ON THE REGULARITY OF VERY WEAK SOLUTIONS FOR LINEAR ELLIPTIC EQUATIONS IN DIVERGENCE FORM

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Abstract. In this paper we consider a linear elliptic equation in divergence form

$$\sum_{i,j} D_j(a_{ij}(x)D_i u) = 0 \qquad \text{in } \Omega. \tag{0.1}$$

Assuming the coefficients a_{ij} in $W^{1,n}(\Omega)$ with a modulus of continuity satisfying a certain Dini-type continuity condition, we prove that any very weak solution $u \in L^{n'}_{\rm loc}(\Omega)$ of (0.1) is actually a weak solution in $W^{1,2}_{\rm loc}(\Omega)$.

1. Introduction

Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded open set. In this paper we study regularity properties of very weak solutions to the linear elliptic equation

$$\sum_{i,j} D_j(a_{ij}(x)D_i u) = 0 \qquad \text{in } \Omega, \tag{1.1}$$

where the matrix-field $A : \Omega \to \mathbb{R}^{n\times n}$, $A(x) = (a_{ij}(x))_{i,j}$, is elliptic and belongs to $W^{1,n}(\Omega, \mathbb{R}^{n\times n}) \cap L^\infty(\Omega, \mathbb{R}^{n\times n})$, i.e.

$$\sup_{i,j=1,\cdots,n} \|a_{ij}\|_{W^{1,n}(\Omega)} \leq M \tag{1.2}$$

and

$$\lambda|\xi|^2 \leq \sum_{i,j} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \qquad \forall \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n, \text{ a.e. in } \Omega, \tag{1.3}$$

for some positive constants λ, Λ, and M. Moreover, the matrix A is symmetric, that is $a_{ij} = a_{ji}$ a.e. in Ω for all $i, j \in \{1, ..., n\}$.

Finally we assume that the coefficients $(a_{ij}(x))_{i,j}$ are double-Dini continuous in Ω, i.e. $a_{ij} \in C^0(\Omega)$ and

$$\bar{A}_\Omega(r) := \sum_{i,j} \sup_{\substack{x,y\in\Omega \\ |x-y|\leq r}} |a_{ij}(x) - a_{ij}(y)|, \quad r > 0,$$

satisfies

$$\int_0^{diam(\Omega)} \frac{1}{t}\int_0^t \frac{\bar{A}_\Omega(s)}{s} ds\, dt < \infty. \tag{1.4}$$

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A common type of double-Dini continuous functions are, of course, $\omega(r) = r^\alpha$, $0 < \alpha \le 1$, thus an example of a matrix-field A satisfying (1.2) and (1.4) is $A \in W^{1,p}(\Omega, \mathbb{R}^{n\times n})$, with $p > n$. On the other hand, condition (1.4) occurs not only for $\omega(r) = r^\alpha$, but more generally for $\omega(r) = \log^\beta\left(\frac{1}{r}\right)$, $\beta < -2$.

Given a measurable matrix $A(x) = (a_{ij}(x))_{i,j}$ satisfying (1.3), a function $u \in W^{1,2}_{\text{loc}}(\Omega)$ is called a *weak solution* of (1.1) if

$$\sum_{i,j} \int a_{ij}(x) D_i u D_j \varphi \, dx = 0, \qquad \forall \varphi \in C_c^\infty(\Omega).$$

The celebrated result by De Giorgi in [5] states that if u is a weak solution of (1.1) then u is locally Hölder continuous.

Subsequently, J. Serrin produced in [13] a famous example, constructing an equation of the form (1.1) which has a solution $u \in W^{1,p}(\Omega)$, with $1 < p < 2$, and $u \notin L^\infty_{\text{loc}}(\Omega)$. Serrin conjectured that if the coefficients a_{ij} are locally Hölder continuous, then any solution (in the sense of distributions) $u \in W^{1,1}_{\text{loc}}(\Omega)$ of (1.1) must be a (usual) weak solution, i.e. $u \in W^{1,2}_{\text{loc}}(\Omega)$. Serrin's conjecture was established by R.A. Hager and J. Ross in [10], and then in full generality by H. Brezis in [2] (see also [1] for a full proof) starting with $u \in W^{1,1}_{\text{loc}}(\Omega)$, or even with $u \in BV_{\text{loc}}(\Omega)$, i.e., $u \in L^1_{\text{loc}}(\Omega)$ and its derivatives (in the sense of distributions) being Radon measures. Let us remark that in Brezis's result the coefficients a_{ij}, satisfying (1.3), are Dini continuous functions in Ω. The Dini continuity of the coefficients is optimal in some sense: for the unit ball B_1 and continuous coefficients, T. Jin, V. Maz'ya, and J.V. Schaftingen in [9] constructed a solution (in the sense of distributions) $u \in W^{1,1}_{\text{loc}}(B_1) \setminus W^{1,p}_{\text{loc}}(B_1)$ for every $p > 1$.

For $A(x) = (a_{ij}(x))_{i,j}$ satisfying (1.2) and (1.3), we will consider a *very weak solution* $u \in L^{n'}_{\text{loc}}(\Omega)$ of (1.1), namely

$$\sum_{i,j} \int u(x) D_i(a_{ij}(x) D_j \varphi(x)) \, dx = 0, \qquad \forall \varphi \in C_c^\infty(\Omega), \tag{1.5}$$

with $n' = \frac{n}{n-1}$.

Remark 1.1. It is not difficult to prove that the test functions φ in (1.5) can be taken in $W^{2,n}(\Omega) \cap W^{1,\infty}(\Omega)$, with $\operatorname{supp} \varphi \Subset \Omega$. Indeed, one can argue by density to show that given a function $\varphi \in W^{2,n}(\Omega) \cap W^{1,\infty}(\Omega)$ with compact support, we may find a sequence $\varphi_k \in C_c^\infty(\Omega)$ such that $\varphi_k \to \varphi$ strongly in $W^{2,n}$ and $\sup_k \|\varphi_k\|_{1,\infty} < \infty$ (so that $D\varphi_k$ converges to $D\varphi$ weakly* in L^∞) and then taking the limit as k goes to infinite in the equation (1.5) for φ_k.

The main result of the paper is the following.

Theorem 1.2. *Let u be a very weak solution of* (1.1)*, with $A(x) = (a_{ij}(x))_{i,j}$ satisfying* (1.2), (1.3) *and* (1.4)*, then u belongs to $W^{1,2}_{\text{loc}}(\Omega)$ and thus it is a weak solution.*

Remark 1.3. It is worth noting that, under hypotheses (1.2) and (1.3), one can consider a very weak solution $u \in L^{n'}_{\text{loc}}(\Omega)$ to (1.1), but when dealing with the regularity properties of u some extra conditions on the coefficients a_{ij} must be considered. The counterexample constructed in [9] provides in fact continuous coefficients a_{ij} which belong also to $W^{1,n}(B_1)$, showing that one can not expect a very weak solution $u \in L^{n'}_{\text{loc}}(\Omega)$ to be a weak solution in $W^{1,2}_{\text{loc}}(\Omega)$ under just conditions (1.2) and (1.3). For the sake of completeness, we will propose the example given in [9] in the Appendix B, underlining that the constructed coefficients belong also to $W^{1,n}(B_1)$.

On the other hand, in Section 4 we propose an alternative to double Dini continuous coefficients which again bypasses the counterexample. In particular, under hypotheses (1.2) and (1.3) we consider a very weak solution in $L^q_{\mathrm{loc}}(\Omega)$, with $q > n'$.

Remark 1.4. In [14] W. Zhang and J. Bao deal with the case of very weak solutions $u \in L^1_{\mathrm{loc}}(\Omega)$ of (1.5), interpreting the coefficients as Lipschitz functions, due to the assumption made on the solutions. Thus our result represents a natural extension from their research.

2. Notation and preliminary results

We collect here the main definitions and notation and some useful results that will be needed in the sequel.

2.1. **Notation.** In the following, we denote by $B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$ the ball of radius r centered at x.

We indicate by $\{e_1, \cdots e_n\}$ the canonical basis of $\mathbb{R}^n$. Given $h \in \mathbb{R} \setminus \{0\}$, for a measurable function $\psi : \mathbb{R}^n \to \mathbb{R}$ and for $\ell = 1, \cdots, n$, we introduce the notation

$$\Delta^\ell_h \psi := \frac{\psi(x + he_\ell) - \psi(x)}{h}$$

for the incremental quotient in the ℓ-th direction. We recall that for every pair of functions φ, ψ, we have

$$\Delta^\ell_h(\varphi\,\psi) = \Delta^\ell_h\varphi\,\psi + \varphi(x + he_\ell)\,\Delta^\ell_h\psi. \tag{2.1}$$

The following result pertaining to difference quotients of functions in Sobolev spaces is well known t(see [8, Proposition 4.8] for example).

Theorem 2.1. *Let $p > 1$; if $\psi \in W^{1,p}(\Omega)$, then $\Delta^\ell_h\psi \in L^p(\Omega')$ for any $\Omega' \Subset \Omega$ satisfying $h < \frac{dist(\Omega', \partial\Omega)}{2}$, and we have*

$$\|\Delta^\ell_h\psi\|_{L^p(\Omega')} \le \|D_\ell\psi\|_{L^p(\Omega)}.$$

If $\psi \in L^p(\Omega)$ and there exists $L \ge 0$ such that, for every $h < dist(\Omega', \partial\Omega)$, $\ell = 1, \cdots, n$, we have

$$\|\Delta^\ell_h\psi\|_{L^p(\Omega')} \le L,$$

then $\psi \in W^{1,p}(\Omega'), \|D_\ell\psi\|_{L^p(\Omega')} \le L$ and $\Delta^\ell_h\psi \to D_\ell\psi$ in $L^p(\Omega')$ as $h \to 0$.

Finally, given $p > 1$, we denote by $p' = \frac{p}{p-1}$ the conjugate exponent of p.

2.2. **Dini continuous functions.** We say that a continuous function f on Ω is Dini continuous if the modulus of continuity $\bar{f}_\Omega : [0, diam(\Omega)] \to \mathbb{R}^+$ defined by

$$\bar{f}_\Omega(r) := \sup_{\substack{x,y\in\Omega \\ |x-y|\le r}} |f(x) - f(y)|$$

satisfies

$$\int_0^{diam(\Omega)} \frac{\bar{f}_\Omega(t)}{t}\,dt < \infty.$$

We also denote by $C^D(\Omega)$ the space of Dini continuous functions; it turns out to be a Banach space equipped with the following norm:

$$\|f\|_{C^D(\Omega)} := \|f\|_\infty + \int_0^{diam(\Omega)} \frac{\bar{f}_\Omega(t)}{t}\,dt,$$

where $\|\cdot\|_\infty$ is the usual uniform norm.

Let us remark that by the uniform continuity, any function in $C^D(\Omega)$ may be extended up to the boundary of Ω with the same modulus of continuity. Moreover,

$$C^{0,\alpha}(\Omega) \subseteq C^D(\Omega),$$

for any $0 < \alpha \leq 1$, where $C^{0,\alpha}(\Omega)$ denotes the space of Hölder continuous functions.

The space $C^D_c(\Omega)$ will denote the set of functions in $C^D(\Omega)$ with compact support in Ω.

Lemma 2.2. *The space $C^\infty_c(\Omega)$ is dense in $C^D_c(\Omega)$.*

Proof. Let $f \in C^D_c(\Omega)$ that we extend to zero on $\mathbb{R}^n \setminus \Omega$ and set $f_\varepsilon(x) = (\rho_\varepsilon * f)(x)$, where ρ_ε is a standard mollifier. Then, if ε is sufficiently small, $f_\varepsilon \in C^\infty_c(\Omega)$; we will prove that

$$f_\varepsilon \to f \qquad \text{in } C^D(\Omega). \tag{2.2}$$

It is easily seen that f_ε uniformly converges to f in Ω, thus in order to prove (2.2) we will just show that

$$\int_0^{diam(\Omega)} \frac{(\overline{f - f_\varepsilon})_\Omega(t)}{t}\, dt \to 0,$$

as ε tends to 0. Observe that

$$(\overline{f - f_\varepsilon})_\Omega(r) = \sup_{\substack{x,y\in\Omega \\ |x-y|<r}} \{|f_\varepsilon(x) - f(x) - f_\varepsilon(y) + f(y)|\} \leq \bar{f}_\Omega(r) + (\bar{f}_\varepsilon)_\Omega(r)$$

and

$$\begin{aligned}(\bar{f}_\varepsilon)_\Omega(r) &= \sup_{\substack{x,y\in\Omega \\ |x-y|<r}} \{|f_\varepsilon(x) - f_\varepsilon(y)|\} \\ &= \sup_{\substack{x,y\in\Omega \\ |x-y|<r}} \left\{\left|\int \rho_\varepsilon(z)\,(f(x-z) - f(y-z))\, dz\right|\right\} \\ &\leq \int \rho_\varepsilon(z)\bar{f}_\Omega(r)dz = \bar{f}_\Omega(r),\end{aligned}$$

which together yield

$$(\overline{f_\varepsilon - f})_\Omega(r) \leq 2\bar{f}_\Omega(r).$$

On the other hand, since $(\overline{f_\varepsilon - f})_\Omega \to 0$ pointwise, the dominated convergence theorem implies

$$\int_0^{diam(\Omega)} \frac{(\overline{f_\varepsilon - f})_\Omega(t)}{t} \to 0,$$

which concludes the proof of (2.2). □

Remark 2.3. The previous result ensures that $C^D_c(\Omega)$ is a separable space, noting that $C^1_c(\Omega)$ is separable with respect to the usual norm $\|f\|_{1,\infty} := \sum_{|\alpha|\leq 1} \|D_\alpha f\|_\infty$, $C^1_c(\Omega) \subseteq C^D_c(\Omega)$ and $\bar{f}_\Omega(r) \leq r\|Df\|_\infty$, for every $f \in C^1_c(\Omega)$.

Lemma 2.4. *Let f, f_ε, and g belonging to $C^D(\Omega)$ such that f_ε converges to f in C^D; then gf_ε converges to gf in C^D.*

Proof. As before, it is enough to prove the convergence of the seminorm since the uniform convergence is immediate. Then, writing the definition of the modulus of continuity, we have

$$\begin{aligned}[\overline{g(f_\varepsilon - f)}]_\Omega(r) &= \sup_{\substack{x,y\in\Omega\\|x-y|<r}} \{|g(x)(f_\varepsilon(x)-f(x)) - g(y)(f_\varepsilon(y)-f(y))|\} \\ &\le \sup_{\substack{x,y\in\Omega\\|x-y|<r}} \{|g(x)|\,|(f_\varepsilon(x)-f(x)) - (f_\varepsilon(y)-f(y))|\} \\ &+ \sup_{\substack{x,y\in\Omega\\|x-y|<r}} \{|g(x)-g(y)||f(y)-f_\varepsilon(y)|\} \\ &\le \|g\|_\infty(\overline{f-f_\varepsilon})_\Omega(r) + \bar{g}_\Omega(r)\|f-f_\varepsilon\|_\infty. \end{aligned} \tag{2.3}$$

Hence,

$$\int_0^{diam(\Omega)} \frac{[\overline{g(f_\varepsilon - f)}]_\Omega(t)}{t}\,dt \le \|g\|_\infty \int_0^{diam(\Omega)} \frac{(\overline{f-f_\varepsilon})_\Omega(t)}{t}dt + \|f-f_\varepsilon\|_\infty \int_0^{diam(\Omega)} \frac{\bar{g}_\Omega(t)}{t}dt,$$

which goes to zero as ε tends to zero. □

2.3. **C^1-Dini regularity of solutions to divergence form elliptic equations with Dini-continuous coefficients.** For the proof of our result, we will need the following extension of the Schauder regularity theory for elliptic equations in divergence form with Dini continuous coefficients (see [11, Theorem 1.1] and [6, Theorem 1.3]). For the L^p-regularity theory we refer to [7], where the general case of VMO coefficients is treated (see also [12, Theorem 5.5.3 (a)] or [3, Theorem 2.2. Chapter 10] for the case of continuous coefficients).

Theorem 2.5. *For $\Omega \subset \mathbb{R}^n$, let a_{ij} satisfy* (1.3) *and* (1.4)*; we consider $f = (f_1, f_2, \dots, f_n)$ with $f_j \in C_c^\infty(\Omega)$ for all $j \in \{1,\dots,n\}$. Assume that $u \in H^1(\Omega)$ is a weak solution of the equation*

$$\sum_{i,j} D_j\left(a_{ij} D_i u\right) = \sum_j D_j f_j \qquad \text{in } \Omega \tag{2.4}$$

Then $u \in C^{1,D}(\Omega')$, for any bounded open set Ω', $\Omega' \Subset \Omega$.

Moreover, let Ω a $C^{1,1}$ bounded open subset of $\mathbb{R}^n$, let a_{ij} satisfy (1.2) *and* (1.3)*, and let $f_j \in L^p(\Omega)$, for every $j \in \{1,\dots,n\}$, with $1 < p < \infty$, then there exists a unique solution $u \in W_0^{1,p}(\Omega)$ to the problem*

$$\sum_{i,j} \int_\Omega a_{ij} D_i u D_j \varphi\, dx = \sum_j \int_\Omega f_j D_j \varphi\, dx \qquad \forall \varphi \in W_0^{1,p'}(\Omega),$$

and

$$\|u\|_{W^{1,p}(\Omega)} \le C \sum_j \|f_j\|_{L^p(\Omega)} \tag{2.5}$$

holds, where C depends on $n, \lambda, \Lambda, p, \partial\Omega$, $\|A\|_{W^{1,n}(\Omega,\mathbb{R}^{n\times n})}$.

Remark 2.6. The first conclusion of Theorem 2.5 comes with an estimate of the Dini modulus of continuity of Du involving the Dini modulus of continuity of a_{ij} and f_j. Actually, in [11, Theorem 1.1] and in [6, Theorem 1.3] only the continuity of Du is proved and these results are obtained with a weaker assumption on the coefficients a_{ij}. Assuming (1.4) for the coefficients we are able to prove also the Dini continuity of the gradient of the solution. In Appendix A we will resume in broad terms the proof of [11, Theorem 1.1], developing it in order to get the needed Dini continuity result.

2.4. **C^2-regularity of solutions to non divergence form elliptic equations with Dini-continuous coefficients.** Let us first recall the $W^{2,p}$-solvability of the Dirichlet problem for non divergence elliptic equations with discontinuous coefficients (see [4, Theorem 4.2 and Theorem 4.4]).

Theorem 2.7. *The Dirichlet problem*

$$\begin{cases} \sum_{i,j} a_{ij}(x) D_{ij} u = f & a.e. \ in \ \Omega \\ u = 0 & on \ \partial\Omega \end{cases} \tag{2.6}$$

where Ω is a $C^{1,1}$ smooth and bounded subset of $\mathbb{R}^n$, $f \in L^p(\Omega)$ with $1 < p < \infty$, and a_{ij} satisfies (1.2) *and* (1.3)*, admits a unique solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and*

$$||u||_{W^{2,p}(\Omega)} \leq C \left(||u||_{L^p(\Omega)} + ||f||_{L^p(\Omega)} \right), \tag{2.7}$$

where the constant C depends on $n, p, \lambda, \Lambda, \partial\Omega, \|A\|_{W^{1,n}(\Omega, \mathbb{R}^{n\times n})}$.

The next result specifies estimate (2.7); its proof is quite standard but we prefer to write it for the sake of completeness.

Proposition 2.8. *Suppose u is a solution of the elliptic Dirichlet problem* (2.6) *with a_{ij}, f, p and Ω as above. Then*

$$||u||_{W^{2,p}(\Omega)} \leq C||f||_{L^p(\Omega)}. \tag{2.8}$$

Proof. Let

$$\mathcal{L} = \left\{ L = \sum_{i,j} a_{ij} D_{ij}, \sup_{i,j} ||a_{ij}||_{W^{1,n}(\Omega)} \leq 2M, \ \lambda|\xi|^2 \leq \sum_{i,j} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \right\};$$

having in mind Theorem 2.7, if we prove that for any operator $L \in \mathcal{L}$ and for any $f \in L^p(\Omega)$, the solution u of

$$\begin{cases} Lu = f & \text{a.e. in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

satisfies

$$||u||_{L^p(\Omega)} \leq C||f||_{L^p(\Omega)},$$

we are done. Suppose it is not the case, then this is equivalent to say that for every $N \in \mathbb{N}$, there exists an operator $L_N = \sum_{i,j} a_{ij}^N D_{ij} \in \mathcal{L}$ and a function $f_N \in L^p(\Omega)$ such that the corresponding solution u_N to the Dirichlet problem

$$\begin{cases} L_N u_N = f_N & \text{a.e. in } \Omega \\ u_N = 0 & \text{on } \partial\Omega, \end{cases}$$

satisfies

$$||u_N||_{L^p(\Omega)} > N||f_N||_{L^p(\Omega)}. \tag{2.9}$$

Let us define $v_N = u_N/\|u_N\|_{L^p(\Omega)}$ and $g_N = f_N/\|u_N\|_{L^p(\Omega)}$, so that v_N solves (2.6) with L_N and g_N. By the $W^{2,p}$ estimate (2.7),

$$||v_N||_{W^{2,p}(\Omega)} \leq C \left(||v_N||_{L^p(\Omega)} + ||g_N||_{L^p(\Omega)} \right) < C \left(1 + \frac{1}{N} \right),$$

where C does not depend on N and hence,

$$||v_N||_{W^{2,p}(\Omega)} \leq C. \tag{2.10}$$

Thus v_N is a precompact sequence: up to a non relabeled subsequence, we can suppose $v_N \rightharpoonup u^*$ weakly in $W^{2,p}(\Omega)$, for some $u^* \in W^{2,p}(\Omega)$, moreover $u^* \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. Similarly, we can also say that, for every $i,j = 1, \cdots, n$, $a_{ij}^N \rightharpoonup a_{ij}^*$ weakly in $W^{1,n}(\Omega)$ and $a_{ij}^N \to a_{ij}^*$ strongly in $L^q(\Omega)$ $\forall\, 1 \leq q < \infty$. Thus, the operator $L^* = \sum_{i,j} a_{ij}^* D_{ij}$ belongs to $\mathcal{L}$ and for $\varphi \in L^{p'}(\Omega)$ we have

$$\begin{aligned}
&\left| \int_\Omega (L_N v_N - L^* u^*)\, \varphi \, dx \right| \\
&\leq \sum_{i,j=1}^n \left\{ \int_\Omega \left| (a_{ij}^N - a_{ij}^*) \frac{\partial^2 v_N}{\partial x_i \partial x_j} \varphi \right| dx + \left| \int_\Omega a_{ij}^* \varphi \left(\frac{\partial^2 v_N}{\partial x_i \partial x_j} - \frac{\partial^2 u^*}{\partial x_i \partial x_j} \right) dx \right| \right\} \\
&\leq C \sum_{i,j=1}^n \|(a_{ij}^N - a_{ij}^*)\varphi\|_{L^{p'}(\Omega)} + \sum_{i,j=1}^n \left\{ \left| \int_\Omega a_{ij}^* \varphi \left(\frac{\partial^2 v_N}{\partial x_i \partial x_j} - \frac{\partial^2 u^*}{\partial x_i \partial x_j} \right) dx \right| \right\}.
\end{aligned}$$

Therefore, $L_N v_N$ converges weakly in $L^p(\Omega)$ to $L^* u^*$. On the other hand, using (2.9), we have

$$||g_N||_{L^p(\Omega)} < \frac{1}{N}.$$

Passing to the limit in the equation satisfied by v_N, we discover that the limit $u^* \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ satisfies $L^* u^* = 0$ a.e. in Ω. By the uniqueness properties of the solutions to (2.6), it follows that $u^* = 0$. Thus v_N converges to zero and the argument becomes contradictory since $\|v_N\|_{L^p(\Omega)} = 1$. □

In [6, Theorem 1.5] it is shown that solutions to elliptic equations in non divergence form with zero Dirichlet boundary conditions are C^2 up to the boundary when the leading coefficients are Dini continuous functions.

Theorem 2.9. *Assume that Ω is a $C^{2,1}$ smooth and bounded open subset of $\mathbb{R}^n$, $f \in C^D(\Omega)$ and a_{ij} satisfies* (1.2), (1.3), *and* (1.4). *Let $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ be a solution of the Dirichlet problem*

$$\begin{cases} \displaystyle\sum_{i,j} a_{ij}(x) D_{ij} u = f & a.e. \ in\ \Omega \\ u = 0 & on\ \partial\Omega, \end{cases} \tag{2.11}$$

then $u \in C^2(\overline{\Omega})$.

Remark 2.10. The assumption in [6] about the coefficients is weaker then (1.4), since they assume that the modulus of continuity

$$\tilde{A}_\Omega(r) := \sum_{i,j} \sup_{x \in \overline{\Omega}} ⨍_{B_r(x) \cap \Omega} |a_{ij}(y) - (a_{ij})_{B_r(x) \cap \Omega}|\, dy$$

with $(a_{ij})_{B_r(x) \cap \Omega} = ⨍_{B_r(x) \cap \Omega} a_{ij}$, satisfies

$$\int_0 \frac{\tilde{A}_\Omega(r)}{r} dr < \infty.$$

3. Proof of the main theorem

We use a duality argument in conjunction with the regularity properties for elliptic equations in divergence and in non divergence form, stated in Theorems 2.5 and 2.9.

Proof. Let $\Omega' \Subset \Omega$ be an open set and choose a $C^{2,1}$ open set Ω_0 with $\Omega' \Subset \Omega_0 \Subset \Omega$; let $d(\Omega', \partial\Omega_0) = d > 0$. Let $h_0 = d/4$, and $0 < |h| < h_0$.
For the sake of clarity, we divide the proof into two steps.

Step 1. For $\ell = 1, \cdots, n$, we claim that $\Delta_h^\ell u$ is bounded in the dual space of Dini continuous functions with compact support $(C_c^D(\Omega'))'$.

Given a Dini continuous function $w \in C_c^D(\Omega')$, according to Theorem 2.9 combined with Theorem 2.7, the solution $v \in W^{2,q}(\Omega_0)$, $\forall q > 1$, to the Dirichlet problem

$$\begin{cases} \sum_{i,j} a_{ij}(x) D_{ij} v = w & \text{a.e in } \Omega_0 \\ v = 0 & \text{on } \partial\Omega_0, \end{cases} \tag{3.1}$$

enjoys the C^2-regularity up to the boundary of Ω_0.

We consider a partition of unity: let $x_1, \cdots, x_J \in \Omega'$ and $\eta_1, \cdots, \eta_J \in C^\infty(\mathbb{R}^n)$ be such that

$$\Omega' \subset \bar{\Omega}' \subset \bigcup_{k=1}^{J} B_{d/8}(x_k),\ 0 \le \eta_k \le 1,\ \forall k = 1, \cdots, J,\ \text{and}\ \sum_{k=1}^{J} \eta_k = 1 \text{ in } \Omega',$$

and

$$\operatorname{supp} \eta_k \text{ is compact and } \operatorname{supp} \eta_k \subset B_{d/8}(x_k).$$

We fix one of these balls and the related function η_k; we omit to indicate the center x_k and the index k for η_k for simplicity.

In view of Remark 1.1, we can insert $\varphi = \eta \Delta_{-h}^\ell v$ in (1.5), getting

$$\begin{aligned} 0 &= \sum_{i,j} \int u D_i (a_{ij} D_j (\eta \Delta_{-h}^\ell v)\, dx \\ &= \sum_{i,j} \int u D_i a_{ij} D_j (\eta \Delta_{-h}^\ell v)\, dx + \sum_{i,j} \int u\, a_{ij} D_{ij} (\eta \Delta_{-h}^\ell v)\, dx \\ &= \sum_{i,j} \int u D_i a_{ij} D_j \eta \Delta_{-h}^\ell v\, dx + \sum_{i,j} \int u\, \eta D_i a_{ij} D_j (\Delta_{-h}^\ell v)\, dx \\ &+ \sum_{i,j} \int \eta\, u\, a_{ij} D_{ij} (\Delta_{-h}^\ell v)\, dx + \sum_{i,j} \int u\, a_{ij} D_j \eta D_i (\Delta_{-h}^\ell v)\, dx \\ &+ \sum_{i,j} \int u\, a_{ij} D_i \eta D_j (\Delta_{-h}^\ell v)\, dx + \sum_{i,j} \int u\, a_{ij} D_{ij} \eta \Delta_{-h}^\ell v\, dx. \end{aligned}$$

We can rearrange the previous equation in order to have

$$\begin{aligned}\sum_{i,j}\int \eta\, u\, a_{ij} D_{ij}(\Delta^{\ell}_{-h}v)\,dx = &-\sum_{i,j}\int uD_i a_{ij} D_j\eta\Delta^{\ell}_{-h}v\,dx\\ &-\sum_{i,j}\int u\,\eta D_i a_{ij} D_j(\Delta^{\ell}_{-h}v)\,dx\\ &-\sum_{i,j}\int u\, a_{ij} D_j\eta D_i(\Delta^{\ell}_{-h}v)\,dx\\ &-\sum_{i,j}\int u\, a_{ij} D_i\eta D_j(\Delta^{\ell}_{-h}v)\,dx\\ &-\sum_{i,j}\int u\, a_{ij} D_{ij}\eta\Delta^{\ell}_{-h}v\,dx.\end{aligned}$$

With a simple change of variables, we get

$$\begin{aligned}\sum_{i,j}\int \eta\, u\, a_{ij} D_{ij}(\Delta^{\ell}_{-h}v)\,dx &= \sum_{i,j}\int_{\mathbb{R}^n} \eta\, u\, a_{ij}\Delta^{\ell}_{-h}(D_{ij}v)\,dx\\ &= \sum_{i,j}\int_{\mathbb{R}^n} \Delta^{\ell}_{h}(\eta\, u\, a_{ij}) D_{ij}v\,dx\\ &= \sum_{i,j}\int \Delta^{\ell}_{h}u\,\eta\, a_{ij} D_{ij}v\,dx\\ &+\sum_{i,j}\int u(x+he_\ell)\Delta^{\ell}_{h}(\eta\, a_{ij}) D_{ij}v\,dx,\end{aligned}$$

where we also used (2.1). Thus, we finally have

$$\begin{aligned}\sum_{i,j}\int \eta\Delta^{\ell}_{h}u\, a_{ij} D_{ij}v\,dx = &-\sum_{i,j}\int uD_i a_{ij} D_j\eta\Delta^{\ell}_{-h}v\,dx\\ &-\sum_{i,j}\int u\,\eta D_i a_{ij} D_j(\Delta^{\ell}_{-h}v)\,dx\\ &-\sum_{i,j}\int u\, a_{ij} D_j\eta D_i(\Delta^{\ell}_{-h}v)\,dx\\ &-\sum_{i,j}\int u\, a_{ij} D_i\eta D_j(\Delta^{\ell}_{-h}v)\,dx\\ &-\sum_{i,j}\int u\, a_{ij} D_{ij}\eta\Delta^{\ell}_{-h}v\,dx\\ &-\sum_{i,j}\int u(x+he_\ell)\Delta^{\ell}_{h}(\eta\, a_{ij}) D_{ij}v\,dx\\ &= \mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4+\mathcal{I}_5+\mathcal{I}_6.\end{aligned}\tag{3.2}$$

Now, we estimate the six terms $\mathcal{I}_m$.

The use of Hölder's inequality gives

$$
\begin{aligned}
|\mathcal{I}_1| &\le \sum_{i,j} \int_{B_{d/8}} |u D_i a_{ij} D_j \eta \Delta^{\ell}_{-h} v|\, dx \\
&\le ||D\eta||_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)} ||A||_{W^{1,n}(\Omega,\mathbb{R}^{n\times n})} ||u||_{L^{n'}(\Omega_0)} ||\Delta^{\ell}_{-h} v||_{L^\infty(B_{d/8})} \\
&\le C||Dv||_{L^\infty(\Omega_0,\mathbb{R}^n)} \le C\|w\|_{L^\infty(\Omega')},
\end{aligned}
$$

combined with Sobolev's embedding and Proposition 2.8 in the last inequality. Analogously

$$
|\mathcal{I}_2| \le ||A||_{W^{1,n}(\Omega,\mathbb{R}^{n\times n})} ||u||_{L^{n'}(\Omega_0)} \|D^2 v\|_{L^\infty(\Omega_0,\mathbb{R}^{n\times n})}.
$$

The terms $\mathcal{I}_3$ and $\mathcal{I}_4$ can be treated in the same way. Using Hölder's inequality, Theorem 2.1 and Proposition 2.8, we have

$$
|\mathcal{I}_3|, |\mathcal{I}_4| \le \Lambda \|u\|_{L^{n'}(\Omega_0)} \|D\eta\|_{L^\infty(\mathbb{R}^n \mathbb{R}^n)} ||v||_{W^{2,n}(\Omega_0)} \le C||w||_{L^n(\Omega')}.
$$

Again, for $\mathcal{I}_5$ we have

$$
|\mathcal{I}_5| \le \Lambda \|u\|_{L^{n'}(\Omega_0)} \|D^2\eta\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^{n\times n})} \|Dv\|_{L^n(\Omega_0,\mathbb{R}^n)} \le C\|w\|_{L^n(\Omega')}.
$$

We finally estimate $\mathcal{I}_6$. From (2.1), we get

$$
\begin{aligned}
\mathcal{I}_6 = &- \sum_{i,j} \int u(x+he_\ell)\, \eta \Delta^{\ell}_h a_{ij} D_{ij} v\, dx \\
&- \sum_{i,j} \int u(x+he_\ell) a_{ij}(x+he_\ell) \Delta^{\ell}_h \eta\, D_{ij} v\, dx.
\end{aligned}
$$

The second term can be estimated as $\mathcal{I}_3$ and $\mathcal{I}_4$, thus:

$$
\begin{aligned}
|\mathcal{I}_6| \le &\sum_{i,j} \int_{B_{d/8}} |u(x+he_\ell)\, \eta \Delta^{\ell}_h a_{ij} D_{ij} v|\, dx + C\|w\|_{L^n(\Omega')} \\
\le &C\|A\|_{W^{1,n}(\Omega,\mathbb{R}^{n\times n})} \|u\|_{L^{n'}(\Omega_0)} \|D^2 v\|_{L^\infty(\Omega_0,\mathbb{R}^{n\times n})} + C\|w\|_{L^n(\Omega')}.
\end{aligned} \tag{3.3}
$$

Here we have used once more Theorem 2.9.

Finally, combining the estimates found for $\mathcal{I}_m$, $m \in \{1, \dots 6\}$, from (3.2) we get

$$
\sum_{i,j} \int_{B_{d/8}} \eta \Delta^{\ell}_h u\, a_{ij} D_{ij} v\, dx \le C,
$$

where C depends on $\lambda, \Lambda, \|D\eta\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)}, \|D^2\eta\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^{n\times n})}, \|u\|_{L^{n'}(\Omega_0)}$, $\|A\|_{W^{1,n}(\Omega,\mathbb{R}^{n\times n})}, \|w\|_{L^\infty(\Omega')}$ and $\|D^2 v\|_{L^\infty(\Omega_0,\mathbb{R}^{n\times n})}$, as well as on the modulus of continuity of the coefficients a_{ij} and of the datum w. Summing over $k = 1, \cdots, J$, since v is the weak solution to the Dirichlet problem (3.1), we finally have

$$
\left| \int_{\Omega'} \eta\, w \Delta^{\ell}_h u\, dx \right| \le C,
$$

and we get

$$
\left| \int_{\Omega'} w \Delta^{\ell}_h u\, dx \right| \le C,
$$

for every $w \in C_c^D(\Omega')$. By the uniform boundedness principle this means that $\{\Delta_h^\ell u\}_h$ is a family of equibounded elements in the dual space of Dini continuous functions $(C_c^D(\Omega'))'$. Since $(C_c^D(\Omega'))'$ is separable, we have that, up to a subsequence,

$$\Delta_h^\ell u \overset{*}{\rightharpoonup} \mu^\ell \in (C_c^D(\Omega'))'.$$

Step 2. We prove that $u \in W^{1,p'}_{\mathrm{loc}}(\Omega)$, with $p > n$.

Using the previous Step we can easily deduce from (1.5) that

$$\sum_{i,j} \langle \mu^i, a_{ij} D_j \varphi \rangle = 0 \qquad \forall \varphi \in C_c^\infty(\Omega'), \tag{3.4}$$

where the duality pairing is between $(C_c^D(\Omega'))'$ and $C_c^D(\Omega')$.

For $j \in \{1, \cdots, n\}$, let $f = (f_1, \cdots, f_n)$ with $f_j \in C_c^\infty(\Omega')$ be such that

$$\sum_j ||f_j||_{L^p(\Omega')} \leq 1,$$

with $p > n$. Introducing as before a regular set Ω_0 between Ω' and Ω we can possibly assume that Ω is a $C^{1,1}$ set. Let $v \in W_0^{1,2}(\Omega)$ be the weak solution of the problem

$$\sum_{i,j} \int a_{ij} D_i v D_j \varphi \, dx = \sum_j \int D_j \varphi f_j \, dx \qquad \forall \varphi \in C_c^\infty(\Omega). \tag{3.5}$$

By Theorem 2.5 we have that $v \in W_0^{1,p}(\Omega)$ and

$$||v||_{W^{1,p}(\Omega)} \leq C||f||_{L^p(\Omega',\mathbb{R}^n)}.$$

Note that, since $p > n$, this means also that the function v is Hölder continuous.

We take $B_{R/2} \subset B_R \subset \Omega'$ a pair of concentric balls centered at $x_0 \in \Omega'$ and we consider $\xi(x) = \xi(|x - x_0|)$ a smooth function such that $\xi(t) = 1$ for $t \in [0, R/2]$ and $\xi(t) = 0$ for $t \geq R$.

We would like to use $\varphi = \xi v$ as test function in (3.4). We first observe that, by Theorem 2.5, the function ξv belongs to $C_c^{1,D}(\Omega')$. Moreover, proving Lemma 2.2, we actually proved that a mollification of a Dini continuous function with compact support strongly converges in C^D to the function itself. Thus, combining this fact with Lemma 2.4, we have that $a_{ij} D_j (\xi v)_\varepsilon$ strongly converges in C^D to $a_{ij} D_j (\xi v)$, where $(\xi v)_\varepsilon(x) = (\rho_\varepsilon * \xi v)(x)$, ρ_ε being a standard mollifier. This in turn implies that the use of $\varphi = \xi v$ as test function in (3.4) is admissible:

$$\sum_{i,j} \langle \mu^i, a_{ij} D_j v \, \xi \rangle + \sum_{i,j} \langle \mu^i, a_{ij} v D_j \xi \rangle = 0. \tag{3.6}$$

Let us come back now to the equation satisfied by v. Let u_ε be a mollification of the solution u, that is $u_\varepsilon = \rho_\varepsilon * u$, with ρ_ε a standard radial mollifier. We use ξu_ε in (3.5):

$$\begin{aligned} &\sum_{i,j} \int a_{ij} \xi D_j u_\varepsilon D_i v \, dx + \sum_{i,j} \int a_{ij} D_j \xi \, u_\varepsilon D_i v \, dx \\ &= \sum_j \int \xi f_j D_j u_\varepsilon \, dx + \sum_j \int u_\varepsilon D_j \xi f_j \, dx. \end{aligned}$$

Now we claim that this implies, when we pass to the limit as $\varepsilon \to 0$, that

$$\sum_{i,j} \langle \mu^j, a_{ij} \xi D_i v \rangle + \sum_{i,j} \int a_{ij} D_i \xi u D_j v \, dx = \sum_j \langle \mu^j, \xi f_j \rangle + \sum_j \int u D_j \xi f_j \, dx. \tag{3.7}$$

Note that the most delicate terms are the two involving the gradient of u_ε. For a Dini continuous function w (the domain of w is not specified since the function will be multiplied by a function with compact support) we will show that

$$\lim_{\varepsilon\to 0}\lim_{h\to 0}\int \Delta_h^j(u_\varepsilon - u)w\,\xi\,dx = 0,$$

or, in other terms, recalling that μ^j is the limit in the weak* topology of $C_c^D(\Omega')$ of the incremental quotient of u

$$\lim_{\varepsilon\to 0}\int D_j u_\varepsilon w\,\xi\,dx = \langle \mu^j, w\,\xi\rangle.$$

We have:

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\lim_{h\to 0}\int \Delta_h^j u_\varepsilon w\,\xi\,dx\\
&=\lim_{\varepsilon\to 0}\lim_{h\to 0}\int \xi(x)w(x)\int \rho_\varepsilon(x-z)\frac{u(z+he_j)-u(z)}{h}\,dz\,dx\\
&=\lim_{\varepsilon\to 0}\lim_{h\to 0}\int \frac{u(z+he_j)-u(z)}{h}\int \rho_\varepsilon(x-z)\xi(x)w(x)\,dx\,dz\\
&=\lim_{\varepsilon\to 0}\lim_{h\to 0}\int \frac{u(z+he_j)-u(z)}{h}\int \rho_\varepsilon(z-x)\xi(x)w(x)\,dx\,dz\\
&=\lim_{\varepsilon\to 0}\lim_{h\to 0}\int \frac{u(z+he_j)-u(z)}{h}(w\,\xi)_\varepsilon(z)\,dz = \lim_{\varepsilon\to 0}\lim_{h\to 0}\int \Delta_h^j u\,(w\,\xi)_\varepsilon\,dz\\
&=\lim_{\varepsilon\to 0}\langle \mu^j, (w\xi)_\varepsilon\rangle = \langle \mu^j, w\,\xi\rangle,
\end{aligned}$$

where in the last equality we used again that a mollified function of a Dini continuous function with compact support strongly converges in C^D to the function itself. Thus we obtain (3.7).

From it, exploiting the symmetry of a_{ij} and using (3.6) we get

$$\begin{aligned}
\sum_j \langle \mu^j, \xi f_j\rangle &= -\sum_{i,j}\langle \mu^i, a_{ij}D_j\xi v\rangle + \sum_{i,j}\int a_{ij}D_i\xi u D_j v\,dx - \sum_j \int u D_j\xi f_j\,dx\\
&= I_1 + I_2 + I_3.
\end{aligned} \tag{3.8}$$

We now estimate the three terms I_m, $m = 1,2,3$. We have

$$|I_1| \le \sum_{i,j}\|\mu^i\|_{(C_c^D(\Omega'))'}\|a_{ij}vD_i\xi\|_{C^D(\Omega')}.$$

By the definition of the norm in the space of Dini continuous functions we have

$$\|a_{ij}vD_i\xi\|_{C^D(\Omega')} \le \Lambda\|v\|_{L^\infty(\Omega')}\|D_i\xi\|_{L^\infty(B_R)} + \int_0^{diam(\Omega')}\frac{(\overline{a_{ij}vD_i\xi})_{\Omega'}(r)}{r}\,dr.$$

By simple computation we have

$$(\overline{a_{ij}vD_i\xi})_{\Omega'}(r) \le \|v\|_{L^\infty(\Omega')}(\overline{a_{ij}D_i\xi})_{\Omega'}(r) + \|a_{ij}D_i\xi\|_{L^\infty(\Omega')}\overline{v}_{\Omega'}(r),$$

and, using the properties of the solution v (recall that $p > n$), the right hand side can be estimated as

$$(\overline{a_{ij} v D_i \xi})_{\Omega'}(r) \leq C(\overline{a_{ij} D_j \xi})_{\Omega'}(r) \|f\|_{L^p(\Omega', \mathbb{R}^n)} + C r^{1-\frac{n}{p}} \|a_{ij} D_j \xi\|_{L^\infty(\Omega')} \|Dv\|_{L^p(\Omega', \mathbb{R}^n)}.$$

To summarize, we have

$$|I_1| \leq C \|f\|_{L^p(\Omega', \mathbb{R}^n)}.$$

The estimate of I_2 and I_3 simply comes by Hölder's inequality and again by the properties of the solution v:

$$|I_2| \leq \sum_{i,j} \left| \int a_{ij} D_i \xi u D_j v \, dx \right| \leq C \|u\|_{L^{n'}(\Omega')} \|D\xi\|_{L^\infty(B_R, \mathbb{R}^n)} \Lambda \|v\|_{W^{1,n}(\Omega')} \leq C \|f\|_{L^n(\Omega', \mathbb{R}^n)},$$

and

$$|I_3| \leq \sum_j \left| \int u D_j \xi f_j dx \right| \leq \|u\|_{L^{n'}(\Omega')} \|D\xi\|_{L^\infty(B_R, \mathbb{R}^n)} \|f\|_{L^n(\Omega', \mathbb{R}^n)}.$$

At the end, the estimates proved for I_1, I_2 and I_3 lead to

$$\sum_j \langle \mu^j, \xi f_j \rangle \leq C \|f\|_{L^p(\Omega', \mathbb{R}^n)},$$

as well

$$\sum_j \langle \mu^j \xi, f_j \rangle \leq C \|f\|_{L^p(\Omega', \mathbb{R}^n)}.$$

Since f is an arbitrary smooth function in $L^p(\Omega', \mathbb{R}^n)$, we conclude

$$\sum_j \|\mu^j \xi\|_{L^{p'}(\Omega')} \leq C,$$

which means, using a finite covering argument, that μ^j is a function in $L^{p'}_{\mathrm{loc}}(\Omega)$ and then $u \in W^{1,p'}_{\mathrm{loc}}(\Omega)$, since, for every $\varphi \in C_c^\infty(\Omega)$ and for h small enough, we have

$$\int \Delta_h^j u \, \varphi \, dx = \int u \Delta_{-h}^j \varphi \, dx;$$

passing to the limit as $h \to 0$, we derive

$$\langle \mu^j, \varphi \rangle = \int \varphi \mu^j \, dx = -\int u D_j \varphi \, dx.$$

Since $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, Brezis's result implies that u is a weak solution of the equation (1.5), i.e. our statement. □

4. Sobolev coefficients

As pointed out in the Introduction, very weak solutions in $L^{n'}_{\text{loc}}(\Omega)$ associated to coefficients in $W^{1,n}(\Omega)$ are not weak solutions, since of the counterexample found in [9]. The quoted references on this problem have suggested us to consider Sobolev coefficients with a modulus of continuity satisfying the double Dini condition.

On the other hand, another way to get around the counterexample is to deal with very weak solutions in $L^q_{\text{loc}}(\Omega)$, with $q > n'$. The result is the following.

Theorem 4.1. *Let $u \in L^q_{\text{loc}}(\Omega)$, $q > n'$, be a very weak solution of* (1.1)*, with $A(x) = (a_{ij}(x))_{i,j}$ satisfying* (1.2) *and* (1.3)*, then u belongs to $W^{1,2}_{\text{loc}}(\Omega)$ and thus it is a weak solution.*

Proof. The proof rests on a duality and a bootstrap argument.

Step 1. We claim that $u \in W^{1,\left(\frac{qn'}{q-n'}\right)'}_{\text{loc}}(\Omega)$.

We proceed as in the Step 1 of the proof of Theorem 1.2 to arrive to (3.2). Now we estimate the six terms $\mathcal{I}_m$. We use Hölder's inequality and Proposition 2.8 to get

$$\begin{aligned}
|\mathcal{I}_1| &\le \|D\eta\|_{L^\infty(\mathbb{R}^n)} \|A\|_{W^{1,n}(\Omega,\mathbb{R}^{n\times n})} \|u\|_{L^q(\Omega_0)} \|Dv\|_{L^{\frac{qn'}{q-n'}}(\Omega_0,\mathbb{R}^n)} \\
&\le C\|w\|_{L^{\frac{qn'}{q-n'}}(\Omega_0)},
\end{aligned}$$

$$|\mathcal{I}_2| \le \|A\|_{W^{1,n}(\Omega,\mathbb{R}^{n\times n})} \|u\|_{L^q(\Omega_0)} \|D^2 v\|_{L^{\frac{qn'}{q-n'}}(\Omega_0,\mathbb{R}^{n\times n})} \le C\|w\|_{L^{\frac{qn'}{q-n'}}(\Omega_0)},$$

$$\begin{aligned}
|\mathcal{I}_3|, |\mathcal{I}_4| \le \Lambda \|u\|_{L^q(\Omega_0)} \|D\eta\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)} \|D^2 v\|_{L^{q'}(\Omega_0,\mathbb{R}^{n\times n})} &\le C\|w\|_{L^{q'}(\Omega')} \\
&\le C\|w\|_{L^{\frac{qn'}{q-n'}}(\Omega')},
\end{aligned}$$

$$\begin{aligned}
|\mathcal{I}_5| \le \Lambda \|u\|_{L^q(\Omega_0)} \|D^2\eta\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^{n\times n})} \|Dv\|_{L^{q'}(\Omega_0,\mathbb{R}^n)} &\le C\|w\|_{L^{q'}(\Omega')} \\
&\le C\|w\|_{L^{\frac{qn'}{q-n'}}(\Omega')},
\end{aligned}$$

and finally, as for (3.3),

$$|\mathcal{I}_6| \le C\|w\|_{L^{\frac{qn'}{q-n'}}(\Omega')} + C\|w\|_{L^{q'}(\Omega')} \le C\|w\|_{L^{\frac{qn'}{q-n'}}(\Omega')}.$$

So, arguing as in the Step 1 of Theorem 1.2, we deduce

$$\left| \int_{\Omega'} w \Delta_h^\ell u \, dx \right| \le C\|w\|_{L^{\frac{qn'}{q-n'}}(\Omega')},$$

which in turn implies, thanks also to Theorem 2.1, that $u \in W^{1,\left(\frac{qn'}{q-n'}\right)'}_{\text{loc}}(\Omega)$. Let us note that thanks to this, the equation satisfied by u may be rewritten as

$$\sum_{i,j} \int a_{ij}(x) D_i u D_j \varphi \, dx = 0, \tag{4.1}$$

where the test functions φ can be taken in $W^{1,\frac{qn'}{q-n'}}(\Omega)$ with compact support. On the other hand, the summability of the solution u is not improved by its belonging to this Sobolev space, since $\left(\frac{qn'}{q-n'}\right)' = \frac{qn'}{qn'-q+n'}$ and the Sobolev conjugate of $\frac{qn'}{qn'-q+n'}$ is q.

Step 2. We prove that $u \in W^{1,q}_{\text{loc}}(\Omega)$.

As in the Step 2 of the proof of Theorem 1.2, for $j \in \{1, \cdots, n\}$ let $f = (f_1, \cdots, f_n)$ with $f_j \in C_c^\infty(\Omega')$ be such that

$$\sum_j ||f_j||_{L^{q'}(\Omega')} \leq 1.$$

For every $p > 1$, let $v \in W_0^{1,p}(\Omega)$ be the weak solution of the problem

$$\sum_{i,j} \int a_{ij} D_i v D_j \varphi \, dx = \sum_j \int D_j \varphi f_j \, dx \qquad \forall \varphi \in W_0^{1,p'}(\Omega). \tag{4.2}$$

By Theorem 2.5 we have in particular that

$$||v||_{W^{1,q'}(\Omega')} \leq C||f||_{L^{q'}(\Omega',\mathbb{R}^n)}.$$

As before, we take $B_{R/2} \subset B_R \subset \Omega'$ a pair of concentric balls centered at $x_0 \in \Omega'$ and we consider $\xi(x) = \xi(|x - x_0|)$ a smooth function such that $\xi(t) = 1$ for $t \in [0, R/2]$ and $\xi(t) = 0$ for $t \geq R$. We can choose $\varphi = v\xi$ in (4.1) and $\varphi = u\xi$ as test function in (4.2), so that

$$\sum_{i,j} \int a_{ij} D_i u D_j v \, \xi \, dx + \sum_{i,j} \int a_{ij} D_i u D_j \xi v \, dx = 0,$$

and

$$\begin{aligned} &\sum_{i,j} \int a_{ij} D_i v D_j u \, \xi \, dx + \sum_{i,j} \int a_{ij} D_i v D_j \xi u \, dx \\ &= \sum_j \int f_j D_j u \, \xi \, dx + \sum_j \int f_j D_j \xi u \, dx. \end{aligned}$$

Subtracting the two equations and using the symmetry of a_{ij} we get

$$\begin{aligned} \sum_j \int f_j D_j u \, \xi \, dx = &- \sum_j \int f_j D_j \xi u \, dx + \sum_{i,j} \int a_{ij} D_i v D_j \xi u \, dx \\ &- \sum_{i,j} \int a_{ij} D_i u D_j \xi v \, dx = I_1 + I_2 + I_3. \end{aligned}$$

We estimate the three terms I_m. We have

$$|I_1| \leq \|u\|_{L^q(\Omega')} \|D\xi\|_{L^\infty(B_R,\mathbb{R}^n)} \|f\|_{L^{q'}(\Omega',\mathbb{R}^n)} \leq C\|f\|_{L^{q'}(\Omega',\mathbb{R}^n)},$$

$$|I_2| \leq \Lambda \|D\xi\|_{L^\infty(B_R,\mathbb{R}^n)} \|u\|_{L^q(\Omega')} \|Dv\|_{L^{q'}(\Omega',\mathbb{R}^n)} \leq C\|f\|_{L^{q'}(\Omega',\mathbb{R}^n)},$$

and finally

$$|I_3| \leq \Lambda \|u\|_{W^{1,\left(\frac{qn'}{q-n'}\right)'}(\Omega')} \|v\|_{L^{\frac{qn'}{q-n'}}(\Omega')} \leq C\|v\|_{W^{1,q'}(\Omega')},$$

where the last inequality derives from the fact that the Sobolev conjugate of q' is $\frac{qn'}{q-n'}$. To sum up we have obtained

$$\Big| \sum_j \int f_j \xi D_j u \, dx \Big| \leq C\|f\|_{L^{q'}(\Omega',\mathbb{R}^n)},$$

as well

$$\|\xi Du\|_{L^q(\Omega',\mathbb{R}^n)} \leq C,$$

and, using a finite covering argument, this implies that $u \in W^{1,q}_{\rm loc}(\Omega)$. Let us observe that this Sobolev regularity improves the summability of u. In particular, $u \in L^{q^*}_{\rm loc}(\Omega)$, where q^* is the Sobolev conjugate of q.

Step 3. We claim that if $q > n$ then u is a weak solution.

By the previous step, we deduce that if $q > n$ then the solution u is in $L^\infty_{\rm loc}(\Omega)$. At this point, it is not difficult to prove, arguing as in Step 1, that $u \in W^{1,n}_{\rm loc}(\Omega)$.

Step 4. We prove that $u \in L^\infty_{\rm loc}(\Omega)$.

We just observed that if $q > n$ we are done. Let us consider now $q \leq n$. The solution u is in $W^{1,q}_{\rm loc}(\Omega)$ and by the Sobolev's embedding $u \in L^{q^*}_{\rm loc}(\Omega)$, where $q^* = \frac{qn}{n-q}$ if $q < n$ and any number greater then 1 if $q = n$. Arguing exactly as in the Step 2 we derive that $u \in W^{1,q^*}_{\rm loc}(\Omega)$, which in turn implies that u is in $L^\infty_{\rm loc}(\Omega)$ if $q^* > n$. We already noticed in Step 3 that this gives the desired result. Let us observe that if $q = n$, q^* is any number greater then 1 and so this can be chosen greater then n, while if $q < n$, $q^* > n$ is equivalent to $q > \frac{n}{2}$. We can iterate this procedure. Given $q > n' = \frac{n}{n-1}$ after (at most) $n-1$ times we deduce that u is locally bounded.

By Step 3 the locally boundedness of the solution gives the desired result.

□

Appendix A. The C^1-Dini regularity of solutions to divergence form elliptic equations with Dini-continuous coefficients

As announced in Remark 2.6, we will specify the modulus of continuity of the gradient of solutions to (2.4) in the proof of [11, Theorem 1.1]. We will consider only the main points of the proof reminding for the rest to [11]. The set Ω is supposed to be the ball B_4 centered at 0 and $\Omega' = B_1$. The improvement regards Proposition 1.1 in [11]: for the sake of completeness we will sketch the proof, modifying the original when needed.

Proposition A.1. *For $B_4 \subset \mathbb{R}^n$, $n \geq 1$, let a_{ij}, defined on B_4, satisfy* (1.3) *and* (1.4) *and let $f = (f_1, f_2, \dots, f_n)$ with $f_j \in C^\infty_c(B_4)$ for all $j \in \{1, \dots, n\}$. Assume that $u \in H^1(B_4)$ is a weak solution of* (2.4)*, then there exist $a \in \mathbb{R}$ and $b \in \mathbb{R}^n$ such that*

$$\fint_{B_r} |u(x) - (a + b \cdot x)|\, dx \leq r\delta(r)[\|u\|_{L^2(B_2)} + \|f\|_{C^1(B_2)}], \qquad \forall r \in (0,1), \tag{A.1}$$

where $\delta(r)$, depending on n, λ, Λ, and on the modulus of continuity of a_{ij} and f, is a monotonically increasing positive function defined on $(0,1)$ satisfying

$$\int_0^1 \frac{\delta(r)}{r}\, dr < +\infty.$$

Remark A.2. As shown in [11, Proposition 1.2], $\delta(r)$ will be the modulus of continuity of Du.

Proof. The proof is carried out for $f = 0$. We use the same notation of [11], denoting by φ the modulus of continuity such that

$$\left(\fint_{B_r} |A - A(0)|^2\right)^{\frac{1}{2}} \leq \varphi(r),$$

where $A=(a_{ij})_{i,j}$. Observe that in our case, assuming (1.4), $\varphi(r)$ has the following form

$$\begin{aligned}\left(\fint_{B_r}|A-A(0)|^2\right)^{\frac{1}{2}} &\le \left(\fint_{B_r}\bar{A}_{B_4}(|x|)^2\,dx\right)^{\frac{1}{2}}\\ &\le C\left(\frac{1}{r^n}\int_0^r \bar{A}_{B_4}(\rho)^2\rho^{n-1}d\rho\right)^{\frac{1}{2}} =: \varphi(r),\end{aligned}$$

which is double-Dini continuous since $\varphi(r)\le \bar{A}_{B_4}(r)$, and satisfies

$$\max_{r/2\le s\le r}\varphi(s)\le \mu\varphi(r),$$

with $\mu>1$. As in [11], by induction, one will find, for $k\ge 0$, $w_k\in H^1(B_{3/4^{k+1}})$ such that

$$\sum_{i,j} D_j(a_{ij}(0)D_i w_k)=0 \qquad \text{in } B_{3/4^{k+1}}, \tag{A.2}$$

$$\|w_k\|_{L^2(B_{2/4^{k+1}})}\le C4^{-\frac{k(n+2)}{2}}\varphi(4^{-k}), \quad \|Dw_k\|_{L^\infty(B_{1/4^{k+1}},\mathbb{R}^n)}C\varphi(4^{-k}), \tag{A.3}$$

$$\|D^2w_k\|_{L^\infty(B_{1/4^{k+1}},\mathbb{R}^{n\times n})}\le C4^k\varphi(4^{-k}), \tag{A.4}$$

$$\|u-\sum_{j=0}^k w_j\|_{L^2(B_{1/4^{k+1}})}\le 4^{-\frac{(k+1)(n+2)}{2}}\varphi(4^{-(k+1)}), \tag{A.5}$$

and

$$\|w_k\|_{L^\infty(B_{1/4^{k+1}})}\le C4^{-k}\varphi(4^{-k}), \tag{A.6}$$

see [11, (14), (15), (16), (17), and (18) of Proposition 1.1]. Here and in the sequel C will denote a universal constant.

For $x\in B_{1/4^{k+1}}$, using (A.3), (A.4), (A.6) and Taylor expansion,

$$\begin{aligned}&|\sum_{j=0}^k w_j(x)-\sum_{j=0}^\infty w_j(0)-\sum_{j=0}^\infty Dw_j(0)\cdot x|\\ &\le \sum_{j=k+1}^\infty(|w_j(0)|+|Dw_j(0)||x|)+\sum_{j=0}^k\|D^2w_j\|_{L^\infty(B_{1/4^{k+1}},\mathbb{R}^{n\times n})}|x|^2\\ &\le C\sum_{j=k+1}^\infty(4^{-j}\varphi(4^{-j})+\varphi(4^{-j})|x|)+C\sum_{j=0}^k 4^j\varphi(4^{-j})|x|^2\\ &\le C4^{-(k+1)}\int_0^{4^{-k}}\frac{\varphi(r)}{r}\,dr+C|x|^2\int_{\frac{|x|}{2}}^1\frac{\varphi(r)}{r^2}\,dr.\end{aligned} \tag{A.7}$$

We then derive from (A.5) and the above, using Hölder's inequality, that

$$
\begin{aligned}
&\int_{B_{1/4^{k+1}}} |u(x) - \sum_{j=0}^{\infty} w_j(0) - \sum_{j=0}^{\infty} Dw_j(0)\cdot x|\,dx \\
&\le \|\sum_{j=0}^{k} w_j(x) - \sum_{j=0}^{\infty} w_j(0) - \sum_{j=0}^{\infty} Dw_j(0)\cdot x\|_{L^1(B_{1/4^{k+1}})} \\
&+ \|u - \sum_{j=0}^{k} w_j(x)\|_{L^1(B_{1/4^{k+1}})} \\
&\le C4^{-(k+1)(n+1)} \int_0^{4^{-k}} \frac{\varphi(r)}{r}\,dr + C\int_0^{1/4^{k+1}} \rho^{n+1} \int_{\frac{\rho}{2}}^1 \frac{\varphi(r)}{r^2}\,dr\,d\rho \\
&+ C\,4^{-(k+1)(n+1)}\varphi(1/4^{k+1}).
\end{aligned} \tag{A.8}
$$

Proposition A.1 follows from the above with $a = \sum_{j=0}^{\infty} w_j(0)$, $b = \sum_{j=0}^{\infty} Dw_j(0)\cdot x$, and

$$\delta(r) \simeq \int_0^r \frac{\varphi(s)}{s}\,ds + \frac{1}{r^{n+1}}\int_0^r \rho^{n+1}\int_{\frac{\rho}{2}}^1 \frac{\varphi(s)}{s^2}\,ds\,d\rho + \varphi(r),$$

the symbol $\simeq$ standing for $=$ up to a constant. It remains to prove that $\delta(r)$ is a Dini modulus of continuity. Thanks to assumption (1.4), it occurs if we show the Dini continuity of the second term in the previous sum. It yields

$$\frac{1}{r^{n+1}}\int_0^r \rho^{n+1}\int_{\frac{\rho}{2}}^1 \frac{\varphi(s)}{s^2}\,ds\,d\rho \le r\int_{\frac{r}{2}}^1 \frac{\varphi(s)}{s^2}ds,$$

so that, integrating by parts,

$$\int_0 \int_{\frac{r}{2}}^1 \frac{\varphi(s)}{s^2}ds\,dr = r\int_{\frac{r}{2}}^1 \frac{\varphi(s)}{s^2}ds\Big|_0 + \int_0 \frac{\varphi(\frac{r}{2})}{r/4}dr.$$

It is easy to see that $\lim_{r\to 0} r\int_{\frac{r}{2}}^1 \frac{\varphi(s)}{s^2}ds = 0$, and thus the thesis follows by the Dini continuity of φ.

□

Appendix B. The counterexample

To construct the example, one first considers, for $r \in (0,1)$ and for $\beta > 1$, the function

$$\alpha(r) = \frac{-\beta n}{(n-1)\log\left(\frac{r_0}{r}\right)} + \frac{\beta(\beta+1)}{(n-1)\log^2\left(\frac{r_0}{r}\right)},$$

for some $r_0 > 1$. One takes then $A(x) = (a_{ij}(x))_{i,j}$ defined by

$$a_{ij}(x) = \delta_{ij} + \alpha(|x|)\left(\delta_{ij} - \frac{x_i x_j}{|x|^2}\right),$$

with r_0 large enough so that $\alpha \ge -\frac{1}{2}$, A being then uniformly elliptic.

Let us check now that $A \in W^{1,n}(B_1, \mathbb{R}^{n\times n})$. Simple computation gives

$$\left|\frac{\partial a_{ij}}{\partial x_\ell}\right| \lesssim |\alpha'(|x|)| + |\alpha(|x|)|\frac{1}{|x|},$$

for every $i, j, \ell = 1, \cdots, n$ (the symbol $\lesssim$ stand for $\leq$ up to a constant). On the other hand

$$|\alpha'(|x|)| \simeq \frac{1}{|x|\log^2\left(\frac{r_0}{|x|}\right)} + \frac{1}{|x|\log^3\left(\frac{r_0}{|x|}\right)},$$

which in turn implies

$$\left|\frac{\partial a_{ij}}{\partial x_\ell}\right| \lesssim \frac{1}{|x|\log\left(\frac{r_0}{|x|}\right)} + \frac{1}{|x|\log^2\left(\frac{r_0}{|x|}\right)} + \frac{1}{|x|\log^3\left(\frac{r_0}{|x|}\right)} \lesssim \frac{1}{|x|\log\left(\frac{r_0}{|x|}\right)},$$

if r_0 is big enough. Thus, the belonging of A to $W^{1,n}(B_1, \mathbb{R}^{n\times n})$ is provided by the estimate

$$\int_{B_1}\left|\frac{\partial a_{ij}}{\partial x_\ell}\right|^n dx \lesssim \int_{B_1}\frac{1}{|x|^n\log^n\left(\frac{r_0}{|x|}\right)}\,dx \simeq \int_0^1 \frac{1}{r\log^n\left(\frac{r_0}{r}\right)}\,dr < +\infty.$$

With such an A, in [9] the authors construct a solution of (1.1) (in the sense of distributions) $u \in W^{1,1}_{\rm loc}(B_1) \setminus W^{1,p}_{\rm loc}(B_1)$ for every $p > 1$. In particular, let us observe that such a solution belongs to $L^{n'}_{\rm loc}(B_1)$.

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