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POHOZAEV-TYPE IDENTITIES FOR DIFFERENTIAL OPERATORS DRIVEN BY HOMOGENEOUS VECTOR FIELDS

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Abstract. We prove Pohozaev-type identities for smooth solutions of Euler-Lagrange equations of second and fourth order that arise from functional depending on homogeneous Hörmander vector fields. We then exploit such integral identities to prove non-existence results for the associated boundary value problems.

1. Introduction

In 1965 in [33] Pohozaev proved an integral identity for solutions of the following elliptic boundary value problem:

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is an open and bounded set with smooth boundary $\partial\Omega$. In particular, under the geometric assumption on the set Ω of being star-shaped, he was able to show that (1.1) does not admit non-trivial solutions in $C^2(\Omega) \cap C^1(\overline{\Omega})$. In 1986 in [34]. Pucci and Serrin extended the approach of Pohozaev proving integral identities for solutions of a large class of variational PDEs coming from functionals possibly depending on the Hessian.

The starting idea of Pohozaev goes actually back to Rellich and Nehari [35, 30] and can be summarized as follows: given a sufficiently smooth solution of (1.1), it suffices to multiply the equation $-\Delta u = f(u)$ by $x \cdot \nabla u$, integrate over Ω and apply the Divergence Theorem. This will lead to the celebrated Pohozaev identity

$$\frac{n-2}{2}\int_\Omega \|\nabla u\|^2 \,\mathrm{d}x - n\int_\Omega F(u)\,\mathrm{d}x + \frac{1}{2}\int_{\partial\Omega}\left|\frac{\partial u}{\partial \nu}\right|^2 \langle x, \nu\rangle \,\mathrm{d}x = 0, \tag{1.2}$$

where F is a primitive of f and ν denotes the unit outward normal of $\partial\Omega$. From (1.2) it is pretty easy to get nonexistence results (of non-trivial solutions) under appropriate assumptions on the nonlinearity f, and therefore on F itself. Assume, e.g., that $F(u) \leq 0$. Then

$$0 \geq n\int_\Omega F(u)\,\mathrm{d}x = \frac{n-2}{2}\int_\Omega \|\nabla u\|^2\,\mathrm{d}x + \frac{1}{2}\int_{\partial\Omega}\left|\frac{\partial u}{\partial \nu}\right|^2 \langle x,\nu\rangle\,\mathrm{d}x.$$

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Since the first integral on the right hand side (r.h.s., in short) is non-negative, it is now clear to the entire sign of the r.h.s. depends on $x\cdot\nu$; the latter is a purely geometric quantity, only depending on the set Ω and not on the solution u. Therefore, assuming for example that Ω is star-shaped, one can achieve the famous non-existence result of Pohozaev.

Since the paper of Pucci and Serrin [34], the intimate connection between integral identities of Pohozaev-type and non-existence results has been the object of study of many papers, who has extended the ideas previously recalled to cover an always wider class of PDEs. The streamline has been definitely interested in extending Pohozaev's results to more general equations, such as quasi-linear elliptic equations, polyharmonic equations and fractional differential equations. Without any attempt of completeness, we refer to [23, 15] for the case of the p-laplacian, to [29] for higher order differential operators and to [18, 36] for more recent contributions dealing with nonlocal operators. We must however remind that there has been a certain interest also in studying non-existence results in case of more general domains, see e.g. [14, 28, 13] and the references therein. We also refer to [37, 38] for slightly different approaches to the proof of the classical Pohozaev identity previously recalled.

Another interesting line of research moved to consider non-Euclidean ambient spaces, like Riemannian manifolds (see e.g. [11, 12]) and Carnot groups, which are the prototypical examples of sub-Riemannian manifolds. In this setting Pohozaev-type identities, and the related non-existence reusults for certain classes of semilinear subelliptic PDEs, have been established in [20, 21, 22, 31]. Among several technical issues to be faced in this setting, we want to stress that one has to understand how to replace the star-shape assumption that naturally appears in the Euclidean case. In this perspective, in [20] the authors introduced the notion of δ_λ-star-shaped set, which is closely related to the anisotropic dilations defined on Carnot groups. Roughly speaking, this choice allows to *give a sign* to the quantity that naturally replaces the term $x\cdot\nu$. We address the interested reader to [16, 19, 17] for further comments on the notion of star-shaped and some applications.

The aim of this note is to continue along the line tracked in [24], where the authors proved similar results for a class of differential operators called Δ_λ laplacians. In particular, we will focus on sufficiently regular solutions of second order variational PDEs that are Euler-Lagrange equations associated to functionals depending on homogeneous Hörmander vector fields (see Section 2 for the details). The setting of the homogeneous Hörmander vector fields has been studied in the series of papers [1, 2, 3, 4, 5, 6, 7], where several *global results* are established: in fact, this setting seems to allow the possibility of developing an interesting global theory, without assuming the existence of a group of translations. A similar context has been also exploited in [32] to study multiplicity results for solutions of possibly degenerate equation in divergence form and in [25, 26, 27] to study the converge of minimizers for integral functionals [25, 26, 27].

Following the spirit of [34] we will also deduce a Pohozaev-type identity for smooth solutions of variational higher-order PDEs. To be more precise, we will start by considering functionals depending on the intrinsic X-Hessian as well, and this choice naturally lead to fourth-order PDEs. As already mentioned, such kind of integral identities should be the good tool to study non-existence of non-trivial solutions for boundary value problems. Quite surprisingly, to conclude this kind of non-existence results it seems necessary to require that the *full* Euclidean gradient vanishes on $\partial\Omega$, and not only the more intrinsic X-gradient (see Section 4 for a more detailed comment on this aspect). We believe that this feature is strictly related to the geometric assumption of δ_λ-star-shaped set. As a part of a future research project, we shall investigate more general geometric assumptions on Ω, possibly allowing us to deal with boundary conditions only involving the intrinsic X-gradient.

Plan of the paper. A short plan of the paper is now in order.

- In Section 2 we recall and discuss all the basic notions needed in the proof of our main theorem
- Section 3 is devoted to the proof of the Pohozaev identity for the second order case and its application to some non-existence results.
- Finally, in Section 4 we will comment on the case in which our operator depends also on the X-Hessian matrix.

2. Main assumptions

Throughout the sequel, we denote by $\mathcal{X}(\mathbb{R}^n)$ the Lie algebra of the smooth vector fields in $\mathbb{R}^n$. Moreover, if $A \subseteq \mathcal{X}(\mathbb{R}^n)$, we let $\mathrm{Lie}(A)$ be the smallest Lie sub-algebra of $\mathcal{X}(\mathbb{R}^n)$ containing A. Finally, if $Y \in \mathcal{X}(\mathbb{R}^n)$ is of the form

$$Y = \sum_{i=1}^{n} a_j(x)\frac{\partial}{\partial x_i} \qquad \text{(for some } a_1, \ldots, a_n \in C^\infty(\mathbb{R}^n)),$$

and if $x \in \mathbb{R}^n$ is arbitrary, we define

$$Y(x) := \begin{pmatrix} a_1(x) \\ \vdots \\ a_n(x) \end{pmatrix} \in \mathbb{R}^n.$$

Assumptions. Let $X := \{X_1, \ldots, X_m\} \subseteq \mathcal{X}(\mathbb{R}^n)$ be a family of *linearly independent* smooth vector fields in $\mathbb{R}^n$ satisfying the following assumptions.

(H.1): $X_1, \ldots, X_m$ are homogeneous of degree 1 with respect to a family of non-isotropic dilations $\{\delta_\lambda\}_{\lambda>0}$ in $\mathbb{R}^n$ of the form

$$\delta_\lambda(x) := (\lambda^{\sigma_1}x_1, \ldots, \lambda^{\sigma_n}x_n), \qquad \begin{array}{c}\text{where } \sigma_1, \ldots, \sigma_n \in \mathbb{N} \text{ and} \\ 1 = \sigma_1 \leq \ldots \leq \sigma_n.\end{array} \tag{2.1}$$

We define the δ_λ-homogeneous dimension of $\mathbb{R}^n$ as

$$q := \sum_{k=1}^{n} \sigma_k \geq n \tag{2.2}$$

(H.2): $X_1, \ldots, X_m$ satisfy Hörmander's condition at $x = 0$, that is,

$$\dim\Big\{Y(0) : Y \in \mathrm{Lie}(X)\Big\} = n. \tag{2.3}$$

As regards assumption (H.1), we remind that a vector field $Y \in \mathcal{X}(\mathbb{R}^n)$ is homogeneous of degree $\alpha \in \mathbb{R}$ with respect to $\{\delta_\lambda\}_{\lambda>0}$ if

$$Y(u \circ \delta_\lambda) = \lambda^\alpha (Yu) \circ \delta_\lambda \quad \text{for all } u \in C^\infty(\mathbb{R}^n) \text{ and } \lambda > 0. \tag{2.4}$$

Writing $Y = \sum_{i=1}^{n} a_i(x)\partial_{x_i}$ (for suitable functions $a_1, \ldots, a_n \in C^\infty(\mathbb{R}^n)$), it is easy to check that (2.4) is equivalent to one of the following conditions:

(a) a_i is δ_λ-homogeneous of degree $\sigma_i - \alpha$, that is,

$$a_i\big(\delta_\lambda(x)\big) = \lambda^{\sigma_i-\alpha}a_i(x) \quad \text{for all } x \in \mathbb{R}^n \text{ and } \lambda > 0; \tag{2.5}$$

(b) for every $x \in \mathbb{R}^n$ and every $\lambda > 0$ one has the identity

$$\delta_\lambda\big(Y(x)\big) = \lambda^\alpha Y\big(\delta_\lambda(x)\big). \tag{2.6}$$

Remark 2.1. We list, for future reference, some easy consequences of assumptions (H.1)-(H.2) which shall be useful in the sequel.

(1) It is not difficult to check that the combination of (H.1) and (H.2) implies the validity of Hörmander's condition at every point $x \in \mathbb{R}^n$, i.e.,

$$\dim\Big\{Y(x) : Y \in \mathrm{Lie}(X)\Big\} = n \qquad \text{for every } x \in \mathbb{R}^n. \tag{2.7}$$

(2) As a consequence of assumption (H.1), and since the σ_i's are increasingly ordered (see (2.1)), we derive from [9, Rem. 1.3.7] that $X_1, \ldots, X_m$ are *pyramid-shaped*. More precisely, if we write (for $i = 1, \ldots, m$)

$$X_i = \sum_{k=1}^n a_{k,i}(x)\,\frac{\partial}{\partial x_k},$$

we have that $a_{k,i}(x)$ *does not depend* on the variables $x_k, \ldots, x_n$.

(3) By crucially exploiting the pyramid-shape of the X_i's, it is easy to see that the following integration-by-parts formula holds: *if $\Omega \subseteq \mathbb{R}^n$ is a bounded open set, regular for the Divergence Theorem , and if $u, v \in C^1(\overline{\Omega})$, then*

$$\int_\Omega u\,X_i v\,\mathrm{d}x = \int_{\partial\Omega} u\,v\,\langle X_i(x), \nu\rangle\,\mathrm{d}H^{n-1} - \int_\Omega v\,X_i u\,\mathrm{d}x, \tag{2.8}$$

where ν is the outward normal to Ω. In particular, choosing $u \equiv 1$, we get

$$\int_\Omega X_i v\,\mathrm{d}x = \int_{\partial\Omega} v\,\langle X_i(x), \nu\rangle\,\mathrm{d}H^{n-1}. \tag{2.9}$$

If $\Omega \subseteq \mathbb{R}^n$ is a bounded open set and $u \in C^1(\Omega)$, we define the *X-gradient of u* as

$$\nabla_X u = \big(X_1 u, \ldots, X_m u\big) \in C(\Omega; \mathbb{R}^m);$$

moreover, if $F = (F_1, \ldots, F_m) \in C^1(\Omega; \mathbb{R}^m)$, we define the *$X$-divergence of F* as

$$\mathrm{div}_X F = -\textstyle\sum_{i=1}^m X_i F_i \in C(\Omega).$$

Example 2.2. Before proceeding, we present some concrete examples of vector fields $X_1, \ldots, X_m$ satisfying the 'structural' assumptions (H.1)-(H.2).

(1) In Euclidean space $\mathbb{R}^n$, let $m = n$ and

$$X_i := \partial_{x_i} \qquad (\text{for } i = 1, \ldots, n).$$

Clearly, $X_1, \ldots, X_n$ are homogeneous of degree 1 with respect to

$$\delta_\lambda(x) = (\lambda x_1, \ldots, \lambda x_n) = \lambda x,$$

so that assumption (H.1) is fulfilled. Moreover, since

$$[X_i, X_j] = 0 \qquad \text{for all } i, j = 1, \ldots, n,$$

we have that $\mathrm{Lie}(X) = \mathrm{span}\{\partial_{x_1}, \ldots, \partial_{x_n}\}$, and assumption (H.2) trivially holds. We explicitly notice that, in this context, we have

- $\nabla_X u = \nabla u$ for every $u \in C^1(\Omega)$;
- $\mathrm{div}_X(F) = -\mathrm{div}(F)$ for every $F = (F_1, \ldots, F_n) \in C^1(\Omega; \mathbb{R}^n)$.

Moreover, the 'sum of squares' naturally associated with X is nothing but

$$\Delta_X = -\textstyle\sum_{i=1}^n \partial_{x_i}^2 = -\Delta.$$

(2) Let $k, n_1, n_2 \in \mathbb{N}$ be arbitrarily fixed, and let $n := n_1 + n_2 \geq 2$. We denote a generic point $x \in \mathbb{R}^n \equiv \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ by

$$x = (y, t), \text{ where } x \in \mathbb{R}^{n_1} \text{ and } t \in \mathbb{R}^{n_2},$$

and we consider the vector fields on $\mathbb{R}^n$ defined as follows:

$$Y_i := \partial_{y_i} \qquad (i = 1, \dots, n_1);$$
$$T_{i,j} := y_i^k\, \partial_{t_j} \qquad (i = 1, \dots, n_1 \text{ and } j = 1, \dots, n_2).$$

Then, it is not difficult to recognize that the family

$$X := \big\{Y_i,\, T_{i,j} \,:\, i = 1, \dots, n_1 \text{ and } j = 1, \dots, n_2\big\}$$

satisfies both assumptions (H.1) and (H.2). In fact, an easy computation based on (2.6) shows that the elements of X are homogeneous of degree 1 with respect to

$$\delta_\lambda(x) = \delta_\lambda(y, t) := (\lambda y, \lambda^{k+1} t),$$

and thus assumption (H.1) is fulfilled. As regards (H.2), since

$$\big[\underbrace{Y_i, [Y_i, \cdots [Y_i}_{k \text{ times}}, T_{i,j}] \cdots]\big] = k!\, \partial_{t_j},$$

we derive that $\mathrm{Lie}(X) \supset \mathrm{span}\{\partial_{y_i},\, \partial_{t_{i,j}}\}$; using this inclusion, it is straightforward to conclude that (2.3) is satisfied. We explicitly notice that the 'sum of squares' naturally associated with X is the sub-elliptic operator

$$\Delta_X := -\textstyle\sum_{i=1}^{n_1} Y_i^2 - \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} T_{i,j}^2 = -\Delta_y - \big(y_1^{2k} + \cdots + y_{n_1}^{2k}\big)\, \Delta_t.$$

(3) In Euclidean space $\mathbb{R}^n$, with $n \geq 2$, we consider the vector fields

$$X_1 := \partial_{x_1}, \qquad X_2 := x_1 \partial_{x_2} + \frac{x_1^2}{2!}\partial_{x_3} + \cdots + \frac{x_1^{n-1}}{(n-1)!}\,\partial_{x_n}.$$

We claim that the family

$$X := \{X_1, X_2\}$$

satisfies both assumptions (H.1) and (H.2). In fact, a direct computation based on (2.6) shows that X_1, X_2 are homogeneous of degree 1 with respect to

$$\delta_\lambda(x) := (\lambda x_1, \lambda^2 x_2, \dots, \lambda^n x_n),$$

and thus assumption (H.1) is fulfilled. As for (H.2), since

$$Y_i := \big[\underbrace{X_1, [X_1, \cdots [X_1}_{i \text{ times}}, X_2] \cdots\big] = \partial_{x_{i+1}} + \sum_{j=2}^{n-i} \frac{x_1^{j-1}}{(j-1)!}\,\partial_{x_{i+j}} \qquad (1 \leq i \leq n-1),$$

we have that $Y_1, \dots, Y_{n-1} \in \mathrm{Lie}(X)$; as a consequence, we get

$$\{Y(0) :\, Y \in \mathrm{Lie}(X)\} \supset \{e_1, \dots, e_n\},$$

and this readily proves that (2.3) is satisfied. We explicitly notice that the 'sum of squares' naturally associated with X is the sub-elliptic operator

$$\Delta_X := -X_1^2 - X_2^2 = -\partial_{x_1}^2 + \Big(x_1\partial_{x_2} + \cdots + \frac{x_1^{n-1}}{(n-1)!}\partial_{x_n}\Big)^2.$$

This operator was first introduced by Bony is his celebrated 1969 paper [10].

With all the above assumptions and notation at hand, we are ready to introduce the main object of our study. Let $\mathcal{D} := \Omega \times \mathbb{R} \times \mathbb{R}^m$ and let

$$\mathcal{F} = \mathcal{F}(x,z,p) : \mathcal{D} \longrightarrow \mathbb{R}$$

satisfy the next two properties:

(P.1) $\mathcal{F}$ is of class C^1 on (an open neighborhood of) $\overline{\mathcal{D}}$;

(P.2) for every $i = 1,\ldots,m$, the function $\partial_{p_i}\mathcal{F}$ is of class C^1 on $\overline{\mathcal{D}}$.

In this paper, we shall be concerned with the functional

$$F : C^2(\Omega) \longrightarrow \mathbb{R}, \qquad F(u) := \int_\Omega \mathcal{F}\big(x, u(x), \nabla_X u(x)\big)\,\mathrm{d}x. \tag{2.10}$$

Remark 2.3. We explicitly notice that the functional F is intimately related with the following *non-linear sub-elliptic* PDE

$$\mathrm{div}_X\Big(\mathcal{F}_p\big(x,u(x),\nabla_X u(x)\big)\Big) + \partial_z\mathcal{F}\big(x,u(x),\nabla_X u(x)\big) = 0, \tag{2.11}$$

where we have used the obvious notation

$$\mathcal{F}_p = \big(\partial_{p_1}\mathcal{F},\ldots,\partial_{p_m}\mathcal{F}\big).$$

In fact, if $u \in C^2(\Omega)$ is a minimum/maximum point of F, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F(u+t\varphi) = 0 \qquad \text{for all } \varphi \in C_0^\infty(\Omega).$$

From this, taking into account properties (P.1)-(P.2) of $\mathcal{F}$, an application of Lebesgue's Dominated Convergence Theorem shows that, for every $\varphi \in C_0^\infty(\Omega)$,

$$\begin{aligned} 0 &= \int_\Omega \Big[\partial_z\mathcal{F}\big(x,u(x),\nabla_X u(x)\big)\,\varphi + \langle \mathcal{F}_p\big(x,u(x),\nabla_X u(x)\big), \nabla_X\varphi\rangle\Big]\mathrm{d}x \\ &\qquad \text{(using (2.8), and since } \varphi \in C_0^\infty(\Omega)) \\ &= \int_\Omega \Big[\partial_z\mathcal{F}\big(x,u(x),\nabla_X u(x)\big) + \mathrm{div}_X\Big(\mathcal{F}_p\big(x,u(x),\nabla_X u(x)\big)\Big)\Big]\varphi\,\mathrm{d}x, \end{aligned}$$

and thus u satisfies (2.11) (point-wise on Ω).

3. Rellich-Pohozaev identity for F

Throughout the sequel, $X = \{X_1,\ldots,X_m\} \subseteq \mathcal{X}(\mathbb{R}^n)$ is a fixed family satisfying assumptions (H.1)-(H.2) and Ω is a bounded open set, regular for the Divergence Theorem. Moreover, F is as in (2.10) with $\mathcal{F}$ fulfilling (P.1)-(P.2).

Our main aim in this section is to prove some integral identities related to F. To this end, if $\sigma_1,\ldots,\sigma_n$ are as in (2.1), we introduce the vector field

$$\mathrm{T} := \sum_{i=1}^n \sigma_i x_i \frac{\partial}{\partial x_i}, \tag{3.1}$$

which shall be referred to as the *infinitesimal generator of* $\{\delta_\lambda\}_{\lambda>0}$.

Remark 3.1. We explicitly observe, for a future reference, that the vector field T is related with the family $\{\delta_\lambda\}_{\lambda>0}$ via the following 'Euler-type' theorem: *a function* $u \in C^\infty(\mathbb{R}^n)$ *is* δ_λ*-homogeneous of degree* $m \geq 0$ *if and only if*

$$\mathrm{T}u = mu.$$

As a consequence, taking into account (2.5), we deduce that a smooth vector field $Y \in \mathcal{X}(\mathbb{R}^n)$ is δ_λ-homogeneous of degree $\alpha \in \mathbb{R}$ if and only if

$$[Y, \mathrm{T}] = \alpha Y.$$

We are now ready to prove the following Pohozaev identity for F. In order to keep our results as clear as possible, in the sequel we shall write

$$\mathcal{F}, \quad \mathcal{F}_x := \big(\partial_{x_1}\mathcal{F}, \ldots, \partial_{x_n}\mathcal{F}\big), \quad \partial_z\mathcal{F}, \quad \mathcal{F}_p = \big(\partial_{p_1}\mathcal{F}, \ldots, \partial_{p_m}\mathcal{F}\big)$$

and, unless otherwise specified, we shall understand that the above functions are evaluated at points $(x, u(x), \nabla_X u(x))$ (with $x \in \Omega$).

Proposition 3.2. *For every $u \in C^2(\overline{\Omega})$, we have the identity*

$$\begin{aligned}(3.2)\qquad &\int_\Omega \big(q\mathcal{F} - \langle \mathcal{F}_p, \nabla_X u\rangle\big)\,\mathrm{d}x + \int_\Omega \mathrm{T}\big(x \mapsto \mathcal{F}(x,z,p)\big)(x, u, \nabla_X u)\,\mathrm{d}x \\ &\qquad + \int_\Omega \mathrm{T}u\,\big(\mathrm{div}_X(\mathcal{F}_p) + \partial_z\mathcal{F}\big)\,\mathrm{d}x \\ &= \int_{\partial\Omega} \mathcal{F}\,\langle \mathrm{T}(x), \nu\rangle\,\mathrm{d}H^{n-1} - \int_{\partial\Omega} \mathrm{T}u\,\langle \mathcal{F}_p, \nu_X\rangle\,\mathrm{d}H^{n-1}\end{aligned}$$

where H^{n-1} is the standard $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$, ν is the outward normal to Ω and ν_X is given by

$$(3.3)\qquad \nu_X := \big(\langle X_1(x), \nu\rangle, \ldots, \langle X_m(x), \nu\rangle\big).$$

Finally, q is the δ_λ-homogeneous dimension of $\mathbb{R}^n$ defined in (2.2).

Proof. Since Ω is regular for the Divergence Theorem, we have

$$\int_\Omega \mathrm{div}\big(\mathcal{F}\cdot \mathrm{T}(x)\big)\,\mathrm{d}x = \int_{\partial\Omega} \mathcal{F}\,\langle \mathrm{T}(x), \nu\rangle\,\mathrm{d}H^{n-1}.$$

From this, observing that $\mathrm{div}(\mathrm{T}(x)) = \sum_{i=1}^n \sigma_i = q$ (see (2.2)), we obtain

$$(3.4)\qquad q\int_\Omega \mathcal{F}\,\mathrm{d}x + \int_\Omega \langle \nabla\mathcal{F}, \mathrm{T}(x)\rangle\,\mathrm{d}x = \int_{\partial\Omega} \mathcal{F}\,\langle \mathrm{T}(x), \nu\rangle\,\mathrm{d}H^{n-1},$$

where we have used the notation

$$\nabla\mathcal{F} = \nabla\big(x \mapsto \mathcal{F}\big(x, u(x), \nabla_X u(x)\big)\big).$$

Now, according to this notation, a direct calculation gives

$$\begin{aligned}(3.5)\qquad \langle \nabla\mathcal{F}, \mathrm{T}(x)\rangle &= \mathrm{T}\big(x \mapsto \mathcal{F}(x,z,p)\big)\big(x, u(x), \nabla_X u(x)\big) \\ &\quad + \mathcal{F}_u \cdot \mathrm{T}u(x) + \sum_{i=1}^m \frac{\partial \mathcal{F}}{\partial p_i}\cdot \mathrm{T}(X_i u).\end{aligned}$$

On the other hand, since the X_i's are δ_λ-homogeneous of degree 1, from Remark 3.1 we derive that $[X_i, \mathrm{T}] = X_i$ (for every $i = 1, \ldots, m$); hence, we can write

$$\begin{aligned}(3.6)\qquad \textstyle\sum_{i=1}^m \partial_{p_i}\mathcal{F}\cdot \mathrm{T}(X_i u) &= \textstyle\sum_{i=1}^m \partial_{p_i}\mathcal{F}\cdot X_i(\mathrm{T}u) + \sum_{i=1}^m \partial_{p_i}\mathcal{F}\cdot [\mathrm{T}, X_i]u \\ &= \langle \mathcal{F}_p, \nabla_X(\mathrm{T}u)\rangle - \textstyle\sum_{i=1}^m \partial_{p_i}\mathcal{F}\cdot X_i u \\ &= \langle \mathcal{F}_p, \nabla_X(\mathrm{T}u)\rangle - \langle \mathcal{F}_p, \nabla_X u\rangle.\end{aligned}$$

By combining (3.4), (3.5) and (3.6), we then get

$$
\begin{aligned}
&\int_\Omega \big(q\mathcal{F} - \langle \mathcal{F}_p, \nabla_X u\rangle\big)\,\mathrm{d}x + \int_\Omega \big[\mathrm{T}\big(x \mapsto \mathcal{F}(x,z,p)\big)(x,u,\nabla_X u) + \partial_z\mathcal{F}\cdot \mathrm{T}u\big]\,\mathrm{d}x \\
&\qquad + \int_\Omega \langle \mathcal{F}_p, \nabla_X(\mathrm{T}u)\rangle\,\mathrm{d}x = \int_{\partial\Omega} \mathcal{F}\,\langle \mathrm{T}(x),\nu\rangle\,\mathrm{d}H^{n-1}.
\end{aligned}
\tag{3.7}
$$

To proceed further, we integrate by parts in the last integral in the left-hand side of (3.7): using formula (2.8) in Remark 2.1-(3), we get

$$
\int_\Omega \partial_{p_i}\mathcal{F}\cdot X_i(\mathrm{T}u)\,\mathrm{d}x = \int_{\partial\Omega} \partial_{p_i}\mathcal{F}\cdot \mathrm{T}u\cdot\langle X_i(x),\nu\rangle\,\mathrm{d}H^{n-1} - \int_\Omega \mathrm{T}u\,X_i(\mathcal{F}_{p_i})\,\mathrm{d}x. \tag{3.8}
$$

Gathering together (3.7), (3.8) and the definition of ν_X in (3.3), we then obtain

$$
\begin{aligned}
&\int_\Omega \big(q\mathcal{F} - \langle \nabla_p\mathcal{F}, \nabla_X u\rangle\big)\,\mathrm{d}x + \int_\Omega \mathrm{T}\big(x \mapsto \mathcal{F}(x,z,p)\big)(x,u,\nabla_X u)\,\mathrm{d}x \\
&\qquad + \int_\Omega \mathrm{T}u\big(\mathrm{div}_X(\mathcal{F}_p) + \partial_z\mathcal{F}\big)\,\mathrm{d}x \\
&= \int_{\partial\Omega} \mathcal{F}\,\langle \mathrm{T}(x),\nu\rangle\,\mathrm{d}H^{n-1} - \int_{\partial\Omega} \mathrm{T}u\cdot\langle \mathcal{F}_p,\nu_X\rangle\,\mathrm{d}H^{n-1},
\end{aligned}
$$

which is exactly the desired (3.2). □

Just about to illustrate the applicability of Proposition 3.2, we write down the integral identity (3.2) for a general $\mathcal{F}$ in two concrete examples.

Example 3.3. In Euclidean space $\mathbb{R}^n$, let $m = n$ and

$$X_1 = \partial_{x_1}, \ldots, X_n = \partial_{x_n}.$$

Moreover, let $\Omega \subseteq \mathbb{R}^n$ be a fixed open set, regular for the Divergence Theorem, and let $k \in (1,n]$ be a fixed real number. We then define

$$\mathcal{F} : \Omega\times\mathbb{R}\times\mathbb{R}^n \longrightarrow \mathbb{R}, \qquad \mathcal{F}(x,z,p) := \frac{|p|^k}{k} - G(z),$$

where $G \in C^1(\mathbb{R})$ is a suitable function satisfying $G(0) = 0$.

We have already recognized in Example 2.2-(1) that $X = \{X_1,\ldots,X_n\}$ satisfies assumption (H.1) and (H.2); in particular, we have

$$\nabla_X u = \nabla u \qquad \text{for all } u \in C^1(\Omega).$$

Moreover, since the X_i's are homogeneous of degree 1 with respect to

$$\delta_\lambda(x) = \big(\lambda x_1,\ldots,\lambda x_n\big),$$

from (2.2) and (3.1) we derive that

$$q = n \qquad \text{and} \qquad \mathrm{T} = \textstyle\sum_{i=1}^n x_i\,\partial_{x_i}.$$

Taking into account all these facts, identity (3.2) boils down to

$$
\begin{aligned}
&\Big(\frac{n}{k} - 1\Big)\int_\Omega |\nabla u|^k\,\mathrm{d}x - n\int_\Omega G(u)\,\mathrm{d}x + \int_\Omega \langle x,\nabla u\rangle\big(\Delta_k u + G'(u)\big)\,\mathrm{d}x \\
&\qquad = \frac{1}{k}\int_{\partial\Omega} \langle x,\nu\rangle\,|\nabla u|^k\,\mathrm{d}H^{n-1} - \int_{\partial\Omega} |\nabla u|^{k-2}\langle x,\nabla u\rangle\,\frac{\partial u}{\partial\nu}\,\mathrm{d}H^{n-1},
\end{aligned}
\tag{3.9}
$$

where $\Delta_k u = -\mathrm{div}(|\nabla u|^{k-2}\,\nabla u)$ is the usual k-Laplacian of u.

Example 3.4. In Euclidean space $\mathbb{R}^n$, let $m < n$ and let $X = \{X_1, \dots, X_m\}$ satisfy assumptions (H.1)-(H.2). Moreover, let $\Omega \subseteq \mathbb{R}^n$ be a fixed open set, regular for the Divergence Theorem, and let $k \in (1, n]$. As in Example 3.3, we then define

$$\mathcal{F} : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}, \qquad \mathcal{F}(x, z, p) := \frac{|p|^k}{k} - G(u),$$

where $G \in C^1(\mathbb{R})$ and $G(0) = 0$. In this scenario, identity (3.2) boils down to

$$\begin{aligned} &\left(\frac{q}{k} - 1\right) \int_\Omega |\nabla_X u|^k \, \mathrm{d}x - q \int_\Omega G(u) \, \mathrm{d}x + \int_\Omega \mathrm{T}u\big(\Delta_{X,k} u + G'(u)\big) \, \mathrm{d}x \\ &\quad = \frac{1}{k} \int_{\partial\Omega} \langle \mathrm{T}(x), \nu \rangle \, |\nabla_X u|^k \, \mathrm{d}H^{n-1} - \int_{\partial\Omega} \mathrm{T}u \, |\nabla u|^{k-2} \, \langle \nabla_X u, \nu_X \rangle \, \mathrm{d}H^{n-1}, \end{aligned} \tag{3.10}$$

where $\Delta_{X,k} u$ is the so-called *horizontal k-Laplacian* of u, that is,

$$\Delta_{X,k} u = \mathrm{div}_X(|\nabla_X u|^{k-2} \, \nabla_X u),$$

and q, T, ν_X are given, respectively, by (2.2), (3.1), (3.3).

3.1. **Applications.** As it happens in the 'classical Euclidean case', Proposition 3.2 is a key tool in the study of the PDE (2.11) naturally associated with F, i.e.,

$$\mathrm{div}_X\big(\mathcal{F}_p\big(x, u(x), \nabla_X u(x)\big)\big) + \partial_z \mathcal{F}\big(x, u(x), \nabla_X u(x)\big) = 0. \tag{3.11}$$

More precisely, we have the following key result.

Theorem 3.5. *Let $u \in C^2(\overline{\Omega})$ be a solution of (3.11). Then*

$$\begin{aligned} &\int_\Omega \big(q\mathcal{F} - \langle \mathcal{F}_p, \nabla_X u \rangle\big) \, \mathrm{d}x + \int_\Omega \mathrm{T}\big(x \mapsto \mathcal{F}(x, z, p)\big)(x, u, \nabla_X u) \, \mathrm{d}x \\ &\quad = \int_{\partial\Omega} \mathcal{F} \, \langle \mathrm{T}(x), \nu \rangle \, \mathrm{d}H^{n-1} - \int_{\partial\Omega} \mathrm{T}u \, \langle \mathcal{F}_p, \nu_X \rangle \, \mathrm{d}H^{n-1}. \end{aligned} \tag{3.12}$$

If, in addition, $u \equiv 0$ on $\partial\Omega$, for every $a \in \mathbb{R}$ we have

$$\begin{aligned} &\int_\Omega \big(q\mathcal{F} - (a+1)\langle \mathcal{F}_p, \nabla_X u \rangle\big) \, \mathrm{d}x + \int_\Omega \mathrm{T}\big(x \mapsto \mathcal{F}(x, z, p)\big)(x, u, \nabla_X u) \, \mathrm{d}x \\ &\quad - a \int_\Omega u \, \partial_z \mathcal{F} \, \mathrm{d}x = \int_{\partial\Omega} \big(\mathcal{F} - \langle \mathcal{F}_p, \nabla_X u \rangle\big) \cdot \langle \mathrm{T}(x), \nu \rangle \, \mathrm{d}H^{n-1}. \end{aligned} \tag{3.13}$$

Proof. As regards identity (3.12), it is an immediate consequence of (3.2) and of the fact that, u being a solution of (3.11), one has

$$\int_\Omega \mathrm{T}u \, \big(\mathrm{div}_X(\mathcal{F}_p) + \partial_z \mathcal{F}\big) \, \mathrm{d}x = 0.$$

We then turn to prove identity (3.13) under the additional assumption that $u \equiv 0$ on $\partial\Omega$. To this end, we first claim that

$$\int_\Omega \big(\langle \mathcal{F}_p, \nabla_X u \rangle + u \, \partial_z \mathcal{F}\big) \, \mathrm{d}x = 0. \tag{3.14}$$

Indeed, since u is a solution of (3.11), we have

$$\textstyle\sum_{i=1}^m X_i\big(u \, \partial_{p_i} \mathcal{F}\big) = \langle \nabla_X u, \mathcal{F}_p \rangle - u \, \mathrm{div}_X(\mathcal{F}_p) = \langle \mathcal{F}_p, \nabla_X u \rangle + u \, \partial_z \mathcal{F};$$

from this, using Divergence's Theorem and the fact that $u \equiv 0$ on $\partial\Omega$, we get

$$\begin{aligned}
\int_\Omega \big(\langle \mathcal{F}_p, \nabla_X u\rangle + u\,\partial_z \mathcal{F}\big)\,\mathrm{d}x &= \sum_{i=1}^m \int_\Omega X_i\big(u\,\partial_{p_i}\mathcal{F}\big)\,\mathrm{d}x \\
&\quad \text{(writing, as usual, } X_i = \textstyle\sum_{k=1}^n a_{k,i}(x)\,\partial_{x_k}) \\
&= \sum_{i=1}^m \sum_{k=1}^n \int_\Omega a_{k,i}(x)\,\partial_{x_k}\big(u\,\partial_{p_i}\mathcal{F}\big)\,\mathrm{d}x \\
&= \sum_{i=1}^m \sum_{k=1}^n \bigg(\int_{\partial\Omega} a_{k,i}(x)\,u\,\partial_{p_i}\mathcal{F}\cdot \nu_k - \int_\Omega u\,\partial_{p_i}\mathcal{F}\cdot \partial_{x_k} a_{k,i}\,\mathrm{d}x\bigg) = 0,
\end{aligned}$$

which is precisely the claimed (3.14) (see also Remark 2.1-(2)). By combining this last identity with (3.12) we then obtain, for every choice of $a \in \mathbb{R}$,

$$\begin{aligned}
&\int_\Omega \big(q\mathcal{F} - (a+1)\langle \mathcal{F}_p, \nabla_X u\rangle\big)\,\mathrm{d}x + \int_\Omega \mathrm{T}\big(x \mapsto \mathcal{F}(x,z,p)\big)(x,u,\nabla_X u)\,\mathrm{d}x \\
&\quad - a\int_\Omega u\,\partial_z\mathcal{F}\,\mathrm{d}x = \int_{\partial\Omega} \mathcal{F}\,\langle \mathrm{T}(x),\nu\rangle\,\mathrm{d}H^{n-1} - \int_{\partial\Omega} \mathrm{T}u\,\langle \mathcal{F}_p, \nu_X\rangle\,\mathrm{d}H^{n-1}.
\end{aligned} \tag{3.15}$$

To complete the demonstration, we now perform some algebraic computations on the term $\mathrm{T}u\,\langle \mathcal{F}_p, \nu_X\rangle$. Using the very definition of T and ν_X, and writing

$$X_k = \textstyle\sum_{j=1}^n a_{j,k}(x)\,\partial_{x_j} \qquad (i = 1,\ldots,m),$$

we derive the following chain of equality:

$$\begin{aligned}
\mathrm{T}u\,\langle \mathcal{F}_p, \nu_X\rangle &= \sum_{i=1}^n \sum_{k=1}^m \sigma_i\,x_i\,\partial_{x_i}u\cdot \partial_{p_k}\mathcal{F}\cdot \langle X_k(x),\nu\rangle \\
&= \sum_{i,j=1}^n \sum_{k=1}^m \sigma_i\,x_i\,\partial_{x_i}u\cdot \partial_{p_k}\mathcal{F}\cdot a_{j,k}(x)\,\nu_j =: (\bigstar).
\end{aligned}$$

On the other hand, since $u \equiv 0$ on $\partial\Omega$, for every $i = 1,\ldots,n$ we have

$$\partial_{x_i} u = \frac{\partial u}{\partial \nu}\cdot \nu_i;$$

as a consequence, we obtain

$$\begin{aligned}
(\bigstar) &= \sum_{i,j=1}^n \sum_{k=1}^m \partial_{p_k}\mathcal{F}\cdot \big(\sigma_i\,x_i\,\nu_i\big)\cdot a_{j,k}(x)\Big(\frac{\partial u}{\partial\nu}\cdot \nu_j\Big) \\
&= \Big(\sum_{i=1}^n \sigma_i\,x_i\,\nu_i\Big)\cdot \sum_{k=1}^m \partial_{p_k}\mathcal{F}\cdot \Big(\sum_{j=1}^n a_{j,k}(x)\,\partial_{x_j}u\Big) \\
&= \langle \mathrm{T}(x),\nu\rangle\cdot \sum_{k=1}^m \partial_{p_k}\mathcal{F}\cdot X_k u = \langle \mathrm{T}(x),\nu\rangle\cdot \langle \mathcal{F}_p, \nabla_X u\rangle.
\end{aligned}$$

By inserting this last identity into (3.15), we finally get

$$\begin{aligned}
&\int_\Omega \big(q\mathcal{F} - (a+1)\langle \mathcal{F}_p, \nabla_X u\rangle\big)\,\mathrm{d}x + \int_\Omega \mathrm{T}\big(x \mapsto \mathcal{F}(x,z,p)\big)(x,u,\nabla_X u)\,\mathrm{d}x \\
&\quad - a\int_\Omega u\,\partial_z\mathcal{F}\,\mathrm{d}x = \int_{\partial\Omega} \langle \mathrm{T}(x),\nu\rangle\big(\mathcal{F} - \langle \mathcal{F}_p, \nabla_X u\rangle\big)\,\mathrm{d}H^{n-1},
\end{aligned}$$

which is exactly the desired (3.13). This ends the proof. □

With Theorem 3.5 at hand, we are ready to prove our non-existence result for the Dirichlet problem associated with (3.11). Before doing this, we fix a definition.

Definition 3.6. Let $\Omega \subseteq \mathbb{R}^n$ be a non-void open set with C^1 boundary. We say that Ω is *δ_λ-star shaped* (with respect to the origin) if

$$\langle \mathrm{T}(x), \nu\rangle \geq 0 \qquad \text{for every } x \in \partial\Omega, \tag{3.16}$$

where T is as in (3.1) and ν is the outward normal to $\partial\Omega$.

Theorem 3.7. *Let $\Omega \subseteq \mathbb{R}^n$ be a* connected *open set, regular for the Divergence Theorem and δ_λ-star shaped with respect to the origin. We assume that*

(i) $\mathcal{F}(x, 0, p) - \langle \mathcal{F}_p(x, 0, p), p\rangle \leq 0$ *for all $x \in \partial\Omega$ and $p \in \mathbb{R}^m$;*

(ii) *there exists a number $a_0 \in \mathbb{R}$ such that*

$$\begin{aligned} &q\mathcal{F}(x, z, p) - (a_0 + 1)\langle \mathcal{F}_p(x, z, p), p\rangle \\ &\qquad + \mathrm{T}\big(x \mapsto \mathcal{F}(x, z, p)\big) - a_0\, z\, \partial_z \mathcal{F}(x, z, p) \geq 0 \end{aligned} \tag{3.17}$$

for every $x \in \Omega$ and every $(z, p) \in \mathbb{R} \times \mathbb{R}^m$;

(iii) *either $z = 0$ or $p = 0$ when* (3.17) *holds.*

Then, the boundary-value problem

$$\begin{cases} \mathrm{div}_X\big(\mathcal{F}_p\big(x, u(x), \nabla_X u(x)\big)\big) + \partial_z \mathcal{F}\big(x, u(x), \nabla_X u(x)\big) = 0 & \text{in } \Omega, \\ u\big|_{\partial\Omega} = 0 \end{cases} \tag{3.18}$$

has no non-trivial solutions $u \in C^2(\overline{\Omega})$.

Proof. Let $u \in C^2(\overline{\Omega})$ be a solution of (3.18). We need to prove that

$$u \equiv 0 \text{ in } \Omega. \tag{3.19}$$

First of all, since $u \equiv 0$ on $\partial\Omega$ and since Ω is δ_λ-star shaped (with respect to the origin), by combining assumption (i) and identity (3.15) we derive that

$$\begin{aligned} &\int_\Omega \big(q\mathcal{F} - (a+1)\langle \mathcal{F}_p, \nabla_X u\rangle\big)\,\mathrm{d}x + \int_\Omega \mathrm{T}\big(x \mapsto \mathcal{F}(x, z, p)\big)(x, u, \nabla_X u)\,\mathrm{d}x \\ &\qquad - a\int_\Omega u\, \partial_z \mathcal{F}\,\mathrm{d}x \leq 0 \qquad (\text{for every } a \in \mathbb{R}); \end{aligned}$$

from this, using assumption (ii) we obtain

$$q\mathcal{F} - (a_0+1)\langle \mathcal{F}_p, \nabla_X u\rangle + \mathrm{T}\big(x \mapsto \mathcal{F}(x, z, p)\big)(x, u, \nabla_X u) - a_0\, u\, \partial_z \mathcal{F} = 0$$

for every $x \in \Omega$. Now, according to assumption (iii), only two cases can occur:

(a) $u \equiv 0$ on Ω. In this case, (3.19) is satisfied and the proof is complete.

(b) $\nabla_X u \equiv 0$ on Ω. In this case, since $X_i u \equiv 0$ on Ω for every $i = 1, \ldots, m$, we see that u *is constant along any integral curve of* $\pm X_1, \ldots, \pm X_m$ *contained in* Ω. On the other hand, since Ω is connected and the X_i's satisfy Hörmander's rank condition, the Chow-Rashevsky Connectivity Theorem holds (see, e.g., [8, Thm. 6.22]): for every $x, y \in \Omega$ there exists a continuous path $\gamma : [0, 1] \to \Omega$, which is piecewise an integral curve of the vector fields $\pm X_1, \ldots, \pm X_m$, such that

$$\gamma(0) = x \qquad \text{and} \qquad \gamma(1) = y.$$

As a consequence, u is constant along γ and, in particular,

$$u(x) = u(y).$$

Due to the arbitrariness of $x, y \in \Omega$, we then conclude that u is constant throughout on Ω. From this, since $u \equiv 0$ on $\partial\Omega$, we derive that $u \equiv 0$ on Ω, and the needed (3.19) is again satisfied. This ends the proof. $\square$

We end this section by 'specializing' the assumptions of Theorem 3.7 in the concrete cases discussed in Examples 3.3 and 3.4.

Example 3.8. In Euclidean space $\mathbb{R}^n$, let $m = n$ and let $X = \{X_1, \dots, X_n\}$, $\mathcal{F}$ be as in Example 3.3. For every $x \in \mathbb{R}^n$ and every $p \in \mathbb{R}^n$, we have

$$\mathcal{F}(x, 0, p) - \langle \mathcal{F}_p(x, 0, p), p\rangle = \frac{|p|^k}{k} - G(0) - |p|^{k-2}\langle p, p\rangle = |p|^k\left(\frac{1}{k} - 1\right) \leq 0$$

(as $k > 1$ and $G(0) = 0$); hence, assumption (i) of Theorem 3.7 is satisfied. Moreover, since in this case $q = n$ and $\mathcal{F}$ is independent of x, one also has

$$\begin{aligned} &q\mathcal{F}(x, z, p) - (a+1)\langle \mathcal{F}_p(x, z, p), p\rangle + \mathrm{T}\big(x \mapsto \mathcal{F}(x, z, p)\big) - a\, z\, \partial_z \mathcal{F}(x, z, p) \\ &\qquad = \frac{n}{k}|p|^k - nG(z) - (a+1)|p|^k + az\, G'(z). \end{aligned}$$

Let us now suppose that G enjoys the following additional properties:

(a) $G'(0) = 0$;

(b) for every $z \in \mathbb{R} \setminus \{0\}$ we have the growth condition

$$G(z) < \varrho z\, G'(z), \qquad \text{where } \varrho := \frac{1}{k} - \frac{1}{n}. \tag{3.20}$$

Then, it is immediate to check that assumptions (ii)-(iii) of Theorem 3.7 are satisfied with the choice $a_0 := \varrho$. As a consequence, the Dirichlet problem

$$\begin{cases} \Delta_k u + G'(u) = 0 & \text{in } \Omega, \\ u\big|_{\partial\Omega} = 0, \end{cases}$$

possesses no non-trivial solutions $u \in C^2(\overline{\Omega})$ if (3.20) holds.

Example 3.9. In Euclidean space $\mathbb{R}^n$, let $m < n$ and let $X = \{X_1, \dots, X_m\}$, $\mathcal{F}$ be as in Example 3.4. By arguing exactly as in Example 3.8, we see that assumption (i) of Theorem 3.7 is satisfied; moreover, since $\mathcal{F}$ is independent of x, we have

$$\begin{aligned} &q\mathcal{F}(x, z, p) - (a+1)\langle \mathcal{F}_p(x, z, p), p\rangle + \mathrm{T}\big(x \mapsto \mathcal{F}(x, z, p)\big) - a\, z\, \partial_z \mathcal{F}(x, z, p) \\ &\qquad = \frac{q}{k}|p|^k - qG(z) - (a+1)|p|^k + az\, G'(z), \end{aligned}$$

where $q \geq n$ is as in (2.2). Again as in Example 3.8, let us assume that

(a) $G'(0) = 0$;

(b) for every $z \in \mathbb{R} \setminus \{0\}$ we have the growth condition

$$G(z) < \varrho_q z\, G'(z), \qquad \text{where } \varrho_q := \frac{1}{k} - \frac{1}{q}. \tag{3.21}$$

Then, it is immediate to check that assumptions (ii)-(iii) of Theorem 3.7 are satisfied with the choice $a_0 := \varrho_q$. As a consequence, the Dirichlet problem

$$\begin{cases} \Delta_{X,k} u + G'(u) = 0 & \text{in } \Omega, \\ u\big|_{\partial\Omega} = 0, \end{cases}$$

possesses no non-trivial solutions $u \in C^2(\overline{\Omega})$ if (3.21) holds.

4. Toward a non-existence result for functionals with X-Hessian dependence

In this last section, we discuss a possible extension of the non-existence result in Theorem 3.7 to functionals F with X-Hessian dependence. In order the clearly describe this setting, we fix a family $X = \{X_1, \ldots, X_m\} \subseteq \mathcal{X}(\mathbb{R}^n)$ satisfying assumptions (H.1)-(H.2), and we inherit all the notation introduced so far. In addition, if $\Omega \subseteq \mathbb{R}^n$ is an open set and if $u \in C^2(\Omega)$, we define

$$\mathcal{H}_X u = \big(X_j(X_i u)\big)_{i,j=1}^m = \begin{pmatrix} X_1^2 u & X_2(X_1 u) & \cdots & X_m(X_1 u) \\ X_1(X_2 u) & X_2^2 u & \cdots & X_m(X_2 u) \\ \vdots & \vdots & \ddots & \vdots \\ X_1(X_m u) & X_2(X_m u) & \cdots & X_m^2 u \end{pmatrix}, \tag{4.1}$$

and we call the matrix $\mathcal{H}_X u$ the *X-Hessian of u*.

Remark 4.1. It should be explicitly noticed that, since the X_i's *do not commute* (in general), the matrix $\mathcal{H}_X u$ *is not symmetric*, i.e., $X_i(X_j u) \not\equiv X_j(X_i u)$.

Let now $\mathcal{O} := \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{m^2}$ and let

$$\mathcal{F} : \mathcal{O} \longrightarrow \mathbb{R}, \qquad \mathcal{F} = \mathcal{F}(x, z, p, r) = \mathcal{F}\big(x, z, p, \{r_{ij}\}_{i,j=1}^m\big),$$

satisfy the next three properties:

(P.1) $\mathcal{F}$ is of class C^1 on (an open neighborhood of) $\overline{\mathcal{O}}$;
(P.2) for every $i = 1, \ldots, m$, the function $\mathcal{F}_{p_i}$ is of class C^1 on $\overline{\mathcal{O}}$;
(P.3) for every $i, j = 1, \ldots, m$, the function $\mathcal{F}_{r_{ij}}$ is of class C^2 on $\overline{\mathcal{O}}$.

We then consider the functional $F : C^4(\Omega) \to \mathbb{R}$ defined as follows:

$$\begin{aligned} F(u) &:= \int_\Omega \mathcal{F}\big(x, u(x), \nabla_X u(x), \mathcal{H}_X u(x)\big)\,\mathrm{d}x \\ &= \int_\Omega \mathcal{F}\big(x, u(x), \nabla_X u(x), \{X_j(X_i u)\}_{i,j=1}^m\big)\,\mathrm{d}x. \end{aligned} \tag{4.2}$$

Remark 4.2. Analogously to what described in Remark 2.3, the functional F defined in (4.2) is deeply related with the following non-linear PDE

$$\begin{aligned} \sum_{i,j=1}^m X_i X_j \Big(\mathcal{F}_{r_{ij}}\big(x, u, \nabla_X u, \mathcal{H}_X u\big)\Big) + \mathrm{div}_X \Big(\mathcal{F}_p\big(x, u(x), \nabla_X u(x), \mathcal{H}_X u\big)\Big) \\ + \mathcal{F}_z\big(x, u, \nabla_X u, \mathcal{H}_X u\big) = 0. \end{aligned} \tag{4.3}$$

In fact, if $u \in C^4(\Omega)$ is a minimum/maximum point of F, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F(u + t\varphi) = 0 \qquad \text{for all } \varphi \in C_0^\infty(\Omega).$$

From this, taking into account properties (P.1)-to-(P.3) of $\mathcal{F}$, an application of Lebesgue's Dominated Convergence Theorem shows that, for every $\varphi \in C_0^\infty(\Omega)$,

$$\begin{aligned}
0 &= \int_\Omega \Big[\partial_z \mathcal{F}\big(x, u(x), \nabla_X u(x), \mathcal{H}_X u(x)\big)\,\varphi + \langle \mathcal{F}_p\big(x, u(x), \nabla_X u(x), \mathcal{H}_X u(x)\big), \nabla_X \varphi\rangle \\
&\qquad + \textstyle\sum_{i,j=1}^m \partial_{r_{ij}} \mathcal{F}\big(x, u(x), \nabla_X u(x), \mathcal{H}_X u(x)\big) \cdot X_j(X_i\varphi)\Big]\,\mathrm{d}x \\
&\quad \text{(using formula (2.8), and since } \varphi \in C_0^\infty(\Omega)) \\
&= \int_\Omega \Big[\partial_z \mathcal{F}\big(x, u(x), \nabla_X u(x), \mathcal{H}_X u(x)\big) + \mathrm{div}_X\Big(\mathcal{F}_p\big(x, u(x), \nabla_X u(x), \mathcal{H}_X u(x)\big)\Big) \\
&\qquad + \textstyle\sum_{i,j=1}^m X_i X_j\Big(\mathcal{F}_{r_{ij}}(x, u, \nabla_X u, \mathcal{H}_X u)\Big)\Big]\,\varphi\,\mathrm{d}x,
\end{aligned}$$

and thus u satisfies (4.3) (point-wise on Ω).

With all these preliminaries at hand, we are ready to prove the analog of Proposition 3.2 in this context. For notational simplicity, in what follows we write

$$\mathcal{F}, \quad \mathcal{F}_x = \big(\partial_{x_1}\mathcal{F}, \ldots, \partial_{x_n}\mathcal{F}\big), \quad \partial_z \mathcal{F},$$
$$\mathcal{F}_p = \big(\partial_{p_1}\mathcal{F}, \ldots, \partial_{p_m}\mathcal{F}\big), \quad \mathcal{F}_{r_{ij}} := \partial_{r_{ij}}\mathcal{F}$$

and, unless otherwise specified, we understand that all the above functions are evaluated at points $(x, u(x), \nabla_X u(x), \mathcal{H}_X u(x))$ (for $x \in \Omega$). Moreover, we tacitly assume that the open set Ω is *bounded and regular for the Divergence Theorem.*

Proposition 4.3. *For every $u \in C^4(\overline{\Omega})$, we have the identity*

$$\begin{aligned}
&\int_\Omega \big(q\mathcal{F} - \langle \nabla_p \mathcal{F}, \nabla_X u\rangle\big)\,\mathrm{d}x + \int_\Omega \mathrm{T}\big(x \mapsto \mathcal{F}(x, z, p, r)\big)(x, u, \nabla_X u, \mathcal{H}_X u)\,\mathrm{d}x \\
&\qquad + \int_\Omega \mathrm{T}u\,\big(\mathrm{div}_X(\mathcal{F}_p) + \textstyle\sum_{i,j=1}^m X_i\big(X_j(\mathcal{F}_{r_{ij}})\big) + \mathcal{F}_u\big)\,\mathrm{d}x \\
&\qquad - 2\sum_{i,j=1}^m \int_\Omega \mathcal{F}_{r_{ij}}\, X_j(X_i u)\,\mathrm{d}x \qquad\qquad (4.4)\\
&\quad = \int_{\partial\Omega} \mathcal{F}\,\langle \mathrm{T}(x), \nu\rangle\,\mathrm{d}H^{n-1} - \int_{\partial\Omega} \mathrm{T}u\,\langle \mathcal{F}_p, \nu_X\rangle\,\mathrm{d}H^{n-1} \\
&\qquad + \sum_{i,j=1}^m \int_{\partial\Omega} \big(X_i(\mathcal{F}_{r_{ji}})\,\mathrm{T}u \;- \mathcal{F}_{r_{ij}}\, X_i(\mathrm{T}u)\big)\langle X_j(x), \nu\rangle\,\mathrm{d}H^{n-1},
\end{aligned}$$

where ν is the outward normal to Ω and ν_X is as in (3.3).

Proof. We argue essentially as in the proof of Proposition 3.2. First of all, since Ω is regular for the Divergence Theorem and $\mathrm{div}(\mathrm{T}(x)) = q$, we can write

$$q\int_\Omega \mathcal{F}\,\mathrm{d}x + \int_\Omega \langle \nabla\mathcal{F}, \mathrm{T}(x)\rangle\,\mathrm{d}x = \int_{\partial\Omega} \mathcal{F}\,\langle \mathrm{T}(x), \nu\rangle\,\mathrm{d}H^{n-1}, \qquad (4.5)$$

where we have used the notation

$$\nabla\mathcal{F} = \nabla\big(x \mapsto \mathcal{F}\big(x, u(x), \nabla_X u(x), \mathcal{H}_X u(x)\big)\big).$$

On the other hand, using identity (3.6) in the proof of Proposition 3.2, we get

$$
\begin{aligned}
\langle \nabla \mathcal{F}, \mathrm{T}(x)\rangle &= \mathrm{T}\big(x \mapsto \mathcal{F}(x,z,p,r)\big)\big(x, u(x), \nabla_X u(x), \mathcal{H}_X u(x)\big) \\
&\quad + \mathcal{F}_u \cdot \mathrm{T}u(x) + \langle \mathcal{F}_p, \nabla_X(\mathrm{T}u)\rangle - \langle \mathcal{F}_p, \nabla_X u\rangle \\
&\quad + \textstyle\sum_{i,j=1}^m \mathcal{F}_{r_{ij}} \cdot \mathrm{T}\big(X_j(X_i u)\big).
\end{aligned} \tag{4.6}
$$

Now, reminding that $[X_i, \mathrm{T}] = X_i$ for every $i = 1, \ldots, m$ (as $X_1, \ldots, X_m$ are δ_λ-homogeneous of degree 1, see Remark 3.1), we have

$$
\begin{aligned}
\mathcal{F}_{r_{ij}} \cdot \mathrm{T}\big(X_j(X_i u)\big) &= \mathcal{F}_{r_{ij}} \cdot \Big(X_j\big(\mathrm{T}(X_i u)\big) + [\mathrm{T}, X_j](X_i u)\Big) \\
&= \mathcal{F}_{r_{ij}} \cdot \Big(X_j\big(X_i(\mathrm{T}u) + [\mathrm{T}, X_i]u\big) - X_j(X_i u)\Big) \\
&= \mathcal{F}_{r_{ij}} \cdot \Big(X_j\big(X_i(\mathrm{T}u)\big) - 2X_j(X_i u)\Big);
\end{aligned} \tag{4.7}
$$

as a consequence, by combining (4.5), (4.6) and (4.7) we obtain

$$
\begin{aligned}
&\int_\Omega \big(q\mathcal{F} - \langle \nabla_p \mathcal{F}, \nabla_X u\rangle\big)\,\mathrm{d}x + \int_\Omega \mathrm{T}\big(x \mapsto \mathcal{F}(x,z,p,r)\big)(x, u, \nabla_X u, \mathcal{H}_X u)\,\mathrm{d}x \\
&\quad + \int_\Omega \mathrm{T}u \cdot \mathcal{F}_u\,\mathrm{d}x + \int_\Omega \big[\langle \mathcal{F}_p, \nabla_X(\mathrm{T}u)\rangle + \textstyle\sum_{i,j=1}^m \mathcal{F}_{r_{ij}} \cdot X_j\big(X_i(\mathrm{T}u)\big)\big]\,\mathrm{d}x \\
&\quad - 2\sum_{i,j=1}^m \int_\Omega \mathcal{F}_{r_{ij}}\, X_j(X_i u)\,\mathrm{d}x = \int_{\partial\Omega} \mathcal{F}\,\langle \mathrm{T}(x), \nu\rangle\,\mathrm{d}H^{n-1}.
\end{aligned} \tag{4.8}
$$

To proceed further, we integrate by parts in the fourth integral in the left-hand side of (4.8). First, using identity (3.8) in the proof of Proposition 3.2, we get

$$
\int_\Omega \langle \mathcal{F}_p, \nabla_X(\mathrm{T}u)\rangle\,\mathrm{d}x = \int_{\partial\Omega} \mathrm{T}u\,\langle \mathcal{F}_p, \nu_X\rangle\,\mathrm{d}H^{n-1} + \int_\Omega \mathrm{T}u \cdot \mathrm{div}_X(\mathcal{F}_p)\,\mathrm{d}x; \tag{4.9}
$$

moreover, by repeatedly exploiting formula (2.8) in Remark 2.1-(3), we have

$$
\begin{aligned}
&\int_\Omega \mathcal{F}_{r_{ij}} \cdot X_j\big(X_i(\mathrm{T}u)\big)\,\mathrm{d}x \\
&\quad = \int_{\partial\Omega} \mathcal{F}_{r_{ij}} \cdot X_i(\mathrm{T}u)\,\langle X_j(x), \nu\rangle\,\mathrm{d}H^{n-1} - \int_\Omega X_j(\mathcal{F}_{r_{ij}})\,X_i(\mathrm{T}u)\,\mathrm{d}x \\
&\quad = \int_{\partial\Omega} \mathcal{F}_{r_{ij}} \cdot X_i(\mathrm{T}u)\,\langle X_j(x), \nu\rangle\,\mathrm{d}H^{n-1} \\
&\qquad - \bigg(\int_{\partial\Omega} X_j(\mathcal{F}_{r_{ij}})\,\mathrm{T}u\,\langle X_i(x), \nu\rangle\,\mathrm{d}H^{n-1} - \int_\Omega \mathrm{T}u \cdot X_i\big(X_j(\mathcal{F}_{r_{ij}})\big)\,\mathrm{d}x\bigg).
\end{aligned} \tag{4.10}
$$

By inserting (4.9) and (4.10) into (4.8), we finally obtain (4.4). □

Example 4.4. In Euclidean space $\mathbb{R}^n$, let $m < n$ and let $X = \{X_1, \ldots, X_m\}$ satisfy assumptions (H.1)-(H.2). Moreover, let $\Omega \subseteq \mathbb{R}^n$ be a fixed open set, regular for the Divergence Theorem. We then consider the function

$$
\mathcal{F} : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{m^2} \to \mathbb{R}, \qquad \mathcal{F}(x,z,p,r) := \frac{1}{2}\big(\textstyle\sum_{i=1}^m r_{ii}\big)^2 - G(z),
$$

where $G \in C^1(\mathbb{R})$ and $G(0) = 0$. In this context, identity (4.4) boils down to

$$\begin{aligned}\Big(\frac{q}{2} - 2\Big) \int_\Omega (\Delta_X u)^2 \,\mathrm{d}x - q \int_\Omega G(u)\,\mathrm{d}x + \int_\Omega \mathrm{T}u \left(\Delta_X^2 u - G'(u)\right) \mathrm{d}x \\ = \int_{\partial\Omega} \mathcal{F}\,\langle \mathrm{T}(x), \nu\rangle \,\mathrm{d}H^{n-1} \\ + \sum_{i=1}^m \int_{\partial\Omega} \big(X_i(\Delta_X u)\,\mathrm{T}u \ - \Delta_X u\, X_i(\mathrm{T}u)\big)\langle X_i(x), \nu\rangle \,\mathrm{d}H^{n-1},\end{aligned}$$

where $\Delta_X u$ is the *horizontal Laplacian* of u, that is,

$$\Delta_X u = -\textstyle\sum_{i=1}^m X_i^2 u,$$

and q, T, ν_X are given, respectively, by (2.2), (3.1), (3.3).

With Proposition 4.3 at hand, we could try to rerun the arguments of the previous section in order to establish a non-existence result for the PDE (4.3), i.e.,

$$\begin{aligned}\sum_{i,j=1}^m X_i X_j \Big(\mathcal{F}_{r_{ij}}\big(x, u, \nabla_X u, \mathcal{H}_X u\big)\Big) + \mathrm{div}_X\Big(\mathcal{F}_p\big(x, u(x), \nabla_X u(x), \mathcal{H}_X u\big)\Big) \\ + \mathcal{F}_z\big(x, u, \nabla_X u, \mathcal{H}_X u\big) = 0.\end{aligned} \tag{4.11}$$

In fact, if $u \in C^4(\overline{\Omega})$ solves (4.11), from identity (4.4) we get

$$\begin{aligned}\int_\Omega \big(q\mathcal{F} - \langle \nabla_p \mathcal{F}, \nabla_X u\rangle\big)\,\mathrm{d}x + \int_\Omega \mathrm{T}\big(x \mapsto \mathcal{F}(x,z,p,r)\big)(x, u, \nabla_X u, \mathcal{H}_X u)\,\mathrm{d}x \\ - 2\sum_{i,j=1}^m \int_\Omega \mathcal{F}_{r_{ij}}\, X_j(X_i u)\,\mathrm{d}x \\ = \int_{\partial\Omega} \mathcal{F}\,\langle \mathrm{T}(x), \nu\rangle\,\mathrm{d}H^{n-1} - \int_{\partial\Omega} \mathrm{T}u\,\langle \mathcal{F}_p, \nu_X\rangle\,\mathrm{d}H^{n-1} \\ + \sum_{i,j=1}^m \int_{\partial\Omega} \big(X_i(\mathcal{F}_{r_{ji}})\,\mathrm{T}u \ - \mathcal{F}_{r_{ij}}\, X_i(\mathrm{T}u)\big)\langle X_j(x), \nu\rangle\,\mathrm{d}H^{n-1}.\end{aligned}$$

Moreover, if we further assume that $u \equiv 0$ on $\partial\Omega$, we already know from the proof of Theorem 3.5 that, for every $x \in \partial\Omega$, the following equality holds:

$$\mathrm{T}u\,\langle \mathcal{F}_p, \nu_X\rangle = \langle \mathrm{T}(x), \nu\rangle \cdot \langle \mathcal{F}_p, \nabla_X u\rangle. \tag{4.12}$$

Gathering together these facts, we then obtain the identity

$$\begin{aligned}\int_\Omega \big(q\mathcal{F} - \langle \nabla_p \mathcal{F}, \nabla_X u\rangle\big)\,\mathrm{d}x + \int_\Omega \mathrm{T}\big(x \mapsto \mathcal{F}(x,z,p,r)\big)(x, u, \nabla_X u, \mathcal{H}_X u)\,\mathrm{d}x \\ - 2\sum_{i,j=1}^m \int_\Omega \mathcal{F}_{r_{ij}}\, X_j(X_i u)\,\mathrm{d}x \\ = \int_{\partial\Omega} \langle \mathrm{T}(x), \nu\rangle \left(\mathcal{F} - \langle \mathcal{F}_p, \nabla_X u\rangle\right)\mathrm{d}H^{n-1} \\ + \sum_{i,j=1}^m \int_{\partial\Omega} \big(X_i(\mathcal{F}_{r_{ji}})\,\mathrm{T}u \ - \mathcal{F}_{r_{ij}}\, X_i(\mathrm{T}u)\big)\langle X_i(x), \nu\rangle\,\mathrm{d}H^{n-1}.\end{aligned} \tag{4.13}$$

Now, following the arguments in the previous section, it is clear that identity (4.13) leads to a non-existence result for (4.11) (which a fourth-order PDE) if we couple the PDE with *two* boundary conditions, namely

$$u \equiv 0 \text{ on } \partial\Omega \qquad \text{and} \qquad \mathcal{B}u \equiv 0 \text{ on } \partial\Omega, \tag{4.14}$$

where $\mathcal{B}$ is a suitable first-order operator. On the other hand, for the 'geometrical' assumption that Ω is δ_λ-star shaped to play a rôle, we need to choose $\mathcal{B}$ in such a way that, for every solution $u \in C^4(\overline{\Omega})$ satisfying (4.14), one has

$$\begin{aligned}\sum_{i,j=1}^{m} \big(X_i(\mathcal{F}_{r_{ji}})\,\mathrm{T}u \; - \mathcal{F}_{r_{ij}}\, X_i(\mathrm{T}u)\big)\langle X_i(x),\nu\rangle \\ = \langle \mathrm{T}(x),\nu\rangle f\big(x,u,\nabla_X u,\mathcal{H}_X u\big),\end{aligned} \tag{4.15}$$

where $f = f(x,z,p,r)$ is a suitable function only depending on $\mathcal{F}$ and on its derivatives. We explicitly point out that, when $\mathcal{F}$ does not depend on r, the analog of (4.15) is identity (4.12) (holding true under the condition $u \equiv 0$ on $\partial\Omega$), where

$$f(x,z,p) = \langle p, \mathcal{F}_p(x,z,p)\rangle.$$

Quite unnaturally, we are able to prove an identity like (4.15) if we assume that $\mathcal{B} = \nabla$, that is, if we consider the boundary condition

$$\nabla u \equiv 0 \text{ on } \partial\Omega \iff \partial_{x_i} u \equiv 0 \text{ on } \partial\Omega \text{ for all } i = 1,\ldots,n. \tag{4.16}$$

The reason why such a boundary condition is 'unnatural' is that it is not intrinsically defined trough the vector fields $X_1,\ldots,X_m$ associated with $\mathcal{F}$; in this spirit, a more 'natural' condition should involve the X-gradient of u, that is,

$$\nabla_X u \equiv 0 \text{ on } \partial\Omega.$$

We believe that the problem of finding adequate *intrinsic* boundary conditions which lead to non-existence results involving the same notion of δ_λ-star shapedness used in the previous section could be an interesting challenge.

For the sake of completeness, we end this section by proving the non-existence result arising from Proposition 4.3 under the 'non-intrinsic' condition (4.16).

Theorem 4.5. *Let $\Omega \subseteq \mathbb{R}^n$ be a* connected *open set, regular for the Divergence Theorem and δ_λ-star shaped with respect to the origin. We assume that*

(i) *for every $x \in \partial\Omega$ and every $r \in \mathbb{R}^{m^2}$ we have*

$$\mathcal{F}(x,0,0,r) - \textstyle\sum_{i,j=1}^m r_{ij}\,\mathcal{F}_{r_{ij}}(x,0,0,r) \le 0; \tag{4.17}$$

(ii) *for every $x \in \Omega$ and every $(z,p,r) \in \mathbb{R}\times\mathbb{R}^m\times\mathbb{R}^{m^2}$ we have*

$$\begin{aligned} &q\mathcal{F}(x,z,p,r) - \langle p, \mathcal{F}_p(x,z,p,r)\rangle \\ &\qquad + \mathrm{T}\big(x \mapsto \mathcal{F}(x,z,p,r)\big) - 2\textstyle\sum_{i,j=1}^m r_{ij}\mathcal{F}_{r_{ij}}(x,z,p,r) \ge 0\end{aligned} \tag{4.18}$$

(iii) *either $z = 0$ or $p = 0$ or $r = 0$ when (4.18) holds.*

Then, the boundary-value problem

$$\begin{cases} \Big(\sum_{i,j=1}^m X_iX_j(\mathcal{F}_{r_{ij}}) + \mathrm{div}_X(\mathcal{F}_p) + \mathcal{F}_z\Big)\big(x,u,\nabla_X u,\mathcal{H}_X u\big) = 0 \quad \text{in } \Omega, \\ u\big|_{\partial\Omega} = 0, \\ \nabla u\big|_{\partial\Omega} = 0, \end{cases} \tag{4.19}$$

has no non-trivial solutions $u \in C^4(\overline{\Omega})$.

Proof. Let $u \in C^4(\overline{\Omega})$ be a solution of (4.19). We need to prove that

$$u \equiv 0 \text{ on } \Omega. \tag{4.20}$$

We first observe that, since $\nabla u \equiv 0$ on $\partial\Omega$, we obviously have

$$\nabla_X u \equiv 0 \quad \text{and} \quad \mathrm{T}u = \langle \mathrm{T}(x), \nabla u\rangle \equiv 0 \qquad \text{on } \partial\Omega.$$

As a consequence, since $u \equiv 0$ on $\partial\Omega$, from identity (4.13) we get

$$\begin{aligned}
&\int_\Omega \big(q\mathcal{F} - \langle \nabla_p \mathcal{F}, \nabla_X u\rangle\big)\,\mathrm{d}x + \int_\Omega \mathrm{T}\big(x \mapsto \mathcal{F}(x,z,p,r)\big)(x,u,\nabla_X u, \mathcal{H}_X u)\,\mathrm{d}x \\
&\qquad - 2\sum_{i,j=1}^m \int_\Omega \mathcal{F}_{r_{ij}}\, X_j(X_i u)\,\mathrm{d}x \\
&\quad = \int_{\partial\Omega} \langle \mathrm{T}(x), \nu\rangle\, \mathcal{F}\,\mathrm{d}H^{n-1} - \sum_{i,j=1}^m \mathcal{F}_{r_{ij}} X_i(\mathrm{T}u)\langle X_j(x),\nu\rangle\,\mathrm{d}H^{n-1}.
\end{aligned} \tag{4.21}$$

We now claim that, for every $x \in \partial\Omega$ and every $i, j = 1, \ldots, m$, one has

$$X_i(\mathrm{T}u)\langle X_j(x), \nu\rangle = \langle \mathrm{T}(x), \nu\rangle \cdot X_j(X_i u). \tag{4.22}$$

In fact, using the very definition of T, and writing

$$X_t = \textstyle\sum_{h=1}^n a_{h,t}(x)\partial_{x_h} \qquad (\text{for all } t = 1, \ldots, m),$$

we obtain the following chain of equality (holding true for all $x \in \partial\Omega$):

$$\begin{aligned}
X_i(\mathrm{T}u)\langle X_j(x), \nu\rangle &= \sum_{h,k=1}^n a_{h,i}(x)\,\partial_{x_h}(\mathrm{T}u)\,a_{k,j}(x)\cdot \nu_k \\
&= \sum_{h,k,l=1}^n a_{h,i}(x)\,\partial_{x_h}\big(\sigma_l\, x_l\, \partial_{x_l} u\big)\, a_{k,j}(x)\cdot\nu_k \\
&= \sum_{h,k=1}^n \sigma_h\, a_{h,i}(x)\,(\partial_{x_h} u)\, a_{k,j}(x)\cdot\nu_k + \sum_{h,k,l=1}^n \sigma_l\, x_l\, a_{h,i}(x)\,\partial^2_{x_h x_l} u\, a_{k,j}(x)\cdot\nu_k \\
&\quad (\text{since } \partial_{x_h} u \equiv 0 \text{ on } \partial\Omega \text{ for every } h = 1, \ldots, n) \\
&= \sum_{h,k,l=1}^n \sigma_l\, x_l\, a_{h,i}(x)\,\partial^2_{x_h x_l} u\, a_{k,j}(x)\cdot\nu_k =: (\bigstar).
\end{aligned}$$

On the other hand, since $u \equiv \nabla u \equiv 0$ on $\partial\Omega$, it is very easy to see that

$$\frac{\partial^2 u}{\partial x_h \partial x_l} u = \frac{\partial^2 u}{\partial \nu^2}\,\nu_h \nu_l \qquad \text{on } \partial\Omega;$$

thus, by crucially exploiting this information, we get

$$\begin{aligned}
(\bigstar) &= \Big(\sum_{l=1}^{n} \sigma_l \, x_l \, \nu_l\Big) \cdot \sum_{h,k=1}^{n} a_{h,i}(x)\, a_{k,j}(x) \Big(\frac{\partial^2 u}{\partial \nu^2}\, \nu_h \nu_k\Big) \\
&= \langle \mathrm{T}(x), \nu \rangle \cdot \sum_{h,k=1}^{n} a_{h,i}(x)\, a_{k,j}(x)\, \partial_{x_h x_k} u \\
&\quad (\text{since } \partial_{x_h} u \equiv 0 \text{ on } \partial\Omega \text{ for all } h = 1, \ldots, n) \\
&= \langle \mathrm{T}(x), \nu \rangle \cdot \sum_{k=1}^{n} a_{k,j}(x)\, \partial_{x_k} \Big(\sum_{h=1}^{n} a_{h,i}(x)\, \partial_{x_h} u \Big) \\
&= \langle \mathrm{T}(x), \nu \rangle \cdot X_j(X_i u),
\end{aligned}$$

which is precisely the claimed (4.22).

With (4.22) at hand, we can proceed with the proof of the theorem. Indeed, by combining the cited (4.22) with (4.21), we get

$$\begin{aligned}
&\int_\Omega \big(q\mathcal{F} - \langle \nabla_p \mathcal{F}, \nabla_X u \rangle\big)\, \mathrm{d}x + \int_\Omega \mathrm{T}\big(x \mapsto \mathcal{F}(x,z,p,r)\big)(x, u, \nabla_X u, \mathcal{H}_X u)\, \mathrm{d}x \\
&\qquad - 2 \sum_{i,j=1}^{m} \int_\Omega \mathcal{F}_{r_{ij}}\, X_j(X_i u)\, \mathrm{d}x \\
&= \int_{\partial\Omega} \langle \mathrm{T}(x), \nu \rangle \big(\mathcal{F} - \textstyle\sum_{i,j=1}^{m} X_j(X_i u)\, \mathcal{F}_{r_{ij}}\big)\, \mathrm{d}H^{n-1};
\end{aligned}$$

from this, using assumptions (i)-(ii) and reminding that Ω is δ_λ-star shaped with respect to the origin (see Definition 3.6), we obtain

$$\begin{aligned}
q\mathcal{F} - \langle \nabla_X, \mathcal{F}_p \rangle &+ \mathrm{T}\big(x \mapsto \mathcal{F}(x,z,p,r)\big)(x, u, \nabla_X, \mathcal{H}_X u) \\
&- 2 \textstyle\sum_{i,j=1}^{m} X_j(X_i u) \mathcal{F}_{r_{ij}} = 0 \qquad \text{for all } x \in \Omega.
\end{aligned}$$

Now, according to assumption (iii), only three cases can occur:

(a) $u \equiv 0$ on Ω. In this case, (4.20) is satisfied and the proof is complete.

(b) $\nabla_X u \equiv 0$ on Ω. In this case, we have already recognized in the demonstration of Theorem 3.7 that u must be constant in Ω. Since, by assumption, $u \equiv 0$ on $\partial\Omega$, we conclude that $u \equiv 0$ throughout Ω, and (4.20) is again satisfied.

(c) $\mathcal{H}_X u \equiv 0$ on Ω. In this case, bearing in mind the definition of $\mathcal{H}_X$, we get

$$\nabla_X(X_i u) \equiv 0 \text{ on } \Omega \text{ for every } i = 1, \ldots, n;$$

from this, by arguing exactly as in the proof of Theorem 3.7, we deduce that $\nabla_X u$ must be constant on Ω. On the other hand, since (by assumption) $\nabla u \equiv 0$ on $\partial\Omega$, we necessarily have that $\nabla_X u \equiv 0$ on $\partial\Omega$ as well; as a consequence,

$$\nabla_X u \equiv 0 \qquad \text{on } \partial\Omega.$$

From (b) we then conclude that $u \equiv 0$ on Ω, and (4.20) is again satisfied. □

Remark 4.6. With reference to discussion before the statement of Theorem 4.5, we point out that identity (4.22) leads to the needed (4.15): in fact, we have

$$\sum_{i,j=1}^m \big(X_i(\mathcal{F}_{r_{ji}})\,\mathrm{T}u \ - \mathcal{F}_{r_{ij}}\,X_j(\mathrm{T}u)\big)\langle X_j(x),\nu\rangle$$
$$= -\sum_{i,j=1}^m \mathcal{F}_{r_{ij}}\,X_j(\mathrm{T}u)\langle X_j(x),\nu\rangle = -\langle \mathrm{T}(x),\nu\rangle\cdot\sum_{i,j=1}^m \mathcal{F}_{r_{ij}}\,X_j(X_i u),$$

and this is precisely identity (4.15) with the choice

$$f(x,z,p,r) := -\sum_{i,j=1}^m r_{ij}\mathcal{F}_{r_{ij}}.$$

References

[1] S. Biagi: *On the Gibbons conjecture for homogeneous Hörmander operators*, NoDEA Nonlinear Differential Equations Appl. **26** (2019), 26–49.

[2] S. Biagi, A. Bonfiglioli: *Global Heat Kernels for Parabolic Homogeneous Hörmander Operators*, arXiv:1910.09907 (2019).

[3] S. Biagi, A. Bonfiglioli: *The existence of a global fundamental solution for homogeneous Hörmander operators via a global lifting method*, Proc. Lond. Math. Soc. **5** (2017), 855-?889.

[4] S. Biagi, A. Bonfiglioli, M. Bramanti: *Global estimates for the fundamental solution of homogeneous Hörmander sums of squares*, arXiv:1906.07836 (2019).

[5] S. Biagi, A. Bonfiglioli, M. Bramanti: *Global estimates in Sobolev spaces for homogeneous Hörmander sums of squares*, arXiv:1906.07835 (2019).

[6] S. Biagi, M. Bramanti: *Global Gaussian estimates for the heat kernel of homogeneous sums of squares*, arXiv:2003.09845 (2020).

[7] S. Biagi, E.Lanconelli: *Large sets at infinity and Maximum Principle on unbounded domains for a class of sub-elliptic operators*, J. Differential Equations **269** (2020), 9680–9719.

[8] S. Biagi, A. Bonfiglioli: 'An introduction to the Geometrical Analysis of Vector Fields. With Applications To Maximum Principles And Lie Groups', World Scientific Publishing Company, 2018.

[9] A. Bonfiglioli, E. Lanconelli, F. Uguzzoni: 'Stratified Lie Groups and Potential Theory for their sub-Laplacians', Springer Monographs in Mathematics **26**, Springer, New York, N.Y., 2007.

[10] J.-M. Bony: *Principe du maximum, inégalite de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés*, Ann. Inst. Fourier (Grenoble) **19** (1969), 277–304.

[11] Y. Bozhkov, E. Mitidieri: *Conformal Killing vector fields and Rellich type identities on Riemannian manifolds. I*, Geometric methods in PDE's, 65–80, Lect. Notes Semin. Interdiscip. Mat., 7, Semin. Interdiscip. Mat. (S.I.M.), Potenza, 2008.

[12] Y. Bozhkov, E. Mitidieri: *Conformal Killing vector fields and Rellich type identities on Riemannian manifolds, II*, Mediterr. J. Math. **9**(1) (2012), 1–20.

[13] C. Cowan: *A short remark regarding Pohozaev-type results on general domains assuming finite Morse index*, Proc. Roy. Soc. Edinburgh Sect. A **147**(2) (2017), 293–297.

[14] E.N. Dancer, K. Zhang: *Uniqueness of solutions for some elliptic equations and systems in nearly star-shaped domains*, Nonlinear Anal. TMA **41**, (2000), 745–761.

[15] G. Dinca, F. Isaia: *Generalized Pohozaev identity and a non-existence result for the p-Laplacian: weak solutions*, Adv. Differential Equations **14** (2009), 497–540.

[16] F. Dragoni, D. Filali: *Starshaped and convex sets in Carnot groups and in the geometries of vector fields*, Journal of Convex Analysis **26**(4) (2019).

[17] F. Dragoni, N. Garofalo, P. Salani: *Starshapedeness for fully-nonlinear equations in Carnot groups*, Journal of the London Mathematical Society 99(3) (2019), 901–918.

[18] M.M. Fall, T. Weth: *Nonexistence results for a class of fractional elliptic boundary value problems*, J. Funct. Anal. **263**(8) (2012), 2205–2227.

[19] M. Fogagnolo, A. Pinamonti: *Strict Starshapedness of solutions to the horizontal p-Laplacian in the Heisenberg group*, Submitted.

[20] N. Garofalo, E. Lanconelli: *Existence and nonexistence results for semilinear differential equations on the Heisenberg group*, Indiana Univ. Math. J. **41** (1992), 71–98.

[21] N. Garofalo, Z. Shen: *Absence of positive eigenvalues for a class of subelliptic operators*, Math. Ann. **304**(4) (1996), 701–715.
[22] N. Garofalo, D. Vassilev: *Regularity near the characteristic set in the non-linear Dirichlet problem and conformal geometry of sub-Laplacians on Carnot groups*, Math. Ann. **318**(3) (2000), 453–516.
[23] M, Guedda, L. Veron: *Quasilinear elliptic equations involving critical exponents*. Nonlinear Anal. **1318**, (1989), 879–902.
[24] A.E. Kogoj, E. Lanconelli: *On semilinear Δ_λ-Laplace equation*, Nonlinear Anal. **75** (2012), 4637–4649.
[25] A. Maione, A. Pinamonti, F. S. Cassano: *Γ-convergence for functionals depending on vector fields I. Integral representation and compactness*, J. Math. Pures Appl. **9** 139 (2020), 109–142.
[26] A. Maione, A. Pinamonti, F. S. Cassano: *Γ-convergence for functionals depending on vector fields II. Convergence of minimizers*, Forthcoming.
[27] A. Maione, E. Vecchi: *Integral representation of local left-invariant functionals in Carnot groups*, Anal. Geom. Metr. Spaces **8** (2020), no. 1, 1–14.
[28] J. McGough, J. Mortensen: *Pohozaev obstructions on non-starlike domains*, Calc. Var. Partial Differential Equations **18**(2) (2003), 189–205.
[29] E. Mitidieri: *A Rellich type identity and applications*, Comm. Partial Differential Equations **18** (1993), 125–151.
[30] Z. Nehari: *On a class of nonlinear second-order differential equations*, Trans. Amer. Math. Soc., **95** (1960), 101–123.
[31] P. Niu, H. Zhang, X. Luo: *Hardy's inequalities and Pohozaev's identities on the Heisenberg group*, Acta Math. Sinica (Chin. Ser.) **46**(2) (2003), 279–290.
[32] A. Pinamonti: *Multiple solutions for possibly degenerate equations in divergence form*, Electronic Journal of Differential Equations, **2016** (2016).
[33] S. Pohozaev: *Eigenfunctions of the equation $\Delta u + \lambda u = 0$*, Sov. Math. Dokl. **6** (1965), 1408–1411.
[34] P. Pucci, J. Serrin: *A general variational identity*, Indiana Univ. Math. J. **35** (1986), 681–703.
[35] F. Rellich: *Darstellung der Eigenwerte von $\Delta u + \lambda u = 0$ durch ein Randintegral*, Math. Z. **46** (1940), 635–636.
[36] X. Ros-Oton, J. Serra: *The Pohozaev identity for the fractional Laplacian*, Arch. Ration. Mech. Anal. **213**(2) (2014), 587–628.
[37] A. Wagner: *Pohozaev's identity from a variational viewpoint*, J. Math. Anal. Appl. **266**(1) (2002), 149–159.
[38] Q. Xu, Y. Lu, L. Wang: *Pohozaev identities from domain deformation*, Internat. J. Math. **21**(9) (2010), 1121-1134.

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