

SYMMETRY RESULTS FOR VARIATIONAL ENERGIES ON CONVEX POLYGONS

DORIN BUCUR, ILARIA FRAGALÀ

ABSTRACT. We prove that, in the class of convex polygons with a given number of sides, the regular n -gon is optimal for some shape optimization problems involving the torsional rigidity, the principal frequency of the Laplacian, or the logarithmic capacity.

1. INTRODUCTION AND MAIN RESULTS

Investigating the symmetry of solutions to free boundary problems is a classic question in shape optimization. Several techniques have been developed towards this purpose, e.g. moving planes, rearrangements, variational methods, etc. A certain number of conjectures inspired by natural phenomena like shapes of drops, soap bubbles, circular drums have been solved. Meanwhile, there are some natural phenomena, like the bees honeycomb or the columnar jointing at Giant's Causeway where the observed structures are of polygonal type (see for instance [5, 19]), and the question is to understand why these structures pop up. The mathematical models may be very different from one problem to another, but usually they involve some variational energies for which the equilibrium is expected to lead to these polygonal structures. A natural associated mathematical question is to understand what are the optimal shapes for those variational energies as soon as the admissible geometries is restricted, for instance, to a class of polygons with a fixed number of sides. This rigidity may simplify some aspects of the mathematical analysis, but makes the search for symmetry a very hard task.

Aim of this paper is to provide some new isoperimetric-type inequalities for variational energies over the class of convex polygons with a given number of sides. The variational energies we consider are classical shape functionals, such as the torsional rigidity, the principal eigenvalue of the Laplacian, or the logarithmic capacity. Recall that, for every K in the class \mathcal{K}_0^2 convex compact sets with nonempty interior, they are defined respectively by

$$(1) \quad \tau(K) := - \inf_{u \in H_0^1(K)} \int_{\Omega} (|\nabla u|^2 - 2u) dx ,$$

$$(2) \quad \lambda_1(K) := \inf_{u \in H_0^1(K) \setminus \{0\}} \frac{\int_K |\nabla u|^2 dx}{\int_K |u|^2 dx} ,$$

$$(3) \quad \text{cap}(K) := \exp\left(- \lim_{|x| \rightarrow +\infty} (u(x) - \log|x|) \right) ,$$

where in the last definition u is the log-equilibrium potential of K (*cf.* eq. (10) below).

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The study of isoperimetric-type inequalities for these functionals is object of a long-lasting interest: started in the mid of last century with the fundamental contribution by Pólya-Szegő [21], it is still a hot topic in modern shape optimization (see for instance the recent monograph [16]).

Among the most relevant open problems in this field, a challenging conjecture by Pólya-Szegő predicts the extremality of the regular n -gon, for any of the functionals above, in the class of n -gons with a given area. While for $n = 3$ and $n = 4$ the conjecture was proved by Pólya-Szegő themselves via the classical tool of Steiner symmetrization (see [21, p.158], for $n \geq 5$ at present the unique solved case is the one of logarithmic capacity (see [24]).

Related contributions include [4, 13, 20], and also the present work is inspired by the conjecture. In fact our aim is to shed some light on the extremality properties of the regular n -gon: we show that it solves some optimization problems for any of the variational energies F introduced in (1)-(2)-(3) over the class \mathcal{P}_n of convex polygons with n sides, subject to different kinds of constraints. Precisely, the constraints we are going to consider are: i) first variation constraint; ii) symmetric content constraint; iii) perimeter constraint.

1.1. First variation constraint. The first problem we consider consists in minimizing, over the class \mathcal{P}_n , what we call the *isoperimetric quotient associated with F* , namely the dilation invariant shape functional

$$(4) \quad \mathcal{I}_F(K) := \frac{d}{dt} F^{\frac{1}{\alpha}}(K + tB) \Big|_{t=0}.$$

Here $K + tB$ denotes the Minkowski addition between K and the unit ball B , and α denotes the homogeneity degree of F under domain dilation, which is given by

$$(5) \quad \alpha = \begin{cases} 4 & \text{if } F = \tau \\ -2 & \text{if } F = \lambda_1 \\ 1 & \text{if } F = \text{cap} . \end{cases}$$

The terminology is justified by the fact that, if F is replaced by the volume functional, \mathcal{I}_F gives back $(1/n)|K|^{\frac{1}{n}-1}|\partial K|$, namely (a multiple of) the classical isoperimetric quotient. To make the expression of \mathcal{I}_F more explicit let us observe that, for $F = \tau$ and $F = \lambda_1$ (in which cases $\alpha \neq 1$), we have

$$(6) \quad \mathcal{I}_F(K) = \frac{1}{\alpha} F^{\frac{1}{\alpha}-1}(K) \frac{d}{dt} F(K + tB) \Big|_{t=0} = \frac{1}{|\alpha|} F^{\frac{1}{\alpha}-1}(K) |\mu_K|,$$

while for $F = \text{cap}$ (in which case $\alpha = 1$), we have

$$(7) \quad \mathcal{I}_F(K) = F(K) \frac{d}{dt} \log F(K + tB) \Big|_{t=0} = \frac{1}{2\pi} F(K) |\mu_K|.$$

As explained more in detail in Section 2.1 below, here $|\mu_K|$ represents the *total mass of the first variation measure* of F (for $F = \tau, \lambda_1$) and of $\log F$ (for $F = \text{cap}$). In all the cases under study, it can be written as

$$(8) \quad |\mu_K| := \int_{\partial K} |\nabla u_K|^2 d\mathcal{H}^1,$$

where u_K is the torsion function, the first Dirichlet eigenfunction of the Laplacian, or the logarithmic-potential of K , namely the unique solution to

$$(9) \quad -\Delta u = 1 \text{ in } K, \quad u = 0 \text{ on } \partial K,$$

$$(10) \quad -\Delta u = \lambda_1(K)u \text{ in } K, \quad u = 0 \text{ on } \partial K, \quad \int_K u^2 dx = 1,$$

$$(11) \quad \Delta u = 0 \text{ in } \mathbb{R}^2 \setminus K, \quad u = 0 \text{ on } \partial K, \quad u(x) \sim \log|x| \text{ as } |x| \rightarrow +\infty.$$

Thus, studying the shape optimization problem

$$(12) \quad \min \left\{ \mathcal{I}_F(P) : P \in \mathcal{P}_n \right\},$$

amounts to determine which polygons in the class \mathcal{P}_n , having a prescribed energy F , minimize the quantity $|\mu_P|$ in (8) (in other words, this amounts to determine polygons of given energy which are the “most stable” when perturbed by addition of a ball).

There are reasons to expect that the regular n -gon P_n^* is optimal. The first one is the well-known fact that P_n^* solves problem (12) in the geometric setting when F is replaced by the volume functional. Moreover, symmetry is suggested by the fact that, if we enlarge the class of competitors in (12) to all convex bodies in \mathcal{K}_0^2 , the solution is given by the ball. This has been proved in [18, Corollary 3.19] for $F = \lambda_1$, in [6, eq. (3.13)] for $F = \tau$, and in [25, Theorems 4.1 and 4.4] for $F = \text{cap}$, see also Remark 8 below.

Actually, we show that the following discrete version of these results holds true:

Theorem 1. *Let F be given by (1), (2), or (3). Then the regular n -gon P_n^* solves problem (12).*

The proof relies on a kind of *Minkowski first inequality for the functionals F* , namely an inequality which transposes to our variational setting the so-called Minkowski first inequality for volume [22, eq. (7.18)] (see Proposition 7).

More precisely, to obtain the optimality of the regular n -gon for problem (12), we proceed in two steps: in the first one we minimize \mathcal{I}_F in the class of polygons with fixed normals, and by means of Proposition 7 we show that the optimal polygon is circumscribed to a disk. Then we observe that, for circumscribed polygons, it holds

$$(13) \quad \mathcal{I}_F(P) = \frac{F^{\frac{1}{\alpha}}(P)}{\rho(P)},$$

where $\rho(P)$ denotes the inradius of P . Hence we conclude by invoking a result due to Solynin which states that, under a fixed inradius, the regular n -gon minimizes τ and cap , while it maximizes λ_1 (see [23, Theorem 2]).

1.2. Symmetric content constraint. The second problem we consider involves instead a purely geometrical constraint. Given $P \in \mathcal{P}_n$, we introduce the *symmetric content of P* as the quantity

$$\sigma(P) := \inf \left\{ \frac{1}{n} \sum_{i=1}^n h_P(\nu_i) : (\nu_1, \dots, \nu_n) \in \mathcal{E}_n(S^1) \right\}.$$

Here h_P is the *support function of P* (whose definition is recalled in Section 2.1), and $\mathcal{E}_n(S^1)$ denotes the class of n -tuples of vectors in S^1 which are *equidistributed*, meaning

that the angle formed by ν_i with ν_{i+1} is independent of $i \in \{1, \dots, n\}$ (with the convention $\nu_{n+1} = \nu_1$). To the best of our knowledge, the geometric quantity $\sigma(P)$ has not been previously considered in the literature. Some bounds for it, in terms of inradius and perimeter, are given in Section 2.1. In view of the homogeneity of degree 1 under dilations of the symmetric content, working under this constraint amounts to consider the shape optimization problem

$$(14) \quad \max \left\{ \frac{F(P)^{\frac{1}{\alpha}}}{\sigma(P)} : P \in \mathcal{P}_n \right\}.$$

Also in this case, we prove that the regular n -gon is optimal:

Theorem 2. *Let F be given by (1), (2), or (3). Then the regular n -gon P_n^* solves problem (14).*

The proof of Theorem 2 is somehow “dual” to that of Theorem 1, meaning that we apply again the variational version of Minkowski first inequality (Proposition 7), but with the roles of the two involved polygons interplayed. However, the analogy is not complete since in case of Theorem 2 we proceed in a single step (*cf.* Remark 15).

1.3. Perimeter constraint. Concerning the optimization problem

$$(15) \quad \max \left\{ \frac{F(P)^{\frac{1}{\alpha}}}{|\partial P|} : P \in \mathcal{P}_n \right\},$$

the regular n -gon is expected to be optimal as a consequence of Pólya-Szegő conjecture. Let us recall that, limited to the case $F = \text{cap}$, this has been established in [23, Theorem 10], but the same proof does not work for τ and λ_1 . In this direction, we move some step forward: inspired by the case of the constraints previously analyzed, we consider problem (15) when the admissible class is restricted to the family of polygons with given normals, or equivalently, with given inner angles.

Firstly we consider problem (15) over the subclass $\widehat{\mathcal{P}}_n$ of polygons having equal inner angles. As a consequence of Theorem 2 we obtain the optimality of the regular n -gon:

Theorem 3. *Let F be given by (1), (2), or (3), and let $\widehat{\mathcal{P}}_n$ denote the class of equiangular polygons in \mathcal{P}_n . Then the regular n -gon P_n^* solves*

$$(16) \quad \max \left\{ \frac{F(P)^{\frac{1}{\alpha}}}{|\partial P|} : P \in \widehat{\mathcal{P}}_n \right\}.$$

Then we consider problem (15) over the subclass $\mathcal{P}_n(\theta_1, \dots, \theta_n)$ of convex polygons having a fixed arbitrary n -uple $(\theta_1, \dots, \theta_n)$ of inner angles. We prove that the solution is unique, and that it can be identified with the only polygon in the admissible class which satisfies an overdetermined boundary value problem involving a condition of integral type on each side. Theorem 4 below should be compared with Theorem 1.3 in [14] and Theorems 5-8-11 in [11], where the analogous overdetermined problem was considered in the extremal cases $n = 3$ (triangles) and $n \rightarrow +\infty$ (arbitrary convex bodies). Note in particular that for $n = 3$ the situation is quite different: indeed, since all the elements of $\mathcal{P}_3(\theta_1, \theta_2, \theta_3)$ are homothetic, the functional $F^{1/\alpha}(P)/|\partial P|$ is constant on such class, and the overdetermined condition (17) is satisfied for all triangles.

Theorem 4. Let F be given by (1), (2), or (3), let $n \geq 4$, and let $\mathcal{P}_n(\theta_1, \dots, \theta_n)$ denote the class of polygons in \mathcal{P}_n with inner angles given by the ordered n -tuple $(\theta_1, \dots, \theta_n)$. Then there exists a unique polygon $\hat{P} \in \mathcal{P}_n(\theta_1, \dots, \theta_n)$ satisfying the family of overdetermined conditions

$$(17) \quad \int_{S_i} |\nabla u_{\hat{P}}|^2 = \kappa[g(\theta_i) + g(\theta_{i+1})] \quad \forall i = 1, \dots, n,$$

where $g(t) = \cot(t/2)$, S_i is the side adjacent to θ_i and θ_{i+1} , and κ is a positive constant (which is uniquely determined as $\kappa = \frac{|\alpha|F(\hat{P})}{|\partial\hat{P}|}$ for $F = \tau, \lambda_1$, and $\kappa = \frac{2\pi}{|\partial\hat{P}|}$ for $F = \text{cap}$).

Moreover, such polygon \hat{P} is the unique solution to

$$(18) \quad \max \left\{ \frac{F(P)^\alpha}{|\partial P|} : P \in \mathcal{P}_n(\theta_1, \dots, \theta_n) \right\}.$$

Remark 5. We address the open problem of finding a geometric characterization of the polygon \hat{P} , which could be quite helpful in order to make further progress towards the solution of problem (15). By now, the only property we are able to provide about \hat{P} is that, in general, it is *not* circumscribed around a disk; the proof of this fact is postponed to Section 3. Such piece of information is to some extent surprising, firstly by comparison with the behaviour of problem (12), and also because we shall see below (*cf.* Corollary 11) that \hat{P} satisfies the following identities holding as well for any circumscribed polygon in $\mathcal{P}_n(\theta_1, \dots, \theta_n)$ (with $g(t) = \cot(t/2)$):

$$(19) \quad \frac{F(P)}{|\partial P||\mu_P|} = \frac{1}{|\alpha| \sum_{i=1}^n [g(\theta_i) + g(\theta_{i+1})]} \quad \text{if } F = \tau, \lambda_1,$$

$$(20) \quad \frac{1}{|\partial P||\mu_P|} = \frac{1}{2\pi \sum_{i=1}^n [g(\theta_i) + g(\theta_{i+1})]} \quad \text{if } F = \text{cap}.$$

Thanks to (19)-(20), it turns out that \hat{P} satisfies

$$\frac{F^{1/\alpha}(\hat{P})}{|\partial\hat{P}|} = \frac{\mathcal{I}_F(\hat{P})}{|\alpha| \sum_{i=1}^n [g(\theta_i) + g(\theta_{i+1})]}.$$

Therefore, establishing the optimality of P_n^* for problem (15) is equivalent to showing that the right hand side of the above equality is maximal at P_n^* .

Eventually, as a consequence of the identities (19)-(20), we derive a further extremal property of the regular n -gon:

Theorem 6. Let F be given by (1), (2), or (3), and let \mathcal{P}_n^o denote the class of circumscribed polygons in \mathcal{P}_n . Then the regular n -gon P_n^* solves

$$(21) \quad \max \left\{ \frac{F(P)}{|\partial P||\mu_P|} : P \in \mathcal{P}_n^o \right\} \quad \text{if } F = \tau, \lambda_1$$

$$(22) \quad \max \left\{ \frac{1}{|\partial P||\mu_P|} : P \in \mathcal{P}_n^o \right\} \quad \text{if } F = \text{cap}.$$

The remaining of the paper is organized as follows. In Section 2 we provide some preliminary results, which concern: the Minkowski first inequality for variational functionals (Section 2.1), a representation formula for perimeter and its consequences (Section 2.2), and some properties of the symmetric content (Section 2.3). The proofs of Theorems 1 to 6 are then given in Section 3.

2. PRELIMINARY RESULTS

2.1. Minkowski first inequality for variational functionals. We start by recalling some properties that any of the functionals F introduced in (1)-(2)-(3) enjoys on the class \mathcal{K}_0^2 of convex bodies with nonempty interior.

– *Brunn-Minkowski inequality.*

For any $K, L \in \mathcal{K}_0^2$, and $t \in [0, 1]$, setting $(1-t)K + tL = \{(1-t)x + ty : x \in K, y \in L\}$, it holds

$$(23) \quad F^{\frac{1}{\alpha}}((1-t)K + tL) \geq (1-t)F^{\frac{1}{\alpha}}(K) + tF^{\frac{1}{\alpha}}(L),$$

with equality if and only if K and L are homothetic. For a proof, see [2, 8] when $F = \tau$, [3, 8] when $F = \lambda_1$, and [1, 9] when $F = \text{cap}$).

– *Integral representation of F and Hadamard variational formula.*

We call *first variation measure* of F (for $F = \tau, \lambda_1$) and of $\log F$ (for $F = \text{cap}$) the measure μ_K defined on S^1 by

$$(24) \quad \mu_K := (\nu_K)_\#(|\nabla u_K|^2 \mathcal{H}^1 \llcorner \partial K).$$

Here ν_K is the unit outer normal to K , which is defined \mathcal{H}^1 -a.e. on ∂K , and u_K denotes the unique solution respectively to problem (9)-(10)-(11), where the PDE has to be understood in the interior of K . (The meaning of the push-forward symbol $(\nu_K)_\#$ is that, for every $\varphi \in C_0(\mathbb{S}^{n-1})$, $\int_{S^1} \varphi d\mu_K = \int_{\partial K} \varphi \circ \nu_K |\nabla u_K|^2 d\mathcal{H}^1$).

In particular, when K is a polygon with sides S_i and normals ν_i , we have

$$(25) \quad \mu_K = \sum_{i=1}^n \left(\int_{S_i} |\nabla u_K|^2 d\mathcal{H}^1 \right) \delta_{\nu_i}.$$

Note that the total mass of μ_K agrees with the quantity introduced in (8):

$$|\mu_K| := \mu_K(S^1) = \int_{S^1} 1 d\mu_K = \int_{\partial K} |\nabla u_K|^2 d\mathcal{H}^1.$$

The reason of the terminology adopted above for the measure μ_K relies in the following integral representation formulae, holding on \mathcal{K}_0^2 for the functionals F . We adopt the standard notation to indicate the support function of K , defined by

$$h_K(\nu) := \sup_{x \in K} (x \cdot \nu) \quad \forall \nu \in \mathbb{S}^1.$$

In case of $F = \tau, \lambda_1$, we have:

$$(26) \quad F(K) = \frac{1}{|\alpha|} \int_{\partial K} h_K(\nu_K(x)) |\nabla u_K|^2(x) d\mathcal{H}^1(x) = \frac{1}{|\alpha|} \int_{S^1} h_K d\mu_K$$

$$(27) \quad \frac{d}{dt} F((1-t)K + tL)|_{t=0+} = (\text{sign } \alpha) \left(\int_{S^1} h_L d\mu_K - \int_{S^1} h_K d\mu_K \right)$$

(for a proof see [10] when $F = \tau$ and [17] when $F = \lambda_1$).

In case of $F = \text{cap}$, we have:

$$(28) \quad 1 = \frac{1}{2\pi} \int_{\partial K} h_K(\nu_K(x)) |\nabla u_K|^2(x) d\mathcal{H}^1(x) = \frac{1}{2\pi} \int_{S^1} h_K d\mu_K$$

$$(29) \quad \frac{d}{dt} \log F((1-t)K + tL)|_{t=0^+} = \frac{1}{2\pi} \left(\int_{S^1} h_L d\mu_K - \int_{S^1} h_K d\mu_K \right)$$

(for a proof see [25]).

We can now state the variational form of Minkowski first inequality. After the proof we outline its usefulness to derive isoperimetric inequalities over \mathcal{K}_0^2 (see Remark 8), while its application in the polygonal setting will appear in Section 3 (see Remarks 14, 15, 16).

Proposition 7. *Let $K, L \in \mathcal{K}_0^2$. We have*

$$\begin{cases} \int_{S^1} h_L d\mu_K \geq |\alpha| F^{1-\frac{1}{\alpha}}(K) F^{\frac{1}{\alpha}}(L) & \text{for } F = \tau, \lambda_1 \\ \int_{S^1} h_L d\mu_K \geq 2\pi F^{-1}(K) F(L) & \text{for } F = \text{cap}. \end{cases}$$

The equality sign holds in the above inequalities if and only if K and L are homothetic.

Proof. By the Brunn-Minkowski inequality (23), the map

$$t \mapsto \Phi(t) := F^{\frac{1}{\alpha}}((1-t)K + tL) - (1-t)F^{\frac{1}{\alpha}}(K) - tF^{\frac{1}{\alpha}}(L)$$

is concave on $[0, 1]$, and satisfies $\Phi(0) = \Phi(1) = 0$. Hence its right derivative at $t = 0$ (which exists by (27) and (29)) satisfies

$$(30) \quad \Phi'_+(0) \geq 0,$$

with equality if and only if Φ is identically 0 (which implies equality in (23)).

In case $F = \tau, \lambda_1$, by using (26)-(27), we obtain

$$\begin{aligned} \Phi'_+(0) &= \frac{\text{sign } \alpha}{\alpha} F^{\frac{1}{\alpha}-1}(K) \left(\int_{S^1} h_L d\mu_K - \int_{S^1} h_K d\mu_K \right) + F^{\frac{1}{\alpha}}(K) - F^{\frac{1}{\alpha}}(L) \\ &= \frac{1}{|\alpha|} F^{\frac{1}{\alpha}-1}(K) \int_{S^1} h_L d\mu_K - F^{\frac{1}{\alpha}}(L). \end{aligned}$$

Inserting the above equality into (30), the result follows.

In case $F = \text{cap}$, by using (28)-(29), we obtain

$$\begin{aligned} \Phi'_+(0) &= F(K) \frac{d}{dt} \log F((1-t)K + tL)|_{t=0^+} + F(K) - F(L) \\ &= \frac{1}{2\pi} F(K) \left(\int_{S^1} h_L d\mu_K - \int_{S^1} h_K d\mu_K \right) + F(K) - F(L) \\ &= \frac{1}{2\pi} F(K) \int_{S^1} h_L d\mu_K - F(L). \end{aligned}$$

Again, inserting the above equality into (30), the result follows. \square

Remark 8. (i) By applying Proposition 7 with K equal to arbitrary body in \mathcal{K}_0^2 , and L equal to the unit ball B , one obtains that balls solve the optimization problem (4) in which the class of competitors is enlarged to \mathcal{K}_0^2 . Indeed, let us consider just to fix the ideas the case when $F = \tau, \lambda_1$. Since $\int h_B d\mu_K = |\mu_K|$, Proposition 7 gives $|\mu_K| \geq |\alpha| F^{1-\frac{1}{\alpha}}(K) F^{\frac{1}{\alpha}}(B)$, or equivalently

$$(31) \quad \mathcal{I}_F(K) = \frac{1}{|\alpha|} F^{\frac{1}{\alpha}-1}(K) |\mu_K| \geq F^{\frac{1}{\alpha}}(B) = \mathcal{I}_F(B) \quad \forall K \in \mathcal{K}_0^2.$$

(ii) In a dual way, by applying Proposition 7 with L equal to arbitrary body in \mathcal{K}_0^2 , and K equal to the unit ball B , one obtains that balls solve the optimization problem

$$(32) \quad \max \left\{ \frac{F(K)^{\frac{1}{\alpha}}}{|\partial K|} : K \in \mathcal{K}_0^2 \right\},$$

Indeed (still arguing for $F = \tau, \lambda_1$, the case $F = \text{cap}$ being analogous), we have $d\mu_B = c d\theta$, where it is easily seen through formula (26) that $c = |\alpha| \frac{F(B)}{|\partial B|}$. Thus Proposition 7 gives $|\alpha| \frac{F(B)}{|\partial B|} |\partial L| \geq |\alpha| F^{\frac{1}{\alpha}}(L) F^{1-\frac{1}{\alpha}}(B)$, or equivalently

$$(33) \quad \frac{F^{\frac{1}{\alpha}}(L)}{|\partial L|} \leq \frac{F^{\frac{1}{\alpha}}(B)}{|\partial B|} \quad \forall L \in \mathcal{K}_0^2.$$

2.2. A representation formula for the perimeter of convex polygons. We recall that $\mathcal{P}_n(\theta_1, \dots, \theta_n)$ denotes the class of polygons in \mathcal{P}_n with inner angles given by the ordered n -tuple $(\theta_1, \dots, \theta_n)$; moreover, for every $P \in \mathcal{P}_n(\theta_1, \dots, \theta_n)$, we let ν_i be the unit outer normal to the side adjacent to θ_i and θ_{i+1} .

Proposition 9. *For every $P \in \mathcal{P}_n(\theta_1, \dots, \theta_n)$, it holds*

$$(34) \quad |\partial P| = \sum_{i=1}^n h_P(\nu_i) [g(\theta_i) + g(\theta_{i+1})].$$

Proof. For every fixed side S of P and every real parameter ε sufficiently small, let us denote by $P_\varepsilon(S)$ the perturbed polygon constructed in the following way: denoting by S' and S'' the two sides of ∂P adjacent to S , we replace the three sides (S, S', S'') by the new sides $(S_\varepsilon, S'_\varepsilon, S''_\varepsilon)$ defined as follows

- S_ε lies on the straight-line parallel to S having signed distance $\varepsilon h_P(\nu_S)$ from S (where ν_S stands for the unit outer normal to ∂P on S);
- S'_ε and S''_ε lie on the same straight-line containing respectively S' and S'' ;
- the lengths of $S_\varepsilon, S'_\varepsilon$ and S''_ε are chosen so that the three sides are consecutive (namely $(S_\varepsilon, S'_\varepsilon)$ and $(S_\varepsilon, S''_\varepsilon)$ have one point in common).

Then we set

$$P_\varepsilon := \bigcup_S P_\varepsilon(S),$$

where the union is taken over the family of sides of P . Note that P_ε is a non-convex polygon, with $n + 2n$ sides, as illustrated in Figure 1 in case of a quadrilateral.

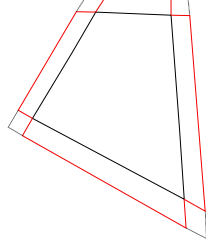


FIGURE 1. Construction of the polygon P_ε in the proof of Proposition 9

It follows from elementary geometric arguments that

$$|\partial P_\varepsilon| = |\partial P| + \varepsilon \sum_{i=1}^n h_P(\nu_i) [g(\theta_i) + g(\theta_{i+1})].$$

On the other hand, by construction, the polygon P_ε has the same perimeter as its convex hull $\text{conv}(P_\varepsilon)$, which agrees with the homothetic polygon $(1 + \varepsilon)P$. Hence,

$$|\partial P_\varepsilon| = |\partial P|(1 + \varepsilon).$$

Then (34) follows by combining the last two equalities. \square

Corollary 10. *For every $P \in \mathcal{P}_n(\theta_1, \dots, \theta_n)$, it holds*

$$(35) \quad \sum_{i=1}^n [g(\theta_i) + g(\theta_{i+1})] \nu_i = 0.$$

Proof. By the invariance of perimeter under translations and the representation formula (34) it follows that, for every $v \in \mathbb{R}^n$,

$$|\partial P| = \sum_{i=1}^n h_P(\nu_i) [g(\theta_i) + g(\theta_{i+1})] = \sum_{i=1}^n h_{P+v}(\nu_i) [g(\theta_i) + g(\theta_{i+1})] = |\partial(P + v)|,$$

Taking into account that

$$h_{P+v}(\nu_i) = h_P(\nu_i) + v \cdot \nu_i,$$

we infer that

$$v \cdot \sum_{i=1}^n [g(\theta_i) + g(\theta_{i+1})] \nu_i = 0,$$

and the statement follows from the arbitrariness of $v \in \mathbb{R}^n$. \square

Corollary 11. *The identities (19)–(20) hold true both for the polygon \widehat{P} given by Theorem 3 and for the polygon in $\mathcal{P}_n(\theta_1, \dots, \theta_n)$ which is circumscribed around a disk.*

Proof. For the polygon \widehat{P} given by Theorem 3, the identities (19)–(20) are obtained by summing the equalities (17) (for $i = 1, \dots, n$), and recalling the value of the constant κ appearing therein. For the circumscribed polygon in $\mathcal{P}_n(\theta_1, \dots, \theta_n)$, one has to use the representation formula (34) for perimeter, and the representation formula for F given in (26)–(28): the required identities follow by taking into account that the support function is equal to the inradius in each of the normal directions ν_i (for $i = 1, \dots, n$). \square

2.3. Some properties of the symmetric content.

Proposition 12. *For every $P \in \mathcal{P}_n$ it holds*

$$(36) \quad \sigma(P) \geq \rho(P), \quad \text{with equality if } P = P_n^* ;$$

$$(37) \quad \frac{\sigma(P)}{\sigma(P_n^*)} \geq \frac{|\partial P|}{|\partial P_n^*|}.$$

Remark 13. By inequality (37), Theorem 2 is weaker than the optimality of P_n^* for problem (15).

Proof. In view of the definition of $\sigma(P)$, inequality (36) follows from the lower bound $h_P(\nu) \geq \rho(P)$ holding for every $\nu \in S^1$. If $P = P_n^*$, the family $(\nu_1^*, \dots, \nu_n^*)$ of its outer normals belongs to $\mathcal{E}_n(S^1)$. So, by choosing $(\nu_1, \dots, \nu_n) = (\nu_1^*, \dots, \nu_n^*)$ in the definition of $\sigma(P_n^*)$, we obtain the converse inequality $\sigma(P_n^*) \leq \rho(P_n^*)$.

To show (37), we argue by contradiction. Assume that the opposite inequality holds. Then one could find $(\nu_1, \dots, \nu_n) \in \mathcal{E}_n(S^1)$ such that

$$\frac{1}{n\rho(P_n^*)} \sum_{i=1}^n h_P(\nu_i) \leq \frac{|\partial P|}{|\partial P_n^*|}$$

Since $|\partial P_n^*| = \rho(P_n^*)2n \cot\left(\frac{\theta_n}{2}\right)$, with $\theta_n := \frac{(n-2)\pi}{n}$, one is led to

$$\sum_{i=1}^n 2h_P(\nu_i) \cot\left(\frac{\theta_n}{2}\right) \leq |\partial P|.$$

But, by Proposition 9, the left hand side of the above inequality is equal to the perimeter of a convex polygon \widetilde{P} having all the inner angles equal to $\theta_n = \frac{(n-2)\pi}{n}$ and containing P , as illustrated in Figure 2 in case of a quadrilateral.

Then the above inequality contradicts the monotonicity of perimeter on convex polygons. \square

3. PROOF OF THE RESULTS IN SECTION 1

3.1. Proof of Theorem 1. We divide the proof in two steps.

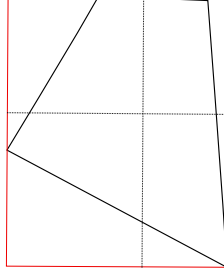


FIGURE 2. Construction of the polygon \tilde{P} in the proof of Proposition 12

Step 1. Let $P \in \mathcal{P}_n$, and let $P_0 \in \mathcal{P}_n$ denote the polygon which is circumscribed around the incircle of P and has the same normals as P . Then it holds $\mathcal{I}_F(P) \geq \mathcal{I}_F(P_0)$.

We are going to apply Proposition 7, with $K = P$ and $L = P_0$.

In case F equals τ or λ_1 , we obtain

$$(38) \quad \int_{S^1} h_{P_0} d\mu_P \geq |\alpha| F^{1-\frac{1}{\alpha}}(P) F^{\frac{1}{\alpha}}(P_0).$$

We observe that the measure μ_P is concentrated on the family of n normal directions to the boundary of P (cf. (25)), and in any of these directions, by the definition of P_0 , the value of the support function h_{P_0} is equal to the inradius of P_0 (which also agrees with the inradius of P). Therefore, denoting such inradius by $\rho(P_0)$, we have

$$(39) \quad \int_{S^1} h_{P_0} d\mu_P = \rho(P_0) |\mu_P|.$$

Inserting (39) into (38), and recalling (6), we obtain

$$\mathcal{I}_F(P) = \frac{1}{|\alpha|} |\mu_P| F^{\frac{1}{\alpha}-1}(P) \geq \frac{1}{\rho(P_0)} F^{\frac{1}{\alpha}}(P_0).$$

The proof of Step 1 is concluded by observing that, since P_0 is circumscribed, the r.h.s. of the above inequality agrees with $\mathcal{I}_F(P_0)$. Indeed, this is easily checked by using (6) and (26):

$$\mathcal{I}_F(P_0) = \frac{1}{|\alpha|} |\mu_{P_0}| F(P_0)^{\frac{1}{\alpha}-1} = \frac{1}{|\alpha|} |\mu_{P_0}| \left(\frac{1}{|\alpha|} \int_{S^1} h_{P_0} d\mu_{P_0} \right)^{\frac{1}{\alpha}-1} = \left(\frac{1}{|\alpha|} \right)^{\frac{1}{\alpha}} |\mu_{P_0}|^{\frac{1}{\alpha}} \rho(P_0)^{\frac{1}{\alpha}-1},$$

$$\frac{1}{\rho(P_0)} F^{\frac{1}{\alpha}}(P_0) = \frac{1}{\rho(P_0)} \left(\frac{1}{|\alpha|} \int_{S^1} h_{P_0} d\mu_{P_0} \right)^{\frac{1}{\alpha}} = \left(\frac{1}{|\alpha|} \right)^{\frac{1}{\alpha}} |\mu_{P_0}|^{\frac{1}{\alpha}} \rho(P_0)^{\frac{1}{\alpha}-1}.$$

In case F equals cap, we argue in a similar way. By applying Proposition 7, still with $K = P$ and $L = P_0$, we obtain

$$(40) \quad \int_{S^1} h_{P_0} d\mu_P \geq 2\pi F^{-1}(P) F(P_0).$$

Inserting (39) into (40), and recalling (7), we obtain

$$\mathcal{I}_F(P) = \frac{1}{2\pi} |\mu_P| F(P) \geq \frac{1}{\rho(P_0)} F(P_0).$$

Again, we conclude by observing that, since P_0 is circumscribed, the r.h.s. of the above inequality agrees with $\mathcal{I}_F(P_0)$. Indeed,

$$\mathcal{I}_F(P_0) = \frac{1}{2\pi} |\mu_{P_0}| F(P_0) = \frac{1}{\rho(P_0)} F(P_0),$$

where the first equality holds by (7) and the second one follows from (28).

Step 2. Let $P_0 \in \mathcal{P}_n$ be a circumscribed polygon, and let P_n^* be a regular n -gon. Then it holds $\mathcal{I}_F(P_0) \geq \mathcal{I}_F(P_n^*)$.

As shown above, since P_0 is circumscribed, in all the cases under study it holds

$$\mathcal{I}_F(P_0) = \frac{1}{\rho(P_0)} F^{\frac{1}{\alpha}}(P_0).$$

Clearly the same equality holds for P_n^* , since P_n^* as well is circumscribed. Then the claim in Step 2 follows from [23, Theorem 2]. \square

Remark 14. We point out that applying Proposition 7 directly with $K = P$ and $L = P_n^*$ does not allow to prove Theorem 1, but leads merely to the following inequalities

$$(41) \quad \begin{cases} \int_{S^1} h_{P_n^*} d\mu_P = \sum_{i=1}^n \gamma_i h_{P_n^*}(\nu_i) \geq |\alpha| F^{1-\frac{1}{\alpha}}(P) F^{\frac{1}{\alpha}}(P_n^*) & \text{for } F = \tau, \lambda_1 \\ \int_{S^1} h_{P_n^*} d\mu_P = \sum_{i=1}^n \gamma_i h_{P_n^*}(\nu_i) \geq 2\pi F^{-1}(P) F(P_n^*) & \text{for } F = \text{cap}, \end{cases}$$

where we have written $\mu_P = \sum_{i=1}^n \gamma_i \delta_{\nu_i}$, being ν_i the normals to the sides S_i of P and $\gamma_i = \int_{S_i} |\nabla u_P|^2 d\mathcal{H}^1$.

3.2. Proof of Theorem 2. Let F equal τ or λ_1 . By Proposition 7, applied with $L = P$ and $K = P_n^*$, we infer

$$(42) \quad \int_{S^1} h_P d\mu_{P_n^*} \geq |\alpha| F^{\frac{1}{\alpha}}(P) F^{1-\frac{1}{\alpha}}(P_n^*).$$

Next we observe that, by the symmetry of P_n^* , we can write

$$(43) \quad \mu_{P_n^*} = \gamma(P_n^*) \sum_{i=1}^n \delta_{\nu_i^*},$$

where $(\nu_1^*, \dots, \nu_n^*)$ are the outer normals to P_n^* and $\gamma(P_n^*)$ is a positive coefficient (equal to the integral over an arbitrary side of P_n^* of $|\nabla u_{P_n^*}|^2$).

By combining (43) with the relation (26) written for P_n^* , *i.e.*,

$$F(P_n^*) = \frac{1}{|\alpha|} \int_{S^1} h_{P_n^*} d\mu_{P_n^*},$$

we see that

$$(44) \quad \gamma(P_n^*) = \frac{|\alpha| F(P_n^*)}{n\rho(P_n^*)}.$$

In view of (43)-(44), the inequality (42) can be rewritten as

$$\frac{|\alpha|F(P_n^*)}{n\rho(P_n^*)} \sum_{i=1}^n h_P(\nu_i^*) \geq |\alpha|F^{\frac{1}{\alpha}}(P)F^{1-\frac{1}{\alpha}}(P_n^*),$$

or

$$\frac{F^{\frac{1}{\alpha}}(P)}{\frac{1}{n} \sum_{i=1}^n h_P(\nu_i^*)} \leq \frac{F^{\frac{1}{\alpha}}(P_n^*)}{\rho(P_n^*)}.$$

Since the outer normals $(\nu_1^*, \dots, \nu_n^*)$ are an arbitrary family of equidistributed unit vectors, by passing to the supremum over $(\nu_1^*, \dots, \nu_n^*) \in \mathcal{E}_n(S^1)$ in the above inequality, and recalling the definition of $\sigma(P)$, we obtain

$$\frac{F^{\frac{1}{\alpha}}(P)}{\sigma(P)} \leq \frac{F^{\frac{1}{\alpha}}(P_n^*)}{\rho(P_n^*)}.$$

To conclude it is enough to observe that $\rho(P_n^*) = \sigma(P_n^*)$ (cf. Proposition 12).

In case F equals cap, we proceed in a similar way. By Proposition 7, still applied with $L = P$ and $K = P_n^*$, we infer this time

$$(45) \quad \int_{S^1} h_P d\mu_{P_n^*} \geq 2\pi F(P)F^{-1}(P_n^*).$$

By writing $\mu_{P_n^*}$ as in (43), and exploiting (28), we obtain that the value of $\gamma(P_n^*)$ is given by

$$(46) \quad \gamma(P_n^*) = \frac{2\pi}{n\rho(P_n^*)}.$$

so that the inequality (45) can be rewritten as

$$\frac{2\pi}{n\rho(P_n^*)} \sum_{i=1}^n h_P(\nu_i^*) \geq 2\pi F(P)F^{-1}(P_n^*),$$

or

$$\frac{F(P)}{\frac{1}{n} \sum_{i=1}^n h_P(\nu_i^*)} \leq \frac{F(P_n^*)}{\rho(P_n^*)}.$$

From this point on, the proof proceeds unaltered as in the cases treated above. \square

Remark 15. Since Theorem 2 is obtained by applying Proposition 7 directly with $K = P_n^*$ and $L = P$, it can be regarded as a dual counterpart to the inequalities (41), rather than of Theorem 1.

3.3. Proof of Theorem 3. Let F be any of the three functionals $\tau, \lambda_1, \text{cap}$ under study, and let $P \in \widehat{\mathcal{P}}_n$. Since P is equiangular, we can choose a regular polygon P_n^* having the same normals $(\nu_1^*, \dots, \nu_n^*)$ as P . Following the same proof of Theorem 2, we arrive at the inequality

$$(47) \quad \frac{F_\alpha^{\frac{1}{\alpha}}(P)}{\frac{1}{n} \sum_{i=1}^n h_P(\nu_i^*)} \leq \frac{F_\alpha^{\frac{1}{\alpha}}(P_n^*)}{\rho(P_n^*)}.$$

Now, since by assumption P is equiangular and has normals $(\nu_1^*, \dots, \nu_n^*)$, the representation formula (34) in Proposition 9 gives

$$(48) \quad |\partial P| = 2g(\theta_n) \sum_{i=1}^n h_P(\nu_i^*), \quad \text{with } \theta_n = \frac{(n-2)\pi}{n}.$$

Inserting (48) into (47) we obtain

$$(49) \quad \frac{F_\alpha^{\frac{1}{\alpha}}(P)}{|\partial P|} \leq \frac{F_\alpha^{\frac{1}{\alpha}}(P_n^*)}{2ng(\theta_n)\rho(P_n^*)} = \frac{F_\alpha^{\frac{1}{\alpha}}(P_n^*)}{|\partial P_n^*|}.$$

□

Remark 16. Since Theorem 3 deals with n -gons having the same normals as the regular n -gon, it can be read as a dual counterpart of Step 1 in the proof of Theorem 1. The missing issue to obtain the optimality of the regular n -gon in problem (15) is understanding a dual counterpart to Step 2 in the proof of the same theorem.

3.4. Proof of Theorem 4. Let us first remark that, denoting by ν_i the unit outer normal to S_i , the unique possible value for the constant κ is easily found by multiplying both sides of (17) by $h_{\widehat{P}}(\nu_i)$, summing over $i = 1, \dots, n$, and exploiting formulas (26)-(28) and (34).

Next we observe that the system of equalities (17) is equivalent to ask that the measure $\mu_{\widehat{P}}$ agrees with the measure

$$\lambda := \kappa \sum_{i=1}^n [g(\theta_i) + g(\theta_{i+1})] \delta_{\nu_i},$$

where δ_{ν_i} is a Dirac mass concentrated at ν_i . Now, searching for a solution \widehat{P} to the equation $\mu_{\widehat{P}} = \lambda$ amounts to prescribe the first variation measure of F , namely corresponds to the so-called Minkowski problem for F . The results proved respectively in [18] (for $F = \lambda_1, \text{cap}$) and in [10] (for $F = \tau$) ensure that there exists a unique solution \widehat{P} , provided the measure λ is not concentrated on an equator of S^{n-1} and is balanced. Both these conditions are readily satisfied. Indeed, the condition that λ is not concentrated on an equator of S^{n-1} is satisfied because ν_i are the unit outer normals to any polygon in $\mathcal{P}_n(\theta_1, \dots, \theta_n)$, while the condition that λ is balanced is satisfied by Corollary 10. Hence there exists a unique polygon \widehat{P} satisfying the overdetermined conditions (17).

Next, the fact that \widehat{P} is a maximizer over $\mathcal{P}_n(\theta_1, \dots, \theta_n)$ of the quotient between $F^{1/\alpha}$ and perimeter readily follows by applying Proposition 7, with $K = \widehat{P}$ and L equal to an arbitrary polygon in $\mathcal{P}_n(\theta_1, \dots, \theta_n)$.

Finally, the uniqueness of a maximizer follows from Brunn-Minkowski inequality: if \widehat{P}_1 and \widehat{P}_2 were two distinct maximizers, not homothetic, one would have

$$\frac{F^{1/\alpha}\left(\frac{\widehat{P}_1+\widehat{P}_2}{2}\right)}{\left|\partial\left(\frac{\widehat{P}_1+\widehat{P}_2}{2}\right)\right|} > \frac{F^{1/\alpha}(\widehat{P}_1) + F^{1/\alpha}(\widehat{P}_2)}{|\partial\widehat{P}_1| + |\partial\widehat{P}_2|} \geq \min_{i=1,2} \frac{F^{1/\alpha}(\widehat{P}_i)}{|\partial\widehat{P}_i|},$$

yielding a contradiction since $\frac{\widehat{P}_1+\widehat{P}_2}{2}$ belongs to the admissible class $\mathcal{P}_n(\theta_1, \dots, \theta_n)$. \square

3.5. Proof of the claim in Remark 5. Here we show that, in general, the polygon \widehat{P} given by Theorem 4 is not circumscribed. We focus on the case of torsional rigidity and we argue by contradiction. For $n \geq 4$, we choose $\theta_1 = \frac{\pi}{2}$, $\theta_2 = \dots = \theta_n = \frac{(2n-5)\pi}{2(n-1)}$, and we denote by \widehat{P}_n the corresponding polygon satisfying the overdetermined condition (17). Assume by contradiction that \widehat{P}_n is circumscribed to a disc for every n . Up to a rescaling, we can assume that it circumscribes a disc of radius 1. Moreover, we can set the position of the vertices so that, for every n , the vertex of the angle θ_1 is fixed at the point $A_1 := (\sqrt{2}, 0)$. Then, as $n \rightarrow +\infty$, the sequence \widehat{P}_n will collapse to the set K which is the convex hull of the point A_1 and the disc of radius 1. Accordingly, the torsion functions u_n of \widehat{P}_n , extended by 0 on the complement of \widehat{P}_n , will converge strongly in $H^1(\mathbb{R}^2)$ to the torsion function u of K . We are going to show that

$$(50) \quad |\nabla u| = \text{constant} \quad \text{on } \{e^{i\theta} : \frac{\pi}{3} \leq \theta \leq 2\pi - \frac{\pi}{3}\} \subset \partial K.$$

Since K is not a disk, this contradicts the symmetry result for partially overdetermined problem proved in [12, Section 3.1] and achieves our proof. Let us prove (50). To that aim, let us fix a point $x = e^{i\theta}$, $\frac{\pi}{3} \leq \theta \leq 2\pi - \frac{\pi}{3}$ and a radius $\varepsilon \in (0, \sin \frac{\pi}{24})$, and let us compute the limiting behaviour, as $n \rightarrow +\infty$, of the sequence

$$\int_{\partial\widehat{P}_n \cap B_{x,\varepsilon}} |\nabla u_n|^2 ds.$$

On one hand, we have that

$$(51) \quad \int_{\partial\widehat{P}_n \cap B_{x,\varepsilon}} |\nabla u_n|^2 ds \rightarrow \int_{\Gamma \cap B_{x,\varepsilon}} |\nabla u|^2 ds$$

where $\Gamma := \{e^{i\theta} : \frac{\pi}{4} \leq \theta \leq 2\pi - \frac{\pi}{4}\} \subset \partial K$. To see this observe that, for any $\varphi \in C_0^\infty(\mathbb{R}^2)$ such that $\text{supp } \varphi \cap \partial K \subset \Gamma$, the sequence of measures $\varphi \cdot \mathcal{H}^1 \llcorner \partial\widehat{P}_n$ converge weakly $*$ to $\varphi \cdot \mathcal{H}^1 \llcorner \Gamma$. Then, by using [7, Proposition 2.2] and the fact that the extension operators from $H^1(\widehat{P}_n)$ to $H^1(\mathbb{R}^2)$ are uniformly bounded, we get (possibly after extracting a subsequence)

$$\int_{\partial\widehat{P}_n} |\nabla u_n|^2 \varphi ds \rightarrow \int_{\Gamma} |\nabla u|^2 \varphi ds.$$

which implies (51).

On the other hand, setting $C = \int_{\Gamma} |\nabla u|^2 ds$, we have that

$$(52) \quad \int_{\partial\widehat{P}_n \cap B_{x,\varepsilon}} |\nabla u_n|^2 ds \rightarrow C \frac{\mathcal{H}^1(\Gamma \cap B_{x,\varepsilon})}{\mathcal{H}^1(\Gamma)}.$$

This follows from the overdetermined condition on $\partial\widehat{P}_n$,

$$\forall 1 < i, j < n - 1, \int_{A_i^n A_{i+1}^n} |\nabla u_n|^2 ds = \int_{A_j^n A_{j+1}^n} |\nabla u_n|^2 ds,$$

taking into account that the L^∞ -norms of ∇u_n are uniformly bounded. This last assertion holds true since the sequence of $H^2(\Omega_n)$ -norms of u_n is bounded, because \widehat{P}_n are convex sets with bounded diameter and inradius bounded from below (see [15, identity 3.1.2.2])

Finally, combining (51) and (52), we obtain (50). \square

3.6. Proof of Theorem 6. It is enough to exploit the identities (19)–(20) holding by Corollary 11, and the convexity of the map $t \mapsto \cot(t/2)$ over $(0, \pi)$. \square

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(Dorin Bucur) UNIVERSITÉ SAVOIE MONT BLANC, LABORATOIRE DE MATHÉMATIQUES CNRS UMR 5127, CAMPUS SCIENTIFIQUE, 73376 LE-BOURGET-DU-LAC (FRANCE)

Email address: `dorin.bucur@univ-savoie.fr`

(Ilaria Fragalà) DIPARTIMENTO DI MATEMATICA, POLITECNICO DI MILANO, PIAZZA LEONARDO DA VINCI, 32, 20133 MILANO (ITALY)

Email address: `ilaria.fragala@polimi.it`