THE ŁOJASIEWICZ–SIMON INEQUALITY FOR THE ELASTIC FLOW

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ABSTRACT. We define the elastic energy of smooth immersed closed curves in $\mathbb{R}^n$ as the sum of the length and the L^2–norm of the curvature, with respect to the length measure. We prove that the L^2–gradient flow of this energy smoothly converges asymptotically to a critical point. One of our aims was to the present the application of a Łojasiewicz–Simon inequality, which is at the core of the proof, in a quite concise and versatile way.

CONTENTS

1. INTRODUCTION

We consider the flow by the gradient of the "classical" elastic energy associated to any regular closed curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^n$

$$\mathscr{E}(\gamma) = \int_{\mathbb{S}^1} \left(1 + \frac{1}{2}|k|^2\right) ds,$$

where k is the curvature vector of γ and $ds = |\gamma'(\theta)|\, d\theta$ denotes the canonical arclength measure of γ. We remark that $\mathscr{E}$ is a "geometric" functional, that is, its value is independent of the parametrization of the curve, moreover it is well defined for every $\gamma \in H^2(\mathbb{S}^1, \mathbb{R}^n) \subseteq C^1(\mathbb{S}^1, \mathbb{R}^n)$.

We will call $\tau = |\gamma'(\theta)|^{-1}\gamma'(\theta)$ the unit tangent vector of γ (which is well defined as γ is regular, that is, $|\gamma'(\theta)| \neq 0$ for every $\theta \in \mathbb{S}^1$) and we will denote by $\partial_s = |\gamma'(\theta)|^{-1}\partial_\theta$ the differentiation with respect to the arclength s of γ (where ∂_θ is the standard derivative with respect to $\theta \in \mathbb{S}^1$, so that $\gamma' = \partial_\theta \gamma$). Recall that then the curvature is given by

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$k = \partial^2_{ss}\gamma = \partial_s\tau$, which is a normal vector field along γ.
If $X : \mathbb{S}^1 \to \mathbb{R}^n$ is a vector field along γ, we define

$$\nabla^\perp X = \partial_s X - \langle \partial_s X, \tau\rangle\tau,$$

that is, the normal projection of the arclength derivative of X (this operator, restricted to normal vector fields along γ, coincides with the canonical connection on the normal bundle of γ in $\mathbb{R}^n$, which is compatible with the metric).

We will see in the next section that the "elastic flow" associated to the functional $\mathscr{E}$, of an initial smooth regular curve $\gamma_0 : \mathbb{S}^1 \to \mathbb{R}^n$, is given by a smooth solution $\gamma : [0,T) \times \mathbb{S}^1 \to \mathbb{R}^n$ of the PDE problem

$$\begin{cases} \partial_t\gamma = -\nabla^\perp\nabla^\perp k - |k|^2 k/2 + k \\ \gamma(0,\cdot) = \gamma_0 \end{cases} \tag{1.1}$$

where $k = k(t,\theta)$ is the curvature vector of the curve $\gamma(t,\theta)$ at time t. It is well known ([12] in codimension one and [6] in any codimension, see also [10]) that for every initial smooth regular curve γ_0, the elastic flow exists smooth uniquely for every positive time (that is, $T = +\infty$) and it "sub–converges" to a smooth critical point $\gamma_\infty : \mathbb{S}^1 \to \mathbb{R}^n$ of the functional $\mathscr{E}$. More precisely, we can state the following sub–convergence result.

Proposition 1.1 ([12, 6]). *Let $\gamma_0 : \mathbb{S}^1 \to \mathbb{R}^n$ be a smooth regular curve. Then there exists a unique smooth solution $\gamma : [0,+\infty) \times \mathbb{S}^1 \to \mathbb{R}^n$ of problem (1.1). Moreover, there exist a smooth critical point $\gamma_\infty : \mathbb{S}^1 \to \mathbb{R}^n$ of $\mathscr{E}$, a sequence of times $t_j \nearrow +\infty$ and a sequence of points $p_j \in \mathbb{R}^n$ such that*

$$\gamma(t_j,\cdot) - p_j \xrightarrow[j\to+\infty]{} \gamma_\infty,$$

in $C^m(\mathbb{S}^1,\mathbb{R}^n)$ for any $m \in \mathbb{N}$, up to reparametrization.

Our aim is to show that actually all the flow converges to a critical point γ_∞, as $t \to +\infty$.

Theorem 1.2. *Let $\gamma : [0,+\infty) \times \mathbb{S}^1 \to \mathbb{R}^n$ be a smooth solution of the elastic flow, then there exists a smooth critical point γ_∞ of $\mathscr{E}$ such that*

$$\gamma(t,\cdot) \xrightarrow[t\to+\infty]{} \gamma_\infty$$

in $C^m(\mathbb{S}^1,\mathbb{R}^n)$ for any $m \in \mathbb{N}$, up to reparametrization. In particular, there exists a compact set $K \subseteq \mathbb{R}^n$ such that $\gamma(t,\mathbb{S}^1) \subseteq K$ for any time $t \geq 0$.

Remark 1.3. We underline that, even if Theorem 1.2 implies that the solution of the elastic flow in $\mathbb{R}^n$ stays in a compact region, it does not tell anything about its shape. We believe it is a nice open question to quantify the size of such compact set, depending on the given initial datum γ_0. We also mention that a related problem proposed by G. Huisken is to determine whether the flow starting from a curve in the upper halfplane of $\mathbb{R}^2$, at some time is instead completely contained in the lower halfplane.

Remark 1.4. We observe that the conclusion of Theorem 1.2 can be extended to the flow by the gradient of the "modified" functional $\mathscr{E}_\lambda(\gamma) \int_{\mathbb{S}^1} \lambda + \frac{1}{2}|k|^2\, ds$, for every $\lambda > 0$. Moreover, we remark that the same result holds also for the elastic flow of curves in the 2–dimensional hyperbolic space or in the 2–dimensional sphere $\mathbb{S}^2$. More generally, we expect that it is possible to prove that in a complete, *homogeneous* Riemannian manifold (M^n, g) (that is, the group of isometries acts transitively on the manifold), the sub–convergence of the elastic flow can be improved to the full convergence. For a proof of these results and further comments we refer to [13]. We remark that the hypothesis of (M^n, g) being an analytic manifold (with analytic metric g) of bounded geometry is not sufficient, see [13, Appendix B].

2. The elastic functional

We first notice that the elastic functional $\mathscr{E}$ can be defined on every regular closed curve in $H^4(\mathbb{S}^1, \mathbb{R}^n)$, since, by Sobolev embedding theorem, such a curve belongs to $C^3(\mathbb{S}^1, \mathbb{R}^n)$, hence its unit tangent and curvature vector fields are well defined and continuous.

Assume that $\gamma : \mathbb{S}^1 \to \mathbb{R}^n$ is a smooth regular closed curve in $\mathbb{R}^n$ and $X \in H^4(\mathbb{S}^1, \mathbb{R}^n)$. If $|\varepsilon|$ is small enough, then $\gamma_\varepsilon = \gamma + \varepsilon X \in H^4(\mathbb{S}^1, \mathbb{R}^n)$ is still a regular curve, being $\gamma_\varepsilon \in C^3(\mathbb{S}^1, \mathbb{R}^n)$ and C^3–converging to γ as $\varepsilon \to 0$, again by Sobolev embedding theorem. Then, denoting with τ_ε and k_ε its unit tangent and curvature vector fields, respectively and letting ds_ε to be the arclength measure associated to γ_ε, we want to compute the first and second derivatives in ε of the function

$$\varepsilon \mapsto \mathscr{E}(\gamma_\varepsilon) = \mathscr{E}(\gamma + \varepsilon X) = \int_{\mathbb{S}^1} (1 + |k_\varepsilon|^2/2)\, ds_\varepsilon,$$

in order to get the first and second variations of $\mathscr{E}$ at γ, with the field X as infinitesimal generator of the "deformation" of γ.

We will denote with ∂_ε the partial derivative in ε, which clearly commutes with ∂_θ but not with ∂_s or $\nabla^\perp$ (see below).

In the next computations, we will need the following straightforward integration by parts formula,

$$\int_{\mathbb{S}^1} \left\langle \nabla^\perp X, Y \right\rangle\, ds = -\int_{\mathbb{S}^1} \left\langle X, \nabla^\perp Y \right\rangle\, ds, \tag{2.1}$$

holding for every couple of normal vector fields $X, Y \in H^1(\mathbb{S}^1, \mathbb{R}^n)$ along γ, coming from the standard formula

$$\int_{\mathbb{S}^1} \langle \partial_s X, Y \rangle\, ds = -\int_{\mathbb{S}^1} \langle X, \partial_s Y \rangle\, ds,$$

for every couple of general vector fields $X, Y \in H^1(\mathbb{S}^1, \mathbb{R}^n)$.

Moreover, if $X : \mathbb{S}^1 \to \mathbb{R}^n$ is a vector field along γ, we will denote with $X^\top$ and $X^\perp$, respectively the projection on the tangent or normal space of γ, that is,

$$X^\top(\theta) = \langle X(\theta), \tau(\theta) \rangle \tau(\theta) \qquad \text{and} \qquad X^\perp(\theta) = X(\theta) - X^\top(\theta).$$

It is easy to compute the variation of the arclength measure ds_ε associated to γ_ε,

$$\begin{aligned}\partial_\varepsilon ds_\varepsilon &= \partial_\varepsilon|\partial_\theta\gamma_\varepsilon|\,d\theta = \frac{\langle\partial_\varepsilon\partial_\theta\gamma_\varepsilon,\partial_\theta\gamma_\varepsilon\rangle}{|\partial_\theta\gamma_\varepsilon|}\,d\theta = \frac{\langle\partial_\theta\partial_\varepsilon\gamma_\varepsilon,\tau_\varepsilon\rangle}{|\partial_\theta\gamma_\varepsilon|}\,ds_\varepsilon = \langle\partial_s X,\tau_\varepsilon\rangle\,ds_\varepsilon\\ &= \big[\partial_s\langle X,\tau_\varepsilon\rangle - \langle X,k_\varepsilon\rangle\big]\,ds_\varepsilon\end{aligned}$$

as $\partial_\varepsilon\gamma_\varepsilon = X$. In order to proceed, we need the following "commutation" formula:

$$\partial_\varepsilon\partial_s f = \partial_\varepsilon\frac{\partial_\theta f}{|\gamma_\varepsilon'|} = \frac{1}{|\gamma_\varepsilon'|}\partial_\varepsilon\partial_\theta f - \Big\langle\frac{\partial_\theta\gamma_\varepsilon}{|\partial_\theta\gamma_\varepsilon|^3},\partial_\varepsilon\partial_\theta\gamma_\varepsilon\Big\rangle\,\partial_\theta f = \partial_s\partial_\varepsilon f - \big\langle\tau_\varepsilon,\partial_s X\big\rangle\,\partial_s f,$$

for every function $f:\mathbb{S}^1\to\mathbb{R}$. Hence, we can write

$$\partial_\varepsilon\partial_s = \partial_s\partial_\varepsilon - \langle\tau_\varepsilon,\partial_s X\rangle\,\partial_s = \partial_s\partial_\varepsilon - \partial_s\langle\tau_\varepsilon,X\rangle\,\partial_s + \langle k_\varepsilon,X\rangle\,\partial_s. \tag{2.2}$$

Then, we compute

$$\begin{aligned}\partial_\varepsilon\tau_\varepsilon &= \partial_\varepsilon\partial_s\gamma_\varepsilon\\ &= \partial_s\partial_\varepsilon\gamma_\varepsilon - \langle\tau_\varepsilon,\partial_s X\rangle\,\partial_s\gamma_\varepsilon\\ &= \partial_s X - \langle\tau_\varepsilon,\partial_s X\rangle\,\tau_\varepsilon\\ &= [\partial_s X]^\perp\\ &= [\partial_s(\langle\tau_\varepsilon,X\rangle\tau_\varepsilon + X^\perp)]^\perp\\ &= \nabla^\perp X^\perp + \langle\tau_\varepsilon,X\rangle k_\varepsilon\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}\partial_\varepsilon k_\varepsilon &= \partial_\varepsilon\partial_s\tau_\varepsilon\\ &= \partial_s\partial_\varepsilon\tau_\varepsilon - \partial_s\langle\tau_\varepsilon,X\rangle\,\partial_s\tau_\varepsilon + \langle k_\varepsilon,X\rangle\,\partial_s\tau_\varepsilon\\ &= \partial_s[\nabla^\perp X^\perp + \langle\tau_\varepsilon,X\rangle k_\varepsilon] - \partial_s\langle\tau_\varepsilon,X\rangle\,k_\varepsilon + \langle k_\varepsilon,X\rangle\,k_\varepsilon\\ &= \nabla^\perp\nabla^\perp X^\perp + \langle\partial_s\nabla^\perp X^\perp,\tau_\varepsilon\rangle\tau_\varepsilon + \langle\tau_\varepsilon,X\rangle\partial_s k_\varepsilon + \langle k_\varepsilon,X\rangle\,k_\varepsilon\\ &= \nabla^\perp\nabla^\perp X^\perp - \langle\nabla^\perp X^\perp,k_\varepsilon\rangle\tau_\varepsilon + \langle\tau_\varepsilon,X\rangle\partial_s k_\varepsilon + \langle k_\varepsilon,X\rangle\,k_\varepsilon,\end{aligned} \tag{2.4}$$

where we canceled the scalar products between orthogonal vectors. We then also get

$$\partial_\varepsilon|k_\varepsilon|^2 = 2\langle k_\varepsilon,\nabla^\perp\nabla^\perp X^\perp\rangle + 2\langle\tau_\varepsilon,X\rangle\,\langle k_\varepsilon,\partial_s k_\varepsilon\rangle + 2\langle k_\varepsilon,X\rangle\,|k_\varepsilon|^2,$$

which implies the first variation formula

$$\begin{aligned}
\delta\mathscr{E}_{\gamma_\varepsilon}(X) &:= \frac{d}{d\varepsilon}\mathscr{E}(\gamma_\varepsilon) \\
&= \int_{\mathbb{S}^1} \Big[\langle k_\varepsilon, \nabla^\perp\nabla^\perp X^\perp\rangle + \langle \tau_\varepsilon, X\rangle \langle k_\varepsilon, \partial_s k_\varepsilon\rangle + \langle k_\varepsilon, X\rangle\, |k_\varepsilon|^2 \\
&\qquad + (1+|k_\varepsilon|^2/2)\big[\partial_s\langle\tau_\varepsilon, X\rangle - \langle k_\varepsilon, X\rangle\big] \Big]\, ds_\varepsilon \\
&= \int_{\mathbb{S}^1} \Big[\langle \nabla^\perp\nabla^\perp k_\varepsilon, X^\perp\rangle + \langle \tau_\varepsilon, X\rangle \langle k_\varepsilon, \partial_s k_\varepsilon\rangle + \langle k_\varepsilon, X\rangle\, |k_\varepsilon|^2/2 \\
&\qquad + \partial_s\langle\tau_\varepsilon, X\rangle |k_\varepsilon|^2/2 - \langle k_\varepsilon, X\rangle \Big]\, ds_\varepsilon \\
&= \int_{\mathbb{S}^1} \Big[\langle \nabla^\perp\nabla^\perp k_\varepsilon, X\rangle + \langle k_\varepsilon, X\rangle\, |k_\varepsilon|^2/2 - \langle k_\varepsilon, X\rangle \Big]\, ds_\varepsilon \\
&= \int_{\mathbb{S}^1} \big\langle \nabla^\perp\nabla^\perp k_\varepsilon + |k_\varepsilon|^2 k_\varepsilon/2 - k_\varepsilon, X \big\rangle\, ds_\varepsilon,
\end{aligned}$$

where we integrated by parts in the second and third step.

In particular, for any smooth regular curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^n$, the $L^2(ds)$–gradient of the functional $\mathscr{E}$, giving rise to the definition of the elastic flow (1.1), is given by

$$\nabla^\perp\nabla^\perp k + |k|^2 k/2 - k$$

simply by evaluating at $\varepsilon = 0$. We notice that the first variation of $\mathscr{E}$ at γ only depends on the normal part $X^\perp$ of the vector field X along γ, being $\nabla^\perp\nabla^\perp k + |k|^2 k/2 - k$ a normal vector field along γ. This well known fact is due to the "geometric nature" of the functional $\mathscr{E}$, in particular to its invariance by reparametrization of the curves.

Remark 2.1. The above computation is also justified if γ is a regular curve in $H^4(\mathbb{S}^1, \mathbb{R}^n)$ and we are considering the first variation $\delta\mathscr{E}_\gamma$ as an element of $H^4(\mathbb{S}^1, \mathbb{R}^n)^\star$, defined by $\delta\mathscr{E}_\gamma(X) = \frac{d}{d\varepsilon}\mathscr{E}(\gamma + \varepsilon X)\big|_{\varepsilon=0}$. Indeed, $\delta\mathscr{E}_\gamma \in L^2(\mathbb{S}^1, \mathbb{R}^n)^\star$ and it is represented by the normal vector field $|\gamma'|\big(\nabla^\perp\nabla^\perp k + |k|^2 k/2 - k\big)$ along γ, with respect to the $L^2(d\theta)$–scalar product (and by $\nabla^\perp\nabla^\perp k + |k|^2 k/2 - k$ with respect to the $L^2(ds)$–scalar product).

A critical point of $\mathscr{E}$ is a regular curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^n$ of class H^4 such that $\delta\mathscr{E}_\gamma = 0$, that is, $\nabla^\perp\nabla^\perp k + |k|^2 k/2 - k = 0$. Standard regularity arguments imply that such a critical point is actually of class C^∞ (see for example the proof of [4, Proposition 4.1]). In particular, an elastic flow (1.1) starting from a critical point simply does not move.

Before dealing with the second variation of $\mathscr{E}$, we work out another commutation formula:

$$\begin{aligned}
\partial_\varepsilon \nabla^\perp Y &= \partial_\varepsilon[\partial_s Y - \langle \partial_s Y, \tau_\varepsilon\rangle \tau_\varepsilon] \\
&= \partial_\varepsilon \partial_s Y - \langle \partial_\varepsilon \partial_s Y, \tau_\varepsilon\rangle\tau_\varepsilon - \langle \partial_s Y, \partial_\varepsilon \tau_\varepsilon\rangle \tau_\varepsilon - \langle \partial_s Y, \tau_\varepsilon\rangle \partial_\varepsilon \tau_\varepsilon \\
&= [\partial_\varepsilon \partial_s Y]^\perp - \langle \partial_s Y, \partial_\varepsilon \tau_\varepsilon\rangle \tau_\varepsilon - \langle \partial_s Y, \tau_\varepsilon\rangle \partial_\varepsilon \tau_\varepsilon \\
&= [\partial_s \partial_\varepsilon Y - \langle \tau_\varepsilon, \partial_s X\rangle\, \partial_s Y]^\perp - \langle \partial_s Y, \partial_\varepsilon \tau_\varepsilon\rangle \tau_\varepsilon - \langle \partial_s Y, \tau_\varepsilon\rangle \partial_\varepsilon \tau_\varepsilon \\
&= \nabla^\perp \partial_\varepsilon Y - \langle \tau_\varepsilon, \partial_s X\rangle\, \nabla^\perp Y - \langle \partial_s Y, \nabla^\perp X^\perp + \langle \tau_\varepsilon, X\rangle k_\varepsilon\rangle \tau_\varepsilon \\
&\quad - \langle \partial_s Y, \tau_\varepsilon\rangle (\nabla^\perp X^\perp + \langle \tau_\varepsilon, X\rangle k_\varepsilon) \\
&= \nabla^\perp \partial_\varepsilon Y - \langle \tau_\varepsilon, \partial_s X\rangle\, \nabla^\perp Y - \langle \nabla^\perp Y, \nabla^\perp X^\perp\rangle \tau_\varepsilon - \langle \tau_\varepsilon, X\rangle \langle \nabla^\perp Y, k_\varepsilon\rangle \tau_\varepsilon \\
&\quad - \langle \partial_s Y, \tau_\varepsilon\rangle \nabla^\perp X^\perp - \langle \partial_s Y, \tau_\varepsilon\rangle \langle \tau_\varepsilon, X\rangle k_\varepsilon
\end{aligned}$$

where we used commutation formula (2.2) and (2.3).

In particular, if $Y = Y(\varepsilon)$ is a normal vector field along γ_ε for any ε, carrying in the last line the ∂_s derivative out of the scalar products, we get

$$\begin{aligned}
\partial_\varepsilon \nabla^\perp Y &= \nabla^\perp \partial_\varepsilon Y - \langle \tau_\varepsilon, \partial_s X\rangle\, \nabla^\perp Y - \langle \nabla^\perp Y, \nabla^\perp X^\perp\rangle \tau_\varepsilon - \langle \tau_\varepsilon, X\rangle \langle \nabla^\perp Y, k_\varepsilon\rangle \tau_\varepsilon \\
&\quad + \langle Y, k_\varepsilon\rangle \nabla^\perp X^\perp + \langle Y, k_\varepsilon\rangle \langle \tau_\varepsilon, X\rangle k_\varepsilon
\end{aligned}$$

and if also X is normal along γ, working analogously we conclude

$$\partial_\varepsilon \nabla^\perp Y = \nabla^\perp \partial_\varepsilon Y + \langle X, k_\varepsilon\rangle\, \nabla^\perp Y - \langle \nabla^\perp Y, \nabla^\perp X\rangle \tau_\varepsilon + \langle Y, k_\varepsilon\rangle \nabla^\perp X \tag{2.5}$$

at $\varepsilon = 0$.

By means of the above conclusion, we can write

$$\delta^2 \mathscr{E}_\gamma(X, X) := \frac{d^2}{d\varepsilon^2} \mathscr{E}(\gamma + \varepsilon X)\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon} \big\langle \nabla^\perp \nabla^\perp k_\varepsilon + |k_\varepsilon|^2 k_\varepsilon/2 - k_\varepsilon, X\big\rangle_{L^2(ds_\varepsilon)}\Big|_{\varepsilon=0},$$

that is,

$$\begin{aligned}
\delta^2 \mathscr{E}_\gamma(X, X) = &\int_{\mathbb{S}^1} \big\langle \partial_\varepsilon \big(\nabla^\perp \nabla^\perp k_\varepsilon + |k_\varepsilon|^2 k_\varepsilon/2 - k_\varepsilon\big)\big|_{\varepsilon=0}, X\big\rangle\, ds \\
&+ \int_{\mathbb{S}^1} \big\langle \nabla^\perp \nabla^\perp k + |k|^2 k/2 - k, X\big\rangle \big[\partial_s \langle X, \tau\rangle - \langle X, k\rangle\big]\, ds.
\end{aligned}$$

Since this is the case we are interested in, we assume that γ is a critical point of $\mathscr{E}$ (that is, $\delta \mathscr{E}_\gamma = 0$) and X is a normal vector field along γ, hence

$$\delta^2 \mathscr{E}_\gamma(X, X) = \int_{\mathbb{S}^1} \big\langle \partial_\varepsilon \big(\nabla^\perp \nabla^\perp k_\varepsilon + |k_\varepsilon|^2 k_\varepsilon/2 - k_\varepsilon\big)\big|_{\varepsilon=0}, X\big\rangle\, ds \tag{2.6}$$

being the second line above equal to zero, as $\nabla^\perp \nabla^\perp k + |k|^2 k/2 - k = 0$.

Assuming that X is a normal vector field along γ, by means of equations (2.5) and (2.4), we have

$$\begin{aligned}\partial_\varepsilon \nabla^\perp\nabla^\perp k_\varepsilon\,\big|_{\varepsilon=0} &= \nabla^\perp\partial_\varepsilon\nabla^\perp k_\varepsilon\,\big|_{\varepsilon=0} + \langle k, X\rangle\,\nabla^\perp\nabla^\perp k - \langle \nabla^\perp\nabla^\perp k, \nabla^\perp X\rangle\tau + \langle \nabla^\perp k, k\rangle\nabla^\perp X\\
&= \nabla^\perp\big[\nabla^\perp\partial_\varepsilon k_\varepsilon\,\big|_{\varepsilon=0} + \langle X, k\rangle\,\nabla^\perp k - \langle\nabla^\perp k, \nabla^\perp X\rangle\tau + \langle k,k\rangle\nabla^\perp X\big]\\
&\quad + \langle k, X\rangle\,\nabla^\perp\nabla^\perp k - \langle \nabla^\perp\nabla^\perp k, \nabla^\perp X\rangle\tau + \langle \nabla^\perp k, k\rangle\nabla^\perp X\\
&= \nabla^\perp\nabla^\perp\big[\nabla^\perp\nabla^\perp X - \langle\nabla^\perp X, k\rangle\tau + \langle k, X\rangle\,k\big]\\
&\quad + \nabla^\perp\big[\langle X,k\rangle\,\nabla^\perp k - \langle\nabla^\perp k, \nabla^\perp X\rangle\tau + |k|^2\nabla^\perp X\big]\\
&\quad + \langle k, X\rangle\,\nabla^\perp\nabla^\perp k - \langle \nabla^\perp\nabla^\perp k, \nabla^\perp X\rangle\tau + \langle \nabla^\perp k, k\rangle\nabla^\perp X.\end{aligned}$$

Hence, dropping the scalar products which are zero by orthogonality, we get

$$\int_{\mathbb{S}^1}\big\langle\partial_\varepsilon\big(\nabla^\perp\nabla^\perp k_\varepsilon\,\big|_{\varepsilon=0}, X\big\rangle\,ds = \int_{\mathbb{S}^1}\big\langle\nabla^\perp\nabla^\perp\nabla^\perp\nabla^\perp X + \Lambda(X), X\big\rangle\,ds$$

where $\Lambda(X)\in L^2(ds)$ is a normal vector field along γ, depending only on k, X and their "normal derivatives" $\nabla^\perp$ up to the third order, moreover the dependence on X is linear. The computation of the remaining term in equation (2.6),

$$\partial_\varepsilon\big(|k_\varepsilon|^2 k_\varepsilon/2 - k_\varepsilon\big)\,\big|_{\varepsilon=0}$$

is easier and follows immediately by equation (2.4), giving rise to another term similar to $\Lambda(X)$, linear in X and containing only "normal derivatives" $\nabla^\perp$ of k and X up to the second order.

Hence, we conclude

$$\delta^2\mathscr{E}_\gamma(X,X) = \int_{\mathbb{S}^1}\big\langle\nabla^\perp\nabla^\perp\nabla^\perp\nabla^\perp X + \Omega(X), X\big\rangle\,ds$$

where $\Omega(X)\in L^2(ds)$ is a normal vector field along γ, linear in X and depending only on k, X and their "normal derivatives" $\nabla^\perp$ up to the order three.

By polarization, we get the symmetric bilinear form on the space of the normal vector fields along γ in $H^4(\mathbb{S}^1,\mathbb{R}^n)$, giving the second variation of the functional $\mathscr{E}$ at γ:

$$\delta^2\mathscr{E}_\gamma(X,Y) = \int_{\mathbb{S}^1}\big\langle\nabla^\perp\nabla^\perp\nabla^\perp\nabla^\perp X + \Omega(X), Y\big\rangle\,ds = \big\langle\mathcal{L}(X), Y\big\rangle_{L^2(\mathbb{S}^1,\mathbb{R}^n)},$$

where we set

$$\mathcal{L}(X) := |\gamma'|\,\big((\nabla^\perp)^4 X + \Omega(X)\big)\,.$$

Remark 2.2. We observe that $\mathcal{L}$ and Ω are linear and continuous maps defined on the space of normal vector fields along γ in $H^4(\mathbb{S}^1,\mathbb{R}^n)$ and taking values in the normal vector fields along γ in $L^2(\mathbb{S}^1,\mathbb{R}^n)$, moreover Ω is a compact operator, by Sobolev embeddings. Therefore, for any normal vector field X in $H^4(\mathbb{S}^1,\mathbb{R}^n)$, we have that $\delta^2\mathscr{E}_\gamma(X,\cdot)$ can be seen as an element of the dual of the space of the normal vector fields along γ in $L^2(\mathbb{S}^1,\mathbb{R}^n)$.

Remark 2.3. We refer to [13, Section 3.1] for the explicit full computation of the first and second variations of $\mathscr{E}$ in the general case of curves on manifolds, even without assuming that γ is a critical point of $\mathscr{E}$ and that X, Y are normal vector fields. For our purpose here, the previous computations are sufficient.

Definition 2.4. Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^n$ be a regular smooth closed curve in $\mathbb{R}^n$ and τ its unit tangent vector field. For $m \in \mathbb{N}$ we define the Sobolev spaces of normal vector fields along γ as

$$H^{m,\perp}_\gamma = \left\{ X \in W^{m,2}(\mathbb{S}^1, \mathbb{R}^n) \, : \, \langle \tau(\theta), X(\theta) \rangle = 0 \text{ for almost every } \theta \in \mathbb{S}^1 \right\},$$

where as usual $W^{0,2}(\mathbb{S}^1, \mathbb{R}^n) = L^2(\mathbb{S}^1, \mathbb{R}^n)$. Moreover, we denote with $L^{2,\perp}_\gamma = H^{0,\perp}_\gamma$ the normal vector fields along γ, belonging to $L^2(\mathbb{S}^1, \mathbb{R}^n)$.
We underline that, unless otherwise stated, the spaces L^p are endowed with the Lebesgue measure $d\theta$. In case it is convenient to consider another measure, like the arclength measure ds of a curve, we will specify $L^p(ds)$. Observe that if $\gamma : \mathbb{S}^1 \to \mathbb{R}^n$ is smooth and regular, then clearly $L^p(d\theta) = L^p(ds)$, for any $p \in [1, +\infty)$.

We conclude this section by showing that the second variation operator $\delta^2 \mathscr{E}_\gamma$ is Fredholm of index zero. We recall that by Remark 2.2, using this definition, we can consider $\delta^2 \mathscr{E}_\gamma : H^{4,\perp}_\gamma \to (L^{2,\perp}_\gamma)^\star$.

Lemma 2.5. *Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^n$ be a smooth regular curve. The operator $(\nabla^\perp)^4 : H^{4,\perp}_\gamma \to L^{2,\perp}_\gamma$ is Fredholm of index zero, and then same holds for the operators $\mathcal{L} : H^{4,\perp}_\gamma \to L^{2,\perp}_\gamma$ and $\delta^2 \mathscr{E}_\gamma : H^{4,\perp}_\gamma \to (L^{2,\perp}_\gamma)^\star$.*

Proof. Since $\delta^2 \mathscr{E}_\gamma(X, Y) = \langle \mathcal{L}(X), Y \rangle_{L^2(\mathbb{S}^1, \mathbb{R}^n)}$ and γ is regular, we have that $\delta^2 \mathscr{E}_\gamma : H^{4,\perp}_\gamma \to (L^{2,\perp}_\gamma)^\star$ is Fredholm of index zero if and only if $(\nabla^\perp)^4 + \Omega : H^{4,\perp}_\gamma \to L^{2,\perp}_\gamma$ is such. The operator $\Omega : H^{4,\perp}_\gamma \to L^{2,\perp}_\gamma$ is compact, thus $(\nabla^\perp)^4 + \Omega : H^{4,\perp}_\gamma \to L^{2,\perp}_\gamma$ is Fredholm of index zero if and only if the same holds for $(\nabla^\perp)^4 : H^{4,\perp}_\gamma \to L^{2,\perp}_\gamma$ (see [7, Section 19.1, Corollary 19.1.8]), and this happens if the operator $\mathrm{Id} + (\nabla^\perp)^4 : H^{4,\perp}_\gamma \to L^{2,\perp}_\gamma$ is invertible, where Id is the identity/inclusion map.

Indeed if $X \in H^{4,\perp}_\gamma$ is in the kernel of $\mathrm{Id} + (\nabla^\perp)^4$, then there must hold

$$0 = \int_{\mathbb{S}^1} \left\langle \nabla^\perp \nabla^\perp \nabla^\perp \nabla^\perp X + X, X \right\rangle ds = \int_{\mathbb{S}^1} |\nabla^\perp \nabla^\perp X|^2 + |X|^2 \, ds,$$

by means of (2.1), which implies that $X = 0$, and then $\mathrm{Id} + (\nabla^\perp)^4$ is injective.

It remains to prove that $\mathrm{Id} + (\nabla^\perp)^4 : H^{4,\perp}_\gamma \to L^{2,\perp}_\gamma$ is surjective. Let $Y \in L^{2,\perp}_\gamma$ and consider the continuous functional $\mathscr{F} : H^{2,\perp}_\gamma \to \mathbb{R}$ defined by

$$\mathscr{F}(X) = \int_{\mathbb{S}^1} \frac{1}{2} \left| (\nabla^\perp)^2 X \right|^2 + \frac{1}{2} |X|^2 - \langle X, Y \rangle \, ds.$$

An explicit computation shows that

$$(\nabla^\perp)^2 X = \partial_s^2 X + \left(2\langle \partial_s X, k \rangle + \langle X, \partial_s k \rangle \right) \tau - \langle \partial_s X, \tau \rangle k, \tag{2.7}$$

hence,

$$\int_{\mathbb{S}^1} |\partial_s^2 X|^2\, ds \le C(\gamma) \int_{\mathbb{S}^1} |(\nabla^\perp)^2 X|^2 + |\partial_s X|^2 + |X|^2\, ds.$$

Then, since

$$\int_{\mathbb{S}^1} |\partial_s X|^2\, ds = -\int_{\mathbb{S}^1} \langle X, \partial_s^2 X\rangle\, ds \le \varepsilon \int_{\mathbb{S}^1} |\partial_s^2 X|^2 + C(\varepsilon) \int_{\mathbb{S}^1} |X|^2,$$

for a suitable constant $C(\varepsilon)$, we conclude

$$\int_{\mathbb{S}^1} |X|^2 + |\partial_s X|^2 + |\partial_s^2 X|^2\, ds \le C(\gamma) \int_{\mathbb{S}^1} \frac{1}{2}|(\nabla^\perp)^2 X|^2 + \frac{1}{2}|X|^2\, ds,$$

which implies that the functional $\mathscr{F}$ is coercive. Therefore, by the direct methods of calculus of variations, it follows that there exists a minimizer Z of $\mathscr{F}$ in $H^{2,\perp}_\gamma$. In particular Z satisfies

$$\int_{\mathbb{S}^1} \big\langle (\nabla^\perp)^2 Z, (\nabla^\perp)^2 X \big\rangle + \langle Z, X\rangle\, ds = \int_{\mathbb{S}^1} \langle Y, X\rangle\, ds,$$

for every $X \in H^{2,\perp}_\gamma$. If we show that $Z \in H^{4,\perp}_\gamma$, then $Z + (\nabla^\perp)^4 Z = Y$, and surjectivity is proved. This follows by standard techniques, simply noticing that once writing the integrand of the functional $\mathscr{F}$ in terms of $\partial_s^2 X$, $\partial_s X$ and X, by means of equation (2.7), its dependence on the highest order term $\partial_s^2 X$ is quadratic and the "coefficients" are given by the geometric quantities of γ, which is smooth. □

3. An abstract Łojasiewicz–Simon gradient inequality

In this section we present a result from [13] collecting some conditions under which a Łojasiewicz–Simon gradient inequality holds (see [8, 9, 15] for a given energy functional. This result is stated in a purely functional analytic setting for an abstract energy functional, and it can be possibly applied to different evolution equations.

Following [3], we assume that V is a Banach space, $U \subseteq V$ is open and $\mathscr{E} : U \to \mathbb{R}$ is a map of class C^2. We denote with $\delta\mathscr{E} : U \to V^\star$ the differential and with $\mathscr{H} : U \to L(V, V^\star)$ the second differential (or Hessian) of $\mathscr{E}$, respectively. We assume that $0 \in U$ and we set $V_0 = \ker \mathscr{H}(0) \subseteq V$.

We recall that a closed subspace $S \subseteq V$ is said to be *complemented* if there exists a continuous *projection* $P : V \to V$ such that $\operatorname{Imm} P = S$ (a continuous projection is a linear and continuous map $P : V \to V$ such that $P \circ P = P$). In such a case, we denote by $P^\star : V^\star \to V^\star$ the *adjoint projection*.

Proposition 3.1 ([3, Corollary 3.11]). *Under the above notation, assume that $\mathscr{E} : U \to \mathbb{R}$ is analytic and $0 \in U$ is a critical point of $\mathscr{E}$, that is, $\delta\mathscr{E}(0) = 0$. Assume that V_0 is finite dimensional (therefore, it is complemented and has a projection map $P : V_0 \to V_0$) and there exists a Banach space $W \hookrightarrow V^\star$ (that is, we identify W with a subset of $V^\star$) such that*

(i) $\operatorname{Imm} \delta\mathscr{E} \subseteq W$ *and the map* $\delta\mathscr{E} : U \to W$ *is analytic (with the norm of* W*),*
(ii) $P^\star(W) \subseteq W$,
(iii) $\mathscr{H}(0)(V) = \ker P^\star \cap W$.

Then, there exist constants $C, \rho > 0$ and $\alpha \in (0, 1/2]$ such that

$$|\mathscr{E}(u) - \mathscr{E}(0)|^{1-\alpha} \le C\|\delta\mathscr{E}(u)\|_W,$$

for any $u \in B_\rho(0) \subseteq U$.

This proposition is a special case of Corollary 3.11 in [3], choosing $X = V$ and $Y = W$ therein. We can then prove the following consequence.

Corollary 3.2 ([13, Corollary 2.6]). *Let $\mathscr{E} : U \subseteq V \to \mathbb{R}$ be an analytic map, where V is a Banach space and $0 \in U$ is a critical point of $\mathscr{E}$. Suppose that we have a Banach space $W = Z^\star \hookrightarrow V^\star$, where $V \hookrightarrow Z$, for some Banach space Z, that $\operatorname{Imm}\delta\mathscr{E} \subseteq W$ and the map $\delta\mathscr{E} : U \to W$ is analytic (with the norm of W). Assume also that $\mathscr{H}(0) \in L(V, W)$ and it is Fredholm of index zero.*
Then the hypotheses of Proposition 3.1 are satisfied, and then there exist constants $C, \rho > 0$ and $\alpha \in (0, 1/2]$ such that

$$|\mathscr{E}(u) - \mathscr{E}(0)|^{1-\alpha} \le C\|\delta\mathscr{E}(u)\|_W,$$

for any $u \in B_\rho(0) \subseteq U$.

Proof. Let us denote $\mathcal{H} := \mathscr{H}(0) : V \to W$. By the hypotheses, the subspace $V_0 := \ker \mathcal{H}$ is finite dimensional, thus it is closed and complemented with a projection $P : V \to V$ such that $\operatorname{Imm} P = V_0$, moreover point (i) of Proposition 3.1 is satisfied.
We can write $V = V_0 \oplus V_1$, where $V_1 = \ker P$, then if $P^\star : V^\star \to V^\star$ is the adjoint projection, we see that also $V^\star = V_0^\star \oplus V_1^\star$ and

$$V_0^\star = \operatorname{Imm} P^\star, \qquad V_1^\star = \ker P^\star.$$

We let $J_0 : Z \to Z^{\star\star}$ to be the canonical isometric injection and we call $J : V \to Z^{\star\star}$ the restriction of J_0 to V. We claim that $\mathcal{H} : V \to W$ satisfies

$$\mathcal{H}^\star \circ J = \mathcal{H}. \tag{3.1}$$

where $\mathcal{H}^\star : W^\star \to V^\star$ is the adjoint of $\mathcal{H}$.
Indeed, since $\mathcal{H}$ is symmetric (it is a second differential), for any $v, u \in V$ and $F := J(u) \in J(V) \subseteq Z^{\star\star}$ we find

$$(\mathcal{H}^\star \circ J)(u)[v] = \mathcal{H}^\star(F)[v] = F(\mathcal{H}v) = J(u)(\mathcal{H}v) = \mathcal{H}v[u] = \mathcal{H}(u)[v].$$

As a general consequence of the fact that $\mathcal{H}$ is Fredholm of index zero, we have

$$\dim\ker\mathcal{H} = \dim\ker\mathcal{H}^\star,$$

indeed, index zero means that $\dim\ker\mathcal{H} = \dim\operatorname{coker}\mathcal{H}$, where we split W as

$$W = \operatorname{Imm}\mathcal{H} \oplus \operatorname{coker}\mathcal{H},$$

and $\operatorname{coker}\mathcal{H}$ is finite dimensional. Therefore, $W^\star = (\operatorname{Imm}\mathcal{H})^\star \oplus (\operatorname{coker}\mathcal{H})^\star$ and since $\ker\mathcal{H}^\star = (\operatorname{Imm}\mathcal{H})^\perp = (\operatorname{coker}\mathcal{H})^\star$, we conclude that

$$\dim\ker\mathcal{H}^\star = \dim(\operatorname{coker}\mathcal{H})^\star = \dim\operatorname{coker}\mathcal{H} = \dim\ker\mathcal{H}.$$

We claim that

$$J(\operatorname{Imm} P) = \ker\mathcal{H}^\star \cap J(V). \tag{3.2}$$

Indeed, by equality (3.1) we see that

$$\ker \mathcal{H} = \ker(\mathcal{H}^\star \circ J) = J^{-1}(\ker \mathcal{H}^\star)$$

and applying then J on both sides, we get $J(\mathrm{Imm}\, P) = \ker \mathcal{H}^\star \cap J(V)$, that is, formula (3.2).
Since $\mathrm{Imm}\, P = \ker \mathcal{H}$ and J is injective, we have $\dim \ker \mathcal{H} = \dim(J(\mathrm{Imm}\, P)) = \dim \ker \mathcal{H}^\star \cap J(V)$. Then, as $\dim \ker \mathcal{H} = \dim \ker \mathcal{H}^\star$, it follows that $\ker \mathcal{H}^\star \cap J(V) = \ker \mathcal{H}^\star$ and

$$J(\mathrm{Imm}\, P) = \ker \mathcal{H}^\star.$$

Therefore, recalling that $V^{\star\star} \hookrightarrow W^\star$ and that $W \hookrightarrow V^\star$, we get

$$\begin{aligned}
(\ker \mathcal{H}^\star)^\perp &= \big\{w \in W \;:\; \langle f, w\rangle_{W^\star, W} = 0 \;\; \forall f \in J(\mathrm{Imm}\, P)\big\} \\
&= \big\{w \in W \;:\; \langle J(v), w\rangle_{W^\star, W} = 0 \;\; \forall v \in \mathrm{Imm}\, P\big\} \\
&= \big\{w \in W \;:\; \langle w, v\rangle_{V^\star, V} = 0 \;\; \forall v \in \mathrm{Imm}\, P\big\} \\
&= (\mathrm{Imm}\, P)^\perp \cap W.
\end{aligned}$$

Finally, as $\mathrm{Imm}\, \mathcal{H}$ is closed, we have

$$\begin{aligned}
\mathrm{Imm}\, \mathcal{H} &= (\ker \mathcal{H}^\star)^\perp \\
&= (\mathrm{Imm}\, P)^\perp \cap W \\
&= \big\{f \in V^\star \;:\; \langle f, Pv\rangle_{V^\star, V} = 0 \;\; \forall v \in V\big\} \cap W \\
&= \ker P^\star \cap W,
\end{aligned}$$

then point (iii) of Proposition 3.1 is verified.
We are just left with proving point (ii), that is, $P^\star(Z^\star) \subseteq Z^\star$. We observe that if we check that $P^\star(Z^\star \cap V_0^\star) \subseteq Z^\star \cap V_0^\star$, then we are done, indeed we would get

$$P^\star(Z^\star) = P^\star(Z^\star \cap V_0^\star \oplus Z^\star \cap V_1^\star) = P^\star(Z^\star \cap V_0^\star) \subseteq Z^\star \cap V_0^\star \subseteq Z^\star.$$

If $f_0 \in Z^\star \cap V_0^\star$, writing any $v \in V$ as $v = v_0 \oplus v_1 \in V_0 \oplus V_1$, we get

$$P^\star(f_0)[v] = f_0(Pv) = f_0(v_0) = f_0(v_0) + f_0(v_1) = f_0(v),$$

indeed,

$$f_0(v_1) = (P^\star f_0)(v_1) = f_0(Pv_1) = f_0(0) = 0.$$

Hence, we proved that $P^\star f_0 = f_0$ for any $f_0 \in Z^\star \cap V_0^\star$, thus we got that $P^\star(Z^\star \cap V_0^\star) \subseteq Z^\star \cap V_0^\star$. □

We mention that a result equivalent to this corollary has been recently proved independently in [14].

4. Convergence of the elastic flow in the Euclidean space

As we said at the beginning of Section 2, if $\gamma : \mathbb{S}^1 \to \mathbb{R}^n$ is a regular closed curve in $H^4(\mathbb{S}^1, \mathbb{R}^n) \hookrightarrow C^3(\mathbb{S}^1, \mathbb{R}^n)$, there exists $\rho > 0$ such that $\gamma + X$ is still a regular curve, for any $X \in B_\rho(0) \subseteq H^4(\mathbb{S}^1, \mathbb{R}^n)$. Moreover, if γ is embedded, choosing such ρ small enough, the open set $U = \{x \in \mathbb{R}^n \, : \, d_\gamma(x) = d(x, \gamma) < \rho\}$ is a tubular neighborhood of γ with the property of *unique orthogonal projection*. The "projection" map $\pi : U \to \gamma(\mathbb{S}^1)$ turns out to be C^2 in U and given by $x \mapsto x - \nabla d^2_\gamma(x)/2$, moreover the vector $\nabla d^2_\gamma(x)$ is orthogonal to γ at the point $\pi(x) \in \gamma(\mathbb{S}^1)$, see [11, Section 4] for instance.
Then, given a curve $\theta \mapsto \sigma(\theta) = \gamma(\theta) + X(\theta)$ with $X \in B_\rho(0) \subseteq H^4(\mathbb{S}^1, \mathbb{R}^n)$, we define a map $\theta' : \mathbb{S}^1 \to \mathbb{S}^1$ as

$$\theta' = \theta'(\theta) = \gamma^{-1}\big[\pi\big(\gamma(\theta) + X(\theta)\big)\big],$$

noticing that it is C^2 and invertible if $\gamma'(\theta) + X'(\theta)$ is never parallel to the unit vector $\nabla d_\gamma(\gamma(\theta) + X(\theta))$, which is true if we have (possibly) chosen a smaller ρ (hence, $|X|$ and $|X'|$ are small and the claim follows as $\langle \gamma'(\theta), \nabla d_\gamma(x)\rangle \to 0$, as $x \to \gamma(\theta)$).
We consider the vector field along γ,

$$Y(\theta') = \nabla d^2_\gamma(\gamma(\theta) + X(\theta))/2$$

which, for every $\theta' \in \mathbb{S}^1$, is orthogonal to γ at the point $\pi(\gamma(\theta) + X(\theta)) = \gamma(\theta')$ by what we said above and the definition of $\theta' = \theta'(\theta)$, hence it is a normal vector field along the curve $\theta' \mapsto \gamma(\theta')$. Thus, we have

$$\begin{aligned}
\gamma(\theta') + Y(\theta') &= \pi\big(\gamma(\theta) + X(\theta)\big) + \nabla d^2_\gamma(\gamma(\theta) + X(\theta))/2 \\
&= \gamma(\theta) + X(\theta) - \nabla d^2_\gamma(\gamma(\theta) + X(\theta))/2 + \nabla d^2_\gamma(\gamma(\theta) + X(\theta))/2 \\
&= \gamma(\theta) + X(\theta)
\end{aligned}$$

and we conclude that the curve $\sigma = \gamma + X$ can be described by the (reparametrized) regular curve $\widetilde{\sigma} = \gamma + Y$, with Y a normal vector field along γ in $H^4(\mathbb{S}^1, \mathbb{R}^n)$ as X, that is, $Y \in H^{4,\perp}_\gamma$. Moreover, it is clear that if $X \to 0$ in $H^4(\mathbb{S}^1, \mathbb{R}^n)$ then also $Y \to 0$ in $H^{4,\perp}_\gamma$.
All this can be done also for a regular curve γ which is only *immersed* (that is, it can have self–intersections), recalling that locally every immersion is an embedding and repeating the above argument a piece at a time along γ, getting also in this case a normal field Y describing a curve σ which is H^4–close enough to γ, that is $\|\sigma - \gamma\|_{H^4(\mathbb{S}^1, \mathbb{R}^n)} < \rho_\gamma$, for some $\rho_\gamma > 0$, as a "normal graph" on γ, as in the embedded case.

We recall now some further details about the sub–convergence of the elastic flow stated in Proposition 1.1. We set $\gamma_t = \gamma(t, \cdot)$ and we let γ_∞, t_j, p_j and $\overline{\gamma}_{t_j} = \overline{\gamma}(t_j, \cdot)$ be the reparametrization of γ_{t_j} as in Proposition 1.1, then

$$\overline{\gamma}_{t_j} - p_j \xrightarrow[j \to +\infty]{} \gamma_\infty$$

in $C^m(\mathbb{S}^1, \mathbb{R}^n)$ for any $m \in \mathbb{N}$. Moreover, there are positive constants $C_L = C_L(\gamma_0)$ and $C(m, \gamma_0)$, for any $m \in \mathbb{N}$, such that

$$\frac{1}{C_L} \leq L(\gamma_t) \leq C_L$$

and
$$\|(\nabla^\perp)^m k(t,\cdot)\|_{L^2(ds)} \leq C(m,\gamma_0) \tag{4.1}$$
for every $t \geq 0$. These facts follow from the results in [6, 12], see [6, Section 3] in particular.
It is then a straightforward computation to see that, if we describe a curve of the flow $\gamma_t = \gamma_\infty + X$, which is H^4–close enough to γ_∞ (precisely, $X \in B_\rho(0) \subseteq H^4(\mathbb{S}^1,\mathbb{R}^n)$, with $\rho = \rho_{\gamma_\infty}$ as above), as a "normal graph" on γ_∞, that is $\widetilde{\gamma} = \gamma_\infty + Y_t$ with $Y \in H^{4,\perp}_{\gamma_\infty}$, we have
$$\|Y_t\|_{H^m} \leq C(m,\gamma_0,\gamma_\infty)\,, \tag{4.2}$$
for every $m \in \mathbb{N}$.

Definition 4.1. Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^n$ be a regular curve of class H^4. We consider $\rho = \rho_\gamma > 0$ as above and we define the functional
$$E : B_\rho(0) \subseteq H^{4,\perp}_\gamma \to \mathbb{R} \qquad\qquad E(X) = \mathscr{E}(\gamma + X)\,.$$

By the conclusions of Section 2, we have
$$\delta E : B_\rho(0) \subseteq H^{4,\perp}_\gamma \to (L^{2,\perp}_\gamma)^\star,$$
given by $X \mapsto \delta E_X = \delta\mathscr{E}_{\gamma+X}$, acting as
$$\delta E_X(Y) = \Big\langle |\gamma' + X'| \Big(\big(\nabla^\perp_{\gamma+X}\big)^2 k_{\gamma+X} + |k_{\gamma+X}|^2 k_{\gamma+X}/2 - k_{\gamma+X}\Big), Y \Big\rangle_{L^2(\mathbb{S}^1,\mathbb{R}^n)}$$
on every $Y \in L^{2,\perp}_\gamma$.
The second variation $\delta^2 E_0$ of E at $0 \in H^{4,\perp}_\gamma$ clearly coincides with the second variation of $\mathscr{E}$ at γ, that is,
$$\delta^2 E_0 = \delta^2 \mathscr{E}_\gamma : H^{4,\perp}_\gamma \to (L^{2,\perp}_\gamma)^\star,$$
and we have
$$\delta^2 E_0(X,Y) = \big\langle \mathcal{L}(X), Y \big\rangle_{L^2(\mathbb{S}^1,\mathbb{R}^n)},$$
where $\mathcal{L}(X) = |\gamma'|\,\big((\nabla^\perp)^4 X + \Omega(X)\big)$.

Proposition 4.2. *Let $\gamma_\infty : \mathbb{S}^1 \to \mathbb{R}^n$ be a critical point of $\mathscr{E}$. Then there exist constants $C, \sigma > 0$ and $\alpha \in (0, 1/2]$ such that*
$$|\mathscr{E}(\gamma_\infty + Y) - \mathscr{E}(\gamma_\infty)|^{1-\alpha} \leq C\|\delta E_Y\|_{(L^{2,\perp}_{\gamma_\infty})^\star} \tag{4.3}$$
for any $Y \in B_\sigma(0) \subseteq H^{4,\perp}_{\gamma_\infty}$, where the functional E at the right hand side is relative to the curve γ_∞.

Proof. We apply Corollary 3.2 to the functional $E : B_\rho(0) \subseteq H^{4,\perp}_{\gamma_\infty} \to \mathbb{R}$, where $\rho > 0$ is as above, with $V = H^{4,\perp}_{\gamma_\infty}$, $W = (L^{2,\perp}_{\gamma_\infty})^\star$, and $Z = L^{2,\perp}_{\gamma_\infty}$. From the above discussion we have that the first variation δE (respectively, the second variation $\delta^2 E_0$, evaluated at 0) is defined on $B_\rho(0) \subseteq V$ (respectively, on V) and it is W–valued (the same for $\delta^2 E_0$). Moreover, $0 \in H^{4,\perp}_{\gamma_\infty}$ is a critical point of E, by assumption and we have that $\delta^2 E_0 : V \to W$ is a Fredholm operator of index zero by Lemma 2.5, as it coincides with $\delta^2 \mathscr{E}_{\gamma_\infty}$. Finally, both E and δE are analytic as maps between $B_\rho(0)$ and $\mathbb{R}$, W (with its norm) respectively (this can be proved directly by noticing that E and δE are compositions and sums of analytic functions – for a detailed proof of this fact we refer to [5, Lemma 3.4]). Therefore, we can apply Corollary 3.2 and we conclude that get that there exist constants $C, \sigma > 0$ and $\alpha \in (0, 1/2]$ such that

$$|E(Y) - E(0)|^{1-\alpha} \leq C\|\delta E_Y\|_{(L^{2,\perp}_{\gamma_\infty})^\star}$$

for any $Y \in B_\sigma(0) \subseteq H^{4,\perp}_{\gamma_\infty}$ and we are done. □

Now we are ready to prove the full convergence of the flow.

Proof of Theorem 1.2. As before, we set $\gamma_t = \gamma(t,\cdot)$ and we let γ_∞, t_j, p_j and $\overline{\gamma}_{t_j} = \overline{\gamma}(t_j,\cdot)$ be as in Proposition 1.1. Moreover, to simplify the notation we denote with $L^2(d\theta)$ the space $L^2(\mathbb{S}^1, \mathbb{R}^n)$.
We start with noticing that along the flow the elastic functional is monotone nonincreasing as

$$\frac{d}{dt}\mathscr{E}(\gamma_t) = -\int_{\mathbb{S}^1} \big|(\nabla^\perp)^2 k - |k|^2 k/2 + k\big|^2\, ds = -\|\partial_t \gamma\|^2_{L^2(ds)}$$

and actually we can assume that it is decreasing in every time interval, otherwise at some time t_0 the curve γ_{t_0} is a critical point, then the flow stops and the theorem clearly follows. As $i \leq j$ implies $t_i \leq t_j$, we have

$$\mathscr{E}(\overline{\gamma}_{t_i} - p_i) = \mathscr{E}(\gamma_{t_i}) \geq \mathscr{E}(\gamma_{t_j}) = \mathscr{E}(\overline{\gamma}_{t_j} - p_j)$$

which clearly implies, as $\overline{\gamma}_{t_j} - p_j \to \gamma_\infty$ in $C^m(\mathbb{S}^1, \mathbb{R}^n)$, that $\mathscr{E}(\gamma_{t_i}) = \mathscr{E}(\overline{\gamma}_{t_i} - p_i) \geq \mathscr{E}(\gamma_\infty)$, for every $i \in \mathbb{N}$ and $\mathscr{E}(\gamma_t) \searrow \mathscr{E}(\gamma_\infty)$, as $t \to +\infty$.
Thus, it is well defined the following positive function

$$H(t) = [\mathscr{E}(\gamma_t) - \mathscr{E}(\gamma_\infty)]^\alpha\,,$$

where $\alpha \in (0, 1/2]$ is given by Proposition 4.2. The function H is monotone decreasing and converging to zero as $t \to +\infty$ (hence, bounded above by $H(0) = [\mathscr{E}(\gamma_0) - \mathscr{E}(\gamma_\infty)]^\alpha$).

Now let $m \geq 6$ be a fixed integer. By Proposition 1.1, for any $\varepsilon > 0$ there exists $j_\varepsilon \in \mathbb{N}$ such that

$$\|\overline{\gamma}_{t_{j_\varepsilon}} - p_{j_\varepsilon} - \gamma_\infty\|_{C^m(\mathbb{S}^1,\mathbb{R}^n)} \leq \varepsilon \qquad \text{and} \qquad H(t_{j_\varepsilon}) \leq \varepsilon.$$

Choosing $\varepsilon > 0$ small enough, in order that

$$(\overline{\gamma}_{t_{j_\varepsilon}} - p_{j_\varepsilon} - \gamma_\infty) \in B_{\rho_{\gamma_\infty}}(0) \subseteq H^4(\mathbb{S}^1, \mathbb{R}^n),$$

by the argument at the beginning of this section (with $\gamma = \gamma_\infty$), for every t in some interval $[t_{j_\varepsilon}, t_{j_\varepsilon} + \delta)$ there exists $Y_t \in H^{4,\perp}_{\gamma_\infty}$ such that the curve $\widetilde{\gamma}_t = \gamma_\infty + Y_t$ is the "normal graph" reparametrization of $\gamma_t - p_{j_\varepsilon}$, hence

$$(\partial_t \widetilde{\gamma})^\perp = -(\nabla^\perp_{\widetilde{\gamma}_t})^2 k_{\widetilde{\gamma}_t} - |k_{\widetilde{\gamma}_t}|^2 k_{\widetilde{\gamma}_t}/2 + k_{\widetilde{\gamma}_t}$$

where $k_{\widetilde{\gamma}_t}$ is the curvature of $\widetilde{\gamma}_t$ (as the flow is invariant by translation and changing the parametrization of the evolving curves only affects the *tangential part* of the velocity). Since $\widetilde{\gamma}_{t_\varepsilon}$ is such reparametrization of $\overline{\gamma}_{t_{j_\varepsilon}} - p_{j_\varepsilon}$ and this latter is close in $C^m(\mathbb{S}^1, \mathbb{R}^n)$ to γ_∞, possibly choosing smaller $\varepsilon, \delta > 0$ above, it easily follows that for every $t \in [t_{j_\varepsilon}, t_{j_\varepsilon} + \delta)$ there holds

$$\|Y_t\|_{H^4} < \sigma,$$

where $\sigma > 0$ is as in Proposition 4.2 applied on γ_∞, and we possibly choose it smaller than the constant ρ_∞.

We want now to prove that if $\varepsilon > 0$ is sufficiently small, then actually we can choose $\delta = +\infty$ and $\|Y_t\|_{H^4} < \sigma$ for every time.

For E as in Proposition 4.2, we have

$$\begin{aligned}
[\mathscr{E}(\gamma_t) - \mathscr{E}(\gamma_\infty)]^{1-\alpha} &= [\mathscr{E}(\widetilde{\gamma}_t) - \mathscr{E}(\gamma_\infty)]^{1-\alpha} \\
&= [E(Y_t) - E(0)]^{1-\alpha} \\
&\leq C_1(\gamma_\infty, \sigma) \|\delta E_{Y_t}\|_{(L^{2,\perp}_{\gamma_\infty})^\star} \\
&= C_1(\gamma_\infty, \sigma) \sup_{\|S\|_{L^{2,\perp}_{\gamma_\infty}}=1} \int_{\mathbb{S}^1} \big\langle |\widetilde{\gamma}_t'| \big((\nabla^\perp_{\widetilde{\gamma}_t})^2 k_{\widetilde{\gamma}_t} + |k_{\widetilde{\gamma}_t}|^2 k_{\widetilde{\gamma}_t}/2 - k_{\widetilde{\gamma}_t} \big), S \big\rangle \, d\theta \\
&\leq C_1(\gamma_\infty, \sigma) \sup_{\|S\|_{L^2(\mathbb{S}^1,\mathbb{R}^n)}=1} \int_{\mathbb{S}^1} \big\langle |\widetilde{\gamma}_t'| \big((\nabla^\perp_{\widetilde{\gamma}_t})^2 k_{\widetilde{\gamma}_t} + |k_{\widetilde{\gamma}_t}|^2 k_{\widetilde{\gamma}_t}/2 - k_{\widetilde{\gamma}_t} \big), S \big\rangle \, d\theta \\
&= C_1(\gamma_\infty, \sigma) \left(\int_{\mathbb{S}^1} |\widetilde{\gamma}_t'|^2 \big| (\nabla^\perp_{\widetilde{\gamma}_t})^2 k_{\widetilde{\gamma}_t} + |k_{\widetilde{\gamma}_t}|^2 k_{\widetilde{\gamma}_t}/2 - k_{\widetilde{\gamma}_t} \big|^2 \, d\theta \right)^{1/2}
\end{aligned} \tag{4.4}$$

where we can assume that $C_1(\gamma_\infty, \sigma) \geq 1$.
Now, $\langle \widetilde{\gamma}_t, \tau_{\gamma_\infty} \rangle = \langle \gamma_\infty, \tau_{\gamma_\infty} \rangle$ is time independent, then $\langle \partial_t \widetilde{\gamma}, \tau_{\gamma_\infty} \rangle = 0$ and possibly taking a smaller $\sigma > 0$, we can suppose that $|\tau_{\gamma_\infty} - \tau_{\widetilde{\gamma}}| \leq \frac{1}{2}$ for any $t \geq t_{j_\varepsilon}$ such that $\|Y_t\|_{H^4} < \sigma$. Hence,

$$|(\partial_t \widetilde{\gamma})^\perp| = |\partial_t \widetilde{\gamma} - \langle \partial_t \widetilde{\gamma}, \tau_{\widetilde{\gamma}} \rangle \tau_{\widetilde{\gamma}}| = |\partial_t \widetilde{\gamma} + \langle \partial_t \widetilde{\gamma}, \tau_{\gamma_\infty} - \tau_{\widetilde{\gamma}} \rangle \tau_{\widetilde{\gamma}}| \geq |\partial_t \widetilde{\gamma}| - |\partial_t \widetilde{\gamma}| |\tau_{\gamma_\infty} - \tau_{\widetilde{\gamma}}| \geq \frac{1}{2} |\partial_t \widetilde{\gamma}|.$$

Differentiating H, we then get

$$\begin{aligned}
\frac{d}{dt} H(t) &= \frac{d}{dt} [\mathscr{E}(\widetilde{\gamma}_t) - \mathscr{E}(\gamma_\infty)]^\alpha \\
&= \alpha H^{\frac{\alpha-1}{\alpha}} \delta \mathscr{E}_{\widetilde{\gamma}_t}(\partial_t \widetilde{\gamma}) \\
&= -\alpha H^{\frac{\alpha-1}{\alpha}} \int_{\mathbb{S}^1} |\widetilde{\gamma}_t'| \big| (\nabla^\perp_{\widetilde{\gamma}_t})^2 k_{\widetilde{\gamma}_t} + |k_{\widetilde{\gamma}_t}|^2 k_{\widetilde{\gamma}_t}/2 - k_{\widetilde{\gamma}_t} \big|^2 \, d\theta \\
&\leq -\alpha H^{\frac{\alpha-1}{\alpha}} C_2(\gamma_\infty, \sigma) \left(\int_{\mathbb{S}^1} \big| (\partial_t \widetilde{\gamma})^\perp \big|^2 \, d\theta \right)^{1/2} \left(\int_{\mathbb{S}^1} |\widetilde{\gamma}_t'|^2 \big| (\nabla^\perp_{\widetilde{\gamma}_t})^2 k_{\widetilde{\gamma}_t} + |k_{\widetilde{\gamma}_t}|^2 k_{\widetilde{\gamma}_t}/2 - k_{\widetilde{\gamma}_t} \big|^2 \, d\theta \right)^{1/2} \\
&\leq -H^{\frac{\alpha-1}{\alpha}} C(\gamma_\infty, \sigma) \|\partial_t \widetilde{\gamma}\|_{L^2(d\theta)} [\mathscr{E}(\widetilde{\gamma}_t) - \mathscr{E}(\widetilde{\gamma}_\infty)]^{1-\alpha} \\
&= -C(\gamma_\infty, \sigma) \|\partial_t \widetilde{\gamma}\|_{L^2(d\theta)},
\end{aligned} \tag{4.5}$$

where $C(\gamma_\infty,\sigma)=\alpha C_2(\gamma_\infty,\sigma)/2C_1(\gamma_\infty,\sigma)$. This inequality clearly implies the estimate

$$C(\gamma_\infty,\sigma)\int_{\xi_1}^{\xi_2}\|\partial_t\widetilde{\gamma}\|_{L^2(d\theta)}\,dt\le H(\xi_1)-H(\xi_2)\le H(\xi_1) \tag{4.6}$$

for every $t_{j_\varepsilon}\le\xi_1<\xi_2<t_{j_\varepsilon}+\delta$ such that $\|Y_t\|_{H^4}<\sigma$. Hence, for such ξ_1,ξ_2 we have

$$\begin{aligned}\|\widetilde{\gamma}_{\xi_2}-\widetilde{\gamma}_{\xi_1}\|_{L^2(d\theta)}&=\left(\int_{\mathbb{S}^1}|\widetilde{\gamma}_{\xi_2}(\theta)-\widetilde{\gamma}_{\xi_1}(\theta)|^2\,d\theta\right)^{1/2}\\&\le\left(\int_{\mathbb{S}^1}\left(\int_{\xi_1}^{\xi_2}\partial_t\widetilde{\gamma}(t,\theta)\,dt\right)^2d\theta\right)^{1/2}\\&=\left\|\int_{\xi_1}^{\xi_2}\partial_t\widetilde{\gamma}\,dt\right\|_{L^2(d\theta)}\\&\le\int_{\xi_1}^{\xi_2}\|\partial_t\widetilde{\gamma}\|_{L^2(d\theta)}\,dt\\&\le\frac{H(\xi_1)}{C(\gamma_\infty,\sigma)}\\&\le\frac{\varepsilon}{C(\gamma_\infty,\sigma)},\end{aligned} \tag{4.7}$$

where we used that $H(\xi_1)\le H(t_{j_\varepsilon})\le\varepsilon$ and the fact that $\left\|\int_{\xi_1}^{\xi_2}v\,dt\right\|_{L^2(d\theta)}\le\int_{\xi_1}^{\xi_2}\|v\|_{L^2(d\theta)}\,dt$, holding for every smooth function $v:[\xi_1,\xi_2]\times\mathbb{S}^1\to\mathbb{R}^n$, indeed

$$\begin{aligned}\left\|\int_{\xi_1}^{\xi_2}v\,dt\right\|_{L^2(d\theta)}^2&\le\int_{\mathbb{S}^1}\left(\int_{\xi_1}^{\xi_2}v(t,\theta)\,dt\right)^2d\theta\\&=\int_{\mathbb{S}^1}\int_{\xi_1}^{\xi_2}v(t,\theta)\left(\int_{\xi_1}^{\xi_2}v(r,\theta)\,dr\right)dt\,d\theta\\&=\int_{\xi_1}^{\xi_2}\int_{\mathbb{S}^1}v(t,\theta)\left(\int_{\xi_1}^{\xi_2}v(r,\theta)\,dr\right)d\theta\,dt\\&\le\int_{\xi_1}^{\xi_2}\|v\|_{L^2(d\theta)}\left\|\int_{\xi_1}^{\xi_2}v\,dt\right\|_{L^2(d\theta)}dt\end{aligned}$$

and such inequality follows.

Therefore, for $t\ge t_{j_\varepsilon}$ such that $\|Y_t\|_{H^4}<\sigma$, we have

$$\|Y_t\|_{L^2(d\theta)}=\|\widetilde{\gamma}_t-\gamma_\infty\|_{L^2(d\theta)}\le\|\widetilde{\gamma}_t-\widetilde{\gamma}_{t_{j_\varepsilon}}\|_{L^2(d\theta)}+\|\widetilde{\gamma}_{t_{j_\varepsilon}}-\gamma_\infty\|_{L^2(d\theta)}\le\frac{\varepsilon}{C(\gamma_\infty,\sigma)}+\varepsilon\sqrt{2\pi}.$$

Then, by means of Gagliardo–Nirenberg interpolation inequalities (see [1] or [2], for instance) and estimates (4.2), for every $l\ge 4$, we have

$$\|Y_t\|_{H^l}\le C\|Y_t\|_{H^{l+1}}^a\|Y_t\|_{L^2(d\theta)}^{1-a}\le C(l,\gamma_0,\gamma_\infty,\sigma)\varepsilon^{1-a},$$

for some $a \in (0,1)$ and any $t \geq t_{j_\varepsilon}$ such that $\|Y_t\|_{H^4} < \sigma$.
In particular setting $l + 1 = m \geq 6$, if $\varepsilon > 0$ was chosen sufficiently small depending only on γ_0, γ_∞ and σ, then $\|Y_t\|_{H^4} < \sigma/2$ for any time $t \geq t_{j_\varepsilon}$, which means that we could have chosen $\delta = +\infty$ in the previous discussion.

Then, from estimate (4.7) it follows that $\widetilde{\gamma}_t$ is a Cauchy sequence in $L^2(d\theta)$ as $t \to +\infty$, therefore $\widetilde{\gamma}_t$ converges in $L^2(d\theta)$ as $t \to +\infty$ to some limit curve $\widetilde{\gamma}_\infty$ (not necessarily coincident with γ_∞). Moreover, by means of the above interpolation inequalities, repeating the argument for higher m we see that such convergence is actually in H^m for every $m \in \mathbb{N}$, hence in $C^m(\mathbb{S}^1, \mathbb{R}^n)$ for every $m \in \mathbb{N}$, by Sobolev embedding theorem. This implies that $\widetilde{\gamma}_\infty$ is a smooth critical point of $\mathscr{E}$. As the original flow γ_t is a fixed translation of $\widetilde{\gamma}_t$ (up to reparametrization) this finally completes the proof. □

Remark 4.3 (Why is the Łojasiewicz–Simon inequality necessary for the conclusion?). We briefly discuss the reason why the Łojasiewicz–Simon inequality gives a very strong improvement on the estimates and leads to the key inequality (4.7), which cannot be obtained simply by the standard variational evolution equation for the energy

$$\frac{d}{dt}\mathscr{E}(\gamma_t) = -\|\partial_t \gamma\|^2_{L^2(ds)},$$

that actually implies (in the notation and hypotheses of the previous proof)

$$\frac{d}{dt}[\mathscr{E}(\widetilde{\gamma}_t) - \mathscr{E}(\gamma_\infty)] \leq -C(\gamma_\infty, \sigma)\|\partial_t \widetilde{\gamma}\|^2_{L^2(d\theta)}.$$

This inequality is very similar to (4.5) (if we choose $\alpha = 1$ in the definition of H) and leads to the estimate

$$C(\gamma_\infty, \sigma)\int_{\xi_1}^{\xi_2} \|\partial_t \widetilde{\gamma}\|^2_{L^2(d\theta)}\, dt \leq H(\xi_1) - H(\xi_2) \leq H(\xi_1) \leq \varepsilon,$$

which differs by (4.6) *only for the exponent "2" on* $\|\partial_t \widetilde{\gamma}\|_{L^2(d\theta)}$ *inside the integral*.
This makes a lot of difference, since in this case we are actually "morally" estimating the integral on an infinite time interval

$$C(\gamma_\infty, \sigma)\int_{\xi_1}^{+\infty} \|\partial_t \widetilde{\gamma}\|^2_{L^2(d\theta)}\, dt \leq \varepsilon,$$

that is the L^2–*in time* norm of $\|\partial_t \widetilde{\gamma}\|_{L^2(d\theta)}$, while in the above proof, by means of the Łojasiewicz–Simon inequality, we got an estimate on the same function in the L^1–*in time* norm. The estimate in the L^1–*in time* norm is stronger because the time interval is unbounded and $\|\partial_t \widetilde{\gamma}\|_{L^2(d\theta)}$ is uniformly bounded by (4.1). Therefore, trying to use such an L^2–in time estimate in order to get an inequality analogous to (4.7), that is, of kind

$$\|\widetilde{\gamma}_{\xi_2} - \widetilde{\gamma}_{\xi_1}\|_{L^2(d\theta)} \leq C(\gamma_\infty, \sigma)\varepsilon,$$

clearly fails (anyway, such L^2–in time estimate is sufficient, and actually essential, in order to show the *sub–convergence* stated in Proposition 1.1). So an L^1–*in time* bound is absolutely needed and this is the reason of the key importance of the Łojasiewicz–Simon inequality in showing the asymptotic *full convergence* of the flow.

We also notice that even assuming to have an inequality (in the notation and hypotheses of Proposition 4.2) as

$$|\mathscr{E}(\gamma_\infty + Y) - \mathscr{E}(\gamma_\infty)| \leq C\|\delta E_Y\|_{(L^{2,\perp}_{\gamma_\infty})^\star},$$

that corresponds to the case $\alpha = 0$ in (4.3), this is not sufficient. Indeed, such an estimate is weaker than the Łojasiewicz–Simon inequality where $\alpha > 0$, because we are interested in the case that the norm of Y is small and so is the left hand side, and if we try to argue as in computations (4.4) and (4.5), choosing any $\beta > 0$ and setting $H = [\mathscr{E}(\widetilde{\gamma}_t) - \mathscr{E}(\gamma_\infty)]^\beta$, we obtain

$$\begin{aligned}
\frac{d}{dt}H(t) &= \frac{d}{dt}[\mathscr{E}(\widetilde{\gamma}_t) - \mathscr{E}(\gamma_\infty)]^\beta \\
&= \beta H^{\frac{\beta-1}{\beta}}\delta\mathscr{E}_{\widetilde{\gamma}_t}(\partial_t\widetilde{\gamma}) \\
&\leq -H^{\frac{\beta-1}{\beta}}C(\gamma_\infty,\sigma)\|\partial_t\widetilde{\gamma}\|_{L^2(d\theta)}[\mathscr{E}(\widetilde{\gamma}_t) - \mathscr{E}(\widetilde{\gamma}_\infty)] \\
&= -C(\gamma_\infty,\sigma)H\|\partial_t\widetilde{\gamma}\|_{L^2(d\theta)},
\end{aligned}$$

and we get the (weak) estimate

$$C(\gamma_\infty,\sigma)\int_{\xi_1}^{\xi_2}\|\partial_t\widetilde{\gamma}\|_{L^2(d\theta)}\,dt \leq \log H(\xi_1) - \log H(\xi_2)$$

which is, as before, clearly not sufficient to produce the necessary estimate on the L^1–*in time* norm of $\|\partial_t\widetilde{\gamma}\|_{L^2(d\theta)}$ on the time interval $[\xi_1,+\infty)$, as $\log H(\xi_2) \to -\infty$ as $\xi_2 \to +\infty$.

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