LUSTERNIK-SCHNIRELMAN AND MORSE THEORY FOR THE VAN DER WAALS-CAHN-HILLIARD EQUATION WITH VOLUME CONSTRAINT

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ABSTRACT. We give a multiplicity result for solutions of the Van der Waals-Cahn-Hilliard two phase transition equation with volume constraints on a closed Riemannian manifold. Our proof employs some results from the classical Lusternik–Schnirelman and Morse theory, together with a technique, the so-called *photography method*, which allows us to obtain lower bounds on the number of solutions in terms of topological invariants of the underlying manifold. The setup for the photography method employs recent results from Riemannian isoperimetry for small volumes.

CONTENTS

1. INTRODUCTION

The Van der Waals-Cahn-Hilliard two phase transition equation (2.1) has attracted the interest of physicists, analysts, and geometers. This is a variational equation, obtained as the Euler-Lagrange equation of the energy functional E_ε defined in equation (2.6), which has the classical form of a kinetic term plus a double well potential. Historically, the energy E_ε was already proposed in 1898 by Van der Waals in [vdW88] for the transition liquid-vapor phase. In 1958, J. W. Cahn and J.E. Hilliard in [CH58] used it to model the transition of phase in some binary alloy. Ginzburg-Landau used the same functional to model ferromagnetic behaviour of materials [Pre09]. A quick search through the literature shows the ubiquitous nature of this equation. For instance, for applications to Biology the interested reader can consult the book of Murray [Mur09].

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From the theoretical physics viewpoint, various attempts were made to derive equation (2.1) as limit of some microscopically consistent model of statistical mechanics, however, this goal is yet to be completely achieved. Along this line, in the papers [GL97], [GL98] a more complicated non-local equation is derived from a microscopical model whose properties have analogies to the properties of equation (2.1), and which constitutes a first order approximation. Let us also mention the paper [BLO97], in which (2.1) is obtained as the hydrodynamic limit of the Ginzburg Landau equation. For a complete account of statistical mechanics studies of these problems the interested reader can also consider the book [Pre09].

A quite complete mathematical analysis of the positive functions realizing the minimum of E_ε was carried out in the work of Modica [Mod87]. Following this seminal paper, many other authors gave results about the minimization of E_ε, see for example [Bal90] for the case of multicomponent mixtures etc.

In the present paper we consider a constrained variational problem for the functional E_ε. More precisely, we develop techniques to determine a family u_ε, $\varepsilon \in \left]0, \varepsilon_0\right]$, of critical points of E_ε under the volume constraint $\int_M u_\varepsilon = V$, with $V > 0$ fixed, i.e., each u_ε is a solution of problem (2.1) below. It follows from [HT00, Theorem 1] that, under mild conditions, u_ε converges as ε goes to 0, in an appropriate geometric measure theoretic sense, to a characteristic function of a finite perimeter set. This set has reduced boundary whose regular part is a constant mean curvature (CMC) smooth hypersurface, and it is relatively open and dense into the boundary. However, in this limit multiplicity issues may occur, and this affects the optimal regularity of the limit. It was only recently that a more advanced regularity theory became available, thanks to the works [BW18], [BCW18] and [CM18]. Under a different perspective, in [PR03] Pacard and Ritoré showed that every smooth constant mean curvature boundary can be suitably approximated by solutions of (2.1). These circumstances open new perspectives as to the search of CMC boundaries using this PDE approach, which is the objective pursued in this research project. We will address the limit procedure and the geometric consequences in a forthcoming paper.

The main results of the present paper are built upon a theory of multiplicity of solutions for semi-linear variational elliptic equations based on topological and nonlinear methods, along the lines of [BC91], [BCP91], [BC94], [Ben95], [BBM07]. More precisely, in order to establish a lower bound on the number of solutions of problem 2.1 we employ a method from Lyusternik–Schnirelman and Morse theory, that will be referred to as the *photography method*, see Section 4 for details. Roughly speaking, a lower bound on the number of solutions that belong to a suitable sublevel of the associated energy is given in terms of topological invariants of the underlying manifold. A correspondence between the topological invariants of the energy sublevel and those of the underlying (finite-dimensional) manifold is produced by two continuous maps going in both ways, and whose composition is a homotopy equivalence of the finite-dimensional manifold. The map from the finite-dimensional manifold to the sublevel is a sort of *photography map*, which associates to each point a bell-shaped function around the point. This map reproduces a copy (the *photography*) of the underlying manifold inside the energy sublevel. The map going backwards, i.e., from the sublevel to the finite-dimensional manifold, is given by a *barycenter* map, which associates to each function, a suitably defined point in the domain around which most of the mass of the function is concentrated. This construction is interesting when it can be made in such a way that the barycenter of a photography map is the identity map, up to homotopies. In this case, by an elementary topological argument the Lysternik–Schnirelman category and each Betti number takes on the energy sublevel a larger value than the value it takes on the domain manifold, and the desired estimate follows from standard variational theories. We use the photography method to prove the existence of multiple solutions of our constrained variational problem; such solutions come in two classes: low energy solutions, and high energy solutions. By high energy solutions, we mean that we do not have an *a priori* estimate of the energy. The result is obtained under suitable assumptions on the potential function W associated to the problem, including a certain growth condition at infinity, see Section 2 for details. It is important to observe that such asymptotic growth assumption can be weakened significantly in order to obtain the existence of low energy solutions. This will be discussed in Section 5.

The paper is organized as follows. Section 2 contains the formulation of the PDE problem, with all technical assumptions on the potential function needed for the variational setup, and the statement of our main results. Section 4 is the core of the paper. After recalling some generalities on Lusternik–Schnirelman theory and Morse theory, we give a detailed description of the photography method, and its concrete application for the variational problem considered here. We prove the Palais–Smale condition, and we establish the properties of the photography map and the barycenter map. As to the photography map, our definition relies heavily on some geometric measure theoretical result proven by Modica in [Mod87] in the case of domains of $\mathbb{R}^n$. For the development of our theory, we will need a formulation of the results in the context of Riemannian manifolds, and the details of this formulation are given in Section 3 and in Proposition 4.19. For the barycenter map, we employ a non-intrinsic approach by resorting to Nash embedding theorem, and we use heavily several extrinsic Riemannian geometry results obtained by the second author in [Nar18]. In particular, our approach requires several technical results from isoperimetric theory that establish an estimate on the diameter of isoperimetric regions of small volume.

2. FORMULATION OF THE PROBLEM AND MAIN RESULTS

In this section we will give the description of the nonlinear PDE problem, and we will formulate the main result concerning the multiplicity of its solutions. Let us assume that $W\colon \mathbb{R} \to [0, +\infty[$ is a function of class C^2 and that (M, g) is an N-dimensional compact Riemannian manifold without boundary; precise assumptions on W will be given below. For fixed $\varepsilon, V > 0$, we are concerned with the existence of multiple pairs $(u_{\varepsilon,V}, \lambda_{\varepsilon,V}) \in H^1(M) \times \mathbb{R}$ such that the following equalities are satisfied:

$$\begin{gathered} -\varepsilon\,\Delta u_{\varepsilon,V} + \tfrac{1}{\varepsilon} W'(u_{\varepsilon,V}) = \lambda_{\varepsilon,V}, \\ \int_M u_{\varepsilon,V}\,\mathrm{d}v_g = V. \end{gathered} \tag{2.1}$$

As to the assumptions on W, we will consider a double well potential, i.e., a map satisfying the following assumptions:

(a) $W(s)$ has two global minima, at $s = 0$ and at $s = 1$, and a unique local maximum at $s = \frac{1}{2}$; moreover

$$W(0) = W'(0) = W(1) = W'(1) = 0; \quad W''(0), W''(1) > 0; \tag{2.2}$$

(b) there exist positive constants A, B such that

$$\big|W'(s)\big| \leqslant A + Bs^{p-1}, \quad \text{for some } p < \frac{2N}{N-2} =: 2^*, \quad (p < \infty \text{ if } N = 1, 2); \tag{2.3}$$

(c) for some $\delta > 0$:

$$W'(s) > 0, \quad \forall\, s \in \left]1, 1+\delta\right]; \tag{2.4}$$

(d) there exists $c_1, c_2, t_0 > 0$ such that:

$$c_1|t|^{p_1} < W(t) < c_2|t|^{p_2}, \quad \text{when } |t| \geqslant t_0, \tag{2.5}$$

and where $2 < p_1 < \hat{2}$, $p_1 \leqslant p_2 \leqslant 2(p_1 - 1)$, with $\hat{2} = \frac{1}{2}\,2^* + 1 \leqslant 2^* + 1$.

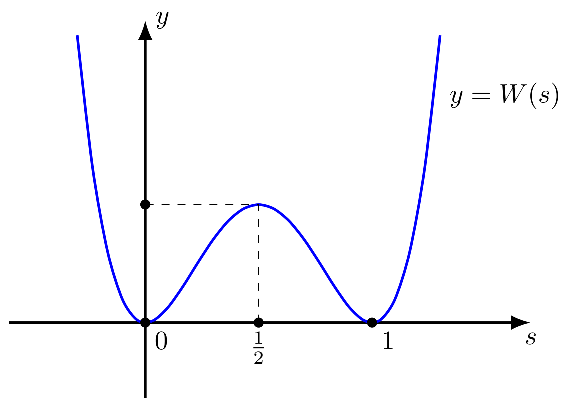

FIGURE 1. The typical shape of the symmetric *double well* potential W considered in Problem $(\mathrm{P}_{\varepsilon,V})$.

The solutions of Problem $(\mathrm{P}_{\varepsilon,V})$ are the critical points of the following energy functional

$$E_\varepsilon : H^1(M) \longrightarrow \mathbb{R},$$

$$E_\varepsilon(u) = \frac{\varepsilon}{2}\int_M |\nabla u|^2 \,\mathrm{d}v_g + \frac{1}{\varepsilon}\int_M W\big(u(x)\big)\,\mathrm{d}v_g, \tag{2.6}$$

under the constraint

$$\int_M u \,\mathrm{d}v_g = V.$$

Here, $\mathrm{d}v_g$ denotes the volume density of the metric g.

Consider the following topological invariants of the manifold M. Given a topological space $\mathcal{X}$, let us recall the definition of some topological invariants of $\mathcal{X}$:

- $\mathrm{cat}(\mathcal{X})$ is the Lusternik–Schnirelman category of $\mathcal{X}$, see Definition 4.1,
- $\beta_k(\mathcal{X})$ is the k-th Betti number[1] of $\mathcal{X}$. Similarly, if $\mathcal{Y} \subset \mathcal{X}$ is a subspace, $\beta_k(\mathcal{X}, \mathcal{Y})$ is the k-th Betti number of the pair;
- $P_1(\mathcal{X})) = \sum_k \beta_k(\mathcal{X})$; this is the value at 1 of the Poincaré polynomial of $\mathcal{X}$ (see Definition 4.5).

The main result of the paper gives a lower bound on the number of solutions of Problem $(\mathrm{P}_{\varepsilon,V})$ in terms of these topological invariants of M.

Theorem 2.1. *Let W satisfy assumptions (a), (b), (c) and (d) above. Then, there exists $V^* = V^*(M,g) > 0$ such that for every $V \in \,]0, V^*[$ there exists $\varepsilon^* = \varepsilon^*(V, M, g) > 0$, such that for every $\varepsilon \in \,]0, \varepsilon^*[$, Problem $(P_{\varepsilon,V})$ admits at least* $\mathrm{cat}(M) + 1$ *distinct solutions. Moreover, if for some given V and ε as above all the solutions of Problem $(P_{\varepsilon,V})$ are* nondegenerate *(i.e., they correspond to nondegenerate critical points of E_ε) then there are at least $2P_1(M) - 1$ solutions.*

[1] Recall that the k-th Betti number of $\mathcal{X}$ is the dimension of the k-th Alexander-Spanier cohomology vector space of $\mathcal{X}$ with coefficients in $\mathbb{R}$

The nondegeneracy assumption in the last part of the statement can be omitted, provided that a suitable notion of multiplicity of solutions is taken into consideration, see Definition 4.8.

Notations and terminology. Given a Riemannian manifold (M, g), we will denote by vol_g the volume function of the metric g, by inj_M the injectivity radius of (M, g) (well defined in the compact case), and by diam_g the diameter function of sets induced by the metric associated to g.

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3. GEOMETRIC MEASURE THEORETICAL PRELIMINARIES

For the development of our theory, we will need a Riemannian counterpart of some results that were originally obtained by Modica in [Mod87] in the case of domains in Euclidean spaces. Although Modica's main ideas carry over to the geometrical setup without major difficulties, for the reader's convenience we will give here a detailed proof of [Mod87, Proposition 2] stated in our Riemannian context. Let us first recall some definitions.

Definition 3.1. Let (M, g) be a Riemannian manifold of dimension n, $U \subseteq M$ an open subset, $\mathfrak{X}_c(U)$ the set of smooth vector fields with compact support on U. Given a function $u \in L^1(M, g)$, define the variation of u by

$$|Du|(M) := \sup\left\{\int_M u\,\mathrm{div}_g(X)\,\mathrm{d}v_g : X \in \mathfrak{X}_c(M), ||X||_\infty \leqslant 1\right\}, \tag{3.1}$$

where $||X||_\infty := \sup\{|X_p|_g : p \in M\}$ and $|X_p|_g$ is the norm of the vector X_p in the metric g on T_pM. We say that a function $u \in L^1(M, g)$, has **bounded variation**, if $|Du|(M) < \infty$ and we define the set of all functions of bounded variations on M by $BV(M, g) := \left\{u \in L^1(M, g) : |Du|(M) < +\infty\right\}$. A function $u \in L^1_{\mathrm{loc}}(M)$ has **locally bounded variation in** M, if for each open set $U \Subset M$,

$$|Du|(U) := \sup\left\{\int_U u\,\mathrm{div}_g(X)\,\mathrm{d}v_g : X \in \mathfrak{X}_c(U), \|X\|_{\infty,g} \leqslant 1\right\} < \infty,$$

and we define the set of all functions of locally bounded variations on M by $BV_{loc}(M) := \{u \in L^1_{loc}(M) : |Du|(U) < +\infty, U \Subset M\}$. So for any $u \in BV(M, g)$, we can associate a vector Radon measure on M $\nabla^g u$ with total variation $|\nabla^g u|$.

Definition 3.2. Let (M, g) be a Riemannian manifold of dimension n, $U \subseteq M$ be an open subset, $\mathfrak{X}_c(U)$ the set of smooth vector fields with compact support in U. Given $E \subset M$ measurable with respect to the Riemannian measure, the **perimeter of** E **in** U, $\mathcal{P}_g(E, U) \in [0, +\infty]$, is

$$\mathcal{P}_g(E, U) := \sup\left\{\int_U \chi_E div_g(X) dv_g : X \in \mathfrak{X}_c(U), ||X||_\infty \leqslant 1\right\}, \tag{3.2}$$

where $||X||_\infty := \sup\{|X_p|_g : p \in M\}$ and $|X_p|_g$ is the norm of the vector X_p in the metric g on T_pM. If $\mathcal{P}_g(E, U) < +\infty$ for every open set $U \subset\subset M$, we call E a **locally finite perimeter set**. Let us set $\mathcal{P}_g(E) := \mathcal{P}_g(E, M)$. Finally, if $\mathcal{P}_g(E) < +\infty$ we say that E **is**

a set of finite perimeter. We will use also the following notation $\mathcal{P}_g(E,F) := |\nabla\chi_E|_g(F)$ for every Borel set $F \subseteq M$.

Proposition 3.3 (Riemannian version of [Mod87, Proposition 2, p. 133])**.** *Let (M^N, g) be a complete smooth Riemannian manifold, let A and Ω be open subsets of M with ∂A a non-empty, compact, smooth hypersurface, and with $\mathcal{H}_g^{N-1}(\partial A \cap \partial\Omega) = 0$. Assume that $\overline{\Omega}$ is compact with smooth boundary (possibly empty). Given real numbers α, β, with $\alpha < \beta$, define the function $v_0 : \Omega \to \mathbb{R}$ by*

$$v_0(x) = \begin{cases} \alpha, & \text{if } x \in A, \\ \beta, & \text{if } x \in \Omega \backslash A. \end{cases}$$

Then there is a family $(v_\varepsilon)_{\varepsilon>0}$ of Lipschitz continuous function on M such that v_ε converges to v_0 in $L^1(M)$ as $\varepsilon \to 0^+$, $\alpha \leqslant v_\varepsilon \leqslant \beta$ for every $\varepsilon > 0$, and

(i) $\int_\Omega v_\varepsilon \, \mathrm{d}v_g = \int_\Omega v_0 \, \mathrm{d}v_g = \alpha|A \cap \Omega| + \beta|\Omega\backslash A|, \quad \forall \varepsilon > 0,$

(ii) $\limsup_{\varepsilon\to 0^+} E_\varepsilon(v_\varepsilon) \leqslant \mathcal{P}_g(A,\Omega)\,\sigma(\alpha,\beta)$*, where* $\sigma(\alpha,\beta) = \int_\alpha^\beta \sqrt{2W(s)}\,\mathrm{d}s$*, and* E_ε *is as in* (2.6).

Proof. Let us define the function d_A as

$$d_A(x) = \begin{cases} -\operatorname{dist}(x, \partial A) \text{ if } x \in A \\ \quad\operatorname{dist}(x, \partial A) \text{ if } x \notin A. \end{cases}$$

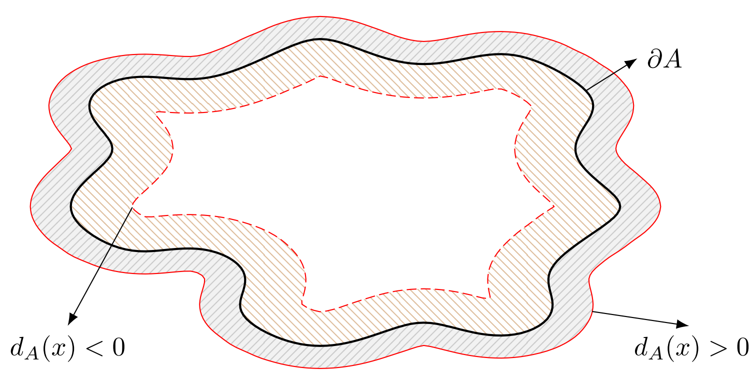

FIGURE 2. Level sets of a typical example of the signed distance function for a subset $A \subseteq M$.

It is well known (see for instance [Mod87, Lemma 4] that d_A is Lipschitz continuous, that $|\nabla_g d_A(x)|_g = 1$ for almost all $x \in M$ and that, if $S_t := \{x \in M : d_A(x) = t\}$, then

$$\lim_{t\to 0} \mathcal{H}_g^{N-1}(S_t \cap \Omega) = \mathcal{H}_g^{N-1}(\partial A \cap \Omega). \tag{3.3}$$

Define the function $q_0 : \mathbb{R} \to \mathbb{R}$ by

$$q_0(t) = \alpha \quad \text{if } t < 0, \qquad q_0(t) = \beta \quad \text{if } t \geqslant 0,$$

and let $v_0(x) = q_0(d_A(x))$.

Now, let us consider functions q_ε satisfying the ordinary differential equation:

$$\frac{\varepsilon}{2} q_\varepsilon'(t)^2 = \frac{1}{\varepsilon} W\left(q_\varepsilon(t)\right) + \frac{\sqrt{\varepsilon}}{2}; \tag{3.4}$$

these functions give an approximation of q_0, and will be employed to define the desired maps (v_ε). Let us explain why this equation. We want approximate the two-valued function q_0 by a Lipschitz continuous function q_ε, which interpolates between α and β and, at the same time, minimizes the one-dimensional Van der Walls-Allen-Cahn-Hilliard gradient phase field energy functional

$$\int_{\mathbb{R}} \left[\frac{\varepsilon}{2} q_\varepsilon'^2 + \frac{1}{\varepsilon} W\left(q_\varepsilon\right)\right] \mathrm{d}t.$$

The corresponding Euler equation is $\varepsilon^2 q_\varepsilon'' = W'\left(q_\varepsilon\right)$; multiplying by q_ε' and integrating, we obtain $\frac{\varepsilon^2}{2} q_\varepsilon'^2 = W\left(q_\varepsilon\right) + c_\varepsilon$. To avoid the constant trivial solutions, the constant c_ε cannot be set equal to zero. On the other hand, we need $c_\varepsilon \gg \varepsilon^2$ to make q_ε fill the gap between α and β as quickly as possible (note that $q_\varepsilon'^2 \geqslant 2c_\varepsilon/\varepsilon^2)$), and for that reason we choose $c_\varepsilon = \varepsilon^{3/2}/2$. To construct the functions q_ε, consider for a fix $\varepsilon > 0$ the function

$$\psi_\varepsilon(t) = \int_\alpha^t \frac{\varepsilon}{\sqrt{\varepsilon^{3/2} + 2W(s)}} \mathrm{d}s, \quad \alpha \leqslant t \leqslant \beta$$

where $\eta_\varepsilon = \psi_\varepsilon(\beta)$.

Let $\widetilde{q}_\varepsilon : [0, \eta_\varepsilon] \to [\alpha, \beta]$ denote the inverse of ψ_ε, see Figure 3.

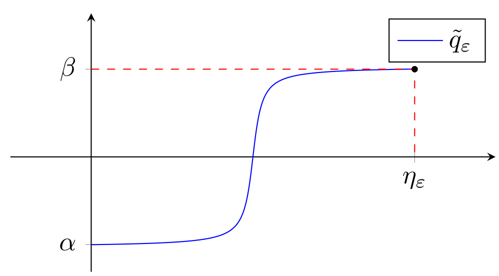

FIGURE 3. Graph of $\widetilde{q}_\varepsilon$.

Since W is non-negative,

$$0 < \eta_\varepsilon \leqslant \varepsilon^{\frac{1}{4}}(\beta - \alpha); \tag{3.5}$$

and, by the continuity of W, $\widetilde{q}_\varepsilon$ is of class C^1 and

$$\varepsilon \widetilde{q}_\varepsilon'(t) = \sqrt{\varepsilon^{3/2} + 2W(\widetilde{q}_\varepsilon)}, \tag{3.6}$$

for $0 \leqslant t \leqslant \eta_\varepsilon$.

We now extend the definition of $\widetilde{q}_\varepsilon$ to the entire real line by setting

$$\widetilde{q}_\varepsilon(t) = \alpha \quad \text{for } t < 0, \quad \widetilde{q}_\varepsilon(t) = \beta \quad \text{for } t > \eta_\varepsilon,$$

so that $\widetilde{q}_\varepsilon$ is a Lipschitz continuous function on $\mathbb{R}$. Note that, for every $t \in \mathbb{R}$, $\widetilde{q}_\varepsilon(t) \leqslant q_0(t)$ and $\widetilde{q}_\varepsilon\left(t + \eta_\varepsilon\right) \geqslant q_0(t)$. Thus, there exists $\delta_{\varepsilon,A,V} \in [0, \eta_\varepsilon]$ such that

$$\int_\Omega \widetilde{q}_\varepsilon\big(d_{A,V}(x) + \delta_{\varepsilon,A,V}\big) \mathrm{d}v_g = \int_\Omega q_0(d_{A,V}(x)) dv_g = \int_\Omega v_0(x) \mathrm{d}v_g. \tag{3.7}$$

Finally, we define $q_\varepsilon(t) = \widetilde{q}_\varepsilon\left(t + \delta_{\varepsilon,A,V}\right)$ for $t \in \mathbb{R}$ and

$$v_{\varepsilon,A,V}(x) = q_\varepsilon(d_{A,V}(x)) = \widetilde{q}_\varepsilon\left(d_{A,V}(x) + \delta_{\varepsilon,A,V}\right),$$

for $x \in \Omega$.

We now prove that $v_{\varepsilon,A,V} \to v_0$ in L^1. Notice that each $v_{\varepsilon,A,V}$ is a Lipschitz continuous function and $\alpha \leqslant v_{\varepsilon,A,V} \leqslant \beta$.

By Lemma 4 of [Mod87] and the co-area formula for Lipschitz functions (see for instance [Fed69])

$$\int_\Omega f(u(x))|\nabla_g u(x)|_g \,\mathrm{d}v_g = \int_{\mathbb{R}} f(t)\mathcal{H}_g^{N-1}\big(\{x \in \Omega : u(x) = t\}\big)\,\mathrm{d}t, \tag{3.8}$$

which holds for any Lebesgue measurable function f and any Lipschitz continuous function u, we get the following

$$\begin{aligned}\int_\Omega |v_{\varepsilon,A,V} - v_0| \,\mathrm{d}v_g &= \int_\Omega |q_\varepsilon(d_A(x)) - q_0(d_A(x))|\,|\nabla_g d_A(x)|_g \mathrm{d}v_g \\ &= \int_{-\delta_{\varepsilon,A,V}}^{\eta_\varepsilon - \delta_{\varepsilon,A,V}} |q_\varepsilon(t) - q_0(t)|\,\mathcal{H}_g^{N-1}\left(S_t \cap \Omega\right)\mathrm{d}t \\ &\leqslant \eta_\varepsilon(\beta-\alpha)\sup_{|t|\leqslant \eta_\varepsilon} \mathcal{H}_g^{N-1}\left(S_t \cap \Omega\right) \\ &\leqslant \varepsilon^{\frac14}(\beta-\alpha)^2 C(M,g),\end{aligned}$$

where $S_t = \{x \in M : d_A(x) = t\}$, and we obtain the last inequality applying (3.5). Then we conclude that $v_{\varepsilon,A,V}$, converges to v_0 in $L^1(\Omega)$ as $\varepsilon \to 0^+$ uniformly with respect to A and V.

To prove (ii) we call

$$\gamma_\varepsilon = \sup_{|t|\leqslant \eta_\varepsilon} \mathcal{H}_g^{N-1}\left(S_t \cap \Omega\right).$$

We again employ the coarea formula (3.8), obtaining

$$\begin{aligned}E_\varepsilon\left(v_{\varepsilon,A,V}\right) &= \int_{\mathbb{R}} \left[\frac{\varepsilon}{2} q_\varepsilon'(t)^2 + \frac{1}{\varepsilon} W\left(q_\varepsilon(t)\right)\right]\mathcal{H}_g^{N-1}\left(S_t \cap \Omega\right)\mathrm{d}t \\ &\leqslant \gamma_\varepsilon \int_{-\delta_{\varepsilon,d_{A,V}}}^{\eta_\varepsilon - \delta_{\varepsilon,A,V}} \left[\frac{\varepsilon}{2}\widetilde{q}_\varepsilon'(t + \delta_{\varepsilon,A,V})^2 + \frac{1}{\varepsilon} W\left(\widetilde{q}_\varepsilon(t + \delta_{\varepsilon,A,V})\right)\right]\mathrm{d}t \\ &\leqslant \gamma_\varepsilon \int_0^{\eta_\varepsilon} \left[\frac{\varepsilon}{2}\widetilde{q}_\varepsilon'(t)^2 + \frac{1}{\varepsilon} W\left(\widetilde{q}_\varepsilon(t)\right) + \frac{\varepsilon^{1/2}}{2}\right]\mathrm{d}t\end{aligned}$$

and, recalling (3.6),

$$\begin{aligned}E_\varepsilon\left(v_{\varepsilon,A,V}\right) &\leqslant \gamma_\varepsilon \int_0^{\eta_\varepsilon} \left(2W(\widetilde{q}_\varepsilon(t)) + \varepsilon^{3/2}\right)^{\frac12} \widetilde{q}_\varepsilon'(t) dt \\ &= \gamma_\varepsilon \int_\alpha^\beta (2W(s) + \varepsilon^{3/2})^{\frac12} ds.\end{aligned}$$

Since Lemma 4 of [Mod87] implies

$$\lim_{\varepsilon \to 0^+} \gamma_\varepsilon = \mathcal{H}_g^{N-1}(\partial A \cap \Omega) = P(A,\Omega),$$

we conclude that

$$\limsup_{\varepsilon \to 0^+} E_\varepsilon\left(v_{\varepsilon,A,V}\right) \leqslant P(A,\Omega)\int_\alpha^\beta \sqrt{2W} ds = P(A,\Omega)\sigma(\alpha,\beta),$$

and the proposition is proved. □

Definition 3.4. Given sets $A, \Omega \subset M$ and a function v_0 as in Proposition 3.3, a family $(v_\varepsilon)_{\varepsilon>0}$ of functions as in the statement of the Proposition will be called a *Modica approximation of* v_0. When the set Ω is not specified, it will be implicitly assumed $\Omega = M$.

4. The photography method

In this section we discuss a technique, originally due to Benci, Cerami, and others, (see [BCP91] or [BC94]) which is a twist over the classical Lusternik-Schnirelmann theory and Morse theory. We will call this technique the *photography method*, for reasons that will be clear along the way. A formal statement of the results generated by this method is given in Theorems 4.4 and 4.9.

4.1. **General setup.** We start off by recalling a few basic definitions.

Definition 4.1. Let (X, τ) be a topological space and $Y \subseteq X$ be a closed subset. We define the ***Lusternik-Schnirelmann category of*** Y ***in*** X , denoted by $\mathrm{cat}_X(Y)$, as the minimum number $k \in \mathbb{N}$ such that there exist $\mathcal{U}_1, ..., \mathcal{U}_k \in \tau$, open subsets $U_i \subseteq X$ contractible in X satisfying $Y \subseteq \bigcup_i \mathcal{U}_i$. If no such finite family exists, then one sets $\mathrm{cat}_X(Y) = +\infty$. Furthermore, one defines $\mathrm{cat}(X) = \mathrm{cat}_X(X)$.

Definition 4.2. Let $\mathfrak{M}$ be a C^2-Hilbert manifold, $J : \mathfrak{M} \to \mathbb{R}$ a C^1 functional, and (u_n) a sequence in $\mathfrak{M}$. We say that u_n is a ***Palais–Smale sequence*** (a ***PS-sequence*** for short), if

$$J(u_n) \to c, \tag{4.1}$$

$$\left\| \mathrm{d}J(u_n) \right\|_{T^*_{u_n}\mathfrak{M}} \to 0. \tag{4.2}$$

Definition 4.3. Let $\mathfrak{M}$ be a C^2-Hilbert manifold, $J : \mathfrak{M} \to \mathbb{R}$ a C^1 functional. We say that J ***satisfies the Palais-Smale condition***, if every Palais-Smale sequence has a convergent subsequence.

Classical results of Calculus of Variations relate the number of critical points in a sublevel of the energy functional with suitable topological/homological/cohomological invariants (category, Betti numbers, cuplength, etc.) of the sublevel. However, it is in principle rather involved to have a good topological description of subleveles of an abstract functional, which typically are the closure of arbitrary open subsets of infinite-dimensional manifolds.

The *photography method* is a technique that allows us to estimate, when the functional space consists of real-valued functions on a manifold, the value of the topological invariants of the sublevels in terms of the analogous invariants associated to the underlying manifold. The estimate is obtained by *reproducing* a copy of the underlying manifold in a given sublevel (the photography); this is done by means of a continuous function which associates to each point of the manifold, a map in the function spaces which concentrates its mass around the given point. The technique works when such operation can be made in such a way that the photography of the underlying manifold is sufficiently *ample* in the sublevel, i.e., when the sublevel can be continuously retracted to the image of the photography. In many situations, such retraction is obtained as a *barycenter map*. This is formalized using continuous maps and homotopies, as follows.

Theorem 4.4. *Let* $\mathfrak{M}$ *be a* C^2*-Hilbert manifold and let* $J : \mathfrak{M} \to \mathbb{R}$ *be a* C^1 *functional. Assume that*

(i) $\inf\{J(u) : u \in \mathfrak{M}\} > -\infty$*,*
(ii) J *satisfies* (PS) *condition,*
(iii) there exist $c > \inf J$*, a topological space* X *and two continuous maps*

$$f : X \to J^c,$$

$$g : J^c \to X$$

such that $g \circ f$ *is homotopic to the identity map of* X*.*

Then, there are at least $\operatorname{cat}(X)$ *critical points in* J^c*. Furthermore, if* $\mathfrak{M}$ *is contractible and* $\operatorname{cat}(X) > 1$*, there is at least one critical point* $u \notin J^c$*.*

Proof. See [BCP91] or [BC94]. □

The above result can be made slightly more accurate (at least in the nondegenerate case) by using Morse theory.

Definition 4.5. Let X be a topological space; the *Poincare's Polynomial* $P_t(X)$ *of* X is defined as the following power series in the variable t

$$P_t(X) = \sum_{n=0}^{\infty} \beta_n(X)\, t^n. \tag{4.3}$$

4.6. *Remark.* If X is a compact manifold, we have that $H^n(X)$ is a finite dimensional vector space and the formal series (4.3) is actually a polynomial.

In the following definition, we give the notion of Morse index of a critical point, which is necessary in our treatment to establish a relation between the Poincare's polynomial $P_t(M)$ and the number of solutions to the Euler equation associated to a given functional J. In this work we use the approach to Morse theory developed in [Ben95], which is suitable in problems arising from PDE's.

Definition 4.7 (Morse Index). Let $\mathfrak{M}$ be a C^2-Hilbert manifold, $J : \mathfrak{M} \to \mathbb{R}$ a C^1 functional and let $u \in \mathfrak{M}$ an isolated critical point of J at level[2] $c \in \mathbb{R}$. We denote by $i_t(u)$ the following formal power series in t

$$i_t(u) = \sum_{k=0}^{+\infty} \beta_k\big(J^c, J^c \backslash \{u\}\big) t^k, \tag{4.4}$$

where $J^c = \{v \in \mathfrak{M} :\ J(v) \leqslant c\}$. We call $i_t(u)$ the ***(polynomial) Morse index of*** u. The number $i_1(u)$ is called the ***multiplicity of*** u.

If J is of class C^2 in a neighborhood of u and $J''[u]$ is not degenerate, we say that u is nondegenerate. In this case we have that

$$i_t(u) = t^{\mu(u)}, \tag{4.5}$$

where $\mu(u)$ is the **(numerical) Morse index of** u, i.e., the dimension of the maximal subspace on which the bilinear form $J''[u](\cdot,\cdot)$ is negative-definite. This fact suggests the following definition.

Definition 4.8. Let $\mathfrak{M}$ be a C^2-Hilbert manifold, $J : \mathfrak{M} \to \mathbb{R}$ be a C^1 functional and let $u \in \mathfrak{M}$ be an isolated critical point of J at level c. We say that u is (***topologically***) ***nondegenerate***, if $i_t(u) = t^{\mu(u)}$, for some natural number $\mu(u) \in \mathbb{N}$.

[2]This means that $J(u) = c$, $\mathrm{d}J(u) = 0$, and there exists a neighborhood $\mathcal{U}$ of u in $\mathfrak{M}$ such that the only critical critical point of J contained in $\mathcal{U}$ coincide with u.

Theorem 4.9. *Under assumptions* (i), (ii), *and* (iii) *of Theorem 4.4, there exists* $c_1 > c$ *such that one of the two following conditions hold:*

(1) J^{c_1} *contains infinitely many critical points.*
(2) J^c *contains* $P_1(X)$ *critical points and* $J^{c_1}\backslash J^c$*, contains* $P_1(X)-1$ *critical points if counted with their multiplicity. More exactly we have the following relation*

$$\sum_{u\in \mathrm{Crit}(J^{c_1})} i_t(u) = P_t(X) + t[P_t(X)-1] + (1+t)Q(t), \tag{4.6}$$

where $Q(t)$ *is a polynomial with nonnegative integer coefficients, and* $\mathrm{Crit}(J^{c_1})$ *denotes the set of critical points of* J *in the sublevel* J^{c_1}*.*

In particular, if all the critical points are nondegenerate there are at least $P_1(X)$ critical points with energy less or equal than c, and at least $P_1(X)-1$ with energy between c and c_1.

4.10. *Remark.* If we count the critical points with their multiplicity, then by Theorem 4.9 follows that there are at least $2P_1(X)-1$ critical points. Namely, when the critical points are isolated, the result follows from the Morse's relation (4.6), otherwise there are infinitely many of them.

4.11. *Remark.* Given topological spaces X and Y, we say that Y *is homotopically superjacent to* X if there exist continuous maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $g\circ f$ is homotopic to the identity map of X. Thus, assumption (iii) of Theorem 4.4 (and of Theorem 4.9) says that the sublevel J^c is homotopically sujacent to X. When Y is homotopically superjacent to X, then the induced map f_* in homotopy or homology is injective, which implies that $\mathrm{cat}(X) \leqslant \mathrm{cat}(Y)$, and that for all $n\in\mathbb{N}$, $\beta_n(X)\leqslant\beta_n(Y)$. This is the reason why the estimates on the number of critical points in Theorem 4.4 and in Theorem 4.9 are given in terms of the topological invariants of X.

4.2. **The Palais–Smale condition.** First of all we need to prove the Palais–Smale condition for the functional E_ε :

Proposition 4.12. *For every* $\varepsilon, V > 0$*, the functional* E_ε *satisfies the Palais-Smale condition on* $\mathfrak{M}^V$*.*

Proof. Assume that (u_n) is a Palais-Smale sequence for E_ε in $\mathfrak{M}^V$; by density, we can suppose that u_n is continuous for all n. Observe that Equations (4.1) and (4.2) written explicitly are

$$\frac{\varepsilon^2}{2}\int_M |\nabla u_n|^2 \,\mathrm{d}x + \int_M W(u_n(x))\,\mathrm{d}x \to c \tag{4.7}$$

$$-\varepsilon^2\Delta u_n + W'(u_n) = \lambda_n + T_n, \tag{4.8}$$

where $(\lambda_n)_n$ is some sequence in $\mathbb{R}$, and $T_n \to 0$ strongly in $H^{-1}(M)$. Then, by (4.7) and using the assumptions (2.2), (2.3) we have

$$\begin{aligned} c+1 &\geqslant \frac{\varepsilon^2}{2}\int |\nabla u_n|^2\,\mathrm{d}x + \int W(u_n(x))\,\mathrm{d}x \\ &\geqslant \frac{\varepsilon^2}{2}\int |\nabla u_n|^2\,\mathrm{d}x - k\int u_n \\ &= \frac{\varepsilon^2}{2}\int |\nabla u_n|^2\,\mathrm{d}x - kV. \end{aligned}$$

Hence, $|\nabla u_n|$ is bounded in L^2. On the other hand, from (4.7) we deduce that $\int_M W(u_n)$ is bounded. By assumption (d) in Section 2, $\lim_{t\to\pm\infty} W(t) = +\infty$, and with this we obtain that $\min\limits_M |u_n|$ is also a bounded sequence. Thus, u_n is bounded in $H^1(M)$, so that there exists $u \in H^1(M)$ such that, up to a subsequence, $u_n \rightharpoonup u$. We have to show that $u_n \to u$ strongly in $\mathfrak{M}^V_{\varepsilon,c}$.

By (2.3), for some $p < \frac{2N}{N-2}$, the map $u \mapsto W'(u)$ of left composition with W' gives a bounded nonlinear operator from $L^p(M)$ to $L^q(M)$, with $1\frac{1}{p} + \frac{1}{q} = 1$; thus $q > \frac{2N}{N+2} \geqslant 2$. By the Sobolev embedding theorem, the inclusion $H^1(M) \hookrightarrow L^p(M)$ is compact, and thus we get a compact nonlinear operator $H^1(M) \ni u \mapsto W'(u) \in L^q(M)$. This implies that, up to subsequences $W'(u_n) \to W'(u)$ strongly in $L^q(M) \subset H^{-1}(M)$. Multiplying (4.8) by u_n, integrating by parts the corresponding identity, and using the constraint $\int u_n = V$, we get that λ_n is a bounded sequence. Whence, up to a subsequence we can assume $\lambda_n \to \lambda$.

Now, recalling that $\Delta^{-1} : H^{-1}(M) \to H^1(M)$ is an isomorphism, we obtain that

$$u_n = \frac{1}{\varepsilon^2}(-\Delta^{-1})\left[\lambda_n - W'(u_n) + T_n\right]$$

is a convergent sequence. □

4.3. **The photography method in our concrete setting.** In this section we will define the objects needed for the setup and the proof of Theorems 4.4 and 4.9; an analysis of these objects will be carried out in the following sections.

The objects $\mathfrak{M}$, J, X, c, $f\colon X \to J^c$ and $g\colon J^c \to X$ that appear in the statement of Theorem 4.4 in our concrete setting are described below.

- $\mathfrak{M} = \mathfrak{M}^V$, where
$$\mathfrak{M}^V = \left\{u \in H^1(M) : \int_M u(x)\,\mathrm{d}v_g = V\right\},$$
- $J = E_\varepsilon|_{\mathfrak{M}^V}$, where
$$E_\varepsilon(u) = \frac{\varepsilon}{2}\int_M |\nabla u|^2\,\mathrm{d}v_g + \frac{1}{\varepsilon}\int_M W(u(x))\,\mathrm{d}v_g,$$
- $X = M$, $f = \Phi_{\varepsilon,V} : M \to E^c_\varepsilon \cap \mathfrak{M}^V =: E^c_\varepsilon$, where $c = c(\varepsilon, V) = \sigma c_N V^{\frac{N-1}{N}} + \delta$ where
$$\sigma = \sigma(0,1) = \int_0^1 \sqrt{2W(s)}\,\mathrm{d}s,$$
c_N is the Euclidean isoperimetric constant, i.e., the best constant in the Euclidean isoperimetric inequality $P_{\mathbb{R}^N}(E) \geqslant c_N V_{\mathbb{R}^N}(E)^{\frac{N-1}{N}}$, $\delta > 0$ is a suitable small constant that will be specified later (Corollary 4.28), and $\Phi_{\varepsilon,V}\colon M \to E^c_\varepsilon \cap \mathfrak{M}^V$ is defined by:

$$\Phi_{\varepsilon,V}(x_0)(x) := U_{\varepsilon,V,x_0}(x)\,, \tag{4.9}$$

where $U_{\varepsilon,V,x_0}\colon M \to \mathbb{R}$ is the function obtained in Proposition 3.3 assuming $\Omega := M$ and $A := M\backslash B_g(x_0, r_V)$ where $B_g(x_0, r_V)$ is the metric ball of volume $vol_g(B_g(x_0, r_V)) = V$. We observe that $U_{\varepsilon,V,x_0}\colon M \to \mathbb{R}$ is a Lipschitz continuous function with the following properties:
  - as it is easy to see, by construction we always have
$$supp\,(U_{\varepsilon,V,x_0}) \Subset B_g(x_0, r_V + \delta_{\varepsilon,M\backslash B_g(x_0,r_V)}),$$

where $\delta_{\varepsilon,M\setminus B_g(x_0,r_V)} > 0$ is defined as in Proposition 3.3. Then for small $V \ll 1$ and small $\varepsilon \ll 1$ we have that for every x_0 it holds

$$B_g(x_0, r_V + \delta_{\varepsilon,M\setminus B_g(x_0,r_V)}) \Subset B_g\left(x_0, \frac{\mathrm{inj}_M}{2}\right).$$

- the family $\left(U_{\varepsilon,V,x_0}\right)_{\varepsilon>0}$ is a Modica approximation (see Definition 3.4) of the characteristic function of the geodesic ball $B_g(x_0, r_V)$ of volume V. Here, for $V \in]0, V_{x_0}[$ with:

$$C_1 \leqslant V_{x_0} = \mathrm{vol}_g\left(B_g(x_0, \tfrac{1}{2}\,\mathrm{inj}_M)\right) \leqslant C_2(M, g),$$

where $C_1 := \mathrm{vol}_{g_{b^-}}(B_{(\mathbb{M}^n_{b^-}, g_{b^-})}(0, \frac{1}{2}\,\mathrm{inj}_M))$ and

$$C_2 := \mathrm{vol}_{g_{b^+}}(B_{(\mathbb{M}^n_{b^+}, g_{b^+})}(0, \tfrac{1}{2}\,\mathrm{inj}_M)),$$

$(\mathbb{M}^n_k, g_k)$ is the simply connected space form of constant sectional curvature $k \in \mathbb{R}$, and $b^-, b^+ \in \mathbb{R}$ are such that $b^- \leqslant Sec_g(\sigma, x) \leqslant b^+$ for every 2-dimensional subspace $\sigma \leqslant T_xM$ and for every $x \in M$, with $Sec_g(\sigma, x)$ being the sectional curvature of the 2-dimensional subspace σ with respect to metric g. The existence of the functions $U_{x_0,V,\varepsilon}$ is proved in Proposition 3.3.

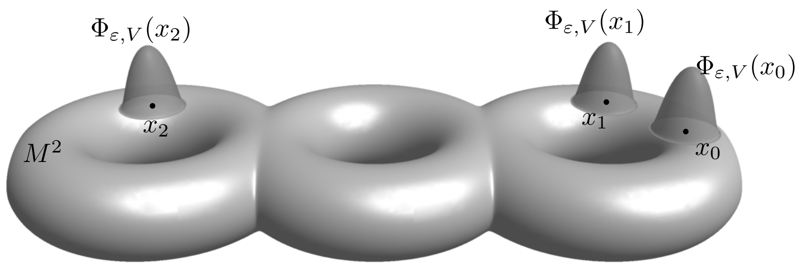

FIGURE 4. The image of the photography map is a (photography) of the underlying (finite-dimensional) manifold in the infinite dimensional functional manifold $H^1(M)$.

- assuming that the Riemannian manifold M is isometrically embedded in some Euclidean space $\mathbb{R}^l$ (Nash embedding theorem) $\hat{g} := \pi \circ \beta : E^c_\varepsilon \cap \mathfrak{M}^V \to M$, where π is the nearest point projection given in Definition 4.14, the barycenter map $\beta, \beta_1 \colon E^c_\varepsilon \cap \mathfrak{M}^V \to \mathbb{R}^l$,

$$\beta_1(u) := \frac{\int_M x u(x)\,\mathrm{d}v_g(x)}{\int_M u(x)\,\mathrm{d}v_g(x)}. \tag{4.10}$$

We will next show that the above objects are well defined, and that they satisfy the assumptions required in the photography method.

4.4. **Continuity of the photography map.** This is the map f that reproduces a copy of the finite-dimensional ambient manifold M^N inside the infinite functional space which is the domain of the energy functional. For the definition of f, see Section 4.3, formula (4.9).

Let us start by looking more closely at its definition and by proving the continuity of the photography map.

Proposition 4.13. *There exists $V_1 = V_1(M,g) > 0$ such that for every $0 < V < V_1$ and for every $\delta > 0$ there exists $\varepsilon_1(V,\delta) > 0$ such that for every $\varepsilon \in]0, \varepsilon_1[$ we have that $\Phi_{\varepsilon,V}$ carries M into the sublevel $E^c_\varepsilon \cap \mathfrak{M}^V$, where $c = \sigma c_N V^{\frac{N-1}{N}} + \delta$, and $\Phi_{\varepsilon,V} : M \to E^c_\varepsilon \cap \mathfrak{M}^V$ is a continuous function.*

Proof. Recall from Section Section 4.3 the map $\Phi_{\varepsilon,V}$ at some point $x_0 \in M$ is defined in terms of Modica approximations for the characteristic functions of balls centered at x_0 with volume equal to V, see formula (4.9). By (ii) in Proposition 3.3 and the asymptotic expansion for small volumes of the area of the geodesic balls with respect to the enclosed volume, it follows that $E_\varepsilon(\Phi_{\varepsilon,V}(x_0)) \lesssim \sigma c_N V^{\frac{N-1}{N}}$ as $\varepsilon \to 0$, uniformly with respect to x_0 and V. Using this and the compactness of M, one proves easily that for any fixed small enough $V < V_1$ it holds $E_\varepsilon(\Phi_{\varepsilon,V}(x_0)) \to \sigma c_N V^{\frac{N-1}{N}}$ as $\varepsilon \to 0$, uniformly with respect to x_0. So the Proposition is proved if we show that $\Phi_{\varepsilon,V}$ is continuous. To this aim, we will first prove the following estimate:

$$||\Phi_{\varepsilon,V}(x_0) - \Phi_{\varepsilon,V}(x_1)||_{W^{1,2}(M)} \leqslant C\big[||h_{x_0} - h_{x_1}||_\infty + |\delta_{\varepsilon,x_0,V} - \delta_{\varepsilon,x_1,V}|\big] + C\big[||\nabla h_{x_0} - \nabla h_{x_1}||_\infty\big], \tag{4.11}$$

where $h_x = \nabla d_g(x,\cdot)$ (see Figure 5), $C = C(\varepsilon, V, M, g, W\big|_{[0,1]}) > 0$ and $\delta_{\varepsilon,x_0,V} := \delta_{\varepsilon,M\setminus B_g(x_0,r_V)}$, where $B_g(x_0,r_V)$ is the small geodesic ball enclosing volume V, see formula (3.7). It is worth to notice here that for small volumes $V \ll 1$, we have that $\partial B_g(x_0,r_V)$ is smooth. The desired continuity property of $\Phi_{\varepsilon,V}$ will follow from this inequality,

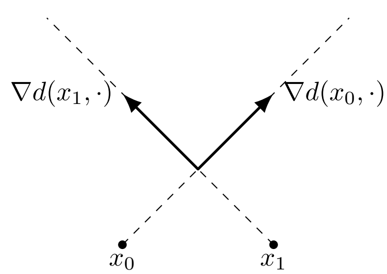

FIGURE 5. Continuity of the photography map, $\nabla d_g(x,\cdot) =: h_x(\cdot)$.

observing that:

- $||h_{x_0} - h_{x_1}||_{C^1(M)} \to 0$, as $x_1 \to x_0$;
- $x \mapsto \delta_{\varepsilon,x,V}$ is a C^1 map, as it can be seen easily applying the implicit function theorem in (3.7).

Thus, $|\delta_{\varepsilon,x_0,V} - \delta_{\varepsilon,x_1,V}| \to 0$ as $x_0 \to x_1$ for any fixed $\varepsilon, V > 0$. In order to prove (4.11), we proceed as follows:

$$
\begin{aligned}
||\Phi_{\varepsilon,V}(x_0) - \Phi_{\varepsilon,V}(x_1)||^2_{W^{1,2}(M)} = \\
&\int_M |\phi'_\varepsilon(h_{x_0,V}(x) + \delta_{\varepsilon,x_0,V})\nabla h_{x_0,V}(x) - \phi'_\varepsilon(h_{x_1,V}(x) + \delta_{x_1})\nabla h_{x_1,V}(x)|^2 \,\mathrm{d}v_g \\
&\quad + \int_M |\phi_\varepsilon(h_{x_0,V}(x) + \delta_{\varepsilon,x_1,V}) - \phi_\varepsilon(h_{x_1,V}(x) + \delta_{x_1})|^2 \,\mathrm{d}v_g \\
&\leqslant 2\|\phi'_\varepsilon\|^2_\infty \int |\nabla h_{x_0}(x) - \nabla h_{x_1}(x)|^2 \,\mathrm{d}v_g \\
&+ 2\|\phi'_\varepsilon\|^2_\infty \int (|h_{x_0}(x) - h_{x_1}(x)|^2 + |\delta_{\varepsilon,h_{x_1,V}} - \delta_{\varepsilon,h_{x_0,V}})|^2) \,\mathrm{d}v_g \\
&\overset{(3.6)}{\leqslant} C\left(\|\nabla h_{x_0} - \nabla h_{x_1}\|^2_\infty + \|h_{x_0} - h_{x_1}\|^2_\infty + |\delta_{\varepsilon,h_{x_1,V}} - \delta_{\varepsilon,h_{x_0,V}})|^2\right),
\end{aligned}
$$

where

$$
\begin{aligned}
C = C(\varepsilon, V, M, g, W|_{[0,1]}) &= 2\|\phi'_\varepsilon\|^2_\infty \operatorname{Vol}_g(B_g(\cdot, \frac{\mathrm{inj}_M}{2})) \\
&\geqslant C^{**}(\varepsilon, W|_{[0,1]}) C^*(M,g) > 0,
\end{aligned}
$$

since $2\|\phi'_\varepsilon\|^2_\infty = C^{**}(\varepsilon, W|_{[0,1]}) > 0$, being ϕ'_ε the solution of the one-dimensional problem. From this estimate the continuity of $\Phi_{\varepsilon,V}$ follows easily and the theorem is proved. □

4.5. **The barycenter map.** In this section we will show that the baricenter map (4.10) is well-defined and continuous. We start with the following:

Definition 4.14. Given an isometric embedding $i : (M^N, g) \to (\mathbb{R}^l, \xi)$. We define the ***normal injectivity radius*** $r_i(M)$ as the largest nonnegative number r such that the normal exponential $exp_{\nu M} : \nu M \to \mathbb{R}^l$ is a diffeomorphism of a neighborhood of the zero section of νM into M_r where $M_r := \{x \in \mathbb{R}^l : d_\xi(x, M) < r\} \subseteq \mathbb{R}^l$ and νM denotes the normal bundle induced by i on M. Let us denote by $\pi : M_{r_i(M)} \to M$ the canonical projection associated with the canonical projection $\widetilde{\pi} : \nu M \to M$.

4.15. *Remark.* Notice that M is a retract of $M_{r_i(M)}$, and $r_i(M) > 0$, since M is compact.

For the reader's convenience, we give a proof of the following simple result.

Lemma 4.16. *The map* $\beta_1 : H^1(M)\backslash\{0\} \to \mathbb{R}^l$ *defined in* (4.10) *is continuous. In particular, their restrictions to* $\mathfrak{M}^V$ *are continuous for every* $V \in \mathbb{R}$.

Proof. Let us prove the continuity of β_1.

For all $w \in H^1(M)$, set $\mu_w := \int_M w(x)\,\mathrm{d}v_g(x)$. We have the following estimate

$$
\left|\frac{\int_M xu(x)\,\mathrm{d}v_g(x)}{\int_M u(x)\,\mathrm{d}v_g(x)} - \frac{\int_M xv(x)\,\mathrm{d}v_g(x)}{\int_M v(x)\,\mathrm{d}v_g(x)}\right| \leqslant \frac{||x||_\infty}{\mu_u} \int_M \left|u - \frac{\mu_u}{\mu_v} v\right| \mathrm{d}v_g \tag{4.12}
$$

where $||x||_\infty := \sup_{x \in M}\{|x|_{\mathbb{R}^l}\} = C(i) < +\infty$, because M is compact. Here $|x|_{\mathbb{R}^l}$ is the Euclidean length of the position vector and i is the isometric embedding of M in $\mathbb{R}^l$. It is easy to show that the right-hand side of (4.12) goes to zero when $v \to u$ in $L^2(M)$ (Lebesgue's dominated convergence, Hölder inequality). □

4.17. *Remark.* In order to apply the abstract theory of Theorems 4.4, 4.9, in our concrete setting, a crucial point to be shown is that for fixed small $\varepsilon, V > 0$ and for c close to the minimum of E_ε in $\mathfrak{M}^V$, the image $\beta(E_\varepsilon^c \cap \mathfrak{M}^V)$ is contained in a tubular neighborhood M_r of M in $\mathbb{R}^l$, whose thinckness $r > 0$ is small enough to make the nearest point projection $M_r \to M$ well-defined and continuous. The proof of this fact is rather involved, and it requires notions and nontrivial results about the isoperimetric problem in Riemannian manifolds.

We recall here a very classical notion of measure theory that will be useful in the proof of Proposition 4.19.

Definition 4.18. Let $u, (u_n)_n$ be measurable functions on a measure space (X, Σ, μ). The sequence f_n is said to ***converge globally in measure to*** u, if for every $\varepsilon > 0$, it holds

$$\lim_{n\to\infty} \mu\big\{x \in X : |u(x) - u_n(x)| \geqslant \varepsilon\big\} = 0.$$

We need a Riemannian formulation of another result from [Mod87].

Proposition 4.19 (Riemannian version of [Mod87, Prop. 1, Prop. 3]). *Under assumption (d) in Section 2 for the potential W (see (2.5)), assume also that there exist constants $E^* > 0$, $t_0 > 0$, $0 < c_1 < c_2$, $2 < p_1 < \hat{2}$, $p_1 \leqslant p_2 \leqslant 2(p_1 - 1)$, with $\hat{2} := \frac{2^*}{2} + 1$, a sequence of positive numbers such that $\varepsilon_i \to 0^+$, and a sequence of functions u_{ε_i} satisfying*

$$E_{\varepsilon_i}(u_{\varepsilon_i}) \leqslant E^*, \forall i \in \mathbb{N}. \tag{4.13}$$

Then, there exists a subsequence still denoted $(\varepsilon_i)_i$ such that $(u_{\varepsilon_i})_i$ converges to a function $u_\infty \in BV(M)$ in $L^1(M)$. Moreover, there exists a finite perimeter set Ω such that $u_\infty = \chi_\Omega$ such that $|Du_\infty|(M) = \int_M |Du_\infty| \leqslant \frac{E^*}{\sigma}$, where $\sigma = \int_0^1 \sqrt{2W(s)}ds$.

Proof. Let ϕ be the primitive function of $(2W)^{\frac{1}{2}}$ with $\phi(0) = 0$, i.e.,

$$\phi(t) = \int_0^t (2W(s))^{\frac{1}{2}}\ \mathrm{d}s,$$

and set $v_{\varepsilon_i}(x) := \phi\left(u_{\varepsilon_i}(x)\right)$. We claim that the family $(v_{\varepsilon_i})_{\varepsilon_i>0}$ is bounded in $L^1(M)$. In fact if (2.5) holds, it is not restrictive to assume that $t_0 \geqslant 1$, and we easily have that

$$\begin{aligned}\phi(t) &= \int_0^{t_0} (2W(s))^{\frac{1}{2}}\, ds + \int_{t_0}^t (2W(s))^{\frac{1}{2}}\ \mathrm{d}s \\ &\leqslant \int_0^{t_0} (2W(s))^{\frac{1}{2}}\, ds + 2\frac{\sqrt{2c_2}}{p_2+2} t^{\frac{p_2}{2}+1}, \quad \forall t \geqslant t_0.\end{aligned}$$

Moreover, $p_2 \leqslant 2(p_1 - 1)$ implies that $\frac{p_2}{2} + 1 \leqslant p_1$; hence

$$\phi(t) \leqslant c_3' + c_4' W(t), \quad \forall t \geqslant 0,$$

for some real constants c_3' and c_4'. One can prove an analogous estimate for $t \leqslant 0$, so we get

$$|\phi(t)| \leqslant c_3 + c_4 W(t), \quad \forall t \in \mathbb{R},$$

for some real constants c_3 and c_4. Then,

$$\begin{aligned}\int_M |v_{\varepsilon_i}| dv_g &\leqslant c_3 \operatorname{Vol}_g(M) + c_4 \int_M W\left(u_{\varepsilon_i}(x)\right)\, \mathrm{d}v_g \\ &\leqslant c_3 \operatorname{Vol}_g(M) + c_4 \varepsilon_i E_{\varepsilon_i}\left(u_{\varepsilon_i}\right),\end{aligned}$$

thus

$$\int_M |v_{\varepsilon_i}| dv_g \leqslant c_3 \operatorname{Vol}_g(M) + \tilde{c}_4 E^*, \forall i \in \mathbb{N},$$

for some real constant $\widetilde{c}_4$, and from this we conclude that (v_{ε_i}) is a bounded sequence in $L^1(M)$. By the chain rule for BV-functions see Theorem 3.96 of [AFP00] we obtain easily that

$$|\nabla v_{\varepsilon_i}| = |\phi'(u_{\varepsilon_i})\nabla u_{\varepsilon_i}| = (2W(u_{\varepsilon_i}))^{\frac{1}{2}}\,|\nabla u_{\varepsilon_i}|.$$

From the elementary inequality $ab \leqslant \frac{\eta a^2}{2} + \frac{b^2}{2\eta}$ valid for every $\eta > 0$ and $a, b \in \mathbb{R}$ but nontrivial only when $a \cdot b \in\,]0, +\infty[$, putting $\eta := \varepsilon_i$, $a = |\nabla u_{\varepsilon_i}|$, $b = \sqrt{2W(u_{\varepsilon_i})}$ we get

$$\begin{aligned}\int_M |\nabla v_{\varepsilon_i}|\,\mathrm{d}v_g &\leqslant \int_M \left(\frac{1}{2}\varepsilon_i|\nabla u_{\varepsilon_i}|^2 + \frac{1}{\varepsilon_i}W(u_{\varepsilon_i})\right)\mathrm{d}v_g \\ &\leqslant E_{\varepsilon_i}(u_{\varepsilon_i}) \overset{(4.13)}{\leqslant} E^*. \end{aligned}\tag{4.14}$$

Applying the compactness theorem for bounded variation functions, (cf. [AFP00, Theorem 3.23] or [MPPP07, Proposition 1.4]), there exists a subsequence also denoted by $(v_{\varepsilon_i})_i$ and an a.e. pointwise limit function $v_\infty \in BV(M,g)$ that is the $L^1(M,g)$ limit of the v_{ε_i}, which satisfies

$$|Dv_\infty|(M) \leqslant \liminf_{i\to\infty} ||\nabla v_{\varepsilon_i}||_{1,M} \leqslant E^*,$$

where as customarily we denote by Dv_∞ the Radon measure representing the distributional derivative of v_∞ and by $|Dv_\infty|$ is its total variation. We now return to the study of the original functions u_{ε_i}. Let ψ be the inverse function of ϕ which always exists because, by our assumption on the double well potential W, $\phi(t)$ is monotone increasing; define $u_\infty(x) = \psi(v_\infty(x))$. By (2.5) then $\phi'(t) \geqslant \sqrt{2c_1}t_0^{p_1/2}$ for every $|t| \geqslant t_0$; hence ψ is Lipschitz continuous on $]-\infty, \phi(-t_0)]\mathring{\bigcup}[\phi(t_0), +\infty[$ and so uniformly continuous on the entire real line. From this combined with Theorem 2 of [BJ61] we infer that $u_{\varepsilon_i} = \psi \circ v_{\varepsilon_i}$ converges in measure on M to u_∞ as $\varepsilon_i \to 0^+$ so *a fortiori* also u_{ε_i} converges pointwise a.e. on M to u_∞ as $\varepsilon_i \to 0^+$; since

$$\begin{aligned}\int_M u_{\varepsilon_i}^{p_1}\,dv_g &\leqslant \int_M t_0^{p_1}\,dv_g + \frac{1}{c_1}\int_M W(u_{\varepsilon_i}(x))\,dv_g \\ &\leqslant t_0^{p_1}\operatorname{Vol}_g(M) + \frac{1}{c_1}\varepsilon_i E_{\varepsilon_i}(u_{\varepsilon_i}) \\ &\leqslant t_0^{p_1}\operatorname{Vol}_g(M) + \frac{\varepsilon_i}{c_1}E^*,\end{aligned}$$

we conclude that (u_{ε_i}) is bounded in $L^{p_1}(M)$ with $p_1 \geqslant 2$. This implies (via Hölder inequality) uniform integrability of the sequence $(u_{\varepsilon_i})_i$. Hence, by the classical theorem of Vitali (compare Theorem 2.18 of [ADPM11] or Theorem 4.5.4 of [Bog07]), we know that uniform integrability and convergence in measure (which implies pointwise convergence a.e.) that (u_{ε_i}) actually converges in $L^1(M)$ to u_∞. The remaining part of the proof goes along the same lines of the proof of [Mod87, Proposition 1]. In fact, by Fatou's Lemma and (4.13) it holds

$$\begin{aligned}0 \leqslant \int_M W(u_\infty)dv_g &\leqslant \liminf_{i\to\infty}\int_M W(u_{\varepsilon_i})dv_g \leqslant \liminf_{i\to\infty}\varepsilon_i E_{\varepsilon_i} \\ &\overset{(4.13)}{\leqslant} \liminf_{i\to\infty}\varepsilon_i E^* = 0.\end{aligned}$$

The last chain of inequalities shows that $W(u_\infty) = 0$ a.e. on M which in turn implies that $u_\infty(M) = \{0,1\}$ a.e. Moreover by Theorem 3.96 of [AFP00] and formula (3.90) of

[AFP00] the sets $u_\infty^{-1}(0)$ and $u_\infty^{-1}(1)$ are of finite perimeter in M, since by the Fleming-Rishel formula

$$\mathcal{H}_g^{N-1}(\partial^*\{u_\infty^{-1}(1)\}) = |Du_\infty|(M) = \int_M |Du_\infty| = \frac{1}{\sigma}\int_M |Dv_\infty| \overset{(4.14)}{\leqslant} \frac{E^*}{\sigma}, \tag{4.15}$$

where $\sigma = \sigma(0,1)$. This yields the proof of Proposition 4.19. □

For our purposes we need the following classical definition.

Definition 4.20. The ***isoperimetric profile function of*** (M^N, g) (or briefly, the isoperimetric profile) $I_{(M,g)} : [0, V(M)[\to [0, +\infty[$, is defined by

$$I_M(V) := \inf\{A_g(\partial\Omega) : \Omega \in \tau_M, V_g(\Omega) = V\},$$

where τ_M denotes the set of relatively compact open subsets of M with smooth boundary, where A_g is the $(N-1)$-volume form of $\partial\Omega$ induced by g.

Lemma 4.21 (Berard-Meyer)**.** *If (M, g) is compact, then*

$$I_M(V) \sim c_N V^{\frac{N-1}{N}}, \text{ as } V \to 0.$$

Lemma 4.22. *Let (M^n, g) be a compact Riemannian manifold. For every $1 > \eta > 0$, $V \in]0, V_g(M)[$, $\delta > 0$ there exists $\varepsilon_0 = \varepsilon_0(g, W, \eta, V, \delta) > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ and for any $u \in E_\varepsilon^c \cap \mathfrak{M}^V$ with $c = c(W, V, \delta) = \sigma I_M(V) + \delta > 0$, there exists $\Omega_{V,u}$ a finite perimeter set of volume V such that $||u - \chi_{\Omega_{V,u}}||_{L^1(M)} \leqslant \eta$ and $\mathcal{P}_g(\Omega_{V,u}) \leqslant \frac{c}{\sigma}$.*

Proof. We argue by contradiction. Suppose that the conclusion does not hold. Then there exist $1 > \eta > 0$, $V \in]0, V_g(M)[$, $\delta > 0$ a sequence $\varepsilon_i \to 0$, $u_{\varepsilon_i} \in E_{\varepsilon_i}^c \cap \mathfrak{M}^V$ such that for every Ω_V finite perimeter set of volume V we have

$$||u_{\varepsilon_i} - \chi_{\Omega_V}||_{L^1(M)} > \eta > 0. \tag{4.16}$$

If we assume furthermore that holds (2.5) we can apply Proposition 4.19 with $E^* := c$. This provides a subsequence still denoted $(\varepsilon_i)_i$, a finite perimeter set Ω_V of volume V such that $\mathcal{P}_g(\Omega_{V,(u_{\varepsilon_i})_i}) \leqslant \frac{c}{\sigma}$ and

$$||u_{\varepsilon_i} - \chi_{\Omega_{V,(u_{\varepsilon_i})_i}}||_{L^1(M)} \to 0, \text{ as } i \to +\infty.$$

This last equation contradicts (4.16) and in turn completes the proof of the lemma. □

Corollary 4.23. *Let (M^n, g) be a compact Riemannian manifold. For every $1 > \eta > 0$, $V \in]0, V_g(M)[$, $\delta > 0$ there exists $\varepsilon_0 = \varepsilon_0(g, W, \eta, V, \delta) > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ and for any $u \in E_\varepsilon^c \cap \mathfrak{M}^V$ with $c = c(W, V, \delta) = \sigma c_N V^{\frac{N-1}{N}} + \delta > 0$ there exists $\Omega_{V,u}$ a finite perimeter set of volume V such that $||u - \chi_{\Omega_{V,u}}||_{L^1(M)} \leqslant \eta$ and $\mathcal{P}_g(\Omega_{V,u}) \leqslant \frac{c}{\sigma}$.*

Observe that we can choose δ sufficiently small and refine the result of Lemma 4.21 in order to have that $\Omega_{u,V}$ as above is actually an isoperimetric region; this yields the following concentration lemma for functions with energy close to the minimum energy level.

Lemma 4.24. *Let (M^n, g) be a compact Riemannian manifold. For every $1 > \eta > 0$, $V \in]0, V_g(M)[$, there exist $\delta_0 = \delta_0(\eta, V, M^n, g, W) > 0$ such that for every $0 < \delta < \delta_0$ there exists $\varepsilon_0 = \varepsilon_0(g, W, \eta, V, \delta) > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ and for any $u \in E_\varepsilon^c \cap \mathfrak{M}^V$ with $c = c(W, V, \delta) = \sigma I_M(V) + \delta$ there exists $\Omega_{V,u}$ isoperimetric region of volume V such that $||u - \chi_{\Omega_{V,u}}||_{L^1(M)} \leqslant \eta$.*

Proof. We argue by contradiction. Suppose that the conclusion does not hold. Then there exist $1 > \eta > 0$, $V \in]0, V_g(M)[$, a sequence $\delta_i \to 0^+$ a second sequence $\varepsilon_i \to 0^+$, a sequence of functions $(u_{\varepsilon_i})_i$ satisfying $u_{\varepsilon_i} \in E_{\varepsilon_i}^{c_i = c(V,\delta_i) = \sigma I_M(V) + \delta_i} \cap \mathfrak{M}^V$ such that for every Ω_V isoperimetric region of volume V we have

$$||u_{\varepsilon_i} - \chi_{\Omega_V}||_{L^1(M)} > \eta > 0. \tag{4.17}$$

If we assume furthermore that (2.5) holds, we can apply Proposition 4.19 with $E^* := c_1$. We can also use the more sophisticated a priori estimates on the Lagrange multiplier as in Proposition 5.3 to show that u_{ε_i} are uniformly bounded with respect to ε_i. This provides a subsequence, still denoted by $(\varepsilon_i)_i$, a finite perimeter set $\Omega_V^{(1)}$ of volume V such that $\mathcal{P}_g(\Omega_V^{(1)}) \leqslant \frac{c_1}{\sigma}$ and

$$||u_{\varepsilon_i} - \chi_{\Omega_V^{(1)}}||_{L^1(M)} \longrightarrow 0, \text{ as } i \to +\infty. \tag{4.18}$$

To this subsequence we apply again Proposition 4.19, now with $E^* := c_2$. In this way we obtain again a new subsequence, still denoted $(\varepsilon_i)_i$, a finite perimeter set $\Omega_V^{(2)}$ of volume V such that $\mathcal{P}_g(\Omega_V^{(2)}) \leqslant c_2/\sigma$ and

$$||u_{\varepsilon_i} - \chi_{\Omega_V^{(2)}}||_{L^1(M)} \to 0, \text{ as } i \to +\infty. \tag{4.19}$$

The sequence appearing in (4.19) being a subsequence of the sequence appearing in (4.18) readily gives that $\Omega_V^{(2)} = \Omega_V^{(1)}$ by the uniqueness of the limit. Continuing this process and applying a standard diagonal argument we get the existence of a subsequence still denoted $(\varepsilon_i)_i$, a finite perimeter set $\Omega_V = \Omega_V^{(1)} = \Omega_V^{(2)} = \Omega_V^{(3)} = ...$, of volume V such that

$$\mathcal{P}_g(\Omega_V) \leqslant \frac{c_i}{\sigma}, \forall i \in \mathbb{N}, \tag{4.20}$$

and

$$||u_{\varepsilon_i} - \chi_{\Omega_V}||_{L^1(M)} \to 0, \text{ as } i \to +\infty. \tag{4.21}$$

From (4.20) we conclude immediately that $\mathcal{P}_g(\Omega_V) \leqslant I_M(V)$ and so a fortiori we can assert that Ω_V is an isoperimetric region of volume V. This last fact combined with equation (4.21) contradicts (4.17) and in turn completes the proof of the lemma. □

Corollary 4.25. *Let (M^n, g) be a compact Riemannian manifold. For every $\eta \in]0, 1[$, $V \in]0, V_g(M)[$, there exist $\delta_0 = \delta_0(\eta, V, M^n, g, W) > 0$ such that for every $\delta \in]0, \delta_0[$ there exists $\varepsilon_0 = \varepsilon_0(g, W, \eta, V, \delta) > 0$ such that for every $\varepsilon \in]0, \varepsilon_0[$ and for any $u \in E_\varepsilon^c \cap \mathfrak{M}^V$ with $c = c(W, V, \delta) = \sigma c_N V^{\frac{N-1}{N}} + \delta$ there exists $\Omega_{V,u}$ isoperimetric region of volume V such that $||u - \chi_{\Omega_{V,u}}||_{L^1(M)} \leqslant \eta$.*

Lemma 4.26 ([MJ00, Theorem 2.2] and [NOA18, Theorem 3.0])**.** *Let (M^n, g) be a compact Riemannian manifold. There exist two positive constants $\mu^* = \mu^*(M) > 0$ and $v^* = v^*(M) > 0$ such that whenever $\Omega \subseteq M$ is an isoperimetric region of volume $0 \leqslant v \leqslant v^*$ it holds that*

$$\operatorname{diam}_g(\Omega) \leqslant \mu^* v^{\frac{1}{n}}.$$

Lemma 4.27. *For any $\eta \in]0, 1[$ sufficiently small and any $r \in \left]0, \frac{1}{2}\mathrm{inj}_M\right[$ there exists $V_2 = V_2(M^n, \eta, r) > 0$ s.t. for all $V \in]0, V_2]$ there exists $\delta_0 = \delta_0(\eta, V, M^n, g, W) > 0$ such that for every $\delta \in]0, \delta_0[$ there exists $\varepsilon_2 = \varepsilon_2(g, W, \eta, V, \delta) > 0$ such that for every $\varepsilon \in]0, \varepsilon_2[$ and for any $u \in E_\varepsilon^c \cap \mathfrak{M}^V$ with $c = c(W, V, \delta) = \sigma I_M(V) + \delta$ there exists*

$\Omega_{V,u}$ isoperimetric region of volume V such that $||u-\chi_{\Omega_{V,u}}||_{L^1(M)} \leqslant \eta$. In particular for any $\widetilde{\eta} \in]0,1[$ (close to 1) there exists $p_u \in M$ such that

$$\int_{B_g(p_u,r/2)} |u| dv_g \geqslant \widetilde{\eta} V. \tag{4.22}$$

Proof. By [NOA18, Lemma 4.9] we know that there exists $v_0^* := v_0^*(n,k,\mathrm{inj}_M,r) > 0$ such that for every isoperimetric region Ω of volume V smaller than V_0^* is contained in a geodesic ball of radius $r/2$. Furthermore, by Lemma 4.24 we get

$$||u-\chi_{\Omega_{V,u}}||_{L^1(M)} \leqslant \eta.$$

From this it is straightforward to deduce the theorem. To see the last fact, if we suppose that for all $p \in M$ we have that $(1-\eta)V \leqslant \int_{B_g(p,r/2)} u \leqslant V$, for some η. But assuming that holds (2.5), and by Lemma 4.24 for all isoperimetric region Ω_V of volume $V \leqslant V_0$

$$\|u-\chi_{\Omega_V}\|_{L^1(M)} > \eta > 0.$$

By lemma 4.26 and for V small, exist p_{Ω_V} such that $\Omega_V \subset B_g(p_{\Omega_V}, r/2)$, and satisfies

$$(1-\eta)V \leqslant \int_{B_g(p_{\Omega_V},r/2)} u \leqslant V = \int_{B_g(p_{\Omega_V},r/2)} \chi_{\Omega_V}$$

then we get the contradiction. □

Corollary 4.28. *For any $\eta \in]0,1[$ sufficiently small and $r \in \left]0, \frac{1}{2}\mathrm{inj}_M\right[$ there exists $V_3 = V_3(n,k,v_0,\eta,r) > 0$ such that for all $V \in]0,V_3]$ there exist $\delta_0 = \delta_0(\eta,V,M^n,g,W) > 0$ such that for every $\delta \in]0,\delta_0[$ there exists $\varepsilon_3 = \varepsilon_3(g,W,\eta,V,\delta) > 0$ such that for every $\varepsilon \in]0,\varepsilon_3[$ and for any $u \in E_\varepsilon^c \cap \mathfrak{M}^V$ with $c = c(W,V,\delta) = \sigma c_N V^{\frac{N-1}{N}} + \delta$ there exists $\Omega_{V,u}$ isoperimetric region of volume V such that*

$$||u-\chi_{\Omega_{V,u}}||_{L^1(M)} \leqslant \eta.$$

In particular for any $\widetilde{\eta} \in]0,1[$ sufficiently close to 1 there exists $p_u \in M$ with

$$\int_{B_g(p_u,r/2)} |u| dv_g \geqslant \widetilde{\eta} V. \tag{4.23}$$

Let us denote by $\mathrm{diam}_{\mathbb{R}^L}(M)$ the diameter of M as subset of $\mathbb{R}^L$.

Lemma 4.29. *For $r \in \left]0, \frac{1}{2}\mathrm{inj}_M\right[$ there exists $V_4 = V_4(n,k,v_0,\mathrm{inj}_M,r,\mathrm{diam}_{\mathbb{R}^L}(M)) > 0$ such that for every $0 < V < V_4$, there exists $\varepsilon_4 = \varepsilon_4(V) > 0$, $0 < \varepsilon < \varepsilon_4$, and every $u \in \mathfrak{M}_V \cap E_\varepsilon^c$ we have $\beta(u) \in M_r$.*

Proof. Define $\rho(u(x)) := \frac{|u(x)|}{\int_M |u(x)| dv_g}$. By (4.23) for every $V \in]0,V_3[$ we obtain $\int_{B_g(p_u,r/2)} \rho(u(x)) dv_g \geqslant \eta V$, where $0 < \eta < 1$ will be chosen later. From this last inequality we deduce

$$\begin{aligned} |\beta(u)-p_u| &= \left| \int_M (x-p_u)\rho(u(x)) dv_g \right| \\ &\leqslant \left| \int_{B_g(p_u,r/2)} (x-p_u)\rho(u(x)) dv_g \right| \\ &+ \left| \int_{M\setminus B_g(p_u,r/2)} (x-p_u)\rho(u(x)) dv_g \right| \\ &\leqslant \frac{r}{2} + 2D(1-\eta), \end{aligned}$$

where $D := \mathrm{diam}_{\mathbb{R}^L}(M)$. Choosing η close to 1 such that $2D(1-\eta) < \frac{r}{2}$ and applying Lemma 4.27 we conclude the proof of the Lemma. $\square$

Corollary 4.30. *There exists $r_0 = r_0(M) > 0$ such that for any $r \in]0, r_0[$, there exists $V_5 = V_5\big(n, k, v_0, \mathrm{inj}_M, r, \mathrm{diam}_{\mathbb{R}^L}(M)\big) > 0$ such that for every $V \in]0, V_5[$, there exists $\varepsilon_5 = \varepsilon_5(V) > 0$ such that for every $\varepsilon \in]0, \varepsilon_5[$, we have $d_g(\pi \circ \beta \circ \Phi_{\varepsilon,V}(x_0), x_0) < \mathrm{inj}_M$. In particular $\pi \circ \beta \circ \Phi_{\varepsilon,V}$ is homotopic to the identity map of M^n.*

Proof. As in Lemma 4.29, if we choose r_0 small enough depending only on the second fundamental form of the isometric immersion of M in $\mathbb{R}^N$ and the injectivity radius of M, it is easy to see that we have $d_g(\pi \circ \beta \circ \Phi_{\varepsilon,V}(x_0), x_0) \leqslant C(||II_M||_\infty) r_0 < \mathrm{inj}_M$, because M is compact. To understand this standard argument of extrinsic Riemannian geometry, the reader can look up [Nar18, Lemma 2.1]. Let us now define the homotopy $F : [0,1] \times M \to M$,

$$F(t, x_0) := \exp_{x_0}(t \exp_{x_0}^{-1}(\pi \circ \beta(\Phi_{\varepsilon,V}(x_0)))).$$

From the very definition of F it is easy to check that $F(0, x_0) = x_0$ and $F(1, x_0) = \pi \circ \beta \circ \Phi_{\varepsilon,V}(x_0)$ for every $x_0 \in M$. Checking the continuity of F with respect to x_0 is a standard fact of Riemannian geometry about the exponential map. $\square$

We are finally in position to prove Theorem 2.1.

Proof. Set $V^* := \min\{V_0, V_1, V_2, V_3, V_4, V_5\} > 0$, then fix $0 < \delta < \delta_0$, with δ_0 as in Lemma 4.27, and set $\varepsilon^* = \min\{\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5\}$, $c = \sigma c_N V^{\frac{N-1}{N}} + \delta$. Then for any $V \in]0, V^*[$ and $\varepsilon \in]0, \varepsilon^*[$, by an easy application of Proposition 4.13, Lemma 4.29 and Corollary 4.30 we obtain the functions $f := \Phi_{\varepsilon,V}$ and $g := \pi \circ \beta$ required to apply Theorem 4.4 to $X = M$, $J = E_\varepsilon|_{\mathfrak{M}^V}$, $\mathfrak{M} = \mathfrak{M}^V$. The conclusion then follows readily.

The last assertion of the theorem follows directly from Theorem 4.9, using the nondegeneracy assumption. $\square$

5. Dropping the subcritical growth condition

We will now show how to deal with the case where one does not assume the subcritical growth of the potential (2.3). The idea is to show some *a priori* estimates on the solutions (and for the corresponding Lagrange multiplier), and then consider a perturbed problem that satisfies the growth condition, whose solutions are also solutions of the original problem. Towards this goal, we need two auxiliary lemmas that have their own interest. Recalling Remark 4.11, a careful inspection of the proof of Theorem 2.1, Corollaries 4.30, and 4.28 reveals that we have:

Lemma 5.1. *Let W satisfying assumptions* (2.2), (2.3)*, and* (2.4)*. Then, there exists $\hat{c} = \hat{c}\big(N, \varepsilon, V, s_0, W|_{[0,s_0]}\big) > \inf_{\mathfrak{M}^V} E_\varepsilon$ such that for all $c \in]\inf_{\mathfrak{M}^V} E_\varepsilon, \hat{c}]$, the sublevel E_ε^c is* homotopically superjacent *to M (see Remark 4.11).* $\square$

The proof of the following result goes along the same lines as the proof of [Che96, Lemma 3.4]. Before giving its statement, it will be useful to make a remark.

5.2. *Remark.* Let us observe that from the quadratic growth condition obtained integrating two times $W''(t) \geqslant c_0 > 0, \forall |t| \geqslant t_0$, on the interval $[t_0, s]$ we obtain $W(s) \geqslant W(t_0) + (s - t_0)W'(t_0) + \frac{1}{2}(s - t_0)^2 c_0$ from which we conclude that there exists $t_1 \geqslant t_0$, $c_0' > c_0$, such that $W(s) \geqslant \frac{1}{2} s^2 c_0'$, for every s such that $|s| \geqslant t_1$.

Proposition 5.3 (Lagrange Multiplier Estimates)**.** *Let $\mathcal{E}_0, V$, and $\bar{\varepsilon} \in]0,+\infty[$ be fixed, and assume that $(u_\varepsilon)_{\varepsilon\in]0,\bar{\varepsilon}]}$ is a family of solutions for the equation*

$$-\operatorname{div}(\varepsilon\nabla u_\varepsilon) + \tfrac{1}{\varepsilon}W'\big(u_\varepsilon\big) = -\lambda_\varepsilon, \quad \textit{in } M, \int_M u_\varepsilon = V > 0, \tag{5.1}$$

where W satisfy (2.2)*, $W''(1) > 0$, $W''(s) \geqslant c_0 > 0$ if $|u| \geqslant t_0$ for some $c_0 > 0$ and $t_0 > s_0 > 0$, i.e., large quadratic or super quadratic growth such that*

$$0 \leqslant E_\varepsilon[u_\varepsilon] \leqslant \mathcal{E}_0, \forall \varepsilon \in]0,\bar{\varepsilon}]\,. \tag{5.2}$$

Then there exist positive constants $c_1 = c_1(N, \mathrm{Vol}_g(M), V, \mathcal{E}_0, t_0, c_0, W\big|_{[-t_1,t_1]}) > 0$ ($c_1 > 0$ large) and $\varepsilon_0 = \varepsilon_0(N, \mathrm{Vol}_g(M), V, \mathcal{E}_0, t_0, W|_{[-t_1,t_1]}) > 0$, ($\varepsilon_0 > 0$ small) such that for any $\hat{\varepsilon} \in]0, \varepsilon_0]$ we have $c_1 \leqslant \widetilde{c}_1$, $\widetilde{c}_1 = \widetilde{c}_1(N, \mathrm{Vol}_g(M), V, \mathcal{E}_0, t_0, c_0, W|_{[-t_1,t_1]}) > 0$ with

$$\begin{aligned} |\lambda_{\hat{\varepsilon}}(u_{\hat{\varepsilon}})| &\leqslant c_1 E_{\hat{\varepsilon}}(u_{\hat{\varepsilon}}) \\ &\leqslant \widetilde{c}_1 E_{\hat{\varepsilon}}(u_{\hat{\varepsilon}}) \\ &\leqslant \widetilde{c}_1 \mathcal{E}_0. \end{aligned}$$

5.4. *Remark.* The assumptions of Proposition 5.3 are satisfied in the case of the classical symmetric Van der Waals-Allen-Cahn-Hilliard potential that is a positive polynomial of fourth order with just two absolute minima at which the potential is zero.

5.5. *Remark.* Roughly speaking, Proposition 5.3 says that the constants involved in the statement of our results depend on the geometry of the problem, on an upper bound of the energy, on the behavior of the potential over a compact interval, and on the index of quadratic and superquadratic growth at infinity, which is represented by the constant $c_0 > 0$.

Proof of Proposition 5.3. We can assume w.l.g. that $0 < \bar{\varepsilon} < 1$. Looking at the equation (5.1) we want to give a uniform estimate with respect to ε of $\lambda_{\varepsilon,V}$ depending only on the energy of the associated solutions. With this aim in mind, we will make use of an auxiliary function $\psi_{\varepsilon,\rho} : M \to \mathbb{R}$ given as the unique solution to

$$\begin{cases} \Delta\psi_{\varepsilon,\rho} = u_{\varepsilon,\rho} - \bar{u}_{\varepsilon,\rho}, \text{ in } M, \\ \displaystyle\int_M \psi_{\varepsilon,\rho} = 0, \end{cases} \tag{5.3}$$

with $u_{\varepsilon,\rho} := u_\varepsilon * \psi_\rho$, where ψ_ρ is the usual mollification kernel satisfying $\int \psi_\rho = 1$ and $\bar{u}_{\varepsilon,\rho} := \mathrm{Vol}_g(M)^{-1}\int_M u_{\varepsilon,\rho}dx$. By a direct computation coming from the very definition of $u_{\varepsilon,\rho}$ we get

$$\begin{aligned} ||u_{\varepsilon,\rho}||_{\infty,M} &= ||[(u_\varepsilon - 1) + 1] * \psi_\rho||_{\infty,M} \\ &\leqslant 1 + \sup_{x\in M_\rho}\int_{B_{\mathbb{R}^N}(x,\rho)} \psi_\rho(y)\,||u_\varepsilon(x-\rho y)| - 1|\;\mathrm{d}y \\ &\overset{\text{Hölder}}{\leqslant} 1 + C\rho^{-\frac{N}{2}}\,|||u_\varepsilon| - 1||_{2,M} \\ &\leqslant 1 + C\sqrt{\mathcal{E}_0}\varepsilon^{\frac{1}{2}}\rho^{-\frac{N}{2}}. \end{aligned}$$

The last inequality is an immediate consequence of

$$\int_M (|u_\varepsilon| - 1)^2 \leqslant C\varepsilon\mathcal{E}_0, \tag{5.4}$$

where $C = C\big(W|_{[-t_0,t_0]}, c_0\big) > 0$.

In order to show (5.4) we start by considering the Taylor expansion of W near the point $s=1$ on the s-axis, which gives the existence of θ between 1 and s such that

$$W(s)+m=W(s)-W(1)=W'(1)(s-1)+\frac{1}{2}(s-1)^2W''(\theta_s).$$

From this we easily obtain that for $|s-1|\leqslant\delta\leqslant\frac{1}{2}$

$$W''(\theta_s)\geqslant\eta_0(\delta,W|_{[1-\delta,1+\delta]})>0,$$

because by assumption (2.2) we have $W''(1)>0$. For $D:=\{s\in\mathbb{R}||s-1|\geqslant\delta, s\in[t_0,t_0]\}$ we have that the function g defined as

$$g(s):=W''(\theta_s)=\frac{W(s)}{(s-1)^2}>0, g:D\to\mathbb{R},$$

is continuous and stays away from zero on an compact interval. Hence $\inf_{s\in D}\{g(s)\}=:\eta_1(\delta,W|_D)>0$. Finally for $|s|\geqslant t_0$ it is immediate to get $W''(\theta_s)>c_0>0$. This argument implies readily that there exists $\eta_2=\eta_2(m,W|_{[-t_0,t_0]},c_0):=\min\{\eta_0,\eta_1,c_0\}$ such that

$$\int_M(|u_\varepsilon|-1)^2\leqslant\int_M(u_\varepsilon-1)^2\leqslant\frac{1}{\eta_2}[\varepsilon E_\varepsilon[u_\varepsilon]-\frac{1}{2}||\nabla u_\varepsilon||^2_{L^2}]\leqslant\frac{1}{\eta_2}\varepsilon\mathcal{E}_0.$$

From the last inequality we infer quickly (5.4) setting $C:=\frac{1}{\eta_2}$. Analogously it is not too hard to see that from the very definition of the mollifier and the theorem of derivation under the integral sign we get

$$||u_{\varepsilon,\rho}||_{C^1(M)}\leqslant C(\nabla\psi,\mathrm{Vol}_g(M),W\big|_{[-t_0,t_0]},c_0)\widetilde{\mathcal{E}}_0\rho^{-1}(1+\varepsilon^{\frac{1}{2}}\rho^{-\frac{N}{2}}),\tag{5.5}$$

where $\widetilde{\mathcal{E}}_0:=\max\{\sqrt{\mathcal{E}_0},1\}$. Thus by classical Schauder's elliptic estimates we conclude

$$\begin{aligned}||\psi_{\varepsilon,\rho}||_{C^2(M)}\leqslant&C||u_{\varepsilon,\rho}||_{C^1(M)}\\&\overset{(5.5)}{\leqslant}(\nabla\psi,\mathrm{Vol}_g(M),W_{|[-t_0,t_0]},c_0)\widetilde{\mathcal{E}}_0\rho^{-1}(1+\varepsilon^{\frac{1}{2}}\rho^{-\frac{N}{2}}).\end{aligned}\tag{5.6}$$

Now we come back to our uniform estimates on λ_ε and multiply (5.1), by the function $\varphi_{\varepsilon,\rho}:=\langle\nabla\psi_{\varepsilon,\rho},\nabla u_\varepsilon\rangle$ then we integrate over M and use the divergence theorem obtaining

$$\begin{aligned}\int_M\varphi_{\varepsilon,\rho}(-\lambda_\varepsilon)dvol_g &= \int_M\varphi_{\varepsilon,\rho}\left(-\operatorname{div}(\varepsilon\nabla u_\varepsilon)+\frac{1}{\varepsilon}W'(u_\varepsilon)\right)\\
&\overset{(5.3)}{\leqslant} \frac{1}{\varepsilon}\int_M W'(u_\varepsilon)\langle\nabla\psi_{\varepsilon,\rho},\nabla u_\varepsilon\rangle_g\\
&= \frac{1}{\varepsilon}\int_M W(u_\varepsilon)\operatorname{div}(\nabla\psi_{\varepsilon,\rho})\\
&= \frac{1}{\varepsilon}\int_M W(u_\varepsilon)\operatorname{div}(\nabla\psi_{\varepsilon,\rho})\\
&\leqslant \frac{1}{\varepsilon}\int_M W(u_\varepsilon)|\operatorname{div}(\nabla\psi_{\varepsilon,\rho})|\\
&\leqslant ||\psi_{\varepsilon,\rho}||_{C^2(M)}E_\varepsilon(u_\varepsilon)\\
&\overset{(5.6)}{\leqslant} C\widetilde{\mathcal{E}}_0\rho^{-1}(1+\varepsilon^{\frac{1}{2}}\rho^{-\frac{N}{2}})E_\varepsilon(u_\varepsilon).\end{aligned}\tag{5.7}$$

An integration by parts on the left-hand side of the above inequality yields

$$\begin{aligned}
\int_M \langle \nabla\psi_{\varepsilon,\rho}, \nabla u_\varepsilon\rangle dx &= -\int_{\partial M} u_\varepsilon \langle \nabla\psi_{\varepsilon,\rho}, \nu_{\partial M}\rangle dx \\
&+ \int_M u_\varepsilon \operatorname{div}(\nabla\psi_{\varepsilon,\rho}) dx \\
&\overset{(5.3)}{=} \int_M u_\varepsilon \Delta\psi_{\varepsilon,\rho} dx \\
&\overset{(5.3)}{=} \int_M u_\varepsilon \left(u_{\varepsilon,\rho} - \bar{u}_{\varepsilon,\rho}\right) dx \\
&= \int_M u_\varepsilon \left(u_{\varepsilon,\rho} - u_\varepsilon\right) dx + \int_M (u_\varepsilon^2 - 1) dx \\
&+ \operatorname{Vol}_g(M)(1-\bar{u}_\varepsilon^2) + \operatorname{Vol}_g(M)\bar{u}_\varepsilon(\bar{u}_\varepsilon - \bar{u}_{\varepsilon,\rho}).
\end{aligned}$$

Recall here that $\bar{u}_\varepsilon = \frac{V}{\operatorname{Vol}_g(M)} \in]0,1[$; using the equality $(x-1)^2 + 2(x-1) = x^2 - 1$ and an application of Hölder inequality we obtain

$$\begin{aligned}
\int_M |u_\varepsilon^2 - 1| dx &\leqslant \int_M |u_\varepsilon^2 - 1|^2 dx + 2\int_M |u_\varepsilon - 1| dx \\
&\overset{\text{by (5.4)}}{\leqslant} C\varepsilon\mathcal{E}_0 + 2\sqrt{C\varepsilon\mathcal{E}_0}\operatorname{Vol}_g(M)^{\frac{1}{2}} \qquad (5.8)\\
&\leqslant C\hat{\mathcal{E}}_0\sqrt{\varepsilon};
\end{aligned}$$

for the last inequality in (5.8) we have taken $\hat{\mathcal{E}}_0 := \mathcal{E}_0 + 2\sqrt{\mathcal{E}_0}|M|^{\frac{1}{2}} > 0$, assuming without losso of generality $\varepsilon \in]0,1[$ and $C > 1$. In order to verify

$$\begin{aligned}
||\bar{u}_{\varepsilon,\rho} - \bar{u}_\varepsilon||_{\infty,M} &\overset{\text{Hölder}}{\leqslant} \operatorname{Vol}_g(M)^{-\frac{1}{2}} ||u_{\varepsilon,\rho} - u_\varepsilon||_{2,M} \qquad (5.9)\\
&\leqslant C\sqrt{\rho}, \qquad (5.10)
\end{aligned}$$

it is convenient to introduce a new auxiliary function w_ε defined by $w_\varepsilon = \widetilde{W} \circ u_\varepsilon$ where

$$\widetilde{W}(s) = \int_0^s \sqrt{2\widetilde{F}(t)} dt,$$

$$\widetilde{F}(t) := \min\left\{W(t)+1, 1+|t|^2\right\} \geqslant 1, \quad \forall t \in \mathbb{R}.$$

Notice that

$$\int_M |\nabla w^\varepsilon| = \int_M \sqrt{2\widetilde{F}\left(u_\varepsilon\right)} |\nabla u_\varepsilon| \leqslant \int_M e_\varepsilon\left(u_\varepsilon\right) = E_\varepsilon[u_\varepsilon] \leqslant \mathcal{E}_0,$$

where $e_\varepsilon(u) = \frac{\varepsilon}{2}|u|^2 + \frac{1}{\varepsilon}W(u)$ is the energy density. Furthermore, by the properties of W, there are positive constants c_1^* and c_2 such that

(5.11)

$$c_1^*\left|s_1 - s_2\right|^2 \leqslant \left|\widetilde{W}\left(s_1\right) - \widetilde{W}\left(s_2\right)\right| \leqslant c_2\left|s_1 - s_2\right|\left(1 + |s_1| + |s_2|\right), \forall s_1, s_2 \in \mathbb{R}.$$

and

$$\begin{aligned}
\int_M \left|u_{\varepsilon,\eta} - u_\varepsilon\right|^2 dx &\leqslant \int_M \int_{B_1} \psi_\rho(y)\left|u_\varepsilon(x-\rho y) - u_\varepsilon(x)\right|^2 \mathrm{d}y\,\mathrm{d}x \\
&\leqslant c_1^* \int_M \int_{B_1} \psi_\rho(y)\left|w_\varepsilon(x-\rho y) - w_\varepsilon(x)\right| \mathrm{d}y\,\mathrm{d}x \\
&\overset{\text{[GT01, Lemma 7.23]+Fubini}}{\leqslant} c_1^*|B_1|\rho\left\|\nabla w_\varepsilon(\cdot)\right\|_{1,M} \leqslant C\rho, \quad (\text{by (5.11)}),
\end{aligned}$$

where $C = C(\mathcal{E}_0, \mathrm{Vol}_g(M)) = c_1^* M_N \mathcal{E}_0 > 0$. From the last inequality and (5.9) we obtain easily (5.10). By the quadratic growth condition obtained integrating two times $W''(t) \geqslant c_0 > 0$, on the interval $[t_0, s]$ we obtain $W(s) \geqslant W(t_0) + (s-t_0)W'(t_0) + \frac{1}{2}(s-t_0)^2 c_0$ from which we conclude that there exists $t_1 \geqslant t_0$, $c_0' > c_0 > 0$, such that $W(s) \geqslant \frac{1}{2}s^2 c_0'$, for every $s \geqslant t_1$. We use this information to give the following estimate

$$\begin{aligned}
\int_M u_\varepsilon^2 &= \int_{|u_\varepsilon|<t_1} u_\varepsilon^2 + \int_{|u_\varepsilon|\geqslant t_1} u_\varepsilon^2 \\
&\leqslant \int_{|u_\varepsilon|\geqslant t_1} \frac{1}{2}c_0' W(u_\varepsilon) + \int_{|u_\varepsilon|<t_1} u_\varepsilon^2 \\
&\leqslant \int_{|u_\varepsilon|\geqslant t_1} \frac{1}{2}c_0' \underbrace{(W(u_\varepsilon))}_{\geqslant 0} + \int_{|u_\varepsilon|<t_1} \frac{1}{2}c_0'(W(u_\varepsilon)) \\
&\leqslant \int_{|u_\varepsilon|\geqslant t_1} \frac{1}{2}c_0'(W(u_\varepsilon)) \\
&\leqslant \frac{1}{2}c_0' \varepsilon E_\varepsilon(u_\varepsilon) \leqslant \frac{1}{2}c_0'\varepsilon\mathcal{E}_0 \overset{\varepsilon<1}{\leqslant} \frac{1}{2}c_0'(\mathcal{E}_0)^+ \leqslant \frac{1}{2}c_0'|\mathcal{E}_0|,
\end{aligned} \tag{5.12}$$

which implies by an application of Hölder inequality that

$$\begin{aligned}
\left|\int_M u_\varepsilon\,(u_{\varepsilon,\rho} - u_\varepsilon)\,dx\right| &\leqslant ||u_\varepsilon||_2||u_{\varepsilon,\rho} - u_\varepsilon||_2 \\
&\overset{(5.9)-(5.12)}{\leqslant} C\sqrt{c_0'|\mathcal{E}_0|}\sqrt{\rho} = C\mathcal{E}_0^*\sqrt{\rho}.
\end{aligned} \tag{5.13}$$

So

$$\begin{aligned}
&\int_M u_\varepsilon \operatorname{div}(\nabla\psi_{\varepsilon,\rho})dx \\
&\overset{(5.8)-(5.13)}{\geqslant} \mathrm{Vol}_g(M)\left(1 - \left(\frac{V}{\mathrm{Vol}_g(M)}\right)^2\right) - C\hat{\mathcal{E}}_0(\sqrt{\varepsilon}) - C(1+\mathcal{E}_0^*)\sqrt{\rho}.
\end{aligned} \tag{5.14}$$

Now combining (5.8) and (5.14) we deduce that

$$|\lambda_{\varepsilon,V}| \leqslant \frac{C\widetilde{\mathcal{E}}_0\rho^{-1}(1+\varepsilon^{\frac{1}{2}}\rho^{-\frac{N}{2}})E_\varepsilon(u_\varepsilon)}{\mathrm{Vol}_g(M)(1-\left(\frac{V}{\mathrm{Vol}_g(M)}\right)^2) - C\hat{\mathcal{E}}_0\sqrt{\varepsilon} - C(1+\mathcal{E}_0^*)\sqrt{\rho}}. \tag{5.15}$$

So taking ε such that

$$\frac{1}{2}\mathrm{Vol}_g(M)\left(1 - \left(\frac{V}{\mathrm{Vol}_g(M)}\right)^2\right) - C\hat{\mathcal{E}}_0\sqrt{\varepsilon} \leqslant \frac{\mathrm{Vol}_g(M)}{4}\left(1 - \left(\frac{V}{\mathrm{Vol}_g(M)}\right)^2\right), \tag{5.16}$$

and ρ such that

$$\frac{1}{2}\mathrm{Vol}_g(M)\left(1 - \left(\frac{V}{\mathrm{Vol}_g(M)}\right)^2\right) - C(1+\mathcal{E}_0^*)\sqrt{\rho} \leqslant \frac{\mathrm{Vol}_g(M)}{4}\left(1 - \left(\frac{V}{\mathrm{Vol}_g(M)}\right)^2\right),$$

we conclude that

$$|\lambda_{\varepsilon,V}| \leqslant \frac{2C\widetilde{\mathcal{E}}_0\rho^{-1}(1+\varepsilon^{\frac{1}{2}}\rho^{-\frac{N}{2}})E_\varepsilon(u_\varepsilon)}{\mathrm{Vol}_g(M)\left(1 - \left(\frac{V}{\mathrm{Vol}_g(M)}\right)^2\right)}. \qquad \square$$

5.6. *Remark.* In Proposition 5.3 we do not require any subcritical growth condition, just that the second derivative of the potential is bounded from below by a positive constant in a neighborhood of infinity.

In our next result we generalize Theorem 2.1 by dropping assumption (2.3), and replacing it with a quite more general one. The price we pay is a weaker estimate on the lower bound on the number of solutions than the one determined in Theorem 2.1. In fact, our proof gives only the existence of low energy solutions. We conjecture that the following result remains true also for high energy solutions, but at the moment we are unable to give a complete proof. Thus under the assumptions of the preceding lemma we state the following theorem.

Theorem 5.7. *For every $W \in C^2(\mathbb{R})$ satisfying* (2.2) *and $W''(1) > 0$, $W''(s) > c_0 > 0, \forall |s| \geqslant t_0 > 1 > 0$, for some large t_0 there exists $V_1 = V_1\big(W|_{[-t_1,t_1]}, ...\big) > 0$, such that for every $V \in \,]0, V_1[$ there exists $\varepsilon_1 = \varepsilon_1(V) > 0$ with the property that for every $\varepsilon \in \,]0, \varepsilon_1[$, Problem $(P_{\varepsilon,V})$ admits at least* $\mathrm{cat}(M)$ *distinct solutions. Moreover, assume additionally that for given $V \in \,]0, V_1[$ and $\varepsilon \in \,]0, \varepsilon_1(V)[$, all solutions of Problem $(P_{\varepsilon,V})$ having energy less than or equal to the constant $c(\varepsilon, V, N, W)$ defined in Lemma 4.27 are nondegenerate* (*see Definition 4.8*). *Then, Problem $(P_{\varepsilon,V})$ has at least $P_1(M)$ distinct solutions.*

Proof. Let V^* be the constant determined in Theorem 2.1, and assume in the rest of the proof that $V \in \,]0, V^*[$. We can suppose that W satisfies

$$\limsup_{s\to+\infty} W'(s) = +\infty, \text{ and } \liminf_{s\to-\infty} W'(s) = -\infty, \tag{5.17}$$

because otherwise W' would be bounded, and so W would satisfy a growth condition as in (2.3), falling under the assumptions of Theorem 2.1. Now, using (5.17) it is easy to see that there exists $\hat{s}^- \leqslant -t_1$, $\hat{s}^+ \geqslant t_1$ such that

$$-\tfrac{1}{\varepsilon}\, W'(\hat{s}^-) - \lambda^* > 0, \tag{5.18}$$

and

$$-\tfrac{1}{\varepsilon}\, W'(\hat{s}^+) + \lambda^* < 0, \tag{5.19}$$

where $0 < V < V_1$, $\lambda^* = \lambda^*(N, \mathrm{Vol}_g(M), \varepsilon, V, s_0, t_0, \hat{c}, W|_{[-t_1,t_1]}) := c_1\hat{c}$ with the notations of Lemma 5.1 and Proposition 5.3. Consider the quadratic truncated problem $(\hat{\mathrm{P}}_{V,\varepsilon})$: for fixed positive constants V and ε, find $u \in H_0^1(M)$, and $\lambda \in \mathbb{R}$ such that

$$\begin{gathered} -\varepsilon^2 \Delta u + \widehat{W}'(u) = \lambda, \\ \int_M u(x)\,\mathrm{d}x = V, \end{gathered} \tag{5.20}$$

with the same M as in the statement of the theorem and $\widehat{W} \in C^2(\mathbb{R})$ satisfying $\widehat{W}(s) := W(s),\ \forall s \in [\hat{s}^-, \hat{s}^+]$, (2.3),

$$-\tfrac{1}{\varepsilon}\widehat{W}'(s) - \lambda^* > 0, \quad \forall s \in \big]{-\infty}, \hat{s}^-\big], \tag{5.21}$$

$$-\tfrac{1}{\varepsilon}\widehat{W}'(s) + \lambda^* < 0, \quad \forall s \in \big[\hat{s}^+, +\infty\big[. \tag{5.22}$$

Observe that it is always possible to find such a $\widehat{W}$. It is straightforward to check that Problem $(\mathrm{P}_{\varepsilon,V})$ satisfies the hypothesis of Theorem 2.1 and Proposition 5.3. Furthermore by the very definition of $\widehat{W}$ we have $\widehat{W}|_{[\hat{s}^+,\hat{s}^-]} = W|_{[\hat{s}^+,\hat{s}^-]}$. We claim that all the solutions with energy (w.r.t. $\widehat{W}$) less than or equal to $\hat{c}$ of Problem $(\widehat{\mathrm{P}}_{\varepsilon,V})$ are also solutions of Problem $(\mathrm{P}_{\varepsilon,V})$ with energy (w.r.t. W) less than or equal to $\hat{c}$. Suppose that $(\hat{u}_1, \hat{\lambda}_1)$ is a solution of Problem $(\widehat{\mathrm{P}}_{\varepsilon,V})$ then again standard elliptic regularity theory (compare Theorem 19 of [GT01]) shows that $\hat{u}_1$ is of class $C^{2,\alpha}_{\mathrm{loc}}(M)$ and using Lemma 5.1, Proposition 5.3, inequalities (5.21), and (5.22) combined with the maximum principle, it is easy to check that $\hat{u}_1 \in [\hat{s}^-, \hat{s}^+]$, so $(\hat{u}_1, \hat{\lambda}_1)$ is also a solution of Problem $(\mathrm{P}_{V,\varepsilon})$, since W and $\widehat{W}$ coincide on the interval $[\hat{s}^-, \hat{s}^+]$. With this last argument, we conclude the proof of the theorem. $\square$

REFERENCES

[ADPM11] Luigi Ambrosio, Giuseppe Da Prato, and Andrea Mennucci. *Introduction to measure theory and integration*, volume 10 of *Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)]*. Edizioni della Normale, Pisa, 2011.

[AFP00] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of Bounded Variation and Free Discontinuity Problems*. Oxford Science Publications. Clarendon Press, 2000.

[Bal90] Sisto Baldo. Minimal interface criterion for phase transitions in mixtures of Cahn-Hilliard fluids. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 7(2):67–90, 1990.

[BBM07] Vieri Benci, Claudio Bonanno, and Anna Maria Micheletti. On the multiplicity of solutions of a nonlinear elliptic problem on Riemannian manifolds. *J. Funct. Anal.*, 252(2):464–489, 2007.

[BC91] Vieri Benci and Giovanna Cerami. The effect of the domain topology on the number of positive solutions of nonlinear elliptic problems. *Arch. Rational Mech. Anal.*, 114(1):79–93, 1991.

[BC94] Vieri Benci and Giovanna Cerami. Multiple positive solutions of some elliptic problems via the Morse theory and the domain topology. *Calc. Var. Partial Differential Equations*, 2(1):29–48, 1994.

[BCP91] V. Benci, G. Cerami, and D. Passaseo. On the number of the positive solutions of some nonlinear elliptic problems. In *Nonlinear analysis*, Sc. Norm. Super. di Pisa Quaderni, pages 93–107. Scuola Norm. Sup., Pisa, 1991.

[BCW18] Costante Bellettini, Otis Chodosh, and Neshan Wickramasekera. Curvature estimates and sheeting theorems for weakly stable cmc hypersurfaces. *arXiv preprint arXiv:1802.08928*, 2018.

[Ben95] Vieri Benci. Introduction to Morse theory: a new approach. In *Topological nonlinear analysis*, volume 15 of *Progr. Nonlinear Differential Equations Appl.*, pages 37–177. Birkhäuser Boston, Boston, MA, 1995.

[BJ61] Robert G. Bartle and James T. Joichi. The preservation of convergence of measurable functions under composition. *Proc. Amer. Math. Soc.*, 12:122–126, 1961.

[BLO97] L. Bertini, C. Landim, and S. Olla. Derivation of Cahn-Hilliard equations from Ginzburg-Landau models. *J. Statist. Phys.*, 88(1-2):365–381, 1997.

[Bog07] Vladimir I Bogachev. *Measure theory*, volume 1. Springer Science & Business Media, 2007.

[BW18] Costante Bellettini and Neshan Wickramasekera. Stable cmc integral varifolds of codimension 1: regularity and compactness. *arXiv preprint arXiv:1802.00377*, 2018.

[CH58] John Werner Cahn and J. E. Hilliard. Free energy of a non-uniform system i:interfacial energy. *J. Chem. Phys.*, 27:258–266, 1958.

[Che96] Xinfu Chen. Global asymptotic limit of solutions of the Cahn-Hilliard equation. *J. Differential Geom.*, 44(2):262–311, 1996.

[CM18] Otis Chodosh and Christos Mantoulidis. Minimal surfaces and the allen-cahn equation on 3-manifolds: index, multiplicity, and curvature estimates. *arXiv preprint arXiv:1803.02716*, 2018.

[Fed69] Herbert Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[GL97] Giambattista Giacomin and Joel L. Lebowitz. Phase segregation dynamics in particle systems with long range interactions. I. Macroscopic limits. *J. Statist. Phys.*, 87(1-2):37–61, 1997.

[GL98] Giambattista Giacomin and Joel L. Lebowitz. Phase segregation dynamics in particle systems with long range interactions. II. Interface motion. *SIAM J. Appl. Math.*, 58(6):1707–1729, 1998.

[GT01] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[HT00] John E. Hutchinson and Yoshihiro Tonegawa. Convergence of phase interfaces in the van der Waals-Cahn-Hilliard theory. *Calc. Var. Partial Differential Equations*, 10(1):49–84, 2000.

[MJ00] Frank Morgan and David L. Johnson. Some sharp isoperimetric theorems for Riemannian manifolds. *Indiana Univ. Math. J.*, 49(3):1017–1041, 2000.

[Mod87] Luciano Modica. The gradient theory of phase transitions and the minimal interface criterion. *Arch. Rational Mech. Anal.*, 98(2):123–142, 1987.

[MPPP07] M. Miranda, Jr., D. Pallara, F. Paronetto, and M. Preunkert. Heat semigroup and functions of bounded variation on Riemannian manifolds. *J. Reine Angew. Math.*, 613:99–119, 2007.

[Mur09] J. D. Murray. *Pre-pattern formation mechanism for animal coat markings*. Theoretical and Mathematical Physics. Springer, Berlin, 2009.

[Nar18] Stefano Nardulli. Regularity of isoperimetric regions that are close to a smooth manifold. *Bull. Braz. Math. Soc. (N.S.)*, 49(2):199–260, 2018.

[NOA18] Stefano Nardulli and Luis Eduardo Osorio Acevedo. Sharp Isoperimetric Inequalities for Small Volumes in Complete Noncompact Riemannian Manifolds of Bounded Geometry Involving the Scalar Curvature. *International Mathematics Research Notices*, 06 2018.

[PR03] Frank Pacard and Manuel Ritoré. From constant mean curvature hypersurfaces to the gradient theory of phase transitions. *J. Differential Geom.*, 64(3):359–423, 2003.

[Pre09] Errico Presutti. *Scaling limits in statistical mechanics and microstructures in continuum mechanics*. Theoretical and Mathematical Physics. Springer, Berlin, 2009.

[vdW88] J. D. van der Waals. *On the continuity of the gaseous and liquid states*. Studies in Statistical Mechanics, XIV. North-Holland Publishing Co., Amsterdam, 1988. Translated from the Dutch, Edited and with an introduction by J. S. Rowlinson.

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