ satisfying (3.1.1a) and (3.1.1b). Differently from [24] we allow the debonding front ℓ to move even with speed one.

We assume that

$$w \in \widetilde{H}^1(0,+\infty), \tag{3.2.1a}$$
$$u_0 \in H^1(0,\ell_0), \quad u_1 \in L^2(0,\ell_0). \tag{3.2.1b}$$

For the initial data we require the compatibility conditions

$$u_0(0) = w(0), \quad u_0(\ell_0) = 0. \tag{3.2.2}$$

We will look for solutions in the space $\widetilde{H}^1(\Omega)$ or, assuming more regular data, in the space $\widetilde{C}^{k,1}(\overline{\Omega})$.

Definition 3.2.1. *We say that a function $u \in \widetilde{H}^1(\Omega)$ (resp. in $H^1(\Omega_T)$) is a solution of (3.0.1) if $u_{tt} - u_{xx} + \nu u_t = 0$ holds in the sense of distributions in Ω (resp. in Ω_T), the boundary conditions are intended in the sense of traces and the initial conditions u_0 and u_1 are satisfied in the sense of $L^2(0,\ell_0)$ and $H^{-1}(0,\ell_0)$, respectively.*

Remark 3.2.2. The definition is well posed, since for a solution $u \in H^1(\Omega_T)$ we have that u_t and u_x belong to $L^2(0,T;L^2(0,\ell_0))$; this implies that both u_t and u_{xx} live in the space $L^2(0,T;H^{-1}(0,\ell_0))$ and so by the wave equation $u_{tt} \in L^2(0,T;H^{-1}(0,\ell_0))$. Therefore $u_t \in H^1(0,T;H^{-1}(0,\ell_0))$, which is contained in $C^0([0,T];H^{-1}(0,\ell_0))$; thus the fact that u_1 is attended in the sense of $H^{-1}(0,\ell_0)$ is meaningful.

One of the standard ways used to deal with the weakly damped wave equation consists in the introduction of the function $v(t,x) := e^{\nu t/2}u(t,x)$ (see for instance [28], Remark 10, pag. 141), which in our setting solves the auxiliary problem

$$\begin{cases} v_{tt}(t,x) - v_{xx}(t,x) - \dfrac{\nu^2}{4}v(t,x) = 0, & t > 0\,,\, 0 < x < \ell(t), \\ v(t,0) = z(t), & t > 0, \\ v(t,\ell(t)) = 0, & t > 0, \\ v(0,x) = v_0(x), & 0 < x < \ell_0, \\ v_t(0,x) = v_1(x), & 0 < x < \ell_0, \end{cases} \tag{3.2.3}$$

where the boundary condition and the initial data are replaced respectively by the functions

$$\begin{gathered} z(t) = e^{\nu t/2} w(t), \\ v_0(x) = u_0(x) \quad \text{and} \quad v_1(x) = u_1(x) + \frac{\nu}{2} u_0(x). \end{gathered} \tag{3.2.4}$$

We notice that z, v_0 and v_1 in (3.2.4) satisfy (3.2.1) and the compatibility conditions (3.2.2) if and only if w, u_0 and u_1 do the same.

Remark 3.2.3. It is easy to see that $u \in \widetilde{H}^1(\Omega)$ (resp. $H^1(\Omega_T)$) is a solution of (3.0.1) if and only if the corresponding function $v(t,x) = e^{\nu t/2} u(t,x) \in \widetilde{H}^1(\Omega)$ (resp. $H^1(\Omega_T)$) is a solution of (3.2.3), according to Definition 3.2.1 (with the obvious changes). The absence of first derivatives in the equation for v makes this second problem more convenient to deal with.

In [24] it has been shown that every solution to the undamped (i.e. $\nu = 0$) wave equation, here and henceforth denoted by $A(t,x)$, satisfies a suitable version of the classical d'Alembert's formula, adapted to the time dependence of the domain; imposing initial data and boundary conditions the authors prove that in Ω' it can be written as $A(t,x) = a_1(t{+}x) + a_2(t{-}x)$, where

$$\begin{aligned} a_1(s) &= \begin{cases} \dfrac{1}{2} v_0(s) + \dfrac{1}{2} \displaystyle\int_0^s v_1(r)\,\mathrm{d}r, & \text{if } s \in (0, \ell_0], \\ -\dfrac{1}{2} v_0(-\omega(s)) + \dfrac{1}{2} \displaystyle\int_0^{-\omega(s)} v_1(r)\,\mathrm{d}r, & \text{if } s \in (\ell_0, 2t^*), \end{cases} \\ a_2(s) &= \begin{cases} \dfrac{1}{2} v_0(-s) - \dfrac{1}{2} \displaystyle\int_0^{-s} v_1(r)\,\mathrm{d}r, & \text{if } s \in (-\ell_0, 0], \\ z(s) - \dfrac{1}{2} v_0(s) - \dfrac{1}{2} \displaystyle\int_0^s v_1(r)\,\mathrm{d}r, & \text{if } s \in (0, \ell_0), \end{cases} \end{aligned} \tag{3.2.5}$$

with $t^* = \inf\{t \in [\ell_0, +\infty) \mid t = \ell(t)\}$ (with the convention $\inf\{\emptyset\} = +\infty$), see Figure 3.1. We notice that by (3.2.1), (3.2.2) and Remark B.0.7, a_1 and a_2 belong to $\widetilde{H}^1(0, 2t^*)$ and $H^1(-\ell_0, \ell_0)$ respectively; this will be used in Lemma 3.2.8.

Remark 3.2.4. We wrote $\widetilde{H}^1(0, 2t^*)$ since t^* can be $+\infty$; if this does not occur, that expression simply stands for $H^1(0, 2t^*)$.

Hence d'Alembert's formula provides an explicit expression of A in Ω':

$$A(t,x) = \begin{cases} \dfrac{1}{2} v_0(x{-}t) + \dfrac{1}{2} v_0(x{+}t) + \dfrac{1}{2} \displaystyle\int_{x-t}^{x+t} v_1(s)\,\mathrm{d}s, & \text{if } (t,x) \in \Omega'_1, \\ z(t{-}x) - \dfrac{1}{2} v_0(t{-}x) + \dfrac{1}{2} v_0(t{+}x) + \dfrac{1}{2} \displaystyle\int_{t-x}^{t+x} v_1(s)\,\mathrm{d}s, & \text{if } (t,x) \in \Omega'_2, \\ \dfrac{1}{2} v_0(x{-}t) - \dfrac{1}{2} v_0(-\omega(x{+}t)) + \dfrac{1}{2} \displaystyle\int_{x-t}^{-\omega(x+t)} v_1(s)\,\mathrm{d}s, & \text{if } (t,x) \in \Omega'_3. \end{cases} \tag{3.2.6}$$

Remark 3.2.5. In $\Omega \setminus \Omega'$ one cannot anymore obtain explicit formulas for a_1, a_2, and hence for A, due to superpositions of forward and backward waves generated by "bouncing" against the endpoints $x = 0$ and $x = \ell(t)$, even though d'Alembert's formula still holds true.

Inspired by the validity of this version of d'Alembert's formula in the undamped and homogeneous case $\nu = 0$, to solve problem (3.2.3) we firstly prove that even the non-homogeneous classical counterpart, the so called Duhamel's principle, holds true in our time-dependent domain setting. Duhamel's principle states that every solution to problem

(3.2.3) can be written (in Ω') as a sum of two terms: the first one is the solution A of the undamped wave equation, while the second one is the integral of the forcing term $\frac{\nu^2}{4}v(t,x)$ over a suitable space-time domain, namely the set $R(t,x)$ defined in (3.1.5).

The precise statement is the following:

Proposition 3.2.6. *A function $v \in \widetilde{H}^1(\Omega')$ is a solution of (3.2.3) in Ω' if and only if*

$$v(t,x) = A(t,x) + \frac{\nu^2}{8}\iint_{R(t,x)} v(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau, \quad \textit{for a.e. } (t,x)\in\Omega', \tag{3.2.7}$$

where A is as in (3.2.6) and R is as in (3.1.5).

Proof. Let $v \in \widetilde{H}^1(\Omega')$ be a solution of (3.2.3) in Ω' and consider the change of variables

$$\begin{cases} \xi = t - x, \\ \eta = t + x. \end{cases} \tag{3.2.8}$$

Then the function $V(\xi,\eta) := v(\frac{\xi+\eta}{2}, \frac{\eta-\xi}{2})$ satisfies (in the sense of distributions)

$$V_{\xi\eta} = \frac{\nu^2}{4}V \qquad \text{in } \Lambda', \tag{3.2.9}$$

where Λ' is the image of Ω' through (3.2.8).

Integrating (3.2.9) over the image of $R(t,x)$ through (3.2.8) and reverting to the original variables (t,x) one gets representation formula (3.2.7) (imposing initial data and boundary conditions).

Now assume that $v \in \widetilde{H}^1(\Omega')$ satisfies (3.2.7); then using Lemma 3.2.9 and recalling that $A_{tt} = A_{xx}$ (weakly) we can conclude. □

Remark 3.2.7. An analogous statement holds true for a solution u of (3.0.1), replacing (3.2.7) by

$$u(t,x) = \widehat{A}(t,x) - \frac{\nu}{2}\iint_{R(t,x)} u_t(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau, \quad \text{for a.e. } (t,x)\in\Omega', \tag{3.2.10}$$

where $\widehat{A}$ is obtained replacing v_0, v_1 and z by u_0, u_1 and w in (3.2.6).

For a better understanding of the function A and of the integral term we state the following two lemmas.

Lemma 3.2.8. *Fix $\ell_0 > 0$ and consider v_0, v_1 and z satisfying (3.2.1) and (3.2.2). Assume that $\ell\colon [0,+\infty) \to [\ell_0,+\infty)$ satisfies (3.1.1).*

Then the function A defined in (3.2.6) is continuous on $\overline{\Omega'}$ and it belongs to $\widetilde{H}^1(\Omega')$; moreover, setting $A \equiv 0$ outside $\overline{\Omega}$, for every $t \in \left[0, \frac{\ell_0}{2}\right]$ it holds true:

$$\frac{A(t+h,\cdot)-A(t,\cdot)}{h} \xrightarrow[h\to 0]{} A_t(t,\cdot), \quad \textit{a.e. in } [0,+\infty) \textit{ and in } L^2(0,+\infty), \tag{3.2.11a}$$

$$\frac{A(t,\cdot+h)-A(t,\cdot)}{h} \xrightarrow[h\to 0]{} A_x(t,\cdot), \quad \textit{a.e. in } [0,+\infty) \textit{ and in } L^2(0,+\infty), \tag{3.2.11b}$$

where for every $t \in \left[0, \frac{\ell_0}{2}\right]$ and for a.e. $x \in [0,+\infty)$

$$A_t(t,x) = \begin{cases} \dot a_1(t+x) + \dot a_2(t-x), & \textit{if } x \in (0,\ell(t)), \\ 0, & \textit{if } x \in (\ell(t),+\infty), \end{cases}$$

$$A_x(t,x) = \begin{cases} \dot a_1(t+x) - \dot a_2(t-x), & \textit{if } x \in (0,\ell(t)), \\ 0, & \textit{if } x \in (\ell(t),+\infty), \end{cases}$$

being a_1 *and* a_2 *as in* (3.2.5).

Furthermore A_t *and* A_x *belong to* $C^0([0,\frac{\ell_0}{2}];L^2(0,+\infty))$ *and hence in particular* A *belongs to* $C^0([0,\frac{\ell_0}{2}];H^1(0,+\infty))\cap C^1([0,\frac{\ell_0}{2}];L^2(0,+\infty))$.

Proof. By the following explicit expression of A,

$$A(t,x)=\begin{cases} a_1(t{+}x)+a_2(t{-}x), & \text{for every } (t,x)\in\overline{\Omega'},\\ 0, & \text{for every } (t,x)\in[0,+\infty)^2\setminus\overline{\Omega},\end{cases}$$

and recalling that a_1 and a_2 belong to $\widetilde{H}^1(0,2t^*)$ and $H^1(-\ell_0,\ell_0)$ respectively, we deduce that $A\in\widetilde{H}^1(\Omega')\cap C^0(\overline{\Omega'})$.

By classical results on Sobolev functions and exploiting the fact that $A(t,\ell(t))=0$ for every $t\in[0,t^*]$ it is easy to see that for every $t\in\left[0,\frac{\ell_0}{2}\right]$ (3.2.11b) holds. Similarly one can show that for every $t\in\left[0,\frac{\ell_0}{2}\right]$ the difference quotient in (3.2.11a) converges to $A_t(t,x)$ for a.e. $x\in(0,+\infty)$; to prove that it converges even in the sense of $L^2(0,+\infty)$ we compute (we assume $h>0$, being the other case analogous):

$$\begin{aligned}&\int_0^{+\infty}\left|\frac{A(t{+}h,x)-A(t,x)}{h}-A_t(t,x)\right|^2\mathrm{d}x\\ &=\int_0^{\ell(t)}\left|\frac{a_1(t{+}h{+}x){-}a_1(t{+}x)}{h}-\dot a_1(t{+}x)+\frac{a_2(t{+}h{-}x){-}a_2(t{-}x)}{h}-\dot a_2(t{-}x)\right|^2\mathrm{d}x\\ &\quad+\frac{1}{h^2}\int_{\ell(t)}^{\ell(t+h)}|A(t{+}h,x)|^2\,\mathrm{d}x.\end{aligned}$$

The first integral tends to zero as $h\to 0^+$ since a_1 and a_2 are Sobolev functions, while for the second one we argue as follows:

$$\begin{aligned}\frac{1}{h^2}\int_{\ell(t)}^{\ell(t+h)}|A(t{+}h,x)|^2\,\mathrm{d}x&=\frac{1}{h^2}\int_{\ell(t)}^{\ell(t+h)}\left|\int_{\ell(t+h)}^{x}\big(\dot a_1(t{+}h{+}s)-\dot a_2(t{+}h{-}s)\big)\,\mathrm{d}s\right|^2\mathrm{d}x\\ &\le\frac{1}{h^2}\int_{\ell(t)}^{\ell(t+h)}(\ell(t{+}h)-\ell(t))\int_{\ell(t)}^{\ell(t+h)}\left|\dot a_1(t{+}h{+}s)-\dot a_2(t{+}h{-}s)\right|^2\mathrm{d}s\,\mathrm{d}x\\ &=\left(\frac{\ell(t{+}h)-\ell(t)}{h}\right)^2\int_{\ell(t)}^{\ell(t+h)}\left|\dot a_1(t{+}h{+}s)-\dot a_2(t{+}h{-}s)\right|^2\mathrm{d}s\\ &\le 2\int_{\ell(t)+t+h}^{\ell(t+h)+t+h}|\dot a_1(y)|^2\,\mathrm{d}y+2\int_{t+h-\ell(t+h)}^{t+h-\ell(t)}|\dot a_2(y)|^2\,\mathrm{d}y,\end{aligned}$$

and by dominated convergence we deduce it goes to zero as $h\to 0^+$ too, so (3.2.11a) is proved.

The fact that A_t and A_x are continuous in $L^2(0,+\infty)$ follows from the continuity of translations in $L^2(0,+\infty)$, arguing as before. □

Next lemma instead is related to the integral term appearing in (3.2.7):

Lemma 3.2.9. *Fix* $\ell_0>0$ *and assume that* $\ell\colon[0,+\infty)\to[\ell_0,+\infty)$ *satisfies* (3.1.1). *Let* $F\in\widetilde{L}^2(\Omega')$ *and for every* $(t,x)\in\Omega'$ *let*

$$H(t,x)=\iint_{R(t,x)}F(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau=\int_0^t\int_{\gamma_1(\tau;t,x)}^{\gamma_2(\tau;t,x)}F(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau. \tag{3.2.12}$$

Then H is continuous on $\overline{\Omega'}$ and it belongs to $\widetilde{H}^1(\Omega')$; moreover, setting $H\equiv 0$ outside $\overline{\Omega}$, for every $t\in\left[0,\frac{\ell_0}{2}\right]$ it holds true:

$$\frac{H(t+h,\cdot)-H(t,\cdot)}{h}\xrightarrow[h\to 0]{} H_t(t,\cdot),\quad \text{a.e. in } [0,+\infty) \text{ and in } L^2(0,+\infty), \tag{3.2.13a}$$

$$\frac{H(t,\cdot+h)-H(t,\cdot)}{h}\xrightarrow[h\to 0]{} H_x(t,\cdot),\quad \text{a.e. in } [0,+\infty) \text{ and in } L^2(0,+\infty), \tag{3.2.13b}$$

where for every $t\in\left[0,\frac{\ell_0}{2}\right]$ and for a.e. $x\in(0,+\infty)$

$$H_t(t,x)=\begin{cases}\displaystyle\int_0^t[F(\tau,\gamma_2(\tau;t,x))(\gamma_2)_t(\tau;t,x)-F(\tau,\gamma_1(\tau;t,x))(\gamma_1)_t(\tau;t,x)]\,\mathrm{d}\tau, & \text{if } x\in(0,\ell(t)),\\ 0, & \text{if } x\in(\ell(t),+\infty).\end{cases}$$

$$H_x(t,x)=\begin{cases}\displaystyle\int_0^t[F(\tau,\gamma_2(\tau;t,x))(\gamma_2)_x(\tau;t,x)-F(\tau,\gamma_1(\tau;t,x))(\gamma_1)_x(\tau;t,x)]\,\mathrm{d}\tau, & \text{if } x\in(0,\ell(t)),\\ 0, & \text{if } x\in(\ell(t),+\infty).\end{cases}$$

Furthermore H_t and H_x belong to $C^0([0,\frac{\ell_0}{2}];L^2(0,+\infty))$ and hence in particular H belongs to $C^0([0,\frac{\ell_0}{2}];H^1(0,+\infty))\cap C^1([0,\frac{\ell_0}{2}];L^2(0,+\infty))$.

Proof. The continuity of H in $\overline{\Omega'}$ follows from the absolute continuity of the integral.

We define $G(\tau;t,x):=\displaystyle\int_{\gamma_1(\tau;t,x)}^{\gamma_2(\tau;t,x)}F(\tau,\sigma)\,\mathrm{d}\sigma$, so that $H(t,x)=\displaystyle\int_0^t G(\tau;t,x)\,\mathrm{d}\tau$, and we notice that for every $t\in\left[0,\frac{\ell_0}{2}\right]$ the function $(x,\tau)\mapsto G(\tau;t,x)$ satisfies the assumptions of Theorem B.0.8; hence, exploiting the fact that $H(t,\ell(t))=0$ for every $t\in[0,t^*]$ and recalling Remark B.0.10, we get that $H(t,\cdot)$ belongs to $H^1(0,+\infty)$ and so (3.2.13b) follows. By direct computations one can show that for every $t\in\left[0,\frac{\ell_0}{2}\right]$ the difference quotient in (3.2.13a) converges to $H_t(t,x)$ for a.e. $x\in(0,+\infty)$; to prove that it converges even in the sense of $L^2(0,+\infty)$ we compute (we assume $h>0$):

$$\int_0^{+\infty}\left|\frac{H(t+h,x)-H(t,x)}{h}-H_t(t,x)\right|^2\mathrm{d}x=\int_0^{\ell(t)}\left|\frac{H(t+h,x)-H(t,x)}{h}-H_t(t,x)\right|^2\mathrm{d}x$$
$$+\frac{1}{h^2}\int_{\ell(t)}^{\ell(t+h)}|H(t+h,x)|^2\,\mathrm{d}x.$$

It is easy to see that the first integral goes to zero as $h\to 0^+$, while for the second one we estimate:

$$\begin{aligned}\frac{1}{h^2}\int_{\ell(t)}^{\ell(t+h)}|H(t+h,x)|^2\,\mathrm{d}x&\le\frac{1}{h^2}\int_{\ell(t)}^{\ell(t+h)}|R(t+h,x)|\left(\iint_{R(t+h,x)}|F(\tau,\sigma)|^2\,\mathrm{d}\sigma\,\mathrm{d}\tau\right)\mathrm{d}x\\&\le\frac{1}{h^2}\int_{\ell(t)}^{\ell(t+h)}h(t+h)\left(\iint_{\widetilde{R}_h(t)}|F(\tau,\sigma)|^2\,\mathrm{d}\sigma\,\mathrm{d}\tau\right)\mathrm{d}x\\&\le(t+h)\left(\frac{\ell(t{+}h)-\ell(t)}{h}\right)\iint_{\widetilde{R}_h(t)}|F(\tau,\sigma)|^2\,\mathrm{d}\sigma\,\mathrm{d}\tau\\&\le(t+h)\iint_{\widetilde{R}_h(t)}|F(\tau,\sigma)|^2\,\mathrm{d}\sigma\,\mathrm{d}\tau=:(*),\end{aligned}$$

where we introduced the set $\widetilde{R}_h(t):=\{(\tau,\sigma)\in\Omega\mid 0<\tau<t{+}h,\ \tau{-}t{-}h{+}\ell(t)<\sigma<\tau{-}t{+}\ell(t)\}$. By dominated convergence $(*)$ goes to zero as $h\to 0^+$, so (3.2.13a) is proved.

We conclude recalling that, arguing as before, the continuity of translations in the space $L^2(0,+\infty)$ ensures that H_t and H_x belong to $C^0([0,\frac{\ell_0}{2}];L^2(0,+\infty))$ (exploiting the definition of γ_1 and γ_2 given by (3.1.6)). In particular this yields $H\in\widetilde{H}^1(\Omega')$. □

Remark 3.2.10. By (3.1.6) one gets that for every $t\in\left[0,\frac{\ell_0}{2}\right]$ more explicit expressions for $H_t(t,\cdot)$ and $H_x(t,\cdot)$, valid for a.e. $x\in(0,\ell(t))$, are respectively

$$H_t(t,x)=\begin{cases}\displaystyle\int_0^t F(\tau,x{+}t{-}\tau)\,\mathrm{d}\tau+\int_0^t F(\tau,x{-}t{+}\tau)\,\mathrm{d}\tau, & \Omega_1',\\ \displaystyle\int_0^t F(\tau,x{+}t{-}\tau)\,\mathrm{d}\tau-\int_0^{t-x} F(\tau,t{-}x{-}\tau)\,\mathrm{d}\tau+\int_{t-x}^t F(\tau,x{-}t{+}\tau)\,\mathrm{d}\tau, & \Omega_2',\\ \displaystyle\int_0^t F(\tau,x{-}t{+}\tau)\,\mathrm{d}\tau-\dot\omega(x{+}t)\int_0^{\psi^{-1}(x+t)} F(\tau,\tau{-}\omega(x{+}t))\,\mathrm{d}\tau+\int_{\psi^{-1}(x+t)}^t F(\tau,x{+}t{-}\tau)\,\mathrm{d}\tau, & \Omega_3',\end{cases} \tag{3.2.14a}$$

$$H_x(t,x)=\begin{cases}\displaystyle\int_0^t F(\tau,x{+}t{-}\tau)\,\mathrm{d}\tau-\int_0^t F(\tau,x{-}t{+}\tau)\,\mathrm{d}\tau, & \Omega_1',\\ \displaystyle\int_0^t F(\tau,x{+}t{-}\tau)\,\mathrm{d}\tau+\int_0^{t-x} F(\tau,t{-}x{-}\tau)\,\mathrm{d}\tau-\int_{t-x}^t F(\tau,x{-}t{+}\tau)\,\mathrm{d}\tau, & \Omega_2',\\ \displaystyle-\int_0^t F(\tau,x{-}t{+}\tau)\,\mathrm{d}\tau-\dot\omega(x{+}t)\int_0^{\psi^{-1}(x+t)} F(\tau,\tau{-}\omega(x{+}t))\,\mathrm{d}\tau+\int_{\psi^{-1}(x+t)}^t F(\tau,x{+}t{-}\tau)\,\mathrm{d}\tau, & \Omega_3'.\end{cases} \tag{3.2.14b}$$

Since by Lemmas 3.2.8 and 3.2.9 the right-hand side in (3.2.7) is continuous on $\overline{\Omega'}$, every solution $v\in\widetilde{H}^1(\Omega')$ of problem (3.2.3) admits a representative, still denoted by v, which is continuous on $\overline{\Omega'}$ and such that (exploiting (3.2.6) and (3.2.12)):

- $v(t,\ell(t))=0$ for every $t\in[0,t^*]$,
- $v(t,0)=z(t)$ for every $t\in[0,\ell_0]$,
- $v(0,x)=v_0(x)$ for every $x\in[0,\ell_0]$.

Moreover (the continuous representative of) the solution v belongs to $C^0([0,\frac{\ell_0}{2}];H^1(0,+\infty))$ and to $C^1([0,\frac{\ell_0}{2}];L^2(0,+\infty))$ and by (3.2.11a), (3.2.13a) and (3.2.6), (3.2.14a) we deduce:

- $v_t(t,\cdot)\xrightarrow[t\to 0^+]{L^2(0,\ell_0)} v_1$,
- $v_t(0,x)=v_1(x)$ for a.e. $x\in[0,\ell_0]$.

By the explicit formulas (3.2.8) and (3.2.14b) we also obtain that for a.e $t\in\left[0,\frac{\ell_0}{2}\right]$ the following equalities hold true:

$$v_x(t,0)=-\dot z(t)+\dot v_0(t)+v_1(t)+\frac{\nu^2}{4}\int_0^t v(\tau,t{-}\tau)\,\mathrm{d}\tau, \tag{3.2.15a}$$

$$v_x(t,\ell(t))=\frac{1}{1+\dot\ell(t)}\left[\dot v_0(\ell(t){-}t)-v_1(\ell(t){-}t)-\frac{\nu^2}{4}\int_0^t v(\tau,\tau+\ell(t){-}t)\,\mathrm{d}\tau\right]. \tag{3.2.15b}$$

This shows that the functions $v_x(\cdot,0)$ and $v_x(\cdot,\ell(\cdot))$, and thus $u_x(\cdot,0)$ and $u_x(\cdot,\ell(\cdot))$, are well-defined for almost every time $t\in\left[0,\frac{\ell_0}{2}\right]$. Reasoning as in Remark 3.3.2 one can extend (3.2.15) to the whole $[0,+\infty)$.

In order to find existence (and uniqueness) of solutions to problem (3.2.3), and hence to problem (3.0.1), we look for a fixed point of the linear operator $\mathfrak{L}\colon C^0(\overline{\Omega'})\to C^0(\overline{\Omega'})$ defined as:

$$\mathfrak{L}v(t,x):=A(t,x)+\frac{\nu^2}{8}\iint_{R(t,x)} v(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau. \tag{3.2.16}$$

Proposition 3.2.11. *Fix $\nu \geq 0$, $\ell_0 > 0$ and consider v_0, v_1 and z satisfying (3.2.1) and (3.2.2). Assume that $\ell\colon [0,+\infty) \to [\ell_0,+\infty)$ satisfies (3.1.1).*

If $T \in \left(0, \frac{\ell_0}{2}\right)$ satisfies $\nu^2\ell_0 T < 4$, then the map $\mathfrak{L}$ in (3.2.16) is a contraction from $C^0(\overline{\Omega_T})$ into itself.

Proof. By Lemmas 3.2.8 and 3.2.9 operator $\mathfrak{L}$ maps $C^0(\overline{\Omega_T})$ into itself. Pick $v^1, v^2 \in C^0(\overline{\Omega_T})$ and let $(t,x) \in \overline{\Omega_T}$, then

$$|\mathfrak{L}v^1(t,x) - \mathfrak{L}v^2(t,x)| \leq \frac{\nu^2}{8}\iint_{R(t,x)} |v^1(\tau,\sigma) - v^2(\tau,\sigma)|\,\mathrm{d}\sigma\,\mathrm{d}\tau \leq \frac{\nu^2}{8}|R(t,x)|\|v^1 - v^2\|_{C^0(\overline{\Omega_T})}$$
$$\leq \frac{\nu^2}{8}|\Omega_T|\|v^1 - v^2\|_{C^0(\overline{\Omega_T})} \leq \frac{\nu^2\ell_0 T}{4}\|v^1 - v^2\|_{C^0(\overline{\Omega_T})}.$$

Since $\nu^2\ell_0 T < 4$ we conclude. □

We are now in a position to state and prove the first main result of the chapter, regarding the existence and uniqueness of solutions of (3.2.3), and hence of (3.0.1) (see Remark 3.2.3), when the debonding front ℓ is assigned:

Theorem 3.2.12. *Fix $\nu \geq 0$, $\ell_0 > 0$ and consider v_0, v_1 and z satisfying (3.2.1) and (3.2.2). Assume that $\ell\colon [0,+\infty) \to [\ell_0,+\infty)$ satisfies (3.1.1).*

Then there exists a unique $v \in \widetilde{H}^1(\Omega)$ solution of (3.2.3). Moreover v has a continuous representative on $\overline{\Omega}$, still denoted by v, and, setting $v \equiv 0$ outside $\overline{\Omega}$, it holds:

$$v \in C^0([0,+\infty); H^1(0,+\infty)) \cap C^1([0,+\infty); L^2(0,+\infty)).$$

Proof. By Proposition 3.2.11 we deduce the existence of a unique continuous function v^1 satisfying (3.2.7) in $\overline{\Omega_{T_1}}$, taking for istance $T_1 = \frac{1}{2}\min\left\{\frac{\ell_0}{2}, \frac{4}{\nu^2\ell_0}\right\}$ $\left(T_1 = \frac{\ell_0}{4} \text{ if } \nu = 0\right)$.

By Lemmas 3.2.8 and 3.2.9 one gets that v^1 is in $H^1(\Omega_{T_1})$ and moreover that it belongs to $C^0([0,T_1]; H^1(0,+\infty)) \cap C^1([0,T_1]; L^2(0,+\infty))$, while Proposition 3.2.6 ensures that v^1 solves problem (3.2.3) in Ω_{T_1}.

Now we can restart the argument from time T_1 replacing ℓ_0 by $\ell_1 := \ell(T_1)$, v_0 by $v^1(T_1,\cdot)$ and v_1 by $v^1_t(T_1,\cdot)$; indeed notice that $v^1(T_1,\cdot) \in H^1(0,\ell_1)$, $v^1_t(T_1,\cdot) \in L^2(0,\ell_1)$ and that they satisfy the compatibility conditions $v^1(T_1,0) = z(T_1)$ and $v^1(T_1,\ell_1) = 0$. Arguing as before we get the existence of a unique solution v^2 of (3.2.3) in $\Omega_{T_2} \setminus \Omega_{T_1}$, with $T_2 = T_1 + \frac{1}{2}\min\left\{\frac{\ell_1}{2}, \frac{4}{\nu^2\ell_1}\right\}$, belonging to $C^0([T_1,T_2]; H^1(0,+\infty)) \cap C^1([T_1,T_2]; L^2(0,+\infty))$.

Then the function $v(t,x) = \begin{cases} v^1(t,x), & \text{if } (t,x) \in \overline{\Omega_{T_1}}, \\ v^2(t,x), & \text{if } (t,x) \in \overline{\Omega_{T_2} \setminus \Omega_{T_1}}, \end{cases}$ is in $C^0([0,T_2]; H^1(0,+\infty))$ and in $C^1([0,T_2]; L^2(0,+\infty))$ and it is easy to see that it is the only solution of (3.2.3) in Ω_{T_2}.

To conclude we need to prove that the sequence of times $\{T_k\}$ defined recursively by

$$\begin{cases} T_k = T_{k-1} + \frac{1}{2}\min\left\{\frac{\ell(T_{k-1})}{2}, \frac{4}{\nu^2\ell(T_{k-1})}\right\}, & \text{if } k \geq 1, \\ T_0 = 0, \end{cases}$$

diverges. This follows easily observing that $\{T_k\}$ is increasing and recalling that $0 < \ell(t) < +\infty$ for every $t \in [0,+\infty)$. □

Remark 3.2.13 (**Regularity**)**.** If we assume $v_0 \in C^{0,1}([0,\ell_0])$, $v_1 \in L^\infty(0,\ell_0)$, $z \in \widetilde{C}^{0,1}([0,+\infty))$ satisfy the compatibility conditions (3.2.2), then by (3.2.6) and (3.2.14) the (continuous representative of the) solution v belongs to $\widetilde{C}^{0,1}(\overline{\Omega})$ and $v_t(t,\cdot) \in L^\infty(0,+\infty)$ for every $t \in [0,+\infty)$.

Remark 3.2.14 (More regularity). If we assume more regularity on v_0, v_1, z and on the debonding front ℓ, in order to get that the solution v possesses the same regularity we need to add more compatibility conditions. For instance, if $\ell \in \widetilde{C}^{1,1}([0,+\infty))$ satisfies (3.1.1b), if $v_0 \in C^{1,1}([0,\ell_0])$, $v_1 \in C^{0,1}([0,\ell_0])$, $z \in \widetilde{C}^{1,1}([0,+\infty))$ satisfy (3.2.2), to get $v \in \widetilde{C}^{1,1}(\overline{\Omega})$ we also need to assume the following first order compatibility conditions:

$$v_1(0) = \dot{z}(0) \quad \text{ and } \quad v_1(\ell_0) + \dot{\ell}(0)\dot{v}_0(\ell_0) = 0. \tag{3.2.17}$$

Indeed, under these assumptions the function A in (3.2.6) belongs to $\widetilde{C}^{1,1}(\overline{\Omega'})$; moreover, exploiting (3.2.14) and the fact that by Remark 3.2.13 we already know that the solution v is in $\widetilde{C}^{0,1}(\overline{\Omega})$, one can deduce that the function $H(t,x) = \iint_{R(t,x)} v(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau$ in (3.2.12) belongs to $\widetilde{C}^{1,1}(\overline{\Omega'})$ too. Hence representation formula (3.2.7) ensures that v belongs to $C^{1,1}(\overline{\Omega_{T_1}})$ for some $T_1 \in \left(0, \frac{\ell_0}{2}\right)$; since $v(t,0) = z(t)$ and $v(t,\ell(t)) = 0$ for every $t \in [0,+\infty)$ we notice that condition (3.2.17) holds at time T_1 too, and reasoning as in the proof of Theorem 3.2.12 one can conclude.

We also notice that, coming back to u_0, u_1 and w, (3.2.17) is equivalent to

$$u_1(0) = \dot{w}(0) \quad \text{ and } \quad u_1(\ell_0) + \dot{\ell}(0)\dot{u}_0(\ell_0) = 0. \tag{3.2.18}$$

We conclude this first section pointing out that the choice of working with H^1 and L^2 functions is only due to the energetic considerations we make in the next Sections in order to formulate the coupled problem. Indeed all the results presented up to now still remains valid in a $W^{1,1}$ and L^1 setting, with the obvious changes.

3.3 Energetic analysis

This section is devoted to the study of the energy of the solution u to problem (3.0.1) given by Theorem 3.2.12 and Remark 3.2.3; this analysis will be used in Section 3.4 to introduce the notion of dynamic energy release rate.

Fix $\nu \geq 0$, $\ell_0 > 0$ and a function $\ell\colon [0,+\infty) \to [\ell_0,+\infty)$ satisfying (3.1.1), and consider u_0, u_1 and w satisfying (3.2.1) and (3.2.2); let u be the solution of (3.0.1) associated with ℓ, u_0, u_1 and w. For $t \in [0,+\infty)$ we first introduce the internal energy of u, namely the sum of kinetic energy:

$$\mathcal{K}(t) := \frac{1}{2}\int_0^{\ell(t)} u_t(t,\sigma)^2\,\mathrm{d}\sigma.$$

and potential energy:

$$\mathcal{E}(t) := \frac{1}{2}\int_0^{\ell(t)} u_x(t,\sigma)^2\,\mathrm{d}\sigma.$$

We then present the energy dissipated by viscosity:

$$\mathcal{V}(t) := \nu\int_0^t\int_0^{\ell(\tau)} u_t(\tau,\sigma)^2\,\mathrm{d}\sigma\,\mathrm{d}\tau,$$

and for the sake of clarity we also consider their sum:

$$\mathcal{S}(t) := \mathcal{K}(t) + \mathcal{E}(t) + \mathcal{V}(t). \tag{3.3.1}$$

As in Section 3.2 we introduce the auxiliary function $v(t,x) = e^{\nu t/2}u(t,x)$ and we consider v_0 and v_1 given by (3.2.4).

Proposition 3.3.1. *The function $\mathcal{S}$ defined in (3.3.1) belongs to $AC([0,+\infty))$ and for a.e. $t\in\left[0,\frac{\ell_0}{2}\right]$ the following formulas hold true:*

$$\begin{aligned}\dot{\mathcal{S}}(t)=&-\frac{\dot\ell(t)}{2}\frac{1-\dot\ell(t)}{1+\dot\ell(t)}\left[\dot u_0(\ell(t){-}t)-u_1(\ell(t){-}t)+\nu\int_0^t u_t(\tau,\tau{-}t{+}\ell(t))\,\mathrm{d}\tau\right]^2\\&+\dot w(t)\left[\dot w(t)-\left(\dot u_0(t)+u_1(t)-\nu\int_0^t u_t(\tau,t{-}\tau)\,\mathrm{d}\tau\right)\right],\end{aligned}\tag{3.3.2a}$$

$$\begin{aligned}\dot{\mathcal{S}}(t)=&-\frac{\dot\ell(t)}{2}\frac{1-\dot\ell(t)}{1+\dot\ell(t)}e^{-\nu t}\left[\dot v_0(\ell(t){-}t)-v_1(\ell(t){-}t)-\frac{\nu^2}{4}\int_0^t v(\tau,\tau{-}t{+}\ell(t))\,\mathrm{d}\tau\right]^2\\&+\dot w(t)\left[\dot w(t)+\frac{\nu}{2}w(t)-e^{-\frac{\nu t}{2}}\left(\dot v_0(t)+v_1(t)+\frac{\nu^2}{4}\int_0^t v(\tau,t{-}\tau)\,\mathrm{d}\tau\right)\right],\end{aligned}\tag{3.3.2b}$$

where the products between $1-\dot\ell(t)$ and the expressions within square brackets are meant as in Remark B.0.2.

Remark 3.3.2. One can obtain similar formulas for $\dot{\mathcal{S}}$ which are valid for a.e. $t\in[0,+\infty)$ arguing in the following way: fix $t_0>0$, then for a.e. $t\in\left[t_0,t_0+\frac{\ell(t_0)}{2}\right]$

$$\begin{aligned}\dot{\mathcal{S}}(t)=&-\frac{\dot\ell(t)}{2}\frac{1-\dot\ell(t)}{1+\dot\ell(t)}\left[u_x(t_0,\ell(t){-}t{+}t_0)-u_t(t_0,\ell(t){-}t{+}t_0)+\nu\int_{t_0}^t u_t(\tau,\tau{-}t{+}\ell(t))\,\mathrm{d}\tau\right]^2\\&+\dot w(t)\left[\dot w(t)-\left(u_x(t_0,t{-}t_0)+u_t(t_0,t{-}t_0)-\nu\int_{t_0}^t u_t(\tau,t{-}\tau)\,\mathrm{d}\tau\right)\right].\end{aligned}$$

and the analogous formula for (3.3.2b) holds.

Remark 3.3.3. We notice that by (3.2.15) we can also write for a.e. $t\in\left[0,\frac{\ell_0}{2}\right]$:

$$\begin{aligned}\dot{\mathcal{S}}(t)&=-\frac{\dot\ell(t)}{2}(1-\dot\ell(t)^2)u_x(t,\ell(t))^2-\dot w(t)u_x(t,0)\\&=-\frac{\dot\ell(t)}{2}(1-\dot\ell(t)^2)e^{-\nu t}v_x(t,\ell(t))^2-\dot w(t)e^{-\frac{\nu t}{2}}v_x(t,0).\end{aligned}$$

Proof of Proposition 3.3.1. Let us define $T:=\ell_0/2$; we notice that by Remark 3.3.2 it is enough to prove the proposition in the time interval $[0,T]$. By (3.2.10) we know that for every $(t,x)\in\overline{\Omega_T}$

$$u(t,x)=\hat a_1(t{+}x)+\hat a_2(t{-}x)-\frac{\nu}{2}\iint_{R(t,x)}u_t(\tau,\sigma)\,\mathrm{d}\tau\,\mathrm{d}\sigma,\tag{3.3.3}$$

where $\hat a_1$ and $\hat a_2$ are as in (3.2.5), replacing v_0, v_1 and z by u_0, u_1 and w, respectively.

Moreover, by (3.3.3), Lemma 3.2.9 and Remark 3.2.10 we get for every $t\in[0,T]$

$$u_t(t,x)=\dot{\hat a}_1(t{+}x)+\dot{\hat a}_2(t{-}x)-\frac{\nu}{2}h_1(t,x)-\frac{\nu}{2}h_2(t,x),\qquad\text{for a.e. } x\in[0,\ell(t)],$$

$$u_x(t,x)=\dot{\hat a}_1(t{+}x)-\dot{\hat a}_2(t{-}x)-\frac{\nu}{2}h_1(t,x)+\frac{\nu}{2}h_2(t,x),\qquad\text{for a.e. } x\in[0,\ell(t)],$$

where

$$h_1(t,x) = \begin{cases} \displaystyle\int_0^t u_t(\tau, t{+}x{-}\tau)\,\mathrm{d}\tau, & \text{if } 0 \le x \le \ell_0{-}t, \\ \displaystyle -\dot\omega(t{+}x)\int_0^{\psi^{-1}(t+x)} u_t(\tau,\tau{-}\omega(t{+}x))\,\mathrm{d}\tau + \int_{\psi^{-1}(t+x)}^t u_t(\tau,t{+}x{-}\tau)\,\mathrm{d}\tau, & \text{if } \ell_0{-}t < x \le \ell(t), \end{cases}$$

$$h_2(t,x) = \begin{cases} \displaystyle\int_0^t u_t(\tau,\tau{-}t{+}x)\,\mathrm{d}\tau, & \text{if } t \le x \le \ell(t), \\ \displaystyle -\int_0^{t-x} u_t(\tau,t{-}x{-}\tau)\,\mathrm{d}\tau + \int_{t-x}^t u_t(\tau,\tau{-}t{+}x)\,\mathrm{d}\tau, & \text{if } 0 \le x < t. \end{cases}$$

Now we compute:

$$\begin{aligned}
\mathcal{K}(t)+\mathcal{E}(t) &= \frac12\int_0^{\ell(t)}\Big(\dot{\hat a}_1(t{+}x)+\dot{\hat a}_2(t{-}x)-\frac{\nu}{2}h_1(t,x)-\frac{\nu}{2}h_2(t,x)\Big)^2\,\mathrm{d}x \\
&\quad + \frac12\int_0^{\ell(t)}\Big(\dot{\hat a}_1(t{+}x)-\dot{\hat a}_2(t{-}x)-\frac{\nu}{2}h_1(t,x)+\frac{\nu}{2}h_2(t,x)\Big)^2\,\mathrm{d}x \\
&= \frac12\int_0^{\ell(t)}\Big[\Big(\dot{\hat a}_1(t{+}x)-\frac{\nu}{2}h_1(t,x)\Big)^2+\Big(\dot{\hat a}_2(t{-}x)-\frac{\nu}{2}h_2(t,x)\Big)^2\Big]\,\mathrm{d}x \\
&= \int_t^{t+\ell(t)}\Big[\dot{\hat a}_1(y)-\frac{\nu}{2}h_1(t,y{-}t)\Big]^2\,\mathrm{d}y + \int_{t-\ell(t)}^t\Big[\dot{\hat a}_2(y)-\frac{\nu}{2}h_2(t,t{-}y)\Big]^2\,\mathrm{d}y \\
&= \int_t^{\ell_0}\Big[\frac{\dot u_0(y)+u_1(y)}{2}-\frac{\nu}{2}\int_0^t u_t(\tau,y{-}\tau)\,\mathrm{d}\tau\Big]^2\,\mathrm{d}y \\
&\quad + \int_{t-\ell(t)}^0\Big[\frac{\dot u_0(-y)-u_1(-y)}{2}+\frac{\nu}{2}\int_0^t u_t(\tau,\tau{-}y)\,\mathrm{d}\tau\Big]^2\,\mathrm{d}y \\
&\quad + \int_{\ell_0}^{t+\ell(t)}\Big[\dot\omega(y)\Big(\frac{\dot u_0(-\omega(y)){-}u_1(-\omega(y))}{2}+\frac{\nu}{2}\int_0^{\psi^{-1}(y)}u_t(\tau,\tau{-}\omega(y))\,\mathrm{d}\tau\Big)-\frac{\nu}{2}\int_{\psi^{-1}(y)}^t u_t(\tau,y{-}\tau)\,\mathrm{d}\tau\Big]^2\mathrm{d}y \\
&\quad + \int_0^t\Big[\dot w(y)-\frac{\dot u_0(y)+u_1(y)}{2}+\frac{\nu}{2}\int_0^y u_t(\tau,y{-}\tau)\,\mathrm{d}\tau-\frac{\nu}{2}\int_y^t u_t(\tau,\tau{-}y)\,\mathrm{d}\tau\Big]^2\,\mathrm{d}y.
\end{aligned}$$

It is easy to check that we can apply Theorem B.0.8 in the Appendix, so we obtain that $\mathcal{K}{+}\mathcal{E}$ belongs to $AC([0,T])$ and that for a.e. $t\in[0,T]$ the following formula for its derivative holds true:

$$\begin{aligned}
\dot{\mathcal{K}}(t)+\dot{\mathcal{E}}(t) &= -\frac{\dot\ell(t)}{2}\frac{1-\dot\ell(t)}{1+\dot\ell(t)}\Big[\dot u_0(\ell(t){-}t)-u_1(\ell(t){-}t)+\nu\int_0^t u_t(\tau,\tau{-}t{+}\ell(t))\,\mathrm{d}\tau\Big]^2 \\
&\quad + \dot w(t)\Big[\dot w(t)-\Big(\dot u_0(t)+u_1(t)-\nu\int_0^t u_t(\tau,t{-}\tau)\,\mathrm{d}\tau\Big)\Big]-\nu\int_0^{\ell(t)}u_t(t,x)^2\,\mathrm{d}x.
\end{aligned}$$

Recalling that $\mathcal{V}$ is absolutely continuous by construction and that $\dot{\mathcal{V}}(t)=\nu\int_0^{\ell(t)}u_t(t,x)^2\,\mathrm{d}x$ for a.e. $t\in[0,T]$, we deduce that $\mathcal{S}$ belongs to $AC([0,T])$ and that formula (3.3.2a) holds.

To get (3.3.2b) one argues in the same way with $v(t,x)=e^{\nu t/2}u(t,x)$, rewriting the internal energy as

$$\mathcal{K}(t)+\mathcal{E}(t)=\frac{e^{-\nu t}}{2}\int_0^{\ell(t)}\Big[\Big(v_t(t,x)-\frac{\nu}{2}v(t,x)\Big)^2+v_x(t,x)^2\Big]\,\mathrm{d}x,$$

and recalling (3.2.7). □

3.4 Principles leading the debonding growth

In the first part of this section we introduce the dynamic energy release rate in the context of our model, following [24]. In the second one we will use it to formulate dynamic Griffith's criterion, namely the energy criterion which rules the evolution of the debonding front.

As before we fix $\nu \geq 0$, $\ell_0 > 0$ and we consider u_0, u_1 and w satisfying (3.2.1) and (3.2.2), but from now on the debonding front will be a function $\ell\colon [0,+\infty) \to [\ell_0,+\infty)$ satisfying (3.1.1a) and (3.1.4).

We want to underline that the requirement of (3.1.4) in place of (3.1.1b) is not merely a technical assumption needed to carry out all the mathematical arguments of the next Sections, although is crucial; it is instead a natural consequence of the Griffith's criterion the debonding front has to fulfill during its evolution, as the reader can check from the final formula (3.4.13).

3.4.1 Dynamic energy release rate

The notion of dynamic energy release has been developed in the framework of Fracture Mechanics to measure the amount of energy spent by the growth of the crack (see [35] for more information); it is defined as the opposite of the derivative of the energy with respect to the measure of the evolved crack.

To define it in the context of our debonding model we argue as in [24]: we fix $\bar t > 0$ and we consider a function $\tilde w \in \widetilde H^1(0,+\infty)$ and a function $\tilde\ell\colon [0,+\infty) \to [\ell_0,+\infty)$ satisfying (3.1.1a) and (3.1.4), and such that

$$\tilde w(t) = w(t) \quad \text{and} \quad \tilde\ell(t) = \ell(t) \quad \text{for every } t \in [0,\bar t\,].$$

Let u and $\tilde u$ be the solutions to problem (3.0.1) corresponding to ℓ, u_0, u_1, w and $\tilde\ell$, u_0, u_1, $\tilde w$, respectively, and for $t \in [0,+\infty)$ let us consider:

$$\mathcal{K}(t;\tilde\ell,\tilde w) := \frac12\int_0^{\tilde\ell(t)} \tilde u_t(t,\sigma)^2\,\mathrm{d}\sigma,$$

$$\mathcal{E}(t;\tilde\ell,\tilde w) := \frac12\int_0^{\tilde\ell(t)} \tilde u_x(t,\sigma)^2\,\mathrm{d}\sigma,$$

$$\mathcal{V}(t;\tilde\ell,\tilde w) := \nu\int_0^t\int_0^{\tilde\ell(\tau)} \tilde u_t(\tau,\sigma)^2\,\mathrm{d}\sigma\,\mathrm{d}\tau,$$

and

$$\mathcal{S}(t;\tilde\ell,\tilde w) := \mathcal{K}(t;\tilde\ell,\tilde w) + \mathcal{E}(t;\tilde\ell,\tilde w) + \mathcal{V}(t;\tilde\ell,\tilde w),$$

where we stressed the dependence on $\tilde\ell$ and on $\tilde w$.

The formal definition of dynamic energy release rate at time $\bar t$ should be:

$$G(\bar t\,) := \lim_{t\to\bar t^+} -\frac{\mathcal{S}(t;\tilde\ell,\bar w) - \mathcal{S}(\bar t;\ell,w)}{\tilde\ell(t) - \ell(\bar t\,)} = -\frac{1}{\dot{\tilde\ell}(\bar t\,)}\lim_{t\to\bar t^+}\frac{\mathcal{S}(t;\tilde\ell,\bar w) - \mathcal{S}(\bar t;\ell,w)}{t-\bar t}, \tag{3.4.1}$$

where $\bar w \in \widetilde H^1(0,+\infty)$ is the constant extension of w after $\bar t$.

Remark 3.4.1. The choice of the particular extension $\bar w$ in (3.4.1) is needed in order to avoid including the work done by the external loading in the energy dissipated to debond the tape.

By Proposition 3.3.1 (see also Remark 3.3.2) for a.e. $t \in \left[0, \frac{\ell_0}{2}\right]$ we have

$$\begin{aligned}\dot{\mathcal{S}}(t;\tilde{\ell},\tilde{w}) = &-\frac{\dot{\tilde{\ell}}(t)}{2}\frac{1-\dot{\tilde{\ell}}(t)}{1+\dot{\tilde{\ell}}(t)}e^{-\nu t}\left[\dot{v}_0(\tilde{\ell}(t)-t)-v_1(\tilde{\ell}(t)-t)-\frac{\nu^2}{4}\int_0^t \tilde{v}(\tau,\tau-t+\tilde{\ell}(t))\,\mathrm{d}\tau\right]^2\\ &+\dot{\tilde{w}}(t)\left[\dot{\tilde{w}}(t)+\frac{\nu}{2}\tilde{w}(t)-e^{-\frac{\nu t}{2}}\left(\dot{v}_0(t)+v_1(t)+\frac{\nu^2}{4}\int_0^t \tilde{v}(\tau,t-\tau)\,\mathrm{d}\tau\right)\right],\end{aligned}$$

where $\tilde{v}(t,x) = e^{\nu t/2}\tilde{u}(t,x)$ and v_0 and v_1 are given by (3.2.4).

Since in (3.4.1) we want to compute the right derivative of $\mathcal{S}(t;\tilde{\ell},\tilde{w})$ precisely at $t=\bar{t}$, we need a slight improvement of Proposition 3.3.1 (see Theorem 3.4.2 below and the analogous Proposition 2.1 in [24]). With this aim we will require that there exist $\alpha,\beta\in\mathbb{R}$ such that

$$\lim_{h\to 0^+}\frac{1}{h}\int_{\bar{t}}^{\bar{t}+h}\left|\dot{\tilde{\ell}}(t)-\alpha\right|\,\mathrm{d}t = 0, \tag{3.4.2a}$$

$$\lim_{h\to 0^+}\frac{1}{h}\int_{\bar{t}}^{\bar{t}+h}\left|\dot{\tilde{w}}(t)-\beta\right|^2\,\mathrm{d}t = 0. \tag{3.4.2b}$$

Theorem 3.4.2. *Fix $\nu\geq 0$, $\ell_0>0$ and consider u_0, u_1 and w satisfying (3.2.1) and (3.2.2). Assume that $\ell\colon[0,+\infty)\to[\ell_0,+\infty)$ satisfies (3.1.1a) and (3.1.4).*

Then there exists a set $N\subseteq[0,+\infty)$ of measure zero, depending only on ℓ, u_0, u_1 and w, such that for every $\bar{t}\in[0,+\infty)\setminus N$ the following statement holds true:
if v_0, v_1, $\tilde{\ell}$, $\tilde{w}$, $\tilde{u}$, $\tilde{v}$, u and v are as above, if $\dot{\tilde{\ell}}$ and $\dot{\tilde{w}}$ satisfy (3.4.2a) and (3.4.2b) respectively, then

$$\dot{\mathcal{S}}_r(\bar{t};\tilde{\ell},\tilde{w}) := \lim_{h\to 0^+}\frac{\mathcal{S}(\bar{t}+h;\tilde{\ell},\tilde{w})-\mathcal{S}(\bar{t};\tilde{\ell},\tilde{w})}{h} \qquad \textit{exists.}$$

Moreover, if $\bar{t}\in\left[0,\frac{\ell_0}{2}\right]\setminus N$, one has the explicit formula

$$\begin{aligned}\dot{\mathcal{S}}_r(\bar{t};\tilde{\ell},\tilde{w}) = &-\frac{\alpha}{2}\frac{1-\alpha}{1+\alpha}e^{-\nu\bar{t}}\left[\dot{v}_0(\ell(\bar{t})-\bar{t})-v_1(\ell(\bar{t})-\bar{t})-\frac{\nu^2}{4}\int_0^{\bar{t}} v(\tau,\tau-\bar{t}+\ell(\bar{t}))\,\mathrm{d}\tau\right]^2\\ &+\beta\left[\beta+\frac{\nu}{2}w(\bar{t})-e^{-\frac{\nu\bar{t}}{2}}\left(\dot{v}_0(\bar{t})+v_1(\bar{t})+\frac{\nu^2}{4}\int_0^{\bar{t}} v(\tau,\bar{t}-\tau)\,\mathrm{d}\tau\right)\right].\end{aligned}$$

Remark 3.4.3. One can obtain a similar formula for $\dot{\mathcal{S}}_r(\bar{t};\tilde{\ell},\tilde{w})$, valid for $\bar{t}\geq\frac{\ell_0}{2}$, reasoning as in Remark 3.3.2.

Proof of Theorem 3.4.2. Let us define $T:=\ell_0/2$; we notice that by Remarks 3.3.2 and 3.4.3 it is enough to prove the theorem in the time interval $[0,T]$.

We call $\rho_1(r):=\dot{v}_0(r)-v_1(r)$ and $\rho_2(r):=\dot{v}_0(r)+v_1(r)$ and we consider the points $\bar{t}\in[0,T]$ with the following properties:

a) $\displaystyle\lim_{h\to 0^+}\frac{1}{h}\int_{\bar{t}-\ell(\bar{t})}^{\bar{t}-\ell(\bar{t})+h}|\,(\rho_1(-r))^2-\big(\rho_1(\ell(\bar{t})-\bar{t})\big)^2|\,\mathrm{d}r = 0$, and
$\displaystyle\lim_{h\to 0^+}\frac{1}{h}\int_{\bar{t}-\ell(\bar{t})}^{\bar{t}-\ell(\bar{t})+h}|\rho_1(-r)-\rho_1(\ell(\bar{t})-\bar{t})|\,\mathrm{d}r = 0;$

b) $\displaystyle\lim_{h\to 0^+}\frac{1}{h}\int_{\bar{t}}^{\bar{t}+h}|\rho_2(r)-\rho_2(\bar{t})|^2\,\mathrm{d}r = 0.$

We call E_T the set of points satisfying a) and b). Since ρ_1 and ρ_2 belong to $L^2(0,\ell_0)$ and since ℓ satisfies (3.1.4) the set $N_T := [0,T]\setminus E_T$ has measure zero (see Corollary B.0.4). Let us fix $\bar t \in E_T$.

In the estimates below the symbol C is used to denote a constant, which may change from line to line, that does not depend on h, although it can depend on $\bar t$. For the sake of clarity we define $I_1(v,\ell)(t) := \frac{\nu^2}{4}\int_0^t v(\tau,\tau-t+\ell(t))\,\mathrm{d}\tau$ and $I_2(v)(t) := \frac{\nu^2}{4}\int_0^t v(\tau,t-\tau)\,\mathrm{d}\tau$, so that

$$\left|\frac{\mathcal{S}(\bar t+h;\tilde\ell,\tilde w)-\mathcal{S}(\bar t;\tilde\ell,\tilde w)}{h}+\frac{\alpha}{2}\frac{1-\alpha}{1+\alpha}e^{-\nu\bar t}\Big[\rho_1(\ell(\bar t)-\bar t)-I_1(v,\ell)(\bar t)\Big]^2-\beta\Big[\beta+\frac{\nu}{2}w(\bar t)-e^{-\frac{\nu\bar t}{2}}\Big(\rho_2(\bar t)+I_2(v)(\bar t)\Big)\Big]\right|$$

$$\leq \frac{1}{2h}\int\limits_{\bar t}^{\bar t+h}\left|\dot{\tilde\ell}(s)\frac{1-\dot{\tilde\ell}(s)}{1+\dot{\tilde\ell}(s)}e^{-\nu s}\Big[\rho_1(\tilde\ell(s)-s)-I_1(\tilde v,\tilde\ell)(s)\Big]^2-\alpha\frac{1-\alpha}{1+\alpha}e^{-\nu\bar t}\Big[\rho_1(\ell(\bar t)-\bar t)-I_1(v,\ell)(\bar t)\Big]^2\right|\mathrm{d}s$$

$$+\frac{1}{h}\int\limits_{\bar t}^{\bar t+h}\left|\dot{\tilde w}(s)\Big[\dot{\tilde w}(s)+\frac{\nu}{2}\tilde w(s)-e^{-\frac{\nu s}{2}}\Big(\rho_2(s)+I_2(\tilde v)(s)\Big)\Big]-\beta\Big[\beta+\frac{\nu}{2}w(\bar t)-e^{-\frac{\nu\bar t}{2}}\Big(\rho_2(\bar t)+I_2(v)(\bar t)\Big)\Big]\right|\mathrm{d}s.$$

We denote by J_1 and J_2 the first and the second integral respectively and we estimate:

$$\begin{aligned}
J_1 &\leq e^{-\nu\bar t}\Big[\rho_1(\ell(\bar t)-\bar t)-I_1(v,\ell)(\bar t)\Big]^2\frac{1}{2h}\int_{\bar t}^{\bar t+h}\left|\dot{\tilde\ell}(s)\frac{1-\dot{\tilde\ell}(s)}{1+\dot{\tilde\ell}(s)}-\alpha\frac{1-\alpha}{1+\alpha}\right|\mathrm{d}s\\
&+\Big[\rho_1(\ell(\bar t)-\bar t)-I_1(v,\ell)(\bar t)\Big]^2\frac{1}{2h}\int_{\bar t}^{\bar t+h}\dot{\tilde\ell}(s)\frac{1-\dot{\tilde\ell}(s)}{1+\dot{\tilde\ell}(s)}|e^{-\nu s}-e^{-\nu\bar t}|\,\mathrm{d}s\\
&+\frac{1}{2h}\int_{\bar t}^{\bar t+h}\dot{\tilde\ell}(s)\frac{1-\dot{\tilde\ell}(s)}{1+\dot{\tilde\ell}(s)}e^{-\nu s}\left|\Big[\rho_1(\tilde\ell(s)-s)-I_1(\tilde v,\tilde\ell)(s)\Big]^2-\Big[\rho_1(\ell(\bar t)-\bar t)-I_1(v,\ell)(\bar t)\Big]^2\right|\mathrm{d}s\\
&\leq\frac{C}{h}\int_{\bar t}^{\bar t+h}|\dot{\tilde\ell}(s)-\alpha|\,\mathrm{d}s+\frac{C}{h}\int_{\bar t}^{\bar t+h}|e^{-\nu s}-e^{-\nu\bar t}|\,\mathrm{d}s\\
&+\frac{1}{2h}\int_{\bar t}^{\bar t+h}(1-\dot{\tilde\ell}(s))\left|\Big[\rho_1(\tilde\ell(s)-s)-I_1(\tilde v,\tilde\ell)(s)\Big]^2-\Big[\rho_1(\ell(\bar t)-\bar t)-I_1(v,\ell)(\bar t)\Big]^2\right|\mathrm{d}s.
\end{aligned}$$

The first two integrals vanish as $h\to 0^+$, so we only need to estimate the last integral, denoted by $\tilde J_1$:

$$\begin{aligned}
\tilde J_1 &\leq \frac{1}{2h}\int_{\bar t}^{\bar t+h}(1-\dot{\tilde\ell}(s))\left|\Big(\rho_1(\tilde\ell(s)-s)\Big)^2-\big(\rho_1(\ell(\bar t)-\bar t)\big)^2\right|\mathrm{d}s\\
&+\frac{1}{2h}\int_{\bar t}^{\bar t+h}\left|\Big(I_1(\tilde v,\tilde\ell)(s)\Big)^2-\big(I_1(v,\ell)(\bar t)\big)^2\right|\mathrm{d}s\\
&+\frac{1}{h}\int_{\bar t}^{\bar t+h}(1-\dot{\tilde\ell}(s))\left|\rho_1(\tilde\ell(s)-s)-\rho_1(\ell(\bar t)-\bar t)\right||I_1(\tilde v,\tilde\ell)(s)|\,\mathrm{d}s\\
&+\frac{|\rho_1(\ell(\bar t)-\bar t)|}{h}\int_{\bar t}^{\bar t+h}\left|I_1(\tilde v,\tilde\ell)(s)-I_1(v,\ell)(\bar t)\right|\mathrm{d}s\\
&\leq\frac{1}{2h}\int\limits_{\bar t-\ell(\bar t)}^{\bar t-\ell(\bar t)+h}\left|(\rho_1(-r))^2-\big(\rho_1(\ell(\bar t)-\bar t)\big)^2\right|\mathrm{d}r+\frac{1}{2h}\int\limits_{\bar t}^{\bar t+h}\left|\Big(I_1(\tilde v,\tilde\ell)(s)\Big)^2-\big(I_1(v,\ell)(\bar t)\big)^2\right|\mathrm{d}s\\
&+\frac{C}{h}\int_{\bar t-\ell(\bar t)}^{\bar t-\ell(\bar t)+h}|\rho_1(-r)-\rho_1(\ell(\bar t)-\bar t)|\,\mathrm{d}r+\frac{C}{h}\int_{\bar t}^{\bar t+h}\left|I_1(\tilde v,\tilde\ell)(s)-I_1(v,\ell)(\bar t)\right|\mathrm{d}s.
\end{aligned}$$

The first and the third integral tend to 0 when $h \to 0^+$ by assumption a), while the other two by the continuity of the function $I_1(\tilde{v}, \tilde{\ell})$. Now we estimate J_2:

$$\begin{aligned}
J_2 \le& \frac{1}{h}\int_{\bar{t}}^{\bar{t}+h} |\dot{\tilde{w}}(s)| \Big|\dot{\tilde{w}}(s) + \frac{\nu}{2}\tilde{w}(s) - e^{-\frac{\nu s}{2}}\Big(\rho_2(s) + I_2(\tilde{v})(s)\Big) - \beta - \frac{\nu}{2}w(\bar{t}) + e^{-\frac{\nu \bar{t}}{2}}\Big(\rho_2(\bar{t}) + I_2(v)(\bar{t})\Big)\Big| \mathrm{d}s \\
&+ \Big|\beta + \frac{\nu}{2}w(\bar{t}) - e^{-\frac{\nu \bar{t}}{2}}\Big(\rho_2(\bar{t}) + I_2(v)(\bar{t})\Big)\Big| \frac{1}{h}\int_{\bar{t}}^{\bar{t}+h} |\dot{\tilde{w}}(s) - \beta| \,\mathrm{d}s \\
\le& \frac{1}{h}\int_{\bar{t}}^{\bar{t}+h} |\dot{\tilde{w}}(s)||\dot{\tilde{w}}(s) - \beta| \,\mathrm{d}s + \frac{\nu}{2h}\int_{\bar{t}}^{\bar{t}+h} |\dot{\tilde{w}}(s)||\tilde{w}(s) - w(\bar{t})| \,\mathrm{d}s + \frac{C}{h}\int_{\bar{t}}^{\bar{t}+h} |\dot{\tilde{w}}(s) - \beta| \,\mathrm{d}s \\
&+ \frac{1}{h}\int_{\bar{t}}^{\bar{t}+h} |\dot{\tilde{w}}(s)| \Big| e^{-\frac{\nu s}{2}}\Big(\rho_2(s) + I_2(\tilde{v})(s)\Big) - e^{-\frac{\nu \bar{t}}{2}}\Big(\rho_2(\bar{t}) + I_2(v)(\bar{t})\Big)\Big| \,\mathrm{d}s.
\end{aligned}$$

The first three integrals tend to 0 as $h \to 0^+$ since $\lim\limits_{h\to 0^+} \frac{1}{h}\int_{\bar{t}}^{\bar{t}+h} |\dot{\tilde{w}}(s) - \beta|^2 \,\mathrm{d}s = 0$ and by the continuity of $\tilde{w}$, so we only need to estimate the last one, denoted by $\tilde{J}_2$:

$$\begin{aligned}
\tilde{J}_2 \le& \frac{1}{h}\int_{\bar{t}}^{\bar{t}+h} e^{-\frac{\nu s}{2}} |\dot{\tilde{w}}(s)||\rho_2(s) - \rho_2(\bar{t})| \,\mathrm{d}s + \frac{1}{h}\int_{\bar{t}}^{\bar{t}+h} e^{-\frac{\nu s}{2}} |\dot{\tilde{w}}(s)||I_2(\tilde{v})(s) - I_2(v)(\bar{t})| \,\mathrm{d}s \\
&+ |\rho_2(\bar{t}) + I_2(v)(\bar{t})| \frac{1}{h}\int_{\bar{t}}^{\bar{t}+h} |\dot{\tilde{w}}(s)||e^{-\frac{\nu s}{2}} - e^{-\frac{\nu \bar{t}}{2}}| \,\mathrm{d}s.
\end{aligned}$$

Exploiting assumption b) and the continuity of $I_2(\tilde{v})$ we conclude. □

Thanks to Theorem 3.4.2 we can give the rigorous definition of dynamic energy release rate:

Definition 3.4.4 (**Dynamic energy release rate**). *Fix $\nu \ge 0$, $\ell_0 > 0$ and consider u_0, u_1 and w satisfying (3.2.1) and (3.2.2). Assume that $\ell\colon [0, +\infty) \to [\ell_0, +\infty)$ satisfies (3.1.1a) and (3.1.4).*

For a.e. $\bar{t} \in [0, +\infty)$ and for every $\alpha \in (0, 1)$ the dynamic energy release rate corresponding to the velocity α of the debonding front is defined as

$$G_\alpha(\bar{t}) := -\frac{1}{\alpha}\dot{\mathcal{S}}_r(\bar{t}; \tilde{\ell}, \bar{w}),$$

where $\tilde{\ell}$ is an arbitrary Lipschitz extension of $\ell_{|[0,\bar{t}]}$ satisfying (3.1.4) and (3.4.2a), while

$$\bar{w}(t) = \begin{cases} w(t) & \text{if } t \in [0, \bar{t}], \\ w(\bar{t}) & \text{if } t \in (\bar{t}, +\infty). \end{cases}$$

By Theorem 3.4.2 for a.e. $\bar{t} \in \left[0, \frac{\ell_0}{2}\right]$ we get

$$G_\alpha(\bar{t}) = \frac{1}{2}\frac{1-\alpha}{1+\alpha} e^{-\nu \bar{t}} \left[\dot{v}_0(\ell(\bar{t}) - \bar{t}) - v_1(\ell(\bar{t}) - \bar{t}) - \frac{\nu^2}{4}\int_0^{\bar{t}} v(\tau, \tau - \bar{t} + \ell(\bar{t})) \,\mathrm{d}\tau\right]^2, \tag{3.4.3}$$

and a similar formula holds true for a.e. $\bar{t} \ge \frac{\ell_0}{2}$ by Remarks 3.3.2 and 3.4.3. Moreover, coming back to the original function u we can alternatively write for a.e. $\bar{t} \in \left[0, \frac{\ell_0}{2}\right)$:

$$G_\alpha(\bar{t}) = \frac{1}{2}\frac{1-\alpha}{1+\alpha} \left[\dot{u}_0(\ell(\bar{t}) - \bar{t}) - u_1(\ell(\bar{t}) - \bar{t}) + \nu\int_0^{\bar{t}} u_t(\tau, \tau - \bar{t} + \ell(\bar{t})) \,\mathrm{d}\tau\right]^2.$$

It is also worth recalling that, if $\alpha = \dot\ell(\bar t)$, by means of Remark (3.3.3) one can write:

$$G_{\dot\ell(\bar t)}(\bar t) = \frac{1}{2}(1-\dot\ell(\bar t)^2)u_x(\bar t, \ell(\bar t))^2, \quad \text{for a.e. } \bar t \in [0,+\infty). \tag{3.4.4}$$

In the case $\nu = 0$ we have the expression

$$G_\alpha(\bar t) = 2\frac{1-\alpha}{1+\alpha}\left[\frac{\dot u_0(\ell(\bar t){-}\bar t) - u_1(\ell(\bar t){-}\bar t)}{2}\right]^2, \qquad \text{for a.e. } \bar t \in \left[0, \frac{\ell_0}{2}\right], \tag{3.4.5}$$

and hence we recover the formula given in [24].

We then extend the dynamic energy release rate to the case $\alpha = 0$ by continuity, so that

$$G_\alpha(\bar t) = \frac{1-\alpha}{1+\alpha}G_0(\bar t), \qquad \text{for a.e. } \bar t \in [0,+\infty). \tag{3.4.6}$$

In particular by (3.4.3) we know that for a.e. $\bar t \in \left[0, \frac{\ell_0}{2}\right]$ we can write

$$G_0(\bar t) = \frac{1}{2}e^{-\nu\bar t}\left[\dot v_0(\ell(\bar t){-}\bar t) - v_1(\ell(\bar t){-}\bar t) - \frac{\nu^2}{4}\int_0^{\bar t} v(\tau, \tau{-}\bar t{+}\ell(\bar t))\,\mathrm{d}\tau\right]^2, \tag{3.4.7}$$

or equivalently

$$G_0(\bar t) = \frac{1}{2}\left[\dot u_0(\ell(\bar t){-}\bar t) - u_1(\ell(\bar t){-}\bar t) + \nu\int_0^{\bar t} u_t(\tau, \tau{-}\bar t{+}\ell(\bar t))\,\mathrm{d}\tau\right]^2. \tag{3.4.8}$$

We want to highlight that in the damped case $\nu > 0$ the dynamic energy release rate depends directly on v and ℓ, see (3.4.3), while in the undamped one it depends only on the debonding front ℓ (at least for small times), see (3.4.5). This is the main reason why the arguments used in [24] become useless if viscosity is taken into account and new ideas have to be developed.

3.4.2 Griffith's criterion

To introduce the criterion which controls the evolution of the debonding front ℓ we need to consider the notion of local toughness of the glue between the substrate and the tape (or the bar). It is a measurable function $\kappa\colon [\ell_0,+\infty) \to (0,+\infty)$ which rules the amount of energy dissipated during the debonding process in the time interval $[0,t]$ via the formula

$$\int_{\ell_0}^{\ell(t)} \kappa(\sigma)\,\mathrm{d}\sigma. \tag{3.4.9}$$

As in [24] and [48] we postulate that our model is governed by an energy-dissipation balance and a maximum dissipation principle; this last one states that the debonding front has to move with the maximum speed allowed by the energy balance. More precisely we assume:

$$\mathcal{K}(t)+\mathcal{E}(t)+\mathcal{V}(t)+\int_{\ell_0}^{\ell(t)} \kappa(\sigma)\,\mathrm{d}\sigma = \mathcal{K}(0)+\mathcal{E}(0)+\mathcal{W}(t), \qquad \text{for every } t \in [0,+\infty), \tag{3.4.10}$$

$$\dot\ell(t) = \max\{\alpha \in [0,1) \mid \kappa(\ell(t))\alpha = G_\alpha(t)\alpha\}, \qquad \text{for a.e. } t \in [0,+\infty), \tag{3.4.11}$$

where $\mathcal{W}$ is the work of the external loading and for $t \in \left[0, \frac{\ell_0}{2}\right]$ it has the form (see also Remark 3.3.2):

$$\mathcal{W}(t) := \int_0^t \dot w(s)\left[\dot w(s){+}\frac{\nu}{2}w(s){-}e^{-\frac{\nu s}{2}}\left(\dot v_0(s){+}v_1(s){+}\frac{\nu^2}{4}\int_0^s v(\tau, s{-}\tau)\,\mathrm{d}\tau\right)\right]\mathrm{d}s.$$

Remark 3.4.5. We notice that by the explicit expressions (3.2.7) and (3.2.10) we can rewrite the work of the external forces in the form

$$\mathcal{W}(t) := -\int_0^t \dot{w}(\tau)u_x(\tau,0)\,\mathrm{d}\tau,$$

which makes sense for every $t \in [0,+\infty)$.

By Proposition 3.3.1, Theorem 3.4.2 and Lemma B.0.1 we deduce that (3.4.10) is equivalent to

$$\kappa(\ell(t))\dot{\ell}(t) = G_{\dot{\ell}(t)}(t)\dot{\ell}(t), \quad \text{for a.e. } t \in [0,+\infty),$$

and we observe that for a.e. $t \in [0,+\infty)$ the set $\{\alpha \in [0,1) \mid \kappa(\ell(t))\alpha = G_\alpha(t)\alpha\}$ has at most one element different from zero by the strict monotonicity of $\alpha \mapsto G_\alpha(t)$ and since $\kappa(x) > 0$ for every $x \geq \ell_0$. Therefore the maximum dissipation principle (3.4.11) simply states that during the evolution of the debonding front ℓ only two phases can occur: if the toughness κ is strong enough, ℓ stops and does not move till the dynamic energy release rate equals κ, otherwise it moves at the only speed which is consistent with the energy-dissipation balance (3.4.10).

Arguing as in [24] we get that (3.4.10)&(3.4.11) are equivalent to the following system, called dynamic Griffith's criterion in analogy to the corresponding criterion in Fracture Mechanics:

$$\begin{cases} 0 \leq \dot{\ell}(t) < 1, \\ G_{\dot{\ell}(t)}(t) \leq \kappa(\ell(t)), \\ \left[G_{\dot{\ell}(t)}(t) - \kappa(\ell(t))\right]\dot{\ell}(t) = 0, \end{cases} \qquad \text{for a.e. } t \in [0,+\infty). \tag{3.4.12}$$

The first row is an irreversibility condition, which ensures that the debonding front can only increase; the second one is a stability condition, and says that the dynamic energy release rate cannot exceed the threshold given by the toughness; the third one is simply the energy-dissipation balance (3.4.10).

Finally, by using (3.4.6) and (3.4.11), it easy to see that Griffith's criterion (3.4.12) is equivalent to the following ordinary differential equation:

$$\dot{\ell}(t) = \max\left\{\frac{G_0(t) - \kappa(\ell(t))}{G_0(t) + \kappa(\ell(t))}, 0\right\}, \qquad \text{for a.e. } t \in [0,+\infty), \tag{3.4.13}$$

which by means of (3.4.7) can also be rewritten for a.e. $t \in \left[0, \frac{\ell_0}{2}\right]$ as

$$\dot{\ell}(t) = \max\left\{\frac{\left[\dot{v}_0(\ell(t){-}t) - v_1(\ell(t){-}t) - \frac{\nu^2}{4}\int_0^t v(\tau,\tau{-}t{+}\ell(t))\,\mathrm{d}\tau\right]^2 - 2e^{\nu t}\kappa(\ell(t))}{\left[\dot{v}_0(\ell(t){-}t) - v_1(\ell(t){-}t) - \frac{\nu^2}{4}\int_0^t v(\tau,\tau{-}t{+}\ell(t))\,\mathrm{d}\tau\right]^2 + 2e^{\nu t}\kappa(\ell(t))}, 0\right\}. \tag{3.4.14}$$

We want to underline again that, differently from [24], the equation for the debonding front (3.4.14) depends also on v (and thus on u) if $\nu > 0$. This will bring the main technical difficulties of the next section.

3.5 Evolution of the debonding front

In this section we couple problem (3.0.1) with the energy-dissipation balance (3.4.10) and the maximum dissipation principle (3.4.11) and we prove existence of a unique pair (u, ℓ) which solves this coupled problem.

We fix $\nu \geq 0$, $\ell_0 > 0$ and we consider u_0, u_1 and w satisfying (3.2.1) and (3.2.2), and a measurable function $\kappa\colon [\ell_0, +\infty) \to (0, +\infty)$.

Since differently from previous Sections the debonding front ℓ is unknown, from now on we will always stress the dependence on ℓ and we shall write A_ℓ, R_ℓ and Ω_ℓ instead of A, R and Ω, and so on. We shall also write $(G_0)_{v,\ell}$ instead of G_0, since by (3.4.7) the dependence of the dynamic energy release rate both on the debonding front ℓ and on the solution v of (3.2.3) is evident. Moreover, as in Lemmas 3.2.8 and 3.2.9, we shall extend the functions A_ℓ and $\iint_{R_\ell(\cdot,\cdot)} v \,\mathrm{d}\sigma\,\mathrm{d}\tau$ setting them to be equal 0 outside $\overline{\Omega_\ell}$.

Definition 3.5.1. *Assume $\ell\colon [0, +\infty) \to [\ell_0, +\infty)$ satisfies (3.1.1a) and (3.1.4) and let $u\colon [0, +\infty)^2 \to \mathbb{R}$ be such that $u \in \widetilde{H}^1(\Omega_\ell)$ (resp. in $H^1((\Omega_\ell)_T)$). We say that the pair (u, ℓ) is a solution of the coupled problem (resp. in $[0, T]$) if:*

i) u solves problem (3.0.1) in Ω_ℓ (resp. in $(\Omega_\ell)_T$) in the sense of Definition 3.2.1,

ii) $u \equiv 0$ outside $\overline{\Omega_\ell}$ (resp. in $([0, T]\times[0, +\infty)) \setminus \overline{(\Omega_\ell)_T}$),

iii) (u, ℓ) satisfies Griffith's criterion (3.4.12) for a.e. $t \in [0, +\infty)$ (resp. for a.e. $t \in [0, T]$).

Using (3.2.3) and (3.4.13) it turns out that the pair (u, ℓ) is a solution of the coupled problem if and only if (v, ℓ), where $v(t, x) = e^{\nu t/2}u(t, x)$, satisfies the following system:

$$\begin{cases} v_{tt}(t,x) - v_{xx}(t,x) - \frac{\nu^2}{4}v(t,x) = 0, & t > 0\,,\ 0 < x < \ell(t), \\ \dot{\ell}(t) = \max\left\{\dfrac{(G_0)_{v,\ell}(t) - \kappa(\ell(t))}{(G_0)_{v,\ell}(t) + \kappa(\ell(t))}, 0\right\}, & t > 0, \\ v(t,x) = 0, & t > 0\,,\ x > \ell(t), \\ v(t,0) = z(t), & t > 0, \\ v(t,\ell(t)) = 0, & t > 0, \\ v(0,x) = v_0(x), & 0 < x < \ell_0, \\ v_t(0,x) = v_1(x), & 0 < x < \ell_0, \\ \ell(0) = \ell_0. \end{cases} \tag{3.5.1}$$

Similarly to Section 3.2 we write the fixed point problem related to (3.5.1). Since representation formula (3.2.7) holds true only in Ω'_ℓ, we fix $T \in \left(0, \frac{\ell_0}{2}\right)$ and we state the problem in $(\Omega_\ell)_T$:

$$\begin{cases} v(t,x) = \left(A_\ell(t,x) + \dfrac{\nu^2}{8}\displaystyle\iint_{R_\ell(t,x)} v(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau\right)\mathbb{1}_{(\Omega_\ell)_T}(t,x), & \text{for a.e. } (t,x) \in (0,T)\times(0,+\infty), \\ \ell(t) = \ell_0 + \displaystyle\int_0^t \max\left\{\dfrac{(G_0)_{v,\ell}(s) - \kappa(\ell(s))}{(G_0)_{v,\ell}(s) + \kappa(\ell(s))}, 0\right\}\mathrm{d}s, & \text{for every } t \in [0,T], \end{cases}$$

where, given a set E, we denoted by $\mathbb{1}_E$ the indicator function of E.

For a reason that will be clear later we prefer to introduce the auxiliary function λ, defined as the inverse of the map $t \mapsto t - \ell(t) = \varphi_\ell(t)$ (see also [24], Theorem 3.5). We notice that λ is absolutely continuous by (3.1.4) and Corollary B.0.5, while in the simpler case in which there exists $\delta_T \in (0, 1)$ such that $0 \leq \dot{\ell}(t) \leq 1 - \delta_T$ for a.e. $t \in [0, T]$, λ is Lipschitz and $1 \leq \dot{\lambda}(y) \leq \frac{1}{\delta_T}$ for a.e. $y \in [-\ell_0, \lambda^{-1}(T)]$. We then consider the equivalent

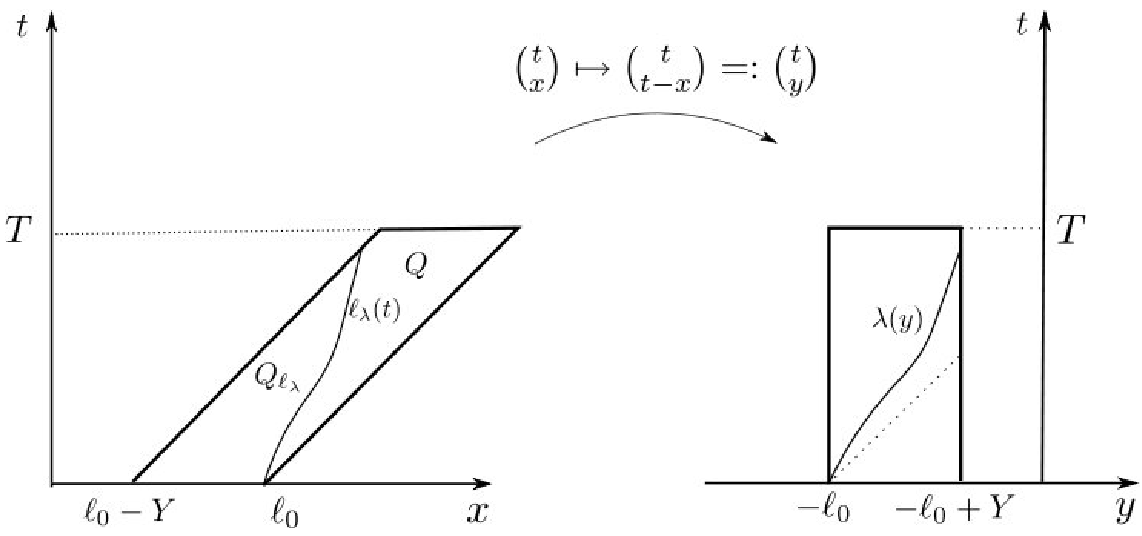

Figure 3.2: The set Q and the functions λ and ℓ_λ.

fixed point problem for the pair (v, λ); exploiting (3.4.14) it takes the form:

$$\begin{cases} v(t,x) = \left(A_{\ell_\lambda}(t,x) + \dfrac{\nu^2}{8}\displaystyle\iint_{R_{\ell_\lambda}(t,x)} v(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau\right)\mathbb{1}_{(\Omega_{\ell_\lambda})_T}(t,x), & \text{for a.e. } (t,x) \in (0,T)\times(0,+\infty), \\ \lambda(y) = \dfrac{1}{2}\displaystyle\int_{-\ell_0}^{y} \Big(1 + \max\{\Theta_{v,\lambda}(s), 1\}\Big)\,\mathrm{d}s, & \text{for every } y \in [-\ell_0, \lambda^{-1}(T)], \end{cases} \tag{3.5.2}$$

where we define for a.e. $y \in [-\ell_0, \lambda^{-1}(T)]$

$$\Theta_{v,\lambda}(y) := \frac{\left[\dot{v}_0(-y) - v_1(-y) - \dfrac{\nu^2}{4}\displaystyle\int_0^{\lambda(y)} v(\tau, \tau - y)\,\mathrm{d}\tau\right]^2}{2e^{\nu\lambda(y)}\kappa(\lambda(y) - y)}, \tag{3.5.3}$$

and where we denoted by ℓ_λ simply the function ℓ, stressing the fact that it depends on λ via the formula $\ell_\lambda(t) = t - \lambda^{-1}(t)$.

As in Section 3.2, we solve problem (3.5.2) showing that a suitable operator is a contraction. We argue as follows: for $T > 0$ and $Y \in (0, \ell_0)$ we consider the sets (see Figure 3.2)

$$Q = Q(T,Y) := \{(t,x) \mid 0 \le t \le T,\ \ell_0 - Y + t \le x \le \ell_0 + t\},$$
$$Q_{\ell_\lambda} := Q \cap \overline{\Omega_{\ell_\lambda}}.$$

Moreover for $M > 0$ and denoting by I_Y the closed interval $[-\ell_0, -\ell_0 + Y]$ we introduce the spaces

$$\mathcal{X}_1 = \mathcal{X}_1(T,Y,M) := \left\{v \in C^0(Q) \mid \|v\|_{C^0(Q)} \le M\right\},$$
$$\mathcal{D}_2 = \mathcal{D}_2(T,Y) := \left\{\lambda \in C^0(I_Y) \mid \lambda(-\ell_0) = 0,\ \|\lambda\|_{C^0(I_Y)} \le T,\ y \mapsto \lambda(y) - y \text{ is nondecreasing}\right\}.$$

Let us define $\mathcal{X} := \mathcal{X}_1 \times \mathcal{D}_2$ and consider the operators:

$$\Psi_1(v,\lambda)(t,x) := \left(A_{\ell_\lambda}(t,x) + \frac{\nu^2}{8}\iint_{R_{\ell_\lambda}(t,x)} v(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau\right)\mathbb{1}_{Q_{\ell_\lambda}}(t,x),$$

$$\Psi_2(v,\lambda)(y) := \frac{1}{2}\int_{-\ell_0}^{y}\left(1 + \max\left\{\frac{\left[\dot{v}_0(-s) - v_1(-s) - \frac{\nu^2}{4}\int_0^{\lambda(s)} v(\tau,\tau - s)\,\mathrm{d}\tau\right]^2}{2e^{\nu\lambda(s)}\kappa(\lambda(s) - s)}, 1\right\}\right)\mathrm{d}s.$$

We then define

$$\Psi(v,\lambda) := \big(\Psi_1(v,\lambda), \Psi_2(v,\lambda)\big). \tag{3.5.5}$$

Remark 3.5.2. From now on we shall write ℓ, ψ and ω instead of ℓ_λ, ψ_λ and ω_λ, being tacit the dependence on λ.

For convenience, we assume for the moment that there exist two positive constants c_1 and c_2 such that

$$0 < c_1 \le \kappa(x) \le c_2 \quad \text{for every } x \ge \ell_0. \tag{3.5.6}$$

Lemma 3.5.3. *Fix $\nu \ge 0$, $\ell_0 > 0$ and consider v_0, v_1 satisfying (3.2.1b) and $v_0(\ell_0) = 0$. Assume that the measurable function $\kappa\colon [\ell_0,+\infty) \to (0,+\infty)$ satisfies (3.5.6).*

Then for every $T > 0$ and $M > 0$ there exists $Y \in (0,\ell_0)$ such that the operator Ψ in (3.5.5) maps $\mathcal{X}$ into itself.

Proof. Fix $T > 0$, $M > 0$ and let $(v,\lambda) \in \mathcal{X}$; by Lemmas 3.2.8 and 3.2.9 we deduce that $\Psi_1(v,\lambda)$ is continuous on Q (indeed notice that $\ell = \ell_\lambda$ satisfies (3.1.1)), while by construction $\Psi_2(v,\lambda)$ is actually absolutely continuous on I_Y and satisfies $\Psi_2(v,\lambda)(-\ell_0) = 0$ and $\frac{\mathrm{d}}{\mathrm{d}y}\Psi_2(v,\lambda)(y) \ge 1$ for a.e. $y \in I_Y$. Hence to conclude it is enough to find $Y \in (0,\ell_0)$ such that

$$\|\Psi_1(v,\lambda)\|_{C^0(Q)} \le M \quad \text{and} \quad \Psi_2(v,\lambda)(-\ell_0{+}Y) \le T.$$

We pick $(t,x) \in Q_\ell$ and using (3.2.6) we estimate:

$$\begin{aligned}
|\Psi_1(v,\lambda)(t,x)| &\le |A_\ell(t,x)| + \frac{\nu^2}{8}\left|\iint_{R_\ell(t,x)} v(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau\right| \\
&\le \int_{\ell_0-Y}^{\ell_0} (|\dot v_0(s)| + |v_1(s)|)\,\mathrm{d}s + \frac{\nu^2}{8} M|Q| \\
&= \int_{\ell_0-Y}^{\ell_0} (|\dot v_0(s)| + |v_1(s)|)\,\mathrm{d}s + \frac{\nu^2}{8} MTY.
\end{aligned}$$

As regards $\Psi_2(v,\lambda)(-\ell_0{+}Y)$ we argue as follows:

$$\begin{aligned}
\Psi_2(v,\lambda)(-\ell_0{+}Y) &= \frac12\int_{-\ell_0}^{-\ell_0+Y} \big(1+\max\{\Theta_{v,\lambda}(s),1\}\big)\,\mathrm{d}s \le \frac12\int_{-\ell_0}^{-\ell_0+Y}\big(2+\Theta_{v,\lambda}(s)\big)\,\mathrm{d}s \\
&\le Y + \frac{1}{2c_1}\left[\int_{-\ell_0}^{-\ell_0+Y} [\dot v_0(-s) - v_1(-s)]^2\,\mathrm{d}s + \frac{\nu^4}{16}\int_{-\ell_0}^{-\ell_0+Y}\left(\int_0^{\lambda(s)} v(\tau,\tau{-}s)\,\mathrm{d}\tau\right)^2\mathrm{d}s\right] \\
&\le Y + \frac{1}{2c_1}\int_{\ell_0-Y}^{\ell_0} [\dot v_0(s) - v_1(s)]^2\,\mathrm{d}s + \frac{\nu^4}{32c_1} M^2T^2Y.
\end{aligned}$$

Since in both estimates the last line tends to 0 when $Y \to 0^+$ we can conclude. □

Lemma 3.5.4. *Fix $\nu \ge 0$, $\ell_0 > 0$ and consider v_0, v_1 satisfying (3.2.1b) and $v_0(\ell_0) = 0$. Fix $T > 0$, $M > 0$ and let $Y \in (0,\ell_0)$ be given by Lemma 3.5.3.*

Then $\Psi_1(\mathcal{X})$ is an equicontinuous family of $\mathcal{X}_1$.

Proof. Let $(v,\lambda) \in \mathcal{X}$ and fix $\delta > 0$.

By simple geometric considerations and by continuity we deduce that

1) $|R_\ell(t_1,x_1) \triangle R_\ell(t_2,x_2)| \le \frac{\sqrt2}{2}(4T+Y)\sqrt{|t_1{-}t_2|^2{+}|x_1{-}x_2|^2}$ for every (t_1,x_1), $(t_2,x_2) \in Q_\ell$,

2) there exists $\delta_1 > 0$ such that for every a, $b \in [0, \ell_0]$ satisfying $|a-b| \le \delta_1$ it holds

$$|v_0(a)-v_0(b)| + \left| \int_a^b v_1(r)\,\mathrm{d}r \right| \le \frac{\delta}{2}.$$

Let us define $\delta := \min\left\{ \frac{\delta_1}{2}, \frac{4\sqrt{2}\,\delta}{\nu^2 M(4T+Y)} \right\}$ $\left(\delta = \frac{\delta_1}{2} \text{ if } \nu = 0\right)$ and take (t_1, x_1), $(t_2, x_2) \in Q$ satisfying $\sqrt{|t_1-t_2|^2+|x_1-x_2|^2} \le \delta$.
For the sake of clarity we define $H_{v,\lambda}(t,x) := \left(\iint_{R_\ell(t,x)} v(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau \right) \mathbb{1}_{Q_\ell}(t,x)$, so that

$$\begin{aligned} &|\Psi_1(v,\lambda)(t_1,x_1) - \Psi_1(v,\lambda)(t_2,x_2)| \\ &\le |A_\ell(t_1,x_1)\mathbb{1}_{Q_\ell}(t_1,x_1) - A_\ell(t_2,x_2)\mathbb{1}_{Q_\ell}(t_2,x_2)| + \frac{\nu^2}{8}|H_{v,\lambda}(t_1,x_1) - H_{v,\lambda}(t_2,x_2)| =: I + II. \end{aligned}$$

We notice that since $A_\ell \mathbb{1}_{Q_\ell}$ and $H_{v,\lambda}$ vanish on $Q \setminus Q_\ell$ and they are continuous on the whole Q, it is enough to consider the case in which both (t_1,x_1) and (t_2,x_2) are in Q_ℓ; in this case to estimate II we use 2):

$$II \le \frac{\nu^2}{8} \iint_{R_\ell(t_1,x_1)\triangle R_\ell(t_2,x_2)} |v(\tau,\sigma)|\,\mathrm{d}\sigma\,\mathrm{d}\tau \le \frac{\nu^2}{8} M |R_\ell(t_1,x_1)\triangle R_\ell(t_2,x_2)| \le \frac{\nu^2}{16} M\sqrt{2}(4T+Y)\delta \le \frac{\delta}{2}.$$

For I we exploit the explicit expression of A_ℓ given by (3.2.6) and we consider three different cases: if (t_1,x_1), $(t_2,x_2) \in Q_\ell \cap \{t+x \le \ell_0\}$ we have

$$I \le \frac{1}{2}|v_0(x_1+t_1) - v_0(x_2+t_2)| + \frac{1}{2}\left| \int_{x_2+t_2}^{x_1+t_1} v_1(r)\,\mathrm{d}r \right| + \frac{1}{2}|v_0(x_1-t_1) - v_0(x_2-t_2)| + \frac{1}{2}\left| \int_{x_2-t_2}^{x_1-t_1} v_1(r)\,\mathrm{d}r \right|,$$

and since $|(x_1 \pm t_1) - (x_2 \pm t_2)| \le 2\sqrt{|t_1-t_2|^2+|x_1-x_2|^2} \le \delta_1$, by 1) we deduce $I \le \delta/2$.
If instead (t_1,x_1), $(t_2,x_2) \in Q_\ell \cap \{t+x \ge \ell_0\}$ we get

$$\begin{aligned} I \le\ & \frac{1}{2}|v_0(-\omega(x_1+t_1)) - v_0(-\omega(x_2+t_2))| + \frac{1}{2}\left| \int_{-\omega(x_2+t_2)}^{-\omega(x_1+t_1)} v_1(r)\,\mathrm{d}r \right| \\ &+ \frac{1}{2}|v_0(x_1-t_1) - v_0(x_2-t_2)| + \frac{1}{2}\left| \int_{x_2-t_2}^{x_1-t_1} v_1(r)\,\mathrm{d}r \right|, \end{aligned}$$

and since $|\omega(x_1+t_1) - \omega(x_2+t_2)| \le |(x_1+t_1) - (x_2+t_2)| \le \delta_1$ (we recall that ω is 1-Lipschitz, see (3.1.3)) again we have $I \le \delta/2$.
Finally if $(t_1,x_1) \in Q_\ell \cap \{t+x \le \ell_0\}$ while $(t_2,x_2) \in Q_\ell \cap \{t+x \ge \ell_0\}$ we get

$$\begin{aligned} I \le\ & \frac{1}{2}|v_0(x_1-t_1) - v_0(x_2-t_2)| + \frac{1}{2}\left| \int_{x_2-t_2}^{x_1-t_1} v_1(r)\,\mathrm{d}r \right| \\ &+ \frac{1}{2}|v_0(x_1+t_1) - v_0(-\omega(x_2+t_2))| + \frac{1}{2}\left| \int_{-\omega(x_2+t_2)}^{x_1+t_1} v_1(r)\,\mathrm{d}r \right|, \end{aligned}$$

and observing that for this configuration of (t_1,x_1) and (t_2,x_2) it holds

$$\begin{aligned} |(x_1+t_1) + \omega(x_2+t_2)| &\le |(x_1+t_1) - \ell_0| + |\omega(x_2+t_2) - \omega(\ell_0)| \\ &\le \ell_0 - (x_1+t_1) + (x_2+t_2) - \ell_0 \\ &\le |t_1-t_2| + |x_1-x_2| \le \delta_1, \end{aligned}$$

we deduce also in this case $I \leq \delta/2$.

These estimates yield

$$|\Psi_1(v,\lambda)(t_1,x_1) - \Psi_1(v,\lambda)(t_2,x_2)| \leq I + II \leq \delta,$$

and so we conclude. □

We now denote by $\mathcal{D}_1$ the closure of $\Psi_1(X)$ with respect to uniform convergence and we define $\mathcal{D} := \mathcal{D}_1 \times \mathcal{D}_2$; we notice that by Lemma 3.5.4 and the Ascoli–Arzelà Theorem (see for istance [81], Theorem 11.28) $\mathcal{D}$ is a complete metric space if endowed with the distance

$$\mathrm{d}\left((v^1,\lambda^1),(v^2,\lambda^2)\right) := \max\{\|v^1 - v^2\|_{L^2(Q)}, \|\lambda^1 - \lambda^2\|_{C^0(I_Y)}\}. \tag{3.5.7}$$

Proposition 3.5.5. *Fix $\nu \geq 0$, $\ell_0 > 0$ and consider v_0, v_1 satisfying (3.2.1b) and $v_0(\ell_0) = 0$. Assume that $\kappa \in C^{0,1}([\ell_0,+\infty))$ satisfies (3.5.6) and fix $T > 0$ and $M > 0$.*

Then there exists $Y \in (0,\ell_0)$ such that the operator Ψ in (3.5.5) is a contraction from $(\mathcal{D}, \mathrm{d})$ into itself.

We prefer to postpone the (long and technical) proof of Proposition 3.5.5 to the end of the section, so that we are at once in a position to state and prove the main result of the chapter, which generalises Theorem 3.5 in [24]:

Theorem 3.5.6. *Fix $\nu \geq 0$, $\ell_0 > 0$ and consider u_0, u_1 and w satisfying (3.2.1) and (3.2.2). Assume that the measurable function $\kappa\colon [\ell_0,+\infty) \to (0,+\infty)$ fulfills the following property:*

$$\text{for every } x \in [\ell_0,+\infty) \text{ there exists } \delta = \delta(x) > 0 \text{ such that } \kappa \in C^{0,1}([x,x+\delta]). \tag{3.5.8}$$

Then there exists a unique pair (u,ℓ) solving the coupled problem in the sense of Definition 3.5.1. Moreover u has a continuous representative on $\overline{\Omega_\ell}$ and it holds:

$$u \in C^0([0,+\infty); H^1(0,+\infty)) \cap C^1([0,+\infty); L^2(0,+\infty)).$$

Remark 3.5.7. Condition (3.5.8) allows for a wide range of left-discontinuous toughnesses, including κ whose limits from the left (at discontinuity points) and to infinity can be 0, $+\infty$ or they cannot even exist. However we point out that the right Lipschitzianity of κ is instead crucial for the validity of the theorem (see Remark 3.5.11).

Proof of Theorem 3.5.6. To conclude we need to prove there exists a unique pair (v,ℓ) solution of (3.5.1). Rearranging Proposition 3.2.11 we firstly deduce there exists a unique v^0 satisfying (3.2.7) in the triangle $\{(t,x) \mid 0 \leq t \leq \ell_0,\ 0 \leq x \leq \ell_0 - t\}$.

Now consider $\delta = \delta(\ell_0)$ given by (3.5.8) and let us introduce a virtual toughness $\tilde\kappa$ which coincides with κ in $[\ell_0,\ell_0+\delta]$ and which is equal to $\kappa(\ell_0+\delta)$ after $\ell_0+\delta$. Since by construction $\tilde\kappa \in C^{0,1}([\ell_0,+\infty))$ and $c_{1\delta} \leq \tilde\kappa(x) \leq c_{2\delta}$ for some $0 < c_{1\delta} \leq c_{2\delta}$, exploiting Proposition 3.5.5 we can find $Y \in (0,\ell_0)$ and $T = T(Y) > 0$ for which there exists a unique pair (v^1,ℓ^1) satisfying

$$\begin{cases} v^1(t,x) = \left(A_{\ell^1}(t,x) + \dfrac{\nu^2}{8}\displaystyle\iint_{R_{\ell^1}(t,x)} v^1(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau\right)\mathbb{1}_{Q_{\ell^1}}(t,x), & \text{for every } (t,x) \in Q,\\ \ell^1(t) = \ell_0 + \displaystyle\int_0^t \max\left\{\frac{(G_0)_{v^1,\ell^1}(s) - \tilde\kappa(\ell^1(s))}{(G_0)_{v^1,\ell^1}(s) + \tilde\kappa(\ell^1(s))}, 0\right\}\mathrm{d}s, & \text{for every } t \in [0,T]. \end{cases} \tag{3.5.9}$$

Since $\ell^1(0) = \ell_0$ and $\tilde\kappa \equiv \kappa$ in $[\ell_0,\ell_0+\delta]$, using the continuity of ℓ^1 we deduce there exists a small time $T_\delta > 0$ such that (v^1,ℓ^1) satisfies (3.5.9) replacing $\tilde\kappa$ by κ and T by T_δ. Gluing together v^0 and v^1 and recalling Lemmas 3.2.8 and 3.2.9 we get the existence of a time $\widetilde{T} \in \left(0, \frac{\ell_0}{2}\right)$ satisfying the following properties:

a) there exists a unique pair $(\tilde{v}, \tilde{\ell})$ solution of (3.5.1) in $[0, \widetilde{T}]$,

b) $\tilde{v}$ belongs to $C^0([0, \widetilde{T}]; H^1(0, +\infty)) \cap C^1([0, \widetilde{T}]; L^2(0, +\infty))$.

Then we define $T^* := \sup\{\widetilde{T} > 0 \mid \widetilde{T} \text{ satisfies a) and b)}\}$. If $T^* = +\infty$ we conclude; so let us assume by contradiction that $T^* < +\infty$ and consider an increasing sequence of times $\{T_k\}$ satisfying a) and b) and converging to T^*. Let (v_k, ℓ_k) be the pair related to T_k by a).

Since by uniqueness $\ell_{k+1}(t) = \ell_k(t)$ for every $t \in [0, T_k]$ and since $0 \le \dot{\ell}_k(t) < 1$ for a.e. $t \in [0, T_k]$, there exists a unique Lipschitz function ℓ defined on $[0, T^*]$ such that $\ell(t) = \ell_k(t)$ for every $t \in [0, T_k]$; hence $\ell(0) = \ell_0$ and $0 \le \dot{\ell}(t) < 1$ for a.e. $t \in [0, T^*]$. Then by Theorem 3.2.12 there exists a unique continuous function v on $\overline{(\Omega_\ell)_{T^*}}$ solution of (3.2.3) in $(\Omega_\ell)_{T^*}$ belonging to $C^0([0, T^*]; H^1(0, +\infty)) \cap C^1([0, T^*]; L^2(0, +\infty))$. Necessarily v and v_k coincide on $\overline{(\Omega_\ell)_{T_k}}$ for every $k \in \mathbb{N}$ and hence (v, ℓ) is the unique solution of (3.5.1) in $[0, T^*]$.

Now we can repeat the contraction argument starting from time T^*: we replace ℓ_0 by $\ell_0^* := \ell(T^*)$, v_0 by $v(T^*, \cdot) \in H^1(0, \ell_0^*)$ and v_1 by $v_t(T^*, \cdot) \in L^2(0, \ell_0^*)$; notice that $v(T^*, 0) = z(T^*)$ and $v(T^*, \ell_0^*) = 0$, so the compatibility conditions (3.2.2) are satisfied. Arguing as before (now with $\delta = \delta(\ell_0^*)$ given by (3.5.8)) and as in the proof of Theorem 3.2.12 we deduce the existence of a time $\widehat{T} > T^*$ satisfying a) and b). This is absurd, being T^* the supremum. □

Remark 3.5.8 (**Regularity**)**.** Arguing as in Remark 3.2.13, if we assume that $u_0 \in C^{0,1}([0, \ell_0])$, $u_1 \in L^\infty(0, \ell_0)$, $w \in \widetilde{C}^{0,1}([0, +\infty))$ satisfy (3.2.2), if the (measurable) toughness κ satisfies (3.5.8), then the solution u belongs to $\widetilde{C}^{0,1}(\overline{\Omega_\ell})$ and $u_t(t, \cdot)$ is in $L^\infty(0, +\infty)$ for every $t \in [0, +\infty)$. If in addition for every $\bar{x} > \ell_0$ there exists a positive constant $c_{\bar{x}}$ such that $\kappa(x) \ge c_{\bar{x}}$ for every $x \in [\ell_0, \bar{x}]$, then for every $T > 0$ there exists $\delta_T \in (0, 1)$ such that $0 \le \dot{\ell}(t) \le 1 - \delta_T$ for a.e. $t \in [0, T]$.

Remark 3.5.9 (**More regularity**)**.** Similarly to Remark 3.2.14, if we assume that $u_0 \in C^{1,1}([0, \ell_0])$, $u_1 \in C^{0,1}([0, \ell_0])$, $w \in \widetilde{C}^{1,1}([0, +\infty))$ satisfy (3.2.2), and if the toughness $\kappa \colon [\ell_0, +\infty) \to (0, +\infty)$ belongs to $\widetilde{C}^{0,1}([\ell_0, +\infty))$, in order to have $\ell \in \widetilde{C}^{1,1}([0, +\infty))$ and $u \in \widetilde{C}^{1,1}(\overline{\Omega_\ell})$ we need to impose a first order compatibility condition:

$$\begin{gathered} u_1(0) = \dot{w}(0), \\ u_1(\ell_0) + \dot{u}_0(\ell_0) \max\left\{ \frac{[\dot{u}_0(\ell_0) - u_1(\ell_0)]^2 - 2\kappa(\ell_0)}{[\dot{u}_0(\ell_0) - u_1(\ell_0)]^2 + 2\kappa(\ell_0)}, 0 \right\} = 0. \end{gathered} \tag{3.5.10}$$

Notice the relationship between (3.5.10) and (3.2.18), given by the equation for ℓ (3.4.14). We want also to point out that the second condition in (3.5.10) is equivalent to:

$$\Big(u_1(\ell_0) = 0,\ \dot{u}_0(\ell_0)^2 \le 2\kappa(\ell_0) \Big) \text{ or } \Big(u_1(\ell_0) \ne 0,\ \dot{u}_0(\ell_0)^2 - u_1(\ell_0)^2 = 2\kappa(\ell_0),\ \frac{\dot{u}_0(\ell_0)}{u_1(\ell_0)} < -1 \Big).$$

Remark 3.5.10 (**Time-dependent toughness**)**.** Proposition 3.5.5, and hence Theorem 3.5.6, holds true even in the case of a time-dependent toughness. To be precise, replacing (3.4.9) by

$$\int_0^t \kappa(\tau, \ell(\tau)) \dot{\ell}(\tau) \, d\tau,$$

where now $\kappa \colon [0, +\infty) \times [\ell_0, +\infty) \to (0, +\infty)$ also depends on time (and is Borel), we obtain that (3.4.13) becomes

$$\dot{\ell}(t) = \max\left\{ \frac{G_0(t) - \kappa(t, \ell(t))}{G_0(t) + \kappa(t, \ell(t))}, 0 \right\}, \qquad \text{for a.e. } t \in [0, +\infty),$$

and in this case the denominator in (3.5.3) reads as $2e^{\nu\lambda(y)}\kappa(\lambda(y),\lambda(y)-y)$.

So, if we assume that $\kappa \in C^{0,1}([0,+\infty)\times[\ell_0,+\infty))$ satisfies $0 < c_1 \le \kappa(t,x) \le c_2$ for every $(t,x)\in[0,+\infty)\times[\ell_0,+\infty)$, we can repeat with no changes the proofs of Lemma 3.5.3 and Proposition 3.5.5 (pay attention to *Step 1*). For Theorem 3.5.6 we replace (3.5.8) by:

$$\text{for every } (t,x)\in[0,+\infty)\times[\ell_0,+\infty) \text{ there exists } \delta=\delta(t,x)>0$$
$$\text{such that } \kappa\in C^{0,1}([t,t+\delta]\times[x,x+\delta]),$$

and we perform a similar proof: in order to start the machinery that leads to the existence of a unique solution to the coupled problem we only need to introduce a virtual toughness $\tilde\kappa$ for which we can apply Proposition 3.5.5; such a $\tilde\kappa$ is obtained by extending κ outside $[0,\delta]\times[\ell_0,\ell_0+\delta]$ (where $\delta=\delta(0,\ell_0)$) in a Lipschitz way and then truncating this extension between two suitable values.

Remark 3.5.11 (**Lack of uniqueness and of existence**)**.** We want to remark that the right Lipschitzianity of the toughness κ is crucial for the validity of Theorem 3.5.6, at least in the undamped case $\nu=0$. Indeed, removing that assumption, the following example shows how the coupled problem can have more than one (actually infinitely many) solution:

fix $\ell_0>0$ and let $\nu=0$; pick $u_0\equiv 0$ and $u_1\equiv 1$ in $[0,\ell_0]$, $w\equiv 0$ in $[0,+\infty)$ and consider $\kappa(x)=\frac{1}{2}\max\left\{\frac{1-\sqrt{x-\ell_0}}{1+\sqrt{x-\ell_0}},\frac{1}{2}\right\}$ for every $x\ge\ell_0$. If the time T is small enough the equation for ℓ in (3.5.1) can be written in the following way:

$$\begin{cases}\dot\ell(t)=\dfrac{1-2\kappa(\ell(t))}{1+2\kappa(\ell(t))}=\sqrt{\ell(t)-\ell_0}, & \text{for a.e. } t\in[0,T],\\ \ell(0)=\ell_0.\end{cases}\tag{3.5.11}$$

It is well known that Cauchy problem (3.5.11) admits infinitely many solutions, for instance two of them are $\ell(t)=\ell_0$ and $\ell(t)=\frac{t^2}{4}+\ell_0$; so coupled problem (3.5.1) admits infinitely many solutions as well.

If instead κ is neither right continuous, we can have no solutions to the coupled problem: under the previous assumptions consider $\kappa(x)=1/6$ if $x=\ell_0$ and $\kappa(x)=1/2$ otherwise, then (for T small enough) the equation for ℓ reads as

$$\dot\ell(t)=\begin{cases}1/2, & \text{if } \ell(t)=\ell_0,\\ 0, & \text{if } \ell(t)>\ell_0.\end{cases}\qquad\text{for a.e. } t\in[0,T].\tag{3.5.12}$$

Since there are no Lipschitz solutions of (3.5.12) satisfying $\ell(0)=\ell_0$ we get that the coupled problem possesses no solutions as well.

This second example can be also adapted to the case of a piecewise constant and left continuous toughness, choosing properly the initial data u_0 and u_1.

Remark 3.5.12 (**Adding a forcing term**)**.** Following the same presentation of the chapter one can also cover the case in which in the model an external force f is present, namely when the equation for the displacement u is

$$u_{tt}(t,x)-u_{xx}(t,x)+\nu u_t(t,x)=f(t,x),\quad t>0\,,\,0<x<\ell(t).$$

For the forcing term f we require

$$f\in L^2_{\text{loc}}((0,+\infty)^2)\quad\text{such that}\quad f\in L^2((0,T)^2)\quad\text{for every } T>0,\tag{3.5.13}$$

and we introduce the function $g(t,x):=e^{\nu t/2}f(t,x)$, so that $v(t,x)=e^{\nu t/2}u(t,x)$ solves

$$v_{tt}(t,x)-v_{xx}(t,x)-\frac{\nu^2}{4}v(t,x)=g(t,x),\quad t>0\,,\,0<x<\ell(t).$$

By Duhamel's principle the representation formula for v now takes the form

$$v(t,x)=A(t,x)+\frac{\nu^2}{8}\iint_{R(t,x)}v(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau+\frac{1}{2}\iint_{R(t,x)}g(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau,\quad \text{for a.e. } (t,x)\in\Omega',$$

and so we can repeat the proofs of Proposition 3.2.11 and Theorem 3.2.12.

For the energetic analysis performed in Section 3.3 we also have to consider the work done by the external forces, namely $\mathcal{F}(t):=\int_0^t\int_0^{\ell(\tau)}f(\tau,\sigma)u_t(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau$; if we take into account the total energy, which now possesses an additional term, i.e. $\mathcal{S}(t)=\mathcal{K}(t)+\mathcal{E}(t)+\mathcal{V}(t)-\mathcal{F}(t)$, then Proposition 3.3.1 holds true modifying formula (3.3.2b) (and analogously (3.3.2a)) to

$$\begin{aligned}\dot{\mathcal{S}}(t)=&-\frac{\dot\ell(t)}{2}\frac{1-\dot\ell(t)}{1+\dot\ell(t)}e^{-\nu t}\Big[\dot v_0(\ell(t)-t)-v_1(\ell(t)-t)-\frac{\nu^2}{4}\int_0^t v(\tau,\tau-t+\ell(t))\mathrm{d}\tau-\int_0^t g(\tau,\tau-t+\ell(t))\mathrm{d}\tau\Big]^2\\&+\dot w(t)\left[\dot w(t)+\frac{\nu}{2}w(t)-e^{-\frac{\nu t}{2}}\left(\dot v_0(t)+v_1(t)+\frac{\nu^2}{4}\int_0^t v(\tau,t-\tau)\,\mathrm{d}\tau+\int_0^t g(\tau,t-\tau)\,\mathrm{d}\tau\right)\right].\end{aligned}$$

We can also repeat the proof of Theorem 3.4.2, obtaining that for a.e. $t\in\left[0,\frac{\ell_0}{2}\right]$ the dynamic energy release rate can be expressed as

$$G_\alpha(t)=\frac{1}{2}\frac{1-\alpha}{1+\alpha}e^{-\nu t}\left[\dot v_0(\ell(t)-t)-v_1(\ell(t)-t)-\frac{\nu^2}{4}\int_0^t v(\tau,\tau-t+\ell(t))\,\mathrm{d}\tau-\int_0^t g(\tau,\tau-t+\ell(t))\,\mathrm{d}\tau\right]^2.$$

Always assuming (3.5.13) we recover Lemma 3.5.3 and Lemma 3.5.4, while for Proposition 3.5.5, and hence for Theorem 3.5.6, we need to require

$$f\in L^\infty_{\mathrm{loc}}((0,+\infty)^2)\quad\text{such that}\quad f\in L^\infty((0,T)^2)\quad\text{for every } T>0; \tag{3.5.14}$$

thanks to (3.5.14) we can perform their proofs replacing operator (3.5.5) by

$$\Psi_1(v,\lambda)(t,x)=\left(A_{\ell_\lambda}(t,x)+\frac{\nu^2}{8}\iint_{R_{\ell_\lambda}(t,x)}v(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau+\frac{1}{2}\iint_{R_{\ell_\lambda}(t,x)}g(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau\right)\mathbb{1}_{Q_{\ell_\lambda}}(t,x),$$

$$\Psi_2(v,\lambda)(y)=\frac{1}{2}\int_{-\ell_0}^{y}\left(1+\max\left\{\frac{\left[\dot v_0(-s)-v_1(-s)-\frac{\nu^2}{4}\int_0^{\lambda(s)}v(\tau,\tau-s)\,\mathrm{d}\tau-\int_0^{\lambda(s)}g(\tau,\tau-s)\,\mathrm{d}\tau\right]^2}{2e^{\nu\lambda(s)}\kappa(\lambda(s)-s)},1\right\}\right)\mathrm{d}s,$$

and arguing in the same way.

We point out that condition (3.5.14) is crucial for the validity of Theorem 3.5.6, as the following example shows: fix $\ell_0>0$ and let $\nu=0$; pick $u_0\equiv 0$ in $[0,\ell_0]$, $w\equiv 0$ in $[0,+\infty)$, $\kappa\equiv 1/2$ in $[\ell_0,+\infty)$ and consider $u_1(x)=\sqrt{2(\ell_0-x)^{\frac{2}{3}}+1}$ and $f(t,x)=\dfrac{2}{3(x-\ell_0)^{\frac{1}{3}}\sqrt{2(x-\ell_0)^{\frac{2}{3}}+1}}$. Notice that f satisfies (3.5.13) but not (3.5.14) and that $f(t,x)=\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{2(x-\ell_0)^{\frac{2}{3}}+1}$. With these data, if $Y>0$ is small enough, the equation for λ becomes

$$\begin{cases}\dot\lambda(y)=1+(\lambda(y)-y-\ell_0)^{\frac{2}{3}} & \text{for a.e. } y\in[-\ell_0,-\ell_0+Y],\\ \lambda(-\ell_0)=0,\end{cases}$$

and so, as in the first example of Remark 3.5.11, we lose uniqueness of solutions to the coupled problem.

We conclude Section 3.5 proving Proposition 3.5.5:

Proof of Proposition 3.5.5. During the proof the symbol C is used to denote a constant, which may change from line to line, that does not depend on the value of Y.

By Lemma 3.5.3 and by the definition of $\mathcal{D}_1$ we know that Ψ maps $\mathcal{D}$ into itself (for suitable small Y), so we only need to show that there exists $Y \in (0, \ell_0)$ for which Ψ is a contraction with respect to the distance d defined in (3.5.7).

Step 1. Lipschitz estimates on Ψ_2.

Fix $(v^1, \lambda^1), (v^2, \lambda^2) \in \mathcal{D}$; let us introduce for a.e. $y \in I_Y$ the function $j(y) := |\dot{v}_0(-y)| + |v_1(-y)| + 1$ and notice that j is in $L^2(-\ell_0, 0)$. For the sake of clarity we also define for $i = 1, 2$ $\rho_{v^i,\lambda^i}(y) := \dot{v}_0(-y) - v_1(-y) - \frac{\nu^2}{4}\int_0^{\lambda^i(y)} v^i(\tau, \tau - y)\,\mathrm{d}\tau$ and we observe that $|\rho_{v^i,\lambda^i}(y)| \leq Cj(y)$ for a.e. $y \in I_Y$; then we compute:

$$
\begin{aligned}
&\|\Psi_2(v^1,\lambda^1) - \Psi_2(v^2,\lambda^2)\|_{C^0(I_Y)} \leq \frac{1}{2}\int_{-\ell_0}^{-\ell_0+Y} |\Theta_{v^1,\lambda^1}(s) - \Theta_{v^2,\lambda^2}(s)|\,\mathrm{d}s \\
&\leq \frac{1}{2}\int_{-\ell_0}^{-\ell_0+Y} \left| \frac{e^{\nu\lambda^2(s)}\kappa(\lambda^2(s)-s)\left(\rho_{v^1,\lambda^1}(s)\right)^2 - e^{\nu\lambda^1(s)}\kappa(\lambda^1(s)-s)\left(\rho_{v^2,\lambda^2}(s)\right)^2}{2e^{\nu(\lambda^1(s)+\lambda^2(s))}\kappa(\lambda^1(s)-s)\kappa(\lambda^2(s)-s)} \right| \mathrm{d}s \\
&\leq C\int_{-\ell_0}^{-\ell_0+Y} e^{\nu\lambda^2(s)}\kappa(\lambda^2(s)-s)\left|\left(\rho_{v^1,\lambda^1}(s)\right)^2 - \left(\rho_{v^2,\lambda^2}(s)\right)^2\right| \mathrm{d}s \\
&\quad + C\int_{-\ell_0}^{-\ell_0+Y} \left(\rho_{v^2,\lambda^2}(s)\right)^2 \left| e^{\nu\lambda^1(s)}\kappa(\lambda^1(s)-s) - e^{\nu\lambda^2(s)}\kappa(\lambda^2(s)-s)\right| \mathrm{d}s \\
&\leq C\left[\int_{-\ell_0}^{-\ell_0+Y} j(s)\left|\int_0^{\lambda^1(s)} v^1(\tau,\tau-s)\,\mathrm{d}\tau - \int_0^{\lambda^2(s)} v^2(\tau,\tau-s)\,\mathrm{d}\tau\right| \mathrm{d}s + \int_{-\ell_0}^{-\ell_0+Y} j^2(s)|\lambda^2(s)-\lambda^1(s)|\,\mathrm{d}s\right] \\
&\leq C\left[\int_{-\ell_0}^{-\ell_0+Y} j(s)\int_0^T |v^1-v^2|(\tau,\tau-s)\,\mathrm{d}\tau\,\mathrm{d}s + \int_{-\ell_0}^{-\ell_0+Y} j(s)\left|\int_{\lambda^2(s)}^{\lambda^1(s)} v^2(\tau,\tau-s)\,\mathrm{d}\tau\right| \mathrm{d}s\right] \\
&\quad + C\left(\int_{-\ell_0}^{-\ell_0+Y} j^2(s)\,\mathrm{d}s\right)\|\lambda^2-\lambda^1\|_{C^0(I_Y)} \\
&\leq C\left[\left(\int_{-\ell_0}^{-\ell_0+Y} j^2(s)\,\mathrm{d}s\right)^{\frac{1}{2}}\|v^1-v^2\|_{L^2(Q)} + \left(\int_{-\ell_0}^{-\ell_0+Y} j(s)\,\mathrm{d}s\right)\|\lambda^2-\lambda^1\|_{C^0(I_Y)}\right] \\
&\quad + C\left(\int_{-\ell_0}^{-\ell_0+Y} j^2(s)\,\mathrm{d}s\right)\|\lambda^2-\lambda^1\|_{C^0(I_Y)} \\
&\leq C\left[\left(\int_{-\ell_0}^{-\ell_0+Y} j^2(s)\,\mathrm{d}s\right)^{\frac{1}{2}} + \int_{-\ell_0}^{-\ell_0+Y}\left(j(s)+j^2(s)\right)\mathrm{d}s\right] \mathrm{d}\left((v^1,\lambda^1),(v^2,\lambda^2)\right).
\end{aligned}
$$

Since j belongs to $L^2(-\ell_0, 0)$ we deduce that choosing Y small enough we get:

$$
\|\Psi_2(v^1,\lambda^1) - \Psi_2(v^2,\lambda^2)\|_{C^0(I_Y)} \leq \frac{1}{2}\mathrm{d}\left((v^1,\lambda^1),(v^2,\lambda^2)\right). \tag{3.5.16}
$$

Step 2. Lipschitz estimates on Ψ_1.

Fix $(v^1, \lambda^1), (v^2, \lambda^2) \in \mathcal{D}$ and let us define for the sake of clarity, as in the proof of Lemma 3.5.4, the function $H_{v,\lambda}(t,x) := \left(\iint_{R_\ell(t,x)} v(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau\right)\mathbb{1}_{Q_\ell}(t,x)$, so that

$$
\|\Psi_1(v^1,\lambda^1) - \Psi_1(v^2,\lambda^2)\|_{L^2(Q)} \leq \|A_{\ell^1}\mathbb{1}_{Q_{\ell^1}} - A_{\ell^2}\mathbb{1}_{Q_{\ell^2}}\|_{L^2(Q)} + \frac{\nu^2}{8}\|H_{v^1,\lambda^1} - H_{v^2,\lambda^2}\|_{L^2(Q)}.
$$

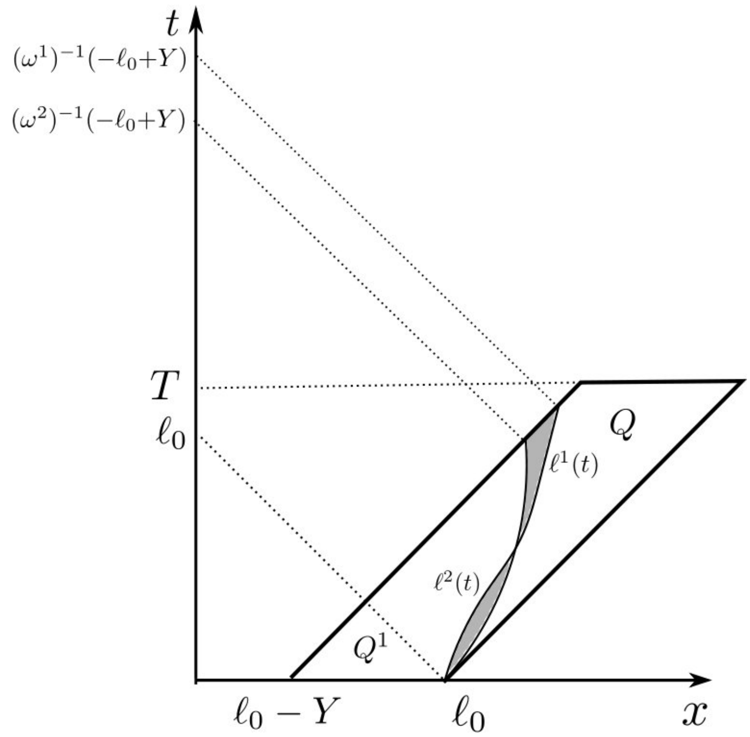

Figure 3.3: The set Q^1 and, in grey, the symmetric difference $Q_{\ell^1} \triangle Q_{\ell^2}$.

We estimate the two norms separately. First of all we rewrite the square of the first term as

$$\begin{aligned}\|A_{\ell^1}\mathbb{1}_{Q_{\ell^1}} - A_{\ell^2}\mathbb{1}_{Q_{\ell^2}}\|^2_{L^2(Q)} = &\iint_{Q_{\ell^1}\setminus Q_{\ell^2}} |A_{\ell^1}(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t + \iint_{Q_{\ell^2}\setminus Q_{\ell^1}} |A_{\ell^2}(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &+ \iint_{Q_{\ell^1}\cap Q_{\ell^2}} |A_{\ell^1}(t,x) - A_{\ell^2}(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{3.5.17}$$

and we notice that for every $s \in [\ell_0, \min\{(\omega^1)^{-1}(-\ell_0+Y), (\omega^2)^{-1}(-\ell_0+Y)\}]$ it holds:

$$\begin{aligned}|\omega^1(s) - \omega^2(s)| &= |\lambda^1(\omega^1(s)) - \lambda^2(\omega^2(s)) - \ell^1(\lambda^1(\omega^1(s))) + \ell^2(\lambda^2(\omega^2(s)))| \\ &= 2|\ell^1(\lambda^1(\omega^1(s))) - \ell^2(\lambda^2(\omega^2(s)))| \\ &\le 2|\ell^1(\lambda^1(\omega^1(s))) - \ell^2(\lambda^2(\omega^1(s)))| \\ &= 2|\lambda^1(\omega^1(s)) - \lambda^2(\omega^1(s))| \\ &\le 2\|\lambda^1 - \lambda^2\|_{C^0(I_Y)}.\end{aligned}$$

This in particular implies (we define $Q^3_{\ell^i} := Q_{\ell^i} \cap \overline{(\Omega_{\ell^i})'_3}$):

$$|\omega^1(x+t) - \omega^2(x+t)| \le 2\|\lambda^1 - \lambda^2\|_{C^0(I_Y)}, \quad \text{if } (t,x) \in Q^3_{\ell^1} \cap Q^3_{\ell^2}, \tag{3.5.18a}$$

$$|(t-x) - \omega^1(x+t)| \le 2\|\lambda^1 - \lambda^2\|_{C^0(I_Y)}, \quad \text{if } (t,x) \in Q_{\ell^1} \setminus Q_{\ell^2}, \tag{3.5.18b}$$

and the same holds interchanging the role of 1 and 2 in (3.5.18b).
Moreover the measure of the symmetric difference of Q_{ℓ^1} and Q_{ℓ^2} can be estimated as

$$|Q_{\ell^1} \triangle Q_{\ell^2}| = \int_{-\ell_0}^{-\ell_0+Y} |\lambda^1(s) - \lambda^2(s)| \,\mathrm{d}s \le Y\|\lambda^1 - \lambda^2\|_{C^0(I_Y)}. \tag{3.5.19}$$

For $(t,x)\in Q_{\ell^1}\setminus Q_{\ell^2}$, exploiting the explicit form of A given by (3.2.6) and using (3.5.18b), we deduce:

$$\begin{aligned}|A_{\ell^1}(t,x)|^2&=\frac{1}{4}\left|\int_{x-t}^{-\omega^1(x+t)}(v_1(s)-\dot v_0(s))\,\mathrm{d}s\right|^2\\&\le\frac{1}{4}|(t-x)-\omega^1(x+t)|\int_0^{\ell_0}|v_1(s)-\dot v_0(s)|^2\,\mathrm{d}s\\&\le C\|\lambda^1-\lambda^2\|_{C^0(I_Y)}.\end{aligned}$$

So, by (3.5.19), we get:

$$\begin{aligned}\iint_{Q_{\ell^1}\setminus Q_{\ell^2}}|A_{\ell^1}(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t+\iint_{Q_{\ell^2}\setminus Q_{\ell^1}}|A_{\ell^2}(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t&\le C\|\lambda^1-\lambda^2\|_{C^0(I_Y)}|Q_{\ell^1}\triangle Q_{\ell^2}|\\&\le CY\|\lambda^1-\lambda^2\|^2_{C^0(I_Y)}.\end{aligned}\tag{3.5.20}$$

To estimate the term in the second line in (3.5.17) we firstly notice that $A_{\ell^1}-A_{\ell^2}$ vanishes on $Q^1:=Q\cap\overline{\Omega_1'}$ (we remark that $Q^1=\overline{Q_{\ell^1}\setminus Q^3_{\ell^1}}=\overline{Q_{\ell^2}\setminus Q^3_{\ell^2}}$ does not depend on ℓ^i, see also Figure 3.3), while for $(t,x)\in Q^3_{\ell^1}\cap Q^3_{\ell^2}$, using (3.5.18a), we have:

$$\begin{aligned}|A_{\ell^1}(t,x)-A_{\ell^2}(t,x)|^2&=\frac{1}{4}\left|\int_{-\omega^2(x+t)}^{-\omega^1(x+t)}(v_1(s)-\dot v_0(s))\,\mathrm{d}s\right|^2\\&\le\frac{|\omega^1(x+t)-\omega^2(x+t)|}{4}\left|\int_{-\omega^2(x+t)}^{-\omega^1(x+t)}|v_1(s)-\dot v_0(s)|^2\,\mathrm{d}s\right|\\&\le\frac{1}{2}\|\lambda^1-\lambda^2\|_{C^0(I_Y)}\left|\int_{-\omega^2(x+t)}^{-\omega^1(x+t)}|v_1(s)-\dot v_0(s)|^2\,\mathrm{d}s\right|.\end{aligned}$$

So we deduce:

$$\begin{aligned}&\iint_{Q_{\ell^1}\cap Q_{\ell^2}}|A_{\ell^1}(t,x)-A_{\ell^2}(t,x)|^2\mathrm{d}x\,\mathrm{d}t\\&\le\frac{1}{2}\|\lambda^1-\lambda^2\|_{C^0(I_Y)}\iint_{Q^3_{\ell^1}\cap Q^3_{\ell^2}}\left|\int_{-\omega^2(x+t)}^{-\omega^1(x+t)}|v_1(s)-\dot v_0(s)|^2\,\mathrm{d}s\right|\mathrm{d}x\,\mathrm{d}t\\&=\frac{1}{2}\|\lambda^1-\lambda^2\|_{C^0(I_Y)}\int_{\ell_0}^{m(Y)}\int_{(\psi^1)^{-1}(b)\vee(\psi^2)^{-1}(b)}^{\frac{b+Y-\ell_0}{2}}\left|\int_{-\omega^2(b)}^{-\omega^1(b)}|v_1(s)-\dot v_0(s)|^2\,\mathrm{d}s\right|\,\mathrm{d}a\,\mathrm{d}b=:(\dagger),\end{aligned}$$

where we performed the change of variables $\begin{cases}a=t,\\b=x+t\end{cases}$, denoted by $m(Y)$ the minimum between $(\omega^1)^{-1}(-\ell_0+Y)$ and $(\omega^2)^{-1}(-\ell_0+Y)$ and used the symbol $\vee$ to denote the maximum between two numbers. We continue the estimate using Fubini's Theorem:

$$\begin{aligned}(\dagger)&\le\frac{1}{2}\|\lambda^1-\lambda^2\|_{C^0(I_Y)}\int_{\ell_0}^{m(Y)}Y\left|\int_{-\omega^2(b)}^{-\omega^1(b)}|v_1(s)-\dot v_0(s)|^2\,\mathrm{d}s\right|\,\mathrm{d}b\\&=\frac{Y}{2}\|\lambda^1-\lambda^2\|_{C^0(I_Y)}\int_{\ell_0-Y}^{\ell_0}|v_1(s)-\dot v_0(s)|^2\left|\int_{(\omega^2)^{-1}(-s)}^{(\omega^1)^{-1}(-s)}\mathrm{d}b\right|\,\mathrm{d}s\\&=\frac{Y}{2}\|\lambda^1-\lambda^2\|_{C^0(I_Y)}\int_{\ell_0-Y}^{\ell_0}|v_1(s)-\dot v_0(s)|^2|\lambda^1(-s)-\lambda^2(-s)+\ell^1(\lambda^1(-s))-\ell^2(\lambda^2(-s))|\,\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&= Y\|\lambda^1-\lambda^2\|_{C^0(I_Y)}\int_{\ell_0-Y}^{\ell_0}|v_1(s)-\dot v_0(s)|^2|\lambda^1(-s)-\lambda^2(-s)|\,\mathrm{d}s\\
&\le CY\|\lambda^1-\lambda^2\|^2_{C^0(I_Y)}.
\end{aligned}
$$

Combining the previous estimate with (3.5.20) and (3.5.17) we get:

$$
\|A_{\ell^1}\mathbb{1}_{Q_{\ell^1}}-A_{\ell^2}\mathbb{1}_{Q_{\ell^2}}\|_{L^2(Q)}\le C\sqrt{Y}\|\lambda^1-\lambda^2\|_{C^0(I_Y)} \tag{3.5.21}
$$

Concerning $\|H_{v^1,\lambda^1}-H_{v^2,\lambda^2}\|_{L^2(Q)}$ we split its square as in (3.5.17):

$$
\begin{aligned}
\|H_{v^1,\lambda^1}-H_{v^2,\lambda^2}\|^2_{L^2(Q)}=&\iint_{Q_{\ell^1}\setminus Q_{\ell^2}}\left|\iint_{R_{\ell^1}(t,x)}v^1(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau\right|^2\mathrm{d}x\,\mathrm{d}t\\
&+\iint_{Q_{\ell^2}\setminus Q_{\ell^1}}\left|\iint_{R_{\ell^2}(t,x)}v^2(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau\right|^2\mathrm{d}x\,\mathrm{d}t\\
&+\iint_{Q_{\ell^1}\cap Q_{\ell^2}}\left|\iint_{R_{\ell^1}(t,x)}v^1(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau-\iint_{R_{\ell^2}(t,x)}v^2(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau\right|^2\mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{3.5.22}
$$

and we denote by $\mathcal{I}$, $\mathcal{II}$ and $\mathcal{III}$ the expressions in the first, second and third line of (3.5.22), respectively. Exploiting (3.5.18b) and (3.5.19) we get:

$$
\begin{aligned}
\mathcal{I}+\mathcal{II}&\le\iint_{Q_{\ell^1}\setminus Q_{\ell^2}}M^2|R_{\ell^1}(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t+\iint_{Q_{\ell^2}\setminus Q_{\ell^1}}M^2|R_{\ell^2}(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\le\iint_{Q_{\ell^1}\setminus Q_{\ell^2}}M^2T^2|(t-x)-\omega^1(x+t)|^2\,\mathrm{d}x\,\mathrm{d}t+\iint_{Q_{\ell^2}\setminus Q_{\ell^1}}M^2T^2|(t-x)-\omega^2(x+t)|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\le 4M^2T^2\|\lambda^1-\lambda^2\|^2_{C^0(I_Y)}|Q_{\ell^1}\triangle Q_{\ell^2}|\le 8M^2T^3Y\|\lambda^1-\lambda^2\|^2_{C^0(I_Y)},
\end{aligned}
$$

while we estimate $\mathcal{III}$ using again (3.5.18a):

$$
\begin{aligned}
\mathcal{III}\le&\iint_{Q^1}\left(\iint_{R(t,x)}|v^1-v^2|(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau\right)^2\mathrm{d}x\,\mathrm{d}t\\
&+\iint_{Q^3_{\ell^1}\cap Q^3_{\ell^2}}\left(\iint_{R_{\ell^1}(t,x)}|v^1-v^2|(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau+\iint_{R_{\ell^1}(t,x)\triangle R_{\ell^2}(t,x)}|v^2(\tau,\sigma)|\,\mathrm{d}\sigma\,\mathrm{d}\tau\right)^2\mathrm{d}x\,\mathrm{d}t\\
\le& C\left[|Q|\|v^1-v^2\|^2_{L^2(Q)}+\iint_{Q^3_{\ell^1}\cap Q^3_{\ell^2}}\left(|Q|\|v^1-v^2\|^2_{L^2(Q)}+|R_{\ell^1}(t,x)\triangle R_{\ell^2}(t,x)|^2\right)\mathrm{d}x\,\mathrm{d}t\right]\\
\le& C\left[|Q|\|v^1-v^2\|^2_{L^2(Q)}+\iint_{Q^3_{\ell^1}\cap Q^3_{\ell^2}}\left(\|v^1-v^2\|^2_{L^2(Q)}+|\omega^1(x+t)-\omega^2(x+t)|^2\right)\mathrm{d}x\,\mathrm{d}t\right]\\
\le& C\left[|Q|\|v^1-v^2\|^2_{L^2(Q)}+|Q^3_{\ell^1}\cap Q^3_{\ell^2}|\left(\|v^1-v^2\|^2_{L^2(Q)}+\|\lambda^1-\lambda^2\|^2_{C^0(I_Y)}\right)\right]\\
\le& CY\,\mathrm{d}\left((v^1,\lambda^1),(v^2,\lambda^2)\right)^2.
\end{aligned}
$$

So we infer:

$$
\|H_{v^1,\lambda^1}-H_{v^2,\lambda^2}\|^2_{L^2(Q)}=\mathcal{I}+\mathcal{II}+\mathcal{III}\le CY\,\mathrm{d}\left((v^1,\lambda^1),(v^2,\lambda^2)\right)^2. \tag{3.5.23}
$$

Using (3.5.21) and (3.5.23) and choosing Y small enough we finally deduce:

$$
\|\Psi_1(v^1,\lambda^1)-\Psi_1(v^2,\lambda^2)\|_{L^2(Q)}\le\frac{1}{2}\,\mathrm{d}\left((v^1,\lambda^1),(v^2,\lambda^2)\right). \tag{3.5.24}
$$

Step 3. $\Psi : \mathcal{D} \to \mathcal{D}$ *is a contraction.*

Combining estimates (3.5.16) and (3.5.24) we obtain:

$$\begin{aligned}
&\mathrm{d}\left(\Psi(v^1,\lambda^1),\Psi(v^2,\lambda^2)\right) \\
&= \max\{\|\Psi_1(v^1,\lambda^1)-\Psi_1(v^2,\lambda^2)\|_{L^2(Q)}, \|\Psi_2(v^1,\lambda^1)-\Psi_2(v^2,\lambda^2)\|_{C^0(I_Y)}\} \\
&\le \frac{1}{2}\,\mathrm{d}\left((v^1,\lambda^1),(v^2,\lambda^2)\right).
\end{aligned}$$

This shows that for a suitable choice of $Y \in (0, \ell_0)$ the operator Ψ is a contraction in $(\mathcal{D}, \mathrm{d})$, and we conclude. □

Chapter 4

Continuous dependence

Contents

Chapter 4 is devoted to the analysis of continuous dependence for the coupled problem (3.0.1) together with dynamic Griffith's criterion (3.0.2).

The chapter is organised as follows. In Section 4.1 we present the continuous dependence problem: we consider sequences of data converging in the natural topologies to some limit data, see (4.1.1), and we wonder whether and in which sense the sequence of solutions to (3.0.1)&(3.0.2) corresponding to these sequences of data, denoted by $\{(u^n,\ell^n)\}_{n\in\mathbb{N}}$, converges to the solution corresponding to the limit ones, denoted by (u,ℓ).

Section 4.2 is devoted to the analysis of the convergence of the sequence of displacements $\{u^n\}_{n\in\mathbb{N}}$ assuming a priori that the sequence of debonding fronts $\{\ell^n\}_{n\in\mathbb{N}}$ converges to ℓ in some suitable topology. The main outcomes of this section are collected in (4.2.4), see also Remark 4.2.12. This is, however, a continuous dependence result for problem (3.0.1), still not coupled with (3.0.2), see Remark 4.1.1.

In Section 4.3 we finally state and prove our continuous dependence result for the coupled problem, see Theorem 4.3.6, showing that the convergence of the sequence of debonding fronts we postulated in Section 4.2 actually happens. The strategy of the proof strongly relies on the representation formula (3.2.7) for solutions to (3.0.1). Furthermore the argument exploits the idea used in Chapter 3 that a certain operator is a contraction with respect to a suitable distance, see (4.3.3) and Propositions 4.3.2 and 4.3.3.

The results contained in this chapter have been published in [74].

4.1 Statement of the problem

4.1.1 Convergence assumptions on the data

We start the section listing all the hypotheses on the limit data and on the sequences of data we will assume in the whole chapter.

The limit data. Let us fix $\nu\geq 0$, $\ell_0>0$, functions u_0, u_1, w satisfying (3.2.1), and a measurable function $\kappa\colon[\ell_0,+\infty)\to(0,+\infty)$ which belongs to $\widetilde{C}^{0,1}([\ell_0,+\infty))$ and so in particular it fulfills property (3.5.8).

We extend u_0, u_1 to the whole $[0,+\infty)$ setting them to be identically zero outside $[0,\ell_0]$ (notice that by compatibility condition u_0 belongs to $H^1(0,+\infty)$) and we extend κ to $[0,+\infty)$ setting $\kappa(x)=\kappa(\ell_0)$ for $x\in[0,\ell_0]$.

The sequences of data. Let us consider a sequence of positive real numbers $\{\ell_0^n\}_{n\in\mathbb{N}}$, a sequence of non negative real numbers $\{\nu^n\}_{n\in\mathbb{N}}$, sequences of functions $\{u_0^n\}_{n\in\mathbb{N}}$, $\{u_1^n\}_{n\in\mathbb{N}}$ and $\{w^n\}_{n\in\mathbb{N}}$ satisfying (3.2.1) replacing ℓ_0 by ℓ_0^n and a sequence of functions $\{\kappa^n\}_{n\in\mathbb{N}}$ such that $\kappa^n\colon[\ell_0^n,+\infty)\to(0,+\infty)$ belongs to $\widetilde{C}^{0,1}([\ell_0^n,+\infty))$ for every $n\in\mathbb{N}$ (and hence it fulfills property (3.5.8), replacing ℓ_0 by ℓ_0^n).

As before we extend u_0^n, u_1^n to the whole $[0,+\infty)$ setting them to be identically zero outside $[0,\ell_0^n]$ and we extend κ^n to $[0,+\infty)$ setting $\kappa^n(x)=\kappa^n(\ell_0^n)$ for $x\in[0,\ell_0^n]$.

The convergence assumptions. As $n\to+\infty$ we assume:

$$\ell_0^n\to\ell_0 \quad\text{and}\quad \nu^n\to\nu; \tag{4.1.1a}$$

$$u_0^n\to u_0 \text{ in } H^1(0,+\infty),\ u_1^n\to u_1 \text{ in } L^2(0,+\infty) \text{ and } w^n\to w \text{ in } \widetilde{H}^1(0,+\infty); \tag{4.1.1b}$$

$$\kappa^n\to\kappa \text{ uniformly in } [0,X] \text{ for every } X>0. \tag{4.1.1c}$$

4.1.2 The main result

Let now (u,ℓ) and (u^n,ℓ^n) be the solutions of the coupled problem given by Theorem 3.5.6 corresponding to the limit data and to the nth term of the sequence of data, respectively. The principal result of the chapter, stated in Theorem 4.3.6, affirms that under the assumptions of this first section the following convergences hold true for every $T>0$:

- $\dot\ell^n\to\dot\ell$ in $L^1(0,T)$, and thus $\ell^n\to\ell$ uniformly in $[0,T]$;
- $u^n\to u$ uniformly in $[0,T]\times[0,+\infty)$;
- $u^n\to u$ in $H^1((0,T)\times(0,+\infty))$;
- $u^n\to u$ in $C^0([0,T];H^1(0,+\infty))$ and in $C^1([0,T];L^2(0,+\infty))$;
- $u_x^n(\cdot,0)\to u_x(\cdot,0)$ and $\sqrt{1-\dot\ell^n(\cdot)^2}u_x^n(\cdot,\ell^n(\cdot))\to\sqrt{1-\dot\ell(\cdot)^2}u_x(\cdot,\ell(\cdot))$ in $L^2(0,T)$.

We recall that the term $\sqrt{1-\dot\ell(\cdot)^2}u_x(\cdot,\ell(\cdot))$ is, up to the constant $1/\sqrt{2}$ and up to the sign, the square root of the dynamic energy release rate $G_{\dot\ell(\cdot)}(\cdot)$, see (3.4.4).

Remark 4.1.1. If instead of considering the coupled problem, we study system (3.0.1) with a prescribed debonding front, then we obtain an analogous continuous dependence result. This analysis will be performed in Section 4.2, see (4.2.4), Remark 4.2.12 and also Propositions 4.2.8, 4.2.9, 4.2.10 and 4.2.11.

As we did in Chapter 3, to prove the theorem we will exploit the sequence of auxiliary functions $v^n(t,x)=e^{\nu^n t/2}u^n(t,x)$, whose boundary and initial data are the functions v_0^n, v_1^n and z^n given by (3.2.4). We recall that for $T<\frac{\ell_0}{2}$ they can be expressed using representation formula (3.2.7) as

$$v^n(t,x)=A^n(t,x)+\frac{(\nu^n)^2}{8}H^n(t,x),\qquad \text{for every } (t,x)\in[0,T]\times[0,+\infty), \tag{4.1.2}$$

where the function A^n is as in (3.2.6) with the obvious changes, while

$$H^n(t,x)=\iint_{R^n(t,x)}v^n(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau. \tag{4.1.3}$$

We stress that they both are extended to zero outside $\overline{\Omega^n}$.

Remark 4.1.2. By (3.2.4) it is easy to see that convergence hypotheses (4.1.1a) and (4.1.1b) yield the same kind of convergence for the functions v_0^n, v_1^n and z^n.

In the next two sections we analyse the convergence of the pair (v^n, ℓ^n) instead of the one of the pair (u^n, ℓ^n). Indeed the transformed pair (v^n, ℓ^n) is easier than (u^n, ℓ^n) to handle with, since in (4.1.3) inside the integral it appears the function itself, and not its time derivative (compare also with (3.4.7) and (3.4.8)). We are able to prove that the convergences listed just above hold true for the auxiliary function v^n, and thus, since it is linked to u^n via the equality $v^n(t,x) = e^{\nu^n t/2} u^n(t,x)$, the result is easily transferred to the solution u^n of the coupled problem.

Remark 4.1.3 (Notation). From now on during all the estimates the symbol C is used to denote a constant, which may change from line to line, which does not depend on n. The symbol δ^n is instead used to denote the nth term of a generic infinitesimal sequence.

4.2 A priori convergence of the debonding front

In this section we prove that if we assume a priori the validity of certain suitable convergence (uniform and in $W^{1,1}$) on the sequence of debonding fronts $\{\ell^n\}_{n\in\mathbb{N}}$ in a time interval $[0,T]$, then the sequence of auxiliary functions $\{v^n\}_{n\in\mathbb{N}}$ converges to v in the natural spaces. First of all we prove an equiboundedness result for the sequence $\{v^n\}_{n\in\mathbb{N}}$:

Proposition 4.2.1. *Assume (4.1.1a), (4.1.1b) and let us denote by N the maximum value of ν^n. If $T < \min\left\{\frac{\ell_0}{2}, \frac{2}{N^2\ell_0}\right\}$, then the functions v^n are uniformly bounded in $C^0([0,T]\times[0,+\infty))$.*

Proof. We exploit representation formula (4.1.2) and we estimate:

$$\begin{aligned}\|v^n\|_{C^0([0,T]\times[0,+\infty))} &\le \|A^n\|_{C^0([0,T]\times[0,+\infty))} + \frac{(\nu^n)^2}{8}\|H^n\|_{C^0([0,T]\times[0,+\infty))}\\ &\le \|A^n\|_{C^0([0,T]\times[0,+\infty))} + \frac{N^2}{8}|\Omega_T^n|\|v^n\|_{C^0([0,T]\times[0,+\infty))}\\ &\le \|A^n\|_{C^0([0,T]\times[0,+\infty))} + \frac{N^2\ell_0 T}{4}\|v^n\|_{C^0([0,T]\times[0,+\infty))}.\end{aligned}$$

Since by hypothesis $T \le \frac{2}{N^2\ell_0}$ we deduce that:

$$\|v^n\|_{C^0([0,T]\times[0,+\infty))} \le 2\|A^n\|_{C^0([0,T]\times[0,+\infty))}.$$

By the explicit expression of A^n given by (3.2.6) and using (4.1.1b) it is easy to get the equiboundedness of A^n in $C^0([0,T]\times[0,+\infty))$ and so we conclude. □

Before starting the analysis of the convergence of the sequence $\{A^n\}_{n\in\mathbb{N}}$ we state several Lemmas regarding the convergence of the sequence $\{\omega^n\}_{n\in\mathbb{N}}$ appearing in formulas (3.1.6), (3.2.6) and (3.2.14).

Lemma 4.2.2. *Let $f^n\colon [a,b]\to\mathbb{R}$ be a sequence of continuous and invertible functions and assume f^n uniformly converges to a continuous and invertible function $f\colon[a,b]\to\mathbb{R}$. Then $\lim\limits_{n\to+\infty}\max\limits_{y\in D_f^n(a,b)}|(f^n)^{-1}(y) - f^{-1}(y)| = 0$, where $D_f^n(a,b) := f^n([a,b])\cap f([a,b])$.*

Proof. For $y\in D_f^n(a,b)$ it holds:

$$|(f^n)^{-1}(y) - f^{-1}(y)| = |f^{-1}(f((f^n)^{-1}(y))) - f^{-1}(y)|. \tag{4.2.1}$$

Since f is continuous, f^{-1} is uniformly continuous on the compact interval $f([a,b])$ and so by (4.2.1) to conclude it is enough to prove that $\max_{y\in f^n([a,b])} |f((f^n)^{-1}(y)) - y| \to 0$ as $n \to +\infty$. So let us take $y \in f^n([a,b])$ and reason as follows:

$$|f((f^n)^{-1}(y)) - y| = |f((f^n)^{-1}(y)) - f^n((f^n)^{-1}(y))| \leq \|f^n - f\|_{C^0([a,b])}.$$

Since by hypothesis f^n uniformly converges to f in $[a,b]$ the proof is complete. □

As we did in Lemma 4.2.2 we now introduce the following notation: given a time $T > 0$ we define $D^n_\psi(0,T) := \psi^n([0,T]) \cap \psi([0,T])$ and $D^n_\varphi(0,T) := \varphi^n([0,T]) \cap \varphi([0,T])$. We notice that we can rewrite them as:

$$D^n_\psi(0,T) = [\ell^n_0 \vee \ell_0, \psi^n(T) \wedge \psi(T)] \quad \text{and} \quad D^n_\varphi(0,T) = [-(\ell_0 \wedge \ell^n_0), \varphi(T) \wedge \varphi^n(T)].$$

Lemma 4.2.3. *If ℓ^n uniformly converges to ℓ in $[0,T]$, then* $\lim_{n\to+\infty} \max_{t\in D^n_\psi(0,T)} |\omega^n(t) - \omega(t)| = 0$*. If (4.1.1a) holds and $\dot\ell^n \to \dot\ell$ in $L^1(0,T)$, then* $\lim_{n\to+\infty} \int_{D^n_\psi(0,T)} |\dot\omega^n(t) - \dot\omega(t)|\,\mathrm{d}t = 0.$

Proof. Assume that $\ell^n \to \ell$ uniformly in $[0,T]$, then obviously $\psi^n \to \psi$ uniformly in $[0,T]$ and so by Lemma 4.2.2 we get $\lim_{n\to+\infty} \max_{t\in D^n_\psi(0,T)} |(\psi^n)^{-1}(t) - \psi^{-1}(t)| = 0$. Take now $t \in D^n_\psi(0,T)$, then

$$\begin{aligned}|\omega^n(t) - \omega(t)| &\leq |\varphi^n((\psi^n)^{-1}(t)) - \varphi((\psi^n)^{-1}(t))| + |\varphi((\psi^n)^{-1}(t)) - \varphi(\psi^{-1}(t))| \\ &\leq \|\ell^n - \ell\|_{C^0([0,T])} + |(\psi^n)^{-1}(t) - \psi^{-1}(t)|,\end{aligned}$$

and hence we deduce $\lim_{n\to+\infty} \max_{t\in D^n_\psi(0,T)} |\omega^n(t) - \omega(t)| = 0$.

Now assume that $\dot\ell^n \to \dot\ell$ in $L^1(0,T)$. Notice that by (4.1.1a) this implies $\ell^n \to \ell$ uniformly in $[0,T]$, and so we have:

$$\begin{aligned}&\int_{D^n_\psi(0,T)} |\dot\omega^n(t) - \dot\omega(t)|\,\mathrm{d}t = \int_{D^n_\psi(0,T)} \left| \frac{1 - \dot\ell^n((\psi^n)^{-1}(t))}{1 + \dot\ell^n((\psi^n)^{-1}(t))} - \frac{1 - \dot\ell(\psi^{-1}(t))}{1 + \dot\ell(\psi^{-1}(t))} \right| \mathrm{d}t \\ &\leq 2 \int_{D^n_\psi(0,T)} \left| \dot\ell^n((\psi^n)^{-1}(t)) - \dot\ell(\psi^{-1}(t)) \right| \mathrm{d}t \\ &\leq 2 \left(\int_{D^n_\psi(0,T)} |\dot\ell^n((\psi^n)^{-1}(t)) - \dot\ell((\psi^n)^{-1}(t))|\,\mathrm{d}t + \int_{D^n_\psi(0,T)} |\dot\ell((\psi^n)^{-1}(t)) - \dot\ell(\psi^{-1}(t))|\,\mathrm{d}t \right) \\ &\leq 2 \left(2 \int_0^T \left| \dot\ell^n(s) - \dot\ell(s) \right| \mathrm{d}s + \int_{D^n_\psi(0,T)} \left| \dot\ell((\psi^n)^{-1}(t)) - \dot\ell(\psi^{-1}(t)) \right| \mathrm{d}t \right).\end{aligned}$$

By assumption the first term in the last line goes to zero as $n \to +\infty$, while for the second term we reason as follows. We fix $\delta > 0$ and we consider $f_\delta \in C^0([0,T])$ such that $\|\dot\ell - f_\delta\|_{L^1(0,T)} \leq \delta$, so we can estimate:

$$\begin{aligned}&\int_{D^n_\psi(0,T)} \left| \dot\ell((\psi^n)^{-1}(t)) - \dot\ell(\psi^{-1}(t)) \right| \mathrm{d}t \\ \leq &\int_{D^n_\psi(0,T)} \left| \dot\ell((\psi^n)^{-1}(t)) - f_\delta((\psi^n)^{-1}(t)) \right| \mathrm{d}t + \int_{D^n_\psi(0,T)} \left| f_\delta((\psi^n)^{-1}(t)) - f_\delta(\psi^{-1}(t)) \right| \mathrm{d}t \\ &+ \int_{D^n_\psi(0,T)} \left| f_\delta(\psi^{-1}(t)) - \dot\ell(\psi^{-1}(t)) \right| \mathrm{d}t\end{aligned}$$

$$\leq 2\|\dot{\ell}-f_\delta\|_{L^1(0,T)}+\int_{D^n_\psi(0,T)}\left|f_\delta((\psi^n)^{-1}(t))-f_\delta(\psi^{-1}(t))\right|\mathrm{d}t+2\|\dot{\ell}-f_\delta\|_{L^1(0,T)}$$
$$\leq 4\delta+\int_{D^n_\psi(0,T)}\left|f_\delta((\psi^n)^{-1}(t))-f_\delta(\psi^{-1}(t))\right|\mathrm{d}t.$$

By dominated convergence the last integral goes to zero as $n\to+\infty$ and so by the arbitrariness of δ we get the result. □

Lemma 4.2.4. *Let f^n be a sequence of $L^2(\mathbb{R})$-functions converging to f strongly in $L^2(\mathbb{R})$. If (4.1.1a) holds and $\dot{\ell}^n\to\dot{\ell}$ in $L^1(0,T)$, then*

$$\lim_{n\to+\infty}\int_{D^n_\psi(0,T)}|f^n(-\omega^n(s))\dot{\omega}^n(s)-f(-\omega(s))\dot{\omega}(s)|^2\,\mathrm{d}s=0.$$

Proof. It is enough to estimate:

$$\begin{aligned}&\int_{D^n_\psi(0,T)}|f^n(-\omega^n(s))\dot{\omega}^n(s)-f(-\omega(s))\dot{\omega}(s)|^2\,\mathrm{d}s\\ &\leq 2\int_{D^n_\psi(0,T)}|f^n(-\omega^n(s))\dot{\omega}^n(s)-f(-\omega^n(s))\dot{\omega}^n(s)|^2\,\mathrm{d}s\\ &\quad+2\int_{D^n_\psi(0,T)}|f(-\omega^n(s))\dot{\omega}^n(s)-f(-\omega(s))\dot{\omega}(s)|^2\,\mathrm{d}s\\ &\leq 2\|f^n-f\|^2_{L^2(\mathbb{R})}+2\int_{D^n_\psi(0,T)}|f(-\omega^n(s))\dot{\omega}^n(s)-f(-\omega(s))\dot{\omega}(s)|^2\,\mathrm{d}s.\end{aligned}$$

Here we used the uniform bound of $\dot{\omega}^n$, see (3.1.3). By assumption the first term in the last line vanishes as $n\to+\infty$, while for the second integral we reason as in the proof of Lemma 4.2.3: for $\delta>0$ fixed let us consider $f_\delta\in C^0_c(\mathbb{R})$ satisfying $\|f-f_\delta\|^2_{L^2(\mathbb{R})}\leq\delta$, then we have:

$$\begin{aligned}&\int_{D^n_\psi(0,T)}|f(-\omega^n(s))\dot{\omega}^n(s)-f(-\omega(s))\dot{\omega}(s)|^2\,\mathrm{d}s\\ &\leq 3\int_{D^n_\psi(0,T)}|f(-\omega^n(s))\dot{\omega}^n(s)-f_\delta(-\omega^n(s))\dot{\omega}^n(s)|^2\,\mathrm{d}s+3\int_{D^n_\psi(0,T)}|f_\delta(-\omega^n(s))\dot{\omega}^n(s)-f_\delta(-\omega(s))\dot{\omega}(s)|^2\,\mathrm{d}s\\ &\quad+3\int_{D^n_\psi(0,T)}|f_\delta(-\omega(s))\dot{\omega}(s)-f(-\omega(s))\dot{\omega}(s)|^2\,\mathrm{d}s\\ &\leq 3\int_{\mathbb{R}}|f(x)-f_\delta(x)|^2\,\mathrm{d}x+3\int_{D^n_\psi(0,T)}|f_\delta(-\omega^n(s))\dot{\omega}^n(s)-f_\delta(-\omega(s))\dot{\omega}(s)|^2\,\mathrm{d}s\\ &\quad+3\int_{\mathbb{R}}|f(x)-f_\delta(x)|^2\,\mathrm{d}x\\ &\leq 6\delta+3\int_{D^n_\psi(0,T)}|f_\delta(-\omega^n(s))\dot{\omega}^n(s)-f_\delta(-\omega(s))\dot{\omega}(s)|^2\,\mathrm{d}s.\end{aligned}$$

By dominated convergence the last integral goes to zero as $n\to+\infty$. Indeed exploiting Lemma 4.2.3 we deduce that, up to subsequences (not relabelled), the function $|\dot{\omega}^n-\dot{\omega}|\mathbb{1}_{D^n_\psi(0,T)}$ (here and henceforth $\mathbb{1}_A$ denotes the indicator function of the set A) vanishes almost everywhere on a bounded interval (the intervals $D^n_\psi(0,T)$ are all contained for instance in $[0,\psi(T)+1]$). By continuity of f_δ and since by assumptions $\mathbb{1}_{D^n_\psi(0,T)}\to\mathbb{1}_{[\ell_0,\psi(T)]}$ almost everywhere as $n\to+\infty$, this implies that also $|f_\delta(-\omega^n)\dot{\omega}^n-f_\delta(-\omega)\dot{\omega}|^2\mathbb{1}_{D^n_\psi(0,T)}$ vanishes almost everywhere on that bounded interval. Since the limit does not depend on the subsequence we conclude.

Thus by the arbitrariness of δ we get the result. □

Now that we have established some convergence results of the sequence $\{\omega^n\}_{n\in\mathbb{N}}$ we can start to study how the sequence $\{A^n\}_{n\in\mathbb{N}}$ behaves under different convergence assumptions on $\{\ell^n\}_{n\in\mathbb{N}}$.

Proposition 4.2.5. *Assume (4.1.1b) and let $T < \frac{\ell_0}{2}$. If ℓ^n uniformly converges to ℓ in $[0,T]$, then A^n uniformly converges to A in $[0,T]\times[0,+\infty)$.*

Proof. We assume without loss of generality that $\ell_0 < \ell_0^n$, the other cases being analogous. As in the whole chapter we exploit explicit formula (3.2.6), so we need to deal with some different cases separately. We thus consider the following partition of $[0,T]\times[0,+\infty)$, see Figure 4.1:

$$\begin{aligned}
&\Lambda_1^n := (\Omega_1')_T, && \Lambda_2^n := (\Omega_2')_T, && \Lambda_3^n := (\Omega_1'^n)_T \cap (\Omega_3')_T,\\
&\Lambda_4^n := (\Omega_1'^n)_T \backslash \Omega_T, && \Lambda_5^n := (\Omega_3'^n)_T \cap (\Omega_3')_T, && \Lambda_6^n := (\Omega_3'^n)_T \backslash \Omega_T,\\
&\Lambda_7^n := (\Omega_3')_T \backslash \Omega_T^n, && \Lambda_8^n := \big([0,T]\times[0,+\infty)\big) \setminus \bigcup_{i=1}^{7} \Lambda_i^n. &&
\end{aligned} \tag{4.2.2}$$

If $(t,x)\in\Lambda_1^n$, then

$$|A^n(t,x)-A(t,x)| \le \|v_0^n - v_0\|_{C^0([0,+\infty))} + \frac{\sqrt{\ell_0}}{2}\|v_1^n - v_1\|_{L^2(0,+\infty)}.$$

If $(t,x)\in\Lambda_2^n$, then

$$|A^n(t,x)-A(t,x)| \le \|z^n - z\|_{C^0([0,T])} + \|v_0^n - v_0\|_{C^0([0,+\infty))} + \frac{\sqrt{\ell_0}}{2}\|v_1^n - v_1\|_{L^2(0,+\infty)}.$$

If $(t,x)\in\Lambda_3^n$, we first notice that $v_0(x{+}t)=0$ and that $-\omega(\ell_0^n)\le -\omega(x{+}t)\le \ell_0 \le x{+}t \le \ell_0^n$, then we estimate:

$$\begin{aligned}
&|A^n(t,x)-A(t,x)|\\
&\le \frac12|v_0^n(x{-}t)-v_0(x{-}t)| + \frac12|v_0^n(x{+}t)+v_0(-\omega(x{+}t))| + \frac12\left|\int_{x-t}^{x+t} v_1^n(s)\,\mathrm{d}s - \int_{x-t}^{-\omega(x+t)} v_1(s)\,\mathrm{d}s\right|\\
&\le \|v_0^n - v_0\|_{C^0([0,+\infty))} + \frac{\sqrt{\ell_0}+\sqrt{\ell_0^n}}{2}\|v_1^n - v_1\|_{L^2(0,+\infty)} + \frac12|v_0(-\omega(x{+}t))| + \frac12\left|\int_{-\omega(x+t)}^{x+t} v_1(s)\,\mathrm{d}s\right|\\
&\le \|v_0^n - v_0\|_{C^0([0,+\infty))} + C\|v_1^n - v_1\|_{L^2(0,+\infty)} + \int_{-\omega(\ell_0^n)}^{\ell_0} (|\dot v_0(s)|+|v_1(s)|)\,\mathrm{d}s.
\end{aligned}$$

If $(t,x)\in\Lambda_4^n$, we notice that $-\omega(\ell_0^n)\le x{-}t\le x{+}t\le \ell_0^n$ and hence we get:

$$\begin{aligned}
|A^n(t,x)-A(t,x)| = |A^n(t,x)| &\le \int_{-\omega(\ell_0^n)}^{\ell_0^n} |\dot v_0^n(s)|\,\mathrm{d}s + \frac12\int_{-\omega(\ell_0^n)}^{\ell_0^n} |v_1^n(s)|\,\mathrm{d}s\\
&\le C\|\dot v_0^n - \dot v_0\|_{L^2(0,+\infty)} + C\|v_1^n - v_1\|_{L^2(0,+\infty)} + \int_{-\omega(\ell_0^n)}^{\ell_0} (|\dot v_0(s)|+|v_1(s)|)\,\mathrm{d}s.
\end{aligned}$$

If $(t,x)\in\Lambda_5^n$, then

$$\begin{aligned}
&|A^n(t,x)-A(t,x)|\\
&\le \frac12\|v_0^n - v_0\|_{C^0([0,+\infty))} + \frac12|v_0^n(-\omega^n(x{+}t)) - v_0(-\omega(x{+}t))| + \frac12\left|\int_{x-t}^{-\omega^n(x+t)} v_1^n(s)\,\mathrm{d}s - \int_{x-t}^{-\omega(x+t)} v_1(s)\,\mathrm{d}s\right|\\
&\le \|v_0^n - v_0\|_{C^0([0,+\infty))} + \frac12|v_0(-\omega^n(x{+}t)) - v_0(-\omega(x{+}t))| + C\|v_1^n - v_1\|_{L^2(0,+\infty)}
\end{aligned}$$

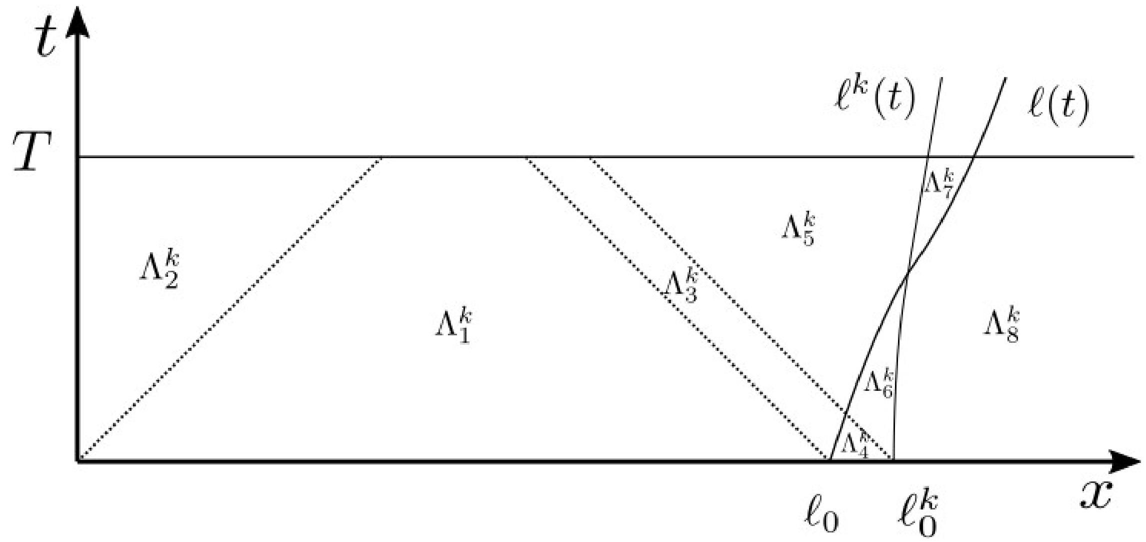

Figure 4.1: The partition of the set $[0,T]\times[0,+\infty)$ via the sets Λ_i^n, for $i=1,\dots,8$, in the case $\ell_0<\ell_0^n$.

$$
\begin{aligned}
&+\frac{1}{2}\left|\int_{-\omega(x+t)}^{-\omega^n(x+t)}|v_1(s)|\,\mathrm{d}s\right|\\
&\le \|v_0^n-v_0\|_{C^0([0,+\infty))}+\max_{r\in D_\psi^n(0,T)}|v_0(-\omega^n(r))-v_0(-\omega(r))|+C\|v_1^n-v_1\|_{L^2(0,+\infty)}\\
&+\max_{r\in D_\psi^n(0,T)}\left|\int_{-\omega(r)}^{-\omega^n(r)}|v_1(s)|\,\mathrm{d}s\right|.
\end{aligned}
$$

If $(t,x)\in\Lambda_6^n$, we notice that $-\omega(x+t)\le x-t\le-\omega^n(x+t)$ and hence we get:

$$
\begin{aligned}
&|A^n(t,x)-A(t,x)|=|A^n(t,x)|\le\frac{1}{2}|v_0^n(x-t)-v_0^n(-\omega^n(x+t))|+\frac{1}{2}\int_{x-t}^{-\omega^n(x+t)}|v_1^n(s)|\,\mathrm{d}s\\
&\le\frac{1}{2}\int_{x-t}^{-\omega^n(x+t)}(|\dot v_0^n(s)|+|v_1^n(s)|)\,\mathrm{d}s\\
&\le C\|\dot v_0^n-\dot v_0\|_{L^2(0,+\infty)}+C\|v_1^n-v_1\|_{L^2(0,+\infty)}+\int_{x-t}^{-\omega^n(x+t)}(|\dot v_0(s)|+|v_1(s)|)\,\mathrm{d}s\\
&\le C\|\dot v_0^n-\dot v_0\|_{L^2(0,+\infty)}+C\|v_1^n-v_1\|_{L^2(0,+\infty)}+\max_{r\in D_\psi^n(0,T)}\int_{-\omega(r)}^{-\omega^n(r)}(|\dot v_0(s)|+|v_1(s)|)\,\mathrm{d}s.
\end{aligned}
$$

If $(t,x)\in\Lambda_7^n$ one reasons just as above, while if $(t,x)\in\Lambda_8^n$ there is nothing to prove since $A^n(t,x)=A(t,x)=0$.

We conclude exploiting Lemma 4.2.3 and using (4.1.1b). □

Proposition 4.2.6. *Assume (4.1.1a), (4.1.1b) and let $T<\frac{\ell_0}{2}$. If $\dot\ell^n\to\dot\ell$ in $L^1(0,T)$, then $A^n\to A$ in $H^1((0,T)\times(0,+\infty))$.*

Proof. First of all we notice that our hypothesis imply ℓ^n uniformly converges to ℓ in $[0,T]$ and hence by Proposition 4.2.5 we deduce that $A^n\to A$ in $L^2((0,T)\times(0,+\infty))$, so we only have to prove that the same kind of convergence holds true for A_t^n and A_x^n. We assume without loss of generality that $\ell_0<\ell_0^n$, the other cases being analogous. We then consider again the partition (4.2.2) used in the proof of previous proposition, see also Figure 4.1. So we have:

$$
\|A_t^n-A_t\|^2_{L^2((0,T)\times(0,+\infty))}=\sum_{i=1}^{7}\iint_{\Lambda_i^n}|A_t^n(t,x)-A_t(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t.
$$

By (4.1.1b) the integrals over Λ_1^n and Λ_2^n goes to zero as $n \to +\infty$. For the others we start to estimate from Λ_3^n:

$$\begin{aligned}
&\iint_{\Lambda_3^n} |A_t^n(t,x) - A_t(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\le C \iint_{\Lambda_3^n} (|\dot v_0^n(x{-}t) - \dot v_0(x{-}t)|^2 + |v_1^n(x{-}t) - v_1(x{-}t)|^2) \,\mathrm{d}x\,\mathrm{d}t \\
&+ C \iint_{\Lambda_3^n} \Big(|\dot v_0^n(x{+}t) - \dot v_0(-\omega(x{+}t))\dot\omega(x{+}t)|^2 + |v_1^n(x{+}t) + v_1(-\omega(x{+}t))\dot\omega(x{+}t)|^2\Big) \,\mathrm{d}x\,\mathrm{d}t \\
&\le C \left(\|\dot v_0^n - \dot v_0\|^2_{L^2(0,+\infty)} + \|v_1^n - v_1\|^2_{L^2(0,+\infty)} + \iint_{\Lambda_3^n} \Big((|\dot v_0|^2 + |v_1|^2)(-\omega(x{+}t))\Big)\dot\omega(x{+}t)^2 \mathrm{d}x\,\mathrm{d}t \right) \\
&\le C \left(\|\dot v_0^n - \dot v_0\|^2_{L^2(0,+\infty)} + \|v_1^n - v_1\|^2_{L^2(0,+\infty)} + \int_{-\omega(\ell_0^n)}^{\ell_0} (|\dot v_0(s)|^2 + |v_1(s)|^2) \,\mathrm{d}s \right).
\end{aligned}$$

As regards Λ_4^n we have:

$$\begin{aligned}
&\iint_{\Lambda_4^n} |A_t^n(t,x) - A_t(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t = \iint_{\Lambda_4^n} |A_t^n(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\le C \left(\iint_{\Lambda_4^n} (|\dot v_0^n(x{-}t)|^2 + |v_1^n(x{-}t)|^2) \,\mathrm{d}x\,\mathrm{d}t + \iint_{\Lambda_4^n} (|\dot v_0^n(x{+}t)|^2 + |v_1^n(x{+}t)|^2) \,\mathrm{d}x\,\mathrm{d}t \right) \\
&\le C \left(\int_{-\omega(\ell_0^n)}^{\ell_0^n} (|\dot v_0^n(s)|^2 + |v_1^n(s)|^2) \,\mathrm{d}s + \int_{\ell_0}^{\ell_0^n} (|\dot v_0^n(s)|^2 + |v_1^n(s)|^2) \,\mathrm{d}s \right) \\
&\le C \left(\|\dot v_0^n - \dot v_0\|^2_{L^2(0,+\infty)} + \|v_1^n - v_1\|^2_{L^2(0,+\infty)} + \int_{-\omega(\ell_0^n)}^{\ell_0} (|\dot v_0(s)|^2 + |v_1(s)|^2) \,\mathrm{d}s \right).
\end{aligned}$$

We then consider $\Lambda_6^n \cup \Lambda_7^n$, so that:

$$\iint_{\Lambda_6^n \cup \Lambda_7^n} |A_t^n(t,x) - A_t(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t = \iint_{\Lambda_6^n} |A_t^n(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t + \iint_{\Lambda_7^n} |A_t(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

Since by assumptions $\ell^n \to \ell$ uniformly in $[0,T]$, we deduce $\Lambda_7^n \to \emptyset$ in measure, and so the second integral goes to zero as $n \to +\infty$, while for the first one we estimate:

$$\begin{aligned}
&\iint_{\Lambda_6^n} |A_t^n(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\le C \iint_{\Lambda_6^n} (|\dot v_0^n(x{-}t)|^2 + |v_1^n(x{-}t)|^2) \,\mathrm{d}x\,\mathrm{d}t \\
&+ C \iint_{\Lambda_6^n} \Big((|\dot v_0^n|^2 + |v_1^n|^2)(-\omega^n(x{+}t))\Big) |\dot\omega^n(x{+}t)|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\le C \max_{r \in D_\varphi^n(0,T)} |(\varphi^n)^{-1}(r) - \varphi^{-1}(r)| \int_0^{\ell_0^n} (|\dot v_0^n(s)|^2 + |v_1^n(s)|^2) \,\mathrm{d}s \\
&\quad + C \max_{r \in D_\psi^n(0,T)} |\omega^n(r) - \omega(r)| \int_{-\omega^n(\psi(T) \wedge \psi^n(T))}^{\ell_0^n} (|\dot v_0^n(s)|^2 + |v_1^n(s)|^2) \,\mathrm{d}s \\
&\le C \left(\max_{r \in D_\varphi^n(0,T)} |(\varphi^n)^{-1}(r) - \varphi^{-1}(r)| + \max_{r \in D_\psi^n(0,T)} |\omega^n(r) - \omega(r)| \right) (\|\dot v_0^n\|^2_{L^2(0,+\infty)} + \|v_1^n\|^2_{L^2(0,+\infty)})
\end{aligned}$$

$$\leq C\left(\max_{r\in D^n_\varphi(0,T)}|(\varphi^n)^{-1}(r)-\varphi^{-1}(r)|+\max_{r\in D^n_\psi(0,T)}|\omega^n(r)-\omega(r)|\right).$$

Applying Lemma 4.2.2 for the sequence of functions $\{\varphi^n\}_{n\in\mathbb{N}}$ and Lemma 4.2.3 we deduce that this last integral vanishes as $n\to+\infty$. The last term to treat is the integral over Λ^n_5:

$$\begin{aligned}
&\iint_{\Lambda^n_5}|A^n_t(t,x)-A_t(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\leq C\iint_{\Lambda^n_5}|\dot v^n_0(x{-}t)-\dot v_0(x{-}t)|^2\,\mathrm{d}x\,\mathrm{d}t+C\iint_{\Lambda^n_5}|v^n_1(x{-}t)-v_1(x{-}t)|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+C\iint_{\Lambda^n_5}\Big|((\dot v^n_0-v^n_1)(-\omega^n(x{+}t)))\,\dot\omega^n(x{+}t)-\Big((\dot v_0-v_1)(-\omega(x{+}t)\Big)\dot\omega(x{+}t)\Big|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\leq C\|\dot v^n_0-\dot v_0\|^2_{L^2(0,+\infty)}+C\|v^n_1-v_1\|^2_{L^2(0,+\infty)}\\
&\quad+C\int_{D^n_\psi(0,T)}\Big|((\dot v^n_0-v^n_1)(-\omega^n(s)))\,\dot\omega^n(s)-\Big((\dot v_0-v_1)(-\omega(s))\Big)\dot\omega(s)\Big|^2\,\mathrm{d}s.
\end{aligned}$$

Applying Lemma 4.2.4 to this last integral and putting together all the previous estimates, by (4.1.1a) and (4.1.1b) we finally conclude that $A^n_t\to A_t$ in $L^2((0,T)\times(0,+\infty))$. Reasoning exactly in the same way one also gets $A^n_x\to A_x$ in $L^2((0,T)\times(0,+\infty))$ and so the Proposition is proved. □

Now we can deal with the convergence of the sequence of auxiliary functions $\{v^n\}_{n\in\mathbb{N}}$. We only need a short lemma. Before the statement we introduce the following notation: here and henceforth by $A\triangle B$ we mean the symmetric difference of the sets A and B; if moreover both sets depend on time and space, we write $(A\triangle B)(t,x)$ instead of $A(t,x)\triangle B(t,x)$.

Lemma 4.2.7. *Let $T<\frac{\ell_0}{2}$ and assume ℓ^n uniformly converges to ℓ in $[0,T]$, then the map $(t,x)\mapsto|(R^n\triangle R)(t,x)|$ uniformly converges to zero in $[0,T]\times[0,+\infty)$.*

Proof. We assume without loss of generality that $\ell_0<\ell^n_0$, the other cases being analogous. We then consider again the partition of $[0,T]\times[0,+\infty)$ given by the sets Λ^n_i, for $i=1,\dots,8$, introduced in the proof of Proposition 4.2.5.
If $(t,x)\in\Lambda^n_1\cup\Lambda^n_2$, then $(R^n\triangle R)(t,x)=\emptyset$ and so $|(R^n\triangle R)(t,x)|=0$.
If $(t,x)\in\Lambda^n_3\cup\Lambda^n_4$, then $(R^n\triangle R)(t,x)\subseteq[0,\psi^{-1}(\ell^n_0)]\times[-\omega(\ell^n_0),\ell^n_0]$ and so

$$|(R^n\triangle R)(t,x)|\leq\psi^{-1}(\ell^n_0)(\ell^n_0+\omega(\ell^n_0)).$$

If finally $(t,x)\in\Lambda^n_5\cup\Lambda^n_6\cup\Lambda^n_7$, then

$$|(R^n\triangle R)(t,x)|\leq T\max_{r\in D^n_\psi(0,T)}|\omega^n(r)-\omega(r)|.$$

We conclude recalling that $\omega(\ell_0)=-\ell_0$ and exploiting Lemma 4.2.3. □

Proposition 4.2.8. *Assume (4.1.1a), (4.1.1b) and let T be as in Proposition 4.2.1. If ℓ^n uniformly converges to ℓ in $[0,T]$, then v^n uniformly converges to v in $[0,T]\times[0,+\infty)$.*

Proof. Exploiting representation formula (4.1.2) we deduce that:

$$\begin{aligned}
&\|v^n-v\|_{C^0([0,T]\times[0,+\infty))}\\
&\leq\|A^n-A\|_{C^0([0,T]\times[0,+\infty))}+\frac{|(\nu^n)^2-\nu^2|}{8}\|H\|_{C^0([0,T]\times[0,+\infty))}\\
&+\frac{(\nu^n)^2}{8}\|H^n-H\|_{C^0([0,T]\times[0,+\infty))}
\end{aligned}$$

$$
\begin{aligned}
&\leq \|A^n - A\|_{C^0([0,T]\times[0,+\infty))} + \frac{|(\nu^n)^2-\nu^2|}{8}\|H\|_{C^0([0,T]\times[0,+\infty))} \\
&\quad + \frac{N^2}{8}\left\| \iint_{R^n} |v^n - v| + \iint_{R^n\triangle R}|v| \right\|_{C^0([0,T]\times[0,+\infty))} \\
&\leq \|A^n - A\|_{C^0([0,T]\times[0,+\infty))} + \frac{|(\nu^n)^2-\nu^2|}{8}\|H\|_{C^0([0,T]\times[0,+\infty))} \\
&+ \frac{N^2}{8}|\Omega^n_T|\|v^n - v\|_{C^0([0,T]\times[0,+\infty))} + \frac{N^2}{8}\||R^n\triangle R|\|_{C^0([0,T]\times[0,+\infty))}\|v\|_{C^0([0,T]\times[0,+\infty))} \\
&\leq \|A^n - A\|_{C^0([0,T]\times[0,+\infty))} + \frac{|(\nu^n)^2-\nu^2|}{8}\|H\|_{C^0([0,T]\times[0,+\infty))} + \frac{1}{2}\|v^n - v\|_{C^0([0,T]\times[0,+\infty))} \\
&\quad + \frac{N^2}{8}\||R^n\triangle R|\|_{C^0([0,T]\times[0,+\infty))}\|v\|_{C^0([0,T]\times[0,+\infty))},
\end{aligned}
$$

and so we get:

$$
\begin{aligned}
\|v^n - v\|_{C^0([0,T]\times[0,+\infty))} \leq 2\|A^n - A\|_{C^0([0,T]\times[0,+\infty))} + \frac{|(\nu^n)^2-\nu^2|}{4}\|H\|_{C^0([0,T]\times[0,+\infty))} \\
+ \frac{N^2}{4}\||R^n\triangle R|\|_{C^0([0,T]\times[0,+\infty))}\|v\|_{C^0([0,T]\times[0,+\infty))}.
\end{aligned}
$$

Letting $n \to +\infty$ we deduce that by Proposition 4.2.5 the first term goes to zero, by (4.1.1a) the second one goes trivially to zero and by Lemma 4.2.7 the third one goes to zero too. So we conclude. □

Proposition 4.2.9. *Assume* (4.1.1a), (4.1.1b) *and let T be as in Proposition 4.2.1. If $\dot\ell^n \to \dot\ell$ in $L^1(0,T)$, then $v^n \to v$ in $H^1((0,T)\times(0,+\infty))$.*

Proof. First of all we notice that our hypothesis imply $\ell^n \to \ell$ uniformly in $[0,T]$ and hence by Proposition 4.2.8 we get $v^n \to v$ uniformly in $[0,T]\times[0,+\infty)$ and so in particular in $L^2((0,T)\times(0,+\infty))$. To get the same result for the sequence of time derivatives $\{v^n_t\}_{n\in\mathbb{N}}$ we estimate:

$$
\begin{aligned}
&\|v^n_t - v_t\|_{L^2((0,T)\times(0,+\infty))} \\
&\leq \|A^n_t - A_t\|_{L^2((0,T)\times(0,+\infty))} + \frac{|(\nu^n)^2-\nu^2|}{8}\|H_t\|_{L^2((0,T)\times(0,+\infty))} \\
&\quad + \frac{N^2}{8}\|H^n_t - H_t\|_{L^2((0,T)\times(0,+\infty))}.
\end{aligned}
$$

By Proposition 4.2.6 we deduce that the first term goes to zero as $n \to +\infty$, by (4.1.1a) the second term goes trivially to zero, while for the third one one gets the same result exploiting the explicit formulas for H^n_t and H_t given by (3.2.14a), the fact that $v^n \to v$ uniformly in $[0,T]\times[0,+\infty)$, and reasoning as in the proof of Proposition 4.2.6.

With the same argument one can show that also $v^n_x \to v_x$ in $L^2((0,T)\times(0,+\infty))$ and so the result is proved. □

Proposition 4.2.10. *Assume* (4.1.1a), (4.1.1b) *and let T be as in Proposition 4.2.1. If $\dot\ell^n \to \dot\ell$ in $L^1(0,T)$, then $v^n \to v$ in $C^0([0,T];H^1(0,+\infty))$ and in $C^1([0,T];L^2(0,+\infty))$.*

Proof. By Proposition 4.2.8 we know that $v^n \to v$ uniformly in $[0,T]\times[0,+\infty)$, so to conclude it is enough to prove that

$$
\lim_{n\to+\infty}\max_{t\in[0,T]}\|v^n_t(t) - v_t(t)\|_{L^2(0,+\infty)} = 0, \quad \text{and} \quad \lim_{n\to+\infty}\max_{t\in[0,T]}\|v^n_x(t) - v_x(t)\|_{L^2(0,+\infty)} = 0.
$$

We actually prove only the validity of the first limit, the other one being analogous. So we fix $t \in [0,T]$ and we assume that $\ell(t) < \ell^n(t)$, being the other cases even easier to deal with, then we estimate:

$$\begin{aligned}\|v_t^n(t) - v_t(t)\|_{L^2(0,+\infty)} &= \int_0^{\ell(t)} |v_t^n(t,x) - v_t(t,x)|^2 \,\mathrm{d}x + \int_{\ell(t)}^{\ell^n(t)} |v_t^n(t,x)|^2 \,\mathrm{d}x \\ &\le 2\int_0^{\ell(t)} |A_t^n(t,x) - A_t(t,x)|^2 \,\mathrm{d}x + 2\int_{\ell(t)}^{\ell^n(t)} |A_t^n(t,x)|^2 \,\mathrm{d}x \quad (4.2.3) \\ &\quad + 2\int_0^{\ell(t)} |H_t^n(t,x) - H_t(t,x)|^2 \,\mathrm{d}x + 2\int_{\ell(t)}^{\ell^n(t)} |H_t^n(t,x)|^2 \,\mathrm{d}x.\end{aligned}$$

Exploiting the explicit formulas (3.2.14a) and Proposition 4.2.1 it is easy to see that the second term in the last line is bounded by $C\|\ell^n - \ell\|_{C^0([0,T])}$; always by (3.2.14a) we deduce that also the first term in the last line goes uniformly to zero in $[0,T]$. We want to remark that the only difficult part to estimate is the following:

$$\begin{aligned}&\int_{\ell_0^n - t}^{\ell(t)} \left| \dot\omega^n(x{+}t) \int_0^{(\psi^n)^{-1}(x+t)} v^n(\tau, \tau - \omega^n(x{+}t))\,\mathrm{d}\tau - \dot\omega(x{+}t)\int_0^{\psi^{-1}(x+t)} v(\tau,\tau-\omega(x{+}t))\,\mathrm{d}\tau \right|^2 \mathrm{d}x \\ &= \int_{\ell_0^n - t}^{\ell(t)} \left| \dot\omega^n(x{+}t) \int_0^{T} v^n(\tau, \tau - \omega^n(x{+}t))\,\mathrm{d}\tau - \dot\omega(x{+}t)\int_0^{T} v(\tau,\tau-\omega(x{+}t))\,\mathrm{d}\tau \right|^2 \mathrm{d}x \\ &= \int_{\ell_0^n}^{\psi(t)} \left| \dot\omega^n(s) \int_0^{T} v^n(\tau, \tau - \omega^n(s))\,\mathrm{d}\tau - \dot\omega(s)\int_0^{T} v(\tau,\tau-\omega(s))\,\mathrm{d}\tau \right|^2 \mathrm{d}s,\end{aligned}$$

which goes uniformly to zero applying Lemma 4.2.4 and recalling that $v^n \to v$ uniformly in $[0,T]\times[0,+\infty)$.

The first term in the second line in (4.2.3) is estimated just as above using hypothesis (4.1.1b), while for the second term we reason as follows:

$$\begin{aligned}&\int_{\ell(t)}^{\ell^n(t)} |A_t^n(t,x)|^2 \,\mathrm{d}x \\ &\le 2\int_{\ell(t)}^{\ell^n(t)} |(\dot v_0^n + v_1^n)(x{-}t)|^2 \,\mathrm{d}x + 2\int_{\ell(t)}^{\ell^n(t)} \left|\left((\dot v_0^n + v_1^n)(-\omega^n(x{+}t))\right)\dot\omega^n(x{+}t)\right|^2 \mathrm{d}x \\ &\le 2\int_{-\varphi(t)}^{-\varphi^n(t)} |(\dot v_0^n + v_1^n)(s)|^2 \,\mathrm{d}s + 2\int_{-\varphi^n(t)}^{-\omega^n(\psi(t))} |(\dot v_0^n + v_1^n)(s)|^2 \,\mathrm{d}s \\ &\le 2\|\dot v_0^n + v_1^n - \dot v_0 - v_1\|^2_{L^2(0,+\infty)} + 2\int_{-\varphi(t)}^{-\omega^n(\psi(t))} |(\dot v_0 + v_1)(s)|^2 \,\mathrm{d}s,\end{aligned}$$

which goes uniformly to zero since $-\omega^n \circ \psi \to -\varphi$ uniformly.

So we have proved that $\lim\limits_{n\to+\infty} \max\limits_{t\in[0,T]} \|v_t^n(t) - v_t(t)\|_{L^2(0,+\infty)} = 0$ and we conclude. □

Proposition 4.2.11. *Assume* (4.1.1a), (4.1.1b) *and let* T *be as in Proposition 4.2.1. If* $\dot\ell^n \to \dot\ell$ *in* $L^1(0,T)$, *then* $v_x^n(\cdot,0) \to v_x(\cdot,0)$ *and* $\sqrt{1{-}\dot\ell^n(\cdot)^2}\, v_x^n(\cdot,\ell^n(\cdot)) \to \sqrt{1{-}\dot\ell(\cdot)^2}\, v_x(\cdot,\ell(\cdot))$ *in* $L^2(0,T)$.

Proof. By (3.2.15a) we recall that for a.e. $t \in [0,T]$ the following equality holds true:

$$v_x^n(t,0) = -\dot z^n(t) + \dot v_0^n(t) + v_1^n(t) + \frac{(\nu^n)^2}{4}\int_0^t v^n(\tau, t{-}\tau)\,\mathrm{d}\tau,$$

and so using (4.1.1b) and Propositions 4.2.1 and 4.2.8 it is easy to deduce $v_x^n(\cdot,0)\to v_x(\cdot,0)$ in $L^2(0,T)$.
Moreover by (3.2.15b) we know that for a.e. $t\in[0,T]$ it holds:

$$\begin{aligned} v_x^n(t,\ell^n(t)) &= \frac{1}{1+\dot\ell^n(t)}\left[\dot v_0^n(\ell^n(t)-t) - v_1^n(\ell^n(t)-t) - \frac{(\nu^n)^2}{4}\int_0^t v^n(\tau,\tau+\ell^n(t)-t)\,\mathrm{d}\tau\right] \\ &= \frac{1}{1+\dot\ell^n(t)}\left[\dot v_0^n(\ell^n(t)-t) - v_1^n(\ell^n(t)-t) - \frac{(\nu^n)^2}{4}\int_0^T v^n(\tau,\tau+\ell^n(t)-t)\,\mathrm{d}\tau\right]. \end{aligned}$$

We denote by $g^n(t-\ell^n(t))$ the expression within the square brackets, i.e. $g^n(t-\ell^n(t)) = (1+\dot\ell^n(t))v_x^n(t,\ell^n(t))$, and we estimate:

$$\begin{aligned} &\int_0^T \left|\sqrt{1-\dot\ell^n(t)^2}v_x^n(t,\ell^n(t)) - \sqrt{1-\dot\ell(t)^2}v_x(t,\ell(t))\right|^2 \mathrm{d}t \\ &= \int_0^T \left|\frac{\sqrt{1-\dot\ell^n(t)^2}}{1+\dot\ell^n(t)}g^n(t-\ell^n(t)) - \frac{\sqrt{1-\dot\ell(t)^2}}{1+\dot\ell(t)}g(t-\ell(t))\right|^2 \mathrm{d}t \\ &\le 2\int_0^T \left|\frac{1}{1+\dot\ell^n(t)}\left(\sqrt{1-\dot\ell^n(t)^2}g^n(t-\ell^n(t)) - \sqrt{1-\dot\ell(t)^2}g(t-\ell(t))\right)\right|^2 \mathrm{d}t \\ &\quad + 2\int_0^T \left|\frac{1}{1+\dot\ell^n(t)} - \frac{1}{1+\dot\ell(t)}\right|^2 (1-\dot\ell(t)^2)g(t-\ell(t))^2\,\mathrm{d}t \\ &\le 2\int_0^T \left|\sqrt{1-\dot\ell^n(t)^2}g^n(t-\ell^n(t)) - \sqrt{1-\dot\ell(t)^2}g(t-\ell(t))\right|^2 \mathrm{d}t \\ &\quad + 2\int_0^T \left|\dot\ell^n(t)-\dot\ell(t)\right|(1-\dot\ell(t)^2)g(t-\ell(t))^2\,\mathrm{d}t. \end{aligned}$$

By dominated convergence the last integral vanishes when $n\to+\infty$, so we conclude if we prove that $\sqrt{1-\dot\ell^n(\cdot)^2}g^n(\cdot-\ell^n(\cdot)) \to \sqrt{1-\dot\ell(\cdot)^2}g(\cdot-\ell(\cdot))$ in $L^2(0,T)$. To this aim we continue to estimate:

$$\begin{aligned} &\int_0^T \left|\sqrt{1-\dot\ell^n(t)^2}g^n(t-\ell^n(t)) - \sqrt{1-\dot\ell(t)^2}g(t-\ell(t))\right|^2 \mathrm{d}t \\ &\le 2\int_0^T (1-\dot\ell^n(t)^2)\left|(g^n-g)(t-\ell^n(t))\right|^2 \mathrm{d}t \\ &\quad + 2\int_0^T \left|\sqrt{1-\dot\ell^n(t)^2}g(t-\ell^n(t))-\sqrt{1-\dot\ell(t)^2}g(t-\ell(t))\right|^2 \mathrm{d}t. \end{aligned}$$

By (4.1.1a), (4.1.1b) and exploiting Proposition 4.2.8 it is easy to see that $g^n(\cdot)\to g(\cdot)$ in $L^2(-\infty,0)$ and so reasoning as in the proof of Lemma 4.2.4 we get both terms go to zero as $n\to+\infty$. Hence we conclude. □

Summarising, in this section we have obtained the following result: if we assume (4.1.1a), (4.1.1b) and if for some $T<\min\left\{\frac{\ell_0}{2},\frac{2}{N^2\ell_0}\right\}$ we know that $\dot\ell^n\to\dot\ell$ in $L^1(0,T)$ (and hence ℓ^n uniformly converges to ℓ in $[0,T]$), then the sequence of auxiliary functions $\{v^n\}_{n\in\mathbb{N}}$ converges to v in the following ways:

- $v^n\to v$ uniformly in $[0,T]\times[0,+\infty)$;
- $v^n\to v$ in $H^1((0,T)\times(0,+\infty))$;
- $v^n\to v$ in $C^0([0,T];H^1(0,+\infty))$ and in $C^1([0,T];L^2(0,+\infty))$; (4.2.4)
- $v_x^n(\cdot,0)\to v_x(\cdot,0)$ and
  $\sqrt{1-\dot\ell^n(\cdot)^2}v_x^n(\cdot,\ell^n(\cdot)) \to \sqrt{1-\dot\ell(\cdot)^2}v_x(\cdot,\ell(\cdot))$ in $L^2(0,T)$.

Remark 4.2.12. We recall that by the formula $u^n(t,x) = e^{-\nu^n t/2} v^n(t,x)$ we deduce that all the convergences in (4.2.4) still remains true replacing v^n and v by the real solutions of the coupled problem u^n and u respectively.

4.3 The continuous dependence result

The goal of this section is proving that under assumptions (4.1.1) there exists a small time $\overline{T} > 0$ such that $\dot{\ell}^n \to \dot{\ell}$ in $L^1(0,\overline{T})$. In this case, by what we proved in Section 4.2, we will deduce as a byproduct that all the convergences in (4.2.4) hold true in $[0,\overline{T}]$. This will lead us to the main theorem of the chapter, namely Theorem 4.3.6, in which we extend the result to arbitrary large time.

To this aim, as in [24], we introduce the functions λ^n and λ as the inverse of φ^n and φ, respectively. By (3.5.2) and (3.5.3) we recall that for $T < \frac{\ell_0}{2}$ we can write:

$$\lambda^n(y) = \frac{1}{2}\int_{-\ell_0^n}^{y} \left(1 + \max\left\{\Theta^n_{v^n,\lambda^n}(s), 1\right\}\right) \mathrm{d}s, \quad \text{for every } y \in [-\ell_0^n, \varphi^n(T)], \tag{4.3.1}$$

where for a.e. $y \in [-\ell_0^n, \varphi^n(T)]$ we considered the function:

$$\Theta^n_{v^n,\lambda^n}(y) = \frac{\left[\dot{v}_0^n(-y) - v_1^n(-y) - \frac{(\nu^n)^2}{4}\int_0^{\lambda^n(y)} v^n(\tau,\tau-y)\,\mathrm{d}\tau\right]^2}{2e^{\nu^n\lambda^n(y)}\kappa^n(\lambda^n(y)-y)}. \tag{4.3.2}$$

Obviously the same formulas without apexes n hold true also for λ.
Furthermore let us define the set (see Figure 4.2):

$$Q^n := \left\{(t,x) \in \mathbb{R}^2 \mid t \in [0,T] \text{ and } x \in [t - (\varphi(T) \wedge \varphi^n(T)), t + (\ell_0 \wedge \ell_0^n)]\right\},$$

and as we did in (3.5.7) let us introduce the distance:

$$\mathrm{d}\left((v^n,\lambda^n),(v,\lambda)\right) := \max\left\{\|v^n - v\|_{L^2(Q^n)}, \max_{y \in D^n_\varphi(0,T)} |\lambda^n(y) - \lambda(y)|\right\}. \tag{4.3.3}$$

First of all let us prove that $D^n_\varphi(0,T) = [-(\ell_0 \wedge \ell_0^n), \varphi(T) \wedge \varphi^n(T)]$ is definitively nondegenerate.

Lemma 4.3.1. *Assume (4.1.1) and let T be as in Proposition 4.2.1. Then there exists $\bar{n} \in \mathbb{N}$ such that for every $n \geq \bar{n}$ the set $D^n_\varphi(0,T)$ is a nondegenerate closed interval.*

Proof. We argue by contradiction. Let us assume that there exists a subsequence (not relabelled) such that $D^n_\varphi(0,T)$ is empty or it is a singleton for every $n \in \mathbb{N}$. Since $\ell_0^n \to \ell_0$ and since $\varphi(T) > -\ell_0$ we can exclude the case $\varphi(T) \leq -\ell_0^n < \varphi^n(T)$ for every n. This means that for every $n \in \mathbb{N}$ we have $-\ell_0^n < \varphi^n(T) \leq -\ell_0$.

CLAIM. We claim that in this case $\lim_{n\to+\infty} \max_{y\in[-\ell_0^n,\varphi^n(T)]} |\lambda^n(y)| = 0$.

If the claim is true we conclude; indeed by definition $\lambda^n(\varphi^n(T)) = T$ and hence we get a contradiction.
To prove the claim we fix $y \in [-\ell_0^n, \varphi^n(T)]$ and we estimate:

$$\begin{aligned}
\lambda^n(y) &\leq \frac{1}{2}\int_{-\ell_0^n}^{\varphi^n(T)} \left(1 + \max\left\{\Theta^n_{v^n,\lambda^n}(s), 1\right\}\right) \mathrm{d}s \leq \int_{-\ell_0^n}^{\varphi^n(T)} \left(1 + \frac{1}{2}\Theta^n_{v^n,\lambda^n}(s)\right) \mathrm{d}s \\
&= \varphi^n(T) + \ell_0^n + \frac{1}{4}\int_{-\ell_0^n}^{\varphi^n(T)} \frac{\left[\dot{v}_0^n(-s) - v_1^n(-s) - \frac{(\nu^n)^2}{4}\int_0^{\lambda^n(s)} v^n(\tau,\tau-s)\,\mathrm{d}\tau\right]^2}{e^{\nu^n\lambda^n(s)}\kappa^n(\lambda^n(s)-s)}\,\mathrm{d}s.
\end{aligned}$$

Since $-\ell_0^n < \varphi^n(T) \le -\ell_0$, by (4.1.1a) we deduce that $\varphi^n(T) + \ell_0^n \to 0$ as $n \to +\infty$. Then we estimate the integral in the last line exploiting Proposition 4.2.1 and hypothesis (4.1.1c):

$$\int_{-\ell_0^n}^{\varphi^n(T)} \frac{\left[\dot v_0^n(-s) - v_1^n(-s) - \frac{(\nu^n)^2}{4}\int_0^{\lambda^n(s)} v^n(\tau,\tau-s)\,\mathrm{d}\tau\right]^2}{e^{\nu^n\lambda^n(s)}\kappa^n(\lambda^n(s)-s)}\,\mathrm{d}s$$
$$\le C\int_{-\ell_0^n}^{\varphi^n(T)} \left(\dot v_0^n(-s)^2 + v_1^n(-s)^2 + N^4M^2T^2\right)\,\mathrm{d}s = C\int_{-\varphi^n(T)}^{\ell_0^n}\left(\dot v_0^n(s)^2 + v_1^n(s)^2 + 1\right)\,\mathrm{d}s.$$

By hypothesis (4.1.1b) and since $\varphi^n(T) + \ell_0^n \to 0$ we conclude. □

To make next proposition more clear let us introduce for $y < 0$ the functions $j^n(y) := |\dot v_0^n(-y)| + |v_1^n(-y)| + \mathbb{1}_{[0,2\ell_0]}(-y)$ and notice that by (4.1.1b) the sequence $\{j^n\}_{n\in\mathbb{N}}$ is equibounded in $L^2(-\infty,0)$. Here $\mathbb{1}_{[0,2\ell_0]}$ stands for the indicator function of $[0,2\ell_0]$; the choice of such an interval is simply related to the fact that definitively $0 < \ell_0^n < 2\ell_0$, since $\ell_0^n \to \ell_0$ as $n \to +\infty$, and thus $D_\varphi^n(0,T) \subseteq [-2\ell_0, 0]$ if the time T is small enough. Moreover, to simplify the expression of $\Theta^n_{v^n,\lambda^n}$ in (4.3.2), we also define the functions $\rho^n(y) := \dot v_0^n(-y) - v_1^n(-y) - \dfrac{(\nu^n)^2}{4}\displaystyle\int_0^{\lambda^n(y)} v^n(\tau,\tau-y)\,\mathrm{d}\tau$ and using Proposition 4.2.1 we observe that

$$|\rho^n(y)| \le Cj^n(y), \quad \text{for a.e. } y \in D_\varphi^n(0,T), \tag{4.3.4}$$

if the time T is sufficiently small. In the same way we define the functions j and ρ. Finally we introduce the nonnegative quantity:

$$\eta^n := \|j\|^2_{L^2(D_\varphi^n(0,T))} + \|j^n\|_{L^2(D_\varphi^n(0,T))} + \|j\|_{L^2(D_\varphi^n(0,T))}. \tag{4.3.5}$$

Proposition 4.3.2. *Assume (4.1.1), let T be as in Proposition 4.2.1 and let $\bar n$ be given by Lemma 4.3.1. Then there exists a constant $C_1 \ge 0$ independent of n and an infinitesimal sequence $\{\delta^n\}_{n\in\mathbb{N}}$ such that for every $n \ge \bar n$ the following estimate holds true:*

$$\max_{y\in D_\varphi^n(0,T)} |\lambda^n(y) - \lambda(y)| \le \delta^n + C_1\eta^n\,\mathrm{d}\left((v^n,\lambda^n),(v,\lambda)\right). \tag{4.3.6}$$

Proof. We assume $\ell_0 < \ell_0^n$, being the other cases even easier, and we estimate by means of (4.3.1) and (4.3.2):

$$\max_{y\in D_\varphi^n(0,T)} |\lambda^n(y) - \lambda(y)|$$
$$\le \int_{-\ell_0^n}^{-\ell_0} \dot\lambda^n(s)\,\mathrm{d}s + \frac14\int_{D_\varphi^n(0,T)}\left|\frac{\rho^n(s)^2}{e^{\nu^n\lambda^n(s)}\kappa^n(\lambda^n(s)-s)} - \frac{\rho(s)^2}{e^{\nu\lambda(s)}\kappa(\lambda(s)-s)}\right|\,\mathrm{d}s. \tag{4.3.7}$$

The first term goes to zero as $n \to +\infty$ reasoning as in the proof of Lemma 4.3.1. For the second one, denoted by I^n, we estimate by using triangular inequality and exploiting assumption (4.1.1c) to get uniform bounds on κ^n:

$$\begin{aligned}
I^n \le\ & C\int_{D_\varphi^n(0,T)} e^{\nu\lambda(s)}\kappa(\lambda(s)-s)\left|\rho^n(s)^2 - \rho(s)^2\right|\mathrm{d}s \\
&+ C\int_{D_\varphi^n(0,T)} \rho(s)^2\left|e^{\nu^n\lambda^n(s)}\kappa^n(\lambda^n(s)-s) - e^{\nu\lambda(s)}\kappa(\lambda(s)-s)\right|\mathrm{d}s \\
\le\ & C\int_{D_\varphi^n(0,T)}\left|\rho^n(s)^2 - \rho(s)^2\right|\,\mathrm{d}s + C\int_{D_\varphi^n(0,T)}\rho(s)^2\left|e^{(\nu^n-\nu)\lambda^n(s)} - 1\right|\,\mathrm{d}s \\
&+ C\max_{y\in[0,\ell_0+T]}|\kappa^n(y) - \kappa(y)|\int_{D_\varphi^n(0,T)}\rho(s)^2\,\mathrm{d}s
\end{aligned}$$

$$+ C \max_{y \in D_\varphi^n(0,T)} |\lambda^n(y) - \lambda(y)| \int_{D_\varphi^n(0,T)} \rho(s)^2 \, \mathrm{d}s.$$

By dominated convergence and by (4.1.1a) and (4.1.1c) the second and the third term go to zero as $n \to +\infty$, while for the first term we estimate by using the explicit expressions of ρ^n and ρ and recalling (4.3.4):

$$\begin{aligned}
&\int_{D_\varphi^n(0,T)} \left|\rho^n(s)^2 - \rho(s)^2\right| \mathrm{d}s \\
&\leq \int_{D_\varphi^n(0,T)} |\dot{v}_0^n(-s) - \dot{v}_0(-s)| \left(|\rho^n(s)| + |\rho(s)|\right) \mathrm{d}s + \int_{D_\varphi^n(0,T)} |v_1^n(-s) - v_1(-s)| \left(|\rho^n(s)| + |\rho(s)|\right) \mathrm{d}s \\
&\quad + \frac{1}{4} \int_{D_\varphi^n(0,T)} \left(|\rho^n(s)| + |\rho(s)|\right) \left| (\nu^n)^2 \int_0^{\lambda^n(s)} v^n(\tau, \tau - s) \, \mathrm{d}\tau - \nu^2 \int_0^{\lambda(s)} v(\tau, \tau - s) \, \mathrm{d}\tau \right| \mathrm{d}s \\
&\leq C \left(\|\dot{v}_0^n - \dot{v}_0\|_{L^2(0,+\infty)} + \|v_1^n - v_1\|_{L^2(0,+\infty)} \right) \left(\|j^n\|_{L^2(-\infty,0)} + \|j\|_{L^2(-\infty,0)} \right) \\
&\quad + C \int_{D_\varphi^n(0,T)} \left(|j^n(s)| + |j(s)|\right) \left| (\nu^n)^2 \int_0^{\lambda^n(s)} v^n(\tau, \tau - s) \, \mathrm{d}\tau - \nu^2 \int_0^{\lambda(s)} v(\tau, \tau - s) \, \mathrm{d}\tau \right| \mathrm{d}s.
\end{aligned}$$

To deal with the last integral we first notice that for every $s \in D_\varphi^n(0,T)$ we have:

$$\begin{aligned}
&\left| (\nu^n)^2 \int_0^{\lambda^n(s)} v^n(\tau, \tau - s) \, \mathrm{d}\tau - \nu^2 \int_0^{\lambda(s)} v(\tau, \tau - s) \, \mathrm{d}\tau \right| \\
&\leq |(\nu^n)^2 - \nu^2| \left| \int_0^{\lambda^n(s)} v^n(\tau, \tau - s) \, \mathrm{d}\tau \right| + \nu^2 \left| \int_0^{\lambda^n(s)} (v^n - v)(\tau, \tau - s) \, \mathrm{d}\tau \right| + \nu^2 \left| \int_{\lambda(s)}^{\lambda^n(s)} v(\tau, \tau - s) \, \mathrm{d}\tau \right| \\
&\leq C \left(|(\nu^n)^2 - \nu^2| + \left| \int_0^T (v^n - v)(\tau, \tau - s) \, \mathrm{d}\tau \right| + \max_{y \in D_\varphi^n(0,T)} |\lambda^n(y) - \lambda(y)| \right),
\end{aligned}$$

and so we deduce:

$$\begin{aligned}
&\max_{y \in D_\varphi^n(0,T)} |\lambda^n(y) - \lambda(y)| \leq \delta^n + I^n \\
&\leq \delta^n + C \left(\|j\|^2_{L^2(D_\varphi^n(0,T))} + \|j^n\|_{L^2(D_\varphi^n(0,T))} + \|j\|_{L^2(D_\varphi^n(0,T))} \right) \mathrm{d}\left((v^n, \lambda^n), (v, \lambda)\right) \\
&= \delta^n + C\eta^n \, \mathrm{d}\left((v^n, \lambda^n), (v, \lambda)\right),
\end{aligned}$$

and we conclude. □

Proposition 4.3.3. *Assume (4.1.1), let T be as in Proposition 4.2.1 and let $\bar{n}$ be given by Lemma 4.3.1. Then there exists a constant $C_2 \geq 0$ independent of n and an infinitesimal sequence $\{\delta^n\}_{n \in \mathbb{N}}$ such that for every $n \geq \bar{n}$ the following estimate holds true:*

$$\|v^n - v\|_{L^2(Q^n)} \leq \delta^n + C_2 \sqrt{|D_\varphi^n(0,T)|} \, \mathrm{d}\left((v^n, \lambda^n), (v, \lambda)\right). \tag{4.3.8}$$

Proof. We use again formula (4.1.2) and we estimate:

$$\begin{aligned}
\|v^n - v\|_{L^2(Q^n)} &\leq \|A^n - A\|_{L^2(Q^n)} + \frac{(\nu^n)^2}{8} \|H^n - H\|_{L^2(Q^n)} + \frac{|(\nu^n)^2 - \nu^2|}{8} \|H\|_{L^2(Q^n)} \\
&\leq \delta^n + \|A^n - A\|_{L^2(Q^n)} + \frac{N^2}{8} \|H^n - H\|_{L^2(Q^n)}.
\end{aligned} \tag{4.3.9}$$

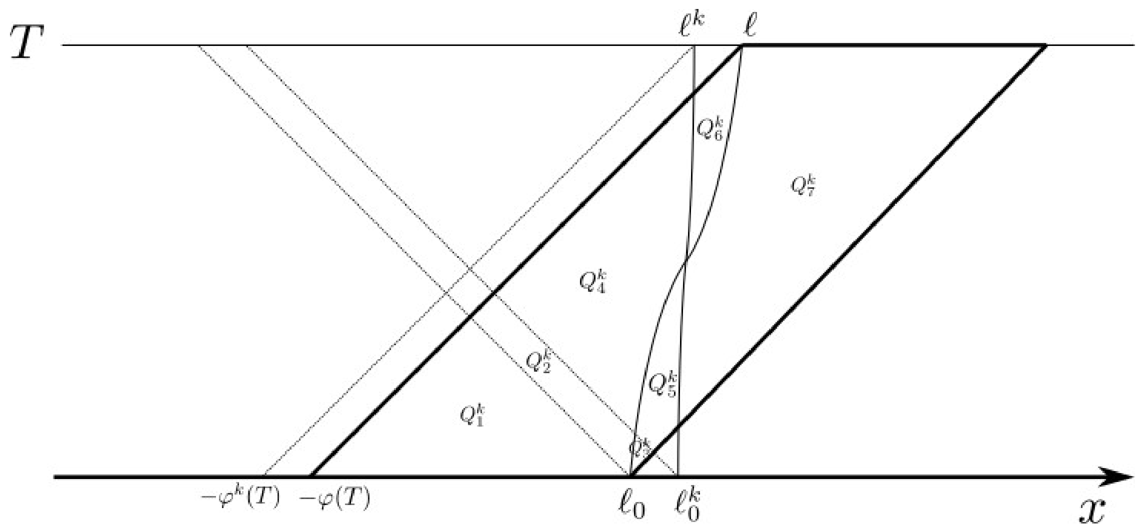

Figure 4.2: The partition of the set Q^n via the sets Q_i^n, for $i = 1, \dots, 7$, in the case $\ell_0 < \ell_0^n$ and $\varphi(T) < \varphi^n(T)$.

Then we split Q^n into seven parts, denoted by Q_i^n for $i = 1, \dots, 7$, as in Figure 4.2, so that:

$$\|A^n - A\|^2_{L^2(Q^n)} = \iint_{Q_1^n \cup Q_2^n \cup Q_4^n} |A^n(t,x) - A(t,x)|^2 \mathrm{d}x\,\mathrm{d}t + \iint_{Q_3^n \cup Q_5^n} A^n(t,x)^2 \mathrm{d}x\,\mathrm{d}t + \iint_{Q_6^n} A(t,x)^2 \mathrm{d}x\,\mathrm{d}t, \tag{4.3.10}$$

and we estimate all the terms.

The integrals over Q_1^n, Q_2^n, Q_3^n go easily to zero as $n \to +\infty$: indeed in Q_1^n we use (4.1.1b), while for the integrals over Q_2^n and Q_3^n we exploit the equiboundedness of the sequence $\{A^n\}_{n\in\mathbb{N}}$ in $C^0([0,T] \times [0,+\infty)$ (see Proposition 4.2.1) and the fact that $Q_2^n \cup Q_3^n$ converges in measure to the empty set. To estimate the remaining terms we reason as in the proof of Proposition 3.5.5 and we recall the validity of the following estimates:

$$|\omega^n(x{+}t) - \omega(x{+}t)| \le 2 \max_{y \in D_\varphi^n(0,T)} |\lambda^n(y) - \lambda(y)|, \quad \text{if } (t,x) \in Q_4^n, \tag{4.3.11a}$$

$$|(t{-}x) - \omega^n(x{+}t)| \le 2 \max_{y \in D_\varphi^n(0,T)} |\lambda^n(y) - \lambda(y)|, \quad \text{if } (t,x) \in Q_5^n, \tag{4.3.11b}$$

$$|(t{-}x) - \omega(x{+}t)| \le 2 \max_{y \in D_\varphi^n(0,T)} |\lambda^n(y) - \lambda(y)|, \quad \text{if } (t,x) \in Q_6^n. \tag{4.3.11c}$$

Moreover we also recall that:

$$|Q_5^n \cup Q_6^n| \le |D_\varphi^n(0,T)| \max_{y \in D_\varphi^n(0,T)} |\lambda^n(y) - \lambda(y)|. \tag{4.3.12}$$

Exploiting (4.3.11b), (4.3.11c), (4.3.12) and reasoning as in the proof of Proposition 3.5.5 one can deduce that:

$$\iint_{Q_5^n} A^n(t,x)^2 \,\mathrm{d}x\,\mathrm{d}t + \iint_{Q_6^n} A(t,x)^2 \,\mathrm{d}x\,\mathrm{d}t \le C|D_\varphi^n(0,T)| \max_{y \in D_\varphi^n(0,T)} |\lambda^n(y) - \lambda(y)|^2.$$

To estimate the integral over Q_4^n we first of all notice that for $(t,x) \in Q_4^n$ we have:

$$|A^n(t,x) - A(t,x)|^2 = \frac{1}{4} \left| \int_{x-t}^{-\omega^n(x+t)} (v_1^n(s) - \dot{v}_0^n(s))\,\mathrm{d}s - \int_{x-t}^{-\omega(x+t)} (v_1(s) - \dot{v}_0(s))\,\mathrm{d}s \right|^2$$

$$\le C\left(\|v_1^n - v_1\|^2_{L^2(0,+\infty)} + \|\dot{v}_0^n - \dot{v}_0\|^2_{L^2(0,+\infty)} \right) + \frac{1}{2} \left| \int_{-\omega(x+t)}^{-\omega^n(x+t)} (v_1(s) - \dot{v}_0(s))\,\mathrm{d}s \right|^2$$

$$\leq \delta^n + \frac{1}{2}|\omega^n(x+t) - \omega(x+t)| \left| \int_{-\omega(x+t)}^{-\omega^n(x+t)} (v_1(s) - \dot{v}_0(s))^2 \, \mathrm{d}s \right|.$$

Using (4.3.11a) we then deduce that for $(t,x) \in Q_4^n$ the following estimate holds true:

$$|A^n(t,x) - A(t,x)|^2 \leq \delta^n + \max_{y \in D_\varphi^n(0,T)} |\lambda^n(y) - \lambda(y)| \left| \int_{-\omega(x+t)}^{-\omega^n(x+t)} (v_1(s) - \dot{v}_0(s))^2 \, \mathrm{d}s \right|.$$

From this inequality, reasoning again as in the proof of Proposition 3.5.5, we conclude that:

$$\iint_{Q_4^n} |A^n(t,x) - A(t,x)|^2 \, \mathrm{d}x \, \mathrm{d}t \leq \delta^n + C|D_\varphi^n(0,T)| \max_{y \in D_\varphi^n(0,T)} |\lambda^n(y) - \lambda(y)|^2.$$

Putting all the previous estimates together we deduce:

$$\begin{aligned} \|A^n - A\|^2_{L^2(Q^n)} &\leq \delta^n + C|D_\varphi^n(0,T)| \max_{y \in D_\varphi^n(0,T)} |\lambda^n(y) - \lambda(y)|^2 \\ &\leq \delta^n + C|D_\varphi^n(0,T)| \, \mathrm{d}\left((v^n, \lambda^n), (v, \lambda)\right)^2. \end{aligned} \tag{4.3.13}$$

Now we estimate $\|H^n - H\|_{L^2(Q^n)}$. As in (4.3.10) we split its square into six integrals and we estimate all of them. With the same argument used before we deduce the integral over $Q_2^n \cup Q_3^n$ goes to zero as $n \to +\infty$, while the integral over Q_1^n is trivially bounded by $C|D_\varphi^n(0,T)| \|v^n - v\|^2_{L^2(Q^n)}$. More work is needed to treat the other three integrals. Exploiting Proposition 4.2.1 we estimate the integrals over Q_5^n and Q_6^n together:

$$\begin{aligned} &\iint_{Q_5^n} H^n(t,x)^2 \, \mathrm{d}x \, \mathrm{d}t + \iint_{Q_6^n} H(t,x)^2 \, \mathrm{d}x \, \mathrm{d}t \\ &\leq C \left(\iint_{Q_5^n} |R^n(t,x)|^2 \, \mathrm{d}x \, \mathrm{d}t + \iint_{Q_6^n} |R(t,x)|^2 \, \mathrm{d}x \, \mathrm{d}t \right) \\ &\leq C \left(\iint_{Q_5^n} |(t-x) - \omega^n(x+t)|^2 \, \mathrm{d}x \, \mathrm{d}t + \iint_{Q_6^n} |(t-x) - \omega(x+t)|^2 \, \mathrm{d}x \, \mathrm{d}t \right). \end{aligned}$$

So, using (4.3.11b) and (4.3.11c) we deduce:

$$\iint_{Q_5^n} H^n(t,x)^2 \, \mathrm{d}x \, \mathrm{d}t + \iint_{Q_6^n} H(t,x)^2 \, \mathrm{d}x \, \mathrm{d}t \leq C|D_\varphi^n(0,T)| \max_{y \in D_\varphi^n(0,T)} |\lambda^n(y) - \lambda(y)|^2.$$

For the integral over Q_4^n we use (4.3.11a) and we reason as follows:

$$\begin{aligned} &\iint_{Q_4^n} |H^n(t,x) - H(t,x)|^2 \, \mathrm{d}x \, \mathrm{d}t \\ &\leq \iint_{Q_4^n} \left(\iint_{R^n(t,x)} |v^n(\tau,\sigma) - v(\tau,\sigma)| \, \mathrm{d}\sigma \, \mathrm{d}\tau + \iint_{R^n(t,x) \triangle R(t,x)} |v(\tau,\sigma)| \, \mathrm{d}\sigma \, \mathrm{d}\tau \right)^2 \mathrm{d}x \, \mathrm{d}t \\ &\leq C \iint_{Q_4^n} \left(|R^n(t,x)| \|v^n - v\|^2_{L^2(Q^n)} + |R^n(t,x) \triangle R(t,x)|^2 \right) \mathrm{d}x \, \mathrm{d}t \\ &\leq C \left(|D_\varphi^n(0,T)| \|v^n - v\|^2_{L^2(Q^n)} + \iint_{Q_4^n} |\omega^n(x+t) - \omega(x+t)|^2 \, \mathrm{d}x \, \mathrm{d}t \right) \\ &\leq C|D_\varphi^n(0,T)| \left(\|v^n - v\|^2_{L^2(Q^n)} + \max_{y \in D_\varphi^n(0,T)} |\lambda^n(y) - \lambda(y)|^2 \right). \end{aligned}$$

Putting together the previous estimates we conclude that:

$$\begin{aligned}\|H^n - H\|^2_{L^2(Q^n)} &\leq \delta^n + C|D^n_\varphi(0,T)| \left(\|v^n - v\|^2_{L^2(Q^n)} + \max_{y \in D^n_\varphi(0,T)} |\lambda^n(y) - \lambda(y)|^2 \right) \\ &\leq \delta^n + C|D^n_\varphi(0,T)| \, \mathrm{d}\left((v^n,\lambda^n),(v,\lambda)\right)^2 ,\end{aligned} \tag{4.3.14}$$

and so by (4.3.9), (4.3.13) and (4.3.14) the Proposition is proved. □

Putting together (4.3.6) and (4.3.8) we deduce that there exists a constant $\overline{C} \geq 0$ independent of n such that for every n large enough it holds:

$$\mathrm{d}\left((v^n,\lambda^n),(v,\lambda)\right) \leq \delta^n + \overline{C} \max\left\{\eta^n, |D^n_\varphi(0,T)|\right\} \mathrm{d}\left((v^n,\lambda^n),(v,\lambda)\right). \tag{4.3.15}$$

By (4.3.15) we are able to improve Lemma 4.3.1:

Lemma 4.3.4. *Assume* (4.1.1) *and let T be as in Proposition 4.2.1. Then there exist $\alpha > 0$ and $\tilde{n} \in \mathbb{N}$ such that for every $n \geq \tilde{n}$ the nondegenerate closed interval $J^n_\alpha = [-(\ell^n_0 \wedge \ell_0), -\ell_0 + \alpha]$ is contained in $D^n_\varphi(0,T)$.*

Proof. Assume by contradiction that there exists a subsequence (not relabelled) such that $D^n_\varphi(0,T)$ goes to the empty set when $n \to +\infty$, namely $\lim_{n\to+\infty} \varphi^n(T) = -\ell_0$. By (4.3.5) we in particular deduce that $\eta^n \to 0$ as $n \to +\infty$.

By (4.3.15) we thus infer $\lim_{n\to+\infty} \mathrm{d}\left((v^n,\lambda^n),(v,\lambda)\right) = 0$, which in particular implies:

$$\lim_{n\to+\infty} \max_{y\in[-(\ell^n_0\wedge\ell_0),\varphi^n(T)]} |\lambda^n(y) - \lambda(y)| = 0.$$

This is absurd, indeed:

$$\lim_{n\to+\infty} |\lambda^n(\varphi^n(T)) - \lambda(\varphi^n(T))| = \lim_{n\to+\infty} |T - \lambda(\varphi^n(T))| = |T - \lambda(-\ell_0)| = T > 0,$$

and we conclude. □

From this lemma, repeating the proofs of Propositions 4.3.2 and 4.3.3 we deduce that (4.3.15) still holds true replacing $D^n_\varphi(0,T)$ by J^n_α, replacing η^n by

$$\eta^n_\alpha := \|j\|^2_{L^2(J^n_\alpha)} + \|j^n\|_{L^2(J^n_\alpha)} + \|j\|_{L^2(J^n_\alpha)},$$

and replacing Q^n by

$$Q^n_\alpha := \left\{(t,x) \in \mathbb{R}^2 \mid t \in [0,T] \text{ and } x \in [t + \ell_0 - \alpha, t + (\ell_0 \wedge \ell^n_0)]\right\}.$$

This means that, choosing α small enough, for every n large enough we have:

$$\mathrm{d}_\alpha\left((v^n,\lambda^n),(v,\lambda)\right) \leq \delta^n + \frac{1}{2}\mathrm{d}_\alpha\left((v^n,\lambda^n),(v,\lambda)\right), \tag{4.3.16}$$

where the new distance d_α is simply as in (4.3.3) replacing $D^n_\varphi(0,T)$ by J^n_α and Q^n by Q^n_α. By (4.3.16) we finally deduce that:

$$\lim_{n\to+\infty} \mathrm{d}_\alpha\left((v^n,\lambda^n),(v,\lambda)\right) = 0. \tag{4.3.17}$$

Furthermore by (4.3.17) we get:

$$\lim_{n\to+\infty} \int_{-(\ell^n_0\wedge\ell_0)}^{-\ell_0+\alpha} |\dot{\lambda}^n(y) - \dot{\lambda}(y)| \, \mathrm{d}y = 0. \tag{4.3.18}$$

To justify the validity of (4.3.18) we reason as follows: in the estimate (4.3.7) at the beginning of the proof of Proposition 4.3.2 we can replace $\max_{y\in J_\alpha^n} |\lambda^n(y)-\lambda(y)|$ by $\int_{-(\ell_0^n\wedge\ell_0)}^{-\ell_0+\alpha} |\dot\lambda^n(y)-\dot\lambda(y)|\,\mathrm{d}y$, obtaining that:

$$\int_{-(\ell_0^n\wedge\ell_0)}^{-\ell_0+\alpha} |\dot\lambda^n(y)-\dot\lambda(y)|\,\mathrm{d}y \le \delta^n + C_1\eta_\alpha^n\,\mathrm{d}_\alpha\left((v^n,\lambda^n),(v,\lambda)\right),$$

and so by (4.3.17) we conclude the argument. This leads to the following corollary:

Corollary 4.3.5. *Assume* (4.1.1)*. Then there exists a small time* $\overline{T}>0$ *such that* $\dot\ell^n\to\dot\ell$ *in* $L^1(0,\overline{T})$.

Proof. Let us take any $\overline{T}\in\big(0,\lambda(-\ell_0+\alpha)\big)$, where α is given by Lemma 4.3.4 and such that (4.3.18) holds true, and for the sake of clarity let us consider the value $m^n:=\lambda^n\big(-(\ell_0^n\wedge\ell_0)\big)\vee\lambda\big(-(\ell_0^n\wedge\ell_0)\big)$. Then we have:

$$\begin{aligned}\|\dot\ell^n-\dot\ell\|_{L^1(0,\overline{T})} &= \int_0^{m^n}|\dot\ell^n(s)-\dot\ell(s)|\,\mathrm{d}s + \int_{m^n}^{\overline{T}}|\dot\ell^n(s)-\dot\ell(s)|\,\mathrm{d}s\\ &\le 2m^n + \int_{m^n}^{\overline{T}}\left|\frac{1}{\dot\lambda^n(\lambda^{n-1}(s))}-\frac{1}{\dot\lambda(\lambda^{-1}(s))}\right|\mathrm{d}s.\end{aligned}$$

By uniform convergence of λ^n to λ and by (4.1.1a) the first term goes to zero as $n\to+\infty$, while for the second one, denoted by I^n, we estimate:

$$\begin{aligned}I^n &\le \int_{m^n}^{\overline{T}}\left|\frac{\dot\lambda^n(\lambda^{n-1}(s))-\dot\lambda(\lambda^{n-1}(s))}{\dot\lambda^n(\lambda^{n-1}(s))}\right|\mathrm{d}s + \int_{m^n}^{\overline{T}}\left|\frac{\dot\lambda(\lambda^{n-1}(s))-\dot\lambda(\lambda^{-1}(s))}{\dot\lambda^n(\lambda^{n-1}(s))\dot\lambda(\lambda^{-1}(s))}\right|\mathrm{d}s\\ &\le \int_{-(\ell_0^n\wedge\ell_0)}^{-\ell_0+\alpha}\left|\dot\lambda^n(y)-\dot\lambda(y)\right|\mathrm{d}y + \int_{m^n}^{\overline{T}}\left|\frac{\dot\lambda(\lambda^{n-1}(s))-\dot\lambda(\lambda^{-1}(s))}{\dot\lambda^n(\lambda^{n-1}(s))\dot\lambda(\lambda^{-1}(s))}\right|\mathrm{d}s.\end{aligned}$$

By (4.3.18) the first term goes to zero as $n\to+\infty$; for the second one, denoted by II^n, we reason as follows: we fix $\delta>0$ and we take $f_\delta\in C^0([-\ell_0,-\ell_0+\alpha])$ such that $\|\dot\lambda-f_\delta\|_{L^1(-\ell_0,-\ell_0+\alpha)}\le\delta$. Then we estimate:

$$\begin{aligned}II^n &\le \int_{m^n}^{\overline{T}}\left|\frac{\dot\lambda(\lambda^{n-1}(s))-f_\delta(\lambda^{n-1}(s))}{\dot\lambda^n(\lambda^{n-1}(s))}\right|\mathrm{d}s + \int_{m^n}^{\overline{T}}|f_\delta(\lambda^{n-1}(s))-f_\delta(\lambda^{-1}(s))|\,\mathrm{d}s\\ &\quad+\int_{m^n}^{\overline{T}}\left|\frac{f_\delta(\lambda^{-1}(s))-\dot\lambda(\lambda^{-1}(s))}{\dot\lambda(\lambda^{-1}(s))}\right|\mathrm{d}s\\ &\le 2\int_{-\ell_0}^{-\ell_0+\alpha}\left|\dot\lambda(y)-f_\delta(y)\right|\mathrm{d}y + \int_{m^n}^{\overline{T}}|f_\delta(\lambda^{n-1}(s))-f_\delta(\lambda^{-1}(s))|\,\mathrm{d}s\\ &\le 2\delta + \int_{m^n}^{\overline{T}}|f_\delta(\lambda^{n-1}(s))-f_\delta(\lambda^{-1}(s))|\,\mathrm{d}s.\end{aligned}$$

By Lemma 4.2.2 and dominated convergence this last integral vanishes as $n\to+\infty$, hence by the arbitrariness of δ we conclude. □

We are now in a position to state and prove the main result of the chapter:

Theorem 4.3.6. *Assume (4.1.1). Then the sequence of pairs $\{(u^n,\ell^n)\}_{n\in\mathbb{N}}$ converges to the solution of the limit problem (u,ℓ) in the following sense: for every $T>0$*

$$\begin{aligned}
&\bullet\ \dot\ell^n \to \dot\ell \text{ in } L^1(0,T), \text{ and thus } \ell^n\to\ell \text{ uniformly in } [0,T];\\
&\bullet\ u^n\to u \text{ uniformly in } [0,T]\times[0,+\infty);\\
&\bullet\ u^n\to u \text{ in } H^1((0,T)\times(0,+\infty));\\
&\bullet\ u^n\to u \text{ in } C^0([0,T];H^1(0,+\infty)) \text{ and in } C^1([0,T];L^2(0,+\infty));\\
&\bullet\ u^n_x(\cdot,0)\to u_x(\cdot,0) \text{ and}\\
&\quad \sqrt{1-\dot\ell^n(\cdot)^2}\,u^n_x(\cdot,\ell^n(\cdot))\to\sqrt{1-\dot\ell(\cdot)^2}\,u_x(\cdot,\ell(\cdot)) \text{ in } L^2(0,T).
\end{aligned}\tag{4.3.19}$$

Proof. As already remarked previously it is enough to prove that (4.3.19) holds true for the sequence of auxiliary functions $v^n(t,x)=e^{\nu^n t/2}u^n(t,x)$. By Corollary 4.3.5 and by the results presented in Section 4.2 we know there exists a small time $\overline{T}>0$ such that all the convergences in (4.3.19) hold true in $[0,\overline{T}]$ for the sequence of pairs $\{(v^n,\ell^n)\}_{n\in\mathbb{N}}$. So we can consider:

$$T^*:=\sup\{\overline{T}>0\mid (v^n,\ell^n)\to(v,\ell) \text{ in the sense of (4.3.19) in } [0,\overline{T}]\}.$$

If $T^*=+\infty$ we conclude. So let us argue by contradiction assuming that T^* is finite. This means there exists an increasing sequence of times $\{T^j\}_{j\in\mathbb{N}}$ converging to T^* and for which $(v^n,\ell^n)\to(v,\ell)$ in the sense of (4.3.19) in $[0,T^j]$ for every $j\in\mathbb{N}$. Since $\dot\ell^n\to\dot\ell$ in $L^1(0,T^j)$ for every $j\in\mathbb{N}$ and $\dot\ell^n(t)<1$ and $\dot\ell(t)<1$ for a.e. $t>0$ it follows that $\dot\ell^n\to\dot\ell$ in $L^1(0,T^*)$ and hence ℓ^n uniformly converges to ℓ in $[0,T^*]$ by (4.1.1a). Moreover, reasoning as in Section 4.2 we also get that $v^n\to v$ in the sense of (4.3.19) in the whole time interval $[0,T^*]$, and hence T^* is a maximum. Now we can repeat the proofs of Propositions 4.3.2 and 4.3.3 starting from time T^*(notice that by (4.3.19) the convergence hypothesis (4.1.1b) is fulfilled by $u^n(T^*,\cdot)$ and $u^n_t(T^*,\cdot)$, while (4.1.1a) is replaced by $\ell^n(T^*)\to\ell(T^*)$) deducing the existence of a time $\hat T>T^*$ for which (4.3.19) holds true. This is absurd being T^* the supremum, so we conclude. □

Remark 4.3.7. Since $\dot\ell^n(t)<1$ for a.e. $t\in[0,+\infty)$, by (4.3.19) we actually deduce that for every $p\geq1$ it holds $\dot\ell^n\to\dot\ell$ in $L^p(0,T)$ for every $T>0$. However this convergence cannot be improved to the case $p=+\infty$. Indeed let us consider $\ell^n_0=\ell_0=1$, $\nu^n=\nu=2$, $w^n\equiv w\equiv0$ in $[0,+\infty)$, $\kappa^n\equiv\kappa\equiv1/2$ in $[\ell_0,+\infty)$, $u^n_0\equiv u_0\equiv u_1\equiv0$ and $u^n_1(x)=3\mathbb{1}_{[1-1/k,1]}(x)$ in $[0,1]$, so that $u^n_1\to0$ in $L^2(0,1)$ but not in $L^\infty(0,1)$. Under these assumptions we have $(v,\ell)\equiv(0,1)$, so by Theorem 4.3.6 we know that $v^n\to0$ uniformly in $[0,T]\times[0,+\infty)$ for every $T>0$. This means that for every n large enough there exists a small time $T_n>0$ such that for a.e. $t\in[0,T_n]$ we have:

$$\begin{aligned}
\dot\ell^n(t)&=\max\left\{0,\frac{\left[u^n_1(\ell^n(t){-}t))+\int_0^t v^n(\tau,\tau{-}t{+}\ell^n(t))\,\mathrm{d}\tau\right]^2-e^{2t}}{\left[u^n_1(\ell^n(t){-}t))+\int_0^t v^n(\tau,\tau{-}t{+}\ell^n(t))\,\mathrm{d}\tau\right]^2+e^{2t}}\right\}\\
&=\max\left\{0,\frac{\left[3+\int_0^t v^n(\tau,\tau{-}t{+}\ell^n(t))\,\mathrm{d}\tau\right]^2-e^{2t}}{\left[3+\int_0^t v^n(\tau,\tau{-}t{+}\ell^n(t))\,\mathrm{d}\tau\right]^2+e^{2t}}\right\}\\
&\geq\frac{[3-1]^2-e}{[3+1]^2+e}=\frac{4-e}{16+e}>0,
\end{aligned}$$

and so $\dot\ell^n$ does not converge to $\dot\ell\equiv0$ in $L^\infty(0,T)$ for any $T>0$.

Remark 4.3.8 (Presence of a forcing term). If in the debonding model we take into account the presence of an external force f, then the equation the displacement u has to satisfy becomes:

$$u_{tt}(t,x) - u_{xx}(t,x) + \nu u_t(t,x) = f(t,x), \quad t > 0\,,\, 0 < x < \ell(t),$$

while the energy-dissipation balance reads as:

$$\mathcal{E}(t) + \mathcal{A}(t) + \int_{\ell_0}^{\ell(t)} \kappa(\sigma)\,\mathrm{d}\sigma = \mathcal{E}(0) + \mathcal{W}(t) + \mathcal{F}(t), \qquad \text{for every } t \in [0,+\infty),$$

where $\mathcal{F}(t) := \int_0^t \int_0^{\ell(\tau)} f(\tau,\sigma)u_t(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau$. As stated in Remark 3.5.12, if the forcing term satisfies:

$$f \in L^\infty_{\mathrm{loc}}((0,+\infty)^2) \quad \text{such that} \quad f \in L^\infty((0,T)^2) \quad \text{for every } T > 0, \tag{4.3.20}$$

then Theorem 3.5.6 still holds true, namely the coupled problem admits a unique solution (u,ℓ).

If now we consider, besides all the assumptions given in Subsection 4.1.1, a sequence of functions $\{f^n\}_{n\in\mathbb{N}}$ satisfying (4.3.20) and we assume that:

$$f^n \to f \quad \text{in } L^\infty((0,T)^2), \quad \text{for every } T > 0, \tag{4.3.21}$$

then we can repeat all the proofs of the chapter, obtaining even in this case the continuous dependence result (4.3.19) stated in Theorem 4.3.6. Indeed in this case the representation formula for the auxiliary function v^n, fixed $T < \frac{\ell_0}{2}$, reads as:

$$v^n(t,x) = A^n(t,x) + \frac{(\nu^n)^2}{8}H^n(t,x) + \frac{1}{2}\iint_{R^n(t,x)} g^n(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau, \qquad \text{for every } (t,x) \in \overline{\Omega^n_T}, \tag{4.3.22}$$

where $g^n(t,x) := e^{\nu^n t/2} f^n(t,x)$. As a byproduct we obtain that for a.e. $y \in [-\ell_0^n, \varphi^n(T)]$ the function $\Theta^n_{v^n,\lambda^n}$ introduced in (4.3.2) becomes:

$$\Theta^n_{v^n,\lambda^n}(y) = \frac{\left[\dot{v}_0^n(-y) - v_1^n(-y) - \frac{(\nu^n)^2}{4}\int_0^{\lambda^n(y)} v^n(\tau,\tau{-}y)\,\mathrm{d}\tau - \int_0^{\lambda^n(y)} g^n(\tau,\tau{-}y)\,\mathrm{d}\tau\right]^2}{2e^{\nu^n\lambda^n(y)}\kappa^n(\lambda^n(y){-}y)}. \tag{4.3.23}$$

Using (4.3.22), (4.3.23) and exploiting (4.3.21) one can perform again the proofs of Sections 4.2 and 4.3, concluding that Theorem 4.3.6 still holds true even in this case.

Chapter 5

Vanishing inertia and viscosity analysis

Contents

In this chapter we finally deal with the quasistatic limit problem for the debonding model under consideration. We thus analyse the limit as $\varepsilon \to 0^+$ of the pair $(u^\varepsilon, \ell^\varepsilon)$ solution of the rescaled dynamical problem:

$$\begin{cases} \varepsilon^2 u^\varepsilon_{tt}(t,x) - u^\varepsilon_{xx}(t,x) + \nu\varepsilon u^\varepsilon_t(t,x) = 0, & t>0\,,\ 0<x<\ell^\varepsilon(t), \\ u^\varepsilon(t,0) = w^\varepsilon(t), & t>0, \\ u^\varepsilon(t,\ell^\varepsilon(t)) = 0, & t>0, \\ u^\varepsilon(0,x) = u^\varepsilon_0(x), & 0<x<\ell_0, \\ u^\varepsilon_t(0,x) = u^\varepsilon_1(x), & 0<x<\ell_0, \end{cases} \tag{5.0.1}$$

coupled with the rescaled dynamic Griffith's criterion:

$$\begin{cases} 0 \le \dot\ell^\varepsilon(t) < 1/\varepsilon, \\ G^\varepsilon_{\varepsilon\dot\ell^\varepsilon(t)}(t) \le \kappa(\ell^\varepsilon(t)), \\ \left[G^\varepsilon_{\varepsilon\dot\ell^\varepsilon(t)}(t) - \kappa(\ell^\varepsilon(t))\right]\dot\ell^\varepsilon(t) = 0, \end{cases} \quad \text{for a.e. } t\in[0,+\infty). \tag{5.0.2}$$

The chapter is organised as follows. Section 5.2 deals with the dynamical model: we first present the rescaled version of the energies introduced in Section 3.3 and we generalise Duhamel's principle (3.2.7) to the whole Ω^ε, see (5.2.10). Then we state the known results, proved in previous chapters, on the rescaled coupled problem (5.0.1)&(5.0.2).

In Section 5.3 we instead analyse the notion of quasistatic evolution in our framework. We first present the different concepts of energetic and quasistatic solutions to our debonding problem (related to global and local minima of the energy, respectively), showing their equivalence under the strongest assumption (K3) presented in the Introduction. We then provide an existence and uniqueness result by writing down explicitely the solution, see Theorems 5.3.8 and 5.3.9.

The last two sections are devoted to the study of the limit of the pair $(u^\varepsilon, \ell^\varepsilon)$ as ε goes to 0^+. In Section 5.4 we exploit the presence of the viscous term in the wave equation to gain uniform bounds and estimates for the displacement u^ε and the debonding front ℓ^ε. Main estimate (5.4.6) is an adaptation to our time-dependent domain setting of the classical estimate used to show exponential stability of the weakly damped wave equation, see for instance [68]. Of course assumption $\nu > 0$ is crucial for its validity , while for the toughness only the minimal assumption (K0) is needed. It is worth noticing that in this section we do not make use of the explicit formula of the displacement u^ε given by Theorem 5.2.3, but we only need the fact that it solves equation (3.0.1).

Finally in Section 5.5 we prove that if $\nu > 0$, namely when viscosity is taken into account, and requiring (K0) and (K2), the limit of dynamic evolutions $(u^\varepsilon, \ell^\varepsilon)$ exists and it coincides with the quasistatic evolution we previously found in Section 5.3, except for a possible discontinuity at time $t = 0$ appearing if the initial position u_0 is too steep; see Theorem 5.5.21. We first make use of the main estimate (5.4.6) proved in Section 5.4 to show the existence of the above limit; then, by means of the explicit representation formula of u^ε, we are able to pass to the limit in the stability condition of Griffith's criterion and in the energy-dissipation balance (5.2.2a), getting a weak formulation of (s2) and (eb) in Definition 5.3.2; see Propositions 5.5.6 and 5.5.7. Up to this point we only need the weakest assumption (K0), while to characterise the limit debonding front as the quasistatic one we need to require (K2) or (K3) and to exploit the equivalence results of Section 5.3. We conclude the chapter by giving a characterisation of the initial jump which might appear; we obtain this characterisation via an asymptotic analysis of the debonding front solving the unscaled coupled problem (5.5.17)&(5.5.18).

The results contained in this chapter have been published in [75].

5.1 Preliminaries

In this preliminary section we collect some notation which generalise the ones gathered in Section 3.1 and which we will use several times throughout the chapter.

Fix $\ell_0 > 0$, $\varepsilon > 0$ and consider a function $\ell^\varepsilon \colon [0, +\infty) \to [\ell_0, +\infty)$ satisfying (3.1.1a) and

$$0 \le \dot{\ell}^\varepsilon(t) < 1/\varepsilon \text{ for a.e. } t \in (0, +\infty). \tag{5.1.1}$$

Likewise (3.1.2), for $t \in [0, +\infty)$ we then introduce:

$$\varphi^\varepsilon(t) := t - \varepsilon\ell^\varepsilon(t)\,, \qquad \psi^\varepsilon(t) := t + \varepsilon\ell^\varepsilon(t), \tag{5.1.2}$$

and we define:

$$\omega^\varepsilon \colon [\varepsilon\ell_0, +\infty) \to [-\varepsilon\ell_0, +\infty), \quad \omega^\varepsilon(t) := \varphi^\varepsilon \circ (\psi^\varepsilon)^{-1}(t).$$

We recall that ψ^ε is a bilipschitz function since by (5.1.1) it holds $1 \le \dot{\psi}^\varepsilon(t) < 2$ for almost every time, while φ^ε turns out to be Lipschitz with $0 < \dot{\varphi}^\varepsilon(t) \le 1$ almost everywhere. Hence φ^ε is invertible and the inverse is absolutely continuous on every compact interval contained in $\varphi^\varepsilon([0, +\infty))$. As a byproduct we get that ω^ε is Lipschitz too and for a.e. $t \in [\varepsilon\ell_0, +\infty)$ we have:

$$0 < \dot{\omega}^\varepsilon(t) = \frac{1 - \varepsilon\dot{\ell}^\varepsilon((\psi^\varepsilon)^{-1}(t))}{1 + \varepsilon\dot{\ell}^\varepsilon((\psi^\varepsilon)^{-1}(t))} \le 1.$$

So also ω^ε is invertible and the inverse is absolutely continuous on every compact interval contained in $\omega^\varepsilon([0, +\infty))$. Moreover, given $j \in \mathbb{N} \cup \{0\}$, and denoting by $(\omega^\varepsilon)^j$ the composition of ω^ε with itself j times (whether it is well defined) one has:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\omega^\varepsilon)^{j+1}(\psi^\varepsilon(t)) = \frac{1-\varepsilon\dot\ell^\varepsilon(t)}{1+\varepsilon\dot\ell^\varepsilon(t)}\frac{\mathrm{d}}{\mathrm{d}t}(\omega^\varepsilon)^{j}(\varphi^\varepsilon(t)),\ \text{for a.e. } t\in((\varphi^\varepsilon)^{-1}((\omega^\varepsilon)^{-j}(-\varepsilon\ell_0)),+\infty). \tag{5.1.3}$$

It will be useful to define the sets:

$$\Omega^\varepsilon := \{(t,x)\mid t>0\,,\, 0<x<\ell^\varepsilon(t)\},$$
$$\Omega^\varepsilon_T := \{(t,x)\in\Omega^\varepsilon \mid t<T\}.$$

For $(t,x)\in\Omega^\varepsilon$ we also introduce:

$$\begin{aligned} R^\varepsilon_+(t,x) &= \bigcup_{j=0}^{m} R^\varepsilon_{2j}(t,x),\\ R^\varepsilon_-(t,x) &= \bigcup_{j=0}^{n} R^\varepsilon_{2j+1}(t,x), \end{aligned} \tag{5.1.4}$$

In order to avoid the cumbersome definitions of $m=m(\varepsilon,t,x)$, $n=n(\varepsilon,t,x)$ and $R^\varepsilon_i(t,x)$ we refer to the very intuitive Figure 5.1.
Finally, for $k\in\mathbb{N}$, let us define the spaces:

$$\widetilde{L}^2(\Omega^\varepsilon) := \{u\in L^2_{\mathrm{loc}}(\Omega^\varepsilon)\mid u\in L^2(\Omega^\varepsilon_T) \text{ for every } T>0\},$$
$$\widetilde{H}^k(\Omega^\varepsilon) := \{u\in H^k_{\mathrm{loc}}(\Omega^\varepsilon)\mid u\in H^k(\Omega^\varepsilon_T) \text{ for every } T>0\},$$
$$\widetilde{H}^k(0,+\infty) := \{u\in H^k_{\mathrm{loc}}(0,+\infty)\mid u\in H^k(0,T) \text{ for every } T>0\},$$
$$\widetilde{C}^{0,1}([\ell_0,+\infty)) := \{u\in C^0([\ell_0,+\infty))\mid u\in C^{0,1}([\ell_0,X]) \text{ for every } X>\ell_0\}.$$

We say that a family $\mathcal{F}$ is bounded in $\widetilde{H}^k(0,+\infty)$ if for every $T>0$ there exists a positive constant C_T such that $\|u\|_{H^k(0,T)}\le C_T$ for every $u\in\mathcal{F}$. We say that a sequence $\{u_n\}_{n\in\mathbb{N}}$ converges strongly (weakly) to u in $\widetilde{H}^k(0,+\infty)$ if for every $T>0$ one has $u_n\to u$ ($u_n\rightharpoonup u$) in $H^k(0,T)$ as $n\to+\infty$.

5.2 Time-rescaled dynamic evolutions

In this section we gather all the known results we proved in previous chapters about the well posedness of the rescaled coupled problem, and we introduce some useful generalisation of Duhamel's formula (3.2.10), see (5.2.10).

We fix $\nu\ge0$, $\ell_0>0$ and we require w^ε, u^ε_0, u^ε_1 satisfy (3.2.1) and (3.2.2) . The rescaled energies, defined for $t\in[0,+\infty)$, involved in the model are simply:

$$\mathcal{K}^\varepsilon(t) = \frac12\int_0^{\ell^\varepsilon(t)}\varepsilon^2 u^\varepsilon_t(t,\sigma)^2\,\mathrm{d}\sigma; \tag{5.2.1a}$$

$$\mathcal{E}^\varepsilon(t) = \frac12\int_0^{\ell^\varepsilon(t)} u^\varepsilon_x(t,\sigma)^2\,\mathrm{d}\sigma; \tag{5.2.1b}$$

$$\mathcal{V}^\varepsilon(t) = \nu\int_0^t\int_0^{\ell^\varepsilon(\tau)}\varepsilon u^\varepsilon_t(\tau,\sigma)^2\,\mathrm{d}\sigma\,\mathrm{d}\tau; \tag{5.2.1c}$$

$$\mathcal{W}^\varepsilon(t) = -\int_0^t \dot w^\varepsilon(\tau)u^\varepsilon_x(\tau,0)\,\mathrm{d}\tau. \tag{5.2.1d}$$

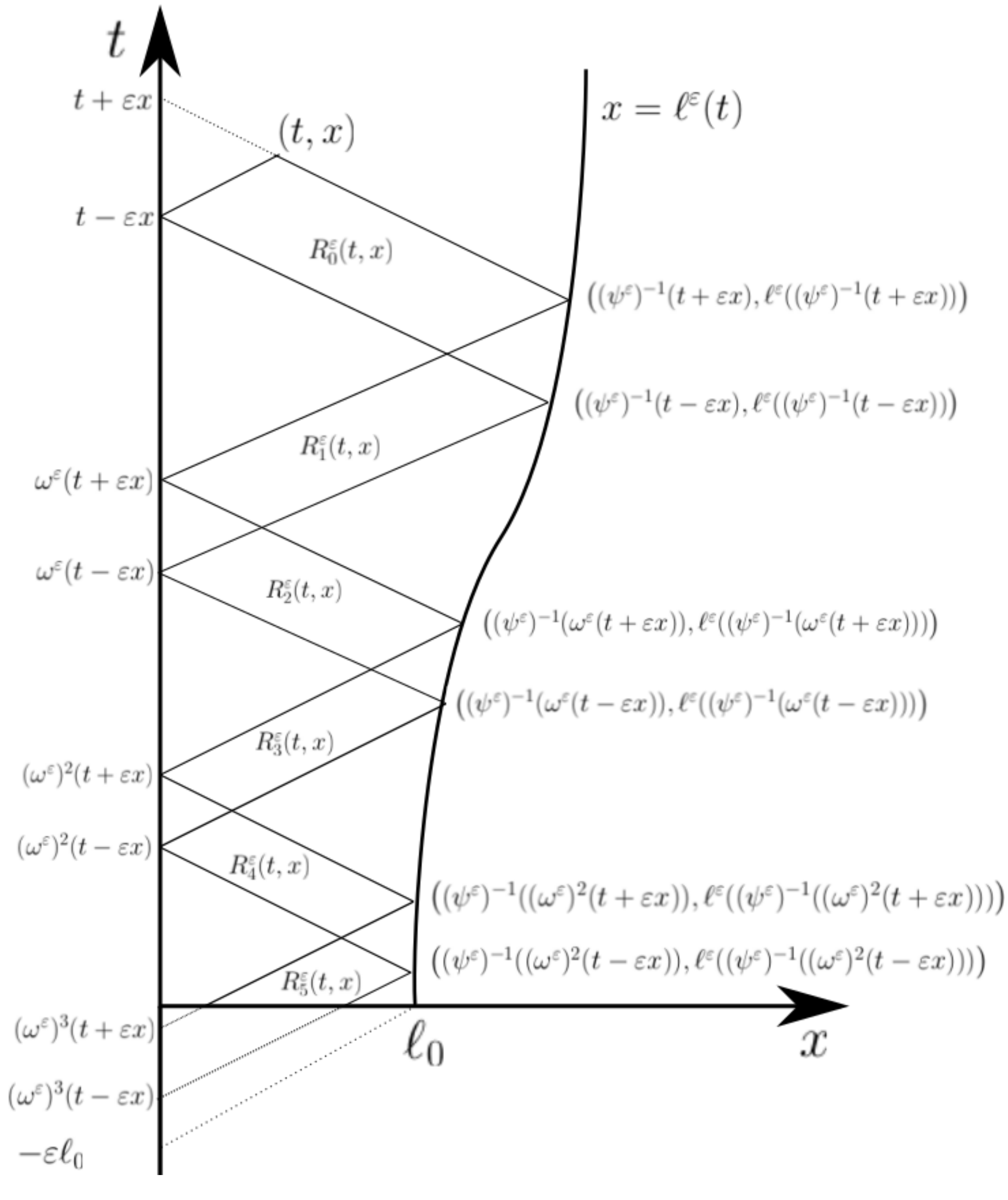

Figure 5.1: The sets $R_i^\varepsilon(t,x)$ in the particular situation $\varepsilon = 1/2$, and with a choice of (t,x) for which $m = 2$, $n = 2$.

They represent the kinetic energy, the (external) potential energy, the energy dissipated by viscosity and the work of the external loading, respectively. The energy-dissipation balance and the maximum dissipation principle in this rescaled context take the form:

$$\mathcal{K}^\varepsilon(t)+\mathcal{E}^\varepsilon(t)+\mathcal{V}^\varepsilon(t)+\int_{\ell_0}^{\ell^\varepsilon(t)}\kappa(\sigma)\,\mathrm{d}\sigma=\mathcal{K}^\varepsilon(0)+\mathcal{E}^\varepsilon(0)+\mathcal{W}^\varepsilon(t),\qquad\text{for every } t\in[0,+\infty), \tag{5.2.2a}$$

where $\kappa\colon[\ell_0,+\infty)\to(0,+\infty)$ is the measurable function representing the toughness of the glue, and:

$$\dot\ell^\varepsilon(t)=\max\left\{\alpha\in[0,1/\varepsilon)\mid \kappa(\ell^\varepsilon(t))\alpha=G^\varepsilon_{\varepsilon\alpha}(t)\alpha\right\},\qquad\text{for a.e. } t\in[0,+\infty), \tag{5.2.2b}$$

where $G^\varepsilon_{\varepsilon\alpha}$ is the (rescaled) dynamic energy release rate at speed $\varepsilon\alpha\in[0,1)$. As in (3.4.6) for every $\alpha\in[0,1/\varepsilon)$ we can express it as:

$$G^\varepsilon_{\varepsilon\alpha}(t)=\frac{1-\varepsilon\alpha}{1+\varepsilon\alpha}G^\varepsilon_0(t),\qquad\text{for a.e. } t\in[0,+\infty).$$

As proved in Chapter 3 the two principles (5.2.2) are equivalent to the (rescaled) dynamic Griffith's criterion:

$$\begin{cases}0\le\dot\ell^\varepsilon(t)<1/\varepsilon,\\ G^\varepsilon_{\varepsilon\dot\ell^\varepsilon(t)}(t)\le\kappa(\ell^\varepsilon(t)),\\ \left[G^\varepsilon_{\varepsilon\dot\ell^\varepsilon(t)}(t)-\kappa(\ell^\varepsilon(t))\right]\dot\ell^\varepsilon(t)=0,\end{cases}\qquad\text{for a.e. } t\in[0,+\infty). \tag{5.2.3}$$

The first row is an irreversibility condition, which ensures that the debonding front can only increase, and moreover states that its velocity is always strictly less than $1/\varepsilon$, namely the speed of internal waves; the second one is a stability condition, and says that the dynamic energy release rate cannot exceed the threshold given by the toughness; the third one is simply the energy-dissipation balance (5.2.2a).

In order to generalise Duhamel's formula (3.2.10) we introduce the following function: given $F\in\widetilde L^2(\Omega^\varepsilon)$ we define

$$H^\varepsilon[F](t,x):=\frac12\left[\iint_{R^\varepsilon_+(t,x)}F(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau-\iint_{R^\varepsilon_-(t,x)}F(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau\right],\ \text{for } (t,x)\in\Omega^\varepsilon, \tag{5.2.4}$$

where $R^\varepsilon_\pm(t,x)$ are as in (5.1.4). Here are listed the main properties of H^ε, under the assumption that ℓ^ε satisfies (3.1.1a) and (5.1.1):

Proposition 5.2.1. *Let $F\in\widetilde L^2(\Omega^\varepsilon)$, then the function $H^\varepsilon[F]$ introduced in (5.2.4) is continuous on $\overline{\Omega^\varepsilon}$ and belongs to $\widetilde H^1(\Omega^\varepsilon)$. Moreover, setting $H^\varepsilon[F]\equiv0$ outside $\overline{\Omega^\varepsilon}$, it belongs to $C^0([0,+\infty);H^1(0,+\infty))$ and to $C^1([0,+\infty);L^2(0,+\infty))$.*
Furthermore for a.e. $t\in[0,+\infty)$ one has:

$$\begin{aligned}H^\varepsilon[F]_x(t,0)=&\sum_{j=0}^{m^\varepsilon-1}\frac{\mathrm{d}}{\mathrm{d}t}(\omega^\varepsilon)^j(t)\int_{(\psi^\varepsilon)^{-1}((\omega^\varepsilon)^j(t))}^{(\omega^\varepsilon)^j(t)}F\left(\tau,\frac{(\omega^\varepsilon)^j(t)-\tau}{\varepsilon}\right)\mathrm{d}\tau\\&-\sum_{j=0}^{m^\varepsilon-1}\frac{\mathrm{d}}{\mathrm{d}t}(\omega^\varepsilon)^{j+1}(t)\int_{(\omega^\varepsilon)^{j+1}(t)}^{(\psi^\varepsilon)^{-1}((\omega^\varepsilon)^j(t))}F\left(\tau,\frac{\tau-(\omega^\varepsilon)^{j+1}(t)}{\varepsilon}\right)\mathrm{d}\tau\\&+I^\varepsilon_1(t),\end{aligned} \tag{5.2.5}$$

where $m^\varepsilon=m^\varepsilon(t)$ is the only natural number (including 0) such that $(\omega^\varepsilon)^{m^\varepsilon}(t)$ belongs to $[0,(\omega^\varepsilon)^{-1}(0))$, while I^ε_1 is defined as follows:

$$I^\varepsilon_1(t)=\frac{\mathrm{d}}{\mathrm{d}t}(\omega^\varepsilon)^{m^\varepsilon}(t)\int_0^{(\omega^\varepsilon)^{m^\varepsilon}(t)}F\left(\tau,\frac{(\omega^\varepsilon)^{m^\varepsilon}(t)-\tau}{\varepsilon}\right)\mathrm{d}\tau,\qquad\textit{if } (\omega^\varepsilon)^{m^\varepsilon}(t)\in[0,\varepsilon\ell_0),$$

while if $(\omega^\varepsilon)^{m^\varepsilon}(t)\in[\varepsilon\ell_0,(\omega^\varepsilon)^{-1}(0))$ *it is defined in this other way:*

$$\begin{aligned}I_1^\varepsilon(t)=&\frac{\mathrm{d}}{\mathrm{d}t}(\omega^\varepsilon)^{m^\varepsilon}(t)\int_{(\psi^\varepsilon)^{-1}((\omega^\varepsilon)^{m^\varepsilon}(t))}^{(\omega^\varepsilon)^{m^\varepsilon}(t)}F\left(\tau,\frac{(\omega^\varepsilon)^{m^\varepsilon}(t)-\tau}{\varepsilon}\right)\mathrm{d}\tau\\&-\frac{\mathrm{d}}{\mathrm{d}t}(\omega^\varepsilon)^{m^\varepsilon+1}(t)\int_0^{(\psi^\varepsilon)^{-1}((\omega^\varepsilon)^{m^\varepsilon}(t))}F\left(\tau,\frac{\tau-(\omega^\varepsilon)^{m^\varepsilon+1}(t)}{\varepsilon}\right)\mathrm{d}\tau.\end{aligned}$$

Finally for a.e. $t\in[0,+\infty)$ *it holds:*

$$H^\varepsilon[F]_x(t,\ell^\varepsilon(t))=\frac{2}{1+\varepsilon\dot\ell^\varepsilon(t)}g^\varepsilon[F](t-\varepsilon\ell^\varepsilon(t)),\tag{5.2.6}$$

where for a.e. $s\in\varphi^\varepsilon([0,+\infty))$ *we define:*

$$\begin{aligned}g^\varepsilon[F](s)=\;&\frac12\sum_{j=0}^{n^\varepsilon-1}\frac{\mathrm{d}}{\mathrm{d}s}(\omega^\varepsilon)^j(s)\int_{(\psi^\varepsilon)^{-1}((\omega^\varepsilon)^j(s))}^{(\omega^\varepsilon)^j(s)}F\left(\tau,\frac{(\omega^\varepsilon)^j(s)-\tau}{\varepsilon}\right)\mathrm{d}\tau\\&-\frac12\sum_{j=0}^{n^\varepsilon-1}\frac{\mathrm{d}}{\mathrm{d}s}(\omega^\varepsilon)^j(s)\int_{(\omega^\varepsilon)^j(s)}^{(\psi^\varepsilon)^{-1}((\omega^\varepsilon)^{j-1}(s))}F\left(\tau,\frac{\tau-(\omega^\varepsilon)^j(s)}{\varepsilon}\right)\mathrm{d}\tau\\&+\frac12\frac{\mathrm{d}}{\mathrm{d}s}(\omega^\varepsilon)^{n^\varepsilon}(s)I_2^\varepsilon(s),\end{aligned}\tag{5.2.7}$$

where $n^\varepsilon=n^\varepsilon(s)$ *is the only natural number (including* 0*) such that* $(\omega^\varepsilon)^{n^\varepsilon}(s)\in[-\varepsilon\ell_0,\varepsilon\ell_0)$*, while*

$$I_2^\varepsilon(s)=\begin{cases}-\displaystyle\int_0^{(\psi^\varepsilon)^{-1}((\omega^\varepsilon)^{n^\varepsilon-1}(s))}F\left(\tau,\frac{\tau-(\omega^\varepsilon)^{n^\varepsilon}(s)}{\varepsilon}\right)\mathrm{d}\tau, & \textit{if } (\omega^\varepsilon)^{n^\varepsilon}(s)\in[-\varepsilon\ell_0,0),\\ \displaystyle\int_0^{(\omega^\varepsilon)^{n^\varepsilon}(s)}F\left(\tau,\frac{(\omega^\varepsilon)^{n^\varepsilon}(s)-\tau}{\varepsilon}\right)\mathrm{d}\tau-\int_{(\omega^\varepsilon)^{n^\varepsilon}(s)}^{(\psi^\varepsilon)^{-1}((\omega^\varepsilon)^{n^\varepsilon-1}(s))}F\left(\tau,\frac{\tau-(\omega^\varepsilon)^{n^\varepsilon}(s)}{\varepsilon}\right)\mathrm{d}\tau, & \textit{otherwise.}\end{cases}$$

Proof. The regularity of $H^\varepsilon[F]$ can be proved in the same way of Lemma 3.2.9. The validity of (5.2.5) is a straightforward matter of computations, see Figure 5.1 for an intuition and also Remark 3.2.10. To get (5.2.6), always referring to Figure 5.1, we compute:

$$\begin{aligned}&H^\varepsilon[F]_x(t,\ell^\varepsilon(t))\\&=\frac12\sum_{j=0}^{n^\varepsilon-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}(\omega^\varepsilon)^j(t-\varepsilon\ell^\varepsilon(t))+\frac{\mathrm{d}}{\mathrm{d}t}(\omega^\varepsilon)^{j+1}(t+\varepsilon\ell^\varepsilon(t))\right)\int_{(\psi^\varepsilon)^{-1}((\omega^\varepsilon)^j(t-\varepsilon\ell^\varepsilon(t)))}^{(\omega^\varepsilon)^j(t-\varepsilon\ell^\varepsilon(t))}F\left(\tau,\frac{(\omega^\varepsilon)^j(t-\varepsilon\ell^\varepsilon(t))-\tau}{\varepsilon}\right)\mathrm{d}\tau\\&-\frac12\sum_{j=0}^{n^\varepsilon-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}(\omega^\varepsilon)^j(t-\varepsilon\ell^\varepsilon(t))+\frac{\mathrm{d}}{\mathrm{d}t}(\omega^\varepsilon)^{j+1}(t+\varepsilon\ell^\varepsilon(t))\right)\int_{(\omega^\varepsilon)^j(t-\varepsilon\ell^\varepsilon(t))}^{(\psi^\varepsilon)^{-1}((\omega^\varepsilon)^{j-1}(t-\varepsilon\ell^\varepsilon(t)))}F\left(\tau,\frac{\tau-(\omega^\varepsilon)^j(t-\varepsilon\ell^\varepsilon(t))}{\varepsilon}\right)\mathrm{d}\tau\\&+\frac12\left(\frac{\mathrm{d}}{\mathrm{d}t}(\omega^\varepsilon)^{n^\varepsilon}(t-\varepsilon\ell^\varepsilon(t))+\frac{\mathrm{d}}{\mathrm{d}t}(\omega^\varepsilon)^{n^\varepsilon+1}(t+\varepsilon\ell^\varepsilon(t))\right)I_2^\varepsilon(t-\varepsilon\ell^\varepsilon(t)),\end{aligned}$$

and we conclude by using (5.1.3). □

Lemma 5.2.2. *Let* $F\in\widetilde{L}^2(\Omega^\varepsilon)$ *and consider* $H^\varepsilon[F]$ *and* $g^\varepsilon[F]$ *given by* (5.2.4) *and* (5.2.7)*, respectively. Then for a.e.* $s\in\varphi^\varepsilon([0,+\infty))\cap(0,+\infty)$ *it holds:*

$$g^\varepsilon[F](s)-\frac12H^\varepsilon[F]_x(s,0)=-\frac12\int_s^{(\varphi^\varepsilon)^{-1}(s)}F\left(\tau,\frac{\tau-s}{\varepsilon}\right)\mathrm{d}\tau.\tag{5.2.8}$$

Proof. We start computing by means of (5.2.5) and (5.2.7):

$$\begin{aligned}
&2g^\varepsilon[F](s) - H^\varepsilon[F]_x(s,0)\\
=&\sum_{j=0}^{n^\varepsilon-1} \frac{\mathrm{d}}{\mathrm{d}s}(\omega^\varepsilon)^j(s) \int_{(\psi^\varepsilon)^{-1}((\omega^\varepsilon)^j(s))}^{(\omega^\varepsilon)^j(s)} F\left(\tau, \frac{(\omega^\varepsilon)^j(s)-\tau}{\varepsilon}\right) \mathrm{d}\tau\\
&-\sum_{j=0}^{n^\varepsilon-1} \frac{\mathrm{d}}{\mathrm{d}s}(\omega^\varepsilon)^j(s) \int_{(\omega^\varepsilon)^j(s)}^{(\psi^\varepsilon)^{-1}((\omega^\varepsilon)^{j-1}(s))} F\left(\tau, \frac{\tau-(\omega^\varepsilon)^j(s)}{\varepsilon}\right) \mathrm{d}\tau\\
&-\sum_{j=0}^{m^\varepsilon-1} \frac{\mathrm{d}}{\mathrm{d}s}(\omega^\varepsilon)^j(s) \int_{(\psi^\varepsilon)^{-1}((\omega^\varepsilon)^j(s))}^{(\omega^\varepsilon)^j(s)} F\left(\tau, \frac{(\omega^\varepsilon)^j(s)-\tau}{\varepsilon}\right) \mathrm{d}\tau\\
&+\sum_{j=0}^{m^\varepsilon-1} \frac{\mathrm{d}}{\mathrm{d}s}(\omega^\varepsilon)^{j+1}(s) \int_{(\omega^\varepsilon)^{j+1}(s)}^{(\psi^\varepsilon)^{-1}((\omega^\varepsilon)^j(s))} F\left(\tau, \frac{\tau-(\omega^\varepsilon)^{j+1}(s)}{\varepsilon}\right) \mathrm{d}\tau\\
&+\frac{\mathrm{d}}{\mathrm{d}s}(\omega^\varepsilon)^{n^\varepsilon}(s) I_2^\varepsilon(s) - I_1^\varepsilon(s) = (\star).
\end{aligned}$$

There are only two cases to consider: $n^\varepsilon(s) = m^\varepsilon(s)$ or $n^\varepsilon(s) = m^\varepsilon(s)+1$. We prove the lemma for the first case, being the other one analogous. So we have:

$$\begin{aligned}
(\star) =& \sum_{j=0}^{n^\varepsilon-1} \frac{\mathrm{d}}{\mathrm{d}s}(\omega^\varepsilon)^{j+1}(s) \int_{(\omega^\varepsilon)^{j+1}(s)}^{(\psi^\varepsilon)^{-1}((\omega^\varepsilon)^j(s))} F\left(\tau, \frac{\tau-(\omega^\varepsilon)^{j+1}(s)}{\varepsilon}\right) \mathrm{d}\tau\\
&-\sum_{j=0}^{n^\varepsilon-1} \frac{\mathrm{d}}{\mathrm{d}s}(\omega^\varepsilon)^j(s) \int_{(\omega^\varepsilon)^j(s)}^{(\psi^\varepsilon)^{-1}((\omega^\varepsilon)^{j-1}(s))} F\left(\tau, \frac{\tau-(\omega^\varepsilon)^j(s)}{\varepsilon}\right) \mathrm{d}\tau\\
&+\frac{\mathrm{d}}{\mathrm{d}s}(\omega^\varepsilon)^{n^\varepsilon}(s) I_2^\varepsilon(s) - I_1^\varepsilon(s) = (\star\star).
\end{aligned} \tag{5.2.9}$$

Exploiting the fact that in (5.2.9) there is now a telescopic sum and by using the explicit formulas of I_1^ε and I_2^ε given by Proposition 5.2.1 we hence deduce:

$$\begin{aligned}
(\star\star) =& \frac{\mathrm{d}}{\mathrm{d}s}(\omega^\varepsilon)^{n^\varepsilon}(s) \int_{(\omega^\varepsilon)^{n^\varepsilon}(s)}^{(\psi^\varepsilon)^{-1}((\omega^\varepsilon)^{n^\varepsilon-1}(s))} F\left(\tau, \frac{\tau-(\omega^\varepsilon)^{n^\varepsilon}(s)}{\varepsilon}\right) \mathrm{d}\tau - \int_s^{(\varphi^\varepsilon)^{-1}(s)} F\left(\tau, \frac{\tau-s}{\varepsilon}\right) \mathrm{d}\tau\\
&+\frac{\mathrm{d}}{\mathrm{d}s}(\omega^\varepsilon)^{n^\varepsilon}(s) \int_0^{(\omega^\varepsilon)^{n^\varepsilon}(s)} F\left(\tau, \frac{(\omega^\varepsilon)^{n^\varepsilon}(s)-\tau}{\varepsilon}\right) \mathrm{d}\tau\\
&-\frac{\mathrm{d}}{\mathrm{d}s}(\omega^\varepsilon)^{n^\varepsilon}(s) \int_{(\omega^\varepsilon)^{n^\varepsilon}(s)}^{(\psi^\varepsilon)^{-1}((\omega^\varepsilon)^{n^\varepsilon-1}(s))} F\left(\tau, \frac{\tau-(\omega^\varepsilon)^{n^\varepsilon}(s)}{\varepsilon}\right) \mathrm{d}\tau\\
&-\frac{\mathrm{d}}{\mathrm{d}s}(\omega^\varepsilon)^{n^\varepsilon}(s) \int_0^{(\omega^\varepsilon)^{n^\varepsilon}(s)} F\left(\tau, \frac{(\omega^\varepsilon)^{n^\varepsilon}(s)-\tau}{\varepsilon}\right) \mathrm{d}\tau\\
=& -\int_s^{(\varphi^\varepsilon)^{-1}(s)} F\left(\tau, \frac{\tau-s}{\varepsilon}\right) \mathrm{d}\tau,
\end{aligned}$$

and we conclude. □

We now recall and restate in a more useful way the main results about dynamic evolutions of the debonding model.

Theorem 5.2.3 (Existence and Uniqueness). *Fix $\nu \geq 0$, $\ell_0 > 0$, $\varepsilon > 0$, assume the functions w^ε, u_0^ε and u_1^ε satisfy (3.2.1), (3.2.2) and let the toughness κ be positive and satisfy (3.5.8). Then there exists a unique pair $(u^\varepsilon, \ell^\varepsilon)$, with:*

- $\ell^\varepsilon \in C^{0,1}([0,+\infty))$, $\ell^\varepsilon(0)=\ell_0$ *and* $0 \le \dot\ell^\varepsilon(t) < 1/\varepsilon$ *for a.e.* $t\in[0,+\infty)$,
- $u^\varepsilon \in \widetilde H^1(\Omega^\varepsilon)$ *and* $u^\varepsilon(t,x)=0$ *for every* (t,x) *such that* $x>\ell^\varepsilon(t)$,

solution of the coupled problem (5.0.1)*&*(5.0.2)*.*

Moreover u^ε *has a continuous representative which fulfils the following representation formula:*

$$u^\varepsilon(t,x)=\begin{cases} w^\varepsilon(t+\varepsilon x)-\dfrac{1}{\varepsilon}f^\varepsilon(t+\varepsilon x)+\dfrac{1}{\varepsilon}f^\varepsilon(t-\varepsilon x)-\nu H^\varepsilon[u^\varepsilon_t](t,x), & \text{if } (t,x)\in\overline{\Omega^\varepsilon},\\ 0, & \text{otherwise,}\end{cases} \tag{5.2.10}$$

where $f^\varepsilon\in\widetilde H^1(-\varepsilon\ell_0,+\infty)$ *is defined by two rules:*

$$(i)\;\; f^\varepsilon(s)=\begin{cases}\varepsilon w^\varepsilon(s)-\dfrac{\varepsilon}{2}u_0^\varepsilon\left(\dfrac{s}{\varepsilon}\right)-\dfrac{\varepsilon^2}{2}\displaystyle\int_0^{s/\varepsilon}u_1^\varepsilon(\sigma)\,\mathrm{d}\sigma-\varepsilon w^\varepsilon(0)+\dfrac{\varepsilon}{2}u_0^\varepsilon(0), & \text{if } s\in(0,\varepsilon\ell_0],\\ \dfrac{\varepsilon}{2}u_0^\varepsilon\left(-\dfrac{s}{\varepsilon}\right)-\dfrac{\varepsilon^2}{2}\displaystyle\int_0^{-s/\varepsilon}u_1^\varepsilon(\sigma)\,\mathrm{d}\sigma-\dfrac{\varepsilon}{2}u_0^\varepsilon(0), & \text{if } s\in(-\varepsilon\ell_0,0],\end{cases}$$

$$(ii)\;\; w^\varepsilon(s+\varepsilon\ell^\varepsilon(s))-\frac{1}{\varepsilon}f^\varepsilon(s+\varepsilon\ell^\varepsilon(s))+\frac{1}{\varepsilon}f^\varepsilon(s-\varepsilon\ell^\varepsilon(s))=0, \qquad \text{for every } s\in(0,+\infty),$$

while H^ε *is as in* (5.2.4)*.*
In particular it holds:

$$u^\varepsilon\in C^0([0,+\infty);H^1(0,+\infty))\cap C^1([0,+\infty);L^2(0,+\infty)).$$

Furthermore for a.e. $t\in[0,+\infty)$ *one has:*

$$u_x^\varepsilon(t,0)=\varepsilon\dot w^\varepsilon(t)-2\dot f^\varepsilon(t)-\nu H^\varepsilon[u_t^\varepsilon]_x(t,0), \tag{5.2.11a}$$

$$u_x^\varepsilon(t,\ell^\varepsilon(t))=-\frac{2}{1+\varepsilon\dot\ell^\varepsilon(t)}\Big[\dot f^\varepsilon(t-\varepsilon\ell^\varepsilon(t))+\nu g^\varepsilon[u_t^\varepsilon](t-\varepsilon\ell^\varepsilon(t))\Big], \tag{5.2.11b}$$

and for $\alpha\in[0,1/\varepsilon)$ *the dynamic energy release rate can be expressed as:*

$$G^\varepsilon_{\varepsilon\alpha}(t)=2\frac{1-\varepsilon\alpha}{1+\varepsilon\alpha}\Big[\dot f^\varepsilon(t-\varepsilon\ell^\varepsilon(t))+\nu g^\varepsilon[u_t^\varepsilon](t-\varepsilon\ell^\varepsilon(t))\Big]^2, \quad \text{for a.e. } t\in[0,+\infty), \tag{5.2.12}$$

where g^ε *has been introduced in* (5.2.7)*.*

Remark 5.2.4 (**Regularity**)**.** If the data are more regular, namely:

$$w^\varepsilon\in\widetilde H^2(0,+\infty),\quad u_0^\varepsilon\in H^2(0,\ell_0),\quad u_1^\varepsilon\in H^1(0,\ell_0),$$

if the (positive) toughness κ belongs to $\widetilde C^{0,1}([\ell_0,+\infty))$ and if besides (3.2.2) also the following first order compatibility conditions are satisfied:

$$u_1^\varepsilon(0)=\dot w^\varepsilon(0),$$

$$\Big(u_1^\varepsilon(\ell_0)=0,\ \dot u_0^\varepsilon(\ell_0)^2\le2\kappa(\ell_0)\Big)\text{ or }\Big(u_1^\varepsilon(\ell_0)\ne0,\ \dot u_0^\varepsilon(\ell_0)^2-\varepsilon^2u_1^\varepsilon(\ell_0)^2=2\kappa(\ell_0),\ \frac{\dot u_0^\varepsilon(\ell_0)}{u_1^\varepsilon(\ell_0)}<-\varepsilon\Big),$$

then the solution u^ε is in $\widetilde H^2(\Omega^\varepsilon)$.

Theorem 5.2.5 (**Continuous Dependence**)**.** *Fix* $\nu\ge0$, $\ell_0>0$, $\varepsilon>0$, *assume the functions* w^ε, u_0^ε *and* u_1^ε *satisfy* (3.2.1), (3.2.2) *and let the toughness* κ *be positive and belong to* $\widetilde C^{0,1}([\ell_0,+\infty))$*. Consider sequences of functions* $\{w_n^\varepsilon\}_{n\in\mathbb N}$, $\{u_{0n}^\varepsilon\}_{n\in\mathbb N}$ *and* $\{u_{1n}^\varepsilon\}_{n\in\mathbb N}$ *satisfying* (3.2.1) *and* (3.2.2)*, and let* $(u_n^\varepsilon,\ell_n^\varepsilon)$ *and* $(u^\varepsilon,\ell^\varepsilon)$ *be the solutions of coupled problem* (5.0.1)*&*(5.0.2) *corresponding to the data with and without the subscript* n*, respectively. If the following convergences hold true as* $n\to+\infty$*:*

$$u_{0n}^\varepsilon\to u_0^\varepsilon \text{ in } H^1(0,\ell_0),\ \ u_{1n}^\varepsilon\to u_1^\varepsilon \text{ in } L^2(0,\ell_0) \text{ and } w_n^\varepsilon\to w^\varepsilon \text{ in } \widetilde H^1(0,+\infty),$$

then for every $T>0$ *one has as* $n\to+\infty$*:*

- $\ell_n^\varepsilon \to \ell^\varepsilon$ *in* $W^{1,1}(0,T)$*;*
- $u_n^\varepsilon \to u^\varepsilon$ *uniformly in* $[0,T]\times[0,+\infty)$*;*
- $u_n^\varepsilon \to u^\varepsilon$ *in* $H^1((0,T)\times(0,+\infty))$*;*
- $u_n^\varepsilon \to u^\varepsilon$ *in* $C^0([0,T];H^1(0,+\infty))$ *and in* $C^1([0,T];L^2(0,+\infty))$*;*
- $(u_n^\varepsilon)_x(\cdot,0) \to u_x^\varepsilon(\cdot,0)$ *in* $L^2(0,T)$.

5.3 Quasistatic evolutions

This section is devoted to the analysis of quasistatic evolutions for the debonding model we are studying. We also refer to [16] and [55] for other adhesion and debonding problems in the static and quasistatic regime. We first introduce and compare three different notions of this kind of evolutions (we refer to [14] or [65] for a wide and complete presentation on the topic), then we prove an existence and uniqueness result under suitable assumptions, see Theorems 5.3.8 and 5.3.9.

Fix $\ell_0>0$; throughout this section we consider a loading term $w\in\widetilde{AC}([0,+\infty))$ and a toughness $\kappa\in C^0([\ell_0,+\infty))$ such that $\kappa(x)>0$ for every $x\geq\ell_0$.

Definition 5.3.1. *Let* $\ell\colon[0,+\infty)\to[\ell_0,+\infty)$ *be a nondecreasing function such that* $\ell(0)=\ell_0$ *and let* $u\colon[0,+\infty)\times[0,+\infty)\to\mathbb{R}$ *be a function which for every* $t\in[0,+\infty)$ *satisfies* $u(t,\cdot)\in H^1(0,+\infty)$, $u(t,0)=w(t)$, $u(t,x)=0$ *for* $x\geq\ell(t)$ *and such that* $u_x(t,0)$ *exists for a.e.* $t\in[0,+\infty)$*. We say that such a pair* (u,ℓ) *is an* ***energetic evolution*** *if for every* $t\in[0,+\infty)$ *it holds:*

(GS) $$\frac12\int_0^{\ell(t)}u_x(t,\sigma)^2\,\mathrm{d}\sigma+\int_{\ell_0}^{\ell(t)}\kappa(\sigma)\,\mathrm{d}\sigma\leq\frac12\int_0^{\hat\ell}\dot{\hat u}(\sigma)^2\,\mathrm{d}\sigma+\int_{\ell_0}^{\hat\ell}\kappa(\sigma)\,\mathrm{d}\sigma,$$
for every $\hat\ell\geq\ell(t)$ *and for every* $\hat u\in H^1(0,\hat\ell)$ *satisfying* $\hat u(0)=w(t)$ *and* $\hat v(\hat\ell)=0$*;*

(EB) $$\frac12\int_0^{\ell(t)}u_x(t,\sigma)^2\,\mathrm{d}\sigma+\int_{\ell_0}^{\ell(t)}\kappa(\sigma)\,\mathrm{d}\sigma=\frac12\int_0^{\ell_0}u_x(0,\sigma)^2\,\mathrm{d}\sigma-\int_0^t\dot w(\tau)u_x(\tau,0)\,\mathrm{d}\tau.$$

Here (GS) stands for global stability, while (EB) for energy(-dissipation) balance. Roughly speaking an energetic evolution is a pair which fulfils an energy-dissipation balance being at every time a global minimiser of the functional $(u,\ell)\mapsto\frac12\int_0^\ell\dot u(\sigma)^2\,\mathrm{d}\sigma+\int_{\ell_0}^\ell\kappa(\sigma)\,\mathrm{d}\sigma$, which is sum of potential energy and energy dissipated in the debonding process.

On the contrary, this two other definitions deal with local minima of the total energy:

Definition 5.3.2. *Given* ℓ *and* u *as in Definition 5.3.1, we say that the pair* (u,ℓ) *is a* ***quasistatic evolution*** *if:*

(o) ℓ *is non decreasing on* $[0,+\infty)$ *and* $\ell(0)=\ell_0$*;*

(s1) $u(t,x)=w(t)\left(1-\dfrac{x}{\ell(t)}\right)\mathbb{1}_{[0,\ell(t)]}(x)$, *for every* $(t,x)\in[0,+\infty)\times[0,+\infty)$*;*

(s2) $\dfrac12\dfrac{w(t)^2}{\ell(t)^2}\leq\kappa(\ell(t))$, *for every* $t\in[0,+\infty)$,

(eb) $$\frac12\frac{w(t)^2}{\ell(t)}+\int_{\ell_0}^{\ell(t)}\kappa(\sigma)\,\mathrm{d}\sigma=\frac12\frac{w(0)^2}{\ell_0}+\int_0^t\dot w(\tau)\frac{w(\tau)}{\ell(\tau)}\,\mathrm{d}\tau,\ \textit{for every } t\in[0,+\infty).$$

Definition 5.3.3. *Given* ℓ *and* u *as in Definition 5.3.1, we say that the pair* (v,ℓ) *is an* ***absolutely continuous quasistatic evolution****, in brief* ***AC-quasistatic evolution****, if:*

(i) ℓ *is absolutely continuous on* $[0,T]$ *for every* $T>0$ *and* $\ell(0)=\ell_0$*;*

(ii) $u(t,x)=w(t)\left(1-\dfrac{x}{\ell(t)}\right)\mathbb{1}_{[0,\ell(t)]}(x)$*, for every* $(t,x)\in[0,+\infty)\times[0,+\infty)$*;*

(iii) the quasistatic version of Griffith's criterion holds true, namely:

$$\begin{cases}\dot\ell(t)\geq 0,\\ \frac12\frac{w(t)^2}{\ell(t)^2}\leq\kappa(\ell(t)),\\ \left[\frac12\frac{w(t)^2}{\ell(t)^2}-\kappa(\ell(t))\right]\dot\ell(t)=0,\end{cases}\qquad \textit{for a.e. } t\in[0,+\infty).$$

Similarities with dynamic Griffith's criterion (5.2.3) are evident, with the exception of the term $\frac12\frac{w(t)^2}{\ell(t)^2}$ which requires some explanations: like in the dynamic case we can introduce the notion of quasistatic energy release rate as $G_{\rm qs}(t)=-\frac{\partial}{\partial\ell}\mathcal{E}(t)$, since kinetic energy is negligible in a quasistatic setting. By means of (ii) we can compute $\mathcal{E}(t)=\frac12\int_0^{\ell(t)}u_x(t,\sigma)^2\,\mathrm{d}\sigma=\frac12\frac{w(t)^2}{\ell(t)}$, from which we recover $G_{\rm qs}(t)=\frac12\frac{w(t)^2}{\ell(t)^2}$. Thus (iii) is the correct formulation of quasistatic Griffith's criterion.

We also notice that in (eb), like in Part I , we recover the formulation of the work of the external loading as the integral of $\frac{\partial}{\partial t}\mathcal{E}$ along the evolution. Indeed if we distinguish between time and space, namely if we consider $\mathcal{E}(t,\ell)=\frac12\frac{w(t)^2}{\ell}$, we easily have $\frac{\partial}{\partial t}\mathcal{E}(t,\ell(t))=\dot w(t)\frac{w(t)}{\ell(t)}$.

For a reason which will be clear during the proof of next proposition we introduce for $x\geq\ell_0$ the function $\phi_\kappa(x):=x^2\kappa(x)$; we recall that ϕ_κ actually appears in the assumptions (K1)-(K3) we listed in the Introduction and which we recall here for the sake of clarity:

(K1) $x\mapsto x^2\kappa(x)$ is nondecreasing on $[\ell_0,+\infty)$;

(K2) $x\mapsto x^2\kappa(x)$ is strictly increasing on $[\ell_0,+\infty)$;

(K3) $x\mapsto x^2\kappa(x)$ is strictly increasing on $[\ell_0,+\infty)$ and its derivative is strictly positive almost everywhere.

We also add the condition

$$\text{(KW)}\quad \lim_{x\to+\infty}x^2\kappa(x)>\frac12\max_{t\in[0,T]}w(t)^2\text{ for every }T>0,\text{ and }\kappa(\ell_0)\geq\frac12\frac{w(0)^2}{\ell_0^2}.$$

It is worth noticing that (K1) ensures local minima of the energy are actually global, as stated in Proposition 5.3.4. Conditions (K2) and (K3) instead imply uniqueness of the minimum, see Proposition 5.3.7. Finally the first assumption in (KW) is related to the existence of such a minimum, replacing the role of coercivity of the energy, which can be missing.

Proposition 5.3.4. *Assume (K1). Then a pair* (u,ℓ) *is an energetic evolution if and only if it is a quasistatic evolution.*

Proof. Let (u,ℓ) be an energetic evolution, then (o) is satisfied by definition. Now fix $t\in[0,+\infty)$ and choose $\hat\ell=\ell(t)$ in (GS). Then we deduce that $u(t,\cdot)$ minimises the functional $\dfrac12\displaystyle\int_0^{\ell(t)}\dot{\hat u}(\sigma)^2\,\mathrm{d}\sigma$ among all functions $\hat u\in H^1(0,\ell(t))$ such that $\hat u(0)=w(t)$ and $\hat u(\ell(t))=0$, and this implies (s1). Choosing now $\hat u(x)=w(t)\left(1-\frac{x}{\hat\ell}\right)\mathbb{1}_{[0,\hat\ell]}(x)$ in (GS) and exploiting (s1) we get:

$$\frac12\frac{w(t)^2}{\ell(t)}+\int_{\ell_0}^{\ell(t)}\kappa(\sigma)\,\mathrm{d}\sigma\leq\frac12\frac{w(t)^2}{\hat\ell}+\int_{\ell_0}^{\hat\ell}\kappa(\sigma)\,\mathrm{d}\sigma,\qquad\text{for every }\hat\ell\geq\ell(t).$$

This means that the energy $E_t\colon [\ell(t),+\infty)\to[0,+\infty)$ defined by $E_t(x):=\frac{1}{2}\frac{w(t)^2}{x}+\int_{\ell_0}^{x}\kappa(\sigma)\,\mathrm{d}\sigma$ has a global minimum in $x=\ell(t)$ and so $\dot{E}_t(\ell(t))\geq 0$, namely (s2) holds true. Finally (eb) follows by (EB) exploiting (s1).

Assume now that (u,ℓ) is a quasistatic evolution. To prove that it is an energetic evolution it is enough to show the validity of (GS), being (EB) trivially implied by (eb) and (s1). So let us fix $t\in[0,+\infty)$ and notice that (s2) is equivalent to $\phi_\kappa(\ell(t))\geq\frac{1}{2}w(t)^2$. By (K1) we hence deduce that $\phi_\kappa(x)\geq\frac{1}{2}w(t)^2$ for every $x\geq\ell(t)$, i.e. $\dot{E}_t(x)\geq 0$ for every $x\geq\ell(t)$. This means that E_t has a global minimum in $x=\ell(t)$ and so we obtain:

$$\frac{1}{2}\frac{w(t)^2}{\ell(t)}+\int_{\ell_0}^{\ell(t)}\kappa(\sigma)\,\mathrm{d}\sigma\leq\frac{1}{2}\frac{w(t)^2}{\hat{\ell}}+\int_{\ell_0}^{\hat{\ell}}\kappa(\sigma)\,\mathrm{d}\sigma,\qquad\text{for every }\hat{\ell}\geq\ell(t),$$

which in particular implies (GS), since affine functions minimise the potential energy. □

If we do not strenghten the assumptions on the toughness κ there is no hope to gain more regularity on ℓ, even in the case of a constant loading term $w>0$. Indeed it is enough to consider $\kappa(x)=\frac{1}{2}\frac{w^2}{x^2}$ (in this case ϕ_κ is constant) to realise that any function satisfying (o) automatically satisfies (s2) and (eb).

Lemma 5.3.5. *Assume (K2). Then any function ℓ satisfying (o), (s2) and (eb) is continuous.*

Proof. Let us assume by contradiction that there exists a time $\bar{t}\in[0,+\infty)$ in which ℓ is not continuous, namely $\ell^-(\bar{t})<\ell^+(\bar{t})$. Here we adopt the convention that $\ell^-(0)=\ell(0)=\ell_0$. Exploiting (s2), (eb) and the continuity of κ and w we deduce that:

$$\frac{1}{2}\frac{w(\bar{t})^2}{\ell^-(\bar{t})^2}\leq\kappa(\ell^-(\bar{t})),\tag{5.3.1a}$$

$$\frac{1}{2}\frac{w(\bar{t})^2}{\ell^+(\bar{t})}+\int_{\ell_0}^{\ell^+(\bar{t})}\kappa(\sigma)\,\mathrm{d}\sigma=\frac{1}{2}\frac{w(\bar{t})^2}{\ell^-(\bar{t})}+\int_{\ell_0}^{\ell^-(\bar{t})}\kappa(\sigma)\,\mathrm{d}\sigma.\tag{5.3.1b}$$

By using (K2), from (5.3.1) we get:

$$\begin{aligned}0&=\int_{\ell^-(\bar{t})}^{\ell^+(\bar{t})}\kappa(\sigma)\,\mathrm{d}\sigma-\frac{1}{2}w(\bar{t})^2\left(\frac{1}{\ell^-(\bar{t})}-\frac{1}{\ell^+(\bar{t})}\right)=\int_{\ell^-(\bar{t})}^{\ell^+(\bar{t})}\frac{\phi_\kappa(\sigma)-w(\bar{t})^2/2}{\sigma^2}\,\mathrm{d}\sigma\\&>\left(\phi_\kappa(\ell^-(\bar{t}))-\frac{1}{2}w(\bar{t})^2\right)\int_{\ell^-(\bar{t})}^{\ell^+(\bar{t})}\frac{1}{\sigma^2}\,\mathrm{d}\sigma\geq 0.\end{aligned}$$

This leads to a contradiction and hence we conclude. □

Lemma 5.3.6. *Assume (K2) and let ℓ be a function satisfying (o), (s2) and (eb). If there exists a time $\bar{t}\in(0,+\infty)$ in which (s2) holds with strict inequality, then ℓ is constant in a neighborhood of $\bar{t}$.*

Proof. Let us consider the function:

$$\Phi(t,x):=\frac{1}{2}\frac{w(t)^2}{x}+\int_{\ell_0}^{x}\kappa(\sigma)\,\mathrm{d}\sigma-\int_0^t\dot{w}(\tau)\frac{w(\tau)}{\ell(\tau)}\,\mathrm{d}\tau,\qquad\text{for }(t,x)\in[0,+\infty)\times[\ell_0,+\infty),$$

which is continuous on its domain. Moreover the derivative of Φ in the direction x exists at every point and it is continuous on $[0,+\infty)\times[\ell_0,+\infty)$, being given by:

$$\Phi_x(t,x)=\kappa(x)-\frac{1}{2}\frac{w(t)^2}{x^2}.$$

Since by assumption $\Phi_x(\bar{t},\ell(\bar{t}))>0$, by continuity we deduce that:

$$\Phi_x(t,x)\geq m>0,\quad \text{for every } (t,x)\in[a,b]\times[c,d], \tag{5.3.2}$$

where $[a,b]\times[c,d]\subset(0,+\infty)\times[\ell_0,+\infty)$ is a suitable rectangle containing the point $(\bar{t},\ell(\bar{t}))$. By continuity of ℓ (given by Lemma 5.3.5), we can assume without loss of generality that $\ell([a,b])\subset[c,d]$. Now we fix t_1, $t_2\in[a,b]$, $t_1\leq t_2$, and by the Mean Value Theorem we deduce:

$$\Phi(t_2,\ell(t_2))-\Phi(t_2,\ell(t_1))=\Phi_x(t_2,\xi)(\ell(t_2)-\ell(t_1)),\quad \text{for some } \xi\in[\ell(t_1),\ell(t_2)]\subset[c,d].$$

From this equality, exploiting (5.3.2) and (eb), we get:

$$\begin{aligned}\ell(t_2)-\ell(t_1)&\leq\frac{1}{m}\big(\Phi(t_2,\ell(t_2))-\Phi(t_2,\ell(t_1))\big)=\frac{1}{m}\big(\Phi(t_1,\ell(t_1))-\Phi(t_2,\ell(t_1))\big)\\&=\frac{1}{m}\left(\frac{1}{2\ell(t_1)}\big(w(t_1)^2-w(t_2)^2\big)+\int_{t_1}^{t_2}\dot{w}(\tau)\frac{w(\tau)}{\ell(\tau)}\,\mathrm{d}\tau\right)\\&=\frac{1}{m}\int_{t_1}^{t_2}\dot{w}(\tau)w(\tau)\left(\frac{1}{\ell(\tau)}-\frac{1}{\ell(t_1)}\right)\mathrm{d}\tau\\&\leq\frac{\ell(t_2)-\ell(t_1)}{m\ell_0^2}\int_a^b|\dot{w}(\tau)w(\tau)|\,\mathrm{d}\tau.\end{aligned}\tag{5.3.3}$$

Since w is absolutely continuous we can also assume that the interval $[a,b]$ is so small that:

$$\frac{1}{m\ell_0^2}\int_a^b|\dot{w}(\tau)w(\tau)|\,\mathrm{d}\tau\leq\frac{1}{2}.$$

From (5.3.3) we hence deduce that $\ell(t_2)=\ell(t_1)$, and so we conclude. □

We now introduce a notation, already adopted in [7] to deal with quasistatic hydraulic fractures: given a continuous function $h\colon[a,b]\to\mathbb{R}$ we define by h_* the smallest nondecreasing function greater or equal than h, namely $h_*(x):=\max\limits_{y\in[a,x]}h(y)$. We refer to [7] for its properties, we only want to recall that if $h\in W^{1,p}(a,b)$ for some $p\in[1,+\infty]$, then also h_* belongs to the same Sobolev space and $\dot{h}_*(x)=\dot{h}(x)\mathbb{1}_{\{h=h_*\}}(x)$ almost everywhere.

Proposition 5.3.7. *Assume (K2) and let ℓ be a function satisfying (o), (s2) and (eb). Then:*

$$\ell(t)=\phi_\kappa^{-1}\left(\max\left\{\frac{1}{2}(w^2)_*(t),\phi_\kappa(\ell_0)\right\}\right),\quad \textit{for every } t\in[0,+\infty). \tag{5.3.4}$$

Proof. Let ℓ satisfy (o), (s2) and (eb). By using (s2) we get $\phi_\kappa(\ell(t))\geq\frac{1}{2}w(t)^2$ for every $t\in[0,+\infty)$, and since the left-hand side is nondecreasing we deduce:

$$\phi_\kappa(\ell(t))\geq\max\left\{\frac{1}{2}(w^2)_*(t),\phi_\kappa(\ell_0)\right\},\quad \text{for every } t\in[0,+\infty).$$

Since by (K2) the function ϕ_κ is invertible, we finally get that $\ell(t)\geq\bar{\ell}(t)$ for every $t\in[0,+\infty)$, where we denoted by $\bar{\ell}$ the function in the right-hand side of (5.3.4).

Since by Lemma 5.3.5 we know ℓ is continuous on $[0,+\infty)$ and since by construction the same holds true for $\bar{\ell}$, we conclude if we prove that $\ell(t)=\bar{\ell}(t)$ for every $t\in(0,+\infty)$. By contradiction let $\bar{t}\in(0,+\infty)$ be such that $\ell(\bar{t})>\bar{\ell}(\bar{t})$. By (K2) this in particular implies that $\kappa(\ell(\bar{t}))>\dfrac{1}{2}\dfrac{w(\bar{t})^2}{\ell(\bar{t})^2}$, and so by Lemma 5.3.6 we get that ℓ is constant around $\bar{t}$. Since $\bar{\ell}$

is nondecreasing we can repeat this argument getting that ℓ is constant on the whole $[0,\bar{t}]$. This is absurd since it implies:

$$\phi_\kappa(\ell_0) = \phi_\kappa(\ell(0)) = \phi_\kappa(\ell(\bar{t})) > \phi_\kappa(\bar{\ell}(\bar{t})) \geq \phi_\kappa(\ell_0),$$

and so we conclude. □

Finally we can state and prove the main results of this section, regarding the equivalence between Definitions 5.3.1, 5.3.2 and 5.3.3 and about existence and uniqueness of quasistatic and AC-quasistatic evolutions. This first theorem only requires condition (K2).

Theorem 5.3.8. *Assume (K2) and (KW). Then there exists a unique quasistatic evolution $(\bar{u},\bar{\ell})$, which furthermore is continuous and is given by:*

$$\begin{aligned} &\bullet\ \bar{u}(t,x) = w(t)\left(1-\frac{x}{\bar{\ell}(t)}\right)\mathbb{1}_{[0,\bar{\ell}(t)]}(x), \quad \textit{for every } (t,x)\in[0,+\infty)\times[0,+\infty),\\ &\bullet\ \bar{\ell}(t) = \phi_\kappa^{-1}\left(\max\left\{\frac{1}{2}(w^2)_*(t),\phi_\kappa(\ell_0)\right\}\right), \quad \textit{for every } t\in[0,+\infty). \end{aligned} \tag{5.3.5}$$

Proof. Uniqueness of quasistatic evolution, its continuity and the validity of explicit formula (5.3.5) follow from Lemma 5.3.5 and Proposition 5.3.7. We only need to check that the pair $(\bar{u},\bar{\ell})$ is actually a quasistatic evolution. By (KW) $\bar{\ell}$ is well defined and (o) is fulfilled. Conditions (s1) and (s2) are satisfied by construction, thus we are left to prove that (eb) holds for $\bar{\ell}$. Since we are not able to prove it directly (even if we think it should be possible), we exploit Theorem 5.3.9, whose proof does not rely on the theorem we are proving now: we approximate the toughness κ by introducing for every $n\in\mathbb{N}$ the function $\kappa_n(x) := \kappa(x)+\frac{x-\ell_0}{nx^2}$, which satisfies (K3) and (KW). Thus the function $\bar{\ell}_n(t) = \phi_{\kappa_n}^{-1}\left(\max\left\{\frac{1}{2}(w^2)_*(t),\phi_\kappa(\ell_0)\right\}\right)$ fulfils (eb), namely:

$$\frac{1}{2}\frac{w(t)^2}{\bar{\ell}_n(t)} + \int_{\ell_0}^{\bar{\ell}_n(t)}\kappa(\sigma)\,\mathrm{d}\sigma = \frac{1}{2}\frac{w(0)^2}{\ell_0} + \int_0^t \dot{w}(\tau)\frac{w(\tau)}{\bar{\ell}_n(\tau)}\,\mathrm{d}\tau, \quad \text{for every } t\in[0,+\infty). \tag{5.3.6}$$

By construction it easy to see that ϕ_{κ_n} converges to ϕ_κ uniformly on compact sets of $[\ell_0,+\infty)$ as $n\to+\infty$, and this implies that $\lim\limits_{n\to+\infty}\phi_{\kappa_n}^{-1}(y) = \phi_\kappa^{-1}(y)$ for every point $y\in[\phi_\kappa(\ell_0),\phi_\kappa(+\infty))$. Hence we obtain that $\bar{\ell}_n$ pointwise converges to $\bar{\ell}$ in $[0,+\infty)$; thus, letting $n\to+\infty$ in (5.3.6), by means of dominated convergence we deduce that $\bar{\ell}$ satisfies (eb) and we conclude. □

If instead we strenghten a bit the assumptions on the toughness, requiring (K3), we are able to improve previous theorem.

Theorem 5.3.9. *Assume (K3). Then a pair (u,ℓ) is an energetic evolution if and only if it is an AC-quasistatic evolution.*
In particular, if we in addition assume (KW), the only AC-quasistatic evolution $(\bar{u},\bar{\ell})$ is given by (5.3.5).

Proof. Let (u,ℓ) be an energetic evolution. By Proposition 5.3.4 we get u satisfies (ii) and ℓ satisfies (o), (s2) and (eb). Moreover by Proposition 5.3.7 ℓ is explicitely given by (5.3.4) and hence by (K3) it is absolutely continuous on $[0,T]$ for every $T>0$, being composition of two nondecreasing absolutely continuous functions. Differentiating (eb) we now conclude that quasistatic Griffith's criterion (iii) holds true and so (u,ℓ) is an AC-quasistatic evolution.

On the other hand checking that any AC-quasistatic evolution is a quasistatic evolution is straightforward, and hence by Proposition 5.3.4 the other implication is proved.

Let us now verify that, assuming (KW), the pair $(\bar{u}, \bar{\ell})$ is actually an AC-quasistatic evolution. By (KW) $\bar{\ell}$ is well defined and (i) is fulfilled. The only nontrivial thing to check is the validity of the third condition in the quasistatic Griffith's criterion (iii). We need to prove that for any differentiability point $\bar{t} \in (0, +\infty)$ of $\bar{\ell}$ such that $\dot{\bar{\ell}}(\bar{t}) > 0$ it holds $\kappa(\bar{\ell}(\bar{t})) = \frac{1}{2}\frac{w(\bar{t})^2}{\bar{\ell}(\bar{t})^2}$. From the explicit expression of $\dot{\bar{\ell}}$, namely:

$$\dot{\bar{\ell}}(t) = \frac{w(t)\dot{w}(t)}{\dot{\phi}_\kappa(\bar{\ell}(t))} \mathbb{1}_{\{w^2=(w^2)_*>2\phi_\kappa(\ell_0)\}}(t), \qquad \text{for a.e. } t \in [0, +\infty),$$

we deduce that in $t = \bar{t}$ we must have $w(\bar{t})^2 = (w^2)_*(\bar{t}) > 2\phi_\kappa(\ell_0)$ and so it holds:

$$\phi_\kappa(\bar{\ell}(\bar{t})) = \max\left\{\frac{1}{2}(w^2)_*(\bar{t}), \phi_\kappa(\ell_0)\right\} = \frac{1}{2}w(\bar{t})^2,$$

and we conclude. □

Remark 5.3.10. In Theorems 5.3.8 and 5.3.9 the first condition of (KW) is needed only to ensure that the quasistatic evolution is defined for every time. If one removes this assumption (but keeps the initial stability condition $\kappa(\ell_0) \geq \frac{1}{2}\frac{w(0)^2}{\ell_0^2}$) then the two Theorems still hold, but $(\bar{u}, \bar{\ell})$ now exists only for times $t \in [0, T^*)$, with

$$T^* := \sup\left\{T > 0 \mid \lim_{x\to+\infty} x^2\kappa(x) > \frac{1}{2}\max_{t\in[0,T]} w(t)^2\right\},$$

and of course one has $\lim\limits_{t\to T^{*-}} \bar{\ell}(t) = +\infty$.

5.4 Energy estimates

In this section we provide useful energy estimates for the pair of dynamic evolutions $(u^\varepsilon, \ell^\varepsilon)$ given by Theorem 5.2.3. These estimates will be used in the next section to analyse the limit as $\varepsilon \to 0^+$ of both u^ε and ℓ^ε. From now on we always assume that the positive toughness κ belongs to $\widetilde{C}^{0,1}([\ell_0, +\infty))$. When needed we will also require the following additional assumptions on the data:

(H1) the families $\{w^\varepsilon\}_{\varepsilon>0}$, $\{u_0^\varepsilon\}_{\varepsilon>0}$, $\{\varepsilon u_1^\varepsilon\}_{\varepsilon>0}$ are bounded in $\widetilde{H}^1(0, +\infty)$, $H^1(0, \ell_0)$ and $L^2(0, \ell_0)$, respectively.

and on the toughness:

(K0) the function κ is not integrable in $[\ell_0, +\infty)$;

Remark 5.4.1. Whenever we assume (H1), we denote by ε_n a subsequence for which we have:

$$w^{\varepsilon_n} \rightharpoonup w \text{ in } \widetilde{H}^1(0, +\infty) \qquad \text{and} \qquad w^{\varepsilon_n} \to w \text{ uniformly in } [0, T] \text{ for every } T > 0, \quad (5.4.1)$$

for a suitable $w \in \widetilde{H}^1(0, +\infty)$. This sequence can be obtained by weak compactness and Sobolev embedding. By abuse of notation we will not relabel further subsequences.

The first step is obtaining an energy bound uniform in ε from the energy-dissipation balance (5.2.2a). As one can see, we must deal with the work of the external loading $\mathcal{W}^\varepsilon$, so we need to find a way to handle the boundary term $u_x^\varepsilon(\cdot, 0)$. Next lemma shows how we can recover it via an integration by parts.

Lemma 5.4.2. *Let the function $h \in C^\infty([0,+\infty))$ satisfy $h(0)=1$, $0 \le h(x) \le 1$ for every $x \in [0,+\infty)$ and $h(x)=0$ for every $x \ge \ell_0$. Then the following equality holds true for every $t \in [0,+\infty)$:*

$$\begin{aligned}
&\frac{1}{2}\int_0^t \Big(\varepsilon^2 \dot{w}^\varepsilon(\tau)^2 + u^\varepsilon_x(\tau,0)^2\Big)\,\mathrm{d}\tau \\
&= -\frac{1}{2}\int_0^t\int_0^{\ell_0} \dot{h}(\sigma)\Big(\varepsilon^2 u^\varepsilon_t(\tau,\sigma)^2 + u^\varepsilon_x(\tau,\sigma)^2\Big)\,\mathrm{d}\sigma\,\mathrm{d}\tau - \nu\int_0^t\int_0^{\ell_0} h(\sigma)\varepsilon u^\varepsilon_t(\tau,\sigma)u^\varepsilon_x(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau \\
&\quad - \varepsilon\left(\int_0^{\ell_0} h(\sigma)\varepsilon u^\varepsilon_t(t,\sigma)u^\varepsilon_x(t,\sigma)\,\mathrm{d}\sigma - \int_0^{\ell_0} h(\sigma)\varepsilon u^\varepsilon_1(\sigma)\dot{u}^\varepsilon_0(\sigma)\,\mathrm{d}\sigma\right).
\end{aligned} \tag{5.4.2}$$

Proof. We start with a formal proof, assuming that all the computation we are doing are allowed, and then we make it rigorous via an approximation argument. Performing an integration by parts we deduce:

$$\begin{aligned}
&\frac{1}{2}\int_0^t \Big(\varepsilon^2 \dot{w}^\varepsilon(\tau)^2 + u^\varepsilon_x(\tau,0)^2\Big)\,\mathrm{d}\tau = \frac{1}{2}\int_0^t \Big(\varepsilon^2 u^\varepsilon_t(\tau,0)^2 + u^\varepsilon_x(\tau,0)^2\Big)\,\mathrm{d}\tau \\
&= -\frac{1}{2}\int_0^t h(0)\Big(\varepsilon^2 u^\varepsilon_t(\tau,0)^2 + u^\varepsilon_x(\tau,0)^2\Big)(-1)\,\mathrm{d}\tau \\
&= -\frac{1}{2}\int_0^t\int_0^{\ell_0} \frac{\partial}{\partial\sigma}\Big[h(\cdot)\Big(\varepsilon^2 u^\varepsilon_t(\tau,\cdot)^2 + u^\varepsilon_x(\tau,\cdot)^2\Big)\Big](\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau \\
&= -\frac{1}{2}\int_0^t\int_0^{\ell_0} \dot{h}(\sigma)\Big(\varepsilon^2 u^\varepsilon_t(\tau,\sigma)^2 + u^\varepsilon_x(\tau,\sigma)^2\Big)\,\mathrm{d}\sigma\,\mathrm{d}\tau \\
&\quad - \int_0^t\int_0^{\ell_0} h(\sigma)\Big(\varepsilon^2 u^\varepsilon_t(\tau,\sigma)u^\varepsilon_{tx}(\tau,\sigma) + u^\varepsilon_x(\tau,\sigma)u^\varepsilon_{xx}(\tau,\sigma)\Big)\,\mathrm{d}\sigma\,\mathrm{d}\tau = (*).
\end{aligned}$$

Exploiting the fact that u^ε solves problem (5.0.1) we hence get:

$$\begin{aligned}
(*) &= -\frac{1}{2}\int_0^t\int_0^{\ell_0} \dot{h}(\sigma)\Big(\varepsilon^2 u^\varepsilon_t(\tau,\sigma)^2 + u^\varepsilon_x(\tau,\sigma)^2\Big)\,\mathrm{d}\sigma\,\mathrm{d}\tau \\
&\quad - \nu\int_0^t\int_0^{\ell_0} h(\sigma)\varepsilon u^\varepsilon_t(\tau,\sigma)u^\varepsilon_x(\tau,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau \\
&\quad - \varepsilon\int_0^t\int_0^{\ell_0} h(\sigma)\Big(\varepsilon u^\varepsilon_{tt}(\tau,\sigma)u^\varepsilon_x(\tau,\sigma) + \varepsilon u^\varepsilon_t(\tau,\sigma)u^\varepsilon_{tx}(\tau,\sigma)\Big)\,\mathrm{d}\sigma\,\mathrm{d}\tau.
\end{aligned}$$

Now we conclude since it holds:

$$\begin{aligned}
&\int_0^t\int_0^{\ell_0} h(\sigma)\Big(\varepsilon u^\varepsilon_{tt}(\tau,\sigma)u^\varepsilon_x(\tau,\sigma) + \varepsilon u^\varepsilon_t(\tau,\sigma)u^\varepsilon_{tx}(\tau,\sigma)\Big)\,\mathrm{d}\sigma\,\mathrm{d}\tau \\
&= \int_0^{\ell_0} h(\sigma)\int_0^t \frac{\partial}{\partial\tau}\Big[\varepsilon u^\varepsilon_t(\cdot,\sigma)u^\varepsilon_x(\cdot,\sigma)\Big](\tau)\,\mathrm{d}\tau\,\mathrm{d}\sigma \\
&= \int_0^{\ell_0} h(\sigma)\varepsilon u^\varepsilon_t(t,\sigma)u^\varepsilon_x(t,\sigma)\,\mathrm{d}\sigma - \int_0^{\ell_0} h(\sigma)\varepsilon u^\varepsilon_1(\sigma)\dot{u}^\varepsilon_0(\sigma)\,\mathrm{d}\sigma.
\end{aligned}$$

All the previous computations are rigorous if u^ε belongs to $\widetilde{H}^2(\Omega^\varepsilon)$, which is not the case. To overcome this lack of regularity we perform an approximation argument, exploiting Remark 5.2.4 and Theorem 5.2.5.

Let us consider a sequence $\{u^\varepsilon_{0n}\}_{n\in\mathbb{N}} \subset H^2(0,\ell_0)$ such that $u^\varepsilon_{0n}(0) = u^\varepsilon_0(0)$, $u^\varepsilon_{0n}(\ell_0) = 0$ and converging to u^ε_0 in $H^1(0,\ell_0)$ as $n \to +\infty$; then we pick a sequence $\{w^\varepsilon_n\}_{n\in\mathbb{N}} \subset \widetilde{H}^2(0,+\infty)$ such that $w^\varepsilon_n(0) = w^\varepsilon(0)$ and converging to w^ε in $\widetilde{H}^1(0,+\infty)$ as $n \to +\infty$;

finally we take another sequence $\{u^\varepsilon_{1n}\}_{n\in\mathbb{N}}\subset H^1(0,\ell_0)$ converging to u^ε_1 in $L^2(0,\ell_0)$ as $n\to+\infty$ and satisfying:

$$u^\varepsilon_{1n}(0)=\dot w^\varepsilon_n(0),\quad u^\varepsilon_{1n}(\ell_0)=\begin{cases}-\frac{\mathrm{sign}\left(\dot u^\varepsilon_{0n}(\ell_0)\right)}{\varepsilon}\sqrt{\dot u^\varepsilon_{0n}(\ell_0)^2-2\kappa(\ell_0)}, & \text{if } \dot u^\varepsilon_{0n}(\ell_0)^2>2\kappa(\ell_0),\\ 0, & \text{otherwise.}\end{cases}$$

Denoting by $(u^\varepsilon_n,\ell^\varepsilon_n)$ the solution of coupled problem (5.0.1)&(5.0.2) related to these data, we deduce by Remark 5.2.4 that u^ε_n belongs to $H^2(\Omega^\varepsilon_T)$, and so by previous computations (5.4.2) holds true for it. By Theorem 5.2.5 equality (5.4.2) passes to the limit as $n\to+\infty$ and hence we conclude. □

Thanks to previous lemma we are able to prove the following energy bound:

Proposition 5.4.3. *Assume (H1). Then for every $T>0$ there exists a positive constant $C_T>0$ such that for every $\varepsilon\in(0,1/2)$ it holds:*

$$\mathcal{K}^\varepsilon(t)+\mathcal{E}^\varepsilon(t)+\mathcal{V}^\varepsilon(t)+\int_{\ell_0}^{\ell^\varepsilon(t)}\kappa(\sigma)\,\mathrm{d}\sigma\le C_T,\qquad \textit{for every } t\in[0,T], \tag{5.4.3}$$

where $\mathcal{K}^\varepsilon$, $\mathcal{E}^\varepsilon$ and $\mathcal{V}^\varepsilon$ are the energies defined in (5.2.1a), (5.2.1b) *and* (5.2.1c).

Proof. We fix $T>0$, $t\in[0,T]$, $\varepsilon\in(0,1/2)$ and by using the energy-dissipation balance (5.2.2a) we estimate:

$$\begin{aligned}&\mathcal{K}^\varepsilon(t)+\mathcal{E}^\varepsilon(t)+\mathcal{V}^\varepsilon(t)+\int_{\ell_0}^{\ell^\varepsilon(t)}\kappa(\sigma)\,\mathrm{d}\sigma=\mathcal{K}^\varepsilon(0)+\mathcal{E}^\varepsilon(0)+\mathcal{W}^\varepsilon(t)\\ &\le\mathcal{K}^\varepsilon(0)+\mathcal{E}^\varepsilon(0)+\frac12\int_0^t\dot w^\varepsilon(\tau)^2\,\mathrm{d}\tau+\frac12\int_0^t u^\varepsilon_x(\tau,0)^2\,\mathrm{d}\tau\\ &=\mathcal{K}^\varepsilon(0)+\mathcal{E}^\varepsilon(0)+\frac{1-\varepsilon^2}{2}\int_0^t\dot w^\varepsilon(\tau)^2\,\mathrm{d}\tau+\frac12\int_0^t\left(\varepsilon^2\dot w^\varepsilon(\tau)^2+u^\varepsilon_x(\tau,0)^2\right)\mathrm{d}\tau=(*).\end{aligned}$$

By Lemma 5.4.2 and by applying Young's inequality, we can continue the estimate getting:

$$\begin{aligned}(*)\le\quad&(1+\varepsilon)\left(\mathcal{K}^\varepsilon(0)+\mathcal{E}^\varepsilon(0)\right)+\frac{1-\varepsilon^2}{2}\int_0^t\dot w^\varepsilon(\tau)^2\,\mathrm{d}\tau\\ &+\left(\max_{x\in[0,\ell_0]}|\dot h(x)|+\nu\right)\int_0^t\left(\mathcal{K}^\varepsilon(\tau)+\mathcal{E}^\varepsilon(\tau)\right)\mathrm{d}\tau+\varepsilon\left(\mathcal{K}^\varepsilon(t)+\mathcal{E}^\varepsilon(t)\right).\end{aligned}$$

We conclude by means of Grönwall Lemma and exploiting (H1). □

As an immediate corollary we have:

Corollary 5.4.4. *Assume (H1) and (K0). Then for every $T>0$ there exists a positive constant $L_T>0$ such that $\ell^\varepsilon(T)\le L_T$ for every $\varepsilon\in(0,1/2)$.*

In order to improve the energy bound given by Proposition 5.4.3 we exploit the classical exponential decay of the energy for a solution to the damped wave equation. Following the ideas of [68] we adapt their argument to our model in which the domain of the equation changes in time. For this aim we introduce the modified shifted potential energy:

$$\widetilde{\mathcal{E}}^\varepsilon(t):=\frac12\int_0^{\ell^\varepsilon(t)}\left(u^\varepsilon_x(t,\sigma)-r^\varepsilon_x(t,\sigma)\right)^2\mathrm{d}\sigma,\quad \text{for } t\in[0,+\infty), \tag{5.4.4}$$

where $r^\varepsilon(t,x)$ is the affine function connecting the points $(0,w^\varepsilon(t))$ and $(\ell^\varepsilon(t),0)$, namely:

$$r^\varepsilon(t,x):=w^\varepsilon(t)\left(1-\frac{x}{\ell^\varepsilon(t)}\right)\mathbb{1}_{[0,\ell^\varepsilon(t)]}(x),\quad \text{for } (t,x)\in[0,+\infty)\times[0,+\infty). \tag{5.4.5}$$

The main result of this section is the following decay estimate:

Theorem 5.4.5. *Assume (H1) and (K0) and let the parameter ν be* ***positive****. Then for every $T>0$ there exists a constant $C_T>0$ such that for every $t\in[0,T]$ and $\varepsilon\in(0,1/2)$ one has:*

$$\mathcal{K}^\varepsilon(t)+\widetilde{\mathcal{E}}^\varepsilon(t)\le 4\left(\mathcal{K}^\varepsilon(0)+\widetilde{\mathcal{E}}^\varepsilon(0)\right)e^{-m\frac{t}{\varepsilon}}+C_T\int_0^t\big(\dot\ell^\varepsilon(\tau)+\dot w^\varepsilon(\tau)^2+u_x^\varepsilon(\tau,0)^2+1\big)e^{-m\frac{t-\tau}{\varepsilon}}\,\mathrm{d}\tau, \tag{5.4.6}$$

where $m=m(\nu,T):=\dfrac{1}{2}\min\left\{\dfrac{1}{2\mu_T^0},\,\dfrac{\nu}{2},\,\dfrac{1}{\mu_T^0+\mu_T^1}\right\}>0$ *and μ_T^0, μ_T^1 are defined as follows:*

$$\mu_T^0:=\frac{L_T}{\pi},\qquad \textit{and}\qquad \mu_T^1:=\nu\left(\frac{L_T}{\pi}\right)^2, \tag{5.4.7}$$

with L_T given by Corollary 5.4.4.

Remark 5.4.6. Estimate (5.4.6) actually still holds true for $\nu=0$, but in this case $m=0$ and so the inequality becomes trivial and useless.

To prove this theorem we will need several lemmas. As before we always assume that $\varepsilon\in(0,1/2)$.

Lemma 5.4.7. *Assume (H1). Then for every $T>0$ the modified internal energy $\mathcal{K}^\varepsilon+\widetilde{\mathcal{E}}^\varepsilon$ is absolutely continuous on $[0,T]$ and the following inequality holds true for a.e. $t\in[0,T]$:*

$$\dot{\mathcal{K}}^\varepsilon(t)+\dot{\widetilde{\mathcal{E}}}^\varepsilon(t)\le-\nu\int_0^{\ell^\varepsilon(t)}\varepsilon u_t^\varepsilon(t,\sigma)^2\,\mathrm{d}\sigma+C_T\big(\dot\ell^\varepsilon(t)+\dot w^\varepsilon(t)^2+u_x^\varepsilon(t,0)^2+1\big), \tag{5.4.8}$$

where C_T is a positive constant depending on T but independent of ε.

Proof. By developing the square in (5.4.4) and exploiting (5.4.5) one can easily show that:

$$\widetilde{\mathcal{E}}^\varepsilon(t)=\mathcal{E}^\varepsilon(t)-\frac{1}{2}\frac{w^\varepsilon(t)^2}{\ell^\varepsilon(t)},\qquad\text{for every } t\in[0,+\infty). \tag{5.4.9}$$

Now fix $T>0$. The modified internal energy $\mathcal{K}^\varepsilon+\widetilde{\mathcal{E}}^\varepsilon$ is absolutely continuous on $[0,T]$ because by (5.4.9) it is sum of two absolutely continuous functions (we recall Proposition 3.3.1). By (5.4.9) and the energy-dissipation balance (5.2.2a) we then compute for a.e. $t\in[0,+\infty)$:

$$\begin{aligned}\dot{\mathcal{K}}^\varepsilon(t)+\dot{\widetilde{\mathcal{E}}}^\varepsilon(t)&=\dot{\mathcal{K}}^\varepsilon(t)+\dot{\mathcal{E}}^\varepsilon(t)-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\frac{w^\varepsilon(t)^2}{\ell^\varepsilon(t)}=\\&=-\kappa(\ell^\varepsilon(t))\dot\ell^\varepsilon(t)-\nu\int_0^{\ell^\varepsilon(t)}\varepsilon u_t^\varepsilon(t,\sigma)^2\,\mathrm{d}\sigma-\dot w^\varepsilon(t)u_x^\varepsilon(t,0)\\&\quad+\frac{\dot\ell^\varepsilon(t)}{2}\frac{w^\varepsilon(t)^2}{\ell^\varepsilon(t)^2}-\dot w^\varepsilon(t)\frac{w^\varepsilon(t)}{\ell^\varepsilon(t)}.\end{aligned}$$

Recalling that $\ell^\varepsilon(t)\ge\ell_0$ and since by (H1) the family $\{w^\varepsilon\}_{\varepsilon>0}$ is uniformly equibounded in $[0,T]$ we conclude by means of Young's inequality. □

Always inspired by [68], for $t\in[0,+\infty)$ we also introduce the auxiliary function:

$$\widetilde{\mathcal{F}}^\varepsilon(t):=\int_0^{\ell^\varepsilon(t)}\varepsilon^2u_t^\varepsilon(t,\sigma)\big(u^\varepsilon(t,\sigma)-r^\varepsilon(t,\sigma)\big)\,\mathrm{d}\sigma+\frac{\nu\varepsilon}{2}\int_0^{\ell^\varepsilon(t)}\big(u^\varepsilon(t,\sigma)-r^\varepsilon(t,\sigma)\big)^2\,\mathrm{d}\sigma.$$

Lemma 5.4.8. *Assume (H1) and (K0). Then for every $T>0$ one has:*

$$-\varepsilon\mu^0_T\left(\mathcal{K}^\varepsilon(t)+\widetilde{\mathcal{E}}^\varepsilon(t)\right)\le\widetilde{\mathcal{F}}^\varepsilon(t)\le\varepsilon(\mu^0_T+\mu^1_T)\left(\mathcal{K}^\varepsilon(t)+\widetilde{\mathcal{E}}^\varepsilon(t)\right),\text{for every } t\in[0,T], \tag{5.4.10}$$

where μ^0_T and μ^1_T have been defined in (5.4.7).

Proof. We fix $t\in[0,T]$ and by means of the sharp Poincarè inequality:

$$\int_a^b f(\sigma)^2\,\mathrm{d}\sigma\le\frac{(b-a)^2}{\pi^2}\int_a^b\dot f(\sigma)^2\,\mathrm{d}\sigma,\ \text{for every } f\in H^1_0(a,b), \tag{5.4.11}$$

together with Young's inequality we get:

$$\begin{aligned}&\left|\varepsilon^2\int_0^{\ell^\varepsilon(t)}u^\varepsilon_t(t,\sigma)\big(u^\varepsilon(t,\sigma)-r^\varepsilon(t,\sigma)\big)\,\mathrm{d}\sigma\right|\\
&\le\frac{\varepsilon}{2}\left[\frac{\ell^\varepsilon(t)}{\pi}\int_0^{\ell^\varepsilon(t)}\varepsilon^2u^\varepsilon_t(t,\sigma)^2\,\mathrm{d}\sigma+\frac{\pi}{\ell^\varepsilon(t)}\int_0^{\ell^\varepsilon(t)}\big(u^\varepsilon(t,\sigma)-r^\varepsilon(t,\sigma)\big)^2\,\mathrm{d}\sigma\right]\\
&\le\varepsilon\frac{\ell^\varepsilon(t)}{\pi}\left[\frac12\int_0^{\ell^\varepsilon(t)}\varepsilon^2u^\varepsilon_t(t,\sigma)^2\,\mathrm{d}\sigma+\frac12\int_0^{\ell^\varepsilon(t)}\big(u^\varepsilon_x(t,\sigma)-r^\varepsilon_x(t,\sigma)\big)^2\mathrm{d}\sigma\right]\le\varepsilon\mu^0_T\left(\mathcal{K}^\varepsilon(t)+\widetilde{\mathcal{E}}^\varepsilon(t)\right).\end{aligned}$$

From the above estimate we hence deduce:

$$\begin{aligned}-\varepsilon\mu^0_T\left(\mathcal{K}^\varepsilon(t)+\widetilde{\mathcal{E}}^\varepsilon(t)\right)&\le-\left|\varepsilon^2\int_0^{\ell^\varepsilon(t)}u^\varepsilon_t(t,\sigma)\big(u^\varepsilon(t,\sigma)-r^\varepsilon(t,\sigma)\big)\,\mathrm{d}\sigma\right|\le\widetilde{\mathcal{F}}^\varepsilon(t)\\
&\le\varepsilon\mu^0_T\left(\mathcal{K}^\varepsilon(t)+\widetilde{\mathcal{E}}^\varepsilon(t)\right)+\frac{\varepsilon\nu}{2}\frac{\ell^\varepsilon(t)^2}{\pi^2}\int_0^{\ell^\varepsilon(t)}\big(u^\varepsilon_x(t,\sigma)-r^\varepsilon_x(t,\sigma)\big)^2\,\mathrm{d}\sigma\\
&\le\varepsilon(\mu^0_T+\mu^1_T)\left(\mathcal{K}^\varepsilon(t)+\widetilde{\mathcal{E}}^\varepsilon(t)\right),\end{aligned}$$

and we conclude. □

Lemma 5.4.9. *Assume (H1) and (K0). Then for every $T>0$ the function $\widetilde{\mathcal{F}}^\varepsilon$ is absolutely continuous on $[0,T]$ and the following inequality holds true for a.e. $t\in[0,T]$:*

$$\dot{\widetilde{\mathcal{F}}}^\varepsilon(t)\le2\int_0^{\ell^\varepsilon(t)}\varepsilon^2u^\varepsilon_t(t,\sigma)^2\,\mathrm{d}\sigma-\mathcal{K}^\varepsilon(t)-\widetilde{\mathcal{E}}^\varepsilon(t)+C_T\varepsilon^2\big(\dot w^\varepsilon(t)^2+\dot\ell^\varepsilon(t)^2\big), \tag{5.4.12}$$

where C_T is a positive constant depending on T but independent of ε.

Proof. Fix $T>0$. By exploiting the fact that u^ε solves problem (5.0.1) we start formally computing the derivative of $\widetilde{\mathcal{F}}^\varepsilon$ at almost every point $t\in(0,T)$:

$$\begin{aligned}\dot{\widetilde{\mathcal{F}}}^\varepsilon(t)&=\int_0^{\ell^\varepsilon(t)}\varepsilon^2u^\varepsilon_t(t,\sigma)\big(u^\varepsilon_t(t,\sigma)-r^\varepsilon_t(t,\sigma)\big)\,\mathrm{d}\sigma+\int_0^{\ell^\varepsilon(t)}\varepsilon^2u^\varepsilon_{tt}(t,\sigma)\big(u^\varepsilon(t,\sigma)-r^\varepsilon(t,\sigma)\big)\,\mathrm{d}\sigma\\
&\quad+\nu\varepsilon\int_0^{\ell^\varepsilon(t)}\big(u^\varepsilon(t,\sigma)-r^\varepsilon(t,\sigma)\big)\big(u^\varepsilon_t(t,\sigma)-r^\varepsilon_t(t,\sigma)\big)\,\mathrm{d}\sigma\\
&=\int_0^{\ell^\varepsilon(t)}\varepsilon^2u^\varepsilon_t(t,\sigma)\big(u^\varepsilon_t(t,\sigma)-r^\varepsilon_t(t,\sigma)\big)\,\mathrm{d}\sigma+\int_0^{\ell^\varepsilon(t)}\big(u^\varepsilon(t,\sigma)-r^\varepsilon(t,\sigma)\big)\big(u^\varepsilon_{xx}(t,\sigma)-r^\varepsilon_{xx}(t,\sigma)\big)\,\mathrm{d}\sigma\\
&\quad-\nu\int_0^{\ell^\varepsilon(t)}\varepsilon r^\varepsilon_t(t,\sigma)\big(u^\varepsilon(t,\sigma)-r^\varepsilon(t,\sigma)\big)\,\mathrm{d}\sigma\\
&=\int_0^{\ell^\varepsilon(t)}\varepsilon^2u^\varepsilon_t(t,\sigma)\big(u^\varepsilon_t(t,\sigma)-r^\varepsilon_t(t,\sigma)\big)\,\mathrm{d}\sigma-\int_0^{\ell^\varepsilon(t)}\big(u^\varepsilon_x(t,\sigma)-r^\varepsilon_x(t,\sigma)\big)^2\,\mathrm{d}\sigma\end{aligned}$$

$$- \nu \int_0^{\ell^\varepsilon(t)} \varepsilon r_t^\varepsilon(t,\sigma)\big(u^\varepsilon(t,\sigma) - r^\varepsilon(t,\sigma)\big)\,\mathrm{d}\sigma.$$

By means of an approximation argument similar to the one adopted in the proof of Lemma 5.4.2 one deduces that $\widetilde{\mathcal{F}}^\varepsilon$ is absolutely continuous on $[0,T]$ and that the formula for $\dot{\widetilde{\mathcal{F}}}^\varepsilon$ found with the previous computation is actually true.

To get (5.4.12) we use the sharp Poincarè inequality (5.4.11) and Young's inequality:

$$\begin{aligned}
&\dot{\widetilde{\mathcal{F}}}^\varepsilon(t)\\
&\le 2\int_0^{\ell^\varepsilon(t)} \varepsilon^2 u_t^\varepsilon(t,\sigma)^2\,\mathrm{d}\sigma - 2\left(\mathcal{K}^\varepsilon(t) + \widetilde{\mathcal{E}}^\varepsilon(t)\right) + \frac{1}{2}\int_0^{\ell^\varepsilon(t)} \varepsilon^2 u_t^\varepsilon(t,\sigma)^2\,\mathrm{d}\sigma + \frac{1}{2}\int_0^{\ell^\varepsilon(t)} \varepsilon^2 r_t^\varepsilon(t,\sigma)^2\,\mathrm{d}\sigma\\
&\quad + \frac{\nu}{2}\left[\frac{1}{\nu}\frac{\pi^2}{\ell^\varepsilon(t)^2}\int_0^{\ell^\varepsilon(t)} \big(u^\varepsilon(t,\sigma) - r^\varepsilon(t,\sigma)\big)^2\,\mathrm{d}\sigma + \nu\frac{\ell^\varepsilon(t)^2}{\pi^2}\int_0^{\ell^\varepsilon(t)} \varepsilon^2 r_t^\varepsilon(t,\sigma)^2\,\mathrm{d}\sigma\right]\\
&\le 2\int_0^{\ell^\varepsilon(t)} \varepsilon^2 u_t^\varepsilon(t,\sigma)^2\,\mathrm{d}\sigma - 2\left(\mathcal{K}^\varepsilon(t) + \widetilde{\mathcal{E}}^\varepsilon(t)\right) + \frac{1}{2}\int_0^{\ell^\varepsilon(t)} \varepsilon^2 u_t^\varepsilon(t,\sigma)^2\,\mathrm{d}\sigma + \frac{1}{2}\int_0^{\ell^\varepsilon(t)} \varepsilon^2 r_t^\varepsilon(t,\sigma)^2\,\mathrm{d}\sigma\\
&\quad + \frac{1}{2}\int_0^{\ell^\varepsilon(t)} \big(u_x^\varepsilon(t,\sigma) - r_x^\varepsilon(t,\sigma)\big)^2\,\mathrm{d}\sigma + \frac{1}{2}\left(\frac{\nu\ell^\varepsilon(t)}{\pi}\right)^2\int_0^{\ell^\varepsilon(t)} \varepsilon^2 r_t^\varepsilon(t,\sigma)^2\,\mathrm{d}\sigma\\
&\le 2\int_0^{\ell^\varepsilon(t)} \varepsilon^2 u_t^\varepsilon(t,\sigma)^2\,\mathrm{d}\sigma - \mathcal{K}^\varepsilon(t) - \widetilde{\mathcal{E}}^\varepsilon(t) + \frac{1}{2}\left(1 + \nu\mu_T^1\right)\varepsilon^2\int_0^{\ell^\varepsilon(t)} r_t^\varepsilon(t,\sigma)^2\,\mathrm{d}\sigma.
\end{aligned}$$

To conclude it is enough to use Corollary 5.4.4, (H1) and to exploit the explicit form of r^ε given by (5.4.5) getting:

$$\int_0^{\ell^\varepsilon(t)} r_t^\varepsilon(t,\sigma)^2\,\mathrm{d}\sigma \le C_T\big(\dot{w}^\varepsilon(t)^2 + \dot{\ell}^\varepsilon(t)^2\big).$$

□

We are now in a position to prove Theorem 5.4.5:

Proof of Theorem 5.4.5. We fix $T > 0$ and we introduce the Lyapunov function:

$$\widetilde{\mathcal{D}}^\varepsilon(t) := \mathcal{K}^\varepsilon(t) + \widetilde{\mathcal{E}}^\varepsilon(t) + \frac{2m}{\varepsilon}\widetilde{\mathcal{F}}^\varepsilon(t), \quad \text{for } t \in [0,T].$$

From (5.4.10) we easily infer:

$$\big(1 - 2m\mu_T^0\big)\left(\mathcal{K}^\varepsilon(t) + \widetilde{\mathcal{E}}^\varepsilon(t)\right) \le \widetilde{\mathcal{D}}^\varepsilon(t) \le \big(1 + 2m(\mu_T^0 + \mu_T^1)\big)\left(\mathcal{K}^\varepsilon(t) + \widetilde{\mathcal{E}}^\varepsilon(t)\right), \text{ for every } t \in [0,T],$$

and so in particular by definition of m we deduce:

$$\frac{1}{2}\left(\mathcal{K}^\varepsilon(t) + \widetilde{\mathcal{E}}^\varepsilon(t)\right) \le \widetilde{\mathcal{D}}^\varepsilon(t) \le 2\left(\mathcal{K}^\varepsilon(t) + \widetilde{\mathcal{E}}^\varepsilon(t)\right), \text{ for every } t \in [0,T]. \tag{5.4.13}$$

Moreover we can estimate the derivative of $\widetilde{\mathcal{D}}^\varepsilon$ for a.e. $t \in [0,T]$ by using (5.4.8) and (5.4.12) and recalling that $\varepsilon\dot{\ell}^\varepsilon(t) < 1$ and that $4m \le \nu$:

$$\begin{aligned}
\dot{\widetilde{\mathcal{D}}}^\varepsilon(t) &= \dot{\mathcal{K}}^\varepsilon(t) + \dot{\widetilde{\mathcal{E}}}^\varepsilon(t) + \frac{2m}{\varepsilon}\dot{\widetilde{\mathcal{F}}}^\varepsilon(t)\\
&\le -(\nu - 4m)\int_0^{\ell^\varepsilon(t)} \varepsilon u_t^\varepsilon(t,\sigma)^2\,\mathrm{d}\sigma - \frac{2m}{\varepsilon}\left(\mathcal{K}^\varepsilon(t) + \widetilde{\mathcal{E}}^\varepsilon(t)\right) + C_T\big(\dot{\ell}^\varepsilon(t) + \dot{w}^\varepsilon(t)^2 + u_x^\varepsilon(t,0)^2 + 1\big)\\
&\le -\frac{2m}{\varepsilon}\left(\mathcal{K}^\varepsilon(t) + \widetilde{\mathcal{E}}^\varepsilon(t)\right) + C_T\big(\dot{\ell}^\varepsilon(t) + \dot{w}^\varepsilon(t)^2 + u_x^\varepsilon(t,0)^2 + 1\big).
\end{aligned}$$

By (5.4.13) we hence deduce:

$$\dot{\widetilde{\mathcal{D}}}^\varepsilon(t) \le -\frac{m}{\varepsilon}\widetilde{\mathcal{D}}^\varepsilon(t) + C_T\big(\dot\ell^\varepsilon(t) + \dot w^\varepsilon(t)^2 + u_x^\varepsilon(t,0)^2 + 1\big), \quad \text{for a.e. } t \in [0,T],$$

from which for every $t \in [0,T]$ we get:

$$\widetilde{\mathcal{D}}^\varepsilon(t) \le \widetilde{\mathcal{D}}^\varepsilon(0)e^{-m\frac{t}{\varepsilon}} + C_T\int_0^t \big(\dot\ell^\varepsilon(\tau) + \dot w^\varepsilon(\tau)^2 + u_x^\varepsilon(\tau,0)^2 + 1\big)e^{-m\frac{t-\tau}{\varepsilon}}\,\mathrm{d}\tau.$$

We conclude by using again (5.4.13). □

5.5 Quasistatic limit

In this section we show how, thanks to the estimates of Section 5.4, dynamic evolutions $(u^\varepsilon, \ell^\varepsilon)$ converge to a quasistatic one as $\varepsilon \to 0^+$, except for a possible initial jump due to a steep initial position u_0. The rigorous result is stated in Theorem 5.5.21. Also in this section we assume that κ belongs to $\widetilde{C}^{0,1}([\ell_0,+\infty))$.

5.5.1 Extraction of convergent subsequences

We first prove that the sequence of debonding fronts ℓ^ε admits a pointwise convergent subsequence.

Proposition 5.5.1. *Assume (H1) and (K0). Then there exists a subsequence $\varepsilon_n \searrow 0$ and there exists a nondecreasing function $\ell\colon [0,+\infty) \to [\ell_0,+\infty)$ such that*

$$\lim_{n\to+\infty} \ell^{\varepsilon_n}(t) = \ell(t), \quad \textit{for every } t \in [0,+\infty).$$

Proof. The result follows by Corollary 5.4.4 and by a simple application of the classical Helly's selection principle. □

In order to deal with the convergence of the displacements u^ε we exploit the energy decay (5.4.5):

Proposition 5.5.2. *Assume (H1) and (K0) and let ν be* ***positive****. Then for every $T > 0$ the modified internal energy $\mathcal{K}^\varepsilon + \widetilde{\mathcal{E}}^\varepsilon$ converges to 0 in $L^1(0,T)$ when $\varepsilon \to 0^+$. Thus there exists a subsequence $\varepsilon_n \searrow 0$ such that:*

$$\lim_{n\to+\infty} \Big(\mathcal{K}^{\varepsilon_n}(t) + \widetilde{\mathcal{E}}^{\varepsilon_n}(t)\Big) = 0, \quad \textit{for almost every } t \in (0,+\infty).$$

Proof. We fix $T > 0$. Theorem 5.4.5 ensures that:

$$\mathcal{K}^\varepsilon(t) + \widetilde{\mathcal{E}}^\varepsilon(t) \le 4\Big(\mathcal{K}^\varepsilon(0) + \widetilde{\mathcal{E}}^\varepsilon(0)\Big)e^{-m\frac{t}{\varepsilon}} + C_T(\rho^\varepsilon * \eta^\varepsilon)(t), \quad \text{for every } t \in [0,T],$$

where the symbol $*$ denotes the convolution product and for a.e. $t \in \mathbb{R}$ we define:

$$\begin{aligned} \rho^\varepsilon(t) &:= \big(\dot\ell^\varepsilon(t) + \dot w^\varepsilon(t)^2 + u_x^\varepsilon(t,0)^2 + 1\big)\mathbb{1}_{[0,T]}(t),\\ \eta^\varepsilon(t) &:= e^{-m\frac{t}{\varepsilon}}\mathbb{1}_{[0,+\infty)}(t). \end{aligned}$$

Furthermore by (5.4.9) and (H1) we get that $\mathcal{K}^\varepsilon(0)$ and $\widetilde{\mathcal{E}}^\varepsilon(0)$ are uniformly bounded in ε, and so by classical properties of convolutions we estimate:

$$\begin{aligned} \|\mathcal{K}^\varepsilon + \widetilde{\mathcal{E}}^\varepsilon\|_{L^1(0,T)} &\le C\int_0^{+\infty} e^{-m\frac{\tau}{\varepsilon}}\,\mathrm{d}\tau + C_T\|\rho^\varepsilon * \eta^\varepsilon\|_{L^1(\mathbb{R})}\\ &\le C\frac{\varepsilon}{m} + C_T\|\rho^\varepsilon\|_{L^1(\mathbb{R})}\|\eta^\varepsilon\|_{L^1(\mathbb{R})} = \frac{\varepsilon}{m}\big(C + C_T\|\rho^\varepsilon\|_{L^1(\mathbb{R})}\big). \end{aligned}$$

Now we bound the L^1-norm of ρ^ε by means of (H1), (K0) and recalling that by Lemma 5.4.2 and Proposition 5.4.3 we know that $\|u_x^\varepsilon(\cdot,0)\|_{L^2(0,T)}$ is uniformly bounded with respect to ε:

$$\|\rho^\varepsilon\|_{L^1(\mathbb{R})} = \ell^\varepsilon(T) - \ell_0 + \|\dot{w}^\varepsilon\|^2_{L^2(0,T)} + \|u_x^\varepsilon(\cdot,0)\|^2_{L^2(0,T)} + T \le C_T.$$

Thus we deduce that $\mathcal{K}^\varepsilon + \widetilde{\mathcal{E}}^\varepsilon \to 0$ in $L^1(0,T)$ when $\varepsilon \to 0^+$ and so we conclude by using a diagonal argument. □

Similarly to what we did in Lemma 5.4.2 we need to understand the behaviour of $u_x^\varepsilon(\cdot,0)$ when $\varepsilon \to 0^+$ before carrying on the analysis of the convergence of u^ε.

Lemma 5.5.3. *Let the function h be as in Lemma 5.4.2. Then the following equality holds true for every $t \in [0,+\infty)$:*

$$\begin{aligned}
&\frac{1}{2}\int_0^t \Big(\varepsilon^2 \dot{w}^\varepsilon(\tau)^2 + \big(u_x^\varepsilon(\tau,0) - r_x^\varepsilon(\tau,0)\big)^2\Big)\,\mathrm{d}\tau \\
&= -\frac{1}{2}\int_0^t\int_0^{\ell_0} \dot{h}(\sigma)\Big(\varepsilon^2 u_t^\varepsilon(\tau,\sigma)^2 + \big(u_x^\varepsilon(\tau,\sigma) - r_x^\varepsilon(\tau,\sigma)\big)^2\Big)\,\mathrm{d}\sigma\,\mathrm{d}\tau \\
&\quad - \nu\int_0^t\int_0^{\ell_0} h(\sigma)\varepsilon u_t^\varepsilon(\tau,\sigma)\big(u_x^\varepsilon(\tau,\sigma) - r_x^\varepsilon(\tau,\sigma)\big)\,\mathrm{d}\sigma\,\mathrm{d}\tau \\
&\quad - \varepsilon\left(\int_0^{\ell_0} h(\sigma)\varepsilon u_t^\varepsilon(t,\sigma)u_x^\varepsilon(t,\sigma)\,\mathrm{d}\sigma - \int_0^{\ell_0} h(\sigma)\varepsilon u_1^\varepsilon(\sigma)\dot{u}_0^\varepsilon(\sigma)\,\mathrm{d}\sigma\right) \\
&\quad - \varepsilon\int_0^{\ell_0} h(\sigma)\left(\frac{w^\varepsilon(t)}{\ell^\varepsilon(t)}\varepsilon u_t^\varepsilon(t,\sigma) - \frac{w^\varepsilon(0)}{\ell_0}\varepsilon u_1^\varepsilon(\sigma)\right)\,\mathrm{d}\sigma \\
&\quad + \varepsilon\int_0^t\int_0^{\ell_0} h(\sigma)\varepsilon u_t^\varepsilon(\tau,\sigma)\frac{\dot{w}^\varepsilon(\tau)\ell^\varepsilon(\tau) - w^\varepsilon(\tau)\dot{\ell}^\varepsilon(\tau)}{\ell^\varepsilon(\tau)^2}\,\mathrm{d}\sigma\,\mathrm{d}\tau.
\end{aligned} \tag{5.5.1}$$

Proof. The proof follows by using exactly the same argument adopted in Lemma 5.4.2, recalling the explicit formula of the affine function r^ε given by (5.4.5). □

Corollary 5.5.4. *Assume (H1) and (K0) and let $\nu > 0$. Then for every $T > 0$ one has:*

$$u_x^\varepsilon(\cdot,0) - r_x^\varepsilon(\cdot,0) \to 0, \quad \text{in } L^2(0,T) \text{ as } \varepsilon \to 0^+.$$

Moreover, considering the subsequence ε_n given by (5.4.1) and Proposition 5.5.1, one gets:

$$u_x^{\varepsilon_n}(\cdot,0) \to -\frac{w}{\ell}, \quad \text{in } L^2(0,T) \text{ as } n \to +\infty, \tag{5.5.2}$$

where w is given by (5.4.1) and ℓ is the function obtained in Proposition 5.5.1.

Proof. We fix $T > 0$ and we simply estimate by using (5.5.1) and recalling that by (H1) the family $\{w^\varepsilon\}_{\varepsilon>0}$ is uniformly equibounded in $[0,T]$:

$$\begin{aligned}
&\int_0^T \big(u_x^\varepsilon(\tau,0) - r_x^\varepsilon(\tau,0)\big)^2\,\mathrm{d}\tau \\
&\le C_T\left[\int_0^T (\mathcal{K}^\varepsilon(\tau) + \widetilde{\mathcal{E}}^\varepsilon(\tau))\,\mathrm{d}\tau + \varepsilon\left(\mathcal{E}^\varepsilon(t) + \mathcal{E}^\varepsilon(0) + \int_0^T \dot{w}^\varepsilon(\tau)^2\,\mathrm{d}\tau + 1 + \int_0^T \varepsilon\dot{\ell}^\varepsilon(\tau)\int_0^{\ell^\varepsilon(\tau)} |u_t^\varepsilon(\tau,\sigma)|\mathrm{d}\sigma\,\mathrm{d}\tau\right)\right].
\end{aligned}$$

By Hölder's inequality and since $\varepsilon\dot{\ell}^\varepsilon(t) < 1$ almost everywhere we then deduce:

$$\int_0^T \varepsilon\dot{\ell}^\varepsilon(\tau)\int_0^{\ell^\varepsilon(\tau)} |u_t^\varepsilon(\tau,\sigma)|\,\mathrm{d}\sigma\,\mathrm{d}\tau \le \sqrt{TL_T}\left(\int_0^T\int_0^{\ell^\varepsilon(\tau)} u_t^\varepsilon(\tau,\sigma)^2\,\mathrm{d}\sigma\,\mathrm{d}\tau\right)^{\frac{1}{2}} = \sqrt{\frac{TL_T}{\varepsilon\nu}}\mathcal{V}^\varepsilon(T)^{\frac{1}{2}}.$$

By means of Proposition 5.4.3 we hence obtain:

$$\int_0^T \left(u_x^\varepsilon(\tau,0)-r_x^\varepsilon(\tau,0)\right)^2 \mathrm{d}\tau \le C_T\left[\int_0^T \left(\mathcal{K}^\varepsilon(\tau)+\widetilde{\mathcal{E}}^\varepsilon(\tau)\right)\mathrm{d}\tau+\varepsilon\left(\|\dot{w}^\varepsilon\|^2_{L^2(0,T)}+1\right)+\sqrt{\varepsilon}\right].$$

We conclude by using (H1) and Proposition 5.5.2.

The proof of (5.5.2) trivially follows by triangular inequality, recalling that by (5.4.5) we know that $r^\varepsilon{}_x(t,0)=-\frac{w^\varepsilon(t)}{\ell^\varepsilon(t)}$ for every $t\in[0,+\infty)$. □

We are now in a position to state our first result about the convergence of u^ε to the proper affine function.

Theorem 5.5.5. *Assume (H1), (K0), $\nu>0$ and let ε_n be the subsequence given by (5.4.1), Propositions 5.5.1 and 5.5.2. Let ℓ be the nondecreasing function obtained in Proposition 5.5.1. Then as $n\to+\infty$ one has:*

- $\varepsilon_n u_t^{\varepsilon_n}(t,\cdot)\to 0$ *strongly in* $L^2(0,+\infty)$, *for every* $t\in(0,+\infty)\setminus J_\ell$,
- $u^{\varepsilon_n}(t,\cdot)\to u(t,\cdot)$ *strongly in* $H^1(0,+\infty)$, *for every* $t\in(0,+\infty)\setminus J_\ell$,

where J_ℓ is the jump set of ℓ and:

$$u(t,x):=w(t)\left(1-\frac{x}{\ell(t)}\right)\mathbb{1}_{[0,\ell(t)]}(x),\quad \textit{for } (t,x)\in[0,+\infty)\times[0,+\infty),$$

with w given by (5.4.1).

Proof. By (5.4.1) and by Proposition 5.5.1 it is easy to see that for every $t\in[0,+\infty)$ one has $r^{\varepsilon_n}(t,\cdot)\to u(t,\cdot)$ strongly in $H^1(0,+\infty)$ as $n\to+\infty$, thus we deduce:

$$\begin{aligned}
&\|\varepsilon_n u_t^{\varepsilon_n}(t,\cdot)\|^2_{L^2(0,+\infty)}+\|u^{\varepsilon_n}(t,\cdot)-u(t,\cdot)\|^2_{H^1(0,+\infty)}\\
&\le C\left(\|\varepsilon_n u_t^{\varepsilon_n}(t,\cdot)\|^2_{L^2(0,+\infty)}+\|u^{\varepsilon_n}(t,\cdot)-r^{\varepsilon_n}(t,\cdot)\|^2_{H^1(0,+\infty)}+\|r^{\varepsilon_n}(t,\cdot)-u(t,\cdot)\|^2_{H^1(0,+\infty)}\right)\\
&\le C\left(\|\varepsilon_n u_t^{\varepsilon_n}(t,\cdot)\|^2_{L^2(0,+\infty)}+\|u_x^{\varepsilon_n}(t,\cdot)-r_x^{\varepsilon_n}(t,\cdot)\|^2_{L^2(0,+\infty)}+\|r^{\varepsilon_n}(t,\cdot)-u(t,\cdot)\|^2_{H^1(0,+\infty)}\right)\\
&= C\left(\mathcal{K}^{\varepsilon_n}(t)+\widetilde{\mathcal{E}}^{\varepsilon_n}(t)+\|r^{\varepsilon_n}(t,\cdot)-u(t,\cdot)\|^2_{H^1(0,+\infty)}\right),
\end{aligned}$$

where we used Poincarè inequality.

To conclude it is enough to show that $\lim\limits_{n\to+\infty}\left(\mathcal{K}^{\varepsilon_n}(t)+\widetilde{\mathcal{E}}^{\varepsilon_n}(t)\right)=0$ for every $t\in(0,+\infty)\setminus J_\ell$. By (5.4.1) and (5.4.9) this is equivalent to prove that:

$$\lim_{n\to+\infty}\left(\mathcal{K}^{\varepsilon_n}(t)+\mathcal{E}^{\varepsilon_n}(t)\right)=\frac{1}{2}\frac{w(t)^2}{\ell(t)},\quad \text{for every } t\in(0,+\infty)\setminus J_\ell. \tag{5.5.3}$$

By Proposition 5.5.2 we know that (5.5.3) holds true for a.e. $t\in[0,+\infty)$. To improve the result we then fix $t\in(0,+\infty)\setminus J_\ell$ and we consider two sequences $\{s_j\}_{j\in\mathbb{N}}$ and $\{t_j\}_{j\in\mathbb{N}}$ such that $0<s_j\le t\le t_j$, the limit in (5.5.3) holds true for s_j and t_j for every $j\in\mathbb{N}$ and $s_j\nearrow t$, $t_j\searrow t$ as $j\to+\infty$. By the energy-dissipation balance (5.2.2a) we hence get:

$$\begin{aligned}
\mathcal{K}^{\varepsilon_n}(t_j)+\mathcal{E}^{\varepsilon_n}(t_j)+\int_t^{t_j}\dot{w}^{\varepsilon_n}(\tau)u_x^{\varepsilon_n}(\tau,0)\,\mathrm{d}\tau&\le\mathcal{K}^{\varepsilon_n}(t)+\mathcal{E}^{\varepsilon_n}(t)\\
\le\mathcal{K}^{\varepsilon_n}(s_j)+\mathcal{E}^{\varepsilon_n}(s_j)+\int_t^{s_j}\dot{w}^{\varepsilon_n}(\tau)u_x^{\varepsilon_n}(\tau,0)\,\mathrm{d}\tau.&
\end{aligned}$$

Passing to the limit as $n \to +\infty$ and exploiting Corollary 5.5.4 together with (5.4.1) we deduce:

$$\frac{1}{2}\frac{w(t_j)^2}{\ell(t_j)} - \int_t^{t_j} \dot{w}(\tau)\frac{w(\tau)}{\ell(\tau)}\,\mathrm{d}\tau \leq \liminf_{n\to+\infty} (\mathcal{K}^{\varepsilon_n}(t) + \mathcal{E}^{\varepsilon_n}(t))$$
$$\leq \limsup_{n\to+\infty} (\mathcal{K}^{\varepsilon_n}(t) + \mathcal{E}^{\varepsilon_n}(t)) \leq \frac{1}{2}\frac{w(s_j)^2}{\ell(s_j)} - \int_t^{s_j} \dot{w}(\tau)\frac{w(\tau)}{\ell(\tau)}\,\mathrm{d}\tau.$$

Passing now to the limit as $j \to +\infty$, recalling that t is a continuity point of ℓ, we finally obtain:

$$\frac{1}{2}\frac{w(t)^2}{\ell(t)} \leq \liminf_{n\to+\infty} (\mathcal{K}^{\varepsilon_n}(t) + \mathcal{E}^{\varepsilon_n}(t)) \leq \limsup_{n\to+\infty} (\mathcal{K}^{\varepsilon_n}(t) + \mathcal{E}^{\varepsilon_n}(t)) \leq \frac{1}{2}\frac{w(t)^2}{\ell(t)},$$

and so we conclude. □

We want to highlight that the viscous term in the wave equation forces the kinetic energy to vanish when $\varepsilon \to 0^+$. Indeed this phenomenon does not happen in [52], where on the contrary the presence of a persistent kinetic energy due to lack of viscosity is the main reason why the convergence of u^ε to an affine function occurs only in a weak sense (see Theorem 3.5 in [52]) and the limit pair (u, ℓ) fails to be a quasistatic evolution.

5.5.2 Characterisation of the limit debonding front

Our aim now is to understand if the limit function ℓ solves quasistatic Griffith's criterion. We thus need to pass to the limit in the dynamic Griffith's criterion (5.2.3). Next proposition deals with the stability condition.

Proposition 5.5.6. *Assume (H1), (K0), $\nu > 0$ and let ℓ be the nondecreasing function obtained in Proposition 5.5.1. Then for every $0 \leq s \leq t$ one has:*

$$\frac{1}{2}\int_s^t \frac{w(\tau)^2}{\ell(\tau)^2}\,\mathrm{d}\tau \leq \int_s^t \kappa(\ell(\tau))\,\mathrm{d}\tau,$$

where w is given by (5.4.1).
In particular the following inequalities hold true:

$$\frac{1}{2}\frac{w(t)^2}{\ell^+(t)^2} \leq \kappa(\ell^+(t)), \quad \textit{for every } t \in [0, +\infty), \tag{5.5.4a}$$

$$\frac{1}{2}\frac{w(t)^2}{\ell^-(t)^2} \leq \kappa(\ell^-(t)), \quad \textit{for every } t \in (0, +\infty), \tag{5.5.4b}$$

where ℓ^+ and ℓ^- are the right limit and the left limit of ℓ, respectively.

Proof. Let ε_n be the subsequence given by (5.4.1) and Proposition 5.5.1. By (5.2.12) we know that for a.e. $t \in [0, +\infty)$ one has:

$$G^{\varepsilon_n}_{\varepsilon_n\dot{\ell}^{\varepsilon_n}(t)}(t) = 2\frac{1-\varepsilon_n\dot{\ell}^{\varepsilon_n}(t)}{1+\varepsilon_n\dot{\ell}^{\varepsilon_n}(t)}F^{\varepsilon_n}(t-\varepsilon_n\ell^{\varepsilon_n}(t))^2 = 2\frac{\dot{\varphi}^{\varepsilon_n}(t)}{\dot{\psi}^{\varepsilon_n}(t)}F^{\varepsilon_n}(\varphi^{\varepsilon_n}(t))^2, \tag{5.5.5}$$

where we introduced the function:

$$F^{\varepsilon_n}(\sigma) = \dot{f}^{\varepsilon_n}(\sigma) + \nu g^{\varepsilon_n}[u^{\varepsilon_n}_t](\sigma), \quad \text{for a.e. } \sigma \in (-\varepsilon_n\ell_0, \varphi^{\varepsilon_n}(+\infty)).$$

Here we adopt the notation $\varphi^{\varepsilon_n}(+\infty) = \lim_{t\to+\infty} \varphi^{\varepsilon_n}(t)$, which exists since φ^{ε_n} is strictly increasing. We want also to remark that $\varphi^{\varepsilon_n}(+\infty) > 0$ for n large enough (actually it diverges to $+\infty$ as $n\to+\infty$), indeed φ^{ε_n} converges locally uniformly to the identity map as $n\to+\infty$ by Corollary 5.4.4. By means of (5.2.11a) and of the explicit form of $\dot f^{\varepsilon_n}$ and $g^{\varepsilon_n}[u^{\varepsilon_n}_t]$ in $(-\varepsilon_n\ell_0,0)$ we deduce that:

$$F^{\varepsilon_n}(\sigma)=\begin{cases}\frac12\varepsilon_n\dot w^{\varepsilon_n}(\sigma)-\frac12 u^{\varepsilon_n}_x(\sigma,0)+\nu\Big(g^{\varepsilon_n}[u^{\varepsilon_n}_t](\sigma)-\frac12 H^{\varepsilon_n}[u^{\varepsilon_n}_t]_x(\sigma,0)\Big), \text{if } \sigma\in(0,\varphi^{\varepsilon_n}(+\infty)),\\ \frac12\varepsilon_n u^{\varepsilon_n}_1\left(-\frac{\sigma}{\varepsilon_n}\right)-\frac12\dot u^{\varepsilon_n}_0\left(-\frac{\sigma}{\varepsilon_n}\right)-\frac{\nu}{2}\int_0^{(\varphi^{\varepsilon_n})^{-1}(\sigma)}u^{\varepsilon_n}_t\left(\tau,\frac{\tau-\sigma}{\varepsilon_n}\right)\mathrm{d}\tau, \text{ if } \sigma\in(-\varepsilon_n\ell_0,0).\end{cases}$$

Thus, thanks to (5.2.8), we obtain:

$$F^{\varepsilon_n}(\sigma)=\begin{cases}\frac12\varepsilon_n\dot w^{\varepsilon_n}(\sigma)-\frac12 u^{\varepsilon_n}_x(\sigma,0)-\frac{\nu}{2}\int_\sigma^{(\varphi^{\varepsilon_n})^{-1}(\sigma)}u^{\varepsilon_n}_t\left(\tau,\frac{\tau-\sigma}{\varepsilon_n}\right)\mathrm{d}\tau, \quad \text{if } \sigma\in(0,\varphi^{\varepsilon_n}(+\infty)),\\ \frac12\varepsilon_n u^{\varepsilon_n}_1\left(-\frac{\sigma}{\varepsilon_n}\right)-\frac12\dot u^{\varepsilon_n}_0\left(-\frac{\sigma}{\varepsilon_n}\right)-\frac{\nu}{2}\int_0^{(\varphi^{\varepsilon_n})^{-1}(\sigma)}u^{\varepsilon_n}_t\left(\tau,\frac{\tau-\sigma}{\varepsilon_n}\right)\mathrm{d}\tau, \text{ if } \sigma\in(-\varepsilon_n\ell_0,0).\end{cases}\tag{5.5.6}$$

By the stability condition in dynamic Griffith's criterion (5.2.3) we hence deduce that for every $0\le s\le t$ one has:

$$\begin{aligned}\int_s^t\kappa(\ell^{\varepsilon_n}(\tau))\,\mathrm{d}\tau&\ge\int_s^t G^{\varepsilon_n}_{\varepsilon_n\dot\ell^{\varepsilon_n}(\tau)}(\tau)\,\mathrm{d}\tau=2\int_s^t\frac{\dot\varphi^{\varepsilon_n}(\tau)}{\dot\psi^{\varepsilon_n}(\tau)}F^{\varepsilon_n}(\varphi^{\varepsilon_n}(\tau))^2\,\mathrm{d}\tau\\&=\int_{\varphi^{\varepsilon_n}(s)}^{\varphi^{\varepsilon_n}(t)}\frac{2}{\dot\psi^{\varepsilon_n}((\varphi^{\varepsilon_n})^{-1}(\sigma))}F^{\varepsilon_n}(\sigma)^2\,\mathrm{d}\sigma=:I^{\varepsilon_n}(s,t).\end{aligned}$$

Thus, by dominated convergence we infer:

$$\int_s^t\kappa(\ell(\tau))\,\mathrm{d}\tau\ge\limsup_{n\to+\infty}I^{\varepsilon_n}(s,t).$$

We actually prove that the limit in the right-hand side exists and it holds:

$$\lim_{n\to+\infty}I^{\varepsilon_n}(s,t)=\frac12\int_s^t\frac{w(\tau)^2}{\ell(\tau)^2}\,\mathrm{d}\tau.\tag{5.5.7}$$

If (5.5.7) is true, then we conclude; to prove it we reason as follows. We first assume $s>0$, so that $\varphi^{\varepsilon_n}(s)>0$ (for n large enough) and we can write:

$$I^{\varepsilon_n}(s,t)=\frac12\int_0^t\frac{\mathbb{1}_{[\varphi^{\varepsilon_n}(s),\varphi^{\varepsilon_n}(t)]}(\sigma)}{\dot\psi^{\varepsilon_n}((\varphi^{\varepsilon_n})^{-1}(\sigma))}\Big(2F^{\varepsilon_n}(\sigma)\Big)^2\mathbb{1}_{[0,\varphi^{\varepsilon_n}(t)]}(\sigma)\,\mathrm{d}\sigma.$$

By means of the properties of φ^{ε_n} and ψ^{ε_n}, see (5.1.2) and the subsequent discussion, and recalling Corollary 5.4.4 it is easy to see that the function $a^{\varepsilon_n}(\sigma):=\dfrac{\mathbb{1}_{[\varphi^{\varepsilon_n}(s),\varphi^{\varepsilon_n}(t)]}(\sigma)}{\dot\psi^{\varepsilon_n}((\varphi^{\varepsilon_n})^{-1}(\sigma))}$ satisfies $\|a^{\varepsilon_n}\|_{L^\infty(0,t)}\le1$ and $a^{\varepsilon_n}\to\mathbb{1}_{[s,t]}$ in $L^1(0,t)$ as $n\to+\infty$. So we conclude if we prove that:

$$2F^{\varepsilon_n}\mathbb{1}_{[0,\varphi^{\varepsilon_n}(t)]}\to\frac{w}{\ell},\ \text{ in } L^2(0,t) \text{ as } n\to+\infty,\tag{5.5.8}$$

since the function w/ℓ belongs to $L^\infty(0,t)$. To prove (5.5.8) we estimate:

$$\begin{aligned}\Big\|2F^{\varepsilon_n}\mathbb{1}_{[0,\varphi^{\varepsilon_n}(t)]}-\frac{w}{\ell}\Big\|_{L^2(0,t)}&\le\varepsilon_n\|\dot w^{\varepsilon_n}\|_{L^2(0,t)}+\Big\|u^{\varepsilon_n}_x(\cdot,0)+\frac{w}{\ell}\Big\|_{L^2(0,t)}\\&\quad+\nu\left(\int_0^{\varphi^{\varepsilon_n}(t)}\left(\int_\sigma^{(\varphi^{\varepsilon_n})^{-1}(\sigma)}u^{\varepsilon_n}_t\left(\tau,\frac{\tau-\sigma}{\varepsilon_n}\right)\mathrm{d}\tau\right)^2\mathrm{d}\sigma\right)^{\frac12}+C\varepsilon_n.\end{aligned}$$

By (H1) and (5.5.2) the first and the second term go to zero as $n \to +\infty$. For the third one we continue the estimate:

$$\begin{aligned}
&\int_0^{\varphi^{\varepsilon_n}(t)} \left(\int_\sigma^{(\varphi^{\varepsilon_n})^{-1}(\sigma)} u_t^{\varepsilon_n}\left(\tau, \frac{\tau-\sigma}{\varepsilon_n}\right) \mathrm{d}\tau \right)^2 \mathrm{d}\sigma \\
&\leq \int_0^{\varphi^{\varepsilon_n}(t)} \left((\varphi^{\varepsilon_n})^{-1}(\sigma) - \sigma\right) \int_\sigma^{(\varphi^{\varepsilon_n})^{-1}(\sigma)} u_t^{\varepsilon_n}\left(\tau, \frac{\tau-\sigma}{\varepsilon_n}\right)^2 \mathrm{d}\tau\, \mathrm{d}\sigma \\
&= \int_0^{\varphi^{\varepsilon_n}(t)} \varepsilon_n \ell^{\varepsilon_n}((\varphi^{\varepsilon_n})^{-1}(\sigma)) \int_\sigma^{(\varphi^{\varepsilon_n})^{-1}(\sigma)} u_t^{\varepsilon_n}\left(\tau, \frac{\tau-\sigma}{\varepsilon_n}\right)^2 \mathrm{d}\tau\, \mathrm{d}\sigma \\
&\leq C_t \int_0^{\varphi^{\varepsilon_n}(t)} \int_0^t \varepsilon_n u_t^{\varepsilon_n}\left(\tau, \frac{\tau-\sigma}{\varepsilon_n}\right)^2 \mathbb{1}_{[\sigma,(\varphi^{\varepsilon_n})^{-1}(\sigma)]}(\tau)\, \mathrm{d}\tau\, \mathrm{d}\sigma \qquad (5.5.9)\\
&= C_t \int_0^t \int_0^{\varphi^{\varepsilon_n}(t)} \varepsilon_n u_t^{\varepsilon_n}\left(\tau, \frac{\tau-\sigma}{\varepsilon_n}\right)^2 \mathbb{1}_{[\sigma,(\varphi^{\varepsilon_n})^{-1}(\sigma)]}(\tau)\, \mathrm{d}\sigma\, \mathrm{d}\tau \\
&\leq C_t \int_0^t \int_0^{\ell^{\varepsilon_n}(\tau)} \varepsilon_n^2 u_t^{\varepsilon_n}(\tau,\sigma)^2\, \mathrm{d}\sigma\, \mathrm{d}\tau = \varepsilon_n \frac{C_t}{\nu} \mathcal{V}^{\varepsilon_n}(t),
\end{aligned}$$

which goes to zero by (5.4.3), and we conclude in the case $s > 0$.

If instead $s = 0$ we can write:

$$\begin{aligned}
I^{\varepsilon_n}(0,t) =& \frac{1}{2} \int_{-\varepsilon_n \ell_0}^{0} \frac{1}{\dot\psi^{\varepsilon_n}((\varphi^{\varepsilon_n})^{-1}(\sigma))} \left[\varepsilon_n u_1^{\varepsilon_n}\left(-\frac{\sigma}{\varepsilon_n}\right) - \dot u_0^{\varepsilon_n}\left(-\frac{\sigma}{\varepsilon_n}\right) - \nu \int_0^{(\varphi^{\varepsilon_n})^{-1}(\sigma)} u_t^{\varepsilon_n}\left(\tau, \frac{\tau-\sigma}{\varepsilon_n}\right) \mathrm{d}\tau \right]^2 \mathrm{d}\sigma \\
&+ \frac{1}{2} \int_0^t \frac{1}{\dot\psi^{\varepsilon_n}((\varphi^{\varepsilon_n})^{-1}(\sigma))} \left(2F^{\varepsilon_n}(\sigma)\right)^2 \mathbb{1}_{[0,\varphi^{\varepsilon_n}(t)]}(\sigma)\, \mathrm{d}\sigma.
\end{aligned}$$

Reasoning as before one can show that the second term goes to $\displaystyle\frac{1}{2}\int_0^t \frac{w(\tau)^2}{\ell(\tau)^2}\mathrm{d}\tau$ as $n \to +\infty$, so we conclude if we prove that the first one, denoted by J^{ε_n}, vanishes in the limit. To this aim we estimate:

$$\begin{aligned}
J^{\varepsilon_n} &\leq C \int_{-\varepsilon_n \ell_0}^0 \left[\varepsilon_n^2 u_1^{\varepsilon_n}\left(-\frac{\sigma}{\varepsilon_n}\right)^2 + \dot u_0^{\varepsilon_n}\left(-\frac{\sigma}{\varepsilon_n}\right)^2 + \nu\varepsilon_n \int_0^{(\varphi^{\varepsilon_n})^{-1}(\sigma)} u_t^{\varepsilon_n}\left(\tau, \frac{\tau-\sigma}{\varepsilon_n}\right)^2 \mathrm{d}\tau \right] \mathrm{d}\sigma \\
&\leq \varepsilon_n C \left(\|\varepsilon_n u_1^{\varepsilon_n}\|^2_{L^2(0,\ell_0)} + \|\dot u_0^{\varepsilon_n}\|^2_{L^2(0,\ell_0)} + \mathcal{V}^{\varepsilon_n}((\varphi^{\varepsilon_n})^{-1}(0)) \right).
\end{aligned}$$

We thus conclude by means of (H1) and (5.4.3), since $(\varphi^{\varepsilon_n})^{-1}(0)$ is uniformly bounded with respect to ε_n thanks to Corollary 5.4.4. □

Now we pass to the limit in the energy-dissipation balance (5.2.2a).

Proposition 5.5.7. *Assume (H1), (K0), $\nu > 0$ and let w and ℓ be given by (5.4.1) and Proposition 5.5.1, respectively. Then there exists a positive measure μ_D on $[0, +\infty)$ for which the following equality holds true for every $t \in [0, +\infty)$:*

$$\frac{1}{2}\frac{w(t)^2}{\ell^+(t)} + \int_{\ell_0}^{\ell^+(t)} \kappa(\sigma)\mathrm{d}\sigma + \mu_D([0,t]) = \liminf_{n\to+\infty} \left(\frac{1}{2}\int_0^{\ell_0} \varepsilon_n^2 u_1^{\varepsilon_n}(\sigma)^2 \mathrm{d}\sigma + \frac{1}{2}\int_0^{\ell_0} \dot u_0^{\varepsilon_n}(\sigma)^2 \mathrm{d}\sigma \right) + \int_0^t \dot w(\tau) \frac{w(\tau)}{\ell(\tau)} \mathrm{d}\tau, \tag{5.5.10}$$

where ε_n is the subsequence given by (5.4.1) and by Propositions 5.5.1 and 5.5.2. Moreover for every $0 < s \leq t$ one has:

$$\frac{1}{2}\frac{w(t)^2}{\ell^+(t)} + \int_{\ell_0}^{\ell^+(t)} \kappa(\sigma)\, \mathrm{d}\sigma + \mu_D([s,t]) = \frac{1}{2}\frac{w(s)^2}{\ell^-(s)} + \int_{\ell_0}^{\ell^-(s)} \kappa(\sigma)\, \mathrm{d}\sigma + \int_s^t \dot w(\tau) \frac{w(\tau)}{\ell(\tau)} \mathrm{d}\tau. \tag{5.5.11}$$

Proof. By classical properties of BV functions in one variable (see for instance [8], Theorem 3.28) it is enough to prove that the function $\rho\colon(-\delta,+\infty)\to\mathbb{R}$ defined as:

$$\rho(t):=\begin{cases}\liminf\limits_{n\to+\infty}\left(\frac12\int_0^{\ell_0}\varepsilon_n^2u_1^{\varepsilon_n}(\sigma)^2\,\mathrm{d}\sigma+\frac12\int_0^{\ell_0}\dot u_0^{\varepsilon_n}(\sigma)^2\,\mathrm{d}\sigma\right), & \text{if } t\in(-\delta,0],\\ \frac12\frac{w(t)^2}{\ell(t)}+\int_{\ell_0}^{\ell(t)}\kappa(\sigma)\,\mathrm{d}\sigma-\int_0^t\dot w(\tau)\frac{w(\tau)}{\ell(\tau)}\,\mathrm{d}\tau, & \text{if } t\in(0,+\infty),\end{cases}$$

belongs to the Lebesgue class of a nonincreasing function. Indeed in that case $\mu_D:=-D\rho$ does the job.

We actually prove that the right limit ρ^+ is nonincreasing. We fix $s,t\in(-\delta,+\infty)$ such that $s<t$ and we consider all the possible cases.

If $s\geq 0$ we pick two sequences $\{s_j\}_{j\in\mathbb{N}}$, $\{t_j\}_{j\in\mathbb{N}}$ such that for every $j\in\mathbb{N}$ one has $s<s_j<t<t_j$, s_j and t_j do not belong to the jump set of ℓ, and $s_j\searrow s$, $t_j\searrow t$ as $j\to+\infty$. By the energy-dissipation balance (5.2.2a) we hence get:

$$\begin{aligned}&\mathcal{K}^{\varepsilon_n}(t_j)+\mathcal{E}^{\varepsilon_n}(t_j)+\int_{\ell_0}^{\ell^{\varepsilon_n}(t_j)}\kappa(\sigma)\,\mathrm{d}\sigma+\int_0^{t_j}\dot w^{\varepsilon_n}(\tau)u_x^{\varepsilon_n}(\tau,0)\,\mathrm{d}\tau\\ &\leq\mathcal{K}^{\varepsilon_n}(s_j)+\mathcal{E}^{\varepsilon_n}(s_j)+\int_{\ell_0}^{\ell^{\varepsilon_n}(s_j)}\kappa(\sigma)\,\mathrm{d}\sigma+\int_0^{s_j}\dot w^{\varepsilon_n}(\tau)u_x^{\varepsilon_n}(\tau,0)\,\mathrm{d}\tau.\end{aligned}$$

Passing to the limit as $n\to+\infty$, by Theorem 5.5.5 and by exploiting Corollary 5.5.4 together with (5.4.1) we deduce:

$$\frac12\frac{w(t_j)^2}{\ell(t_j)}+\int_{\ell_0}^{\ell(t_j)}\kappa(\sigma)\,\mathrm{d}\sigma-\int_0^{t_j}\dot w(\tau)\frac{w(\tau)}{\ell(\tau)}\,\mathrm{d}\tau\leq\frac12\frac{w(s_j)^2}{\ell(s_j)}+\int_{\ell_0}^{\ell(s_j)}\kappa(\sigma)\,\mathrm{d}\sigma-\int_0^{s_j}\dot w(\tau)\frac{w(\tau)}{\ell(\tau)}\,\mathrm{d}\tau.$$

Passing now to the limit as $j\to+\infty$ we get $\rho^+(t)\leq\rho^+(s)$.

If $s\in(-\delta,0)$ and $t\geq 0$ we consider a sequence $\{t_j\}_{j\in\mathbb{N}}$ as before and by means of the energy-dissipation balance we infer:

$$\begin{aligned}&\mathcal{K}^{\varepsilon_n}(t_j)+\mathcal{E}^{\varepsilon_n}(t_j)+\int_{\ell_0}^{\ell^{\varepsilon_n}(t_j)}\kappa(\sigma)\,\mathrm{d}\sigma+\int_0^{t_j}\dot w^{\varepsilon_n}(\tau)u_x^{\varepsilon_n}(\tau,0)\,\mathrm{d}\tau\\ &\leq\mathcal{K}^{\varepsilon_n}(0)+\mathcal{E}^{\varepsilon_n}(0)=\frac12\int_0^{\ell_0}\varepsilon_n^2u_1^{\varepsilon_n}(\sigma)^2\,\mathrm{d}\sigma+\frac12\int_0^{\ell_0}\dot u_0^{\varepsilon_n}(\sigma)^2\,\mathrm{d}\sigma.\end{aligned}$$

Passing to the limit as $n\to+\infty$ and then $j\to+\infty$ we hence deduce that also in this case $\rho^+(t)\leq\rho^+(s)$.

If finally both s and t belong to $(-\delta,0)$, then trivially $\rho^+(t)=\rho^+(s)$ and so we conclude. □

The measure μ_D introduced in the previous proposition somehow represents the amount of energy dissipated by viscosity which still is present in the limit. Indeed it can be seen as a weak*-limit of $\mathcal{V}^\varepsilon$ as $\varepsilon\to0^+$. The rise of such a limit measure occurs also in [79] in a model of contact between two visco-elastic bodies. Of course, to obtain the desired quasistatic energy-dissipation balance (eb) we need to prove that $\mu_D\equiv0$, namely that $\mathcal{V}^\varepsilon$ vanishes as $\varepsilon\to0^+$. To this aim we assume (K1) and we exploit the following lemma:

Lemma 5.5.8. *Assume (H1), (K0), (K1), $\nu>0$ and let w and ℓ be given by (5.4.1) and Proposition 5.5.1, respectively. Then for every $0\leq s\leq t$ the following inequality holds true:*

$$\frac12\frac{w(t)^2}{\ell^+(t)}+\int_{\ell_0}^{\ell^+(t)}\kappa(\sigma)\,\mathrm{d}\sigma-\int_0^t\dot w(\tau)\frac{w(\tau)}{\ell(\tau)}\,\mathrm{d}\tau\geq\frac12\frac{w(s)^2}{\ell^+(s)}+\int_{\ell_0}^{\ell^+(s)}\kappa(\sigma)\,\mathrm{d}\sigma-\int_0^s\dot w(\tau)\frac{w(\tau)}{\ell(\tau)}\,\mathrm{d}\tau.\quad(5.5.12)$$

Furthermore for every $0 < s \le t$ we have:

$$\frac{1}{2}\frac{w(t)^2}{\ell^-(t)}+\int_{\ell_0}^{\ell^-(t)}\kappa(\sigma)\,\mathrm{d}\sigma-\int_0^t\dot w(\tau)\frac{w(\tau)}{\ell(\tau)}\,\mathrm{d}\tau\ge\frac{1}{2}\frac{w(s)^2}{\ell^-(s)}+\int_{\ell_0}^{\ell^-(s)}\kappa(\sigma)\,\mathrm{d}\sigma-\int_0^s\dot w(\tau)\frac{w(\tau)}{\ell(\tau)}\,\mathrm{d}\tau. \quad (5.5.13)$$

If in addition $\frac{1}{2}\frac{w(0)^2}{\ell_0^2}\le\kappa(\ell_0)$, then (5.5.13) holds true even in $s=0$.

Proof. If $s=t$ (5.5.12) is trivially satisfied, so let us fix $0\le s<t$ and consider a sequence of partitions of $[s,t]$ of the form $s=t_0^j<t_1^j<\dots<t_{k(j)}^j=t$ such that:

(1) $\lim\limits_{j\to+\infty}\max\limits_{k=1,\dots,k(j)}\left|t_k^j-t_{k-1}^j\right|=0.$

It is worth noticing that by (1) and the absolute continuity of the integral we can assume without loss of generality that:

(2) for every $j\in\mathbb{N}$ it holds $\displaystyle\int_{t_{k-1}^j}^{t_k^j}|\dot w(\tau)w(\tau)|\,\mathrm{d}\tau\le\frac{1}{j}$ for every $k=1,\dots,k(j)$.

Fix one of these partitions and, since by (5.5.4a) we know that $\ell^+(t)$ satisfies (s2) for every $t\in[0,+\infty)$, arguing as in the proof of Proposition 5.3.4 and exploiting (K1) we deduce that for every $k=1,\dots,k(j)$ we have:

$$\frac{1}{2}\frac{w(t_{k-1}^j)^2}{\ell^+(t_{k-1}^j)}+\int_{\ell_0}^{\ell^+(t_{k-1}^j)}\kappa(\sigma)\,\mathrm{d}\sigma\le\frac{1}{2}\frac{w(t_{k-1}^j)^2}{\ell^+(t_k^j)}+\int_{\ell_0}^{\ell^+(t_k^j)}\kappa(\sigma)\,\mathrm{d}\sigma,$$

and thus we obtain:

$$\begin{aligned}&\frac{1}{2}\frac{w(t_k^j)^2}{\ell^+(t_k^j)}-\frac{1}{2}\frac{w(t_{k-1}^j)^2}{\ell^+(t_{k-1}^j)}+\int_{\ell^+(t_{k-1}^j)}^{\ell^+(t_k^j)}\kappa(\sigma)\,\mathrm{d}\sigma\\ \ge&\ \frac{1}{2}\frac{w(t_k^j)^2}{\ell^+(t_k^j)}-\frac{1}{2}\frac{w(t_{k-1}^j)^2}{\ell^+(t_k^j)}=\int_{t_{k-1}^j}^{t_k^j}\dot w(\tau)\frac{w(\tau)}{\ell^+(t_k^j)}\,\mathrm{d}\tau.\end{aligned}$$

By summing the above inequality from $k=1$ to $k(j)$ we get:

$$\frac{1}{2}\frac{w(t)^2}{\ell^+(t)}-\frac{1}{2}\frac{w(s)^2}{\ell^+(s)}+\int_{\ell^+(s)}^{\ell^+(t)}\kappa(\sigma)\,\mathrm{d}\sigma\ge\sum_{k=1}^{k(j)}\int_{t_{k-1}^j}^{t_k^j}\dot w(\tau)\frac{w(\tau)}{\ell^+(t_k^j)}\,\mathrm{d}\tau=:I_j.$$

We prove (5.5.12) if we show that $\lim\limits_{j\to+\infty}I_j=\displaystyle\int_s^t\dot w(\tau)\frac{w(\tau)}{\ell(\tau)}\,\mathrm{d}\tau$. For this aim we split it into two parts:

$$I_j=\int_s^t\dot w(\tau)\frac{w(\tau)}{\ell(\tau)}\,\mathrm{d}\tau+\sum_{k=1}^{k(j)}\int_{t_{k-1}^j}^{t_k^j}\dot w(\tau)w(\tau)\left(\frac{1}{\ell^+(t_k^j)}-\frac{1}{\ell^+(\tau)}\right)\mathrm{d}\tau,$$

and thus we conclude if we show that the above sum, denoted by II_j, vanishes as $j\to+\infty$. So we estimate by using (2):

$$\begin{aligned}|II_j|&\le\sum_{k=1}^{k(j)}\int_{t_{k-1}^j}^{t_k^j}|\dot w(\tau)w(\tau)|\frac{\ell^+(t_k^j)-\ell^+(\tau)}{\ell^+(t_k^j)\ell^+(\tau)}\,\mathrm{d}\tau\le\sum_{k=1}^{k(j)}\frac{\ell^+(t_k^j)-\ell^+(t_{k-1}^j)}{\ell_0^2}\int_{t_{k-1}^j}^{t_k^j}|\dot w(\tau)w(\tau)|\,\mathrm{d}\tau\\&\le\frac{1}{j}\sum_{k=1}^{k(j)}\frac{\ell^+(t_k^j)-\ell^+(t_{k-1}^j)}{\ell_0^2}=\frac{1}{j}\frac{\ell^+(t)-\ell^+(s)}{\ell_0^2},\end{aligned}$$

and hence (5.5.12) is proved.

Recalling that (5.5.4b) holds true only for $t\in(0,+\infty)$ and reasoning in the same way one gets (5.5.13). If in addition $\frac{1}{2}\frac{w(0)^2}{\ell_0^2}\le\kappa(\ell_0)$, then (5.5.4b) is also valid in $t=0$, and again with the same argument one obtains (5.5.13) even in $s=0$. □

As a simple byproduct we obtain the following proposition, which states that assuming (K1) the measure μ_D appearing in Proposition 5.5.7 is identically zero in $(0,+\infty)$:

Proposition 5.5.9. *Assume (H1), (K0), (K1), $\nu>0$ and let w and ℓ be given by (5.4.1) and Proposition 5.5.1, respectively. Then for every $0<s\le t$ the following energy balance holds true:*

$$\frac{1}{2}\frac{w(t)^2}{\ell^+(t)}+\int_{\ell_0}^{\ell^+(t)}\kappa(\sigma)\,\mathrm{d}\sigma=\frac{1}{2}\frac{w(s)^2}{\ell^-(s)}+\int_{\ell_0}^{\ell^-(s)}\kappa(\sigma)\,\mathrm{d}\sigma+\int_s^t\dot w(\tau)\frac{w(\tau)}{\ell(\tau)}\,\mathrm{d}\tau. \tag{5.5.14}$$

If in addition $\frac{1}{2}\frac{w(0)^2}{\ell_0^2}\le\kappa(\ell_0)$, then for every $t\in[0,+\infty)$ we also have:

$$\frac{1}{2}\frac{w(0)^2}{\ell_0}\le\frac{1}{2}\frac{w(t)^2}{\ell^+(t)}+\int_{\ell_0}^{\ell^+(t)}\kappa(\sigma)\mathrm{d}\sigma-\int_0^t\dot w(\tau)\frac{w(\tau)}{\ell(\tau)}\mathrm{d}\tau\le\liminf_{n\to+\infty}\left(\frac{1}{2}\int_0^{\ell_0}\varepsilon_n^2 u_1^{\varepsilon_n}(\sigma)^2\mathrm{d}\sigma+\frac{1}{2}\int_0^{\ell_0}\dot u_0^{\varepsilon_n}(\sigma)^2\mathrm{d}\sigma\right), \tag{5.5.15}$$

where ε_n is the subsequence given by (5.4.1) and by Propositions 5.5.1 and 5.5.2.

Proof. Let us fix $0<s\le t$. By (5.5.11) we know that the left-hand side of (5.5.14) is less or equal than the right-hand side. Let us now consider a sequence $\{s_j\}_{j\in\mathbb{N}}$ such that $0<s_j<s$ for every $j\in\mathbb{N}$ and $s_j\nearrow s$ as $j\to+\infty$. By means of (5.5.12) we deduce that

$$\frac{1}{2}\frac{w(t)^2}{\ell^+(t)}+\int_{\ell_0}^{\ell^+(t)}\kappa(\sigma)\,\mathrm{d}\sigma\ge\frac{1}{2}\frac{w(s_j)^2}{\ell^+(s_j)}+\int_{\ell_0}^{\ell^+(s_j)}\kappa(\sigma)\,\mathrm{d}\sigma+\int_{s_j}^t\dot w(\tau)\frac{w(\tau)}{\ell(\tau)}\,\mathrm{d}\tau,$$

and letting $j\to+\infty$ we prove the other inequality, thus (5.5.14) holds true.

If we assume $\frac{1}{2}\frac{w(0)^2}{\ell_0^2}\le\kappa(\ell_0)$, reasoning in a similar way by using (5.5.10) and (5.5.13) we also prove (5.5.15) and we conclude. □

Previous proposition ensures that, assuming (K1), the measure μ_D introduced in Proposition 5.5.7 is concentrated on the singleton $\{0\}$. This means that viscosity dissipates all the initial energy at the initial time $t=0$. Unfortunately, the fact that μ_D is concentrated on $t=0$ gives us no informations about the limit debonding front. Let us indeed consider the following example: we take $w^\varepsilon(t)\equiv w>0$, $\kappa(x)=\frac{1}{2}\frac{w^2}{x^2}$ if $x\in[\ell_0,L]$, where $L>>\ell_0$, and $\kappa(x)=\frac{1}{2}\frac{w^2}{L^2}$ if $x\ge L$. Moreover we pick $u_1^\varepsilon\equiv 0$ and $u_0^\varepsilon(x)=w\left(1-\frac{x}{\ell_0}\right)$. Then any nondecreasing function ℓ for which $t^*:=\inf\{t>0\mid \ell^+(t)\ge L\}$ is positive satisfies (5.5.4) with equality for every $t\in[0,t^*)$; furthermore in this case (5.5.14) and (5.5.15) are trivially satisfied in $[0,t^*)$.

To overcome this problem and to give a characterisation of the limit debonding front ℓ we are forced to strenghten the assumptions on the toughness κ. As we did in Section 5.3 to show equivalence between energetic and AC-quasistatic evolutions, we first prove that ℓ is a continuous function; this is, however, a crucial step for getting (eb) from (5.5.14).

Corollary 5.5.10. *Assume (H1), (K0), (K2) and let ν be positive. Then the nondecreasing function ℓ given by Proposition 5.5.1 is continuous in $(0,+\infty)$.*

Proof. The result follows arguing as in the proof of Lemma 5.3.5 by means of (5.5.4b) and (5.5.14). □

Proposition 5.5.11. *Assume (H1), (K0), (K2) and let ν be positive. Then the following energy-dissipation balance holds true for the nondecreasing function ℓ obtained in Proposition 5.5.1:*

$$\frac{1}{2}\frac{w(t)^2}{\ell(t)}+\int_{\ell^+(0)}^{\ell(t)}\kappa(\sigma)\,\mathrm{d}\sigma=\frac{1}{2}\frac{w(0)^2}{\ell^+(0)}+\int_0^t\dot w(\tau)\frac{w(\tau)}{\ell(\tau)}\,\mathrm{d}\tau,\quad \textit{for every } t\in(0,+\infty),\tag{5.5.16}$$

where w is given by (5.4.1).

Proof. By Corollary 5.5.10 we know ℓ is continuous on $(0,+\infty)$, so (5.5.16) follows from (5.5.14). □

Up to now we have thus proved that, under suitable assumptions, the limit pair (u,ℓ) is a quasistatic evolution starting from the point $\ell^+(0)$. The aim of the next subsection will be characterise the value $\ell^+(0)$.

5.5.3 The initial jump

In this subsection we show that the (possible) initial jump of the limit debonding front ℓ is characterised by the equality $\ell^+(0)=\lim\limits_{t\to+\infty}\tilde\ell(t)$, where $\tilde\ell$ is the debonding front related to the unscaled dynamical coupled problem:

$$\begin{cases}\tilde u_{tt}(t,x)-\tilde u_{xx}(t,x)+\nu\tilde u_t(t,x)=0, & t>0\,,\,0<x<\tilde\ell(t),\\ \tilde u(t,0)=w(0), & t>0,\\ \tilde u(t,\tilde\ell(t))=0, & t>0,\\ \tilde u(0,x)=u_0(x), & 0<x<\ell_0,\\ \tilde u_t(0,x)=0, & 0<x<\ell_0,\end{cases}\tag{5.5.17}$$

$$\begin{cases}0\le\dot{\tilde\ell}(t)<1,\\ G_{\dot{\tilde\ell}(t)}(t)\le\kappa(\tilde\ell(t)),\\ \Big[G_{\dot{\tilde\ell}(t)}(t)-\kappa(\tilde\ell(t))\Big]\dot{\tilde\ell}(t)=0,\end{cases}\qquad\text{for a.e. } t\in[0,+\infty).\tag{5.5.18}$$

Here we are assuming that $u_0\in H^1(0,\ell_0)$ satisfies $u_0(0)=w(0)$ and $u_0(\ell_0)=0$. Moreover, as before, we consider $\nu>0$ and a positive toughness κ which belongs to $\widetilde C^{0,1}([\ell_0,+\infty))$. We also need to introduce stronger conditions than (H1):

(H2) the family $\{w^\varepsilon\}_{\varepsilon>0}$ is bounded in $\widetilde H^1(0,+\infty)$, $u_0^\varepsilon\to u_0$ strongly in $H^1(0,\ell_0)$, $\varepsilon u_1^\varepsilon\to 0$ strongly in $L^2(0,\ell_0)$ as $\varepsilon\to0^+$.

(H3) $w^\varepsilon\rightharpoonup w$ weakly in $\widetilde H^1(0,+\infty)$, $u_0^\varepsilon\to u_0$ strongly in $H^1(0,\ell_0)$, $\varepsilon u_1^\varepsilon\to0$ strongly in $L^2(0,\ell_0)$ as $\varepsilon\to0^+$.

Remark 5.5.12. Assuming (H3), by the compact embedding of $H^1(0,T)$ in $C^0([0,T])$ we deduce that for every $T>0$ we have $w^\varepsilon\to w$ uniformly in $[0,T]$ as $\varepsilon\to0^+$.

Remark 5.5.13. As explained in Section 5.2 the pair $(\tilde u,\tilde\ell)$ solution of (5.5.17)&(5.5.18) fulfils the energy-dissipation balance:

$$\mathcal K(t)+\mathcal E(t)+\mathcal V(t)+\int_{\ell_0}^{\tilde\ell(t)}\kappa(\sigma)\,\mathrm{d}\sigma=\frac12\int_0^{\ell_0}\dot u_0(\sigma)^2\,\mathrm{d}\sigma,\quad\text{for every } t\in[0,+\infty),\tag{5.5.19}$$

where $\mathcal K$, $\mathcal E$ and $\mathcal V$ are as in (5.2.1a), (5.2.1b) and (5.2.1c) with $\varepsilon=1$ and $\tilde u$, $\tilde\ell$ in place of u^ε and ℓ^ε.

We want to notice that, assuming (H2) and considering the subsequence ε_n given by Remark 5.4.1, one can apply Theorem 5.2.5 deducing that actually the pair $(\tilde{u}, \tilde{\ell})$ is the limit as $n \to +\infty$ (in the sense of Theorem 5.2.5) of $(u_{\varepsilon_n}, \ell_{\varepsilon_n})$, where this last pair is the dynamic evolution related to the unscaled problem

$$\begin{cases} (u_\varepsilon)_{tt}(t,x) - (u_\varepsilon)_{xx}(t,x) + \nu (u_\varepsilon)_t(t,x) = 0, & t > 0\,,\, 0 < x < \ell_\varepsilon(t), \\ u_\varepsilon(t,0) = w^\varepsilon(\varepsilon t), & t > 0, \\ u_\varepsilon(t, \ell^\varepsilon(t)) = 0, & t > 0, \\ u_\varepsilon(0,x) = u_0^\varepsilon(x), & 0 < x < \ell_0, \\ (u_\varepsilon)_t(0,x) = \varepsilon u_1^\varepsilon(x), & 0 < x < \ell_0, \end{cases}$$

coupled with dynamic Griffith's criterion.

We denote by ℓ_1 the limit of $\tilde{\ell}(t)$ when t goes to $+\infty$. Before studying the relationship between ℓ_1 and $\ell^+(0)$ we perform an asymptotic analysis of the pair $(\tilde{u}, \tilde{\ell})$ as $t \to +\infty$.

Lemma 5.5.14. *Assume (K0). Then for every $\delta > 0$ there exists a time $T_\delta > 0$ and a measurable set $N_\delta \subseteq (T_\delta, +\infty)$ such that $|N_\delta| \le \delta$ and $\dot{\tilde{\ell}}(t) \le \delta$ for every $t \in (T_\delta, +\infty) \setminus N_\delta$.*

Proof. First of all we notice that by (K0) we deduce from the energy-dissipation balance (5.5.19) that ℓ_1 is finite. Then we fix $\delta > 0$ and we consider $T_\delta > 0$ in such a way that $\ell_1 - \tilde{\ell}(T_\delta) \le \delta^2$. Introducing the sets:

$$ND_\delta := \{t > T_\delta \mid \tilde{\ell} \text{ is not differentiable at } t\},$$

$$M_\delta := \{t > T_\delta \mid \tilde{\ell} \text{ is differentiable at } t \text{ and } \dot{\tilde{\ell}}(t) > \delta\},$$

we then define $N_\delta := ND_\delta \cup M_\delta$. By construction $\dot{\tilde{\ell}}(t) \le \delta$ for every $t \in (T_\delta, +\infty) \setminus N_\delta$, while by means of Čebyšëv inequality we deduce:

$$|N_\delta| = |M_\delta| \le \frac{1}{\delta} \int_{T_\delta}^{+\infty} \dot{\tilde{\ell}}(\tau)\, \mathrm{d}\tau = \frac{\ell_1 - \tilde{\ell}(T_\delta)}{\delta} \le \delta,$$

and we conclude. □

All the next propositions trace what we have done in the previous sections to deal with the analysis of the limit of the pair $(u^\varepsilon, \ell^\varepsilon)$ when $\varepsilon \to 0^+$. For this reason the proofs are only sketched.

Proposition 5.5.15. *Assume (K0). Then one has* $\lim\limits_{t \to +\infty} (\mathcal{K}(t) + \mathcal{E}(t)) = \dfrac{1}{2} \dfrac{w(0)^2}{\ell_1}$.

Proof. As in Section 5.4 we introduce the modified shifted potential energy:

$$\widetilde{\mathcal{E}}(t) := \frac{1}{2} \int_0^{\tilde{\ell}(t)} \big(\tilde{u}_x(t,\sigma) - \tilde{r}_x(t,\sigma)\big)^2 \, \mathrm{d}\sigma, \quad \text{for } t \in [0, +\infty),$$

where

$$\tilde{r}(t,x) := w(0) \left(1 - \frac{x}{\tilde{\ell}(t)}\right) \mathbb{1}_{[0,\tilde{\ell}(t)]}(x), \quad \text{for } (t,x) \in [0,+\infty) \times [0,+\infty).$$

Repeating the proof of Theorem 5.4.5 we deduce that the following estimate holds true:

$$\mathcal{K}(t) + \widetilde{\mathcal{E}}(t) \le 4 \widetilde{\mathcal{E}}(0) e^{-mt} + C e^{-mt} \int_0^t \dot{\tilde{\ell}}(\tau) e^{m\tau} \, \mathrm{d}\tau, \quad \text{for every } t \in [0, +\infty), \tag{5.5.20}$$

where m is a suitable positive value and C is a positive constant independent of t. By means of Lemma 5.5.14 now we show that the second term in (5.5.20) goes to 0 when $t \to +\infty$. Indeed let us fix $\delta > 0$ and consider T_δ, N_δ as in Lemma 5.5.14; then for every $t \geq T_\delta$ we can estimate:

$$\begin{aligned} e^{-mt}\int_0^t \dot{\tilde{\ell}}(\tau)e^{m\tau}\,\mathrm{d}\tau &= e^{-mt}\left(\int_0^{T_\delta}\dot{\tilde{\ell}}(\tau)e^{m\tau}\,\mathrm{d}\tau + \int\limits_{(T_\delta,t)\cap N_\delta}\dot{\tilde{\ell}}(\tau)e^{m\tau}\,\mathrm{d}\tau + \int\limits_{(T_\delta,t)\setminus N_\delta}\dot{\tilde{\ell}}(\tau)e^{m\tau}\,\mathrm{d}\tau\right) \\ &\leq e^{-mt}\left(\int_0^{T_\delta}\dot{\tilde{\ell}}(\tau)e^{m\tau}\,\mathrm{d}\tau + e^{mt}|N_\delta| + \delta\int_{T_\delta}^t e^{m\tau}\,\mathrm{d}\tau\right) \\ &\leq e^{-mt}\int_0^{T_\delta}\dot{\tilde{\ell}}(\tau)e^{m\tau}\,\mathrm{d}\tau + \delta\left(1+\frac{1}{m}\right). \end{aligned}$$

Letting first $t \to +\infty$ and then $\delta \to 0^+$ we hence deduce that $\lim\limits_{t\to+\infty} e^{-mt}\int_0^t \dot{\tilde{\ell}}(\tau)e^{m\tau}\,\mathrm{d}\tau = 0$ and so we get $\lim\limits_{t\to+\infty}\left(\mathcal{K}(t)+\widetilde{\mathcal{E}}(t)\right) = 0$. Now we conclude since like in (5.4.9) we have:

$$\mathcal{E}(t) = \widetilde{\mathcal{E}}(t) + \frac{1}{2}\frac{w(0)^2}{\tilde{\ell}(t)}, \quad \text{for every } t \in [0,+\infty).$$

□

Lemma 5.5.16. *Assume (K0). Then the following limit holds true:*

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t\left(\tilde{u}_x(\sigma,0)+\frac{w(0)}{\tilde{\ell}(\tau)}\right)^2\mathrm{d}\tau = 0.$$

Proof. The proof is analogous to the one of Corollary 5.5.4. By using (5.5.1) with the obvious changes, for every $t > 0$ we obtain the estimate:

$$\begin{aligned} &\int_0^t\left(\tilde{u}_x(\sigma,0)+\frac{w(0)}{\tilde{\ell}(\tau)}\right)^2\mathrm{d}\tau \leq C\left[\int_0^t\left(\mathcal{K}(\tau)+\widetilde{\mathcal{E}}(\tau)\right)\mathrm{d}\tau + \mathcal{K}(t)+\mathcal{E}(t)+\tilde{\ell}(t)\right] \\ &\leq C\left[\int_0^t\left(\mathcal{K}(\tau)+\widetilde{\mathcal{E}}(\tau)\right)\mathrm{d}\tau + \mathcal{E}(0)+\ell_1\right]. \end{aligned}$$

From this we conclude by applying de l'Hôpital's rule since in Proposition 5.5.15 we proved that $\lim\limits_{t\to+\infty}\left(\mathcal{K}(t)+\widetilde{\mathcal{E}}(t)\right) = 0$. □

Proposition 5.5.17. *Assume (K0). Then ℓ_1 satisfies the stability condition at time $t = 0$, namely:*

$$\frac{1}{2}\frac{w(0)^2}{\ell_1^2} \leq \kappa(\ell_1).$$

Proof. The idea is to pass to the limit as $t \to +\infty$ in the stability condition in Griffith's criterion (5.5.18), as we did in Proposition 5.5.6. Since here we want to compute a limit when t grows to $+\infty$, as in Lemma 5.5.16 we need to average the stability condition, getting:

$$\frac{1}{t}\int_0^t \kappa(\tilde{\ell}(\sigma))\,\mathrm{d}\sigma \geq \frac{1}{t}\int_0^t G_{\dot{\tilde{\ell}}(\sigma)}(\sigma)\,\mathrm{d}\sigma, \qquad \text{for every } t \in (0,+\infty). \tag{5.5.21}$$

By de l'Hôpital's rule the left-hand side in (5.5.21) converges to $\kappa(\ell_1)$ as $t \to +\infty$, while to deal with the right-hand side we argue as in the proof of Proposition 5.5.6. For the sake of simplicity we introduce the time $t^* > 0$ which satisfies $t^* = \tilde{\ell}(t^*)$, so that for every $t \geq t^*$ we can write:

$$\begin{aligned}\frac{1}{t}\int_0^t G_{\dot{\tilde{\ell}}(\sigma)}(\sigma)\,\mathrm{d}\sigma &\geq \frac{1}{t}\int_{t^*}^t G_{\dot{\tilde{\ell}}(\sigma)}(\sigma)\,\mathrm{d}\sigma\\ &= \frac{1}{t}\int_0^{\tilde{\varphi}(t)} \frac{1}{\dot{\tilde{\psi}}(\tilde{\varphi}^{-1}(\sigma))}\frac{1}{2}\left(\tilde{u}_x(\sigma,0)+\nu\int_\sigma^{\tilde{\varphi}^{-1}(\sigma)}\tilde{u}_t(\tau,\tau-\sigma)\,\mathrm{d}\tau\right)^2\mathrm{d}\sigma,\end{aligned} \tag{5.5.22}$$

where we used the explicit formula for $G_{\dot{\tilde{\ell}}(\sigma)}(\sigma)$ given by (5.5.5) and (5.5.6), with the obvious changes. By means of Lemma 5.5.16 and since $\lim\limits_{t\to+\infty}\frac{\tilde{\varphi}(t)}{t} = \lim\limits_{t\to+\infty}\frac{t-\tilde{\ell}(t)}{t} = 1$ it is easy to infer:

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^{\tilde{\varphi}(t)}\frac{1}{\dot{\tilde{\psi}}(\tilde{\varphi}^{-1}(\sigma))}\frac{1}{2}\tilde{u}_x(\sigma,0)^2\,\mathrm{d}\sigma = \frac{1}{2}\frac{w(0)^2}{\ell_1^2}. \tag{5.5.23}$$

Moreover, by using estimate (5.5.9) in the proof of Proposition 5.5.6 and recalling that the dissipated energy $\mathcal{V}$ is bounded by (5.5.19), we deduce:

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^{\tilde{\varphi}(t)}\frac{1}{\dot{\tilde{\psi}}(\tilde{\varphi}^{-1}(\sigma))}\left(\int_\sigma^{\tilde{\varphi}^{-1}(\sigma)}\tilde{u}_t(\tau,\tau-\sigma)\,\mathrm{d}\tau\right)^2\mathrm{d}\sigma = 0. \tag{5.5.24}$$

From (5.5.23) and (5.5.24) we can pass to the limit in (5.5.22) deducing that:

$$\liminf_{t\to+\infty}\frac{1}{t}\int_0^t G_{\dot{\tilde{\ell}}(\sigma)}(\sigma)\,\mathrm{d}\sigma \geq \frac{1}{2}\frac{w(0)^2}{\ell_1^2},$$

and so we conclude. □

We are now in a position to compare the value of $\ell^+(0)$ with ℓ_1.

Lemma 5.5.18. *Assume (H2) and (K0). Then* $\ell_1 \leq \ell^+(0)$.

Proof. We fix $t > 0$ and we consider the subsequence $\varepsilon_n \searrow 0$ given by Remark 5.4.1 and Proposition 5.5.1. Then one has:

$$\ell(t) = \lim_{n\to+\infty}\ell^{\varepsilon_n}(t) = \lim_{n\to+\infty}\ell_{\varepsilon_n}\left(\frac{t}{\varepsilon_n}\right).$$

Now we fix $T > 0$ and by monotonicity we deduce $\ell_{\varepsilon_n}\left(\frac{t}{\varepsilon_n}\right) \geq \ell_{\varepsilon_n}(T)$ for n large enough. Thus, by means of Theorem 5.2.5, we get:

$$\lim_{n\to+\infty}\ell_{\varepsilon_n}\left(\frac{t}{\varepsilon_n}\right) \geq \lim_{n\to+\infty}\ell_{\varepsilon_n}(T) = \tilde{\ell}(T).$$

Hence $\ell(t) \geq \tilde{\ell}(T)$ and by the arbitrariness of $t > 0$ and $T > 0$ we conclude. □

Proposition 5.5.19. *Assume (H2), (K0) and (K2). Then the following inequality holds true:*

$$\frac{1}{2}\frac{w(0)^2}{\ell^+(0)} + \int_{\ell_0}^{\ell^+(0)}\kappa(\sigma)\,\mathrm{d}\sigma \leq \frac{1}{2}\frac{w(0)^2}{\ell_1} + \int_{\ell_0}^{\ell_1}\kappa(\sigma)\,\mathrm{d}\sigma.$$

Proof. By the energy-dissipation balance (5.2.2a), Corollary 5.5.4, Theorem 5.5.5 and Corollary 5.5.10 we know that for every $t > 0$ it holds:

$$\lim_{n\to+\infty}\mathcal{V}^{\varepsilon_n}(t) = \frac{1}{2}\int_0^{\ell_0}\dot{u}_0(\sigma)^2\,\mathrm{d}\sigma - \frac{1}{2}\frac{w(t)^2}{\ell(t)} - \int_{\ell_0}^{\ell(t)}\kappa(\sigma)\,\mathrm{d}\sigma + \int_0^t\dot{w}(\tau)\frac{w(\tau)}{\ell(\tau)}\,\mathrm{d}\tau,$$

where ε_n is the subsequence given by (5.4.1) and by Propositions 5.5.1 and 5.5.2. By means of (5.5.16) we hence deduce:

$$\lim_{n\to+\infty} \mathcal{V}^{\varepsilon_n}(t) = \frac{1}{2}\int_0^{\ell_0} \dot{u}_0(\sigma)^2\,\mathrm{d}\sigma - \frac{1}{2}\frac{w(0)^2}{\ell^+(0)} - \int_{\ell_0}^{\ell^+(0)} \kappa(\sigma)\,\mathrm{d}\sigma. \tag{5.5.25}$$

By a simple change of variable we now notice that:

$$\mathcal{V}^{\varepsilon_n}(t) = \nu\int_0^{t/\varepsilon_n}\int_0^{\ell_{\varepsilon_n}(\tau)} (u_{\varepsilon_n})_t(\tau,\sigma)^2\,\mathrm{d}\sigma\,\mathrm{d}\tau \geq \nu\int_0^{t}\int_0^{\ell_{\varepsilon_n}(\tau)} (u_{\varepsilon_n})_t(\tau,\sigma)^2\,\mathrm{d}\sigma\,\mathrm{d}\tau,$$

and so, by Theorem 5.2.5, we get:

$$\lim_{n\to+\infty} \mathcal{V}^{\varepsilon_n}(t) \geq \nu\int_0^{t}\int_0^{\tilde{\ell}(\tau)} \tilde{u}_t(\tau,\sigma)^2\,\mathrm{d}\sigma\,\mathrm{d}\tau. \tag{5.5.26}$$

Putting together (5.5.25) and (5.5.26) we finally deduce:

$$\frac{1}{2}\int_0^{\ell_0} \dot{u}_0(\sigma)^2\,\mathrm{d}\sigma - \frac{1}{2}\frac{w(0)^2}{\ell^+(0)} - \int_{\ell_0}^{\ell^+(0)} \kappa(\sigma)\,\mathrm{d}\sigma \geq \lim_{t\to+\infty}\nu\int_0^{t}\int_0^{\tilde{\ell}(\tau)} \tilde{u}_t(\tau,\sigma)^2\,\mathrm{d}\sigma\,\mathrm{d}\tau = \lim_{t\to+\infty}\mathcal{V}(t).$$

To conclude it is enough to recall that by energy-dissipation balance (5.5.19) we have:

$$\mathcal{V}(t) = \frac{1}{2}\int_0^{\ell_0} \dot{u}_0(\sigma)^2\,\mathrm{d}\sigma - \mathcal{K}(t) - \mathcal{E}(t) - \int_{\ell_0}^{\tilde{\ell}(t)} \kappa(\sigma)\,\mathrm{d}\sigma, \qquad \text{for every } t\in[0,+\infty),$$

and so by Proposition 5.5.15 we obtain:

$$\lim_{t\to+\infty}\mathcal{V}(t) = \frac{1}{2}\int_0^{\ell_0} \dot{u}_0(\sigma)^2\,\mathrm{d}\sigma - \frac{1}{2}\frac{w(0)^2}{\ell_1} - \int_{\ell_0}^{\ell_1} \kappa(\sigma)\,\mathrm{d}\sigma.$$

□

Corollary 5.5.20. *Assume (H2), (K0) and (K2). Then* $\ell_1 = \ell^+(0)$.

Proof. By Lemma 5.5.18 we already know that $\ell_1 \leq \ell^+(0)$. As in Proposition 5.3.4 we introduce the energy:

$$E_0(x) := \frac{1}{2}\frac{w(0)^2}{x} + \int_{\ell_0}^{x} \kappa(\sigma)\,\mathrm{d}\sigma, \qquad \text{for } x\in[\ell_0,+\infty).$$

By Proposition 5.5.19 we get $E_0(\ell^+(0)) \leq E_0(\ell_1)$, while by Proposition 5.5.17 and (K2) we deduce that $\dot{E}_0(x) > 0$ for every $x > \ell_1$, namely E_0 is strictly increasing in $(\ell_1,+\infty)$. Thus we finally obtain $\ell_1 = \ell^+(0)$. □

Putting together all the results obtained up to now we can finally deduce our main theorem:

Theorem 5.5.21. *Fix* $\nu > 0$, $\ell_0 > 0$ *and assume the functions* w^ε, u_0^ε *and* u_1^ε *satisfy* (3.2.1) *and* (3.2.2) *for every* $\varepsilon > 0$. *Let the positive toughness* κ *belong to* $\widetilde{C}^{0,1}([\ell_0,+\infty))$ *and assume (H2), (K0) and (K2). Let* $(u^\varepsilon,\ell^\varepsilon)$ *be the pair of dynamic evolutions given by Theorem 5.2.3. Let* ε_n *and* w *be the subsequence and the function given by Remark 5.4.1 and let* ℓ_1 *be defined as* $\ell_1 := \lim_{t\to+\infty}\tilde{\ell}(t)$, *with* $(\tilde{u},\tilde{\ell})$ *solution of* (5.5.17)&(5.5.18). *Then for every* $t\in(0,+\infty)$ *one has:*

(a) $\lim_{n\to+\infty}\ell^{\varepsilon_n}(t) = \ell(t)$,

(b) $\varepsilon_n u_t^{\varepsilon_n}(t,\cdot) \to 0$ *strongly in* $L^2(0,+\infty)$ *as* $n \to +\infty$,

(c) $u^{\varepsilon_n}(t,\cdot) \to u(t,\cdot)$ *strongly in* $H^1(0,+\infty)$ *as* $n \to +\infty$,

where (u,ℓ) *is the quasistatic evolution given by* (5.3.5) *starting from* ℓ_1 *and with external loading* w.

Moreover, if we assume (H3), then we do not need to pass to a subsequence and the whole sequence $(u^\varepsilon,\ell^\varepsilon)$ *converges to* (u,ℓ) *in the sense of* (a), (b), (c) *for every* $t \in (0,+\infty)$ *as* $\varepsilon \to 0^+$.

If finally we assume (K3), then the limit pair (u,ℓ) *is the AC-quasistatic evolution given by Theorem 5.3.9 starting from* ℓ_1 *and with external loading* w.

Remark 5.5.22. Of course stability condition at time $t=0$, namely $\frac{1}{2}\frac{w(0)^2}{\ell_0^2} \le \kappa(\ell_0)$, is a necessary condition to have $\ell_1=\ell_0$, due to Proposition 5.5.17; however it is not sufficient, indeed it does not involve the initial position u_0, which can produce the initial jump if steep enough, as the following example shows. Let us consider the case of a constant toughness $\kappa = 1/2$, a loading term satisfying $0 \le w(0) \le \ell_0$ (so that initial stability holds) and a smooth (C^1 is enough) initial position u_0 fulfilling compatibility conditions $u_0(0)=w(0)$ and $u_0(\ell_0)=0$. By means of the explicit equation solved by the debonding front $\tilde\ell$, namely (3.4.13), and thanks to (3.4.8) we deduce that under our assuptions $\tilde\ell$ is actually $C^1([0,+\infty))$. Moreover we can compute:

$$\dot{\tilde\ell}(0) = \max\left\{\frac{\dot u_0(\ell_0)^2-1}{\dot u_0(\ell_0)^2+1},0\right\},$$

from which we get $\dot{\tilde\ell}(0)>0$, and thus $\ell_1>\ell_0$, if $\dot u_0(\ell_0)^2>1$.

On the other hand, if the initial position is affine, namely $u_0(x)=w(0)\left(1-\frac{x}{\ell_0}\right)$, then stability condition at time $t=0$ becomes equivalent to the absence of initial jump. Indeed in this case if $\frac{1}{2}\frac{w(0)^2}{\ell_0^2}\le\kappa(\ell_0)$, then the pair $(\tilde u,\tilde\ell)$ is explicitely given by $\tilde u(t,x)=w(0)\left(1-\frac{x}{\ell_0}\right)$ and $\tilde\ell(t)=\ell_0$, since by (5.2.12) we have $G_0(t)\equiv\frac{1}{2}\frac{w(0)^2}{\ell_0^2}$, and thus trivially $\ell_1=\ell_0$.

Remark 5.5.23. Under the same assumptions of the above theorem the convergence of the debonding fronts can be slightly improved by classical arguments. Indeed, since ℓ^ε are nondecreasing continuous functions and since the pointwise limit ℓ is continuous in $(0,+\infty)$, we can infer that the convergence stated in (a) is actually uniform on compact sets contained in $(0,+\infty)$.

We want also to recall that for every $T>0$ the convergences in (b) and (c) holds true respectively in $L^2(0,T;L^2(0,+\infty))$ and $L^2(0,T;H^1(0,+\infty))$ too, as we proved in Proposition 5.5.2 under weaker assumptions.

Appendix A

Proofs on $AC_{\mathcal{R}}$ and $BV_{\mathcal{R}}$ functions

In this Appendix we collect all the proofs of the results of Section 1.3 about $\mathcal{R}$–absolutely continuous functions and functions of bounded $\mathcal{R}$–variation.

We recall that we are considering a reflexive Banach space X and a ψ–regular function $\mathcal{R}\colon [a,b]\times X\to[0,+\infty]$ in the sense of Definition 1.3.1, which we rewrite for the sake of clarity:

Definition A.0.1. *Given an* admissible *function* $\psi\colon X\to[0,+\infty]$, *namely satisfying*

(ψ0) $\psi(0)=0$*;*

(ψ1) ψ *is convex;*

(ψ2) ψ *is positively homogeneous of degree one;*

(ψ3) ψ *is lower semicontinuous;*

(ψ4) *there exists a positive constant* $c>0$ *such that* $c|\cdot|\le\psi(\cdot)$,

we say that $\mathcal{R}\colon[a,b]\times X\to[0,+\infty]$ *is* ψ–regular *if:*

- *for every* $t\in[a,b]$, $\mathcal{R}(t,\cdot)$ *is convex, positively homogeneous of degree one, lower semicontinuous, and satisfies* $\mathcal{R}(t,0)=0$*;*
- *there exist two positive constants* $\alpha^*\ge\alpha_*>0$ *for which*

$$\alpha_*\psi(v)\le\mathcal{R}(t,v)\le\alpha^*\psi(v),\qquad \text{for every } (t,v)\in[a,b]\times X; \tag{A.0.1}$$

- *there exists a nonnegative and nondecreasing function* $\sigma\in C^0([0,b-a])$ *satisfying* $\sigma(0)=0$ *and for which*

$$|\mathcal{R}(t,v)-\mathcal{R}(s,v)|\le\psi(v)\sigma(t-s),\ \text{for every } a\le s\le t\le b \text{ and } v\in\{\psi<+\infty\}. \tag{A.0.2}$$

We also recall the definitions of $\mathcal{R}$–absolutely continuous functions and functions of bounded $\mathcal{R}$–variation:

Definition A.0.2. *We say that a function* $f\colon[a,b]\to X$ *is* $\mathcal{R}$*–absolutely continuous, and we write* $f\in AC_{\mathcal{R}}([a,b];X)$ *if* f *is absolutely continuous and* $\displaystyle\int_a^b\mathcal{R}(\tau,\dot f(\tau))\,\mathrm{d}\tau<+\infty$.

Definition A.0.3. *Given a function $f\colon [a,b]\to X$, we define its $\mathcal{R}$–variation in $[s,t]$, with $a\le s<t\le b$, as:*

$$V_{\mathcal{R}}(f;s,t):=\lim_{n\to+\infty}\sum_{k=1}^{n}\mathcal{R}(t_{k-1},f(t_k)-f(t_{k-1})), \tag{A.0.3}$$

where $\{t_k\}_{k=1}^{n}$ is a fine *sequence of partitions of $[s,t]$, namely it is of the form $s=t_0<t_1<\dots<t_n=t$ and satisfies*

$$\lim_{n\to+\infty}\sup_{k=1,\dots,n}(t_k-t_{k-1})=0. \tag{A.0.4}$$

We also set $V_{\mathcal{R}}(f;t,t):=0$, for every $t\in[a,b]$.

We say that f is a function of bounded $\mathcal{R}$–variation in $[a,b]$ if its $\mathcal{R}$–variation in $[a,b]$ is finite, i.e. $V_{\mathcal{R}}(f;a,b)<+\infty$. In this case we write $f\in BV_{\mathcal{R}}([a,b];X)$,

We can now start to prove all the results of Section 1.3. The first proof regards the relationship between $\mathcal{R}$–absolutely continuous functions and its classical counterpart.

Proof of Proposition 1.3.4. The equivalence between (1) and (2) easily follows by means of (A.0.1).

Now assume (2). Then for every $a\le s\le t\le b$ we have:

$$\psi(f(t)-f(s))=\psi\left(\int_s^t\dot f(\tau)\,\mathrm{d}\tau\right)\le\int_s^t\psi(\dot f(\tau))\,\mathrm{d}\tau,$$

where in the last step we used Jensen's inequality together with (ψ2). Since $\psi(\dot f(\cdot))$ is summable we obtain (3) with $m(t)=\psi(\dot f(t))$.

If instead we assume (3), then by (ψ4) we get that f is absolutely continuous, so $\dot f$ is well defined almost everywhere in $[a,b]$ as a (strong) limit of differential quotients. By means of (ψ2) and (ψ3) we thus deduce:

$$\psi(\dot f(\tau))\le\liminf_{h\searrow0}\frac{\psi(f(\tau+h)-f(\tau))}{h}\le\liminf_{h\searrow0}\frac1h\int_\tau^{\tau+h}m(\theta)\,\mathrm{d}\theta=m(\tau),\ \text{ for a.e. }\tau\in[a,b],$$

which implies $\displaystyle\int_a^b\psi(\dot f(\tau))\,\mathrm{d}\tau\le\int_a^b m(\tau)\,\mathrm{d}\tau<+\infty.$ □

Then, we prove the properties of the $\mathcal{R}$–variation we exploited in the first part of the thesis.

Proof of Proposition 1.3.11. For (a) it is enough to take a fine sequence of partions of $[r,t]$ containing s. The proof of (b) follows easily by (a).

The only nontrivial part in (c) are the two equalities:

$$V_{\mathcal{R}}(f;t,t+)=V_{\mathcal{R}(t)}(f;t,t+),\qquad\text{and}\qquad V_{\mathcal{R}}(f;t-,t)=V_{\mathcal{R}(t)}(f;t-,t). \tag{A.0.5}$$

We prove only the first one, the other being analogous. Exploiting (A.0.2) we deduce that for every $t'>t$ we have:

$$\begin{aligned}|V_{\mathcal{R}}(f;t,t')-V_{\mathcal{R}(t)}(f;t,t')|&\le\limsup_{n\to+\infty}\sum_{k=1}^{n}|\mathcal{R}(t_{k-1},f(t_k)-f(t_{k-1}))-\mathcal{R}(t,f(t_k)-f(t_{k-1}))|\\&\le\limsup_{n\to+\infty}\sum_{k=1}^{n}\psi(f(t_k)-f(t_{k-1}))\sigma(t_{k-1}-t)\\&\le V_\psi(f;t,t')\sigma(t'-t),\end{aligned}$$

where $\{t_k\}_{k=1}^n$ is a fine sequence of partitions of $[t,t']$. Letting now $t' \searrow t$ we get (A.0.5).

As regards the first inequality of (d), it is enough to prove

$$V_{\mathcal{R}}(f;s',t') \geq \max\left\{V_{\mathcal{R}}(f^+;s',t'), V_{\mathcal{R}}(f^-;s',t')\right\}, \tag{A.0.6}$$

where $s' < s \leq t < t'$ are continuity points of f. So we fix $\delta > 0$ and a fine sequence of partition of $[s',t']$. Then, exploiting lower semicontinuity and (A.0.2), for any of these partitions there exists another partition of $[s',t']$, made of continuity points of f and such that each point $\tilde{t}_{k-1}$ belongs to $[t_{k-1},t_k)$, which satisfies:

$$\begin{aligned}
&\sum_{k=1}^n \mathcal{R}(t_{k-1}, f^+(t_k) - f^+(t_{k-1})) \leq \sum_{k=1}^n \mathcal{R}(t_{k-1}, f(\tilde{t}_k) - f(\tilde{t}_{k-1})) + \delta \\
&\leq \sum_{k=1}^n \mathcal{R}(\tilde{t}_{k-1}, f(\tilde{t}_k) - f(\tilde{t}_{k-1})) + \sum_{k=1}^n \psi(f(\tilde{t}_k) - f(\tilde{t}_{k-1}))\sigma(\tilde{t}_{k-1} - t_{k-1}) + \delta \\
&\leq \sum_{k=1}^n \mathcal{R}(\tilde{t}_{k-1}, f(\tilde{t}_k) - f(\tilde{t}_{k-1})) + V_\psi(f;s',t') \sup_{k=1,\ldots n} \sigma(t_k - t_{k-1}) + \delta.
\end{aligned}$$

By letting first $n \to +\infty$ and then $\delta \to 0$, recalling (A.0.4) and the uniform continuity of σ, we get $V_{\mathcal{R}}(f;s',t') \geq V_{\mathcal{R}}(f^+;s',t')$, and arguing in a similar way we also obtain $V_{\mathcal{R}}(f;s',t') \geq V_{\mathcal{R}}(f^-;s',t')$, thus the first inequality in (d) is proved.

We now prove the second inequality of (d). We fix $t' > t$ a continuity point of f, we consider $\delta > 0$ and a fine sequence of partitions of $[s,t']$. As before, for any of these partitions there exist continuity points of f such that each point $\tilde{t}_{k-1}$ belongs to (t_{k-1},t_k) and they satisfy:

$$\begin{aligned}
&\sum_{k=1}^n \mathcal{R}(t_{k-1}, f^+(t_k) - f^+(t_{k-1})) \\
&\leq \sum_{k=1}^n \mathcal{R}(\tilde{t}_{k-1}, f(\tilde{t}_k) - f(\tilde{t}_{k-1})) + V_\psi(f;s',t') \sup_{k=1,\ldots n} \sigma(t_k - t_{k-1}) + \delta \\
&= \sum_{k=1}^n \mathcal{R}(\tilde{t}_{k-1}, f(\tilde{t}_k) - f(\tilde{t}_{k-1})) + \mathcal{R}(s, f(\tilde{t}_0) - f(s)) - \mathcal{R}(s, f(\tilde{t}_0) - f(s)) \\
&\quad + V_\psi(f;s',t') \sup_{k=1,\ldots n} \sigma(t_k - t_{k-1}) + \delta.
\end{aligned}$$

Letting $n \to +\infty$, thanks to (A.0.4), we deduce

$$V_{\mathcal{R}}(f^+;s,t') \leq V_{\mathcal{R}}(f;s,t') - V_{\mathcal{R}(s)}(f;s,s+) + \delta = V_{\mathcal{R}}(f;s+,t') + \delta.$$

Letting now $\delta \to 0$ and $t' \searrow t$ we deduce $V_{\mathcal{R}}(f;s+,t+) \geq V_{\mathcal{R}}(f^+;s,t+)$.

The third inequality in (d) follows in a similar way, thus we conclude. □

Next proof deals with the inclusion of the space of $\mathcal{R}$–absolutely continuous functions into the space of functions of bounded $\mathcal{R}$–variation.

Proof of Proposition 1.3.12. Assume f is $\mathcal{R}$–absolutely continuous. We fix $a \leq s \leq t \leq b$ and we consider a fine sequence of partitions of $[s,t]$. Thanks to (A.0.1) and (A.0.2) we estimate:

$$\begin{aligned}\sum_{k=1}^{n}\mathcal{R}(t_{k-1},f(t_k)-f(t_{k-1}))&\leq\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\mathcal{R}(t_{k-1},\dot f(\tau))\,\mathrm{d}\tau\\&\leq\sum_{k=1}^{n}\left(\int_{t_{k-1}}^{t_k}\mathcal{R}(\tau,\dot f(\tau))\,\mathrm{d}\tau+\int_{t_{k-1}}^{t_k}\psi(\dot f(\tau))\sigma(\tau-t_{k-1})\,\mathrm{d}\tau\right)\\&\leq\int_s^t\mathcal{R}(\tau,\dot f(\tau))\,\mathrm{d}\tau+\sup_{k=1,\dots n}\sigma(t_k-t_{k-1})\int_s^t\psi(\dot f(\tau))\,\mathrm{d}\tau.\end{aligned}$$

Letting $n\to+\infty$ (we again recall (A.0.4)) we deduce

$$V_{\mathcal{R}}(f;s,t)\leq\int_s^t\mathcal{R}(\tau,\dot f(\tau))\,\mathrm{d}\tau,\tag{A.0.7}$$

thus f is of bounded $\mathcal{R}$–variation and the $\mathcal{R}$-variation is absolutely continuous.

To obtain also the other implication and the opposite inequality in (A.0.7) we argue as follows: first of all we notice that (A.0.1) implies:

$$V_{\mathcal{R}}(f;s,t)\geq\alpha_*V_\psi(f;s,t)\geq\alpha_*\psi(f(t)-f(s)),\qquad\text{for every } a\leq s\leq t\leq b,\tag{A.0.8}$$

and thus f is $\mathcal{R}$–absolutely continuous by applying Proposition 1.3.4 (thus ($\psi4$) here is needed). To conclude, introducing the notation $v_{\mathcal{R}}(t):=V_{\mathcal{R}}(f;a,t)$, we only need to prove that $\dot v_{\mathcal{R}}(\tau)\geq\mathcal{R}(\tau,\dot f(\tau))$ for almost every $\tau\in[a,b]$.

With this aim we fix τ differentiability point for both $v_{\mathcal{R}}$ and f, and we consider $h>0$. By using (A.0.2) we obtain:

$$v_{\mathcal{R}}(\tau+h)-v_{\mathcal{R}}(\tau)=V_{\mathcal{R}}(f;\tau,\tau+h)\geq\mathcal{R}(\tau,f(\tau+h)-f(\tau))-V_\psi(f;\tau,\tau+h)\sigma(h).$$

Hence, letting $h\to0$ we deduce:

$$\begin{aligned}\dot v_{\mathcal{R}}(\tau)&\geq\liminf_{h\to0}\mathcal{R}\left(\tau,\frac{f(\tau+h)-f(\tau)}{h}\right)-\lim_{h\to0}\frac{1}{h}V_\psi(f;\tau,\tau+h)\sigma(h)\\&\geq\mathcal{R}(\tau,\dot f(\tau)),\end{aligned}$$

where the limit vanishes if we pick τ which is also a differentiability point of $V_\psi(f;a,\cdot)$, which is absolutely continuous by (A.0.8). Hence the proof is complete. □

Subsequently, we show the weak lower semicontinuity property of the $\mathcal{R}$–variation.

Proof of Lemma 1.3.13. We only sketch the proof, see the Appendix of [39] for more details. If $s=t$ the inequality is trivial, thus let us fix $a\leq s<t\leq b$ and without loss of generality we assume $\liminf\limits_{j\to+\infty}V_{\mathcal{R}}(f_n;s,t)<+\infty$. We now consider a fine sequence of partitions of $[s,t]$ and, recalling that convexity plus lower semicontinuity implies weak lower semicontinuity, we obtain :

$$\sum_{k=1}^{N}\mathcal{R}(t_{k-1},f(t_k)-f(t_{k-1}))\leq\liminf_{n\to+\infty}\sum_{k=1}^{N}\mathcal{R}(t_{k-1},f_n(t_k)-f_n(t_{k-1})).\tag{A.0.9}$$

We now fix $n\in\mathbb{N}$ and we notice that by subadditivity (ensured by convexity and one-homogeneity), (A.0.1) and (A.0.2) we have

$$\begin{aligned}\sum_{k=1}^{N}\mathcal{R}(t_{k-1},f_n(t_k)-f_n(t_{k-1}))&\leq V_{\mathcal{R}}(f_n;s,t)+V_\psi(f_n;s,t)\sup_{k=1,\dots N}\sigma(t_k-t_{k-1})\\&\leq V_{\mathcal{R}}(f_n;s,t)\left(1+\frac{1}{\alpha_*}\sup_{k=1,\dots N}\sigma(t_k-t_{k-1})\right).\end{aligned}\tag{A.0.10}$$

Combining (A.0.9) and (A.0.10) we hence deduce:

$$\sum_{k=1}^{N} \mathcal{R}(t_{k-1}, f(t_k) - f(t_{k-1})) \leq \liminf_{n\to+\infty} V_{\mathcal{R}}(f_n; s, t) \left(1 + \frac{1}{\alpha_*} \sup_{k=1,\dots N} \sigma(t_k - t_{k-1})\right).$$

Letting $N \to +\infty$ and recalling (A.0.4) we conclude. □

Finally, we prove the lemma stating uniform convergence of a sequence of pointwise converging functions whose $\mathcal{R}$–variation is continuous and convergent.

Proof of Lemma 1.3.14. We denote for simplicity

$$v^n_{\mathcal{R}}(t) := V_{\mathcal{R}}(f_n; a, t), \qquad \text{and} \qquad v_{\mathcal{R}}(t) := V_{\mathcal{R}}(f; a, t).$$

By assumptions and since the $\mathcal{R}$–variation is nondecreasing, we deduce that $\{v^n_{\mathcal{R}}\}_{n\in\mathbb{N}}$ is a sequence of nondecreasing and continuous functions pointwise converging to the nondecreasing continuous function $v_{\mathcal{R}}$; this implies that the convergence is actually uniform in $[a, b]$.

We now fix $s, t \in [a, b]$ and we estimate by using $(\psi 4)$ and (A.0.1):

$$\begin{aligned} c\alpha_*|f_n(t) - f_n(s)| &\leq |v^n_{\mathcal{R}}(t) - v^n_{\mathcal{R}}(s)| \\ &\leq |v_{\mathcal{R}}(t) - v_{\mathcal{R}}(s)| + |v^n_{\mathcal{R}}(t) - v_{\mathcal{R}}(t)| + |v^n_{\mathcal{R}}(s) - v_{\mathcal{R}}(s)| \\ &\leq |v_{\mathcal{R}}(t) - v_{\mathcal{R}}(s)| + 2 \max_{\tau\in[a,b]} |v^n_{\mathcal{R}}(\tau) - v_{\mathcal{R}}(\tau)|. \end{aligned}$$

Since $v^n_{\mathcal{R}}$ uniformly converges to $v_{\mathcal{R}}$ and $v_{\mathcal{R}}$ is (uniformly) continuous on $[a, b]$, we get that for every $\varepsilon > 0$ there exist $n_\varepsilon \in \mathbb{N}$ and $\delta_\varepsilon > 0$ such that, assuming $|t - s| \leq \delta_\varepsilon$, it holds:

$$|f_n(t) - f_n(s)| \leq \frac{\varepsilon}{3}, \qquad \text{for every } n > n_\varepsilon. \tag{A.0.11}$$

So we fix $\varepsilon > 0$ and we consider a finite partition of $[a, b]$ of the form $a = \tau_0 < \tau_1 < \dots < \tau_{N_\varepsilon} = b$ such that $\max\limits_{k=1,\dots N_\varepsilon} (\tau_k - \tau_{k-1}) \leq \delta_\varepsilon$. This means that for every $t \in [a, b]$ there exists a point of this partition, denoted by $\tau(t)$, for which $|t - \tau(t)| \leq \delta_\varepsilon$. Without loss of generality we can assume that δ_ε is also the treshold given by the (uniform) continuity of f (indeed notice that f is continuous since $v_{\mathcal{R}}$ is continuous by assumption). Thus by means of (A.0.11) we deduce that for every $n > n_\varepsilon$ and for every $t \in [a, b]$ we have:

$$\begin{aligned} |f_n(t) - f(t)| &\leq |f_n(t) - f_n(\tau(t))| + |f_n(\tau(t)) - f(\tau(t))| + |f(t) - f(\tau(t))| \\ &\leq \frac{\varepsilon}{3} + \max_{k=0,\dots,N_\varepsilon} |f_n(\tau_k) - f(\tau_k)| + \frac{\varepsilon}{3}. \end{aligned}$$

Since the maximum in the above estimate involves only a finite number of terms, by means of the assumption of pointwise convergence and by considering a possibly greater $\hat{n}_\varepsilon \geq n_\varepsilon$ we conclude that for every $t \in [a, b]$ it holds

$$|f_n(t) - f(t)| \leq \varepsilon, \qquad \text{for every } n > \hat{n}_\varepsilon,$$

and we conclude. □

Appendix B

Chain rule and Leibniz differentiation rule

In this Appendix we gather some results about the Chain rule and the Leibniz differentiation rule under low regularity assumptions. These results have been used throughout the thesis and they are of some interest on their own.

For the sake of brevity we assume that in all the statements the function φ is nondecreasing (or strictly increasing), although they are still valid if φ is nonincreasing (or strictly decreasing), with little changes in the proofs.

Lemma B.0.1 (Change of variables formula). *Let $\varphi \in C^{0,1}([a,b])$ be nondecreasing. Then for every nonnegative and measurable function g on $[\varphi(a), \varphi(b)]$ (and hence for every $g \in L^1(\varphi(a), \varphi(b))$ it holds*

$$\int_{\varphi(a)}^{\varphi(b)} g(y)\,\mathrm{d}y = \int_a^b g(\varphi(s))\dot{\varphi}(s)\,\mathrm{d}s. \tag{B.0.1}$$

Remark B.0.2. In general the expression $g(\varphi(s))\dot{\varphi}(s)$ in (B.0.1) has to be meant replacing g by a Borel function $\tilde{g}$ equal to g a.e. in $[\varphi(a), \varphi(b)]$ and finite everywhere (if g is finite a.e.); in the particular case in which $\dot{\varphi}(t) > 0$ for a.e. $t \in [a,b]$ that expression is meaningful without modifications on sets of measure zero (see Corollary B.0.4).

Proof of Lemma B.0.1. If φ is strictly increasing, hence injective, the result is well known. If not, by the Area Formula for Lipschitz maps (see [45], Corollary 5.1.13) we have

$$\int_{\varphi(a)}^{\varphi(b)} g(y)\#\varphi^{-1}(\{y\})\,\mathrm{d}y = \int_a^b g(\varphi(s))\dot{\varphi}(s)\,\mathrm{d}s. \tag{B.0.2}$$

We conclude if we prove that $\#\varphi^{-1}(\{y\}) = 1$ for a.e. $y \in [\varphi(a), \varphi(b)]$.

Since φ is nondecreasing and continuous, for every $y \in [\varphi(a), \varphi(b)]$ the set $\varphi^{-1}(\{y\})$ can be either a singleton either a closed interval, so $\#\varphi^{-1}(\{y\}) \in \{1, +\infty\}$. Taking $g \equiv 1$ in (B.0.2) we deduce

$$+\infty > \varphi(b) - \varphi(a) = \int_a^b \dot{\varphi}(s)\,\mathrm{d}s = \int_{\varphi(a)}^{\varphi(b)} \#\varphi^{-1}(\{y\})\,\mathrm{d}y.$$

This yields $\#\varphi^{-1}(\{y\}) < +\infty$ for a.e. $y \in [\varphi(a), \varphi(b)]$ and so necessarily $\#\varphi^{-1}(\{y\}) = 1$ a.e..

As an alternative proof we notice that the set $\{y \in [\varphi(a), \varphi(b)] \mid \#\varphi^{-1}(\{y\}) = +\infty\}$ is in bijection with a subset of rational numbers, so it is countable and hence of measure zero. □

Remark B.0.3. Formula (B.0.1) still holds true only assuming that φ is absolutely continuous on $[a,b]$ (and nondecreasing), see Theorem 7.26 in [81]. This ensures that every result in this Appendix is valid replacing the assumption $\varphi \in C^{0,1}([a,b])$ by $\varphi \in AC([a,b])$; indeed the reader can easily check that the only ingredient needed to carry out all the proofs is (B.0.1).

Corollary B.0.4. *Let $\varphi \in C^{0,1}([a,b])$ be nondecreasing and let $N \subset [\varphi(a), \varphi(b)]$ be a set of measure zero. Then the set $M = \{t \in \varphi^{-1}(N) \mid \dot\varphi(t)$ exists and $\dot\varphi(t) > 0\}$ has measure zero as well. In particular, if $\dot\varphi(t) > 0$ for a.e. $t \in [a,b]$, then φ^{-1} maps sets of measure zero in sets of measure zero.*

Proof. Let $N \subset [\varphi(a), \varphi(b)]$ be a set of measure zero; then by Lemma B.0.1

$$0 = \int_{\varphi(a)}^{\varphi(b)} \mathbb{1}_N(y)\,\mathrm{d}y = \int_a^b \mathbb{1}_N(\varphi(s))\dot\varphi(s)\,\mathrm{d}s = \int_{\varphi^{-1}(N)} \dot\varphi(s)\,\mathrm{d}s = \int_M \dot\varphi(s)\,\mathrm{d}s.$$

Since by construction $\dot\varphi(t) > 0$ for every $t \in M$, we deduce that the set M has measure zero. □

Corollary B.0.5. *Let $\varphi \in C^{0,1}([a,b])$ be a strictly increasing function such that $\dot\varphi(t) > 0$ for a.e. $t \in [a,b]$. Then φ^{-1} belongs to $AC([\varphi(a), \varphi(b)])$ and $\frac{\mathrm{d}}{\mathrm{d}x}(\varphi^{-1})(x) = \frac{1}{\dot\varphi(\varphi^{-1}(x))}$ for a.e. $x \in [\varphi(a), \varphi(b)]$.*

Proof. Firstly we notice that Lemma B.0.1 ensures that $\frac{1}{\dot\varphi \circ \varphi^{-1}}$ belongs to $L^1(\varphi(a), \varphi(b)$:

$$\int_{\varphi(a)}^{\varphi(b)} \frac{1}{\dot\varphi(\varphi^{-1}(y))}\,\mathrm{d}y = \int_a^b \frac{1}{\dot\varphi(s)}\dot\varphi(s)\,\mathrm{d}s = b - a < +\infty.$$

Moreover for every $x \in [\varphi(a), \varphi(b)]$

$$\varphi^{-1}(x) - \varphi^{-1}(\varphi(a)) = \int_a^{\varphi^{-1}(x)} \mathrm{d}s = \int_a^{\varphi^{-1}(x)} \frac{\dot\varphi(s)}{\dot\varphi(s)}\,\mathrm{d}s = \int_{\varphi(a)}^x \frac{1}{\dot\varphi(\varphi^{-1}(y))}\,\mathrm{d}y,$$

so we conclude. □

Lemma B.0.6 (Chain rule). *Let $\varphi \in C^{0,1}([a,b])$ be nondecreasing and let ϕ belong to $AC([\varphi(a), \varphi(b)])$. Then $\phi \circ \varphi$ belongs to $AC([a,b])$ and $\frac{\mathrm{d}}{\mathrm{d}t}(\phi \circ \varphi)(t) = \dot\phi(\varphi(t))\dot\varphi(t)$ for a.e. $t \in [a,b]$, where the right-hand side is meant as in Remark B.0.2.*

Proof. Since $\phi \in AC([\varphi(a), \varphi(b)])$, Lemma B.0.1 ensures that $\dot\phi(\varphi(\cdot))\dot\varphi(\cdot)$ is in $L^1(a,b)$. Moreover for every $t \in [a,b]$

$$\phi(\varphi(t)) - \phi(\varphi(a)) = \int_{\varphi(a)}^{\varphi(t)} \dot\phi(y)\,\mathrm{d}y = \int_a^t \dot\phi(\varphi(s))\dot\varphi(s)\,\mathrm{d}s,$$

so we conclude. □

Remark B.0.7. With a similar proof one can show that if $\phi \in W^{1,p}(\varphi(a), \varphi(b))$ for $p \in [1, +\infty]$, then $\phi \circ \varphi \in W^{1,p}(a,b)$ and the same formula for the derivative holds. In contrast with Remark B.0.3, for the validity of this fact we cannot replace $\varphi \in C^{0,1}([a,b])$ by $\varphi \in AC([a,b])$.

Theorem B.0.8 (Leibniz differentiation rule). *Let $\varphi \in C^{0,1}([0,T])$ be nondecreasing and let $a \leq \varphi(0)$. Consider the set $\Omega_T^\varphi := \{(t,y) \mid 0 \leq t \leq T,\ a \leq y \leq \varphi(t)\}$ and let $f\colon \Omega_T^\varphi \to \mathbb{R}$ be a measurable function such that:*

a) for every $t \in [0,T]$ it holds $f(t,\cdot) \in L^1(a,\varphi(t))$,

b) for a.e. $y \in [a,\varphi(T)]$ it holds $f(\cdot,y) \in AC(I_y)$, where $I_y = \{t \in [0,T] \mid y \le \varphi(t)\}$,

c) the partial derivative $\dfrac{\partial f}{\partial t}(t,y) := \lim\limits_{h\to 0} \dfrac{f(t+h,y)-f(t,y)}{h}$ (which for a.e. $y \in [a,\varphi(T)]$ is well defined for a.e. $t \in I_y$) is summable in Ω_T^φ.

Then the function $F(t) := \displaystyle\int_a^{\varphi(t)} f(t,y)\,\mathrm{d}y$ belongs to $AC([0,T])$ and moreover for a.e. $t \in [0,T]$

$$\dot F(t) = f(t,\varphi(t))\dot\varphi(t) + \int_a^{\varphi(t)} \frac{\partial f}{\partial t}(t,y)\,\mathrm{d}y. \tag{B.0.3}$$

Proof. To conclude we need to prove two things :

1) The right-hand side in (B.0.3) belongs to $L^1(0,T)$.

2) $F(t) = \displaystyle\int_a^{\varphi(T)} f(T,y)\,\mathrm{d}y - \int_t^T f(s,\varphi(s))\dot\varphi(s)\,\mathrm{d}s - \int_t^T\int_a^{\varphi(s)} \frac{\partial f}{\partial t}(s,y)\,\mathrm{d}y\,\mathrm{d}s$, for every $t \in [0,T]$.

To prove 1) notice that the integral part in the formula belongs to $L^1(0,T)$ by c) and Fubini's Theorem. To ensure that also $f(\cdot,\varphi(\cdot))\dot\varphi(\cdot) \in L^1(0,T)$ we argue as follows:

- By b) we know that for a.e. $y \in [a,\varphi(T)]$ it holds $f(t,y) = f(T,y) - \displaystyle\int_t^T \frac{\partial f}{\partial t}(s,y)\,\mathrm{d}s$ for every $t \in I_y$,
- since φ is continuous and nondecreasing we know that for a.e. $y \in [\varphi(0),\varphi(T)]$ there exists a unique element of $[0,T]$, denoted by $\varphi^{-1}(y)$, such that $\varphi(\varphi^{-1}(y)) = y$ (see the proof of Lemma B.0.1).

Hence $f(\varphi^{-1}(y),y) = f(T,y) - \displaystyle\int_{\varphi^{-1}(y)}^T \frac{\partial f}{\partial t}(s,y)\,\mathrm{d}s$ for a.e $y \in [\varphi(0),\varphi(T)]$, and so

$$\begin{aligned}\int_{\varphi(0)}^{\varphi(T)} |f(\varphi^{-1}(y),y)|\,\mathrm{d}y &\le \int_{\varphi(0)}^{\varphi(T)} |f(T,y)|\,\mathrm{d}y + \int_{\varphi(0)}^{\varphi(T)}\int_{\varphi^{-1}(y)}^T \left|\frac{\partial f}{\partial t}\right|(s,y)\,\mathrm{d}s\,\mathrm{d}y \\ &\le \|f(T,\cdot)\|_{L^1(a,\varphi(T))} + \left\|\frac{\partial f}{\partial t}\right\|_{L^1(\Omega_T^\varphi)} < +\infty.\end{aligned}$$

Using Lemma B.0.1 and recalling Corollary B.0.4 we deduce:

$$+\infty > \int_{\varphi(0)}^{\varphi(T)} |f(\varphi^{-1}(y),y)|\,\mathrm{d}y = \int_0^T |f(s,\varphi(s))|\dot\varphi(s)\,\mathrm{d}s.$$

Now we prove 2). Fix $t \in [0,T]$, then

$$\begin{aligned}F(t) &= \int_a^{\varphi(t)} f(t,y)\,\mathrm{d}y = \int_a^{\varphi(t)} f(T,y)\,\mathrm{d}y - \int_a^{\varphi(t)}\int_t^T \frac{\partial f}{\partial t}(s,y)\,\mathrm{d}s\,\mathrm{d}y \\ &= \int_a^{\varphi(T)} f(T,y)\,\mathrm{d}y - \int_{\varphi(t)}^{\varphi(T)} f(T,y)\,\mathrm{d}y - \int_t^T\int_a^{\varphi(t)} \frac{\partial f}{\partial t}(s,y)\,\mathrm{d}y\,\mathrm{d}s \\ &= \int_a^{\varphi(T)} f(T,y)\,\mathrm{d}y - \int_t^T\int_a^{\varphi(s)} \frac{\partial f}{\partial t}(s,y)\,\mathrm{d}y\,\mathrm{d}s - \int_{\varphi(t)}^{\varphi(T)} f(T,y)\,\mathrm{d}y + \int_t^T\int_{\varphi(t)}^{\varphi(s)} \frac{\partial f}{\partial t}(s,y)\,\mathrm{d}y\,\mathrm{d}s.\end{aligned}$$

So we conclude if we prove that

$$-\int_{\varphi(t)}^{\varphi(T)} f(T,y)\,\mathrm{d}y + \int_t^T \int_{\varphi(t)}^{\varphi(s)} \frac{\partial f}{\partial t}(s,y)\,\mathrm{d}y\,\mathrm{d}s = -\int_t^T f(s,\varphi(s))\dot{\varphi}(s)\,\mathrm{d}s.$$

This is true by the following computation:

$$\int_t^T \int_{\varphi(t)}^{\varphi(s)} \frac{\partial f}{\partial t}(s,y)\,\mathrm{d}y\,\mathrm{d}s = \int_{\varphi(t)}^{\varphi(T)} \int_{\varphi^{-1}(y)}^{T} \frac{\partial f}{\partial t}(s,y)\,\mathrm{d}s\,\mathrm{d}y$$
$$= \int_{\varphi(t)}^{\varphi(T)} f(T,y)\,\mathrm{d}y - \int_{\varphi(t)}^{\varphi(T)} f(\varphi^{-1}(y),y)\,\mathrm{d}y = \int_{\varphi(t)}^{\varphi(T)} f(T,y)\,\mathrm{d}y - \int_t^T f(s,\varphi(s))\dot{\varphi}(s)\,\mathrm{d}s.$$

All the equalities are justified by part 1), Lemma B.0.1 and Corollary B.0.4. □

Remark B.0.9. We can replace assumption a) in Theorem B.0.8 by the weaker

a′) $f(T,\cdot) \in L^1(a,\varphi(T))$.

Indeed exploiting b) and c) one can recover a) from a′).

Remark B.0.10. If for some $p \in [1,+\infty]$ the function f in Theorem B.0.8 satisfies

α) $f(t,\cdot) \in L^p(a,\varphi(t))$ for every $t \in [0,T]$,

β) $f(\cdot,y) \in W^{1,p}(I_y)$ for a.e. $y \in [a,\varphi(T)]$,

γ) $\dfrac{\partial f}{\partial t} \in L^p(\Omega_T^{\varphi})$,

then the function F belongs to $W^{1,p}(0,T)$ and the same formula for the derivative holds. As in Remark B.0.7, for the validity of this fact we cannot replace $\varphi \in C^{0,1}([a,b])$ by $\varphi \in AC([a,b])$.

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