THE ISOPERIMETRIC PROBLEM FOR REGULAR AND CRYSTALLINE NORMS IN $\mathbb{H}^1$

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Abstract. We study the isoperimetric problem for anisotropic left-invariant perimeter measures on $\mathbb{R}^3$, endowed with the Heisenberg group structure. The perimeter is associated with a left-invariant norm ϕ on the horizontal distribution. We first prove a representation formula for the ϕ-perimeter of regular sets and, assuming some regularity on ϕ and on its dual norm ϕ^*, we deduce a foliation property by sub-Finsler geodesics of C^2-smooth surfaces with constant ϕ-curvature. We then prove that the characteristic set of C^2-smooth surfaces that are locally extremal for the isoperimetric problem is made of isolated points and horizontal curves satisfying a suitable differential equation. Based on such a characterization, we characterize C^2-smooth ϕ-isoperimetric sets as the sub-Finsler analogue of Pansu's bubbles. We also show, under suitable regularity properties on ϕ, that such sub-Finsler candidate isoperimetric sets are indeed C^2-smooth. By an approximation procedure, we finally prove a conditional minimality property for the candidate solutions in the general case (including the case where ϕ is crystalline).

1. Introduction

Let $\phi : \mathbb{R}^n \to [0, \infty)$ be a norm in $\mathbb{R}^n$, $n \geq 2$. The associated *Finsler* or *anisotropic perimeter* of a Lebesgue measurable set $E \subset \mathbb{R}^n$ is defined as

$$P_\phi(E) = \sup\left\{ \int_E \mathrm{div}(V)\, dp : V \in \mathrm{C}^\infty_c(\mathbb{R}^n;\mathbb{R}^n) \text{ with } \max_{p\in\mathbb{R}^n} \phi(V(p)) \leq 1 \right\}.$$

If E is regular, $P_\phi(E)$ can be represented as a surface integral as follows

$$P_\phi(E) = \int_{\partial E} \phi^*(\nu_E)\, d\mathscr{H}^{n-1},$$

where ν_E is the inner unit normal to ∂E and $\phi^* : \mathbb{R}^n \to [0, \infty)$ is the dual norm defined by

$$\phi^*(w) = \max_{\phi(v)=1} \langle w, v\rangle, \quad w \in \mathbb{R}^n.$$

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Here, $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean scalar product in $\mathbb{R}^n$ and $|\cdot|$ the Euclidean norm. In the theory of crystals, ϕ^* is the surface tension of the interface between an anisotropic material E and a fluid, and $P_\phi(E)$ is the total free energy.

In the case where $\phi = |\cdot|$, P_ϕ is the standard De Giorgi perimeter and isoperimetric sets (i.e., sets of fixed volume that minimize perimeter) are Euclidean balls. For a general norm ϕ, isoperimetric sets are translations and dilations of the *Wulff shape*, first considered by G. Wulff in [31]. In our notation it corresponds to the unit ball of the ϕ-norm. The first complete proof of the isoperimetric property of Wulff shapes in the class of Lebesgue measurable sets with given volume is contained in [11, 12], and based on the Brunn-Minkowski inequality. We refer to [10] for a quantitative version.

In this paper, we study the isoperimetric problem for sub-Finsler perimeter measures in the Heisenberg group $\mathbb{H}^1$. The latter is $\mathbb{R}^3$ endowed with the non-commutative group law

$$(\xi, z) * (\xi', z') = (\xi + \xi', z + z' + \omega(\xi, \xi')), \quad \xi, \xi' \in \mathbb{R}^2, \quad z, z' \in \mathbb{R}, \tag{1.1}$$

where $\omega : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is the symplectic form

$$\omega(\xi, \xi') = \frac{1}{2}(xy' - x'y), \quad \xi = (x, y),\ \xi' = (x', y') \in \mathbb{R}^2. \tag{1.2}$$

The vector fields

$$X = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z} \qquad \text{and} \qquad Y = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}$$

are left-invariant for the group action and span a two-dimensional distribution $\mathscr{D}(\mathbb{H}^1)$ in $T\mathbb{H}^1$, called the *horizontal distribution*. We denote by $\mathscr{D}(p)$ the fiber of $\mathscr{D}$ at $p \in \mathbb{H}^1$.

Given a norm $\phi : \mathbb{R}^2 \to [0, \infty)$, the associated anisotropic perimeter measure in $\mathbb{H}^1$ is introduced in Definition 2.1 and takes into account only horizontal directions. For a regular set $E \subset \mathbb{R}^3$ it can be represented as

$$\mathscr{P}_\phi(E) = \int_{\partial E} \phi^*(N_E)\, d\mathscr{H}^2,$$

where N_E is obtained by projecting the inner unit normal ν_E onto the horizontal distribution. A set $E \subset \mathbb{H}^1$ is said to be *ϕ-isoperimetric* if there exists $m > 0$ such that E minimizes

$$\inf\left\{\mathscr{P}_\phi(E) : E \subset \mathbb{H}^1 \text{ measurable}, \mathscr{L}^3(E) = m\right\}. \tag{1.3}$$

If $\phi = |\cdot|$ is the Euclidean norm in $\mathbb{R}^2$, then $\mathscr{P}_\phi$ corresponds to the standard horizontal perimeter in $\mathbb{H}^1$, introduced and studied in [7, 17, 16]. In this case, the problem of characterizing ϕ-isoperimetric sets in the class of Lebesgue measurable sets in $\mathbb{H}^1$ is open. According to Pansu's conjecture [24], $|\cdot|$-isoperimetric sets are obtained through *left-translations* and *anisotropic dilations* $\delta_\lambda : \mathbb{H}^1 \to \mathbb{H}^1$, $\lambda > 0$,

$$\delta_\lambda(\xi, z) = (\lambda\xi, \lambda^2 z), \tag{1.4}$$

of the so-called *Pansu's bubble*.

An absolutely continuous curve $\gamma : I \to \mathbb{H}^1$ is said to be *horizontal* if $\dot\gamma(t) \in \mathscr{D}(\gamma(t))$ for a.e. $t \in I$ and we call *horizontal lift* of an absolutely continuous curve $\xi : I \to \mathbb{R}^2$ any horizontal curve $\gamma = (\xi, z)$ with

$$\dot z = \omega(\xi, \dot\xi).$$

Pansu's bubble is the bounded set whose boundary is foliated by horizontal lifts of planar circles of a given radius, passing through the origin. Such horizontal curves are length minimizing between their endpoints for the sub-Riemannian distance in $\mathbb{H}^1$, so that Pansu's conjecture in $\mathbb{H}^1$ explicits a relation between isoperimetric sets and the geometry of the ambient space. The conjecture is supported by several results contained in [27, 22, 23, 26, 15, 14], but it is still unsolved in its full generality. A quantitative version of the Heisenberg isoperimetric inequality has been proposed in [13].

Very little is known on the isoperimetric problem when $\phi : \mathbb{R}^2 \to [0, \infty)$ is a general norm in $\mathbb{R}^2$. While preparing the final version of this article, we became aware that J. Pozuelo and M. Ritoré have recently obtained several results on the problem, considering also the case where ϕ is convex and homogeneous, but not necessarily a norm (see [25]).

Existence of ϕ-isoperimetric sets follows by the arguments of [19], see Section 3. The construction of the Pansu's bubble can be generalized to the sub-Finsler context in the following way. We call *ϕ-circle* of radius $r > 0$ and center $\xi_0 \in \mathbb{R}^2$ the set

$$C_\phi(\xi_0, r) = \{\xi \in \mathbb{R}^2 : \phi(\xi - \xi_0) = r\}, \tag{1.5}$$

and we call *ϕ-bubble* the bounded set E_ϕ whose boundary is foliated by horizontal lifts of ϕ-circles in the plane of a given radius, passing through the origin.

In Figure 1 we represent two ϕ-bubbles, corresponding to $\phi = \ell^p$, with $\ell^p(x, y) = (|x|^p + |y|^p)^{\frac{1}{p}}$, in the cases $p = 3$ and $p = 100$. The latter can be seen as an approximation of the crystalline case.

Our main result is the characterization of C^2-smooth ϕ-isoperimetric sets when ϕ and ϕ^* are C^2-smooth. This result suggests that the ϕ-bubble is the solution to the isoperimetric problem for $\mathscr{P}_\phi$. Here and in the following, if $\phi \in \mathrm{C}^k(\mathbb{R}^2 \setminus \{0\})$ we say that ϕ is of class C^k, for $k \in \mathbb{N}$.

Theorem 1.1. *Let ϕ be a norm of class C^2 such that ϕ^* is of class C^2. If $E \subset \mathbb{H}^1$ is a ϕ-isoperimetric set of class C^2 then we have $E = E_\phi$, up to left-translations and anisotropic dilations.*

The proof of Theorem 1.1 is presented in Section 8 and is based on a fine study of the *characteristic set* of isoperimetric sets. The characteristic set of a set $E \subset \mathbb{H}^1$ of

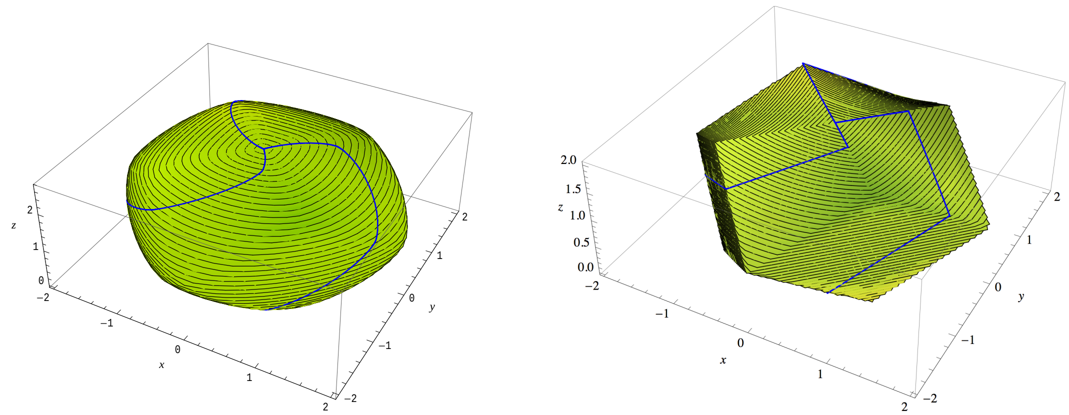

Figure 1. The ℓ^p-bubbles with $p = 3$ (left) and $p = 100$ (right). In blue we outlined three horizontal lifts of ℓ^p-circles foliating the ℓ^p-bubble.

class C^1 (equivalently, of its boundary ∂E) is

$$\mathscr{C}(E)= \mathscr{C}(\partial E) = \{p \in \partial E : T_p\partial E = \mathscr{D}(p)\}. \tag{1.6}$$

In Section 7 we characterize the structure of $\mathscr{C}(E)$ for a C^2-smooth ϕ-isoperimetric set $E \subset \mathbb{H}^1$, proving that $\mathscr{C}(E)$ is made of isolated points. For the more general case of ϕ-critical surfaces we obtain the following result, that we prove by adapting to the sub-Finsler case the theory of Jacobi fields of [27]. Any ϕ-critical surface has constant ϕ-curvature and the definition is presented in Section 7.

Theorem 1.2. *Let ϕ and ϕ^* be of class C^2 and let $\Sigma \subset \mathbb{H}^1$ be a complete and oriented surface of class C^2. If Σ is ϕ-critical with non-vanishing ϕ-curvature then $\mathscr{C}(\Sigma)$ consists of isolated points and C^2 curves that are either horizontal lines or horizontal lifts of simple closed curves.*

The simple closed curves of Theorem 1.2 are described by a suitable ordinary differential equation. We expect that these curves are $\phi^\dagger$-circles, where $\phi^\dagger$ is the norm defined as

$$\phi^\dagger(\xi) = \phi^*(\xi^\perp), \quad \xi \in \mathbb{R}^2.$$

Here and hereafter, $\perp\colon \mathbb{R}^2 \to \mathbb{R}^2$ denotes the perp-operator $\perp(\xi) = \xi^\perp$, with

$$\xi^\perp = (x, y)^\perp = (-y, x), \quad \xi = (x, y) \in \mathbb{R}^2.$$

Theorem 1.1 then follows by combining the results of Sections 4.2, 5, and 7. In particular, starting from a first variation analysis, we establish a foliation property outside the characteristic set for C^2-smooth ϕ-isoperimetric sets (and more generally for constant ϕ-curvature surfaces). Theorem 1.2 is a key step for concluding the proof.

We also identify an explicit relation between ϕ-isoperimetric sets and geodesics in the ambient space. In Section 6, we show that, outside the characteristic set, ϕ-isoperimetric sets are foliated by *sub-Finsler geodesics* in $\mathbb{H}^1$ relative to the norm $\phi^\dagger$. We refer to Corollary 6.4 for a statement of the result. Notice that when $\phi = |\cdot|$ is the Euclidean norm, $\phi^\dagger$ reduces to $|\cdot|$, and we recover the foliation by sub-Riemannian geodesics of C^2-smooth $|\cdot|$-isoperimetric sets.

The *regularity* of the candidate isoperimetric sets E_ϕ is a major issue that we treat in Section 8. While it is rather easy to check that ϕ-Pansu bubbles have the same regularity as ϕ outside the characteristic set (at least if ϕ-circles are strictly convex, see Lemma 8.1), it is not clear what regularity is inherited from ϕ at characteristic points. In Section 8.3 we prove the following.

Theorem 1.3. *Assume that ϕ is of class C^4 and that ϕ-circles have strictly positive curvature. Then ∂E_ϕ is an embedded surface of class C^2.*

In the case where ϕ or ϕ^* are not differentiable, Theorems 1.1 and 1.3 cannot be applied in a direct way. In Corollary 5.5 we show that the foliation property by horizontal lifts of ϕ-circles outside the characteristic set can be recovered when ϕ^* is only *piecewise* C^2, thus allowing to cover the case $\phi = \ell^p$ for $p > 2$. For general non-differentiable norms, our next result is conditioned to the validity of the following conjecture.

Conjecture 1.4. For any norm ϕ of class C^∞_+, ϕ-isoperimetric sets are of class C^2.

Here, a norm ϕ in $\mathbb{R}^2$ is said to be of class C^∞_+ if $\phi \in \mathrm{C}^\infty(\mathbb{R}^2 \setminus \{0\})$ and ϕ-circles have strictly positive curvature. The proof of the following result is presented in Section 9.

Theorem 1.5. *Assume that Conjecture 1.4 holds true. Then for any norm ϕ in $\mathbb{R}^2$ the ϕ-bubble $E_\phi \subset \mathbb{H}^1$ is ϕ-isoperimetric.*

Of a particular interest is the case of a *crystalline norm.* A norm $\phi : \mathbb{R}^2 \to [0, \infty)$ is called crystalline if the ϕ-circle $C_\phi = C_\phi(0, 1)$ is a convex polygon centrally symmetric with respect to the origin. Let $v_1, \dots, v_{2N} \in C_\phi$ be the ordered vertices of this polygon, and denote by $e_i = v_i - v_{i-1}$, $i = 1, \dots, 2N$, the edges of C_ϕ, where $v_0 = v_{2N}$. We consider the left-invariant vector fields

$$X_i := e_{i,1}X + e_{i,2}Y, \quad i = 1, \dots, 2N, \tag{1.7}$$

where $e_i = (e_{i,1}, e_{i,2})$, and we notice that $X_{i+N} = -X_i$ for $i = 1, \dots, N$. By a first variation argument, we deduce a foliation property for ϕ-isoperimetric sets by integral curves of the X_i, see Section 4.3.

Theorem 1.6. *Let $E \subset \mathbb{H}^1$ be ϕ-isoperimetric for a crystalline norm ϕ. Let $A \subset \mathbb{H}^1$ be an open set such that $\partial E \cap A$ is a connected z-graph of class C^2. Then there exists $i = 1, \dots, N$ such that $\partial E \cap A$ is foliated by integral curves of X_i.*

Geodesics of sub-Finsler structures on the Heisenberg group and other Carnot groups have been studied in several papers (see, in particular, [2, 5, 6, 20, 29]). Unfortunately, Theorem 1.6 does not provide enough information in order to establish the global foliation property by $\phi^\dagger$-geodesics in the crystalline case.

1.1. **Structure of the paper.** In Section 2 we introduce the notion of sub-Finsler perimeter and we deduce a representation formula for Lipschitz sets (see Proposition 2.2), holding for any norm ϕ in $\mathbb{R}^2$. In Section 3 we prove existence of ϕ-isoperimetric sets for a general norm ϕ, following the arguments in [19]. In Section 4 we derive first-variation necessary conditions for ϕ-isoperimetric sets, both when ϕ^* is of class C^1 (see Section 4.2) and when ϕ^* is not differentiable (see Section 4.3). In the former case, we introduce the notion of ϕ-curvature of a C^2-smooth surfaces (when ϕ^* is C^2) and of ϕ-critical surface. In the latter case, we deduce the (partial) foliation property stated in Theorem 1.6 for crystalline norms. In Section 5 we deduce a foliation property outside the characteristic set for ϕ-isoperimetric sets of class C^2, assuming ϕ and ϕ^* to be regular enough. We then study such a foliation from the point of view of geodesics in the ambient space in Section 6, and in Section 7 we study the characteristic set of C^2-smooth ϕ-critical surfaces and of ϕ-isoperimetric sets, assuming ϕ and ϕ^* to be C^2 (Theorem 1.2). In Section 8 we then prove Theorem 1.1 and we study regularity of ϕ-bubbles, summarized in Theorem 1.3. Finally, Section 9 is dedicated to general norms and contains the proof of Theorem 1.5.

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2. Sub-Finsler perimeter

In this section, we introduce the notion of ϕ-perimeter in $\mathbb{H}^1$ for a norm ϕ in $\mathbb{R}^2$. We start by fixing the notation relative to horizontal vector fields and sub-Finsler norms in $\mathbb{H}^1$.

A smooth horizontal vector field is a vector field V on $\mathbb{R}^3$ that can be written as $V = aX + bY$ where $a, b \in \mathrm{C}^\infty(\mathbb{H}^1)$. When $A \subset \mathbb{H}^1$ is an open set and $a, b \in \mathrm{C}^\infty_c(A)$ have compact support in A we shall write $V \in \mathscr{D}_c(A)$. We fix on $\mathscr{D}(\mathbb{H}^1)$ the scalar product $\langle\cdot,\cdot\rangle_{\mathscr{D}}$ that makes X, Y pointwise orthonormal. Then each fiber $\mathscr{D}(p)$ can be identified with the Euclidean plane $\mathbb{R}^2$.

Let $\phi : \mathbb{R}^2 \to [0,\infty)$ be a norm. We fix on $\mathscr{D}(\mathbb{H}^1)$ the left-invariant norm associated with ϕ. Namely, with a slight abuse of notation, for any $p \in \mathbb{H}^1$ and with $a, b \in \mathbb{R}$ we define

$$\phi(aX(p) + bY(p)) = \phi\big((a, b)\big).$$

Since the Haar measure of $\mathbb{H}^1$ is the Lebesgue measure of $\mathbb{R}^3$, the divergence in $\mathbb{H}^1$ is the standard divergence. Therefore, for a smooth horizontal vector field $V = aX + bY$ we have $\operatorname{div}(V) = Xa + Yb$.

Definition 2.1. The *ϕ-perimeter* of a Lebesgue measurable set $E \subset \mathbb{H}^1$ in an open set $A \subset \mathbb{H}^1$ is

$$\mathscr{P}_\phi(E; A) = \sup\left\{\int_E \operatorname{div}(V)dp : V \in \mathscr{D}_c(A) \text{ with } \max_{\xi\in A}\phi(V(\xi)) \le 1\right\}.$$

When $\mathscr{P}_\phi(E; A) < \infty$ we say that E has finite perimeter in A. When $A = \mathbb{H}^1$, we let $\mathscr{P}_\phi(E) = \mathscr{P}_\phi(E; \mathbb{H}^1)$.

Since all the left-invariant norms in the horizontal distribution are equivalent, we have $\mathscr{P}_\phi(E) < \infty$ if and only if the set E has finite horizontal perimeter in the sense of [7, 16, 17].

For regular sets, we can represent $\mathscr{P}_\phi(E)$ integrating on ∂E a kernel related to the normal. Let ν_E be the Euclidean unit inner normal to ∂E. We define the horizontal vector field $N_E : \partial E \to \mathscr{D}(\mathbb{H}^1)$ by

$$N_E = \langle \nu_E, X\rangle X + \langle \nu_E, Y\rangle Y,$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclidean scalar product in $\mathbb{R}^3$.

Proposition 2.2 (Representation formula)**.** *Let $E \subset \mathbb{H}^1$ be a set with Lipschitz boundary. Then for every open set $A \subset \mathbb{H}^1$ we have*

$$\mathscr{P}_\phi(E; A) = \int_{\partial E\cap A} \phi^*(N_E)\, d\mathscr{H}^2, \tag{2.1}$$

where $\mathscr{H}^2$ is the standard 2-Hausdorff measure in $\mathbb{R}^3$.

Proof. Let $V \in \mathscr{D}_c(A)$ be such that $\phi(V) \le 1$. By the standard divergence theorem and by the definition of dual norm, we have

$$\begin{aligned}\int_E \operatorname{div}(V)\, d\xi &= -\int_{\partial E} \langle V, N_E\rangle_{\mathscr{D}}\, d\mathscr{H}^2 \le \int_{\partial E\cap A} \phi(V)\phi^*(N_E) d\mathscr{H}^2 \\ &\le \int_{\partial E\cap A} \phi^*(N_E) d\mathscr{H}^2.\end{aligned}$$

By taking the supremum over all admissible V we then obtain

$$\mathscr{P}_\phi(E;A) \le \int_{\partial E\cap A} \phi^*(N_E)d\mathscr{H}^2.$$

To get the opposite inequality it is sufficient to prove that for every $\varepsilon > 0$ there exists $V \in \mathscr{D}_c(A)$ such that $\phi(V) \le 1$ and

$$-\int_{\partial E} \langle V, N_E\rangle_{\mathscr{D}}\, d\mathscr{H}^2 \ge \int_{\partial E\cap A} \phi^*(N_E)d\mathscr{H}^2 - \varepsilon.$$

Here, without loss of generality, we assume that A is bounded. We will construct such a V with continuous coefficients and with compact support in A. The smooth case $V \in \mathscr{D}_c(A)$ will follow by a standard regularization argument.

Let us define the sets

$$\mathscr{U} = \big\{p \in \partial E \cap A : N_E(p) \text{ is defined}\big\}, \quad \mathscr{Z} = \big\{p \in \mathscr{U} : N_E(p) = 0\big\}.$$

From the results of [4] it follows that $\mathscr{Z}$ has vanishing $\mathscr{H}^2$-measure. For any $p \in \mathscr{U} \setminus \mathscr{Z}$ we take $V \in \mathscr{D}(p)$ such that $\phi(V) = 1$ and

$$\langle V, N_E\rangle_{\mathscr{D}} = \phi^*(N_E).$$

In general, this choice is not unique. However, there is a selection $p \mapsto V(p)$ that is measurable (this follows since the coordinates are measurable, see for instance [3, Theorem 8.1.3]). We extend V to $\mathscr{Z}$ letting $V = 0$ here. This extension is still measurable.

Since $\partial E \cap A$ has finite $\mathscr{H}^2$-measure, by Lusin's theorem there exists a compact set $K_\varepsilon \subset \partial E \cap A$ such that $\mathscr{H}^2\big((\partial E \cap A) \setminus K_\varepsilon\big) < \varepsilon$ and the restriction of V to K_ε is continuous. Now, by Tietze–Uryshon theorem we extend V from K_ε to A in such a way that the extended map, still denoted by V, is continuous with compact support in A and satisfies $\phi(V) \le 1$ everywhere.

Our construction yields the following

$$\begin{aligned}\int_{\partial E\cap A} \phi^*(N_E)d\mathscr{H}^2 &= \int_{K_\varepsilon} \langle V, N_E\rangle_{\mathscr{D}}\, d\mathscr{H}^2 + \int_{(\partial E\cap A)\setminus K_\varepsilon} \phi^*(N_E)d\mathscr{H}^2 \\ &= \int_{\partial E\cap A} \langle V, N_E\rangle_{\mathscr{D}}\, d\mathscr{H}^2 - \int_{(\partial E\cap A)\setminus K_\varepsilon} \left(\langle V, N_E\rangle_{\mathscr{D}} - \phi^*(N_E)\right) d\mathscr{H}^2 \\ &\le \int_{\partial E\cap A} \langle V, N_E\rangle_{\mathscr{D}}\, d\mathscr{H}^2 + C\varepsilon.\end{aligned}$$

In the last inequality we used the fact that $\langle V, N_E\rangle_{\mathscr{D}} - \phi^*(N_E)$ is bounded and $\mathscr{H}^2\big((\partial E \cap A) \setminus K_\varepsilon\big) < \varepsilon$. The claim follows. $\square$

3. Existence of isoperimetric sets

For a measurable set $E \subset \mathbb{H}^1$ with positive and finite measure and a given norm ϕ on $\mathbb{R}^2$ we define the *ϕ-isoperimetric quotient* as

$$\operatorname{Isop}_\phi(E) = \frac{\mathscr{P}_\phi(E)}{\mathscr{L}^3(E)^{\frac{3}{4}}},$$

where $\mathscr{L}^3$ denotes the Lebesgue measure of $\mathbb{R}^3$.

The isoperimetric quotient is invariant under left-translation (w.r.t. the operation in (1.1)), i.e., $\operatorname{Isop}_\phi(p * E) = \operatorname{Isop}_\phi(E)$ for any $p \in \mathbb{H}^1$ and $E \subset \mathbb{H}^1$ admissible, and it is 0-homogeneous with respect to the one-parameter family of automorphisms (1.4), i.e., $\operatorname{Isop}_\phi(\lambda E) = \operatorname{Isop}_\phi(E)$, where $\lambda E = \delta_\lambda(E)$.

The isoperimetric problem (1.3) is then equivalent to minimizing the isoperimetric quotient among all admissible sets. Namely, given $m \in (0, \infty)$, any isoperimetric set $E \subset \mathbb{H}^1$ with $\mathscr{L}^3(E) = m$ is a solution to

$$C_I = \inf \left\{ \operatorname{Isop}_\phi(E) : E \subset \mathbb{H}^1 \text{ measurable, } 0 < \mathscr{L}^3(E) < \infty \right\}, \tag{3.1}$$

and, *vice versa*, any solution $E \subset \mathbb{H}^1$ to (3.1) solves (1.3) within its volume class, i.e., with $m = \mathscr{L}^3(E)$. In particular, we have

$$C_I = \inf \left\{ \mathscr{P}_\phi(E) : E \subset \mathbb{H}^1 \text{ measurable, } \mathscr{L}^3(E) = 1 \right\}. \tag{3.2}$$

The constant C_I depends on ϕ.

Since $\mathscr{P}_\phi$ is equivalent to the standard horizontal perimeter, the isoperimetric inequality in [17] implies that $C_I > 0$ and the validity of the following inequality for any measurable set E with finite measure:

$$\mathscr{P}_\phi(E) \geq C_I \mathscr{L}^3(E)^{\frac{3}{4}}. \tag{3.3}$$

The constant C_I is the largest one making true the above inequality and isoperimetric sets are precisely those for which (3.3) is an equality.

Theorem 3.1 (Existence of isoperimetric sets)**.** *Let ϕ be any norm on $\mathbb{R}^2$. There exists a set $E \subset \mathbb{H}^1$ with non-zero and finite ϕ-perimeter such that*

$$\mathscr{P}_\phi(E) = C_I \mathscr{L}^3(E)^{\frac{3}{4}}. \tag{3.4}$$

Theorem 3.1 follows by applying the strategy of [19, Section 4]. In the sequel we denote the left-invariant homogeneous ball centered at $p \in \mathbb{H}^1$ of radius $r > 0$ by $B(p, r)$.

Proof of Theorem 3.1. We give a sketch of the proof. By perimeter and volume homogeneity with respect to $\{\delta_\lambda\}_{\lambda \in \mathbb{R}}$ it is enough to prove the existence of a minimizing set in the class of volume $\mathscr{L}^3(E) = 1$. Let $\{E_k\}_{k\in\mathbb{N}}$ be a minimizing sequence for (3.2) such that for $k \in \mathbb{N}$ we have

$$\mathscr{L}^3(E_k) = 1, \qquad \mathscr{P}_\phi(E_k) \leq C_I\left(1 + \frac{1}{k}\right).$$

Assume that there exists $m_0 \in (0, 1/2)$ such that for any $k \in \mathbb{N}$ there exists $p_k \in \mathbb{H}^1$ satisfying

$$\mathscr{L}^3(E_k \cap B(p_k, 1)) \geq m_0. \tag{3.5}$$

Then, the translated sequence $\{-p_k * E_k\}_{k\in\mathbb{N}}$, still denoted $\{E_k\}_{k\in\mathbb{N}}$, is also minimizing for (3.2) and satisfies $\mathscr{L}^3(E_k \cap B(0,1)) \geq m_0$.

Since $\mathscr{P}_\phi$ is equivalent to the standar horizontal perimeter, we have a compactness theorem for sets of finite ϕ-perimeter as in [17, Theorem 1.28]. Then, we can extract a sub-sequence, still denoted $\{E_k\}_{k\in\mathbb{N}}$, converging in the $L^1_{\mathrm{loc}}(\mathbb{H}^1)$ sense to a set $E \subset \mathbb{H}^1$ of finite ϕ-perimeter. The lower semi-continuity of $\mathscr{P}_\phi$ therefore implies

$$\mathscr{P}_\phi(E) \leq \liminf_{k\to\infty} \mathscr{P}_\phi(E_k) \leq C_I.$$

Moreover, we have

$$\begin{aligned} &\mathscr{L}^3(E) \leq \liminf_{k\to\infty} \mathscr{L}^3(E_k) = 1 \quad \text{ and} \\ &\mathscr{L}^3(E \cap B(0,1)) = \lim_{k\to\infty} \mathscr{L}^3(E_k \cap B(0,1)) \geq m_0. \end{aligned} \tag{3.6}$$

To prove (3.4) we are left to show that $\mathscr{L}^3(E) = 1$, which follows by applying a sub-Finsler version of [19, Lemma 4.2], ensuring existence of a radius $R > 0$ such that $\mathscr{L}^3(E \cap B(0,R)) = 1$. This is based on (3.6) and on a canonical relation between perimeter and derivative of volume in balls with respect to the radius, which is valid in quite general metric structures, including sub-Finsler ones, see [1, Lemma 3.5].

We conclude by justifying the assumption (3.5). This follows by a sub-Finsler version of [19, Lemma 4.1]. Using once more the equivalence of $\mathscr{P}_\phi$ with the standard horizontal perimeter, we deduce from [17, Theorem 1.18] the validity of the following *relative isoperimetric inequality* holding for a constant $C > 0$ and any measurable set E with finite measure

$$\min\left\{\mathscr{L}^3(B \cap E)^{\frac{3}{4}}, \mathscr{L}^3(B \setminus E)^{\frac{3}{4}}\right\} \leq C\mathscr{P}_\phi(E, \lambda B),$$

where $\lambda \geq 1$ is a constant depending only on ϕ, and B is any left-invariant homogeneous ball. Together with the fact that the family $\{B(p,\lambda) : p \in \mathbb{H}^1\}$ has bounded overlap, we can reproduce the argument of [19, Lemma 4.1] and prove the claim. □

Remark 3.2. Following the arguments of [19, Lemma 4.2], one also shows that any isoperimetric set is equivalent to a bounded and connected one (i.e., it is bounded and connected up to sets of zero Lebesgue measure).

4. First variation of the isoperimetric quotient

In this section we derive a first order necessary condition for ϕ-isoperimetric sets, both when ϕ is regular or not.

4.1. **Notation.** We now introduce some notation that will be used throughout the paper.

Let $E, A \subset \mathbb{H}^1$ be sets such that E is closed, A is open and there exists a function $g \in \mathrm{C}^1(A)$, called *defining function for* $\partial E \cap A$, such that $\partial E \cap A = \{p \in A : g(p) = 0\}$ and $\nabla g(p) \neq 0$ for every $p \in \partial E \cap A$. We say that $E \cap A$ is a *z-subgraph* if there exist an open set $D \subset \mathbb{R}^2$ and a function $f \in \mathrm{C}^1(D)$, called *graph function* for $\partial E \cap A$, such that

$$E \cap A = \{(\xi, z) \in A : \xi \in D \text{ and } z \leq f(\xi)\}.$$

In this case, $g(\xi, z) = f(\xi) - z$ is a defining function for $\partial E \cap A$.

The definition of *z-epigraph* is analogous and all results given below for z-subgraphs have a straightforward counterpart for z-epigraphs. In a similar way, one can also define *x-subgraphs*, *y-subgraphs*, *x-epigraphs*, and *y-epigraphs*.

Given a function $g \in \mathrm{C}^1(A)$, we denote by $\mathscr{G} = (Xg)X + (Yg)Y$ the *horizontal gradient* of g and we define the *projected horizontal gradient* as

$$G = (Xg, Yg) \in \mathbb{R}^2. \tag{4.1}$$

If $\partial E \cap A$ is a z-graph with graph function $f \in \mathrm{C}^1(D)$, we define $F : D \to \mathbb{R}^2$ by

$$F(\xi) = G(\xi, f(\xi)) = \nabla f(\xi) - \frac{1}{2}\xi^{\perp}, \tag{4.2}$$

and

$$\mathscr{C}(f) = \{\xi \in D : F(\xi) = 0\}. \tag{4.3}$$

Hence $\mathscr{C}(E) \cap A = \{(\xi, f(\xi)) : \xi \in \mathscr{C}(f)\}$, where $\mathscr{C}(E)$ is the characteristic set of E, defined in (1.6). The set $\mathscr{C}(f)$ has zero Lebesgue measure in D.

If $E \cap A$ is the z-subgraph of a function $f \in \mathrm{C}^1(D)$, by the representation formula (2.1) we have

$$\mathscr{P}_\phi(E; A) = \int_D \phi^*(F(\xi))d\xi.$$

When the dual norm ϕ^* is of class C^1, starting from a graph function $f \in \mathrm{C}^1(D)$ we define the vector field $\mathscr{X}f : D \to \mathbb{R}^2$ by

$$\mathscr{X}f(\xi) = \nabla\phi^*(F(\xi)), \quad \xi \in D.$$

The geometric meaning of the vector field $\mathscr{X}f$ will be clarified in the next section, see (5.2).

Remark 4.1. At any point $\xi \in D$ such that $F(\xi) \neq 0$ the vector field $\mathscr{X}f$ satisfies

$$\phi(\mathscr{X}f(\xi)) = 1, \tag{4.4}$$

since the gradient of ϕ^* at any nonzero point has norm ϕ equal to one (even when ϕ^* is not regular, by replacing the gradient by any element of the subgradient; see, for instance, [18, Example 3.6.5]).

4.2. **Regular norms.**

Proposition 4.2 (First variation for isoperimetric sets)**.** *Let ϕ be a norm such that ϕ^* is of class C^1. Let $E \subset \mathbb{H}^1$ be a ϕ-isoperimetric set such that, for some open set $A \subset \mathbb{H}^1$, $E \cap A$ is a z-subgraph of class C^1. Then the graph function $f \in \mathrm{C}^1(D)$ satisfies in the weak sense the partial differential equation*

$$\operatorname{div}\big(\mathscr{X}f\big) = -\frac{3}{4}\frac{\mathscr{P}_\phi(E)}{\mathscr{L}^3(E)} \quad \text{in } D. \tag{4.5}$$

Proof. For small $\varepsilon \in \mathbb{R}$ and $\varphi \in \mathrm{C}_c^\infty(D)$ let $E_\varepsilon \subset \mathbb{H}^1$ be the set such that

$$E_\varepsilon \cap A = \{(\xi, z) \in A : z \leq f(\xi) + \varepsilon\varphi(\xi),\ \xi \in D\},$$

and $E_\varepsilon \setminus A = E \setminus A$. Starting from the representation formula

$$\mathscr{P}_\phi(E_\varepsilon; A) = \int_{\partial E_\varepsilon \cap A} \phi^*(N_{E_\varepsilon}) d\mathscr{H}^2 = \int_D \phi^*(F + \varepsilon(X\varphi, Y\varphi)) d\xi, \tag{4.6}$$

we compute the derivative

$$\mathscr{P}_\phi{}' = \frac{d}{d\varepsilon}\mathscr{P}_\phi(E_\varepsilon; A)\bigg|_{\varepsilon=0} = \int_D \langle \mathscr{X}f, (X\varphi, Y\varphi)\rangle d\xi = \int_D \langle \mathscr{X}f, \nabla\varphi\rangle d\xi. \tag{4.7}$$

On the other hand, the derivative of the volume is

$$\mathscr{V}' = \frac{d}{d\varepsilon}\mathscr{L}^3(E_\varepsilon)\bigg|_{\varepsilon=0} = \int_D \varphi\, d\xi.$$

Inserting these formulas into

$$0 = \frac{d}{d\varepsilon}\frac{\mathscr{P}_\phi(E_\varepsilon)^4}{\mathscr{L}^3(E_\varepsilon)^3}\bigg|_{\varepsilon=0} = \frac{\mathscr{P}_\phi(E)^3}{\mathscr{L}^3(E)^4}\big(4\mathscr{P}_\phi{}'\mathscr{L}^3(E) - 3\mathscr{V}'\mathscr{P}_\phi(E)\big),$$

we obtain

$$\int_D \langle \mathscr{X}f, \nabla\varphi\rangle d\xi = \frac{3}{4}\frac{\mathscr{P}_\phi(E)}{\mathscr{L}^3(E)}\int_D \varphi\, d\xi$$

for any test function $\varphi \in \mathrm{C}_c^\infty(D)$. This is our claim. $\square$

Proposition 4.2 still holds if we only have $f \in \mathrm{Lip}(D)$. If ϕ^* is of class C^2 and $f \in \mathrm{C}^2(D)$ then we have $\mathscr{X}f \in \mathrm{C}^1(D \setminus \mathscr{C}(f); \mathbb{R}^2)$. So equation (4.5) is satisfied pointwise in $D \setminus \mathscr{C}(f)$ in the strong sense.

Definition 4.3. Let $f \in \mathrm{C}^2(D)$. We call the function $H_\phi : D \setminus \mathscr{C}(f) \to \mathbb{R}$

$$H_\phi(\xi) = \operatorname{div}\big(\mathscr{X} f(\xi)\big), \quad \xi \in D \setminus \mathscr{C}(f), \tag{4.8}$$

the *ϕ-curvature of the graph* $\mathrm{gr}(f)$. We say that $\mathrm{gr}(f)$ has *constant ϕ-curvature* if there exists $h \in \mathbb{R}$ such that $H_\phi = h$ on $D \setminus \mathscr{C}(f)$. Finally, we say that $\mathrm{gr}(f)$ is *ϕ-critical* if there exists $h \in \mathbb{R}$ such that

$$\int_D \langle \mathscr{X} f, \nabla \varphi \rangle d\xi = -h \int_D \varphi \, d\xi \tag{4.9}$$

is satisfied for every $\varphi \in C_c^\infty(D)$.

Proposition 4.2 then asserts that the part of the boundary of a ϕ-isoperimetric set of class C^2 that can be represented as a z-graph is ϕ-critical and in particular it has constant ϕ-curvature at noncharacteristic points.

Remark 4.4. Let us discuss how the proof of Proposition 4.2 can be adapted to the case where $E \cap A$ is a x-subgraph of class C^2. The case of y-subgraphs is analogous. We have a defining function for $\partial E \cap A$ of the type $g(x, y, z) = f(y, z) - x$ with $f \in \mathrm{C}^2(D)$. The projected horizontal gradient in (4.1) reads

$$G(y, z) = \Big(-1 - \frac{1}{2} y f_z, f_y + \frac{1}{2} f f_z \Big).$$

For $\varepsilon \in \mathbb{R}$ and $\varphi \in \mathrm{C}_c^\infty(D)$ let E_ε be the x-subgraph in A of $f + \varepsilon\varphi$. Then the derivative of the ϕ-perimeter of E_ε is

$$\begin{aligned} \frac{d}{d\varepsilon} \mathscr{P}_\phi(E_\varepsilon; A) \Big|_{\varepsilon=0} &= \int_D \big\langle \nabla \phi^*(G), \big(-y\varphi_z/2, \varphi_y + (\varphi f)_z/2 \big) \big\rangle dy dz \\ &= -\int_D \varphi(y, z) \, \mathscr{L} f(y, z) \, dy dz, \end{aligned}$$

where $\mathscr{L} : \mathrm{C}^2(D) \to \mathrm{C}(D)$ is the partial differential operator

$$\mathscr{L} f = \Big(\frac{\partial}{\partial y} + \frac{f}{2} \frac{\partial}{\partial z} \Big) \phi_b^*(G) - \frac{y}{2} \frac{\partial}{\partial z} \phi_a^*(G), \tag{4.10}$$

with $\nabla \phi^* = (\phi_a^*, \phi_b^*)$.

The statement for x-graphs is then that if $E \subset \mathbb{H}^1$ is ϕ-isoperimetric and $E \cap A$ is a x-subgraph with graph function $f \in \mathrm{C}^2(D)$, then

$$\mathscr{L} f = -\frac{3}{4} \frac{\mathscr{P}_\phi(E)}{\mathscr{L}^3(E)} \quad \text{in } D.$$

When we only have $f \in \operatorname{Lip}(D)$, $\mathscr{L} f$ is well-defined in the distributional sense.

4.3. **Crystalline norms.** In this section we focus on a norm ϕ having non-differentiability points, and in particular on the case where it is crystalline. Recall that the dual norm ϕ^* to a non-differentiable one is not strictly convex, so that $\nabla\phi^*$ is constant on subsets of $\mathbb{R}^2$ having nonempty interior.

Lemma 4.5. *Let O be a subset of $\mathbb{R}^2$ where $\nabla\phi^*$ exists and is constant. Let $E \subset \mathbb{H}^1$ be such that $E \cap A = \{(\xi, z) \in A : z \le f(\xi),\ \xi \in D\}$ for some open set $A \subset \mathbb{H}^1$ and $f \in \mathrm{Lip}(D)$. If $F(\xi) \in O$ for almost every $\xi \in D$ then E is not ϕ-isoperimetric.*

Proof. As in the proof of Proposition 4.2, consider $\varphi \in \mathrm{C}^\infty_c(D)$ and, for $\varepsilon \in \mathbb{R}$ small, let $E_\varepsilon \subset \mathbb{H}^1$ be the set such that

$$E_\varepsilon \cap A = \{(\xi, z) \in A : z \le f(\xi) + \varepsilon\varphi(\xi),\ \xi \in D\},$$

and $E_\varepsilon \setminus A = E \setminus A$. Then, as in (4.7),

$$\mathscr{P}_\phi' = \frac{d}{d\varepsilon}\mathscr{P}_\phi(E_\varepsilon; A)\bigg|_{\varepsilon=0} = \int_D \langle \nabla\phi^*(F), \nabla\varphi\rangle d\xi.$$

By hypothesis, $\nabla\phi^*(F)$ is constant on D, so that $\mathscr{P}_\phi' = 0$.

Now, choosing $\varphi \ne 0$ with constant sign, we deduce that

$$\frac{d}{d\varepsilon}\frac{\mathscr{P}_\phi(E_\varepsilon)^4}{\mathscr{L}^3(E_\varepsilon)^3}\bigg|_{\varepsilon=0} = -3\frac{\mathscr{P}_\phi(E)^4}{\mathscr{L}^3(E)^4}\int_D \varphi(\xi)d\xi \ne 0,$$

contradicting the extremality of E for the isoperimetric quotient. □

We are ready for the proof of Theorem 1.6. This theorem disproofs Conjecture 8.0.1 in [30], where Pansu's bubble was conjectured to solve the isoperimetric problem for crystalline norms.

Let ϕ be a crystalline norm and denote by $v_1, \dots, v_{2N} \in \mathbb{R}^2$ the ordered vertices of the polygon $C_\phi = C_\phi(0, 1)$. Notice that $v_{i+N} = -v_i$ for $i = 1, \dots, N$. The dual norm ϕ^* is also crystalline and the vertices of $C_{\phi^*}(0, 1)$ are in one-to-one correspondence with the edges $e_i = v_i - v_{i-1}$ of $C_\phi(0, 1)$ (with $v_0 = v_{2N}$). Namely, $C_{\phi^*}(0, 1)$ is the convex hull of $v_1^*, \dots, v_{2N}^*$ where, for $i = 1, \dots, 2N$, the vertex v_i^* is the unique vector of $\mathbb{R}^2$ such that

$$\langle v_i^*, e_i\rangle = 0 \tag{4.11}$$

and $\langle v_i^*, v_i\rangle = \langle v_i^*, v_{i-1}\rangle = 1$. In particular, $v_{i+N}^* = -v_i^*$ for $i = 1, \dots, N$.

Along the lines $L_i = \mathbb{R}v_i^*$, the norm ϕ^* is not differentiable. In the positive convex cone bounded by $\mathbb{R}^+v_i^*$ and $\mathbb{R}^+v_{i+1}^*$ the gradient $\nabla\phi^*$ exists and is constant, and we have $\nabla\phi^* = v_i$. For piecewise C^1-smooth ϕ-isoperimetric sets the projected horizontal gradient F takes values in $L_1 \cup \ldots \cup L_N$, by Lemma 4.5.

Proof of Theorem 1.6. Let $f \in \mathrm{C}^2(D)$ be the graph function of $\partial E \cap A$. For $i = 1, \dots, N$, we let

$$D_i = \{\xi \in D : F(\xi) \in L_i = \mathbb{R}v_i^*\}.$$

If $\xi \in D_i$ then by (4.11) we have

$$F(\xi)^\perp \in \mathbb{R}(v_i^*)^\perp = \mathbb{R}e_i.$$

This implies that the vector field X_i in (1.7) is tangent to $\partial E \cap A$ at the point $(\xi, f(\xi))$.

We are going to prove the theorem by showing that $D = D_i$ for some $i \in \{1, \dots, N\}$. Notice that, for $i, j \in \{1, \dots, N\}$ and $i \neq j$, v_i and v_j are linearly independent. By Lemma 4.5 we have that $D = \cup_{i=1}^N D_i$. We claim, moreover, that

$$\overline{D} = \cup_{i=1}^N \overline{\operatorname{int} D_i}. \tag{4.12}$$

In order to check the claim, pick $\xi \in D$ and assume by contradiction that $\xi \notin \overline{\operatorname{int} D_i}$ for $i = 1, \dots, N$. Let i_1 be such that $\xi \in D_{i_1}$. Since $\xi \notin \operatorname{int} D_{i_1}$, for every $\varepsilon > 0$ the set $D \setminus D_{i_1}$ intersects the disc of radius ε centered at ξ. Hence, there exist $i_2 \neq i_1$, and a sequence $(\xi_n)_{n\in\mathbb{N}}$ in $D_{i_2} \setminus D_{i_1}$ converging to ξ. Now, either $\xi_n \in \operatorname{int} D_{i_2}$ for infinitely many n or $\xi_n \notin \operatorname{int} D_{i_2}$ for n large enough. In the first case $\xi \in \overline{\operatorname{int} D_{i_2}}$, leading to a contradiction. In the second case, we repeat the reasoning leading to $(\xi_n)_{n\in\mathbb{N}}$, replacing D_{i_1} by D_{i_2} and ξ by ξ_n for every $n \in \mathbb{N}$, and, by a diagonal argument, we obtain $i_3 \neq i_1, i_2$, and a sequence $(\hat{\xi}_n)_{n\in\mathbb{N}}$ in $D_{i_3} \setminus (D_{i_1} \cup D_{i_2})$ converging to ξ. Repeating the argument finitely many times, we end up with $i_N \in \{1, \dots, N\}$ and a sequence $(\tilde{\xi}_n)_{n\in\mathbb{N}}$ in $D_{i_N} \setminus (\cup_{j=1}^{N-1} D_{i_j})$ converging to ξ with $D = D_{i_1} \cup \dots \cup D_{i_N}$. Since $D_{i_N} \setminus (\cup_{j=1}^{N-1} D_{i_j}) = D \setminus (\cup_{j=1}^{N-1} D_{i_j})$ is open, we deduce that $\xi \in \overline{\operatorname{int} D_{i_N}}$. This concludes the contradiction argument, proving (4.12).

Let v_i and v_j be linearly independent. We claim that

$$\overline{\operatorname{int}(D_i)} \cap \overline{\operatorname{int}(D_j)} = \emptyset. \tag{4.13}$$

Consider the vector field X' on $D \times \mathbb{R}$ defined by $X'(\xi, z) = (e_i, e_{i,1} f_x(\xi) + e_{i,2} f_y(\xi))$. Then X' is C^1 and both X' and X_j are tangent to $\partial E \cap A$ in a neighborhood of any point of S_j, where

$$S_k = \{(\xi, f(\xi)) : \xi \in \operatorname{int}(D_k)\}, \quad k = 1, \dots, N.$$

Hence $[X', X_j] \in T_\xi(\partial E \cap A)$ for every $\xi \in S_j$. On the other hand, X' coincides with X_i on $S_i \times \mathbb{R}$, and therefore $[X', X_j] = [X_i, X_j] = c_{ij} Z$ on S_i, with $c_{ij} \in \mathbb{R} \setminus \{0\}$. Assume by contradiction that $\overline{S}_i \cap \overline{S}_j$ contains at least one point ξ. By continuity of $[X', X_j]$, we deduce from the above reasoning that $Z(\xi) \in T_\xi(\partial E \cap A)$. The contradiction comes from the remark that, by definition of S_i and S_j, also $X_i(\xi)$ and $X_j(\xi)$ are in $T_\xi(\partial E \cap A)$. We proved (4.13).

We deduce from (4.12) and (4.13) that $\{\overline{\operatorname{int}(D_1)}, \dots, \overline{\operatorname{int}(D_N)}\}$ is an open disjoint cover of $\overline{D}$. We conclude by connectedness of D. $\square$

5. Integration of the curvature equation

Throughout this section ϕ^* is a norm of class C^2, unless explicitly mentioned otherwise.

Let $A \subset \mathbb{H}^1$ be open and $g \in \mathrm{C}^{1,1}(A)$ be such that $\nabla g(p) \neq 0$ for every p in $\Sigma = \{p \in A : g(p) = 0\}$. The projected horizontal gradient $G : A \to \mathbb{R}^2$ introduced in (4.1) is Lipschitz continuous. Assume that Σ has no characteristic points, that is, $G(p) \neq 0$ for every $p \in \Sigma$. We use the coordinates $G = (a, b)$ with $a, b \in \mathrm{Lip}(A)$ and we consider $G^\perp = (-b, a)$. The horizontal vector field $\mathscr{G}^\perp = -bX + aY$ is tangent to Σ.

Definition 5.1. A curve $\gamma \in \mathrm{C}^1(I; \Sigma)$ is said to be a *Legendre curve* of Σ if $\dot\gamma(t) = \mathscr{G}^\perp(\gamma(t))$ for all $t \in I$.

In coordinates, a curve $\gamma = (\xi, z)$ in Σ is a Legendre curve if and only if

$$\dot\xi = G^\perp(\gamma) \quad \text{and} \quad \dot z = \omega(\xi, \dot\xi).$$

Since $\mathscr{G}^\perp$ is Lipschitz continuous, the graph Σ is foliated by Legendre curves: for any $p \in \Sigma$ there exists a unique (maximal) Legendre curve passing through p.

Consider now the case where Σ is a z-graph with graph function $f \in \mathrm{C}^{1,1}(D)$, where D is an open subset of $\mathbb{R}^2$. Then $G(\xi, f(\xi)) = F(\xi)$, where F is defined as in (4.2), and a Legendre curve $\gamma = (\xi, z)$ satisfies

$$\dot\xi = F^\perp(\xi) \quad \text{and} \quad \dot z = \omega(\xi, \dot\xi). \tag{5.1}$$

The domain D is foliated by integral curves of $F^\perp$. On D we define the vector field $\mathscr{N} \in \mathrm{Lip}(D; \mathbb{R}^2)$ by

$$\mathscr{N}(\xi) = \mathscr{X} f(\xi) = \nabla\phi^*(F(\xi)), \quad \xi \in D. \tag{5.2}$$

We know that $\phi(\mathscr{N}) = 1$, by (4.4). We may call $\mathscr{N}$ the *ϕ-normal* to the foliation of D by integral curves of $F^\perp$. We denote by $H_\phi = \mathrm{div}(\mathscr{N})$ the divergence of $\mathscr{N}$.

Theorem 5.2. *Let ϕ^* be of class C^2. Let Σ be the z-graph of a function $f \in \mathrm{C}^2(D)$ with $\mathscr{C}(f) = \emptyset$. Then any Legendre curve $\gamma \in \mathrm{C}^1(I; \Sigma)$, with $\gamma = (\xi, z)$, satisfies*

$$\frac{d}{dt}\mathscr{N}(\xi) = H_\phi(\xi)\dot\xi \quad \textit{and} \quad \dot z = \omega(\xi, \dot\xi). \tag{5.3}$$

Proof. The second equality in (5.3) is part of the definition of a Legendre curve. We prove the first equality.

We identify $\mathscr{N}(\xi)$ and $\dot\xi = F^\perp(\xi)$ with column vectors and we denote by Jg the Jacobian matrix of a differentiable mapping g. By the chain rule, using the coordinates $F = (a, b)$ and $\dot\xi = (-b(\xi), a(\xi))$ we obtain

$$\frac{d}{dt}\mathscr{N}(\xi) = \mathscr{H}\phi^*(F(\xi)) JF(\xi)\dot\xi = \begin{pmatrix} -ba_x\phi^*_{aa} - bb_x\phi^*_{ab} + aa_y\phi^*_{aa} + ab_y\phi^*_{ab} \\ -ba_x\phi^*_{ab} - bb_x\phi^*_{bb} + aa_y\phi^*_{ab} + ab_y\phi^*_{bb} \end{pmatrix}, \tag{5.4}$$

where $\mathscr{H}\phi^*$ is the Hessian matrix of ϕ^* and the second order derivatives of ϕ^* are evaluated at $F(\xi)$. Since ϕ^* is of class C^2, we identified $\phi^*_{ab} = \phi^*_{ba}$. By Euler's homogeneous function theorem, since $\nabla\phi^*$ is 0-positively homogeneous there holds $\langle\nabla\phi^*_a(F), F\rangle = 0$ and $\langle\nabla\phi^*_b(F), F\rangle = 0$. These formulas read

$$a\phi^*_{aa} + b\phi^*_{ab} = 0 \quad \text{and} \quad a\phi^*_{ab} + b\phi^*_{bb} = 0.$$

Plugging these relations into (5.4), we obtain

$$\frac{d}{dt}\mathscr{N}(\xi) = (a_x\phi^*_{aa} + b_x\phi^*_{ab} + a_y\phi^*_{ab} + b_y\phi^*_{bb})\,\dot{\xi}. \tag{5.5}$$

On the other hand, we have

$$\operatorname{div}(\mathscr{N}) = \operatorname{div}(\mathscr{X}f) = a_x\phi^*_{aa} + b_x\phi^*_{ab} + a_y\phi^*_{ab} + b_y\phi^*_{bb},$$

so that (5.5) yields the claim. □

Remark 5.3. An analogue of Theorem 5.2 holds true for x-graphs. Let Σ be a x-graph Σ without characteristic points and with defining function $g(x,y,z) = f(y,z) - x$ for some f of class C^2. Let $\gamma \in \mathrm{C}^1(I;\Sigma)$ be a Legendre curve with coordinates $\gamma(t) = (f(\zeta(t)), \zeta(t))$ for $t \in I$ and consider the vector $\mathscr{N}(y,z) = \nabla\phi^*(G(y,z))$. Following the same steps as in the proof of Theorem 5.2, one gets

$$\frac{d}{dt}\mathscr{N}(\zeta) = \mathscr{L}f(\zeta)G^{\perp}(\zeta) \quad \text{on } I.$$

Hence, the conclusion of Theorem 5.2 holds with $H_\phi = \operatorname{div}\big(\mathscr{X}f\big)$ replaced by the quantity $\mathscr{L}f$ defined in Remark 4.4. Notice that H_ϕ and $\mathscr{L}f$ coincide on surfaces that are both x-graphs and z-graphs.

An analogous remark can be made for y-graphs.

Corollary 5.4. *Let ϕ^* be of class C^2. Let Σ be the z-graph of a function $f \in \mathrm{C}^2(D)$ with $\mathscr{C}(f) = \emptyset$. If Σ has constant ϕ-curvature $h \neq 0$ then it is foliated by Legendre curves that are horizontal lift of ϕ-circles in D with radius $1/|h|$, followed in clockwise sense if $h > 0$ and in anti-clockwise sense if $h < 0$.*

Proof. Having constant ϕ-curvature h means that

$$\operatorname{div}(\mathscr{N}) = \operatorname{div}(\mathscr{X}f) = h \quad \text{in } D.$$

By Theorem 5.2, for any Legendre curve $\gamma = (\xi, z)$ we have

$$\frac{d}{dt}\mathscr{N}(\xi) - H(\xi)\dot{\xi} = 0.$$

We may than integrate this equation and deduce that there exists $\xi_0 \in \mathbb{R}^2$ such that along ξ we have

$$\mathscr{N}(\xi) - h\xi = -h\xi_0. \tag{5.6}$$

From (4.4) and (5.2) we conclude that

$$|h|\phi(\xi-\xi_0)=\phi(h(\xi-\xi_0))=\phi(\mathscr{N})=1.$$

Finally, notice that $\langle \mathscr{N}(\xi), F(\xi)\rangle > 0$ if $F(\xi)\neq 0$, so that $t\mapsto F(\xi(t))$ rotates clockwise if $h>0$ and anti-clockwise if $h<0$, according to (5.3). Hence, $t\mapsto F(\xi(t))^\perp$ and $t\mapsto \xi(t)$ also rotate clockwise if $h>0$, and anti-clockwise if $h<0$. □

Let us discuss an extension of Corollary 5.4 to the case in which we replace the assumption that ϕ^* is C^2 by the weaker assumption that ϕ^* is *piecewise* C^2, in the following sense: there exist $k\in\mathbb{N}$ and $A_1,\dots,A_k\in\mathbb{R}^2$ such that ϕ^* is C^2 on $\mathbb{R}^2\setminus\cup_{j=1}^k \mathrm{span}(A_j)$.

A relevant case where this assumption holds true is when ϕ is the ℓ^p norm

$$\ell^p(x,y)=(|x|^p+|y|^p)^{\frac{1}{p}},\qquad x,y\in\mathbb{R},$$

with $p>2$. Indeed, the dual norm $(\ell^p)^*$ coincides with the norm ℓ^q, with $q=p/(p-1)<2$, which is C^2 out of the coordinate axes, but not on the whole punctured plane $\mathbb{R}^2\setminus\{0\}$. We can prove the following.

Corollary 5.5. *Let ϕ^* be piecewise C^2. Let Σ be the z-graph of a function $f\in\mathrm{C}^2(D)$ with $\mathscr{C}(f)=\emptyset$. If Σ has constant ϕ-curvature $h\neq 0$ then it is foliated by Legendre curves that are horizontal lifts of ϕ-circles in D with radius $1/|h|$, followed in clockwise sense if $h>0$ and in anti-clockwise sense if $h<0$.*

Proof. Under the assumptions of the corollary, the projected horizontal gradient is C^1 on D and Legendre curves can be introduced as in Definition 5.1.

Consider any Legendre curve $\gamma=(\xi,z)$ on Σ. Let us denote by $I\subset\mathbb{R}$ the maximal interval of definition of γ and by J the open subset of I defined as follows: $t\in J$ if and only if $F(\xi(t))$ is in the region where ϕ^* is C^2. For the restriction of γ to a connected component J_0 of J, Theorem 5.2 can be recovered. In particular, since Σ has constant ϕ-curvature $h\neq 0$, then $\gamma|_{J_0}$ is the lift of a ϕ-circle of radius $1/|h|$, followed clockwise or anti-clockwise depending on the sign of h. If $t\in I\setminus J$, then $F(\xi(t))$ belongs to one of the lines $\mathrm{span}(A_1),\dots,\mathrm{span}(A_k)$ on which ϕ^* may lose the C^2 regularity. Notice that the restriction of ξ to a connected component of J compactly contained in I follows an arc of ϕ-circle connecting two lines of the type $\mathrm{span}(A_j)$. In particular, it cannot have arbitrarily small length.

If $I\setminus J$ is made of isolated points, then $\gamma: I\to\Sigma$ is the lift of a ϕ-circle of radius $1/|h|$. Indeed, an arc of ϕ-circle of prescribed radius followed in a prescribed sense is only determined by its initial point and its tangent line there. Since γ is an arbitrary Legendre curve on Σ, the proof is complete if show that $I\setminus J$ does not contain intervals of positive length.

Assume by contradiction that $[t_0, t_1]$ is contained in J with $t_0 < t_1$. Then $F(\xi(t))$ is constantly equal to some $A \in \mathbb{R}^2$ for $t \in [t_0, t_1]$. Let $\delta > 0$ and $\kappa : (-\delta, \delta) \to \Sigma$ be a C^1 curve such that $\kappa(0) = \gamma(t_0)$ and $\kappa'(0)$ is not proportional to $\gamma'(t_0)$. Write $\kappa(s) = (\xi_s, z_s)$ and notice that $F(\xi_s)$ converges to A as $s \to 0$. Consider for each $s \in (-\delta, \delta)$ the Legendre curve γ_s such that $\gamma_s(t_0) = \kappa(s)$. Then γ_s converges to γ and $F \circ \gamma_s$ converges to $F \circ \gamma$, uniformly on $[t_0, t_1]$, as $s \to 0$. Hence, for $\varepsilon > 0$ and $|s|$ small enough, the restriction of γ_s to $(t_0 + \varepsilon, t_1 - \varepsilon)$ cannot contain the lift of any arc of ϕ-circle of radius $1/|h|$. This implies that there exists a nonempty open region of Σ of the form $\{\gamma_s(t) : t \in (t_0 + \varepsilon, t_1 - \varepsilon),\ |s| < \bar{\delta}\}$ on which $F(\xi) = A$, contradicting the assumption that Σ has constant nonzero ϕ-curvature. □

6. Foliation property with geodesics

In this section we prove that the Legendre foliation of a surface (a z-graph) with constant ϕ-curvature consists of length minimizing curves in the ambient space (geodesics) relative to the norm $\phi^\dagger$ in $\mathbb{R}^2$ defined by

$$\phi^\dagger(\xi) = \phi^*(\xi^\perp), \quad \xi \in \mathbb{R}^2.$$

We consider a general norm ψ in $\mathbb{R}^2$ and for $T \geq 0$ we introduce the class of curves

$$\mathscr{A}_T = \big\{\gamma = (\xi, z) \in \mathrm{AC}([0, T]; \mathbb{H}^1) : \dot{z} = \omega(\xi, \dot{\xi}) \text{ and } \psi(\dot{\xi}) \leq 1 \text{ a.e.}\big\},$$

where ω is the symplectic form introduced in (1.2). In the sequel, we denote by $u = \dot{\xi} \in L^1([0, T]; \mathbb{R}^2)$ the *control* of γ. For given points $p_0, p_1 \in \mathbb{H}^1$ we consider the *optimal time problem*

$$\inf \big\{T \geq 0 : \text{ there exists } \gamma \in \mathscr{A}_T \text{ such that } \gamma(0) = p_0 \text{ and } \gamma(T) = p_1\big\}. \tag{6.1}$$

We call a curve γ realizing the minimum in (6.1) a *ψ-time minimizer* between p_0 and p_1. In this case, we call the pair (γ, u) with $u = \dot{\xi}$ an *optimal pair.* A ψ-time minimizer is always parameterized by ψ-arclength, i.e., $\psi(u) = 1$. So, ψ-time minimizers are ψ-length minimizers parameterized by ψ-arclength.

An optimal pair (γ, u) satisfies the necessary conditions given by Pontryagin's Maximum Principle. As observed in [6], it necessarily is a *normal extremal*, whose definition is recalled below. The Hamiltonian associated with the optimal time problem (6.1) is $\mathfrak{H} : \mathbb{H}^1 \times \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}$

$$\mathfrak{H}(p, \lambda, u) = \Big(\lambda_x - \frac{y}{2}\lambda_z\Big) u_1 + \Big(\lambda_y + \frac{x}{2}\lambda_z\Big) u_2 = \langle \lambda_\xi + \frac{1}{2}\lambda_z \xi^\perp, u\rangle,$$

where $\lambda = (\lambda_\xi, \lambda_z) \in \mathbb{R}^2 \times \mathbb{R}$.

Definition 6.1. The pair $(\gamma, u) \in \mathrm{AC}([0, T]; \mathbb{H}^1) \times L^1([0, T]; \mathbb{R}^2)$ is a *normal extremal* if there exists a nowhere vanishing curve $\lambda \in \mathrm{AC}([0, T]; \mathbb{R}^3)$ such that (γ, λ) solves a.e. the Hamiltonian system

$$\begin{cases} \dot\gamma = \mathfrak{H}_\lambda(\gamma, \lambda, u) \\ \dot\lambda = -\mathfrak{H}_p(\gamma, \lambda, u), \end{cases}$$

and for every $t \in [0, T]$ we have

$$1 = \mathfrak{H}(\gamma(t), \lambda(t), u(t)) = \max_{\psi(u)\le 1} \mathfrak{H}(\gamma(t), \lambda(t), u). \tag{6.2}$$

In the coordinates $\gamma = (\xi, z)$ and $\lambda = (\lambda_\xi, \lambda_z)$, the Hamiltonian system reads

$$\begin{cases} \dot\xi = u, \\ \dot z = \omega(\xi, u), \end{cases} \qquad \begin{cases} \dot\lambda_\xi = \frac{1}{2}\lambda_z u^\perp, \\ \dot\lambda_z = 0. \end{cases} \tag{6.3}$$

Theorem 6.2. *Let ψ be of class C^1 and let $\gamma = (\xi, z) \in \mathrm{AC}([0, T]; \mathbb{H}^1)$ be a horizontal curve. The following statements (i) and (ii) are equivalent:*

(i) *γ is a local ψ-length minimizer parametrized by ψ-arclength;*
(ii) *the pair (γ, u) with $u = \dot\xi$ is a normal extremal.*

Moreover, if ψ is of class C^2 then each of (i) and (ii) is equivalent to

(iii) *γ is of class C^2 and parameterized by ψ-arclength, and there is $\lambda_0 \in \mathbb{R}$ such that*
$$\mathscr{H}\psi(\dot\xi)\ddot\xi = \lambda_0 \dot\xi^\perp, \tag{6.4}$$
where $\mathscr{H}\psi$ is the Hessian matrix of ψ.

Proof. The equivalence between (i) and (ii) is [6, Theorem 1].

Let us show that (ii) implies (iii). We set

$$\mathscr{M}(t) = \lambda_\xi(t) + \frac{1}{2}\lambda_z(t)\xi(t)^\perp, \quad t \in [0, T], \tag{6.5}$$

where $\lambda = (\lambda_\xi, \lambda_z)$ is the curve given by the definition of extremal. Then the maximality condition in (6.2) for normal extremals reads

$$1 = \langle \mathscr{M}(t), u(t)\rangle = \max_{\psi(u)\le 1} \langle \mathscr{M}(t), u\rangle = \psi^*(\mathscr{M}(t)). \tag{6.6}$$

This is equivalent to the identity

$$\mathscr{M}(t) = \nabla\psi(u(t)). \tag{6.7}$$

When ψ is of class C^2, from (6.7), (6.5), and (6.3) we obtain the differential equation for $u = \dot\xi$

$$\mathscr{H}\psi(u)\dot u = \dot{\mathscr{M}} = \dot\lambda_\xi + \frac{1}{2}\dot\lambda_z\xi + \frac{1}{2}\lambda_z u^\perp = \lambda_z u^\perp. \tag{6.8}$$

This is (6.4) with $\lambda_0 := \lambda_z$.

Now we show that (ii) is implied by (iii). Consistently with (6.7), we define $\mathscr{M}(t) = \nabla\psi(u(t))$, for $t \in [0, T]$. Then $\psi^*(\mathscr{M}) = 1$.

We define the curve $\lambda = (\lambda_\xi, \lambda_z)$ letting $\lambda_z = \lambda_0$ and $\lambda_\xi = \mathscr{M} - \frac{1}{2}\lambda_z \xi^\perp$. When ψ is of class C^2, we obtain

$$\dot{\lambda}_\xi = \dot{\mathscr{M}} - \frac{1}{2}\lambda_z \dot{\xi}^\perp = \mathscr{H}\psi(\dot{\xi})\ddot{\xi} - \frac{1}{2}\lambda_z \dot{\xi}^\perp = \frac{1}{2}\lambda_z u^\perp.$$

Hence, all equations in (6.3) are satisfied, showing that the pair (γ, u) is a normal extremal. This proves that (iii) implies (ii). $\square$

Remark 6.3. When $\lambda_0 \neq 0$, equation (6.4) can be integrated in the following way. Using (6.8), the equation is equivalent to $\dot{\mathscr{M}} = \lambda_0 \dot{\xi}^\perp$, that implies $\mathscr{M} = \lambda_0(\xi^\perp - \xi_0^\perp)$ for some constant $\xi_0 \in \mathbb{R}^2$. So from (6.6) we deduce that $|\lambda_0|\psi^*(\xi^\perp - \xi_0^\perp) = 1$. If we choose $\psi = \phi^\dagger$ then we have $\psi^*(\xi^\perp) = \phi(\xi)$. So the previous equation becomes the equation for a ϕ-circle

$$\phi(\xi - \xi_0) = 1/|\lambda_0|.$$

Corollary 6.4. *Let ϕ be a norm with dual norm ϕ^* of class piecewise C^2 and let $f \in C^2(D)$ be such that $\mathscr{C}(f) = \emptyset$. If $\mathrm{gr}(f)$ has constant ϕ-curvature, then it is foliated by geodesics of $\mathbb{H}^1$ relative to the norm $\phi^\dagger$.*

The proof is Corollary 5.5, combined with Remark 6.3 and Theorem 6.2.

7. Characteristic set of ϕ-critical surfaces

In this section we study the characteristic set of ϕ-critical surfaces and then apply the results to ϕ-isoperimetric sets. For a C^2 surface $\Sigma \subset \mathbb{H}^1$, the characteristic set is

$$\mathscr{C}(\Sigma) = \{p \in \partial E : T_p\Sigma = \mathscr{D}(p)\}. \tag{7.1}$$

Note that any C^2 surface $\Sigma \subset \mathbb{H}^1$ is a z-graph around any of its characteristic points $p \in \mathscr{C}(\Sigma)$.

When Σ is oriented, the ϕ-curvature H_ϕ of Σ can be defined in a globally coherent way. When Σ is a z-graph at the point $p = (\xi, z) = (x, y, z) \in \Sigma$ we let $H_\phi(p) = \mathrm{div}(\mathscr{X}f)(\xi)$ where f is a z-graph function; when Σ is a x-graph, we let $H_\phi(p) = \mathscr{L}f(y, z)$, where now f is a x-graph function and $\mathscr{L}f$ is defined in (4.10); when Σ is a y-graph we proceed analogously.

We say that Σ is *ϕ-critical* if it is closed, has constant ϕ-curvature and it is ϕ-critical in a neighborhood of any characteristic point.

Our goal is to prove Theorem 1.2. The proof is obtained combining Lemma 7.1 and Theorem 7.2 below.

In this section, ϕ and ϕ^* are two norms of class C^2. We will omit to mention this assumptions in the various statements.

7.1. **Qualitative structure of the characteristic set.**

Lemma 7.1. *Let $\Sigma \subset \mathbb{H}^1$ be a C^2 surface with constant ϕ-curvature. Then $\mathscr{C}(\Sigma)$ consists of isolated points and C^1 curves. Moreover, for every isolated point $p_0 = (\xi_0, z_0) \in \mathscr{C}(\Sigma)$ and every f such that $p_0 \in \mathrm{gr}(f) \subset \Sigma$, we have* $\mathrm{rank}(JF(\xi_0)) = 2$, *where F is the projected horizontal gradient introduced in* (4.2).

Proof. We let $\mathscr{C}(f)$ be as in (4.3). For any $\xi_0 \in \mathscr{C}(f)$, the Jacobian matrix $JF(\xi_0)$ has rank 1 or 2. Indeed, an explicit calculation shows that $JF(\xi_0) \neq 0$ for all $\xi_0 \in D$. If $\mathrm{rank}(JF(\xi_0)) = 2$ then ξ_0 is an isolated point of $\mathscr{C}(f)$.

We study the case $\mathrm{rank}(JF(\xi_0)) = 1$. We claim that in this case $\mathscr{C}(f)$ is a curve of class C^1 in a neighborhood of ξ_0. The argument that we use here is inspired by [8].

For $b \in \mathbb{R}^2$ we define $F_b : D \to \mathbb{R}$, $F_b = \langle F, b\rangle$. When $b \notin \ker(JF(\xi_0))$, the equation $F_b = 0$ defines a C^1 curve Γ_b near and through ξ_0. We have $\mathscr{C}(f) \subset \Gamma_b$. Since $\nabla F_b(\xi_0)$ is in the image of $JF(\xi_0)$, which is a line independent of b, the normal direction to Γ_b at ξ_0 does not depend on b. We choose one of the two unit normals and we call it $N \in \mathbb{R}^2$.

We claim that there exist $a, b \in \mathbb{S}^1$, where $\mathbb{S}^1 = \{w \in \mathbb{R}^2 : |w| = 1\}$, such that

$$a \notin \{b, -b\}, \quad a, b \notin \ker(JF(\xi_0)), \quad |\langle \nabla\phi^*(b^\perp), N\rangle| \neq |\langle \nabla\phi^*(a^\perp), N\rangle|. \tag{7.2}$$

To prove the claim pick $b \in \mathbb{S}^1 \setminus \ker(JF(\xi_0))$ (this is possible since $\mathrm{rank}(JF(\xi_0)) \neq 0$), and define the set

$$K_b := \left\{ v \in C_\phi : |\langle v, N\rangle| = |\langle \nabla\phi^*(b^\perp), N\rangle| \right\}.$$

Since the map $\nabla\phi^* : \mathbb{S}^1 \to C_\phi$ is continuous, the set $(\nabla\phi^*)^{-1}(K_b) \subset \mathbb{S}^1$ is closed in $\mathbb{S}^1$. Moreover, $\nabla\phi^* : \mathbb{S}^1 \to C_\phi$ is surjective, since for every $w \in C_\phi$ and every v in the subgradient of ϕ^* at w, we have $w = \nabla\phi^*(v)$ (see, e.g., [28, Theorem 23.5]). As a consequence, $(\nabla\phi^*)^{-1}(K_b) \neq \mathbb{S}^1$, since otherwise we would have $K_b = C_\phi$, which is impossible. The set

$$\Upsilon = \ker(JF(\xi_0))^\perp \cup (\nabla\phi^*)^{-1}(K_b) \cup \{b^\perp, -b^\perp\}$$

is therefore a proper closed subset of $\mathbb{S}^1$, and the claim follows by choosing $a^\perp \in \mathbb{S}^1 \setminus \Upsilon$.

Fix $a, b \in \mathbb{S}^1$ such that (7.2) holds and, for $\alpha \in (0,1)$, let $C_\alpha := \{v \in \mathbb{R}^2 : |\langle N, v\rangle| < |v| \sin\alpha\}$ be the cone centered at ξ_0 with axis parallel to $N^\perp$ and aperture 2α. Since Γ_a, Γ_b are C^1, there exists $\delta \in (0,1)$ such that

$$\{\xi \in \Gamma_a \cup \Gamma_b : |\xi - \xi_0| < \delta\} \subset C_{\alpha,\delta}, \tag{7.3}$$

where we set $C_{\alpha,\delta} = \{\xi \in C_\alpha : |\xi - \xi_0| < \delta\}$.

Let us assume by contradiction that $\mathscr{C}(f)$ is not a C^1 curve near ξ_0. Then there exists a nonempty connected component A of $C_{\alpha,\delta} \setminus (\Gamma_a \cup \Gamma_b)$ such that, letting

$$\Lambda_a = \Gamma_a \cap \partial A, \quad \Lambda_b := \Gamma_b \cap \partial A, \quad \Lambda_\partial := \partial\{|\xi - \xi_0| < \delta\} \cap \partial A,$$

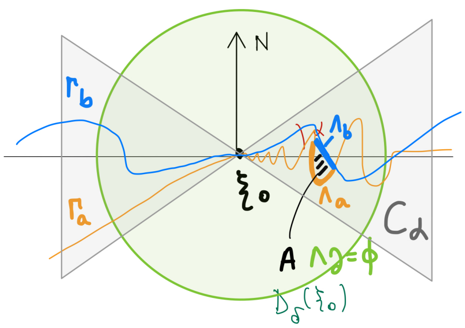

FIGURE 2. The cone C_α and the region A. On the left, A does not touch $\partial\{|\xi-\xi_0|<\delta\}$, while it does on the right. We can always restrict our attention to the case on the left when ξ_0 is a density point of $\mathscr{C}(f)$.

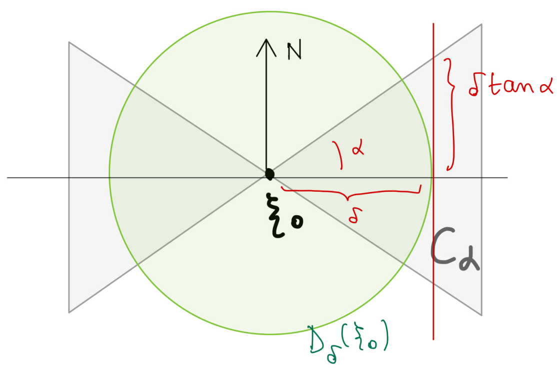

FIGURE 3. Proportions in $C_{\alpha,\delta}$.

we have

$$\Lambda_a \neq \emptyset, \quad \Lambda_b \neq \emptyset, \quad \partial A = \Lambda_a \cup \Lambda_b \cup \Lambda_\partial, \quad \sharp(\Lambda_a \cap \Lambda_b) \leq 2. \tag{7.4}$$

See Figure 2. Notice that A, Λ_a, Λ_b, and Λ_∂ depend on δ. By (7.3) (see also Figure 3), we have

$$\mathscr{L}^2(A) \leq \delta^2 \tan(\alpha). \tag{7.5}$$

By (7.4) and since $\mathscr{C}(f) \subset \Lambda_a \cap \Lambda_b$, for $\xi \in \mathrm{int}(\Lambda_a) \cup \mathrm{int}(\Lambda_b)$ we have $F(\xi) \neq 0$, where we endow Λ_a and Λ_b with their relative topologies. We deduce that $F(\xi) = c_a(\xi)a^\perp$ with $c_a(\xi) \neq 0$ for $\xi \in \mathrm{int}(\Lambda_a)$ and $F(\xi) = c_b(\xi)b^\perp$ with $c_b(\xi) \neq 0$ for $\xi \in \mathrm{int}(\Lambda_b)$. Using the fact that $\nabla\phi^*$ is positively 0-homogeneous it then follows that the vector field $\mathscr{N} : D \setminus \mathscr{C}(f) \to \mathbb{R}^2$, $\mathscr{N}(\xi) = \nabla\phi^*(F(\xi))$, is constant along Λ_a and Λ_b. Namely,

$$\mathscr{N}(\xi) = \mathrm{sgn}(c_a)\nabla\phi^*(a^\perp) =: \mathscr{N}_a, \quad \xi \in \mathrm{int}(\Lambda_a),$$
$$\mathscr{N}(\xi) = \mathrm{sgn}(c_b)\nabla\phi^*(b^\perp) =: \mathscr{N}_b, \quad \xi \in \mathrm{int}(\Lambda_b).$$

By assumption, and since $\phi^* \in \mathrm{C}^2$, there exists a constant $h \in \mathbb{R}$ such that

$$\operatorname{div}(\mathscr{N}(\xi)) = h, \quad x \in D \setminus \mathscr{C}(f),$$

in the strong sense. Then by the divergence theorem, and since $A \cap \mathscr{C}(f) = \emptyset$, we have

$$h\mathscr{L}^2(A) = \int_A \operatorname{div}(\mathscr{N}) dxdy = \int_{\Lambda_a} \langle \mathscr{N}_a, N_a \rangle d\mathscr{H}^1 + \int_{\Lambda_b} \langle \mathscr{N}_b, N_b \rangle d\mathscr{H}^1 + \int_{\Lambda_\partial} \langle \mathscr{N}, N_\partial \rangle d\mathscr{H}^1,$$

where N_a, N_b, and N_∂ are, respectively, the normals to Λ_a, Λ_b, and Λ_∂, exterior with respect to A. For $\alpha \to 0^+$ we have

$$\begin{aligned}
&\int_{\Lambda_a} N_a \mathscr{H}^1 = \delta(-N + o(1)),\\
&\int_{\Lambda_b} N_b \mathscr{H}^1 = \delta(N + o(1)),\\
&\left| \int_{\Lambda_\partial} \langle \mathscr{N}, N_\partial \rangle d\mathscr{H}^1 \right| \leq C\delta\alpha,
\end{aligned}$$

where $o(1) \to 0$ as $\alpha \to 0^+$ and $C > 0$ denotes a suitable constant. Now from (7.5) we deduce that

$$|\delta \tan(\alpha) h| \geq |\langle \mathscr{N}_b - \mathscr{N}_a, N \rangle + o(1)| - C\alpha,$$

that implies $\langle \mathscr{N}_b - \mathscr{N}_a, N \rangle = 0$ in contradiction with (7.2).

This proves that $\mathscr{C}(f)$ is a C^1 curve around any point ξ_0 with $\operatorname{rank}(JF(\xi_0)) = 1$. □

7.2. **Characteristic curves in ϕ-critical surfaces.** Given a surface $\Sigma \subset \mathbb{H}^1$, we call a *characteristic curve on* Σ any (nontrivial) curve $\Gamma \subset \mathscr{C}(\Sigma)$. In this section we prove the following result.

Theorem 7.2. *Let Σ be a complete and oriented surface of class C^2. If Σ is ϕ-critical with non-vanishing ϕ-curvature $h \neq 0$ then any characteristic curve on Σ is either a horizontal line or the horizontal lift of a simple closed curve.*

For a characteristic curve Γ in Σ we denote its coordinates by $\Gamma = (\Xi, \zeta) \in \mathbb{R}^2 \times \mathbb{R}$. For any $p_0 = (\xi_0, z_0)$ on Γ, let $\delta > 0$ be small enough to have

$$\{\xi \in \mathbb{R}^2 : |\xi - \xi_0| < \delta\} \setminus \operatorname{supp}(\Xi) = B^+ \cup B^-, \tag{7.6}$$

where $B^+, B^- \subset \mathbb{R}^2$ are disjoint open connected sets. The ϕ-normal $\mathscr{N}$ in (5.2) is well-defined in $B^+ \cup B^-$.

Lemma 7.3. *Let Σ be a C^2 surface with constant ϕ-curvature. With the above notation, the following limits exist*

$$\mathscr{N}^\pm(\xi_0) := \lim_{B^\pm \ni \xi \to \xi_0} \mathscr{N}(\xi) \tag{7.7}$$

and satisfy $\mathscr{N}^+(\xi_0) = -\mathscr{N}^-(\xi_0)$.

Proof. This is a straightforward corollary of [8, Proposition 3.5]. □

Proposition 7.4. *Let Σ be a ϕ-critical surface of class C^2 and let $\Gamma = (\Xi, \zeta)$ be a characteristic curve on Σ. Then for every $p_0 = (\xi_0, z_0)$ in Γ we have*

$$\mathscr{N}^{\pm}(\xi_0) \in T_{\xi_0}\Xi, \tag{7.8}$$

where $\mathscr{N}^{\pm}$ is defined as in Lemma 7.3.

Proof. Let $f \in C^2(D)$ be a graph function for Σ with $\xi_0 \in D \subset \mathbb{R}^2$. Without loss of generality we assume $D = \{|\xi - \xi_0| < \delta\}$ and let $D^{\pm} := D \cap B^{\pm}$, where $B^{\pm}$ are as in (7.6). Let $h \in \mathbb{R}$ be the ϕ-curvature of Σ. Since Σ is ϕ-critical, for any $\varphi \in C_c^{\infty}(D)$ we have

$$\int_D \langle \mathscr{X} f, \nabla\varphi\rangle \, d\xi = -\int_D h\varphi \, d\xi$$

and $\operatorname{div}(\mathscr{X} f) = h$ pointwise in $D^+ \cup D^-$. Then, denoting by N_Ξ the normal to Ξ pointing towards D^-, by the divergence theorem we have

$$\begin{aligned}
\int_D h\varphi \, d\xi &= \int_{D^+} \operatorname{div}(\mathscr{X} f)\varphi \, d\xi + \int_{D^-} \operatorname{div}(\mathscr{X} f)\varphi \, d\xi \\
&= -\int_{D^+\cup D^-} \langle \mathscr{X} f, \nabla\varphi\rangle \, d\xi + \int_\Xi \varphi\langle \mathscr{N}^+, N_\Xi\rangle \, d\mathscr{H}^1 - \int_\Xi \varphi\langle \mathscr{N}^-, N_\Xi\rangle \, d\mathscr{H}^1 \\
&= \int_D h\varphi \, d\xi + \int_\Xi \varphi\langle \mathscr{N}^+ - \mathscr{N}^-, N_\Xi\rangle \, d\mathscr{H}^1.
\end{aligned}$$

By Lemma 7.3, this implies that

$$\int_\Xi \varphi\langle \mathscr{N}^+, N_\Xi\rangle \, d\mathscr{H}^1 = 0$$

and since φ is arbitrary, this yields the claim. □

Remark 7.5. Under the assumptions of the previous proposition, the characteristic curves $\Gamma = (\Xi, \zeta)$ of ∂E are of class C^2. This can be proved exactly as in Proposition 4.20 of [27] using condition (7.8). In particular, Ξ is of class C^2.

7.2.1. *Parametrization of constant ϕ-curvature surfaces around characteristic curves.* In this section, we study a ϕ-critical surface Σ of class C^2 having constant ϕ-curvature $h \neq 0$ near a characteristic curve. Without loss of generality we assume $h > 0$.

We assume ϕ to be normalized in such a way that $\phi(1, 0) = 1$ and we fix a parametrization $\mu : [0, M] \to \mathbb{R}^2$ of C_ϕ such that $\phi^\dagger(\dot\mu) = 1$, $\mu([0, M]) = C_\phi$, with initial and end-point $\mu(0) = \mu(M)$. We choose the clockwise orientation and we extend μ to the whole $\mathbb{R}$ by M-periodicity. We have $\mu \in C^2(\mathbb{R}; \mathbb{R}^2)$ and

$$\mu(\tau) = \nabla\phi^*(\dot\mu(\tau)^\perp), \qquad \text{for all } \tau \in \mathbb{R}. \tag{7.9}$$

In fact, letting $\mathscr{N}(t) = \nabla\phi^*(\dot\mu(t)^\perp)$, we have $\dot{\mathscr{N}} = \dot\mu$ as in (5.3). Equation (7.9) then follows by integration using the fact that 0 is the center of C_ϕ.

Let $\Gamma = (\Xi, \zeta) \in \mathrm{C}^2(I; \Sigma)$ be a characteristic curve parameterized in such a way that

$$\phi(\dot{\Xi}) = 1 \quad \text{on } I. \tag{7.10}$$

Locally, Γ disconnects Σ and there are no other characteristic points of Σ close to Γ, by Lemma 7.1.

According to Corollary 5.4, $\Sigma \setminus \mathscr{C}(\Sigma)$ admits near Γ a Legendre foliation made of horizontal lifts of ϕ-circles of radius $1/h$, followed in the clockwise sense. Hence, given a point $(\xi_0, z_0) \in \Sigma \setminus \mathscr{C}(\Sigma)$ near Γ, there exist $c \in \mathbb{R}^2$ and $\tau \in [0, M]$ such that the horizontal lift of

$$\xi(s) = c + h^{-1}\mu(\tau + hs)$$

passing through (ξ_0, z_0) at $s = 0$ stays in Σ until it meets a characteristic point. Here, c is the center of the ϕ-circle. Notice that $\nabla\phi^*(\dot{\xi}(s)^\perp) = \mathscr{N}(\xi(s))$, so that, by Lemma 7.3 and (7.8), $\nabla\phi^*(\dot{\xi}(0)^\perp)$ converges to a vector collinear to $\dot{\Xi}(t)$ as ξ_0 approaches $\Xi(t)$ for some $t \in I$. By (4.4) and (7.10), $\nabla\phi^*(\dot{\xi}(0)^\perp)$ converges either to $\dot{\Xi}(t)$ or to $-\dot{\Xi}(t)$ as ξ_0 approaches $\Xi(t)$. Since Ξ locally disconnects the plane, we can fix a side from where ξ_0 approaches Ξ and, up to reversing the parameterization of Γ, we can assume that $\nabla\phi^*(\dot{\xi}(0)^\perp)$ converges to $\dot{\Xi}(t)$ as ξ_0 converges to $\Xi(t)$. Thanks to (7.9) and since $\dot{\xi}(0) = \dot{\mu}(\tau)$, we deduce that $\mu(\tau) = \nabla\phi^*(\dot{\xi}(0)^\perp)$ converges to $\dot{\Xi}(t)$ as $\xi_0 \to \Xi(t)$. In particular, the limit direction of $\dot{\xi}(0)$ as $\xi_0 \to \Xi(t)$ is transversal to Ξ.

By local compactness of the set of ϕ-circles with radius $1/h$, the horizontal lift passing through $\Gamma(t)$ at $s = 0$ of a curve $c + h^{-1}\mu(\tau + hs)$ with $\mu(\tau) = \dot{\Xi}(t)$ is a Legendre curve contained in Σ, for s either in a positive or a negative neighborhood of 0. To fix the notations, we assume that s is in a positive neighborhood of 0, the computations being equivalent in the other case. Moreover, there is no other Legendre curve having $\Gamma(t)$ in its closure and whose projection on the xy-plane stays in the chosen side of Ξ, since $\tau \in [0, M)$ and $c \in \mathbb{R}^2$ are uniquely determined by

$$\mu(\tau) = \dot{\Xi}(t), \qquad c = \Xi(t) - h^{-1}\mu(\tau) = \Xi(t) - h^{-1}\dot{\Xi}(t).$$

It is then possible to parameterize locally near Γ one of the two connected components of $\Sigma \setminus \Gamma$ by Legendre curves using the function

$$(t, s) \mapsto \gamma(t, s) = (\xi(t, s), z(t, s)) \tag{7.11}$$

where

$$\xi(t, s) = h^{-1}\mu(\tau(t) + hs) + \Xi(t) - h^{-1}\dot{\Xi}(t), \quad t \in I, \ s > 0, \tag{7.12}$$

with τ uniquely defined via the equation

$$\mu(\tau(t)) = \dot{\Xi}(t), \quad t \in I, \tag{7.13}$$

and z defined by

$$z(t,s) = \zeta(t) + \int_0^s \omega(\xi(t,\sigma), \xi_s(t,\sigma))d\sigma. \tag{7.14}$$

As discussed above, we have

$$\nabla\phi^*(\xi_s(t,0)^\perp) = \dot{\Xi}(t), \tag{7.15}$$

$$\phi^\dagger(\xi_s) = 1. \tag{7.16}$$

For $t \in I$, we define the *characteristic time* $s(t)$ as the first positive time $s > 0$ such $\gamma(t, s(t)) \in \mathscr{C}(\Sigma)$. We will prove later that such a $s(t)$ exists. Finally, we let $S := \{(t,s) : t \in I,\ 0 \le s \le s(t)\}$ and we consider the surface $\gamma(S) \subset \Sigma$.

Lemma 7.6. *We have $\gamma \in \mathrm{C}^1(S;\Sigma)$ with $\gamma(\cdot, 0) = \Gamma$. Moreover, the second order derivatives $\gamma_{ss}, \gamma_{ts}, \gamma_{st}$ are well-defined and*

$$\gamma_{ts} = \gamma_{st}. \tag{7.17}$$

Proof. By (7.12) and (7.14), we see that γ_{ss} exists and that $\xi_{ts} = \xi_{st}$. Moreover,

$$\begin{aligned} z_{st} &= \omega(\xi_t(t,\cdot), \xi_s(t,\cdot)) + \omega(\xi(t,\cdot), \xi_{st}(t,\cdot)) \\ &= \omega(\xi_t(t,\cdot), \xi_s(t,\cdot)) + \omega(\xi(t,\cdot), \xi_{ts}(t,\cdot)) = z_{ts}. \end{aligned}$$

□

On the surface $\gamma(S)$ we consider the vector field

$$V(t,s) := \gamma_t(t,s) = (\xi_t(t,s), z_t(t,s)) \in \mathbb{R}^3. \tag{7.18}$$

It plays the role of the Jacobi vector field V in [27, Lemma 6.2]. The characteristic time $s(t)$ is precisely the first positive time such that $\langle V(s(t),t), Z\rangle_{\mathscr{D}} = 0$. Here, with a slight abuse of notation, $\langle\cdot,\cdot\rangle_{\mathscr{D}}$ denotes the scalar product that makes X, Y, Z orthonormal. The following computation is crucial in what follows. We recall that we are assuming the ϕ-curvature to be a constant $h \neq 0$.

Lemma 7.7. *We have the identity*

$$\langle V(t,s), Z\rangle_{\mathscr{D}} = 2\big[h^{-2}\omega(\ddot{\Xi}, \dot{\Xi}) + \omega(\dot{\Xi} - h^{-1}\ddot{\Xi}, h^{-1}\mu(\tau + hs))\big].$$

Proof. First notice that

$$\langle V, Z\rangle_{\mathscr{D}} = z_t + \omega(\xi_t, \xi), \tag{7.19}$$

where

$$z_t(t,s) = z_t(t,0) + \int_0^s \omega(\xi_t(t,\sigma), \xi_s(t,\sigma))\, d\sigma + \int_0^s \omega(\xi(t,\sigma), \xi_{st}(t,\sigma))\, d\sigma.$$

Using (7.12), (7.13), and the skew-symmetry of ω, the above implies

$$\begin{aligned}
z_t(\cdot,s) &= \omega(\Xi,\dot\Xi) + \int_0^s \omega(\dot\Xi - h^{-1}\ddot\Xi + h^{-1}\dot\tau\dot\mu(\tau+h\sigma), \dot\mu(\tau+h\sigma))\,d\sigma \\
&\quad + \int_0^s \omega(\Xi - h^{-1}\dot\Xi + h^{-1}\mu(\tau+h\sigma), \dot\tau\ddot\mu(\tau+h\sigma))\,d\sigma \\
&= \omega(\Xi,\dot\Xi) + h^{-1}\omega(\dot\Xi - h^{-1}\ddot\Xi, \mu(\tau+hs)-\mu(\tau)) \\
&\quad + h^{-1}\omega(\Xi - h^{-1}\dot\Xi, \dot\tau\dot\mu(\tau+hs) - \dot\tau\dot\mu(\tau)) \\
&\quad + h^{-2}\omega(\mu(\tau+hs),\dot\tau\dot\mu(\tau+hs)) - h^{-2}\omega(\mu(\tau),\dot\tau\dot\mu(\tau)) \\
&= \omega(\Xi,\dot\Xi) + h^{-1}\omega(\dot\Xi - h^{-1}\ddot\Xi, \mu(\tau+hs)) - h^{-1}\omega(\dot\Xi - h^{-1}\ddot\Xi, \dot\Xi) \\
&\quad + h^{-1}\omega(\Xi - h^{-1}\dot\Xi, \dot\tau\dot\mu(\tau+hs)) - h^{-1}\omega(\Xi - h^{-1}\dot\Xi, \ddot\Xi) \\
&\quad + h^{-2}\omega(\mu(\tau+hs),\dot\tau\dot\mu(\tau+hs)) - h^{-2}\omega(\dot\Xi,\ddot\Xi) \\
&= \omega(\Xi,\dot\Xi) - h^{-1}\omega(\Xi,\ddot\Xi) + h^{-2}\omega(\ddot\Xi,\dot\Xi) + \omega(\dot\Xi - h^{-1}\ddot\Xi, h^{-1}\mu(\tau+hs)) \\
&\quad + h^{-1}\omega(\Xi - h^{-1}\dot\Xi + h^{-1}\mu(\tau+hs), \dot\tau\dot\mu(\tau+hs)).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\omega(\xi_t,\xi) &= \omega(\dot\Xi - h^{-1}\ddot\Xi + h^{-1}\dot\tau\dot\mu(\tau+hs), \Xi - h^{-1}\dot\Xi + h^{-1}\mu(\tau+hs)) \\
&= h^{-1}\omega(\dot\tau\dot\mu(\tau+hs), \Xi - h^{-1}\dot\Xi + h^{-1}\mu(\tau+hs)) + \omega(\dot\Xi,\Xi) \\
&\quad - h^{-1}\omega(\ddot\Xi,\Xi) + \omega(\dot\Xi - h^{-1}\ddot\Xi, h^{-1}\mu(\tau+hs)) + h^{-2}\omega(\ddot\Xi,\dot\Xi).
\end{aligned}$$

Summing up, we obtain the claim. □

We show next that for every $t \in I$, the Legendre curve $s \mapsto \gamma(t,s)$ meets a characteristic point before that $\xi(t,s)$ comes back to the point $\xi(t,0) = \Xi(t)$, i.e., $hs(t) < M$.

Lemma 7.8. *For any $t \in I$, there exists $s(t) \in (0, M/h)$ such that $\langle V(t,s(t)), Z\rangle_{\mathscr{D}} = 0$.*

Proof. For fixed t, consider the function $\theta : [0,M] \to \mathbb{R}$, defined by

$$\theta(s) = \omega(\dot\Xi - h^{-1}\ddot\Xi, h^{-1}\mu(\tau+hs)).$$

By Lemma 7.7, we have that $\langle V(t,s), Z\rangle_{\mathscr{D}} = 0$ if and only if $\theta(s) = b$ with $b := h^{-2}\omega(\dot\Xi,\ddot\Xi)$. The equation $\theta(s) = b$ is certainly satisfied for $hs = nM$, $n \in \mathbb{N}$. This follows by the M-periodicity of μ and the fact that $V(t,0) = \dot\Gamma(t)$ is horizontal.

It is enough to consider the case $b \geq 0$, the case $b < 0$ being analogous. By (7.13) we have

$$\dot\theta(0) = \omega(\dot\Xi - h^{-1}\ddot\Xi, \dot\mu(\tau)) = \omega(\mu(\tau),\dot\mu(\tau)).$$

By the fact that C_ϕ is a convex curve around 0, it follows that $\dot\theta(0) \neq 0$.

If $\dot\theta(0) > 0$ there exists $s^* \in (0, M/(2h))$ such that $\theta(s^*) > \theta(0) = b$. In this case, by symmetry of C_ϕ we have $\mu(\tau + h(s^* + M/(2h))) = -\mu(\tau + hs^*)$, thus implying

$\theta(s^*+M/(2h)) = -\theta(s^*) < -b \le 0$. By continuity of θ, we deduce the existence of $\bar{s} \in (0, M/h)$ satisfying $\theta(\bar{s}) = b$. We argue in the same way in the case $\dot{\theta}(0) < 0$. $\square$

We now determine a quantity that remains constant along the Legendre curves $s \mapsto \gamma(t,s)$.

Proposition 7.9. *For any $t \in I$ and for all $s \in [0, s(t)]$ we have*

$$\langle V(t,s), Z\rangle_{\mathscr{D}} + h\langle \nabla\phi^\dagger(\xi_s(t,s)), \xi_t(t,s)\rangle = 0. \tag{7.20}$$

Proof. By (7.19), (7.14) and (7.17), we have

$$\begin{aligned}\frac{\partial}{\partial s}\langle V, Z\rangle_{\mathscr{D}} &= z_{ts} + \omega(\xi_{ts}, \xi) + \omega(\xi_t, \xi_s) = \frac{\partial}{\partial t}\omega(\xi, \xi_s) + \omega(\xi_{st}, \xi) + \omega(\xi_t, \xi_s) \\ &= \omega(\xi_t, \xi_s) + \omega(\xi, \xi_{st}) + \omega(\xi_{st}, \xi) + \omega(\xi_t, \xi_s) = 2\omega(\xi_t, \xi_s). \end{aligned} \tag{7.21}$$

We claim that

$$h\frac{\partial}{\partial s}(\langle \nabla\phi^\dagger(\xi_s), \xi_t\rangle) = 2\omega(\xi_s, \xi_t). \tag{7.22}$$

Indeed, by Theorem 6.2 and Remark 6.3, we have

$$\frac{\partial}{\partial s}\nabla\phi^\dagger(\xi_s) = \mathscr{H}\phi^\dagger(\xi_s)\xi_{ss} = \frac{1}{h}\xi_s^\perp, \tag{7.23}$$

and therefore

$$\frac{\partial}{\partial s}\langle \nabla\phi^\dagger(\xi_s(t,s)), \xi_t(t,s)\rangle = \frac{1}{h}\langle \xi_s^\perp, \xi_t\rangle + \langle \nabla\phi^\dagger(\xi_s), \xi_{st}\rangle$$

On differentiating (7.16) w.r.t. t we see that $\langle \nabla\phi^\dagger(\xi_s), \xi_{st}\rangle = 0$. This is (7.22).

Summing up (7.21) and (7.22), we deduce that the function $\Lambda_t(s) = \langle V(t,s), Z\rangle_{\mathscr{D}} + h\langle \nabla\phi^\dagger(\xi_s(t,s)), \xi_t(t,s)\rangle$ is constant. To conclude the proof it is enough to check that $\Lambda_t(0) = 0$. On the one hand, we have $\langle V(t,0), Z\rangle_{\mathscr{D}} = \langle \dot{\Gamma}(t), Z\rangle_{\mathscr{D}} = 0$, since Γ is horizontal. On the other hand, since $\nabla\phi^\dagger(v) = -\nabla\phi^*(v^\perp)^\perp$ for any $v \ne 0$, using (7.15) we finally obtain

$$\langle \nabla\phi^\dagger(\xi_s(t,0)), \xi_t(t,0)\rangle = -\langle \nabla\phi^*(\xi_s(t,0)^\perp)^\perp, \dot{\Xi}(t)\rangle = 0. \qquad \square$$

Since the set $\Gamma_1 := \{\gamma(t, s(t)) : t \in I\}$ is made of characteristic points, it is either an isolated point or a nontrivial characteristic curve (Lemma 7.1). We will see in the proof of Theorem 1.1, contained in Section 8.2, that if Γ_1 were an isolated characteristic point, then the same would be true for Γ. We stress that the argument leading to such a conclusion does not rely on the characterization of Γ provided in this section. We then have that $\Gamma_1 := \{\gamma(t, s(t)) : t \in I\}$ is a nontrivial characteristic curve.

Proposition 7.10. *The function $t \mapsto s(t)$ is constant.*

Proof. Let $t \in I$. Since $\langle V(t, s(t)), Z\rangle_{\mathscr{D}} = 0$, the point $\gamma(t, s(t))$ is characteristic for Σ. Then, by Lemma 7.1 and Remark 7.5, Γ_1 is a C^2 characteristic curve. By the implicit function theorem, the function $t \mapsto s(t)$ is C^1-smooth and for $t \in I$ we have

$$\dot{\Gamma}_1(t) = V(t, s(t)) + \dot{s}(t)\gamma_s(t, s(t)).$$

The curve Ξ_1 obtained by projecting Γ_1 on the xy-plane then satisfies

$$\dot{\Xi}_1(t) = \xi_t(t, s(t)) + \dot{s}(t)\xi_s(t, s(t)).$$

Since $\gamma(t, s(t)) \in \mathscr{C}(\Sigma)$, by Proposition 7.4, and using the fact that $\nabla\phi^\dagger(v) = -\nabla\phi^*(v^\perp)^\perp$ for any $v \neq 0$ we have

$$\langle \nabla\phi^\dagger(\xi_s(t, s(t))), \dot{\Xi}_1(t)\rangle = -\langle \nabla\phi^*(\xi_s(t, s(t))^\perp)^\perp, \dot{\Xi}_1(t)\rangle = 0.$$

Therefore we obtain

$$0 = \langle \nabla\phi^\dagger(\xi_s(t, s(t))), \xi_t(t, s(t))\rangle + \dot{s}(t)\langle \nabla\phi^\dagger(\xi_s(t, s(t))), \xi_s(t, s(t))\rangle, \tag{7.24}$$

where, by Proposition 7.9,

$$\langle \nabla\phi^\dagger(\xi_s(t, s(t))), \xi_t(t, s(t))\rangle = 0,$$

and moreover, by (7.16),

$$\langle \nabla\phi^\dagger(\xi_s(t, s(t))), \xi_s(t, s(t)) = \phi^\dagger(\xi_s(t, s(t))) = 1.$$

Equation (7.24) thus implies $\dot{s} = 0$, which concludes the proof. □

We are now ready to prove Theorem 7.2.

Proof of Theorem 7.2. Without loss of generality we assume $h > 0$. By Remark 7.5, Γ is of class C^2 and we denote by I an interval of parametrization of $\Gamma = (\Xi, \zeta)$ satisfying (7.10). We consider the parametrization γ given by Lemma 7.6. By Proposition 7.10 the characteristic time $s(t)$ is constant on I and we let $s(t) = \bar{s} \in \mathbb{R}$. Since $\langle V(t, \bar{s}), Z\rangle_{\mathscr{D}} = 0$, by Lemma 7.7 we thus have

$$h^{-2}\omega(\ddot{\Xi}(t), \dot{\Xi}(t)) + \omega(\dot{\Xi}(t) - h^{-1}\ddot{\Xi}(t), h^{-1}\mu(\tau(t) + h\bar{s})) = 0.$$

Using (7.13), the last equation reads

$$\dot{\tau}\omega(\dot{\mu}(\tau), \mu(\tau) - \mu(\tau + h\bar{s})) = h\omega(\mu(\tau + h\bar{s}), \mu(\tau)). \tag{7.25}$$

If the right-hand side is 0 at some $t \in I$, then $\mu(\tau(t))$ and $\mu(\tau(t) + h\bar{s})$ are parallel by definition of ω (cf. (1.2)). Since $h\bar{s} \in (0, M)$ by Lemma 7.8, the only possible choice is $h\bar{s} = M/2$. Plugging such choice into the left-hand side and using the fact that $\mu(\tau + M/2) = -\mu(\tau)$, we obtain

$$2\dot{\tau}\omega(\dot{\mu}(\tau), \mu(\tau)) = 0 \quad \text{on} \quad I.$$

This implies that $\dot{\tau} = 0$ on I and therefore that τ is constant on I. By (7.13) we deduce that $\dot{\Xi}$ is constant on I implying that Ξ is a straight line.

We are now left to consider the case $h\bar{s} \in (0, M)$, $h\bar{s} \neq M/2$, so that $\omega(\mu(\tau(t) + h\bar{s}), \mu(\tau(t))) \neq 0$ for every $t \in I$. Equation (7.25) reads

$$\dot{\tau} = f(\tau) \quad \text{with} \quad f(\tau) := \frac{h\omega(\mu(\tau + h\bar{s}), \mu(\tau))}{\omega(\dot{\mu}(\tau), \mu(\tau) - \mu(\tau + h\bar{s}))}.$$

For the sake of simplicity, assume $0 \in I$. Notice that f is $M/2$-periodic and of class C^1 as a function of τ. Hence, given $\tau_0 \in \mathbb{R}$ satisfying $\mu(\tau_0) = \dot{\Xi}(0)$, there is a unique maximal solution τ to the differential equation with the initial condition $\tau(0) = \tau_0$. Since $h\bar{s} \in (0, M)$, $h\bar{s} \neq M/2$, we have $f(\tau) \neq 0$, yielding that $\dot{\tau}$ has constant sign. To fix the ideas, assume that $\operatorname{sign}(\dot{\tau}) = 1$. Then, there exists $T_0 > 0$ such that $\tau(T_0) = \tau_0 + M/2$. We claim that

$$\tau(t + T_0) = \tau(t) + \frac{M}{2} \qquad \text{for all } t \in \mathbb{R}. \tag{7.26}$$

This follows from the fact that $\tau_1(t) := \tau(T_0 + t)$ and $\tau_2(t) := \tau(t) + M/2$ for $t \in \mathbb{R}$ solve the same Cauchy problem $\dot{\tau}(t) = f(\tau)$, $\tau(0) = \tau_0 + M/2$. Then, by (7.26), M-periodicity of μ, and (7.13), we have for every $t \in \mathbb{R}$

$$\dot{\Xi}(t + 2T_0) = \mu(\tau(t + 2T_0)) = \mu(\tau(t) + M) = \mu(\tau(t)) = \dot{\Xi}(t),$$

i.e., $\dot{\Xi}$ is $2T_0$-periodic. This implies that Ξ is also $2T_0$-periodic. Indeed, for $t \in \mathbb{R}$ we have

$$\begin{aligned}
\Xi(t + 2T_0) - \Xi(t) &= \int_t^{t+2T_0} \dot{\Xi}(\sigma)\, d\sigma = \int_t^{t+2T_0} \mu(\tau(\sigma))\, d\sigma \\
&= \int_t^{t+T_0} \mu(\tau(\sigma))\, d\sigma + \int_t^{t+T_0} \mu(\tau(\sigma + T_0))\, d\sigma \\
&= \int_t^{t+T_0} \mu(\tau(\sigma))\, d\sigma - \int_t^{t+T_0} \mu(\tau(\sigma))\, d\sigma = 0,
\end{aligned}$$

where we have used again the symmetry of C_ϕ and (7.26).

We are left to show that $\Xi(\bar{\sigma}) \neq \Xi(\bar{t})$ for any $0 \leq \bar{\sigma} < \bar{t} < 2T_0$. Assume that $\Xi(\bar{\sigma}) = \Xi(\bar{t})$ for some $0 \leq \bar{\sigma} < \bar{t} \leq 2T_0$. Then we have $0 = \int_{\bar{\sigma}}^{\bar{t}} \dot{\Xi}(t)\, dt = \int_{\bar{\sigma}}^{\bar{t}} \mu(\tau(t))\, dt$. Now, letting $v := \mu(\tau(\bar{\sigma}))$, by the symmetry of C_ϕ the function

$$\sigma \mapsto \int_{\bar{\sigma}}^{\sigma} \langle \mu(\tau(t), v) \rangle\, dt$$

is monotone increasing for $\sigma \in [\bar{\sigma}, \bar{\sigma} + T_0]$ and decreasing for $\sigma \in [\bar{\sigma} + T_0, \bar{\sigma} + 2T_0]$. Hence, the equation $\int_{\bar{\sigma}}^{\bar{t}} \mu(\tau(t))\, dt = 0$ implies $\bar{\sigma} = 0$ and $\bar{t} = 2T_0$. □

7.3. **Characteristic set of isoperimetric sets.** In this section we apply the previous results to the study of the characteristic set of ϕ-isoperimetric sets. As a Corollary of Theorem 7.2 we have the following

Corollary 7.11. *Let ϕ^* be of class C^2 and let $E \subset \mathbb{H}^1$ be a ϕ-isoperimetric set of class C^2. Then $\mathscr{C}(E)$ consists of isolated points. Moreover, for every $p_0 = (\xi_0, z_0) \in \mathscr{C}(E)$ and every f such that $p_0 \in \mathrm{gr}(f) \subset \partial E$, we have* $\mathrm{rank}(JF(\xi_0)) = 2$.

Proof. By Remark 3.2, we know that ∂E is bounded. Therefore we exclude the possibility that $\mathscr{C}(\partial E)$ contains complete (unbounded) lifts of simple curves. □

Lemma 7.12. *Let ϕ^* be of class C^2 and $E \subset \mathbb{H}^1$ be a ϕ-isoperimetric set of class C^2. Let $p_0 \in \mathscr{C}(E)$. There exists $r > 0$ such that for $p \in \partial E \cap B(p_0, r)$, $p \neq p_0$, the maximal horizontal lift of the ϕ-circle in ∂E through p meets p_0.*

Proof. The surface $\partial E \cap B(p_0, r)$ is the z-graph of $f \in \mathrm{C}^2(D)$ and $p_0 = (\xi_0, f(\xi_0))$ with $\mathscr{C}(f) \cap \{|\xi - \xi_0| < r\} = \{\xi_0\}$. Let $\Theta_\xi \subset D$ be the maximal ϕ-circle (integral curve of $F^\perp$) passing though $\xi \in D \setminus \{\xi_0\}$. Notice that the radius of Θ_ξ does not depend on ξ, as it follows from Corollary 5.4. If $\xi_0 \notin \Theta_\xi$, then the normal vector $\mathscr{N}_\xi = \nabla\phi^*(F)$ is continuously defined on Θ_ξ.

Assume that there exists a sequence of such ξ with $\xi \to \xi_0$. By an elementary compactness argument it follows that there exists a ϕ-circle Θ passing through ξ_0 and there exists a normal $\mathscr{N}$ that is continuously defined along Θ and, in particular, through ξ_0. Outside ξ_0 we have $\mathscr{N} = \nabla\phi^*(F)$.

Let $b \in \mathbb{R}^2$ the unit vector tangent to Θ at ξ_0. Then we have

$$F(\xi_0 + tb) = F(\xi_0) + tJF(\xi_0)b + o(t) = tJF(\xi_0)b + o(t),$$

with $JF(\xi_0)b \neq 0$, because $JF(\xi_0)$ has rank 2 by Lemma 7.1. Since $\nabla\phi(-v) = -\nabla\phi(v)$, for $v \in \mathbb{R}^2 \setminus \{0\}$, it follows that

$$\begin{aligned}\lim_{t\to 0^+} \nabla\phi^*(F(\xi_0 + tb)) &= \nabla\phi^*(JF(\xi_0)b),\\ \lim_{t\to 0^-} \nabla\phi^*(F(\xi_0 + tb)) &= -\nabla\phi^*(JF(\xi_0)b).\end{aligned}$$

This contradicts the continuity of $\mathscr{N}$ along Θ at ξ_0. □

8. Classification of ϕ-isoperimetric sets of class C^2

8.1. **Construction of ϕ-bubbles.** Let ϕ be a norm in $\mathbb{R}^2$ that we normalize by $\phi(1, 0) = 1$. For $\xi_0 \in \mathbb{R}^2$ and $r > 0$, ϕ-circles are defined in (1.5) and we let the *ϕ-disk* of radius r and center ξ_0 be

$$D_\phi(\xi_0, r) = \{\xi \in \mathbb{R}^2 : \phi(\xi - \xi_0) < r\}.$$

We also let $C_\phi(r) = C_\phi(0, r)$, $C_\phi = C_\phi(1)$ and $D_\phi(r) = D_\phi(0, r)$, $D_\phi = D_\phi(1)$.

The circle C_ϕ is a Lipschitz curve and we denote by $L = L_\phi > 0$ its Euclidean length. We parametrize C_ϕ by arc-length through $\kappa \in \mathrm{Lip}\big([0, L]; \mathbb{R}^2\big)$ such that $\kappa([0, L]) = C_\phi$ with initial and end-point $\kappa(0) = \kappa(L) = (-1, 0)$. We choose the anti-clockwise orientation and we extend κ to the whole $\mathbb{R}$ by L-periodicity. Then we have $\kappa \in \mathrm{Lip}(\mathbb{R};\mathbb{R}^2)$.

The map $\xi : \mathbb{R}^2 \to \mathbb{R}^2$, $\xi(t,\tau) = \kappa(t) + \kappa(\tau)$, is in $\mathrm{Lip}(\mathbb{R}^2;\mathbb{R}^2)$. We restrict ξ to the domain

$$D = \big\{(t,\tau) \in \mathbb{R}^2 : \tau \in [0, L], t \in [\tau + L/2, \tau + 3L/2]\big\}.$$

Notice that $\xi(\tau + L/2, \tau) = \xi(\tau + 3L/2, \tau) = 0$ for any $\tau \in [0, L]$. We define the function $z \in \mathrm{Lip}(D)$,

$$z(t,\tau) = \int_{\tau+L/2}^{t} \omega\big(\xi(s,\tau), \xi_s(s,\tau)\big)ds. \tag{8.1}$$

The map $\Phi : D \to \mathbb{R}^3$ defined by $\Phi = (\xi, z)$ is Lipschitz continuous. Moreover, Φ is C^k if ϕ is C^k.

We define the Lipschitz surface $\Sigma_\phi = \Phi(D) \subset \mathbb{R}^3$ and call $S = \Phi(\tau + L/2, \tau) = 0 \in \Sigma_\phi$ the south pole of Σ_ϕ and $N = \Phi(\tau + 3L/2, \tau) = (0, 0, z(\tau + 3L/2, \tau))$ the north pole. We call the bounded region $E_\phi \subset \mathbb{R}^3$ enclosed by Σ_ϕ the ϕ-bubble. E_ϕ is a topological ball and it is the candidate solution to the ϕ-isoperimetric problem. When ϕ is the Euclidean norm in the plane, the set E_ϕ is the well-known Pansu's ball.

8.2. **Classification of ϕ-isoperimetric sets of class C^2.** We are ready to prove the main theorem of the paper.

Proof of Theorem 1.1. The set E is bounded and connected, by Remark 3.2. We may also assume that it is open. It follows from Corollary 5.4 (and from the analogous result for x-graphs and y-graphs based on Remark 5.3) that, out of the characteristic set $\mathscr{C}(E)$, the surface ∂E is foliated by horizontal lifts of ϕ-circles. Then $\mathscr{C}(E)$ contains at least one point, since otherwise, ∂E would contain an unbounded curve, contradicting the boundedness of E.

Let $f \in \mathrm{C}^2(D)$, with $D \subset \mathbb{R}^2$ open, be a maximal function such that $\mathrm{gr}(f) \subset \partial E$ and $\mathscr{C}(f) \neq \emptyset$. We may assume that $0 \in \mathscr{C}(f)$, $f(0) = 0$ and that E lies above the graph of f near 0. Around the characteristic point 0, the function f must have the structure described in Lemma 7.12. It follows that, up to a dilation, we have $\mathrm{gr}(f) \subset \partial E_\phi$.

The maximal domain for f must be $D = D_\phi(2)$. Otherwise, at each point $\xi \in \partial D \setminus \partial D_\phi(2)$ the space $T_{(\xi,f(\xi))}\partial E = T_{(\xi,f(\xi))}\partial E_\phi$ is not vertical, contradicting the maximality of D. This shows that the graph of f is the 'lower hemisphere' of ∂E_ϕ.

Up to extending f by continuity to ∂D, we have $(\xi, f(\xi)) \notin \mathscr{C}(E)$ for each $\xi \in \partial D$. Hence there exists a ϕ-circle passing through 0 whose horizontal lift stays in ∂E and passes through $(\xi, f(\xi))$. The collection of all the maximal extensions of such horizontal lifts completes the upper hemisphere of ∂E_ϕ, thus implying that $\partial E_\phi \subset \partial E$. Moreover, since ∂E is C^2, we deduce that ∂E_ϕ is a connected component of ∂E.

In conclusion we have proved that ∂E is the finite union of boundaries of ϕ-bubbles having the same curvature. By connectedness of E this concludes the proof. $\square$

In general, ϕ-bubbles are not of class C^2 and not even of class C^1, e.g., in the case of a crystalline norm. Even when ϕ is regular, there may be a loss of regularity at the poles of E_ϕ.

8.3. **Regularity of ϕ-bubbles.** We first show that ϕ-bubbles have the same regularity as ϕ outside the poles.

Lemma 8.1. *If ϕ is strictly convex and of class C^k, for some $k \geq 1$, then the set $\Sigma_\phi \setminus \{S, N\}$ is an embedded surface of class C^k.*

Proof. If the Jacobian of Φ has rank 2 at the point $(t,\tau) \in D$, then Σ_ϕ is an embedded surface of class C^k around the point $\Phi(t,\tau)$. A sufficient condition for this is $\det J\xi(t,\tau) \neq 0$. The Jacobian of $\xi : D \to \mathbb{R}^2$ satisfies

$$\det J\xi(t,\tau) = 0 \quad \text{if and only if} \quad \dot\kappa(t) = \pm\dot\kappa(\tau).$$

The case $\dot\kappa(t) = -\dot\kappa(\tau)$ is equivalent to $\kappa(t) = -\kappa(\tau)$, by the strict convexity of the norm. This is in turn equivalent to $t = \tau + L/2$ or $t = \tau + 3L/2$. In the former case we have $\Phi(t,\tau) = S$, in the latter $\Phi(t,\tau) = N$.

We are left to consider the case $\dot\kappa(t) = \dot\kappa(\tau)$. By strict convexity of ϕ, this implies $\kappa(t) = \kappa(\tau)$, that is equivalent to $t = \tau + L$. In this case, we have $\xi(t,\tau) = 2\kappa(\tau) \in C_\phi(2)$ The point $\Phi(t,\tau)$ is on the 'equator' of Σ_ϕ.

We study the regularity of Σ_ϕ at points $\Phi(\tau + L, \tau)$. The height $z(\tau + L, \tau)$ does not depend on τ because it is half the area of the disk D_ϕ. It follows that $0 = \partial_\tau\big(z(\tau + L, \tau)\big) = z_t(\tau + L, \tau) + z_\tau(\tau + L, \tau)$ and this implies that

$$z_t(\tau + L, \tau) \neq z_\tau(\tau + L, \tau), \tag{8.2}$$

as soon as we prove that the left-hand side does not vanish. Indeed, differentiating (8.1) we obtain

$$z_t(\tau + L, \tau) = 2\omega\big(\kappa(\tau), \dot\kappa(\tau)\big) \neq 0,$$

because $\kappa(\tau)$ and $\dot\kappa(\tau)$ are not proportional.

From $\dot\kappa(\tau + L) = \dot\kappa(\tau) \neq 0$ and (8.2), we deduce that the Jacobian matrix $J\Phi(\tau + L, \tau)$ has rank 2. This shows that Σ_ϕ is of class C^k also around the 'equator'. $\square$

The regularity of Σ_ϕ at the poles is much more subtle. We study the problem in Theorem 1.3, whose proof is presented below.

Proof of Theorem 1.3. We study the regularity at the south pole. By Lemma 8.1 there exists a function $f \in \mathrm{C}^2(D_\phi(2) \setminus \{0\})$ such that the graph of f is the lower hemisphere of Σ_ϕ without the south pole. We shall show that f can be extended to a function $f \in \mathrm{C}^2(D_\phi(2))$ satisfying $\nabla f(0) = 0$ and $\mathscr{H} f(0) = 0$. Here and in the sequel, we denote by $\mathscr{H} f$ the Hessian matrix of f. Differentiating the identity

$$z(t,\tau) = f(\xi(t,\tau)), \qquad \tau \in [0,L],\ t \in (\tau + L/2, \tau + L),$$

we find the identities

$$z_t(t,\tau) = \langle \nabla f, \dot\kappa(t)\rangle, \tag{8.3}$$

$$z_\tau(t,\tau) = \langle \nabla f, \dot\kappa(\tau)\rangle, \tag{8.4}$$

$$z_{tt}(t,\tau) = \langle \mathscr{H} f \dot\kappa(t), \dot\kappa(t)\rangle + \langle \nabla f, \ddot\kappa(t)\rangle, \tag{8.5}$$

$$z_{\tau\tau}(t,\tau) = \langle \mathscr{H} f \dot\kappa(\tau), \dot\kappa(\tau)\rangle + \langle \nabla f, \ddot\kappa(\tau)\rangle, \tag{8.6}$$

$$z_{t\tau}(t,\tau) = z_{\tau t}(t,\tau) = \langle \mathscr{H} f \dot\kappa(t), \dot\kappa(\tau)\rangle, \tag{8.7}$$

where, above and in the following, $\mathscr{H} f$ and ∇f are evaluated at $\xi(t,\tau)$.

On the other hand, from (8.1) we compute the derivatives

$$z_t(t,\tau) = \omega(\kappa(t) + \kappa(\tau), \dot\kappa(t)), \tag{8.8}$$

$$z_\tau(t,\tau) = \omega(\dot\kappa(\tau), \kappa(t) + \kappa(\tau)), \tag{8.9}$$

$$z_{tt}(t,\tau) = \omega(\kappa(t) + \kappa(\tau), \ddot\kappa(t)), \tag{8.10}$$

$$z_{\tau\tau}(t,\tau) = \omega(\ddot\kappa(\tau), \kappa(t) + \kappa(\tau)), \tag{8.11}$$

$$z_{t\tau}(t,\tau) = \omega(\dot\kappa(\tau), \dot\kappa(t)). \tag{8.12}$$

In formulas (8.3)–(8.12), we will replace $\kappa(\tau), \dot\kappa(\tau)$, and $\ddot\kappa(\tau)$ with their Taylor expansions at the point $t - L/2$.

By assumption, the arc-length parameterization of the circle C_ϕ satisfies $\kappa \in \mathrm{C}^4(\mathbb{R};\mathbb{R}^2)$ and

$$\ddot\kappa(t) = \lambda(t)\dot\kappa(t)^\perp, \quad t \in [0,L], \tag{8.13}$$

for a function (the curvature) $\lambda \in \mathrm{C}^2(\mathbb{R})$ that is L-periodic and strictly positive. So there exist $0 < \lambda_0 \le \Lambda_0 < \infty$ such that

$$0 < \lambda_0 \le \lambda \le \Lambda_0, \quad |\dot\lambda| \le \Lambda_0, \quad |\ddot\lambda| \le \Lambda_0.$$

The third and fourth derivatives of κ have the representation:

$$\kappa^{(3)} = \dot\lambda\dot\kappa^\perp - \lambda^2\dot\kappa \qquad \text{and} \qquad \kappa^{(4)} = (\ddot\lambda - \lambda^3)\dot\kappa^\perp - 3\lambda\dot\lambda\dot\kappa. \tag{8.14}$$

In the following, we let $\delta = t - \tau - L/2 > 0$. The third order Taylor expansion for $\kappa(\tau)$ at $t - L/2$ is

$$\begin{aligned}
\kappa(\tau) &= \kappa(t-L/2) - \delta\dot\kappa(t-L/2) + \frac{\delta^2}{2}\ddot\kappa(t-L/2) - \frac{\delta^3}{6}\kappa^{(3)}(t-L/2) + o(\delta^3) \\
&= -\kappa(t) + \delta\dot\kappa(t) - \frac{\delta^2}{2}\ddot\kappa(t) + \frac{\delta^3}{6}\kappa^{(3)}(t) + o(\delta^3) \\
&= -\kappa(t) + \delta\dot\kappa(t) - \frac{\delta^2}{2}\lambda(t)\dot\kappa(t)^\perp + \frac{\delta^3}{6}(\dot\lambda(t)\dot\kappa(t)^\perp - \lambda(t)^2\dot\kappa(t)) + o(\delta^3). \qquad (8.15)
\end{aligned}$$

Hereafter, when not explicit, the functions κ and λ and their derivatives are evaluated at t. The little-o remainders are uniform with respect to the base point $t - L/2$. By a similar computation, using (8.14) we also obtain

$$\dot\kappa(\tau) = -\dot\kappa + \delta\lambda\dot\kappa^\perp - \frac{\delta^2}{2}\big(\dot\lambda\dot\kappa^\perp - \lambda^2\dot\kappa\big) + \frac{\delta^3}{6}(\ddot\lambda - \lambda^3)\dot\kappa^\perp - \frac{\delta^3}{2}\lambda\dot\lambda\dot\kappa + o(\delta^3), \qquad (8.16)$$

$$\ddot\kappa(\tau) = -\lambda\dot\kappa^\perp + \delta(\dot\lambda\dot\kappa^\perp - \lambda^2\dot\kappa) - \frac{\delta^2}{2}\big((\ddot\lambda - \lambda^3)\dot\kappa^\perp - 3\lambda\dot\lambda\dot\kappa\big) + o(\delta^2). \qquad (8.17)$$

We are ready to start the proof. We will use the identities

$$\omega(\dot\kappa, \dot\kappa^\perp) = -\omega(\dot\kappa^\perp, \dot\kappa) = \frac{1}{2}. \qquad (8.18)$$

Recall our notation $\delta = t - \tau - L/2$.

Step 1. We claim that there exists $C > 0$ such that

$$|\nabla f(\xi(t,\tau))| \le C\delta^2 \quad \text{for all } \tau \in [0, L],\ t \in (\tau + L/2, \tau + L). \qquad (8.19)$$

This estimate implies that f can be extended to a function $f \in \mathrm{C}^1(D_\phi(2))$ satisfying $\nabla f(0) = 0$.

Inserting (8.15) and (8.16) into (8.8) and (8.9) yields

$$z_t(t,\tau) = \omega\big((-\tfrac{\delta^2}{2}\lambda + \tfrac{\delta^3}{6}\dot\lambda)\dot\kappa(t)^\perp, \dot\kappa(t)\big) = \frac{\delta^2}{4}\lambda - \frac{\delta^3}{12}\dot\lambda + o(\delta^3), \qquad (8.20)$$

$$\begin{aligned}
z_\tau(t,\tau) &= \omega\big((-1 + \tfrac{\delta^2}{2}\lambda^2)\dot\kappa, (-\tfrac{\delta^2}{2}\lambda + \tfrac{\delta^3}{6}\dot\lambda)\dot\kappa^\perp\big) + \omega\big((\delta\lambda - \tfrac{\delta^2}{2}\dot\lambda)\dot\kappa^\perp, (\delta - \tfrac{\delta^3}{6}\lambda^2)\dot\kappa\big) + o(\delta^3) \\
&= -\frac{\delta^2}{4}\lambda + \frac{\delta^3}{6}\dot\lambda + o(\delta^3). \qquad (8.21)
\end{aligned}$$

Now, plugging (8.16) and (8.21) into (8.4) and then using (8.3) and (8.20) we obtain

$$\begin{aligned}
-\frac{\delta^2}{4}\lambda + \frac{\delta^3}{6}\dot\lambda &= \langle \nabla f, \dot\kappa(t)\rangle \left(-1 + \tfrac{\delta^2}{2}\lambda^2 - \tfrac{\delta^3}{2}\lambda\dot\lambda\right) \\
&\quad + \langle \nabla f, \dot\kappa(t)^\perp\rangle \left(\delta\lambda - \tfrac{\delta^2}{2}\dot\lambda + \tfrac{\delta^3}{6}(\ddot\lambda - \lambda^3)\right) + o(\delta^3) \\
&= -\frac{\delta^2}{4}\lambda + \frac{\delta^3}{12}\dot\lambda + \langle \nabla f, \dot\kappa(t)^\perp\rangle \left(\delta\lambda - \tfrac{\delta^2}{2}\dot\lambda + \tfrac{\delta^3}{6}(\ddot\lambda - \lambda^3)\right) + o(\delta^3).
\end{aligned}$$

Dividing the last equation by $\lambda\delta > 0$, we get

$$\langle \nabla f, \dot\kappa^\perp \rangle = \frac{\delta^2}{12}\frac{\dot\lambda}{\lambda} + o(\delta^2). \tag{8.22}$$

Thus, there exists $C > 0$ such that

$$\big|\langle \nabla f, \dot\kappa^\perp \rangle\big| \le C\delta^2.$$

On the other hand, by (8.3) and (8.20), possibly changing $C > 0$ we also obtain

$$\big|\langle \nabla f, \dot\kappa \rangle\big| = |z_t(t,\tau)| \le C\delta^2,$$

thus yielding (8.19).

Step 2. We claim that the norm of the Hessian matrix $\mathscr{H}f$ satisfies

$$|\mathscr{H}f(\xi(t,\tau))| = o(1) \quad \text{for } \tau \in [0,L],\ t \in (\tau + L/2, \tau + L), \tag{8.23}$$

where $o(1) \to 0$ as $\delta = t - \tau - L/2 \to 0$. This implies that f can be extended to a function $f \in \mathrm{C}^2(D_\phi(2))$ satisfying $\mathscr{H}f(0) = 0$.

Plugging (8.10) and (8.15) into (8.5), and then using (8.13), (8.18), and (8.22) yields

$$\begin{aligned}
\langle \mathscr{H}f\dot\kappa, \dot\kappa \rangle &= z_{tt}(t,\tau) - \langle \nabla f, \ddot{k} \rangle = \omega\big(\kappa(t) + \kappa(\tau), \ddot\kappa\big) - \langle \nabla f, \ddot\kappa \rangle \\
&= \omega\big((\delta - \tfrac{\delta^3}{6}\lambda^2)\dot\kappa + (-\tfrac{\delta^2}{2}\lambda + \tfrac{\delta^3}{6}\dot\lambda)\dot\kappa^\perp, \lambda\dot\kappa^\perp\big) - \langle \nabla f, \lambda\dot\kappa^\perp \rangle + o(\delta^3) \\
&= \frac{\delta}{2}\lambda - \frac{\delta^2}{12}\dot\lambda + o(\delta^2).
\end{aligned} \tag{8.24}$$

On the other hand, plugging (8.16) into (8.7), and then using (8.24) we get

$$\begin{aligned}
z_{t\tau}(t,\tau) &= \langle \mathscr{H}f\dot\kappa(t), \dot\kappa(t) \rangle \left(-1 + \frac{\delta^2}{2}\lambda^2\right) + \langle \mathscr{H}f\dot\kappa, \dot\kappa^\perp \rangle \left(\delta\lambda - \frac{\delta}{2}\dot\lambda\right) + o(\delta^2) \\
&= -\frac{\delta}{2}\lambda + \frac{\delta^2}{12}\dot\lambda + \langle \mathscr{H}f\dot\kappa, \dot\kappa^\perp \rangle \left(\delta\lambda - \frac{\delta}{2}\dot\lambda\right) + o(\delta^2),
\end{aligned}$$

while from (8.12), (8.16) and (8.18) we get

$$z_{t\tau}(t,\tau) = -\frac{1}{2}\left(\delta\lambda - \frac{\delta^2}{2}\dot\lambda\right) + o(\delta^2).$$

Therefore we obtain the identity

$$\begin{aligned}
\left(\delta\lambda - \frac{\delta^2}{2}\dot\lambda\right)\langle \mathscr{H}f\dot\kappa, \dot\kappa^\perp \rangle &= z_{t\tau}(t,\tau) + \frac{\delta}{2}\lambda - \frac{\delta^2}{12}\dot\lambda + o(\delta^2) \\
&= -\frac{1}{2}\big(\delta\lambda - \frac{\delta^2}{2}\dot\lambda\big) + \frac{\delta}{2}\lambda - \frac{\delta^2}{12}\dot\lambda + o(\delta^2) \\
&= \frac{\delta^2}{6}\dot\lambda + o(\delta^2),
\end{aligned}$$

and dividing by $\lambda\delta > 0$ we get

$$\langle \mathscr{H} f \dot\kappa, \dot\kappa^\perp\rangle = \frac{\delta}{6}\frac{\dot\lambda}{\lambda} + o(\delta). \tag{8.25}$$

By symmetry of the Hessian matrix, we also have

$$\langle \mathscr{H} f \dot\kappa^\perp, \dot\kappa\rangle = \frac{\delta}{6}\frac{\dot\lambda}{\lambda} + o(\delta). \tag{8.26}$$

We are left to estimate $\langle \mathscr{H} f \dot\kappa^\perp, \dot\kappa^\perp\rangle$. By (8.17), (8.22), (8.3), and (8.20) we obtain

$$\begin{aligned}\langle \nabla f, \ddot\kappa(\tau)\rangle &= (-\lambda + \delta\dot\lambda - \tfrac{\delta^2}{2}(\ddot\lambda - \lambda^3)\langle \nabla f, \dot\kappa^\perp\rangle + (-\delta\lambda^2 + \tfrac{3}{2}\delta^2\lambda\dot\lambda)\langle \nabla f, \dot\kappa\rangle + o(\delta^2)\\ &= (-\lambda + \delta\dot\lambda - \tfrac{\delta^2}{2}(\ddot\lambda - \lambda^3))\tfrac{\delta^2}{12}\tfrac{\dot\lambda}{\lambda} + (-\delta\lambda^2 + \tfrac{3}{2}\delta^2\lambda\dot\lambda)\tfrac{\delta^2}{4}\lambda + o(\delta^2)\\ &= -\frac{\delta^2}{12}\dot\lambda + o(\delta^2).\end{aligned} \tag{8.27}$$

On the other hand, by (8.16), (8.24), (8.25), (8.26) we have

$$\begin{aligned}\langle \mathscr{H} f \dot\kappa(\tau), \dot\kappa(\tau)\rangle &= (-1 + \tfrac{\delta^2}{2}\lambda^2)^2\langle \mathscr{H} f\dot\kappa, \dot\kappa\rangle + (\delta\lambda - \tfrac{\delta^2}{2}\dot\lambda)^2\langle \mathscr{H} f \dot\kappa^\perp, \dot\kappa^\perp\rangle)\\ &\quad + 2(-1 + \tfrac{\delta^2}{2}\lambda^2)(\delta\lambda - \tfrac{\delta^2}{2}\dot\lambda)\langle \mathscr{H} f\dot\kappa, \dot\kappa^\perp\rangle + o(\delta^2)\\ &= (-1 + \tfrac{\delta^2}{2}\lambda^2)^2(\tfrac{\delta}{2}\lambda - \tfrac{\delta^2}{12}\dot\lambda + o(\delta^2)) + (\delta\lambda - \tfrac{\delta^2}{2}\dot\lambda)^2\langle \mathscr{H} f\dot\kappa^\perp, \dot\kappa^\perp\rangle\\ &\quad + 2(-1 + \tfrac{\delta^2}{2}\lambda^2)(\delta\lambda - \tfrac{\delta^2}{2}\dot\lambda)(\tfrac{\delta}{6}\tfrac{\dot\lambda}{\lambda} + o(\delta))\\ &= \frac{\delta}{2}\lambda - \frac{5}{12}\delta^2\dot\lambda + \delta^2\lambda^2\langle \mathscr{H} f\dot\kappa^\perp, \dot\kappa^\perp\rangle + o(\delta^2).\end{aligned} \tag{8.28}$$

Plugging (8.27) and (8.28) into (8.6) we get

$$\begin{aligned}z_{\tau\tau}(t,\tau) &= \frac{\delta}{2}\lambda - \frac{5}{12}\delta^2\dot\lambda + \delta^2\lambda^2\langle \mathscr{H} f\dot\kappa^\perp, \dot\kappa^\perp\rangle - \frac{\delta^2}{12}\dot\lambda + o(\delta^2)\\ &= \frac{\delta}{2}\lambda - \frac{\delta^2}{2}\dot\lambda + \delta^2\lambda^2\langle \mathscr{H} f\dot\kappa^\perp, \dot\kappa^\perp\rangle + o(\delta^2).\end{aligned} \tag{8.29}$$

Moreover, plugging (8.15) and (8.17) into (8.11), and using (8.18), we get

$$z_{\tau\tau}(t,\tau) = \omega\big(-\delta\lambda^2\dot\kappa + (-\lambda + \delta\dot\lambda)\dot\kappa^\perp, \delta\dot\kappa - \tfrac{\delta^2}{2}\lambda\dot\kappa^\perp\big) + o(\delta^2) = -\frac{\delta}{2}(-\lambda + \delta\dot\lambda) + o(\delta^2). \tag{8.30}$$

Comparing (8.29) and (8.30) we therefore obtain

$$\delta^2\lambda^2\langle \mathscr{H} f\dot\kappa^\perp, \dot\kappa^\perp\rangle = -\frac{\delta}{2}\lambda + \frac{\delta^2}{2}\dot\lambda - \frac{\delta}{2}(-\lambda + \delta\dot\lambda) + o(\delta^2) = o(\delta^2).$$

This yields $\langle \mathscr{H} f\dot\kappa^\perp, \dot\kappa^\perp\rangle = o(1)$ as $\delta \to 0$. Together with (8.24), (8.25), and (8.26) this implies (8.23) and concludes the proof of the theorem. □

9. The isoperimetric problem for general norms

In the case of crystalline norms, the first order necessary conditions satisfied by an isoperimetric set are not sufficient to reconstruct its structure, even assuming sufficient regularity. In this section we show that the ϕ-isoperimetric problem for a general norm – in particular for a crystalline norm – can be approximated by the isoperimetric problem for smooth norms.

By Theorem 1.3, we know that if ϕ is of class C^∞_+ then the ϕ-bubble E_ϕ is of class C^2. In this section, we show that the validity of Conjecture 1.4 implies the ϕ-isoperimetric property for the ϕ-bubble of any (crystalline) norm.

9.1. Smooth approximation of norms in the plane.

We start with the mollification of a norm.

Proposition 9.1. *Let ϕ be a norm in $\mathbb{R}^2$. Then, for any $\varepsilon > 0$ there exists a norm ϕ_ε of class C^∞_+ with dual norm of class C^∞, such that for all $\xi \in \mathbb{R}^2$ we have*

$$(1-\eta(\varepsilon))\phi_\varepsilon(\xi) \leq \phi(\xi) \leq (1+\eta(\varepsilon))\phi_\varepsilon(\xi), \tag{9.1}$$

and $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0^+$.

Proof. For $\varepsilon > 0$, we introduce the smooth mollifiers $\varrho_\varepsilon : \mathbb{R} \to \mathbb{R}$, supported in $[-\varepsilon\pi, \varepsilon\pi]$ defined by

$$\varrho_\varepsilon(t) = \begin{cases} c_\varepsilon \exp\left(\frac{\pi^2\varepsilon^2}{t^2-\pi^2\varepsilon^2}\right) & \text{if } |t| < \pi, \\ 0 & \text{if } |t| \geq \pi, \end{cases}$$

where c_ε is chosen in such a way that $\int_{\mathbb{R}} \varrho_\varepsilon(t)\,dt = 1$. Following [9, 21], we define the function $\psi_\varepsilon : \mathbb{R}^2 \to [0,\infty)$ letting

$$\psi_\varepsilon(\xi) := \int_{\mathbb{R}} \varrho_\varepsilon(t)\phi(R_t\xi)\,dt,$$

where R_t denotes the anti-clockwise rotation matrix of angle t. The function ψ_ε is a C^∞ norm. On the circle $\mathbb{S}^1 = \{\xi \in \mathbb{R}^2 : |\xi| = 1\}$, the norms ψ_ε converge uniformly to ϕ as $\varepsilon \to 0^+$. So our claim (9.1) with $\eta(\varepsilon) \to 0$ holds with ψ_ε replacing ϕ_ε, by the positive 1-homogeneity of norms.

We let $\phi_\varepsilon : \mathbb{R}^2 \to [0,\infty)$ be defined by

$$\phi_\varepsilon(\xi) := \sqrt{\psi_\varepsilon(\xi)^2 + \varepsilon|\xi|^2}, \quad \xi \in \mathbb{R}^2.$$

This is a C^∞ norm in $\mathbb{R}^2$ and (9.1) is satisfied with $\eta(\varepsilon) \to 0$. The unit ϕ_ε-circle centered at the origin is the 0-level set of the function

$$F_\varepsilon(\xi) = \psi_\varepsilon^2(\xi) + \varepsilon|\xi|^2 - 1, \quad \xi \in \mathbb{R}^2.$$

Since the Hessian matrix of the squared Euclidean norm is proportional to the identity matrix I_2 and ψ_ε^2 is convex, we have that $\mathscr{H}F_\varepsilon \geq 2\varepsilon I_2$ in the sense of matrices. Then the curvature λ_ε of a unit ϕ_ε-circle satisfies

$$\lambda_\varepsilon = \frac{\langle \mathscr{H}F_\varepsilon \nabla F_\varepsilon^\perp, \nabla F_\varepsilon^\perp \rangle}{|\nabla F_\varepsilon|^3} \geq \frac{2\varepsilon}{|\nabla F_\varepsilon|} > 0.$$

The proof that the dual norm of a norm of class C^∞_+ is itself of class C^∞ is standard and we omit it. □

9.2. **Crystalline ϕ-bubbles as limits of smooth isoperimetric sets.** Let ϕ be any norm in $\mathbb{R}^2$ and let $\{\phi_\varepsilon\}_{\varepsilon>0}$ be the smooth approximating norms found in Proposition 9.1.

Given a Lebesgue measurable set $F \subset \mathbb{R}^2$, from (9.1) and from the definition of perimeter (Definition 2.1), we have

$$(1-\eta(\varepsilon))\mathscr{P}_\phi(F) \leq \mathscr{P}_{\phi_\varepsilon}(F) \leq (1+\eta(\varepsilon))\mathscr{P}_\phi(F). \tag{9.2}$$

The ϕ_ε-circles C_{ϕ_ε} converge in Hausdorff distance to the circle C_ϕ. This implies that the ϕ_ε-bubbles E_{ϕ_ε} converge in the Hausdorff distance to the limit bubble E_ϕ. This in turn implies the convergence in $L^1(\mathbb{H}^1)$, namely,

$$\lim_{\varepsilon\to 0^+} \mathscr{L}^3(E_{\phi_\varepsilon} \Delta E_\phi) = 0, \tag{9.3}$$

where Δ denotes the symmetric difference of sets.

Proof of Theorem 1.5. Let $F \subset \mathbb{H}^1$ be any Lebesgue measurable set with $0 < \mathscr{L}^3(F) < \infty$. Assuming the validity of Conjecture 6.4, E_{ϕ_ε} is isoperimetric for any $\varepsilon > 0$. So using twice (9.2) we find

$$\mathrm{Isop}_\phi(F) \geq \frac{\mathrm{Isop}_{\phi_\varepsilon}(F)}{1+\eta(\varepsilon)} \geq \frac{\mathrm{Isop}_{\phi_\varepsilon}(E_{\phi_\varepsilon})}{1+\eta(\varepsilon)} \geq \frac{1-\eta(\varepsilon)}{1+\eta(\varepsilon)} \mathrm{Isop}_\phi(E_{\phi_\varepsilon}).$$

By the lower semicontinuity of the perimeter with respect to the L^1 convergence and from (9.3), we deduce that

$$\liminf_{\varepsilon\to 0^+} \mathrm{Isop}_\phi(E_{\phi_\varepsilon}) \geq \mathrm{Isop}_\phi(E_\phi),$$

and using the fact that $\eta(\varepsilon) \to 0$ we conclude that $\mathrm{Isop}_\phi(F) \geq \mathrm{Isop}_\phi(E_\phi)$. □

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