EQUILIBRIUM MEASURE FOR A NONLOCAL DISLOCATION ENERGY WITH PHYSICAL CONFINEMENT

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Abstract. In this paper we characterize the equilibrium measure for a family of nonlocal and anisotropic energies I_α that describe the interaction of particles confined in an elliptic subset of the plane. The case $\alpha = 0$ corresponds to purely Coulomb interactions, while the case $\alpha = 1$ describes interactions of positive edge dislocations in the plane. The anisotropy into the energy is tuned by the parameter α and favors the alignment of particles. We show that the equilibrium measure is completely unaffected by the anisotropy and always coincides with the optimal distribution in the case $\alpha = 0$ of purely Coulomb interactions, which is given by an explicit measure supported on the boundary of the elliptic confining domain. Our result seems to be in constrast with the mechanical conjecture that positive edge dislocation at equilibrium tend to arrange themselves along "wall-like" structures. Moreover, this is one of the very few examples of explicit characterization of the equilibrium measure for nonlocal interaction energies outside the radially symmetric case.

1. Introduction

In this paper we consider the interaction energies

$$J_\alpha(\mu) := \iint_{\mathbb{R}^2\times\mathbb{R}^2} V_\alpha(x-y)\,d\mu(x)\,d\mu(y) + \int_{\mathbb{R}^2} F(x)\,d\mu(x), \tag{1.1}$$

defined on the class of probability measures $\mu \in \mathcal{P}(\mathbb{R}^2)$, where

$$V_\alpha(x) := \begin{cases} -\log|x| + \alpha\dfrac{(x\cdot\tau)^2}{|x|^2} & \text{if } x \neq 0,\\ +\infty & \text{if } x = 0, \end{cases} \tag{1.2}$$

$\tau = (\tau_1, \tau_2)$ is a given vector in $\mathbb{S}^1$ and $\alpha \in \mathbb{R}$. Here F denotes a confining potential, that is, a lower semicontinuous function with the property that

$$\lim_{|x|\to\infty} (F(x) - \log|x|) = +\infty.$$

The interaction kernel V_α is an anisotropic perturbation of the classical $2d$ Coulomb interaction kernel, which corresponds to the case $\alpha = 0$. The purely Coulomb case has been widely studied in several contexts, from $2d$ electrostatic phenomena, to Coulomb gases, random matrix theory, Ginzburg-Landau vortices, etc. (see, e.g., [11]). The case $\alpha = 1$ has a physical motivation in dislocation theory, since V_1 corresponds to the interaction kernel of positive edge dislocations in the plane with Burgers vector τ (see, e.g., [5]). The parameter α has the role of tuning the anisotropy in the interaction potential: When $\alpha \neq 0$, the anisotropy is turned on, favoring alignment in the direction orthogonal to τ (if $\alpha > 0$; parallel to τ if $\alpha < 0$).

In the Coulomb case $\alpha = 0$ the minimizer of J_0 can be characterized explicitly for several choices of F. For instance, if $F(x) = |x|^2$, it is well-known that the unique minimizer of J_0 is the *circle law*, i.e., the measure $\mu_0 = \frac{1}{\pi}\chi_{B_1(0)}$ (see, e.g., [4]); if F is the indicator function of the ball $B_r(0)$, the unique minimizer is the measure $\mu_r = \frac{1}{2\pi r}\mathcal{H}^1 \llcorner \partial B_r(0)$ (see, e.g., [11]). Those explicit characterizations are strongly based on the fact that V_0 is the fundamental solution of the Laplacian and on the radial symmetry of the problem. In the presence of anisotropy (i.e., for $\alpha \neq 0$) finding the minimizers explicitly is an extremely hard task and only very recently some characterizations have been proved in the case of the quadratic confinement $F(x)=|x|^2$, see [1, 2, 7, 8, 10]. The main result in these works is that for this choice of the confinement the values $\alpha=\pm1$ are *critical* values of the parameter, at which a sudden drop of dimensionality of the minimizer occurs. Indeed, for $\alpha\in(-1,1)$ and $\tau=e_1$ it is proved that the unique minimizer of J_α is the normalized characteristic function of the region surrounded by an ellipse of semi-axes $\sqrt{1-\alpha}$ and $\sqrt{1+\alpha}$. On the other hand, for every $\alpha\geq1$ the only minimizer is the *semi-circle law*

$$\mu_1 := \frac{1}{\pi}\delta_0\otimes\sqrt{2-x_2^2}\,\mathcal{H}^1 \llcorner (-\sqrt{2},\sqrt{2}) \tag{1.3}$$

on the vertical axis, while for $\alpha\leq-1$ it is the semi-circle law on the horizontal axis.

In this paper we characterize explicitly the minimizer of J_α in the case of a *physical confinement*, that is, we assume the confining potential F to be the indicator function of an elliptic domain of $\mathbb{R}^2$. More precisely, for $a,b>0$ we define

$$\Omega(a,b) := \left\{ x=(x_1,x_2)\in\mathbb{R}^2 :\ \frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}\leq 1\right\},$$

which is the region surrounded by an ellipse centered at the origin with horizontal semi-axis a and vertical semi-axis b. We consider F to be the function that is equal to 0 in $\Omega(a,b)$ and $+\infty$ otherwise. The functional J_α in (1.1) can thus be rewritten as

$$I_\alpha(\mu) := \iint_{\Omega(a,b)\times\Omega(a,b)} V_\alpha(x-y)\,d\mu(x)\,d\mu(y) \tag{1.4}$$

for every $\mu\in\mathcal{P}(\Omega(a,b))$, where $\mathcal{P}(\Omega(a,b))$ denotes the class of all positive Borel measures on $\mathbb{R}^2$, with support contained in $\Omega(a,b)$ and with unitary mass.

We note that, since $\Omega(a,b)$ is a compact set, $V_\alpha(x-y)$ is bounded from below for $(x,y)\in\Omega(a,b)\times\Omega(a,b)$. Therefore, the energy (1.4) is well defined on $\mathcal{P}(\Omega(a,b))$, possibly equal to $+\infty$.

The main result of the paper is the following.

Theorem 1.1. *Let $\alpha\in[-1,1]$. Then, the measure*

$$\mu_{a,b} := \frac{1}{2\pi ab}\frac{1}{\sqrt{\frac{x_1^2}{a^4}+\frac{x_2^2}{b^4}}}\mathcal{H}^1 \llcorner \partial\Omega(a,b) \tag{1.5}$$

is the unique minimizer of I_α on $\mathcal{P}(\Omega(a,b))$ and satisfies the Euler-Lagrange condition

$$(V_\alpha * \mu_{a,b})(x) = -\log\Big(\frac{a+b}{2}\Big)+\alpha\frac{a\tau_1^2+b\tau_2^2}{a+b} \qquad \textit{for every } x\in\Omega(a,b). \tag{1.6}$$

The theorem shows, in particular, that for $\alpha\in[-1,1]$ the equilibrium measure is independent of α and τ, that is, is completely unaffected by the presence of the anisotropy. This is a surprising result, when compared with the behavior found in [1, 10] for a quadratic confinement. Moreover, for $\alpha=1$ it seems to contradict the mechanical conjecture that the optimal configurations of dislocations should align in the orthogonal direction to the Burgers vector τ, producing wall-like structures (see, e.g., [3, 5, 6]). Theorem 1.1 shows that, in general, this may be not the case in presence of a physical confinement.

The plan of the paper is the following. In Section 2 we prove Theorem 1.1. In the last section we extend this result to a more general family of even anisotropies.

2. Proof of Theorem 1.1

To prove Theorem 1.1 we need some preliminary results. We first introduce the notion of capacity. For any compact set $K\subset\Omega(a,b)$ we define the *logarithmic capacity* of K as

$$\mathrm{cap}(K) := \Phi\left(\inf_{\mu\in\mathcal{P}(K)}\iint_{K\times K} V_0(x-y)\,d\mu(x)\,d\mu(y)\right), \qquad \Phi(t)=e^{-t},$$

where $\mathcal{P}(K)$ is the class of all probability measures with support in K. For any Borel set $B\subset\Omega(a,b)$ the capacity of B is defined as the supremum of the capacity of compact sets

$K \subset B$. Finally, any set (not necessarily Borel) contained in a Borel set of zero capacity, is considered to have zero capacity.

In the following we say that a property holds *quasi everywhere* (q.e.) in a set A if the set of points in A where the property is not satisfied has zero capacity. Note that if B is a Borel set with zero capacity and $\mu \in \mathcal{P}(\Omega(a,b))$ is a measure with $I_\alpha(\mu) < +\infty$ for some α, then $\mu(B) = 0$. In other words, any measure with finite interaction energy does not charge sets of zero capacity.

In the following theorem we establish existence and uniqueness of the minimizer of the energy (1.4) and we characterize it via Euler-Lagrange conditions.

Theorem 2.1. *Let $\alpha \in [-1,1]$. Then the energy I_α is strictly convex on the class of measures with finite energy and has a unique minimizer $\mu_\alpha \in \mathcal{P}(\Omega(a,b))$. Moreover, μ_α is uniquely characterized by the Euler-Lagrange conditions: there exists $c_\alpha \in \mathbb{R}$ such that*

$$(V_\alpha * \mu_\alpha)(x) = c_\alpha \qquad \text{for } \mu_\alpha\text{-a.e. } x \in \operatorname{supp}\mu_\alpha, \tag{2.1}$$

$$(V_\alpha * \mu_\alpha)(x) \geq c_\alpha \qquad \text{for q.e. } x \in \Omega(a,b). \tag{2.2}$$

Proof. The case $\alpha = 0$ is well-known. For $\alpha \in [-1,1]$, $\alpha \neq 0$, the proof follows the lines of [10, Section 2]. Here we just recall that the strict convexity of I_α on the class of measures with finite energy is equivalent to the condition

$$\int_{\mathbb{R}^2} V_\alpha * (\nu_1 - \nu_2)\, d(\nu_1 - \nu_2) > 0$$

for every $\nu_1, \nu_2 \in \mathcal{P}(\Omega(a,b))$, $\nu_1 \neq \nu_2$, with finite energy. This condition is proved by showing that the Fourier transform of V_α is a positive distribution on test functions vanishing at zero. Indeed, the heuristic idea is to rewrite the energy of $\nu_1 - \nu_2$ in Fourier space as

$$\int_{\mathbb{R}^2} V_\alpha * (\nu_1 - \nu_2)\, d(\nu_1 - \nu_2) = \int_{\mathbb{R}^2} \widehat{V}_\alpha(\xi) |\widehat{\nu}_1(\xi) - \widehat{\nu}_2(\xi)|^2\, d\xi$$

and note that $\widehat{\nu}_1 - \widehat{\nu}_2$ vanishes at $\xi = 0$ since $\nu_1 - \nu_2$ is a neutral measure.

Since $V_\alpha \in L^1_{loc}(\mathbb{R}^2)$ and has a logarithmic growth at infinity, it is a tempered distribution, namely, $V_\alpha \in \mathcal{S}'$, where $\mathcal{S}$ is the Schwartz space; hence, $\widehat{V}_\alpha \in \mathcal{S}'$. We recall that $\widehat{V}_\alpha$ is defined as

$$\langle \widehat{V}_\alpha, \varphi \rangle := \langle V_\alpha, \widehat{\varphi} \rangle$$

for every $\varphi \in \mathcal{S}$, where

$$\widehat{\varphi}(\xi) := \int_{\mathbb{R}^2} \varphi(x) e^{-2\pi i \xi \cdot x}\, dx.$$

By [1, eq. (2.4)], setting $\tau^\perp := (-\tau_2, \tau_1)$, we have that

$$\langle \widehat{V}_\alpha, \varphi \rangle = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(1-\alpha)(\xi \cdot \tau)^2 + (1+\alpha)(\xi \cdot \tau^\perp)^2}{|\xi|^4} \varphi(\xi)\, d\xi \tag{2.3}$$

for every $\varphi \in \mathcal{S}$ with $\varphi(0) = 0$. From this formula we immediately deduce that for $\alpha \in [-1,1]$, $\widehat{V}_\alpha$ is a positive tempered distribution on test functions that are vanishing at zero. □

Owing to the last theorem, we are reduced to prove that the probability measure $\mu_{a,b}$ defined in (1.5) satisfies the Euler-Lagrange conditions (2.1)–(2.2) for every $\alpha \in [-1,1]$. Our strategy consists in showing that the function $V_\alpha * \mu_{a,b}$ is constant on $\partial\Omega(a,b)$ and harmonic in the interior of $\Omega(a,b)$, so that by the maximum principle it has to be constant on $\Omega(a,b)$. In the following lemma we first compute the distributional Laplacian of the anisotropic term of the interaction kernel.

Lemma 2.2. *Let $W : \mathbb{R}^2 \to [0, +\infty]$ be defined by*

$$W(x) := \frac{(x \cdot \tau)^2}{|x|^2}$$

if $x \in \mathbb{R}^2$, $x \neq 0$, and $W(0) := 0$. Then, the distributional Laplacian of W is given by

$$\Delta W = 2D^2 V_0\, \tau^\perp \cdot \tau^\perp + 2\pi \delta_0, \tag{2.4}$$

where D^2V_0 is the distributional Hessian of V_0.

Proof. Assume first that $\tau = e_1$, the unit vector along the x_1-axis. In this case we have that

$$\Delta W(x) = \frac{2(x_2^2 - x_1^2)}{|x|^4} = 2\frac{\partial^2 V_0}{\partial x_2^2}(x) \qquad \text{for } x \neq 0. \tag{2.5}$$

Let $\varphi \in C_c^\infty(\mathbb{R}^2)$. For every $\varepsilon > 0$ we can write

$$\langle \Delta W, \varphi\rangle = \int_{B_\varepsilon(0)} \frac{x_1^2}{|x|^2}\Delta\varphi(x)\,dx + \int_{\mathbb{R}^2\setminus B_\varepsilon(0)} \frac{x_1^2}{|x|^2}\Delta\varphi(x)\,dx. \tag{2.6}$$

We note that the first term of (2.6) is infinitesimal as $\varepsilon \to 0$. Thus, we focus on the last term of (2.6). Integrating by parts and using (2.5) yield

$$\begin{aligned}
&\int_{\mathbb{R}^2\setminus B_\varepsilon(0)} \frac{x_1^2}{|x|^2}\Delta\varphi(x)\,dx \\
&= 2\int_{\mathbb{R}^2\setminus B_\varepsilon(0)} \frac{\partial^2 V_0}{\partial x_2^2}(x)\varphi(x)\,dx - \frac{1}{\varepsilon^3}\int_{\partial B_\varepsilon(0)} x_1^2 \nabla\varphi(x)\cdot x\,d\mathcal{H}^1(x) \\
&= 2\int_{\mathbb{R}^2\setminus B_\varepsilon(0)} V_0(x)\frac{\partial^2 \varphi}{\partial x_2^2}(x)\,dx + \frac{2}{\varepsilon^3}\int_{\partial B_\varepsilon(0)} x_2^2\varphi(x)\,d\mathcal{H}^1(x) \\
&\quad - \frac{2\log\varepsilon}{\varepsilon}\int_{\partial B_\varepsilon(0)} x_2\frac{\partial\varphi}{\partial x_2}(x)\,d\mathcal{H}^1(x) - \frac{1}{\varepsilon^3}\int_{\partial B_\varepsilon(0)} x_1^2\nabla\varphi(x)\cdot x\,d\mathcal{H}^1(x)
\end{aligned} \tag{2.7}$$

where we have used that $\nabla W(x)\cdot x = 0$ for $x \neq 0$. As $\varepsilon \to 0$, we have that

$$\frac{2}{\varepsilon^3}\int_{\partial B_\varepsilon(0)} x_2^2\varphi(x)\,d\mathcal{H}^1(x) \to 2\pi\varphi(0),$$

whereas the other boundary terms in (2.7) converge to zero. Passing to the limit as $\varepsilon \to 0$ in (2.7), we conclude that

$$\Delta W = 2\frac{\partial^2 V_0}{\partial x_2^2} + 2\pi\delta_0 \tag{2.8}$$

in the case $\tau = e_1$.

Swapping the role of x_1 and x_2, we immediately see that

$$\Delta\Big(\frac{x_2^2}{|x|^2}\Big) = 2\frac{\partial^2 V_0}{\partial x_1^2} + 2\pi\delta_0. \tag{2.9}$$

Using that

$$\Delta\Big(\frac{x_1x_2}{|x|^2}\Big) = -\frac{4x_1x_2}{|x|^4} = -2\frac{\partial^2 V_0}{\partial x_1 x_2}(x) \qquad \text{for } x \neq 0,$$

a similar argument shows that

$$\Delta\Big(\frac{x_1x_2}{|x|^2}\Big) = -2\frac{\partial^2 V_0}{\partial x_1 x_2} \tag{2.10}$$

in the sense of distributions.

Combining (2.8)–(2.10), we obtain the thesis. $\square$

We now state a lemma that will be useful to compute the potential $V_\alpha * \mu_{a,b}$ on the boundary of the domain $\Omega(a,b)$.

Lemma 2.3. *Let $b_1 > b_2 > 0$. Then,*

$$\frac{1}{\pi}\int_0^\pi \log(b_1 \pm b_2\cos\theta)\,d\theta = \log\Big(b_1 + \sqrt{b_1^2 - b_2^2}\Big) - \log 2,$$

$$\frac{1}{\pi}\int_0^\pi \frac{1}{b_1 \pm b_2\cos\theta}\,d\theta = \frac{1}{\sqrt{b_1^2 - b_2^2}},$$

and

$$\frac{1}{\pi}\int_0^\pi \frac{\cos\theta}{b_1 \pm b_2\cos\theta}d\theta = \pm\frac{\sqrt{b_1^2-b_2^2}-b_1}{b_2\sqrt{b_1^2-b_2^2}}.$$

Proof. See [11, Lemma IV.1.15]. □

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. For every $\alpha\in[-1,1]$ we set

$$F_\alpha(x) := (V_\alpha * \mu_{a,b})(x) \qquad \text{for } x\in\Omega(a,b).$$

We first compute F_0 on the boundary of $\Omega(a,b)$. Let $x=(a\cos\varphi, b\sin\varphi)\in\partial\Omega(a,b)$ with $\varphi\in[0,2\pi]$. By a change of variable we have

$$\begin{aligned} F_0(x) &= -\frac{1}{4\pi ab}\int_{\partial\Omega(a,b)} \log\Big((a\cos\varphi-y_1)^2+(b\sin\varphi-y_2)^2\Big)\frac{1}{\sqrt{\frac{y_1^2}{a^4}+\frac{y_2^2}{b^4}}}\,d\mathcal{H}^1(y) \\ &= -\frac{1}{4\pi}\int_0^{2\pi}\log\Big(a^2(\cos\varphi-\cos\theta)^2+b^2(\sin\varphi-\sin\theta)^2\Big)\,d\theta. \end{aligned}$$

Using the identities

$$\cos\varphi-\cos\theta = 2\sin\Big(\frac{\theta-\varphi}{2}\Big)\sin\Big(\frac{\theta+\varphi}{2}\Big), \tag{2.11}$$

$$\sin\varphi-\sin\theta = -2\sin\Big(\frac{\theta-\varphi}{2}\Big)\cos\Big(\frac{\theta+\varphi}{2}\Big), \tag{2.12}$$

we obtain

$$\begin{aligned} F_0(x) =& -\frac{1}{4\pi}\int_0^{2\pi}\log\Big(a^2\sin^2\Big(\frac{\theta+\varphi}{2}\Big)+b^2\cos^2\Big(\frac{\theta+\varphi}{2}\Big)\Big)\,d\theta \\ &-\frac{1}{4\pi}\int_0^{2\pi}\log\Big(4\sin^2\Big(\frac{\theta-\varphi}{2}\Big)\Big)\,d\theta. \end{aligned}$$

By the identity $4\sin^2\big((\theta-\varphi)/2\big)=2\big(1-\cos(\theta-\varphi)\big)$ we deduce that

$$-\frac{1}{4\pi}\int_0^{2\pi}\log\Big(4\sin^2\Big(\frac{\theta-\varphi}{2}\Big)\Big)\,d\theta = -\frac{1}{4\pi}\int_0^{2\pi}\log\big(1-\cos\theta\big)\,d\theta-\frac{1}{2}\log 2 = 0,$$

where the last equality follows from Lemma 2.3. Therefore, we have

$$\begin{aligned} F_0(x) &= -\frac{1}{4\pi}\int_0^{2\pi}\log\Big(\frac{a^2+b^2}{2}+\frac{a^2-b^2}{2}\Big(\sin^2\Big(\frac{\theta+\varphi}{2}\Big)-\cos^2\Big(\frac{\theta+\varphi}{2}\Big)\Big)\Big)\,d\theta \\ &= -\frac{1}{4\pi}\int_0^{2\pi}\log\Big(\frac{a^2+b^2}{2}-\frac{a^2-b^2}{2}\cos(\theta+\varphi)\Big)\,d\theta \\ &= -\frac{1}{4\pi}\int_0^{2\pi}\log\big(a^2+b^2-(a^2-b^2)\cos\theta\big)\,d\theta+\frac{1}{2}\log 2 \\ &= -\log\Big(\frac{a+b}{2}\Big), \end{aligned}$$

where we used again Lemma 2.3 (if $a\neq b$; otherwise, the conclusion is trivial). We thus conclude that F_0 is constant on the boundary of $\Omega(a,b)$. Since the function F_0 is continuous on $\Omega(a,b)$ and harmonic in the interior of $\Omega(a,b)$, we deduce by the maximum principle that

$$F_0(x) = -\log\Big(\frac{a+b}{2}\Big) \qquad \text{for every } x\in\Omega(a,b). \tag{2.13}$$

In particular, by Theorem 2.1 the statement is proved for $\alpha=0$.

To compute F_α, we introduce the function

$$G(x) := (W * \mu_{a,b})(x) \qquad \text{for } x\in\Omega(a,b)$$

and we first compute its value on the boundary of $\Omega(a,b)$. Let $x=(a\cos\varphi, b\sin\varphi)\in\partial\Omega(a,b)$ with $\varphi\in[0,2\pi]$. By a change of variable we have

$$\begin{aligned} G(x) &= \frac{1}{2\pi ab}\int_{\partial\Omega(a,b)} \frac{\big(\tau_1(a\cos\varphi-y_1)+\tau_2(b\sin\varphi-y_2)\big)^2}{(a\cos\varphi-y_1)^2+(b\sin\varphi-y_2)^2}\,\frac{1}{\sqrt{\frac{y_1^2}{a^4}+\frac{y_2^2}{b^4}}}\,d\mathcal{H}^1(y) \\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{a^2\tau_1^2(\cos\varphi-\cos\theta)^2}{a^2(\cos\varphi-\cos\theta)^2+b^2(\sin\varphi-\sin\theta)^2}\,d\theta \\ &\quad+\frac{1}{\pi}\int_0^{2\pi}\frac{ab\tau_1\tau_2(\cos\varphi-\cos\theta)(\sin\varphi-\sin\theta)}{a^2(\cos\varphi-\cos\theta)^2+b^2(\sin\varphi-\sin\theta)^2}\,d\theta \\ &\quad+\frac{1}{2\pi}\int_0^{2\pi}\frac{b^2\tau_2^2(\sin\varphi-\sin\theta)^2}{a^2(\cos\varphi-\cos\theta)^2+b^2(\sin\varphi-\sin\theta)^2}\,d\theta. \end{aligned}$$

Using again the identities (2.11)–(2.12) and a change of variable, we obtain

$$\begin{aligned} G(x) &= \frac{1}{2\pi}\int_0^{2\pi}\frac{a^2\tau_1^2\sin^2\left(\frac{\theta+\varphi}{2}\right)+b^2\tau_2^2\cos^2\left(\frac{\theta+\varphi}{2}\right)}{a^2\sin^2\left(\frac{\theta+\varphi}{2}\right)+b^2\cos^2\left(\frac{\theta+\varphi}{2}\right)}\,d\theta \\ &\quad-\frac{1}{\pi}\int_0^{2\pi}\frac{ab\tau_1\tau_2\sin\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta+\varphi}{2}\right)}{a^2\sin^2\left(\frac{\theta+\varphi}{2}\right)+b^2\cos^2\left(\frac{\theta+\varphi}{2}\right)}\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{a^2\tau_1^2+b^2\tau_2^2-(a^2\tau_1^2-b^2\tau_2^2)\cos\theta}{a^2+b^2-(a^2-b^2)\cos\theta}\,d\theta \\ &\quad-\frac{1}{2\pi}\int_0^{2\pi}\frac{ab\tau_1\tau_2\sin\theta}{a^2+b^2-(a^2-b^2)\cos\theta}\,d\theta. \end{aligned} \tag{2.14}$$

Since the last expression is independent of φ, we can already conclude that G is constant for $x\in\partial\Omega(a,b)$. We compute the value of G for completeness. If $a\neq b$, we have that

$$\int_0^{2\pi}\frac{ab\tau_1\tau_2\sin\theta}{a^2+b^2-(a^2-b^2)\cos\theta}\,d\theta=\frac{ab\tau_1\tau_2}{a^2-b^2}\big[\log\left(a^2+b^2-(a^2-b^2)\cos\theta\right)\big]_0^{2\pi}=0.$$

Therefore, by Lemma 2.3 we deduce that

$$G(x)=\frac{a}{a+b}\tau_1^2+\frac{b}{a+b}\tau_2^2 \qquad \text{for every } x\in\partial\Omega(a,b). \tag{2.15}$$

By a direct integration of (2.14) one can check that the same equality holds true if $a=b$.

On the other hand, by (2.13) and Lemma 2.2 the function G is harmonic in the interior of $\Omega(a,b)$. Since it is continuous on $\Omega(a,b)$, by the maximum principle we conclude that

$$G(x)=\frac{a}{a+b}\tau_1^2+\frac{b}{a+b}\tau_2^2 \qquad \text{for every } x\in\Omega(a,b). \tag{2.16}$$

This proves (1.6) and, in turn, by Theorem 2.1 the minimality of $\mu_{a,b}$ for every $\alpha\in[-1,1]$. □

Remark 2.4. The previous proof shows that the measures $\mu_{a,b}$ satisfies the Euler-Lagrange conditions (2.1)–(2.2) for every $\alpha\in\mathbb{R}$. However, one cannot conclude that $\mu_{a,b}$ is a minimizer of I_α for $\alpha\notin[-1,1]$, since the functional is not convex for this range of parameters. Moreover, one can show that $\mu_{a,b}$ is not a minimizer of I_α for $|\alpha|$ large enough. Indeed, let $\nu\in\mathcal{P}(\Omega(a,b))$ be any measure with $I_0(\nu)<+\infty$ and support contained in a straight line orthogonal to τ. Then, $I_\alpha(\nu)=I_0(\nu)$ for every $\alpha\in\mathbb{R}$. On the other hand, by (1.6)

$$I_\alpha(\mu_{a,b})=\int_{\Omega(a,b)}(V_\alpha*\mu_{a,b})\,d\mu_{a,b}=-\log\Big(\frac{a+b}{2}\Big)+\alpha\frac{a\tau_1^2+b\tau_2^2}{a+b}.$$

Therefore, for $\alpha\gg1$ we have $I_\alpha(\mu_{a,b})>I_\alpha(\nu)$.

A similar argument can be applied to the case $\alpha\ll-1$, using that

$$V_\alpha(x)=V_0(x)-\alpha\frac{(x\cdot\tau^\perp)^2}{|x|^2}+\alpha.$$

3. An extension to more general even anisotropies

In this section we discuss the extension of Theorem 1.1 to more general potentials. More precisely, for $k \in \mathbb{N}$ we define

$$W_k(x) := \frac{x_1^{2k}}{|x|^{2k}}$$

if $x \in \mathbb{R}^2$, $x \neq 0$, and $W_k(0) := 0$. Here we are assuming for simplicity that $\tau \in \mathbb{S}^1$ coincides with e_1, the unit vector along the x_1-axis. For $\alpha \in \mathbb{R}$ we consider the interaction kernel

$$V_{k,\alpha}(x) := -\log|x| + \alpha W_k(x)$$

if $x \in \mathbb{R}^2$, $x \neq 0$, and $V_{k,\alpha}(0) := +\infty$. For every $\mu \in \mathcal{P}(\Omega(a,b))$ we define the interaction energy

$$I_{k,\alpha}(\mu) := \iint_{\Omega(a,b)\times\Omega(a,b)} V_{k,\alpha}(x-y)\, d\mu(x)\, d\mu(y). \tag{3.1}$$

We are now in a position to prove the following theorem, which is the analog of Theorem 2.1 and Theorem 1.1.

Theorem 3.1. *For every $k \geq 2$ there exists an interval J_k, containing 0, such that for every $\alpha \in J_k$ the functional $I_{k,\alpha}$ is strictly convex on the class of measures in $\mathcal{P}(\Omega(a,b))$ with finite energy. The unique minimizer is the measure*

$$\mu_{a,b} = \frac{1}{2\pi ab} \frac{1}{\sqrt{\frac{x_1^2}{a^4} + \frac{x_2^2}{b^4}}} \mathcal{H}^1 \llcorner \partial\Omega(a,b)$$

and satisfies the Euler-Lagrange condition

$$(V_{k,\alpha} * \mu_{a,b})(x) = c_{k,\alpha}(a,b) \qquad \text{for every } x \in \Omega(a,b), \tag{3.2}$$

where $c_{k,\alpha}(a,b)$ is a suitable constant (possibly depending on α, k, a, b, but not on x).

Proof. The proof is subdivided into several steps.

Step 1: Computation of the Fourier transform of $V_{k,\alpha}$. A direct computation shows that for $k \geq 2$

$$\Delta W_k(x) = \frac{k}{k-1} \frac{\partial^2 W_{k-1}}{\partial x_2^2}(x) \qquad \text{for } x \neq 0. \tag{3.3}$$

Moreover, arguing as in the proof of (2.8), one can prove that the distributional Laplacian of W_k for $k \geq 2$ is given by the formula

$$\Delta W_k = \frac{k}{k-1} \frac{\partial^2 W_{k-1}}{\partial x_2^2} + 4k\pi a_{k-1}\delta_0, \tag{3.4}$$

where

$$a_m := \frac{1}{2\pi} \int_0^{2\pi} (\cos\theta)^{2m} (\sin\theta)^2\, d\theta = \frac{(2m-1)!!}{(2m+2)!!}, \tag{3.5}$$

for $m \geq 1$. In the equality above, which can be proved by induction, $n!!$ denotes the double factorial, that is, the product of all the integers from 1 up to n that have the same parity (even or odd) as n.

Passing to the Fourier transforms in (3.4), we obtain

$$\widehat{W}_k(\xi) = \frac{k}{k-1} \frac{\xi_2^2}{|\xi|^2} \widehat{W}_{k-1}(\xi) - \frac{k}{\pi} a_{k-1} \frac{1}{|\xi|^2}$$

for $k \geq 2$.

By induction one can show that

$$\widehat{W}_k(\xi) = k \frac{\xi_2^{2k-2}}{|\xi|^{2k-2}} \widehat{W}_1(\xi) - \frac{k}{\pi} \sum_{j=1}^{k-1} a_j \frac{\xi_2^{2k-2j-2}}{|\xi|^{2k-2j}} \tag{3.6}$$

for every $k \geq 2$. Combining (2.3) and (3.6), we deduce that

$$\langle \widehat{V}_{k,\alpha}, \psi \rangle = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{|\xi|^{2k} + \alpha P_k(\xi)}{|\xi|^{2k+2}} \psi(\xi)\, d\xi \tag{3.7}$$

for every $\psi \in \mathcal{S}$ with $\psi(0) = 0$, where P_k is the $2k$-homogeneous polynomial

$$P_k(\xi) := k\xi_2^{2k} - k\xi_1^2\xi_2^{2k-2} - 2k \sum_{j=1}^{k-1} a_j \xi_2^{2k-2j-2} |\xi|^{2j+2}$$

for every $\xi \in \mathbb{R}^2$.

Step 2: Strict convexity of $I_{k,\alpha}$. Arguing as in the proof of Theorem 2.1, the strict convexity of $I_{k,\alpha}$ on the class of measures with finite energy is equivalent to the condition

$$\int_{\mathbb{R}^2} V_{k,\alpha} * (\nu_1 - \nu_2)\, d(\nu_1 - \nu_2) > 0 \tag{3.8}$$

for every $\nu_1, \nu_2 \in \mathcal{P}(\Omega(a,b))$, $\nu_1 \neq \nu_2$, with finite energy. This condition, in turn, can be proved by checking the sign of the Fourier transform $\widehat{V}_{k,\alpha}$ on test functions vanishing at zero. More precisely, by continuity for every $k \geq 2$ there exists an interval $\tilde{J}_k$ containing zero such that for $\alpha \in \tilde{J}_k$

$$|\xi|^{2k} + \alpha P_k(\xi) > 0 \qquad \text{for every } \xi \neq 0. \tag{3.9}$$

By standard properties of the Fourier transform and by (3.7) we have that for $\alpha \in \tilde{J}_k$

$$\int_{\mathbb{R}^2} (V_{k,\alpha} * \varphi)\varphi\, dx = \langle \widehat{V}_{k,\alpha}, |\widehat{\varphi}|^2 \rangle = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{|\xi|^{2k} + \alpha P_k(\xi)}{|\xi|^{2k+2}} |\widehat{\varphi}(\xi)|^2\, d\xi \geq 0$$

for every $\varphi \in \mathcal{S}$ with $\int_{\mathbb{R}^2} \varphi(x)\, dx = 0$. Here we used that $\widehat{\varphi}(0) = \int_{\mathbb{R}^2} \varphi(x)\, dx = 0$. Let now $\nu_1, \nu_2 \in \mathcal{P}(\Omega(a,b))$, $\nu_1 \neq \nu_2$, with finite energy, let $\nu := \nu_1 - \nu_2$, and let $\alpha \in \tilde{J}_k$. Using the same approximation argument as in [10, Proof of Theorem 1.1], we deduce that

$$\int_{\mathbb{R}^2} (V_{k,\alpha} * \nu)\, d\nu \geq \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{|\xi|^{2k} + \alpha P_k(\xi)}{|\xi|^{2k+2}} |\widehat{\nu}(\xi)|^2\, d\xi \geq 0. \tag{3.10}$$

Moreover, if the left-handside of (3.10) is equal to zero, then

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{|\xi|^{2k} + \alpha P_k(\xi)}{|\xi|^{2k+2}} |\widehat{\nu}(\xi)|^2 = 0 \qquad \text{for a.e. } \xi \in \mathbb{R}^2.$$

By (3.9) this implies that $\widehat{\nu} = 0$ a.e. on $\mathbb{R}^2$. By continuity of $\hat{\nu}$ this implies $\hat{\nu}(\xi) = 0$ for every $\xi \in \mathbb{R}^2$. Thus, $\nu = 0$, hence $\nu_1 = \nu_2$. This proves (3.8) for every $\alpha \in \tilde{J}_k$ and $k \geq 2$.

For every $k \geq 2$ we denote by J_k the maximal interval such that $I_{k,\alpha}$ is strictly convex on the class of measures with finite energy for every $\alpha \in J_k$.

Step 3: Characterization of the unique minimizer. Let now $k \geq 2$ and $\alpha \in J_k$. We introduce the function

$$G_k(x) := (W_k * \mu_{a,b})(x) \qquad \text{for } x \in \Omega(a,b).$$

We want to show that G_k is constant on the boundary of $\Omega(a,b)$. Let $x = (a\cos\varphi, b\sin\varphi) \in \partial\Omega(a,b)$ with $\varphi \in [0, 2\pi]$. By a change of variable we have

$$\begin{aligned} G_k(x) &= \frac{1}{2\pi ab} \int_{\partial\Omega(a,b)} \frac{\big(a\cos\varphi - y_1\big)^{2k}}{\big((a\cos\varphi - y_1)^2 + (b\sin\varphi - y_2)^2\big)^k} \frac{1}{\sqrt{\frac{y_1^2}{a^4} + \frac{y_2^2}{b^4}}}\, d\mathcal{H}^1(y) \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{a^{2k}(\cos\varphi - \cos\theta)^{2k}}{\big(a^2(\cos\varphi - \cos\theta)^2 + b^2(\sin\varphi - \sin\theta)^2\big)^k}\, d\theta \end{aligned}$$

Using the identities (2.11)–(2.12) and arguing as in the proof of (2.14), we obtain

$$G_k(x) = \frac{1}{2\pi} \int_0^{2\pi} \frac{a^{2k}(1 - \cos\theta)^k}{\big(a^2 + b^2 - (a^2 - b^2)\cos\theta\big)^k}\, d\theta =: d_k(a,b), \tag{3.11}$$

that is, G_k is independent of x on the boundary of $\Omega(a,b)$.

We now prove that for every $k \geq 2$

$$G_k(x) = d_k(a,b) \qquad \text{for every } x \in \Omega(a,b). \tag{3.12}$$

We argue by induction on k. Let $k=2$. By (3.4) and (2.16)

$$\Delta G_2 = 2\frac{\partial^2 G}{\partial x_2^2}(x) + 8\pi a_1 \mu_{a,b} = 0$$

in the interior of $\Omega(a,b)$. Since G_2 is continuous on $\Omega(a,b)$ and constant on $\partial\Omega(a,b)$, by the maximum principle we obtain (3.12) for $k=2$. Assume now by induction that (3.12) holds for $k-1$. By (3.11) we have that G_k is constant on $\partial\Omega(a,b)$ and by (3.4) and the inductive hypothesis that G_k is harmonic in the interior of $\Omega(a,b)$. Since G_k is continuous on $\Omega(a,b)$, by the maximum principle we conclude that (3.12) holds.

For completeness we note that the value of the constant $d_k(a,b)$ can be computed explicitly using the formula

$$G_k(x) = \frac{a^{2k-1}}{2(1-k)}\frac{d}{da}\Big(\frac{1}{a^{2(k-1)}}G_{k-1}(x)\Big)$$

for every $x \in \partial\Omega(a,b)$ and every $k \geq 2$, where we set $G_1 := G$.

Equation (3.12), together with (2.13), proves (3.2) and, in turn, the minimality of $\mu_{a,b}$ for every $k \geq 2$ and every $\alpha \in J_k$. □

Acknowledgements. The first author acknowledges support by the Università degli Studi di Pavia through the 2017 Blue Sky Research Project "Plasticity at different scales: micro to macro" and by GNAMPA–INdAM. The second author acknowledges support by GNAMPA–INdAM through the 2016 "Borsa di Avviamento alla Ricerca".

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