Geometric properties of planar BV-extension domains

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Dedicated to Professor Vladimir G. Maz'ya on his 70th birthday.

Abstract

In this note we investigate geometric properties of those planar domains that are extension for functions with bounded variation. We start from a characterization of such domains given by Burago–Maz'ya [BM] and prove that a bounded, simply connected domain is a BV-extension domain if and only if its complement is quasiconvex. We further prove that the extension property is a bi-Lipschitz invariant and give applications to Sobolev extension domains.

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a domain and $1 \le p \le \infty$. Recall that

$$BV(\Omega) = \{u \in L^1(\Omega) : \ |Du|(\Omega) < \infty\},$$

where

$$|Du|(\Omega) = \sup\Big\{ \int_\Omega u \text{ div } v \; dx : \ v = (v_1, v_2) \in C_0^\infty(\Omega; \mathbb{R}^2), |v| \le 1\},$$

and

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) : \ \nabla u \in L^p(\Omega, \mathbb{R}^2)\}.$$

Here ∇u is the distributional gradient of u. We employ these spaces with the norms $\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + |Du|(\Omega)$ and $\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}$. From the discussion in [EG] and [M],

$$|Du|(\Omega) = \inf\{\liminf_{k\to\infty} \int_\Omega |\nabla u_k| dx \ : \ u_k \in W^{1,1}_{loc}(\Omega), u_k \to u \text{ in } L^1(\Omega)\}, \qquad (1)$$

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where we also may replace $W^{1,1}(\Omega)$ with $C^\infty(\Omega)$.

In this note, we study geometric properties of those bounded, simply connected planar domains Ω that are extension domains for BV or for $W^{1,1}$. We say that a domain $\Omega \subset \mathbb{R}^2$ is a BV-extension domain if there exists a constant c and an extension operator $T : BV(\Omega) \to BV(\mathbb{R}^2)$, not necessarily linear, so that $Tu|_\Omega = u$ and $\|Tu\|_{BV(\mathbb{R}^2)} \le c\|u\|_{BV(\Omega)}$ for each $u \in BV(\Omega)$. Replacing BV by $W^{1,p}$ above gives the definition of a $W^{1,p}$-extension domain. For $p > 1$, $W^{1,p}$-extension domains admit a linear extension operator, but it appears to be unknown if this holds for $p = 1$ or for BV-extension domains. For other possible definitions of extension domains see Section 2 below.

The geometry of bounded, simply connected $W^{1,p}$-extension domains for $p = 2$ is well understood. Indeed, this class of domains coincides with the thoroughly investigated class of quasidisks (cf. [GLV], [GR], [GV], [J]) that allows for a number of geometric characterizations. For $p > 2$, one also has rather good geometric criteria for the extension property [K]. In the remaining range $1 \le p < 2$ for bounded simply connected domains, it is known that Ω has to be a so-called John domain (cf. [GR], [NV]) but no geometric characterization is available. Finally, Burago and Maz'ya [BM] have given a characterization for an extension property related to BV in terms of extendability of sets of finite perimeter in the domain. In fact, this seminal result by Burago and Maz'ya was the first characterization for Sobolev-type extensions and should be viewed as the predecessor of all the results mentioned above.

Our first result that partly relies on the work of Burago and Maz'ya [BM] (see also [M, Section 6.3.5]) gives a concrete characterization for bounded, simply connected BV-extension domains.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected domain. Then Ω is a BV-extension domain if and only if there exists a constant $C > 0$ such that for all $x, y \in \mathbb{R}^2 \setminus \Omega$ there is a rectifiable curve $\gamma \subset \mathbb{R}^2 \setminus \Omega$ connecting x and y with length $\ell(\gamma) \le C\,|x - y|$. That is, Ω is a BV-extension domain if and only if the complement of Ω is quasiconvex.*

As a corollary of this Theorem and Lemma 2.4 we obtain a new necessary condition for a bounded, simply connected domain to be a $W^{1,1}$-extension domain.

Corollary 1.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected domain that is a $W^{1,1}$-extension domain. Then the complement of Ω is quasiconvex.*

Simple examples such as a slit disk show that quasiconvexity of the complement does not characterize $W^{1,1}$-exendability. However, it is easy to check that quasiconvexity of the complement of Ω is a stronger requirement than Ω being a John domain or the complement of Ω being of bounded turning [GR]. Notice also that the complement of a quasidisk is quasiconvex. Consequently, the claim of Corollary 1.2 holds also in the $W^{1,2}$-extension setting. We conjecture that it, in fact, holds for all $1 \le p \le 2$.

Our second corollary deals with the invariance of the extension property under bi-Lipschitz mappings of Ω onto Ω'. This may seem trivial as bi-Lipschitz mappings preserve the spaces in question. The novelty here is that our bi-Lipschitz mapping is *a priori* only defined in the domain in question and extendability requires information in the entire plane.

Corollary 1.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected domain that is a BV-extension domain (or a $W^{1,1}$-extension domain) and let $f : \Omega \to \Omega' \subset \mathbb{R}^n$ be a bi-Lipschitz mapping. Then Ω' is also a BV-extension domain (or a $W^{1,1}$-extension domain).*

Corollary 1.3 leaves open the case $1 < p \leq \infty$, but the analog holds also in this case by a recent result from [HKT]. We conjecture that the assumption that Ω be simply connected in Corollary 1.3 is superfluous.

This note is organized as follows. In Section 2 we give the necessary preliminaries and discuss an alternative definition for an extension domain. Section 3 contains the proofs of the main results stated above. Finally, in Section 4, we discuss the meaning of Theorem 1.1 in a special case and briefly comment on possible generalizations of our result.

2 Preliminaries

The notation used in this note is as follows. Given $x \in \mathbb{R}^2$ and $r > 0$, the (open) disk centered at x with radius r will be denoted $B_r(x)$, and $S(x, r)$ denotes its boundary $\partial B_r(x)$. The 2-dimensional Lebesgue measure of a measurable set $A \subset \mathbb{R}^2$ is denoted $|A|$.

Burago and Maz'ya, in [BM], consider extension operators for

$$BV_l(\Omega) = \{u \in L^1_{\mathrm{loc}}(\Omega) : |Du|(\Omega) < +\infty\}.$$

They provide a necessary and sufficient condition for the existence of an extension operator $T_l : BV_l(\Omega) \to BV_l(\mathbb{R}^2)$ such that for all $u \in BV_l(\Omega)$,

$$|DT_l(u)|(\mathbb{R}^2) \leq c|Du|(\Omega). \tag{2}$$

We call such a domain a BV_l-extension domain. If $E \subset \mathbb{R}^2$ is a measurable set whose characteristic function χ_E lies in $BV_l(\Omega)$, then we say that E has finite perimeter in Ω, and denote $P(E, \Omega) := |D\chi_E|(\Omega)$. It follows from the Burago-Maz'ya characterization and the subadditivity property of the perimeter measure of a given set that it is necessary and sufficient to know that for every set $E \subset \Omega$ of finite perimeter $P(E, \Omega)$ in Ω there is a set $F \subset \mathbb{R}^2$ of finite perimeter such that $F \cap \Omega = E$ and $P(F, \mathbb{R}^2) \leq C\, P(E, \Omega)$.

Let us begin by pointing out that a bounded domain is a BV-extension domain in our sense if and only if it is a BV_l-extension domain. This can be seen e.g. via a modification of an argument of Herron and Koskela [HerK].

Lemma 2.1. *A bounded domain $\Omega \subset \mathbb{R}^2$ is a BV-extension domain if and only if it is a BV_l-extension domain.*

Towards the proof, we record a Poincaré type inequality resulting from compacness of a suitable embedding. It can be obtained by combining some results in [M] (see 6.1.7, 3.2.3 and 3.5.2). For convenience of the reader we give a simple proof below. Recall that a normed space X is said to embed compactly into another normed space Y if there is a bounded embedding map $\iota : X \to Y$ such that whenever $(a_k)_k$ is a norm-bounded sequence in X, the limit $\lim_j \iota(a_{k_j})$ exists in Y, for some subsequence $(a_{k_j})_j$. We call this embedding natural, if ι can be taken to be the identity map.

We continue with a simple observation.

Lemma 2.2. *Suppose that $\Omega \subset \mathbb{R}^2$ is a domain such that $BV(\Omega)$ embeds naturally compactly in $L^1(\Omega)$. Then $|\Omega| < \infty$.*

Proof. We define a function $m_\Omega : [0, \infty) \to \mathbb{R}$ by setting

$$m_\Omega(r) = |\Omega \cap B_r(0)|. \tag{3}$$

Then $m_\Omega \in Lip_{\mathrm{loc}}([0, +\infty))$ (with $m_\Omega(r) \leq \pi r^2$). Therefore m_Ω is differentiable almost everywhere and, by the coarea formula applied to the function $u(x) = (|x| - r)/h$ in the annular region $\Omega \cap B_{r+h}(0) \setminus B_r(0)$, at almost all points r of differentiability of m_Ω we have

$$m'_\Omega(r) = P(B_r(0), \Omega). \tag{4}$$

Let I be the set of all $r > 0$ that are points of differentiability of m_Ω and for which (4) holds true. We claim that

$$\liminf_{I \ni r \to +\infty} \frac{m'_\Omega(r)}{m_\Omega(r)} = 0.$$

In fact, if there are positive numbers M and r_M so that $m'_\Omega(r)/m_\Omega(r) \geq M$ for all $r \geq r_M$, then $m_\Omega(r) \geq m_\Omega(r_M) e^{M(r - r_M)}$, contradicting (3).

It follows from the above discussion that there exist $C > 0$ and a sequence $(r_n)_n$ from I with $r_n \to \infty$ such that $P(B_{r_n}(0), \Omega) = m'_\Omega(r_n) \leq C m_\Omega(r_n)$. We define a sequence of functions by setting

$$u_n = \frac{1}{m_\Omega(r_n)} \chi_{\Omega \cap B_{r_n}(0)}.$$

Then $\|u_n\|_{L^1(\Omega)} = 1$ and

$$|Du_n|(\Omega) = \frac{1}{m_\Omega(r_n)} P(B_{r_n}, \Omega) \leq C.$$

If the area of Ω were infinite, the sequence $(u_n)_n$ would converge uniformly to the zero function, and so this would be the only potential L^1-limit of a subsequence of $(u_n)_n$. Since $\|u_n\|_{L^1(\Omega)} = 1$, we would conclude that there is no subsequence that converges in $L^1(\Omega)$, contradicting our assumption. $\square$

In the next result and in what follows, for sets A with $0 < |A| < +\infty$ we write

$$u_A = \fint_A u\,dx = \frac{1}{|A|}\int_A u\,dx$$

whenever $u \in L^1(A)$.

Lemma 2.3. *If $\Omega \subset \mathbb{R}^2$ is a domain and $BV(\Omega)$ embeds naturally compactly into $L^1(\Omega)$, then there is a constant $C > 0$ such that whenever $u \in BV(\Omega)$,*

$$\int_\Omega |u - u_\Omega|\,dx \le C\,|Du|(\Omega). \tag{5}$$

Proof. By Lemma 2.2 and the hypothesis of this lemma, the measure of Ω must necessarily be finite. Suppose that for each positive integer n there is a function $u_n \in BV(\Omega)$ such that

$$\int_\Omega |u_n - (u_n)_\Omega|\,dx \ge n\,|Du_n|(\Omega).$$

By replacing u_n with $\left(\int_\Omega |u_n - (u_n)_\Omega|\,dx\right)^{-1}(u_n - (u_n)_\Omega)$, we may also assume that $\|u_n\|_{L^1(\Omega)} = 1$ and $(u_n)_\Omega = \fint_\Omega u_n\,dx = 0$. Then, by the above assumption, $|Du_n|(\Omega) \le n^{-1}$, and so the sequence (u_n) is bounded in $BV(\Omega)$, and hence there exists $w \in L^1(\Omega)$ such that, for some subsequence $(u_{n_j})_j$, $u_{n_j} \to w$ in $L^1(\Omega)$. Because

$$\lim_{n\to\infty} |Du_n|(\Omega) = 0,$$

we have that $w \in BV(\Omega)$ with $|Dw|(\Omega) = 0$. As Ω is connected, it follows (using the Poincaré inequality for BV, see [EG],[M]) that w is constant on Ω. On the other hand, $\int_\Omega w\,dx = \lim_n \int_\Omega u_n\,dx = 0$, but $\int_\Omega |w|\,dx = \lim_n \int_\Omega |u_n|\,dx = 1$, which is not possible if w is a constant function. This leads to a contradiction. $\square$

Proof of Lemma 2.1. First suppose that Ω is a bounded BV_l-extension domain, and let $T_l : BV_l(\Omega) \to BV_l(\mathbb{R}^2)$ be the bounded extension operator. Since $BV(\Omega) \subset BV_l(\Omega)$, for every $f \in BV(\Omega)$ the function $T_l f$ belongs to $BV_l(\mathbb{R}^2)$, with $|DT_l f|(\mathbb{R}^2) \le C|Df|(\Omega)$. Let B be a ball in $\mathbb{R}^2$ such that Ω is a relatively compact subdomain of B. Let $c_0 = (T_l f)_B$. By the Poincaré inequality

$$\int_B |T_l f - c_0|\,dx \le C\mathrm{diam}(B)|DT_l f|(B) \le C\mathrm{diam}(B)|Df|(\Omega).$$

Thus

$$\begin{aligned}|c_0| &\le \fint_\Omega |f - c_0|\,dx + \fint_\Omega |f|\,dx \le \frac{1}{|\Omega|}\int_B |T_l f - c_0|\,dx + \fint_\Omega |f|\,dx \\ &\le C|\Omega|^{-1}\mathrm{diam}(B)\left(|Df|(\Omega) + \int_\Omega |f|\,dx\right).\end{aligned}$$

Fix a Lipschitz function $\eta : \mathbb{R}^2 \to [0,1]$ with compact support in B such that $\eta = 1$ on Ω. We define our extension operator $E : BV(\Omega) \to BV(\mathbb{R}^2)$ by setting $Ef = \eta\, T_l f$. Now

$$\begin{aligned}\int_{\mathbb{R}^2} |Ef|\, dx &\le \int_B |T_l f|\, dx \le \int_B |T_l f - c_0|\, dx + |B|\,|c_0| \\ &\le C\,\mathrm{diam}(B)\left(|Df|(\Omega) + |B||\Omega|^{-1}|Df|(\Omega) + |B| \fint_\Omega |f|\, dx \right) \\ &\le C_0\, \|f\|_{BV(\Omega)}.\end{aligned}$$

Furthermore,

$$\begin{aligned}|DEf|(\mathbb{R}^2) &\le |DT_l f|(B) + \int_{\mathbb{R}^2} |T_l f||\nabla\eta|\, dx \\ &\le C\,|Df|(\Omega) + C\int_B |T_l f - c_0|\, dx + C|B|\,|c_0| \\ &\le C_1\, \|f\|_{BV(\Omega)}.\end{aligned}$$

This proves that E is bounded, and hence Ω is a BV-extension domain.

Now suppose that Ω is a bounded BV-extension domain. Let $T : BV(\Omega) \to BV(\mathbb{R}^2)$ be an extension operator. Fix a ball B so that $\Omega \subset B$. By the Rellich theorem for BV (see [EG],[M]), $T(BV(\Omega))\big|_B$ embeds naturally compactly into $L^1(B)$. Especially, $BV(\Omega)$ embeds naturally compactly into $L^1(\Omega)$. Hence, by Lemma 2.3, we have a constant $C > 0$ for which inequality (5) is satisfied by every $u \in BV(\Omega)$. For $u \in BV_l(\Omega)$ and each positive integer n, let $u_n(x) = \max\{-n, \min\{n, u(x)\}\}$. Then $u_n \in BV(\Omega)$ with $|Du_n|(\Omega) \le |Du|(\Omega)$ and $u_n \to u$ pointwise. Let $c_n = \fint_\Omega u_n\, dx$. Then $u_n - c_n \in BV(\Omega)$, and by inequality (5), $\|u_n - c_n\|_{BV(\Omega)} \le C\,|Du_n|(\Omega) \le C\,|Du|(\Omega)$. Hence, by the compactness of the embedding $BV(\Omega)$ into $L^1(\Omega)$, there is a subsequence, $(u_{n_k} - c_{n_k})_k$, converging in $L^1(\Omega)$ to a function $w \in L^1(\Omega)$. By passing to a further subsequence if necessary, we may also assume that $u_{n_k} - c_{n_k} \to w$ pointwise almost everywhere in Ω as well. Since $u_{n_k} \to u$ pointwise in Ω, it follows that the sequence $(c_{n_k})_k$ of real numbers converges to some $c_0 \in \mathbb{R}$. Therefore $w = u - c_0$, $u \in L^1(\Omega)$ and hence $u \in BV(\Omega)$, and $u_{n_k} - c_{n_k} \to u - c_0$ in $L^1(\Omega)$. Furthermore, $c_0 = \fint_\Omega u\, dx$

Because $u \in BV(\Omega)$, we have that $Tu \in BV(\mathbb{R}^2)$, but it is not clear if we can control the BV_l norm of Tu purely in terms of the BV_l norm of u. To fix this, we modify our extension operator by setting by $E(u) = T(u - c_0) + c_0$, where $c_0 = \fint_\Omega u\, dx$; then $E : BV_l(\Omega) \to BV_l(\mathbb{R}^2)$. Moreover,

$$|DE(u)|(\mathbb{R}^2) = |DT(u - c_0)|(\mathbb{R}^2) \le C\,\|u - c_0\|_{BV(\Omega)} \le C\,|Du|(\Omega),$$

where we used inequality (5) again to obtain the last inequality. This completes the proof. □

For completeness we include a simple proof for the following connection between Sobolev- and BV-extension domains.

Lemma 2.4. *A $W^{1,1}$-extension domain is necessarily a BV-extension domain.*

Proof. Let Ω be a $W^{1,1}$-extension domain, with a bounded extension operator $T : W^{1,1}(\Omega) \to W^{1,1}(\mathbb{R}^2)$, and let $u \in BV(\Omega)$. Then there is a sequence $(u_k)_k \subset W^{1,1}(\Omega)$ such that $u_k \to u$ in $L^1(\Omega)$, $\int_\Omega |\nabla u_k| dx \leq 2|Du|(\Omega)$, $\int_\Omega |u_k| dx \leq 2\int_\Omega |u| dx$, and $\lim_k \int_\Omega |\nabla u_k| dx = |Du|(\Omega)$. Let $v_k = Tu_k \in W^{1,1}(\mathbb{R}^2)$.

Since $\|u_k\|_{W^{1,1}(\Omega)} \leq 2\|u\|_{BV(\Omega)}$, we see that $\|v_k\|_{W^{1,1}(\mathbb{R}^2)} \leq C\,\|u\|_{BV(\Omega)}$. Again, fix a ball $B_j(0)$ so that Ω is a relatively compact subdomain of $B_j(0)$. By the Rellich theorem, there is a subsequence $(v_k^{(j)})_k$ that converges in $L^1(B_j(0))$ and almost everywhere in $B_j(0)$ to some function $w_j \in L^1(B_j(0))$. We repeat the argument for this subsequence and $B_{j+1}(0)$, and continue by induction. Then the diagonal sequence $(v_k^{(k)})_k$ converges almost everywhere to a function w with $w = w_l$ on $B_l(0), l > j$, and the convergence holds also with respect to $L^1(B_l(0))$. It follows that $\|v_k\|_{L^1(B_l(0))} \leq C\,\|u\|_{BV(\Omega)}$ for all $l \geq j$ and consequently, $w \in L^1(\mathbb{R}^2)$ with the same bound. Secondly,

$$\int_{\mathbb{R}^2} |\nabla v_k^k| \leq 2\|u\|_{BV(\Omega)},$$

and it thus easily follows that $w \in BV(\mathbb{R}^2)$ with the desired norm bound. The claim follows when we set $E(u) = w$.

$w \in BV(\mathbb{R}^2)$ with the desired norm bound. □

In [BM], Burago and Maz'ya gave a characterization of BV_l-extension domains. A general Euclidean spaces version of the following result can be found in Burago–Maz'ya [BM] or page 314 of Maz'ya [M]. A more general metric space version of this statement has recently been given in [BaMo].

Theorem 2.5 (Burago–Maz'ya). *A domain $\Omega \subset \mathbb{R}^2$ is a BV_l-extension domain if and only if there is a constant $C > 0$ such that whenever $E \subset \Omega$ is a Borel set of finite perimeter in Ω,*

$$\tau_\Omega(E) \leq C\,P(E, \Omega), \tag{6}$$

where

$$\tau_\Omega(E) = \inf\{P(F, \mathbb{R}^2 \setminus \Omega) : F \cap \Omega = E\}.$$

Note that

$$P(F, \mathbb{R}^2 \setminus \Omega) = \inf\{P(F, U) \,:\, U \text{ open and } \mathbb{R}^2 \setminus \Omega \subset U\}.$$

The following lemma of Burago–Maz'ya [BM] gives an analogous characterization for a variant of bounded BV-extension domains; see also Section 6.3.5 of [M]. For a self-contained proof of this lemma in a more general setting, also see [BaMo].

Lemma 2.6. *If $\Omega \subset \mathbb{R}^2$ is a bounded domain, then there is a bounded extension map $T : BV(\Omega) \to BV_l(\mathbb{R}^2)$ if and only if there exist constants $C, \delta > 0$ such that for all Borel sets $E \subset \Omega$ of finite perimeter in Ω with* $\operatorname{diam}(E) \leq \delta$,

$$\tau_\Omega(E) \leq C\,P(E, \Omega).$$

The next lemma allows us to approximate sets of finite perimeter by smooth sets of finite perimeter. The statement and proof of this theorem for domains in $\mathbb{R}^n$ can be found in 6.1.3 of [M]. Recall that given sets F, G, their symmetric difference is denoted $F\Delta G$.

Lemma 2.7. *If $F \subset \mathbb{R}^2$ is a set of finite perimeter, then there exist sets $F_k \subset \mathbb{R}^2$ such that ∂F_k is smooth, $\chi_{F_k} \to \chi_E$ in $L^1_{loc}(\mathbb{R}^2)$, and $\lim_k P(F_k, \mathbb{R}^2) = P(F, \mathbb{R}^2)$. Furthermore, this sequence can be chosen so that*

$$F_k \Delta F \subset \bigcup_{x \in \partial F} B_{1/k}(x). \tag{7}$$

Recall that by the isoperimetric inequality in $\mathbb{R}^2$, if F is a set of finite perimeter then either $|F|$ or $|\mathbb{R}^2 \setminus F|$ is finite. If $|F|$ is finite, the expression (7) follows from the construction in [M] of F_k as certain level sets of smooth convolution approximations of χ_F. If $|\mathbb{R}^2 \setminus F|$ is finite, then (7) follows from setting F_k to be the complement of the construction in [M] that approximates $\mathbb{R}^2 \setminus F$.

Suppose that ∂F_k is smooth in $\mathbb{R}^2$. If F_k is bounded (or it's complement is bounded), then the number of connected components of ∂F_k is finite. If both F_k and it's complement are unbounded, then there can be infinitely (but countably) many components, but only finitely many can intersect any given disc. If ∂F_k is only assumed to be smooth in a domain Ω, then the corresponding analog is that the connected components cannot accumulate in any compact part of Ω, though they could accumulate toward $\partial\Omega$.

We will from now on use the abbreviation $\partial F \cap \Omega \in \mathcal{C}^\infty$ for the statement that $\partial F \cap \Omega$ be smooth.

Lemma 2.8. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Suppose that there is a constant $C > 0$ such that for every closed set $F \subset \mathbb{R}^2$ with $\partial F \cap \Omega \in \mathcal{C}^\infty$ there exists a set $\widehat{F} \subset \mathbb{R}^2$ with $\widehat{F} \cap \Omega = F \cap \Omega$ and*

$$|D\chi_{\widehat{F}}|(\mathbb{R}^2) \le C|D\chi_F|(\Omega).$$

Then Ω is a BV-extension domain.

Proof. By Lemma 2.1, it suffices to show that Ω is a BV_l-extension domain; that is, Ω satisfies the Burago-Maz'ya condition of Lemma 2.5.

Let E be any set such that $\chi_\Omega \ne \chi_E \in BV(\Omega)$. Then, by 6.1.3 in [M] there exists a sequence $(F_k)_k$ of sets in Ω so that $\partial F_k \cap \Omega \in \mathcal{C}^\infty$ and

$$\chi_{F_k} \to \chi_E \text{ in } L^1(\Omega),\ |D\chi_{F_k}|(\Omega) \to |D\chi_E|(\Omega). \tag{8}$$

By the regularity of F_k, we may assume that F_k is closed. Now, by hypothesis, there exist sets $\widehat{F_k}$ so that $\widehat{F_k} \cap \Omega = F_k \cap \Omega$ and

$$|D\chi_{\widehat{F_k}}|(\mathbb{R}^2) \le C|D\chi_{F_k}|(\Omega). \tag{9}$$

By (8) and (9), we get that

$$\limsup_k |D\chi_{\widehat{F_k}}|(\mathbb{R}^2) \le C|D\chi_E|(\Omega).$$

By the Rellich theorem applied to balls containing Ω and an application of a diagonalization argument, we may assume that there is F_∞ such that $\chi_{\widehat{F}_k} \to \chi_{F_\infty}$ in $L^1_{loc}(\mathbb{R}^2)$. For this set we have by the lower semicontinuity of the BV_l norm that

$$|D\chi_{F_\infty}|(\mathbb{R}^2) \leq \limsup_k |D\chi_{\hat{F}_k}|(\mathbb{R}^2) \leq C|D\chi_E|(\Omega).$$

Since for every k we have $\widehat{F}_k \cap \Omega = F_k \cap \Omega$, we conclude that $\chi_{F_\infty \cap \Omega} = \chi_E$ almost everywhere. Thus such an extension χ_{F_∞} of χ_E proves that Ω satisfies the Burago-Maz'ya condition (6). □

We will later need a lower bound for the perimeters of certain sets. The following lemma provides a suitable one.

Lemma 2.9. *Let $E \subset \mathbb{R}^2$ be an open set with finite perimeter and suppose there exist two curves $\gamma_1, \gamma_2 : [0,1] \to \mathbb{R}^2$ with $\gamma_1([0,1]) \subset E$ and $\gamma_2([0,1]) \subset \mathbb{R}^2 \setminus \overline{E}$ with*

$$\min\{|\gamma_1(1) - \gamma_1(0)|, |\gamma_2(1) - \gamma_2(0)|\} \geq \tau.$$

Then $P(E, \mathbb{R}^2) \geq 2\tau$.

Proof. Let us first assume that E is an open, bounded, connected smooth subset of $\mathbb{R}^2$ and let $x, y \in E$; then $P(E) \geq 2|x - y|$. In fact, if we consider

$$t_1 = \inf\{t \in \mathbb{R} : x + t(y - x) \in E\}, t_2 = \sup\{t \in \mathbb{R} : x + t(y - x) \in E\},$$

with our hypothesis on E, the points $x_i = x + t_i(y - x)$, $i = 1, 2$, are on the same connected component β of ∂E and they divide it into two curves β_1 and β_2 each with length $l(\beta_i) \geq |x_1 - x_2|$, and then

$$P(E) \geq l(\beta_1) + l(\beta_2) \geq 2|x_1 - x_2| \geq 2|x - y|.$$

If now E is any open set with finite perimeter, then either E or $\mathbb{R}^2 \setminus E$ has finite area; let us assume that $|\mathbb{R}^2 \setminus E| < +\infty$. We then consider $F = \mathbb{R}^2 \setminus E$ and the curve γ_2 (in case $|E| < +\infty$, we have to consider γ_1); by assumption, $\delta = \text{dist}(\gamma_2, \overline{E}) > 0$. Let F_k be an approximation of F obtained as in Lemma 2.7, with $k > 2/\delta$; with this choice, the curve γ_2 is eventually contained in one of the connected components $\tilde{F}_\varepsilon$ of F_ε. Now by the discussion in the previous paragraph,

$$P(E) = P(F) = \lim_{k\to\infty} P(F_k) \geq \limsup_{k\to\infty} P(\tilde{F}_k) \geq 2\tau.$$

□

Lemma 2.10. *Let $\Omega \subset \mathbb{R}^2$ be a BV_l-extension domain. Then there exist constants $c, c_1, c_2 \in (0,1)$ and $r_0 > 0$ such that for any $x \in \partial\Omega$ and $0 < r < r_0$ we have*

$$|\Omega \cap B_r(x)| \geq c|B_r(x)|. \tag{10}$$

Moreover, for each connected component E of $\Omega \cap B_r(x)$ that intersects $B_{r/5}(x)$ we have

$$|E| \geq c_1|B_r(x)| \quad \text{and} \quad \mathcal{H}^1(\Omega \cap \partial E) \geq c_2\, r.$$

Proof. We choose $r_0 > 0$ such that whenever $x \in \partial\Omega$ and $0 < r < r_0$, $\Omega \setminus \overline{B}_r(x)$ contains a connected subset of diameter at least r_0.

Suppose that there exists a sequence $(x_k)_k \subset \partial\Omega$, $0 < r_k < r_0$, and a sequence $\varepsilon_k \to 0$ such that there is a connected component E_k of $\Omega \cap B_{r_k}(x_k)$ intersecting $B_{r_k/5}(x_k)$ with

$$|E_k| = \varepsilon_k |B_{r_k}(x_k)| = \pi\varepsilon_k r_k^2.$$

Since

$$|E_k| = \int_0^{r_k} \mathcal{H}^1(E_k \cap \partial B_t(x_k))dt,$$

there exists $\bar{t} \in [r_k/2, r_k]$ such that

$$P(B_{\bar{t}}(x_k) \cap E_k, \Omega) = \mathcal{H}^1(E_k \cap \partial B_{\bar{t}}(x_k)) \leq 2\pi\varepsilon_k r_k. \tag{11}$$

Observe that as E_k contains a curve connecting a point in $S(x_k, r_k)$ to some point in $S(x_k, r_k/5)$, it is clear that $E_k \cap B_{\bar{t}}(x_k)$ contains a curve connecting some point in $S(x_k, \bar{t})$ to a point in $S(x_k, r_k/5)$. Hence the extension $\hat{E}_k$ of $E_k \cap B_{\bar{t}}(x_k)$ has a connected component of diameter at least $3r_k/10$. Furthermore, as $r_k < r_0$ and $\hat{E}_k \cap \Omega = E_k \cap B_{\bar{t}}(x_k) \subset B_r(x_k)$, it follows that $\mathbb{R}^2 \setminus \overline{\hat{E}_k}$ also contains a connected set of diameter at least $3r_k/10$. It therefore follows by Lemma 2.9 that $P(\hat{E}_k, \mathbb{R}^2) \geq 3r_k/10$. This means that

$$\frac{P(\hat{E}_k, \mathbb{R}^2)}{P(E_k \cap B_{\bar{t}}(x_k), \Omega)} = \frac{P(\hat{E}_k, \mathbb{R}^2)}{\mathcal{H}^1(E_k \cap \partial B_{\bar{t}}(x_k))} \geq \frac{3r_k}{20\pi\varepsilon_k r_k} = \frac{3}{20\pi\varepsilon_k}.$$

Letting $k \to \infty$ and recalling that $\varepsilon_k \to 0$, gives us a contradiction with the extension property.

Now, fix a connected component E of $B_r(x) \cap \Omega$ that intersects $B_{r/5}(x)$. Then by the above argument and the BV-extension property, with $\hat{E}$ an extension of E given by the BV-extension property,

$$C \geq \frac{P(\hat{E}, \mathbb{R}^2)}{\mathcal{H}^1(\Omega \cap \partial E)} = \frac{P(\hat{E}, \mathbb{R}^2)}{P(E, \Omega)} \geq \frac{3r}{10P(E, \Omega)},$$

completing the proof. □

We end this section by pointing out that the Lemmas 2.1, 2.3, 2.4, 2.5, 2.6, 2.7, and 2.8, as well as their proofs given here, hold true in higher dimensional Euclidean spaces as well.

3 Proofs of the results

Proof of Theorem 1.1. We first prove quasiconvexity of a bounded, simply connected BV_l-extension domain. The same for BV-extension domains then follows from Lemma 2.1.

Suppose that Ω is a bounded, simply connected BV_l-extension domain. It suffices to prove the quasiconvexity estimate for all $x, y \in \partial\Omega$ such that $d(x,y) \leq r_0$ for some fixed $r_0 > 0$ (recall that we assume the domain to be bounded). Let $\delta_0 > 0$ be the constant from Lemma 2.6, and let $r_0 = \min\{\delta_0, \mathrm{diam}(\Omega)\}/(2C)$, where C is the maximum of all the constants from the previous section. We denote the line segment joining x and y by L_{xy}. If $L_{xy} \cap \Omega$ is empty, then we can set $\gamma = L_{xy}$. Hence we may assume that L_{xy} intersects Ω.

Since Ω is an open set, $L_{xy} \cap \Omega$ is the disjoint union of countably many line segments $L_{x_iy_i}$, $i \in I \subset \mathbb{N}$, with end points $x_i, y_i \in \partial\Omega$. Let $L_{x_iy_i}$ be one of them. Because Ω is simply connected, $\Omega \setminus L_{x_iy_i}$ has exactly two components, say E_1, E_2. Assume that $|E_1| \leq |E_2|$. Since $\Omega \cap \partial E_1 = L_{x_iy_i}$ and hence $P(E_1, \Omega) = \mathcal{H}^1(L_{x_iy_i}) = |x_i - y_i|$, by Theorem 2.5 and subadditivity of the perimeter measure, there is a set $F \subset \mathbb{R}^2$ of finite perimeter such that $F \cap \Omega = E_1$ and

$$P(F, \mathbb{R}^2) \leq C\,|x_i - y_i|. \tag{12}$$

By Lemma 2.7, there is a sequence of smooth sets F_k with $\chi_{F_k} \to \chi_F$ both in $L^1_{loc}(\mathbb{R}^2)$ and pointwise almost everywhere, $P(F_k, \mathbb{R}^2) \to P(F, \mathbb{R}^2)$, $F_k \Delta F \subset \bigcup_{x\in\partial F} B(x, 1/k)$, and as vector-valued signed Radon measures, $D\chi_{F_k}$ converge weakly to $D\chi_F$.

Since F_k is smooth, ∂F_k consists of countably many smooth simple loops $\beta_{k,1}, \ldots$ (these curves are loops because they are of finite length). Recall from the discussion following the statement of Lemma 2.7 that the sets F_k are certain level sets of convolution approximations to χ_F. Hence for sufficiently large k (by passing to a subsequence if necessary), we may assume that $\partial F_k \subset \bigcup_{x\in\partial F} B(x, 1/k)$ and that one of the loops $\beta_{k,1}, \ldots$, say $\beta_{k,1}$, has the property that all of the line segment L_{x_i,y_i} except perhaps a $1/k$-neighborhood of x_i and y_i lies in a $1/k$-neighborhood of $\beta_{k,1}$, that is,

$$\beta_{k,1} \subset \bigcup_{x\in\partial F} B(x, 1/k) \tag{13}$$

and

$$L_{x_i,y_i} \setminus (B(x_i, 1/k) \cup B(y_i, 1/k)) \subset \bigcup_{x\in\beta_{k,1}} B(x, 1/k). \tag{14}$$

Furthermore,

$$\ell(\beta_{k,1}) \leq P(F_k, \mathbb{R}^2) \leq 2\,P(F, \mathbb{R}^2),$$

and so we can use Arzela-Ascoli's theorem (and pass to a further subsequence if necessary) to obtain a loop β such that $\beta_{k,1} \to \beta$ uniformly and $\ell(\beta) \leq 2P(F, \mathbb{R}^2)$. By (13) and (14) it follows that $L_{x_i,y_i} \subset \beta$, and that $\beta \subset \partial F$. Hence $\beta \cap \Omega = L_{x_i,y_i}$. Furthermore, by inequality (12),

$$\ell(\beta) \leq 2P(F, \mathbb{R}^2) \leq 2C\,P(E, \Omega) = 2C\,|x_i - y_i|.$$

Since β is a loop containing E_1 and not containing E_2, by [Kur, Theorem 5 of page 513]) there is a simple subloop β_0 containing L_{x_i,y_i}. The curve $\gamma_i :=$

$\beta_0 \setminus L_{x_i,y_i} \subset \mathbb{R}^2 \setminus \Omega$ with $\ell(\gamma_i) \le (2C-1)|x_i - y_i|$ is a curve in $\mathbb{R}^2 \setminus \Omega$ connecting x_i to y_i.

The concatenated curve $\gamma = (L_{xy} \setminus \Omega) *_{i\in I} \beta_i$ is a curve in $\mathbb{R}^2 \setminus \Omega$ connecting x and y, with

$$\ell(\gamma) \le \ell(L_{xy}) + \sum_{i\in I} \ell(\beta_i) \le |x-y| + \sum_{i\in I} C\,|x_i - y_i| \le (1+C)\,|x-y|.$$

Next suppose that $\mathbb{R}^2 \setminus \Omega$ is quasiconvex. By Lemma 2.8 we only need to verify the extension property for characteristic functions of sets $E \subset \mathbb{R}^2$ such that $\partial E \cap \Omega$ is smooth. Therefore $P(E,\Omega) = P(\overline{E},\Omega) = P(\mathrm{int}(\overline{E}),\Omega)$, and so without loss of generality we may assume that $\mathrm{int}(\overline{E}) \cap \Omega = E \cap \Omega$ is open. Again, without loss of generality we may assume that $E \subset \Omega$ and that E is connected; recall that only a finite number of the components of $\partial E \cap \Omega$ can intersect a given relatively compact open set $U \subset \Omega$ and that $P(E,\Omega)$ can be computed as the supremum of the perimeters $P(E,U)$ over all such U. It follows from the smoothness of E that $\Omega \cap \partial E$ consists of a collection of closed curves in Ω and a collection of at most a countable union of smooth curves γ_i, $i \in I \subset \mathbb{N}$, with end points $x_i, y_i \in \partial\Omega$ (indeed, if $\partial E \cap \partial\Omega$ is empty, that is, no such points x_i, y_i exist, then E or $\mathbb{R}^2 \setminus E$ is the extension of E or $\Omega \setminus E$ respectively, and we need not do anything). Again, without loss of generality, we may assume that $|E| \le |\Omega \setminus E|$, since otherwise we replace E with $\Omega \setminus E$. By assumption, there is a curve $\beta_i \subset \mathbb{R}^2 \setminus \Omega$ connecting x_i and y_i with $\ell(\beta_i) \le C|x_i - y_i|$. The concatenated curve $\gamma_i * \beta_i$ is a simple loop (Jordan curve) in $\mathbb{R}^2$; let F_i be the bounded subset of $\mathbb{R}^2$ enclosed by this loop. Since E is connected, if $E \cap F_i \neq \emptyset$, then $E \subset F_i$; let J be the collection of all indices $i \in I$ for which this holds. If J is not empty, then we define

$$F := \left(\bigcap_{i\in J} F_i\right) \setminus \left(\bigcup_{i\in I\setminus J} F_i\right) \setminus (\text{all regions bounded by loops lying in } \Omega).$$

If J is empty, then we set

$$F := \mathbb{R}^2 \setminus \left(\bigcup_{i\in I} F_i\right) \setminus (\text{all regions bounded by loops lying in } \Omega).$$

With the above selection of F, we see that $F \cap \Omega = E$, and by the construction of the curves β_i we have that

$$\begin{aligned} P(F,\mathbb{R}^2) \le \sum_{i\in I} \ell(\gamma_i * \beta_i) &= \sum_{i\in I} \ell(\gamma_i) + \sum_{i\in I} \ell(\beta_i) \\ &\le \sum_{i\in I} \ell(\gamma_i) + C \sum_{i\in I} |x_i - y_i| \\ &\le \sum_{i\in I} \ell(\gamma_i) + C \sum_{i\in I} \ell(\gamma_i) = (1+C) P(E,\Omega), \end{aligned}$$

concluding the proof. □

Proof of Corollary 1.2. The claim follows from Lemma 2.4 and Theorem 1.1. □

We record the following recent result by Väisälä, see 2.8 in [V].

Lemma 3.1 (**Väisälä, 2008**). *If Ω is a bounded simply connected planar domain whose complement is quasiconvex, and if Ω' is a planar domain with $f : \Omega \to \Omega'$ a bi-Lipschitz mapping, then there are open sets $U \supset \overline{\Omega}$, $V \supset \overline{\Omega'}$, and a bi-Lipschitz mapping $F : U \to V$ such that $F = f$ on Ω.*

Proof of Corollary 1.3. By Theorem 1.1 and Corollary 1.2, the complement of Ω is quasiconvex. Hence, by the above lemma, the bi-Lipschitz map f on Ω can be extended to a bi-Lipschitz map F on a neighborhood U of the compact set $\overline{\Omega}$. Hence if Ω is a BV-extension domain (or $W^{1,1}$-extension domain) and u is a function in $BV(\Omega')$ (or $W^{1,1}(\Omega')$ respectively), then $u \circ f$ is in the class $BV(\Omega)$ (or $W^{1,1}(\Omega)$ respectively), and hence can be extended to a function $T(u \circ f)$ that lies in the class $BV(\mathbb{R}^2)$ (or $W^{1,1}(\mathbb{R}^2)$ respectively), with norm controlled by the norm of u. Thus $T(u \circ f) \circ F^{-1}$ lies in the class $BV(V)$ (or $W^{1,1}(V)$ respectively), where $V = F(U)$ is a neighborhood of the compact set $\overline{\Omega'}$, with norm controlled by the norm of $T(u \circ f)$, and hence by the norm of u.

Let $\eta : \mathbb{R}^2 \to [0, 1]$ be an L-Lipschitz function with compact support in V such that $\eta = 1$ on Ω'. Let $E(u) := \eta\, T(u \circ f) \circ F^{-1}$; then $E(u) \in BV(\mathbb{R}^2)$ (or in $W^{1,1}(\mathbb{R}^2)$ respectively). Note that

$$\|E(u)\|_{L^1(\mathbb{R}^2)} \leq \|T(u \circ f) \circ F^{-1}\|_{L^1(V)} \leq C\|u\|_X,$$

where $X = BV(\Omega')$ (or $X = W^{1,1}(\Omega')$ respectively). Furthermore,

$$|DE(u)|(V) \leq \mathrm{Lip}(\eta)\|T(u \circ f) \circ F^{-1}\|_{L^1(V)} + |DT(u \circ f) \circ F^{-1}|(V) \leq C\|u\|_X,$$

where $\mathrm{Lip}\eta = \sup |\eta(x) - \eta(y)|/|x - y|$, the supremum taken over all distinct pairs of points $x, y \in \mathbb{R}^2$. This completes the proof. □

4 Examples

The characterization given by Theorem 1.1 is easy to verify for planar Jordan domains. We now explore some specific examples of bounded simply connected planar BV-extension domains by answering the following question: suppose $\Omega \subset \mathbb{R}^2$ is a bounded BV_l-extension domain, and let $\gamma \subset \Omega$ be a curve with $\Omega \setminus \gamma$ also a domain. When is $\Omega \setminus \gamma$ also a BV-extension domain? It follows from Theorem 1.1 that γ has to be a rectifiable curve. However, the rectifiability of γ by itself does not guarantee the BV-extension property of $\Omega \setminus \gamma$, as the following example demonstrates.

Example 4.1. *Let $\Omega = (-a, a) \times (-2, 2)$ be a rectangular region centered at the origin where $a = \sum_{j=1}^{\infty} \dfrac{1}{j^3}$. Further, let $\gamma : [0, 1) \to \Omega$ with $\gamma(0) = (1, 0)$ be defined as follows: for each $n \in \mathbb{N}$ with $n \geq 2$, $\gamma(1 - 1/n) = (\sum_{j=1}^{n} 1/j^3, 0)$,* *and the open interval $(1 - 1/n, 1 - 1/(n+1)) \subset [0,1]$ is mapped to the curve obtained by joining two segments, the line segment joining $(\sum_{j=1}^{n} 1/j^3, 0)$ and $(1/(2n+2)^3 + \sum_{j=1}^{n} 1/j^3, 1/n^2)$, and the line segment joining $(1/(2n+2)^3 + \sum_{j=1}^{n} 1/j^3, 1/n^2)$ and $(\sum_{j=1}^{n+1} 1/j^3, 0)$. This γ is a saw-tooth curve for which the height of the n-th tooth, $1/n^2$, is substantially larger than the width $1/n^3$ of the tooth. It can be seen that γ is rectifiable and that $\Omega \setminus \gamma$ is* not *a BV-extension domain.*

Example 4.2. *Let $\Omega = (0,2) \times (-2,2)$, and let γ be the curve given by $\gamma : (0,1] \to \Omega$, $\gamma(t) = (t^2, t)$. Again it can be seen, via the use of sets*

$$E_t = \{(x,y) \in \Omega \,:\, 0 < y < t, 0 < x < y^2\},$$

that $\Omega \setminus \gamma$ is not a BV-extension domain, even though γ is rectifiable.

The following answer to the above question is a corollary to Theorem 1.1. Here, $\delta_\Omega(x) = \mathrm{dist}(x, \partial\Omega)$ for $x \in \Omega$.

Corollary 4.3. *Suppose $\Omega \subset \mathbb{R}^2$ is a simply connected bounded BV-extension domain and that γ is a curve in Ω so that $\Omega \setminus \gamma$ is also a simply connected domain. Then $\Omega \setminus \gamma$ is a BV-extension domain if and only if the following two conditions hold for γ.*

- *(i) There is a constant $C > 0$ such that for all $x, y \in \gamma$ and for all subcurves γ_{xy} of γ with end points x and y, we have $\ell(\gamma_{xy}) \le C\,|x-y|$ (that is, γ is quasiconvex).*
- *(ii) There is a constant $C > 0$ such that for all $x, y \in \gamma$ and a subcurve γ_{xy} of γ with end points x and y, we have $|x-y| \le C \max\{\delta_\Omega(x), \delta_\Omega(y)\}$ (that is, γ satisfies a double cone condition in Ω).*

If γ is not rectifiable then $\Omega \setminus \gamma$ is not a BV-extension domain as $\mathbb{R}^2 \setminus (\Omega \setminus \gamma)$, and hence γ, has to be quasiconvex. This is in contrast to the fact that $\partial\Omega$ need not be rectifiable even if Ω is a BV-extension domain; as shown by the von Koch snowflake domain, which is a uniform domain and hence (see [J]) is a $W^{1,1}$- and further a BV-extension domain. It should be noted that the assumption that $\Omega \setminus \gamma$ is a domain ensures that γ does not have loops. The first condition above ensures that γ is quasiconvex. Observe that the curve in Example 4.1 fails to satisfy this condition, though it does satisfy the second condition. Note also that the curve in Example 4.2 fails to satisfy the second condition of the above corollary, but does satisfy the first condition. Hence both conditions above are essential in the above result, though if γ does not intersect the boundary of Ω the second condition will follow from the first condition.

The conclusion of the corollary remains valid if we replace the condition that $\Omega \setminus \gamma$ is a simply connected domain with the condition that $\Omega \setminus \gamma$ is a domain; however, in this case the result is not a direct consequence of Theorem 1.1.

Remark 4.4. Lemma 2.1 fails for some unbounded domains; for example, the domain

$$\Omega = \{(x,y) \in \mathbb{R}^2 \, : \, |y| > x \text{ if } x \geq 0 \text{ and } |y| + 1 > -x \text{ if } x \leq -1\}$$

is a BV-extension domain because it has uniformly Lipschitz boundary, but is not a BV_l-extension domain as the set $E = \{(x,y) \in \Omega \, : \, y > 0\}$ has no extension satisfying the Burago–Mazya characterization. Therefore Theorem 1.1 might fail for unbounded simply connected domains. However, the actual proof of this theorem demonstrates that the complement of a planar simply connected domain is quasiconvex if and only if the domain itself is a BV_l-extension domain. The above example also shows that if Ω is an unbounded simply connected planar domain, the conclusion of Corollary 1.2 may fail. The domain in the above counterexample has the property that the complement of the domain in $\mathbb{R}^2$ is not connected; however, the example $\Omega = (0,\infty) \times (0,1) \subset \mathbb{R}^2$ also is a BV-extension domain, but is not a BV_l-extension domain, even though $\mathbb{R}^2 \setminus \Omega$ is indeed connected. If $\Omega \subset \mathbb{R}^2$ is such that $\mathbb{R}^2 \setminus \Omega$ is connected, then the proof of Theorem 1.1 also shows that $\mathbb{R}^2 \setminus \Omega$ is quasiconvex if and only if Ω is a BV_l-extension domain. We point out here that in this case, there is no reason to assume that Ω needs to be simply connected.

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