Ornstein-Uhlenbeck semigroups in infinite dimension

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Abstract

This is a survey paper about Ornstein-Uhlenbeck semigroups in infinite dimension, and their generators. We start from the classical Ornstein-Uhlenbeck semigroup in Wiener spaces and then discuss the general case in Hilbert spaces. Finally, we present some results for O-U semigroups in Banach spaces.

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Introduction

In this article we present the basic results on Ornstein-Uhlenbeck semigroups on infinite dimensional spaces. We refer to the survey *The Ornstein-Uhlenbeck semigroup in finite dimension* by A. Lunardi, G. Metafune and D. Pallara in this volume for a general introduction to the finite dimensional case.

Infinite dimensional Ornstein-Uhlenbeck operators, semigroups and processes find their motivations in statistical mechanics, quantum theory, analysis of partial differential equations, control theory, random processes and stochastic PDEs. In quantum field theory the classical O-U operator is the "number operator", whose eigenvalues represent the number of bosons in a quantum field, and indeed classical results like hypercontractivity and logarithmic Sobolev inequalities have their origins in the quantum theory community. In analysis, the O-U operator appears as the generator of Chebyshev-Hermite orthogonal polynomials, which eventually led to the Wiener chaos decomposition mentioned in Section 2. The classical Ornstein-Uhlenbeck semigroup plays an essential role in Malliavin Calculus. This theory began in order to provide a probabilistic proof of Hörmander hypoellipticity theorem and found important applications in the regularity theory of probability distributions of functionals of underlying Gaussian processes and of solutions of stochastic differential equations, as well as multiple stochastic integrals.

The principal motivation to study nonsymmetric OU semigroups comes from stochastic evolution equations. The connection is explained in Subsection 33.1, see (3.6), (3.7).

The paper is organized as follows. After an introductory section with preliminaries and notation, the classical O-U semigroup on separable Banach spaces is discussed in Section 2; we refer to the survey paper [2] for many details and historical notes.

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The main body of this paper is Section 3, where we describe the theory of Ornstein-Uhlenbeck semigroups on separable Hilbert spaces. The reference book is [15], where one can find the basic ideas, many examples and applications, and connections with stochastic analysis in Hilbert spaces.

In the last section we consider Ornstein-Uhlenbeck semigroups on separable Banach spaces. There are many more technicalities and far fewer examples than in the Hilbert setting, and in this short survey we have not room to give details, so we only briefly list some extensions of the results of Section 3 to the Banach case.

1 Preliminaries

Throughout the paper X is a separable real Banach space, with norm $\|\cdot\|$. $\mathcal{B}_b(X)$, $C_b(X)$ and $BUC(X)$ denote the spaces of Borel measurable, continuous, uniformly continuous and bounded functions from X to $\mathbb{R}$, respectively, endowed with the sup norm $\|\cdot\|_\infty$. Occasionally, we will be concerned also with the *mixed topology* in $C_b(X)$, which is the finest locally convex topology that agrees on every bounded set in $C_b(X)$ with the topology of uniform convergence on compact sets. As we are concerned with Gaussian measures on X and the relative Cameron-Martin Hilbert space $H \subset X$ is separable, we state the standing assumption that X itself is separable: in fact, Gaussian measures are always concentrated on the closure of H in X. See Subsection 1.4.

If Y is any Banach space, $\mathscr{L}(X,Y)$ is the space of linear bounded operators from X to Y; as usual if $Y = X$ it is denoted by $\mathscr{L}(X)$ and if $X = \mathbb{R}$ it is denoted by X^*. For $2 \leq h \in \mathbb{N}$, $\mathcal{L}^h(X)$ is the space of continuous h-linear operators from X^h to $\mathbb{R}$.

The Borel σ-algebra $\mathcal{B}(X)$ coincides with the σ-algebra $\mathscr{E}(X)$ generated by the *cylindrical* sets, i.e, the sets of the form $C = \{x \in X : (f_1(x), \ldots, f_n(x)) \in C_0\}$, where $f_1, \ldots, f_n \in X^*$ and $C_0 \in \mathcal{B}(\mathbb{R}^n)$, see e.g. [43, Ch. 1]. Accordingly, a function $f : X \to \mathbb{R}$ is called *cylindrical* if there are $f_1, \ldots, f_n \in X^*$ and $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $f(x) = \varphi(f_1(x), \ldots, f_n(x))$.

If X is a Hilbert space, we denote by $\langle \cdot, \cdot \rangle$ its inner product. $\mathscr{L}_1(X)$ and $\mathscr{L}_2(X)$ denote the subspaces of $\mathscr{L}(X)$ of nuclear self-adjoint and Hilbert-Schmidt operators, respectively.

1.1 Symmetric and positive operators.

An operator $Q \in \mathscr{L}(X^*, X)$ is called *symmetric* if $g(Qf) = f(Qg)$ for every f, $g \in X^*$, and *positive* if $f(Qf) \geq 0$ for every $f \in X^*$ (in fact, the right word should be "nonnegative" but we adopt the common terminology). As usual, if X is a Hilbert space we identify X and X^*, and the above notions correspond respectively to a self-adjoint and nonnegative $Q \in \mathscr{L}(X)$.

If Q is symmetric and positive, there exists a unique Hilbert space H_Q continuously embedded in X such that $Q(X^*)$ is dense in H_Q and $\langle Qf, Qg \rangle_{H_Q} = g(Qf)$, for every f, $g \in X^*$, see [43, Prop. III.1.6]. Denoting by $i : H_Q \to X$ the embedding we have $\|i\|_{\mathscr{L}(H_Q,X)} = \|Q\|^{1/2}_{\mathscr{L}(X^*,X)}$ and $i \circ i^* = Q$; see [43, Chapter III]. H_Q may be equivalently constructed by completing $Q(X^*)$ with respect to the norm associated with the inner product $(Qf, Qg) \mapsto g(Qf)$. It is easily seen that every Cauchy sequence (Qf_n) in such norm converges in X, and two equivalent Cauchy sequences converge in X to the same limit. Identifying (the equivalence class of) any Cauchy sequence (Qf_n) with its X-limit h, the completion is identified with H_Q.

If X is a Hilbert space and $Q \in \mathscr{L}(X)$ is self-adjoint and nonnegative, H_Q is just $Q^{1/2}(X)$ with the inner product $\langle Q^{1/2}x, Q^{1/2}y\rangle_{H_Q} = \langle x, y\rangle$ for every x, $y \in X$, or equivalently $\langle h, k\rangle_{H_Q} = \langle Q^{-1/2}h, Q^{-1/2}k\rangle$. Here, if $Q^{1/2}$ is not one to one, $Q^{-1/2}$ denotes its pseudo-inverse[(1)].

The space H_Q is sometimes called *reproducing kernel Hilbert space* associated with Q, but since the expression "reproducing kernel Hilbert space" has several different meanings in the literature, we will not use it.

1.2 Regular functions.

Let Y be any Banach space, $\alpha \in (0,1)$, $k \in \mathbb{N}$.

$C^\alpha_b(X;Y)$ is the space of bounded and α-Hölder continuous functions from X to Y, endowed with its standard norm $\|f\|_{C^\alpha_b(X;Y)} := \|f\|_\infty + [f]_{C^\alpha(X;Y)}$, where $[f]_{C^\alpha(X;Y)} = \sup_{x,\, y\in X;\, x\neq y} \|f(x) - f(y)\|_Y/\|x-y\|^\alpha$. If $Y = \mathbb{R}$ we set $C^\alpha_b(X;\mathbb{R}) =: C^\alpha_b(X)$.

$C^k_b(X)$ is the space of k times Fréchet differentiable functions $F : X \to \mathbb{R}$, with continuous and bounded derivatives $D^j f : X \to \mathcal{L}^j(X)$ for every $j = 1, \ldots, k$. The first order Fréchet derivative D^1 is denoted by D.

If X is a Hilbert space and $f : X \to \mathbb{R}$ is Fréchet differentiable at x, by the Riesz isometry there is a unique $z \in X$ such that $Df(x)(h) = \langle z, h\rangle$ for every $h \in X$. Such z is denoted by $\nabla f(x)$.

$C^{k+\alpha}_b(X)$ is the space of functions $f \in C^k_b(X)$ such that $D^k f \in C^\alpha(X;\mathcal{L}^k(X))$, endowed with the norm $\|f\|_{C^{k+\alpha}_b(X)} := \|f\|_\infty + \sum_{j=1}^k \sup_{x\in X} \|D^j f(x)\|_{\mathcal{L}^j(X)} + [D^k f]_{C^\alpha(X;\mathcal{L}^k(X))}$.

Let now $H \subset X$ be a Hilbert space continuously embedded in X, with inner product $\langle\cdot,\cdot\rangle_H$. A function $\varphi : X \to Y$ is H-Hölder continuous if there is $\alpha \in (0,1)$ such that $[\varphi]_{C^\alpha_H(X,Y)} := \sup_{x\in X,\, h\in H\setminus\{0\}}\{\|\varphi(x+h) - \varphi(x)\|_Y/\|h\|^\alpha_H\} < +\infty$. $C^\alpha_H(X,Y)$ is the space of the functions in $C_b(X,Y)$ that are H-Hölder continuous with exponent α, with norm $\|\varphi\|_{C^\alpha_H(X,Y)} := \|\varphi\|_\infty + [\varphi]_{C^\alpha_H(X,Y)}$.

A function $\varphi : X \to Y$ is H-differentiable at $x \in X$ if there exists $G \in \mathscr{L}(H,Y)$ such that $\|\varphi(x+h) - \varphi(x) - G(h)\|_Y = o(\|h\|_H)$, as $h \to 0$ in H. In this case the operator G is unique, and denoted by $D_H\varphi(x)$. Again, if $Y = \mathbb{R}$ there is a unique $y \in H$ such that $G(h) = \langle y, h\rangle_H$ for each $h \in H$. Such y is denoted by $\nabla_H\varphi(x)$. If φ is differentiable at x it is also H-differentiable at x, and

$$\frac{\partial\varphi}{\partial h}(x) := Y - \lim_{t\to 0}\frac{\varphi(x+th) - \varphi(x)}{h} = \langle\nabla_H\varphi(x), h\rangle_H = D_H\varphi(x)(h) = D\varphi(x)(h), \quad h \in H.$$

If $\varphi : X \to \mathbb{R}$ is H-differentiable at every point, and in its turn $D_H : X \to H^*$ is H-differentiable at $x \in X$, we set $D^2_H\varphi(x) := D_H(D_H\varphi)(x) \in \mathcal{L}^2(H)$ (after identifying $\mathscr{L}(H,H^*)$ with $\mathcal{L}^2(H)$). The space $C^1_H(X)$ (resp. $C^2_H(X)$) consists of the continuous, bounded and (resp. twice) H-differentiable functions such that $\nabla_H\varphi \in C_b(X,H)$ (resp. $\nabla_H\varphi \in C_b(X,H)$ and $D^2_H f \in C_b(X,\mathcal{L}^2(H))$).

1.3 Semigroups of bounded operators on $C_b(X)$

Let $T(t)$ be a semigroup of bounded operators on $C_b(X)$, such that $\|T(t)\|_{\mathscr{L}(C_b(X))} \leq Me^{\omega t}$ for some $M > 0$, $\omega \in \mathbb{R}$ and for every $t \geq 0$. Assume in addition that the function

[1]If $T \in \mathscr{L}(X)$ is self-adjoint and nonnegative, for every $h \in T(X)$ $T^{-1}h$ is the element of minimal norm in the set $T^{-1}(\{h\})$. We have $T^{-1}h = Py$ for every $y \in T(\{h\})$, where P is the orthogonal projection on $\overline{T(X)} = (\operatorname{Ker} T)^\perp$.

$(t, x) \mapsto T(t)f(x)$ is continuous in $[0, +\infty) \times X$.

Since we are going to deal with resolvent and spectrum, it is convenient to extend $T(t)$ to the space $C_b(X; \mathbb{C})$, setting $T(t)(f + ig) = T(t)f + iT(t)g$ for f, $g \in C_b(X)$.

This allows to define a generator through its resolvent,

$$R_\lambda f(x) := \int_0^\infty e^{-\lambda t} T(t)f(x)dt, \quad \operatorname{Re}\lambda > \omega,\ f \in C_b(X; \mathbb{C}), x \in X. \tag{1.1}$$

Indeed, in the space $C_b(X, \mathbb{C})$ the family $\{R_\lambda : \text{Re } \lambda > \omega\}$ satisfies the resolvent identity $R_\lambda R_\mu = (R_\mu - R_\lambda)/(\lambda - \mu)$ in the half-plane $\Pi := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega\}$, since $T(t)$ is a semigroup. Moreover, such identity implies that if $R_\mu f = 0$ for some $\mu \in \Pi$ then $R_\lambda f = 0$ for every $\lambda \in \Pi$. In particular, for every $x \in X$ the Laplace transform G of the function $g(t) := e^{-\omega t}T(t)f(x)$ vanishes for $\operatorname{Re}\lambda > 0$; since $g \in C_b([0, +\infty))$ then $g(0) = \lim_{\lambda\to\infty} \lambda G(\lambda) = 0$, so that $f(x) = g(0) = 0$. Therefore, R_μ is one to one for every $\mu \in \Pi$, and by e.g. [45, §VIII.4] there exists a unique closed operator whose resolvent operator is R_μ for every μ with $\operatorname{Re}\mu > \omega$. The part L of such operator in $C_b(X)$ preserves $C_b(X)$ and it is called *generator* of $T(t)$ in $C_b(X)$, although it is not an infinitesimal generator in the classical sense.

From the definition it follows $T(t)L = LT(t)$ on $D(L)$. For every $x \in X$, the continuity of $T(\cdot)f(x)$ in $[0, +\infty)$ yields easily its differentiability provided $f \in D(L)$, see e.g. [7, Prop. 4.2].

We recall that a Borel probability measure μ in X is called *invariant* for $T(t)$ if

$$\int_X T(t)f\, d\mu = \int_X f\, d\mu, \quad t > 0,\ f \in C_b(X). \tag{1.2}$$

1.4 Gaussian measures

We list here notation and results that will be used in the paper, referring to [1] for their proofs and for the general theory.

A probability measure γ on $(X, \mathcal{B}(X))$ is *Gaussian* if $\gamma \circ f^{-1}$ (defined as $\gamma \circ f^{-1}(B) = \gamma(f^{-1}(B))$ for every $B \in \mathcal{B}(\mathbb{R})$) is a Gaussian measure on $\mathbb{R}$ for every $f \in X^*$. The measure γ is called *centered* if all the measures $\gamma \circ f^{-1}$ have zero mean, and it is called *nondegenerate* if for any $f \neq 0$ the measure $\gamma \circ f^{-1}$ is absolutely continuous with respect to the Lebesgue measure.

We fix a centered Gaussian measure γ. By the Fernique Theorem, see [1, Thm. 2.8.5], γ has finite moments of any order. For every $g \in X^*$ the mapping $R : X^* \to \mathbb{R}$, $Rf := \int_X f(x)g(x)\, \gamma(dx)$ belongs to X^{**}, and even if X is not reflexive there exists a unique $y \in X$ such that $Rf = f(y)$, for every $f \in X^*$. We set $y = Qg$. The operator $Q \in \mathscr{L}(X^*, X)$ is called *covariance operator*, it is symmetric and positive and it is represented by the Bochner integral

$$Qf = \int_X f(x)\, x\, \gamma(dx), \quad f \in X^*.$$

(Such a formula may be used as an equivalent definition of Q). If X is a Hilbert space we identify as usual X and X^*, and therefore $Q \in \mathscr{L}(X)$ is defined by

$$\langle Qx_0, y_0\rangle = \int_X \langle x_0, x\rangle \langle y_0, x\rangle\, \gamma(dx), \quad x_0,\ y_0 \in X.$$

Moreover, Q belongs to $\mathscr{L}_1(X)$. Conversely, if a linear self-adjoint nonnegative operator Q is nuclear, then it is the covariance of a centered Gaussian measure, called $\mathcal{N}_{0,Q}$.

Let us go back to general Banach spaces. The closure of X^* in $L^2(X,\gamma)$ is denoted by X^*_γ. For every $g \in X^*_\gamma$, the mapping R defined above still has the representation $Rg = g(y)$, for a suitable (unique) $y \in X$, and we set $y =: R_\gamma g$. So, R_γ is the natural extension of Q to the whole X^*_γ.

The *Cameron-Martin space* H consists of the elements $h \in X$ such that the measure $\gamma_h(B) := \gamma(B-h)$, $B \in \mathcal{B}(X)$, is absolutely continuous with respect to γ. An important characterization of H, that yields a Hilbert space structure in it, is the following: we have $H = R_\gamma(X^*_\gamma)$, namely $h \in X$ belongs to H if and only if there is $\widehat{h} \in X^*_\gamma$ such that $\int_X \widehat{h}(x)g(x)\,\gamma(dx) = g(h)$ for every $g \in X^*$. In this case, $\|h\|_H = \|\widehat{h}\|_{L^2(X,\gamma)}$. Therefore $R_\gamma : X^*_\gamma \to H$ is an isometry, and H is a Hilbert space with the inner product $\langle h,k\rangle_H := \langle \widehat{h},\widehat{k}\rangle_{L^2(X,\gamma)}$ whenever $h = R_\gamma\widehat{h}$, $k = R_\gamma\widehat{k}$.

Remark 1.1 *The triplet (X,γ,H) is usually referred to as* abstract Wiener space. *In our discussion we have followed the presentation in [1]. As we have seen, γ (or equivalently, the covariance operator Q), determines H, but it is possible to go the other way around as follows. If a separable Hilbert space H is given together with a continuous inclusion mapping $i : H \to X$, setting $Q = i \circ i^*$, it turns out that $Q : X^* \to X$ is a positive symmetric operator. If Q is the covariance operator of a Gaussian measure γ, then*

$$\|i^*f\|_H^2 = f(Qf) = \int_X (f(x))^2\gamma(dx) = \|f\|^2_{L^2(X,\gamma)}, \quad f \in X^*.$$

Since the range of i^* is dense in H, this shows that the mapping $i^*f \mapsto f$ has a unique extension to an isometric embedding of H into $L^2(X,\gamma)$. The image of every $h \in H$ under this embedding is just $\hat{h}$, so that the range of this embedding is the space X^*_γ defined above.

For every $h \in H$, the density of γ_h with respect to γ is given by $e^{-\|h\|_H^2/2+\widehat{h}}$. It yields the integration by parts formula

$$\int_X \frac{\partial\varphi}{\partial h}\,\psi\,\gamma(dx) = -\int_X \varphi\frac{\partial\psi}{\partial h}\,\gamma(dx) + \int_X \varphi\,\psi\,\widehat{h}\,\gamma(dx), \quad \varphi,\ \psi \in C^1_b(X). \tag{1.3}$$

Moreover for every $h \in H$ the function $\widehat{h}$ is a real Gaussian random variable with law $\mathcal{N}_{0,\|h\|_H^2}$. In particular, $\widehat{h} \in L^q(X,\gamma)$ for every $q \in [1,\infty)$ and $\|\widehat{h}\|_{L^q(X,\gamma)} = (\int_{\mathbb{R}} |\xi|^q \mathcal{N}_{0,1}(d\xi))^{1/q}\|h\|_H =: c_q\|h\|_H$.

Recalling that for $f \in X^*$ we have $\int_X f(x)g(x)\,\gamma(dx) = g(Qf)$ for every $g \in X^*$, we see that $H = H_Q$ (the space introduced in Subsection (i)), with the same inner product. More precisely, referring to the construction of H_Q in [43, Chapter III] and the operators A involved there, we can take $A : X^* \to X^*_\gamma$, $Af = f$, so that $A^* = R_\gamma$.

If X is a Hilbert space, the Cameron-Martin space is the range of $Q^{1/2}$, and we have precisely $\langle Q^{1/2}x, Q^{1/2}y\rangle_H = \langle x,y\rangle$ for every $x,\ y \in X$, or equivalently $\langle h,k\rangle_H = \langle Q^{-1/2}h, Q^{-1/2}k\rangle$.

If $\{e_j :\ j \in \mathbb{N}\}$ is any orthonormal basis of X such that $Qe_j = \lambda_j e_j$ for every $j \in \mathbb{N}$, then for every $h \in H$ the function $\widehat{h}$ may be represented as $\widehat{h}(x) = \sum_{j:\,\lambda_j\neq 0} \lambda_j^{-1}\langle h,e_j\rangle\langle x,e_j\rangle$. The series converges in $L^p(X,\gamma)$ for every $p \in [1,+\infty)$ and it converges pointwise only for $x \in H$, in which case we have $\widehat{h}(x) = \langle h,x\rangle_H$. For this reason $\widehat{h}$ is called $\langle Q^{-1/2}h, Q^{-1/2}\cdot\rangle$ in [16, 15].

We warn the reader that in the literature about Gaussian measures the expression "reproducing kernel Hilbert space" is used both for H and for X^*_γ.

We denote by $\mathcal{F}\mathcal{C}_b^k(X)$ the space of the cylindrical functions $f : X \to \mathbb{R}$ such that $f(x) = \varphi(f_1(x), \dots, f_n(x))$ with $f_1, \dots, f_n \in X^*$ and $\varphi \in C_b^k(\mathbb{R}^n)$. Any such functions is k times Fréchet differentiable, and we have $Df(x) = \sum_{j=1}^n D_j\varphi(f_1(x), \dots, f_n(x))f_j$, $\nabla_H f(x) = QDf(x)$. Using (1.3), one proves that for every $p \in [1, \infty)$ and $k \in \mathbb{N}$, the operator $\nabla_H : \mathcal{F}C_b^k(X) \subset L^p(X, \gamma) \to L^p(X, \gamma; H)$ is closable, and the domain of its closure (still denoted by ∇_H) is a Banach space endowed with the graph norm, independent of k, called $W^{1,p}(X, \gamma)$. Moreover for $k \geq 2$ the operator $(\nabla_H, D_H^2) : \mathcal{F}C_b^k(X) \subset L^p(X, \gamma) \to L^p(X, \gamma; H) \times L^p(X, \gamma; \mathscr{L}_2(H))$ is closable too, and the domain of its closure, endowed with the graph norm, is independent of k and called $W^{2,p}(X, \gamma)$.

The *Gaussian divergence* is defined as minus the formal adjoint of ∇_H and is denoted by div_H. More precisely, a vector field $F \in L^1(X, \gamma; H)$ has Gaussian divergence if there exists (a unique) $\beta \in L^1(X, \gamma)$ such that $\int_X \langle \nabla_H \varphi, F \rangle_H \, \gamma(dx) = \int_X \varphi(x)\beta(x) \, \gamma(dx)$, for every $\varphi \in \mathcal{F}C_b^1(X)$. In this case we set $\text{div}_H f := -\beta$.

2 The classical O-U semigroup

Here X is a separable Banach space endowed with a centered Gaussian measure γ. The proofs of the statements of this section may be found in the book [1], unless otherwise specified.

The Ornstein–Uhlenbeck semigroup is defined through the Mehler formula by

$$T(0)f = f, \quad T(t)f(x) := \int_X f(e^{-t}x + \sqrt{1 - e^{-2t}}y)\gamma(dy), \quad t > 0, \ f \in C_b(X). \tag{2.1}$$

It is a contraction semigroup on $C_b(X)$, and γ is its unique invariant measure. It is not strongly continuous, not even on $BUC(X)$. In fact, it is easily seen that for every $f \in BUC(X)$ we have $\lim_{t\to 0^+} \|T(t)f - f\|_\infty = 0$ if and only if $\lim_{t\to 0^+} \|f(e^{-t}\cdot) - f\|_\infty = 0$. However, for every $f \in C_b(X)$ the function $(t, x) \mapsto T(t)f(x)$ is continuous on $[0, \infty) \times X$ by the Dominated Convergence Theorem, and this allows to define the generator L as in Section 1(iii). Moreover, $T(t)$ is strongly continuous in the mixed topology, see [25, 26].

Coming back to the norm topology, $T(t)$ is not analytic and even not continuous in norm on $(0, \infty)$, since $\|T(t) - T(s)\|_{\mathscr{L}(C_b(X))} \geq 2$ for $t \neq s$, as a consequence of [41, Prop. 2.4]. The semigroup $T(t)$ is smoothing along the Cameron-Martin space H. More precisely, for every $f \in C_b(X)$ and $t > 0$, $T(t)f$ is H-differentiable at every $x \in X$, and we have

$$\langle \nabla_H T(t)f(x), h \rangle_H = \frac{e^{-t}}{\sqrt{1 - e^{-2t}}} \int_X f(e^{-t}x + \sqrt{1 - e^{-2t}}y)\hat{h}(y)\gamma(dy), \quad h \in H. \tag{2.2}$$

Therefore, using the Hölder inequality and recalling that $\|\hat{h}\|_{L^1(X,\gamma)} \leq \|h\|_H$, $\|\hat{h}\|_{L^q(X,\gamma)} = c_q\|h\|_H$, for every $f \in C_b(X)$ and $x \in X$ we get

$$\begin{aligned} &(i) \quad \|\nabla_H T(t)f(x)\|_H \leq e^{-t}(\sqrt{1 - e^{-2t}})^{-1/2}\|f\|_\infty, \\ &(ii) \quad \|\nabla_H T(t)f(x)\|_H \leq c_{p'}e^{-t}(\sqrt{1 - e^{-2t}})^{-1/2}[(T(t)|f|^p)(x)]^{1/p}, \quad p \in (1, \infty), \end{aligned} \tag{2.3}$$

and moreover $\nabla_H T(t)f : X \to H$ is continuous. If in addition $f \in C_b^1(X)$, then $T(t)f \in C_b^1(X)$ for any $t \geq 0$, and

$$\frac{\partial T(t)f}{\partial h}(x) = DT(t)f(x)(h) = e^{-t}T(t)(Df(\cdot)(h)), \quad x, \ h \in X, \tag{2.4}$$

so that $\sup_{x\in X}\|DT(t)f(x)\|_{X^*} \le e^{-t}\sup_{x\in X}\|Df(x)\|_{X^*}$. Iterating, we get $T(t)C_b^k(X) \subset C_b^k(X)$ for any $t \ge 0$, $k \in \mathbb{N}$, and $\sup_{x\in X}\|D^kT(t)f(x)\|_{\mathcal{L}^k(X)} \le e^{-kt}\sup_{x\in X}\|D^kf(x)\|_{\mathcal{L}^k(X)}$.

Notice that (2.2) and (2.3) describe a smoothing property of $T(t)$, while the subsequent statements assert that $T(t)$ preserves the spaces $C_b^k(X)$ and it is contractive there. However, $T(t)$ regularizes only along H and it does not map $C_b(X)$ into $C^1(X)$.

The continuity of $\nabla_H T(t)f$ for $f \in C_b(X)$ and estimate (2.3)(i) yield the embedding $D(L) \subset C_H^1(X)$ through the representation formula (1.1) for $R(\lambda, L)$. Here, L is the generator of $T(t)$ defined in Section 2(1.3). Moreover, for every $f \in D(L)$, $D_H f \in C_b^\theta(X, H)$ for every $\theta \in (0, 1)$, and it also satisfies a Zygmund condition along H, see [10]. A Schauder type theorem was proved in [10] for H-Hölder continuous functions, and precisely: for every $\alpha \in (0, 1)$, $\lambda > 0$ and $f \in C_H^\alpha(X)$, $R(\lambda, L)f \in C_H^2(X)$ and $D_H^2 R(\lambda, L)f \in C_H^\alpha(X, \mathcal{L}^2(H))$.

The semigroup $T(t)$ is readily extended to $L^p(X, \gamma)$, for every $p \in [1, \infty)$. Indeed, we have

$$\int_X |T(t)f(x)|^p\, \gamma(dx) \le \int_X T(t)(|f|^p)\, \gamma(dx) = \int_X |f|^p \gamma(dx), \quad t > 0,\ f \in C_b(X), \tag{2.5}$$

by the Hölder inequality and the invariance of γ. Hence $\{T(t) :\ t \ge 0\}$ is uniquely extendable to a contraction semigroup $\{T_p(t) :\ t \ge 0\}$ on $L^p(X, \gamma)$. Moreover,

(i) $\{T_p(t) :\ t \ge 0\}$ is strongly continuous on $L^p(X, \gamma)$, for every $p \in [1, \infty)$;

(ii) $T_2(t)$ is self-adjoint and nonnegative on $L^2(X, \gamma)$ for every $t > 0$;

(iii) $\int_X T_p(t)f\, \gamma(dx) = \int_X f\, \gamma(dx)$, for every $f \in L^p(X, \gamma)$;

(iv) (hypercontractivity) for any $p, q > 1$ and $t > 0$ such that $q \le 1 + (p - 1)e^{2t}$, $T(t)$ maps $L^p(X, \gamma)$ into $L^q(X, \gamma)$ and $\|T(t)f\|_{L^q(X,\gamma)} \le \|f\|_{L^p(X,\gamma)}$ for every $f \in L^p(X, \gamma)$. For $q > 1 + (p - 1)e^{2t}$, $T(t)(L^p(X, \gamma))$ is not contained in $L^q(X, \gamma)$.

For $p \in (1, \infty)$, L^p estimates for $\|\nabla_H T_p(t)f\|_H$ are obtained similarly to (2.5). For every $f \in C_b(X)$, (2.3)(ii) yields

$$\int_X \|\nabla_H T(t)f(x)\|_H^p\, \gamma(dx) \le \frac{c_{p'}e^{-t}}{\sqrt{1 - e^{-2t}}} \int_X T(t)(|f|^p)\, \gamma(dx) = \frac{c_{p'}e^{-t}}{\sqrt{1 - e^{-2t}}} \int_X |f|^p \gamma(dx).$$

This argument fails for $p = 1$, since (2.3)(ii) holds only for $p > 1$. Indeed, $T(t)$ does not map $L^1(X, \gamma)$ into $W^{1,1}(X, \gamma)$ for $t > 0$, even in the simplest case $X = \mathbb{R}$ where γ is the standard Gaussian measure (see for instance [35, Corollary 5.1]). For $1 \le p < \infty$, using formulae (2.4) in $C_b^1(X)$, one obtains that $T_p(t)$ preserves $W^{1,p}(X, \gamma)$ for every $t > 0$, and $\|T_p(t)f\|_{W^{1,p}(X,\gamma)} \le \|f\|_{W^{1,p}(X,\gamma)}$ for every $f \in W^{1,p}(X, \gamma)$.

Let us denote by L_p the infinitesimal generator of $T_p(t)$ in $L^p(X, \gamma)$. It is not hard to see that every $f \in \mathcal{F}C_b^2(X)$ belongs to $D(L_p)$, and using (1.3) we get

$$L_p f(x) = \mathrm{div}_\gamma \nabla_H f(x) = \sum_{j=1}^\infty \Big(\partial^2_{h_j h_j} f(x) - \hat{h}_j(x)\partial_{h_j} f(x)\Big), \quad \gamma\text{–a.e. } x \in X, \tag{2.6}$$

where $\{h_j :\ j \in \mathbb{N}\}$ is any orthonormal basis of H. Moreover, $\mathcal{F}C_b^2(X)$ is a core of L_p for every $p \in [1, \infty)$. In other words, $D(L_p)$ consists of all $f \in L^p(X, \gamma)$ such that there exists a sequence (f_n) in $\mathcal{F}C_b^2(X)$ which converges to f in $L^p(X,\gamma)$ and such that $L_p f_n = \mathrm{div}_H \nabla_H f_n$ converges in $L^p(X,\gamma)$. The Meyer inequalities, see [37], yield

$$D(L_p) = W^{2,p}(X,\gamma), \quad 1 < p < \infty, \tag{2.7}$$

with equivalence of the respective norms (an independent analytic proof is in [1, Section 5.5]). For $p = 2$, L_2 is the operator associated with the Dirichlet form

$$\mathcal{D}(f,g) = \int_X \langle \nabla_H f, \nabla_H g\rangle_H \, d\gamma, \qquad f,g \in W^{1,2}(X,\gamma), \tag{2.8}$$

namely we have

$$\begin{aligned} D(L_2) = \{u \in W^{1,2}(X,\gamma) : \exists\, f \in L^2(X,\gamma) \ \text{ s. t.} \\ \mathcal{D}(u,g) = -\langle f, g\rangle_{L^2(X,\gamma)} \ \forall\, g \in W^{1,2}(X,\gamma)\}, \qquad L_2 u = f. \end{aligned}$$

In particular, $\langle L_2 u, u\rangle_{L^2(X,\gamma)} = -\|\nabla_H u\|^2_{L^2(X,\gamma;H)} \le 0$ for every $u \in D(L_2)$. Having a self-adjoint and dissipative generator, $T_2(t)$ is an analytic semigroup with angle of analyticity $\pi/2$; classical results about Markov semigroups (e.g., [17, Thm. 1.4.2]) yield that $T_p(t)$ is an analytic semigroup on $L^p(X,\gamma)$ with angle of analyticity $\ge \pi(1 - |2/p - 1|)/2$, for every $p \in (1, +\infty)$. The optimal analyticity angle angle in finite dimension $\theta_p = \pi/2 - \arctan(|\pi - 2|/2\sqrt{p-1})$ was shown to be optimal also in infinite dimension in the paper [31]. Functional calculus for L_p in the sector $\{z \in \mathbb{C} : \ z \neq 0, \ |\arg z| < \theta_p\}$ was considerd in [5, 6].

A complete description of the spectral properties of L_2 is available. Even more, there is an explicit orthonormal basis of $L^2(X,\gamma)$ made by eigenfunctions of L_2, that are the Hermite polynomials, defined for every multiindex $\alpha \in \Lambda := \{\alpha \in (\mathbb{N} \cup \{0\})^{\mathbb{N}}, \ \alpha = (\alpha_j), \ |\alpha| = \sum_{j=1}^\infty \alpha_j < \infty\}$, by

$$H_\alpha(x) = \prod_{j=1}^\infty H_{\alpha_j}(\hat{h}_j(x)), \quad x \in X, \tag{2.9}$$

where for $k \in \mathbb{N} \cup \{0\}$, $H_k : \mathbb{R} \to \mathbb{R}$ is the polynomial $H_k(\xi) = \frac{(-1)^k}{\sqrt{k!}} \exp\{\xi^2/2\} \frac{d^k}{d\xi^k} \exp\{-\xi^2/2\}$, for every $\xi \in \mathbb{R}$.

All the polynomials H_α belong to $L^p(X,\gamma)$ for every $p \in [1,\infty)$, and the set $\{H_\alpha :\ \alpha \in \Lambda\}$ is an orthonormal basis of $L^2(X,\gamma)$. Moreover, denoting by $\mathcal{X}_k$ the closure of span $\{H_\alpha : \ \alpha \in \Lambda, \ |\alpha| = k\}$ in $L^2(X,\gamma)$, we have the *Wiener chaos decomposition*,

$$L^2(X,\gamma) = \bigoplus_{k \in \mathbb{N} \cup \{0\}} \mathcal{X}_k.$$

The spectrum of L_2 is equal to $-\mathbb{N} \cup \{0\}$. For every $k \in \mathbb{N} \cup \{0\}$, $\mathcal{X}_k$ is the eigenspace of L_2 with eigenvalue $-k$. $\mathcal{X}_0$ is the kernel of L_2, consisting of constant functions, and $\mathcal{X}_1 = X^*_\gamma$.

3 Ornstein-Uhlenbeck semigroups in Hilbert spaces

Here X is a separable real Hilbert space, $Q \in \mathscr{L}(X)$ is a self-adjoint nonnegative operator, and $A : D(A) \subset X \to X$ is the infinitesimal generator of a strongly continuous semigroup e^{tA} on X.

We consider the Ornstein-Uhlenbeck operator formally defined by

$$\mathcal{L}u(x) = \frac{1}{2}\mathrm{Tr}[QD^2u(x)] + \langle Ax, \nabla u(x)\rangle. \tag{3.1}$$

The standing assumption of this section is that the linear operators Q_t defined by

$$Q_t x = \int_0^t e^{sA}Qe^{sA^*}x\,ds, \quad t > 0,\ x \in X, \tag{3.2}$$

are nuclear (Q itself does not need to be nuclear). Under such assumption, in [15, Ch. 6] it was proved that for very good initial data, namely $f \in BUC^2(X)$ such that $QD^2f \in BUC(X; \mathscr{L}_1(X))$, the initial value problem

$$u_t(t, x) = \mathcal{L}u(t, \cdot)(x),\ \ t \geq 0,\ x \in D(A); \quad u(0, x) = f(x),\ \ x \in X, \tag{3.3}$$

has a unique strict solution, which is a continuous function $u : [0, +\infty) \times X \to \mathbb{R}$ such that $u(t, \cdot) \in BUC^2(X)$ for every $t \geq 0$, $QD^2u(t, x) \in \mathscr{L}_1(X)$ for every $t \geq 0$ and $x \in X$, $u(\cdot, x)$ is continuously differentiable in $[0, +\infty)$ for every $x \in D(A)$, and satisfies (3.3). Moreover, the solution is given by

$$u(t, x) = \int_X f(e^{tA}x + y)\mu_t(dy), \quad t \geq 0,\ x \in X, \tag{3.4}$$

where μ_t is the centered Gaussian measure $\mathcal{N}_{0,Q_t}$ with covariance Q_t for $t > 0$, and $\mu_0 = \delta_0$.

3.1 Ornstein-Uhlenbeck semigroups on spaces of continuous functions

The right hand side of (3.4) is meaningful for every $f \in \mathcal{B}_b(X)$. Setting

$$T(t)f(x) := \int_X f(e^{tA}x + y)\mu_t(dy), \quad t \geq 0,\ f \in \mathcal{B}_b(X),\ x \in X, \tag{3.5}$$

$T(t)$ is a contraction semigroup on $\mathcal{B}_b(X)$. The fact that $T(t)$ maps $\mathcal{B}_b(X)$ into itself and it is a contraction is obvious. The fact that $T(t)$ is a semigroup is less obvious. It can be proved rewriting $T(t + s)$, for t, $s > 0$, as

$$T(t+s)f(x) = \int_X f(e^{(t+s)A}x + w)(\mu_t \circ (e^{sA})^{-1} \star \mu_s)(dw), \quad f \in \mathcal{B}_b(X),\ x \in X,$$

and checking that $\mu_t \circ (e^{sA})^{-1} \star \mu_s = \mu_{t+s}$, or else recalling that $T(t)$ is the transition semigroup of the stochastic differential equation

$$dX_t = AX_t\,dt + \sqrt{Q}\,dW_t,\ \ t > 0, \quad X(0) = x, \tag{3.6}$$

where W_t is any cylindrical Wiener process on X. Indeed, for every $x \in X$ the unique mild solution of (3.6) is $X_t = e^{tA}x + \int_0^t e^{(t-s)A}Q^{1/2}dW_s$, and the law of the stochastic convolution $\int_0^t e^{(t-s)A}Q^{1/2}dW_s$ is precisely $\mathcal{N}_{0,Q_t}$, see [16, Ch. 5]. Therefore,

$$T(t)f(x) = \mathbb{E}(f(X_t)), \quad t \geq 0,\ f \in \mathcal{B}_b(X),\ x \in X. \tag{3.7}$$

If $A=-I$ and Q is nuclear, setting $\gamma := \mathcal{N}_{0,2Q}$, $T(t)$ coincides with the classical Ornstein-Uhlenbeck semigroup considered in Section 2. If $A=0$, $T(t)$ may be called *heat semigroup*. In this case, $Q_t = tQ$ so that setting $y=\sqrt{t}z$ in the right hand side of (3.5) we get a simpler representation formula for $T(t)$,

$$T(t)f(x) := \int_X f(x+\sqrt{t}z)\mu(dz), \quad t \geq 0,\ f \in \mathcal{B}_b(X),\ x \in X,$$

where $\mu := \mathcal{N}_{0,Q}$.

Going back to the general case, the representation formula (3.5) yields that $T(t)$ is a Feller semigroup, namely it maps $C_b(X)$ into itself and, in fact, it maps the subspaces $BUC(X)$, $C_b^\alpha(X)$, $C_b^k(X)$, $C_b^{\alpha+k}(X)$ into themselves, for $\alpha \in (0,1)$, $k \in \mathbb{N}$. In particular, for $f \in C_b^1(X)$ we have

$$\langle \nabla T(t)f(x), h\rangle = \int_X \langle e^{tA^*}\nabla f(e^{tA}x+y), h\rangle\, \mu_t(dy), \quad x,\ h \in X. \tag{3.8}$$

$T(t)$ is strong-Feller (namely, it maps $\mathcal{B}_b(X)$ into $C_b(X)$) iff (see also [16, Remark 9.20])

$$e^{tA}(X) \subset Q_t^{1/2}(X), \quad t>0. \tag{3.9}$$

In this case, $T(t)$ maps $\mathcal{B}_b(X)$ into $C_b^k(X)$ for every $k \in \mathbb{N}$, see [15, Ch. 6]), and the operators

$$\Lambda_t = Q_t^{-1/2}e^{tA}, \quad t>0, \tag{3.10}$$

play an important role in the rest of the theory. First, $\Lambda_t \in \mathscr{L}(X)$ for every $t>0$. Moreover, for every $k \in \mathbb{N}$ there exists $C_k>0$ such that

$$\|D^kT(t)f(x)\|_{\mathcal{L}^k(X)} \leq C_k\|\Lambda_t\|^k_{\mathscr{L}(X)}\|f\|_\infty, \quad t>0,\ f \in \mathcal{B}_b(X),\ x \in X. \tag{3.11}$$

A proof for $k=1,2$ is in [15, Ch. 6]. For general k, (3.11) follows e.g. from [30, Sect. 5.1, Prop. 3.3(ii)].

Condition (3.9) is called *controllability condition* since it is equivalent to null controllability in any time t of an associated linear evolution equation in X, see e.g. [16, Appendix B] and [15, Chapter III]. It is not satisfied if $A=-I$, and, more generally, if A generates a strongly continuous group. Instead, it is satisfied if $Q=I$, and in this case $\|\Lambda_t\|_{\mathscr{L}(X)} \leq Me^{\omega t}t^{-1/2}$ for some $M>0$, $\omega \in \mathbb{R}$, and for every $t>0$. See [15, Appendix B], [26, Thm. 3.5(3)].

Anyway, smoothing properties along $H := Q^{1/2}(X)$ are available also in the case where H is properly contained in X, provided that e^{tA} maps H into itself, and that $S_H(t) := e^{tA}_{|H} : H \to H$ is a strongly continuous semigroup on H. In this case e^{tA} maps H into $Q_t^{1/2}(X)$ for every $t>0$, and $\sup_{0<t<1}\|e^{tA}\|_{\mathscr{L}(H,Q_t^{1/2}(X))} < \infty$, by [26, Thm. 3.5]. This allows to prove that $T(t)$ is smoothing along H, by arguments similar to the ones that led to (2.3)(ii). See [34, Sect. 2], and [30] for representation formulae and estimates for any order H-derivatives of $T(t)f$ when $f \in C_b(X)$.

Let us discuss strong continuity. Even in the case $X=\mathbb{R}$, $T(t)$ is not strongly continuous on $BUC(X)$ unless $A=0$ (let alone on $C_b(X)$). However, it is not hard to show that μ_t converges weakly to δ_0 as $t \to 0$ (namely, $\lim_{t\to 0}\int_X f(y)\mu_t(dy) = f(0)$ for every $f \in C_b(X)$) and this implies

$$\lim_{t\to 0}\|T(t)f - f(e^{tA}\cdot)\|_\infty = 0, \quad f \in BUC(X).$$

So, the subspace $BUC_S(X)$ of strong continuity of $T(t)$ on $BUC(X)$ is $\{f \in BUC(X) : \|T(t)f - f(e^{tA}\cdot)\|_\infty \to 0$ as $t \to 0\}$. If (3.11) holds, $T(t)(C_b(X)) \subset BUC(X)$ for every $t > 0$ and therefore $BUC_S(X)$ coincides with the subspace of strong continuity of $T(t)$ in $C_b(X)$. In the general case, the subspace of strong continuity of $T(t)$ in $C_b(X)$ is not known. However, $T(t)$ is strongly continuous on $C_b(X)$ with respect to the mixed topology, see [7, 25]. In particular, the function $(t, x) \mapsto T(t)f(x)$ is continuous on $[0, +\infty) \times X$ for every $f \in C_b(X)$, and this allows to define a generator L as in Section 1(iii). Moreover, setting $\Delta_h f = (T(h)f - f)/h$ for $h > 0$, we have

$$
\begin{aligned}
D(L) =\{& f \in C_b(X) :\ \limsup_{h\to 0} \|\Delta_h f\|_\infty < +\infty,\ \exists g \in C_b(X) \text{s.t.} \\
& \lim_{h\to 0} \Delta_h f(x) = g(x) \text{ uniformly on compact sets}\}, \qquad Lf = g.
\end{aligned} \tag{3.12}
$$

See [25, 26]. An analogous characterization with the space $C_b(X)$ replaced by $BUC(X)$ is in [40]. Still in [25] it was proved that (similarly to the case of strongly continuous semigroups on Banach spaces) any subspace $D \subset D(L)$ which is dense in $C_b(X)$ in the mixed topology and such that $T(t)(D) \subset D$, is a core for L, namely for every $f \in D(L)$ there exists a net $(f_\alpha) \subset D(L)$ such that $f_\alpha \to f$ and $Lf_\alpha \to Lf$ in the mixed topology. In [26, Thm. 6.6], see also [25, Thm. 4.5], it is proved that

$$
\begin{aligned}
\mathcal{F}_0 := \{& f \in C_b(X) : f = \varphi(\langle \cdot, a_1\rangle, \ldots, \langle \cdot, a_n\rangle);\ \varphi \in C^2_b(\mathbb{R}^n), \\
& n \in \mathbb{N},\ a_i \in D(A^*),\ \langle \cdot, A^*\nabla f\rangle \in C_b(X)\}
\end{aligned} \tag{3.13}
$$

and its subspace $\mathcal{F}C^\infty$ (whose members are the functions f represented as in (3.13) with $\varphi \in C^\infty_c(\mathbb{R}^n)$) are cores for L and that

$$
Lf(x) = \frac{1}{2}\text{Tr}[QD^2 f(x)] + \langle x, A^*\nabla f(x)\rangle, \quad f \in \mathcal{F}_0,\ x \in X, \tag{3.14}
$$

where the right-hand side is equal to $\mathcal{L}f(x)$ for every $x \in D(A)$. Related results with $BUC(X)$ replacing $C_b(X)$ are in [7, 9, 40]. In some papers, see e.g. [25], also the realization of $T(t)$ in the *weighted* spaces $C_m(X) = \{f \in C(X; \mathbb{R}) :\ \|f\|_{C_m(X)} := \sup_{x\in X} |f(x)|/(1 + \|x\|^m) < \infty\}$ has been studied.

In finite dimension $T(t)$ is analytic iff $A = 0$. Instead, if X is infinite dimensional, we have $\|T(t) - T(s)\|_{\mathscr{L}(BUC_S(X))} = 2$ and therefore $\|T(t) - T(s)\|_{\mathscr{L}(C_b(X)} \geq 2$ whenever μ_t and μ_s are singular (which is the case for every t, $s > 0$ if $A = 0$). The same equality holds if $e^{tA} \neq e^{sA}$, see [41, 38]. Therefore, $T(t)$ is not norm continuous, and hence not analytic, both in the case $A = 0$ and in the case $A \neq 0$. See [26, 41, 38].

An alternative proof of norm discontinuity in the case $A = 0$ comes from [36], where it has been proved that the spectrum of the part of L in $BUC(X; \mathbb{C})$ is the halfplane $\{\lambda \in \mathbb{C} : \text{Re}\,\lambda \leq 0\}$, and for every $t > 0$ the spectrum of $T(t)$ in $BUC(X; \mathbb{C})$ is the whole closed unit disk.

Schauder type results in the usual Hölder spaces are available if (3.9) holds, under the further assumption

$$
\exists\ M,\ \theta > 0,\ \omega \in \mathbb{R} : \quad \|\Lambda_t\|_{\mathscr{L}(X)} \leq \frac{Me^{\omega t}}{t^\theta}, \quad t > 0. \tag{3.15}
$$

Easy examples such that (3.2) and (3.15) hold (with any $\theta > 0$) are given in [15, Ex. 6.2.11]. The following theorem is taken from [30, Sect. 5.1].

Theorem 3.1 *Let (3.2) and (3.15) hold. For every $f \in C_b(X)$ and $\lambda > 0$, let $u = R(\lambda, L)f$. Then*

(i) If $1/\theta \notin \mathbb{N}$, then $u \in C_b^{1/\theta}(X)$, and there is $C > 0$, independent of f, such that $\|u\|_{C_b^{1/\theta}(X)} \leq C\|f\|_\infty$.

(ii) If in addition $f \in C_b^\alpha(X)$ with $\alpha \in (0,1)$ and $\alpha + 1/\theta \notin \mathbb{N}$, then $u \in C_b^{\alpha+1/\theta}(X)$ and there is $C > 0$, independent of f, such that $\|u\|_{C_b^{\alpha+1/\theta}(X)} \leq C\|f\|_{C_b^\alpha(X)}$.

Statement (ii) was already proved in [8] in the case that A is the realization of a second order elliptic system with general boundary conditions in $X = L^2(\Omega)$, Ω being a bounded open set in $\mathbb{R}^n$, and suitable assumptions on Q that yield $\theta = 1/2$. See also [4] for an earlier result.

Statement (i) implies that $D(L) \subset C_b^{1/\theta}(X)$ if $1/\theta \notin \mathbb{N}$, with continuous embedding. Statement (ii) implies that the domain of the part of L in $C_b^\alpha(X)$ is continuously embedded in $C_b^{\alpha+1/\theta}(X)$ if $\alpha + 1/\theta \notin \mathbb{N}$. In both cases, we gain "$1/\theta$ degrees" of regularity.

Both for $\alpha = 0$ and for $\alpha > 0$, in the critical cases $\alpha + 1/\theta = k \in \mathbb{N}$ we cannot expect that $u \in C^k(X)$; in [30] it is proved that u belongs to a suitable Zygmund space, which is continuously embedded in all spaces $C_b^{k-\varepsilon}(X)$ with $\varepsilon \in (0,1)$. This difficulty arises even in finite dimension, for instance if $X = \mathbb{R}^n$, $A = 0$, $Q = 2I$ we have $\mathcal{L} = \Delta$, $Q_t = 2tI$ and (3.15) holds with $\theta = 1/2$, but if $\lambda u - \Delta u = f \in C_b(\mathbb{R}^n)$ with $n \geq 2$, u is not necessarily a C^2 function.

If e^{tA} maps $H = Q^{1/2}(X)$ into itself, and $S_H(t) = e^{tA}_{|H} : H \to H$ is a strongly continuous semigroup on H, Schauder theorems similar to the ones stated in Sect. 2 were proved in [30]: for every $\alpha \in (0,1)$, $\lambda > 0$ and $f \in C_H^\alpha(X)$, $R(\lambda, L)f \in C_H^2(X)$ and $D_H^2 R(\lambda, L)f \in C_H^\alpha(X, \mathcal{L}^2(H))$.

Schauder type regularity results are available also for evolution equations with bounded and continuous data, see [30].

The asymptotic behavior of $T(t)$ is well understood if

$$\sup_{t>0} \mathrm{Tr}\,(Q_t) = \int_0^\infty \mathrm{Tr}(e^{sA}Qe^{sA^*})ds < +\infty. \tag{3.16}$$

Next statements are taken from [16, Sect. 11.3], [15, Sect. 10.1]. If (3.16) holds there exists a nuclear self-adjoint operator Q_∞, given by

$$Q_\infty x = \int_0^\infty e^{sA}Qe^{sA^*}x\,ds, \quad x \in X, \tag{3.17}$$

which maps $D(A^*)$ into $D(A)$ and satisfies the identity (called Lyapunov equation)

$$Q_\infty A^* x + AQ_\infty x = -Qx, \quad x \in D(A^*), \tag{3.18}$$

Such identity is easily obtained recalling that $\langle Q_\infty e^{tA^*}x, e^{tA^*}y\rangle = -\langle Q_\infty x, y\rangle - \langle Q_t x, y\rangle$ for every x, $y \in X$. Indeed, taking x, $y \in D(A^*)$, differentiating in time and taking $t = 0$ we get $\langle Q_\infty A^* x, y\rangle + \langle Q_\infty x, A^* y\rangle = \langle Qx, y\rangle$ and (3.18) follows by the density of $D(A^*)$.

Moreover, the Gaussian measure $\mu_\infty := \mathcal{N}_{0,Q_\infty}$ is invariant for $T(t)$, namely

$$\int_X T(t)f(x)\,\mu_\infty(dx) = \int_X f(x)\,\mu_\infty(dx), \quad t > 0,\ f \in C_b(X).$$

In fact, it is possible to show that (3.16) holds iff there exists a probability invariant measure for $T(t)$ iff there exists a self-adjoint nonnegative nuclear operator P mapping $D(A^*)$ into $D(A)$ and such that $PA^*x + APx = -Qx$ for every $x \in D(A^*)$ (which is equivalent to $2\langle PA^*x, x\rangle + \langle Qx, x\rangle = 0$ for every $x \in D(A^*)$). Moreover, any invariant measure is given by $\nu \star \mu_\infty$, ν being a probability invariant measure for the semigroup $R(t)$ defined by $R(t)f(x) = f(e^{tA}x)$ (e.g. [46], [16, Thm. 11.17]). So, if $R(t)$ has no invariant measures except δ_0, μ_∞ is the unique invariant measure for $T(t)$. In particular, this happens if $\lim_{t\to\infty} e^{tA}x = 0$ for every x.

If $\|e^{tA}\|_{\mathscr{L}(X)}$ vanishes as $t \to \infty$, namely if there are M, $\omega > 0$ such that

$$\|e^{tA}\|_{\mathscr{L}(X)} \le Me^{-\omega t}, \quad t > 0, \tag{3.19}$$

it is not hard to see that (3.16) holds (e.g., [16, Thm. 11.20]), and therefore μ_∞ is well defined and it is the unique invariant measure for $T(t)$. Moreover, if (3.19) holds then A is invertible.

Notice that if Q commutes with e^{tA} for every t and in addition A is self-adjoint then $Q_\infty = -QA^{-1}/2 = -A^{-1}Q/2$. The equality $Q_\infty = -A^{-1}Q/2$ holds even in a more general situation, see the remarks after Theorem 3.2.

It is interesting to compare kernels and ranges of $Q^{1/2}$, $Q_t^{1/2}$ and $Q_\infty^{1/2}$ for $t > 0$, that play an important role in the theory. We set

$$H := Q^{1/2}(X), \quad H_t := Q_t^{1/2}(X), \quad H_\infty := Q_\infty^{1/2}(X),$$

endowing them with their natural inner products, described in Sect. 1(i). Using the Lyapunov equation (3.18) one gets easily (e.g., [22, Lemma 2.1])

$$e^{tA}H_\infty \subset H_\infty, \quad \|Q_\infty^{-1/2}e^{tA}Q_\infty^{1/2}\|_{\mathscr{L}(X)} \le 1, \quad t > 0.$$

Therefore, $e^{tA}_{|H_\infty} : H_\infty \to H_\infty$ is a contraction semigroup, called $S_\infty(t)$. Its infinitesimal generator is the part A_∞ of A in H_∞. Since $\langle Q_t x, x\rangle \le \langle Q_\infty x, x\rangle$ for every $t > 0$ and $x \in X$, then $\operatorname{Ker} Q_\infty = \operatorname{Ker} Q_\infty^{1/2} \subset \operatorname{Ker} Q_t^{1/2} \subset \operatorname{Ker} Q^{1/2} = \operatorname{Ker} Q$, and $H_t \subset H_\infty$ (we recall that, given self-adjoint operators T_1, $T_2 \in \mathscr{L}(X)$, we have $T_1(X) \subset T_2(X)$ iff there exists $C > 0$ such that $\|T_1x\| \le C\|T_2x\|$ for every $x \in X$). Instead, the converse inclusion $H_\infty \subset H_t$ is not necessarily satisfied, and by [22, Prop. 4.1] or [11, Lemma 4] it is equivalent to

$$\|Q_\infty^{-1/2}e^{tA}Q_\infty^{1/2}\|_{\mathscr{L}(X)} < 1, \tag{3.20}$$

namely, to $\|S_\infty(t)\|_{\mathscr{L}(H_\infty)} < 1$.

In the proof of Theorem 11.22 of [16] it was shown that if (3.9) holds, then $H_\infty \subset H_t$ (so that (3.20) holds) and moreover the operators $Q_t^{-1/2}Q_\infty Q_t^{-1/2} - I$ are Hilbert-Schmidt on H_∞ for every $t > 0$ and therefore μ_t and μ_∞ are equivalent measures, for every $t > 0$, by the Feldman-Hájek Theorem (see e.g. [16, Thm. 2.25]).

If (3.16) holds, we have (see [16, Thm. 11.20])

$$\lim_{t\to\infty} T(t)f(x) = \int_X f(y)\,\mu_\infty(dy), \quad f \in C_b(X),\ x \in X. \tag{3.21}$$

We notice that if $A = 0$, then $\operatorname{Tr} Q_t = t \operatorname{Tr} Q$, so that (3.16) does not hold, and the heat semigroup has no invariant measure. Instead, if $A = -\omega I$ with $\omega > 0$, then $\operatorname{Tr} Q_t = (1 - e^{-2\omega t}) \operatorname{Tr} Q/(2\omega)$, so that (3.16) holds with $Q_\infty = Q/(2\omega)$. In particular, as we already mentioned in Section 2, the classical Ornstein-Uhlenbeck semigroup has γ itself as unique invariant measure (we recall that the covariance of γ is $2Q$).

3.2 Ornstein-Uhlenbeck semigroups on L^p spaces with respect to invariant measures

Throughout this section we assume that (3.16) holds, and we consider L^p spaces with respect to the invariant measure μ_∞, $1 \le p < \infty$.

For every $f \in C_b(X)$ and $t > 0$, the Hölder inequality and the invariance of μ_∞ yield

$$\int_X |T(t)f(x)|^p \mu_\infty(dx) \le \int_X T(t)(|f|^p)\mu_\infty(dx) = \int_X |f|^p \mu_\infty(dx)$$

and therefore, since $C_b(X)$ is dense in $L^p(X, \mu_\infty)$, $T(t)$ has a bounded extension to $L^p(X, \mu_\infty)$, denoted by $T_p(t)$. The above inequality implies that $T_p(t)$ is a contraction semigroup on $L^p(X, \mu_\infty)$. By the Dominated Convergence Theorem, $\lim_{t\to 0} \|T(t)f - f\|_{L^p(X,\mu_\infty)} = 0$ for every $f \in C_b(X)$, and this yields $\lim_{t\to 0} \|T_p(t)f - f\|_{L^p(X,\mu_\infty)} = 0$ for every $f \in L^p(X, \mu_\infty)$.

The generator of $T_p(t)$ is denoted by L_p. Since $T_p(t)f = T_q(t)f$ for $p \le q$ and $f \in L^q(X, \mu_\infty)$, then L_q is the part of L_p in $L^q(X, \mu_\infty)$, and the subindex p will be written only if needed.

Notice that, for every $f \in D(L_p)$, letting $t \to 0$ in the equality $\int_X [(T(t)f - f)/t] d\mu_\infty = 0$ we obtain $\int_X L_p f \, d\mu_\infty = 0$.

Concerning asymptotic behavior, for every $f \in L^p(X, \mu_\infty)$ we have

$$\lim_{t\to\infty} \left\| T_p(t)f - \int_X f(y)\, \mu_\infty(dy) \right\|_{L^p(X,\mu_\infty)} = 0. \tag{3.22}$$

If $f \in C_b(X)$, (3.22) is a consequence of (3.21) through the Dominated Convergence Theorem; if $f \in L^p(X, \mu_\infty)$ it follows approximating f by a sequence of continuous and bounded functions.

Using the Dominated Convergence Theorem, it is easy to see that the space $\mathcal{F}_0$ defined in (3.13) is contained in $D(L_p)$ for every $p \in [1, \infty)$, and it is a core for L_p since it is invariant under $T(t)$ and dense in $L^p(X, \mu_\infty)$. Another convenient core, used in [15], is the subspace of $\mathcal{F}_0$ defined by

$$\mathcal{E}_A(X) := \text{span}\,\{\cos(\langle \cdot, h\rangle),\ \sin(\langle \cdot, k\rangle);\ h,\, k \in D(A^*)\}.$$

Necessary and sufficient conditions for $T_2(t)$ be self-adjoint for every $t > 0$ (or, equivalently, for L_2 be self-adjoint) were given in [13] under the assumption that Q_∞ is one to one, that was later removed in [26]. In both papers, the key tool was the representation of $T_2(t)$ as the second quantization operator of the operator $S_\infty(t)^*$, that goes back to [11].

Theorem 3.2 *The following conditions are equivalent.*

(i) $T_2(t) = T_2(t)^*$ *for every* $t > 0$*;*

(ii) $Q(D(A^*)) \subset D(A)$*, and* $AQx = QA^*x$ *for every* $x \in D(A^*)$*;*

(iii) $e^{tA}Q = Qe^{tA^*}$*, for every* $t > 0$*;*

(iv) $e^{tA}Q_\infty = Q_\infty e^{tA^*}$*, for every* $t > 0$*;*

(v) $e^{tA}(H) \subset H$*, and* $S_H(t) := e^{tA}_{|H} : H \to H$ *is a self-adjoint strongly continuous semigroup on* H*.*

We refer to the conditions of Theorem 3.2 as "the symmetric case". In such a case, by the general theory of semigroups the infinitesimal generator L_2 of $T_2(t)$ is self-adjoint too. Moreover $T_2(t)$ is a symmetric Markov semigroup on $L^2(X,\mu_\infty)$, according to the terminology of [17], and therefore $T_p(t)$ is an analytic semigroup on $L^p(X,\mu_\infty)$ for every $p \in (1,+\infty)$ with angle of analyticity $\geq \pi(1-|2/p-1|)/2$, by [17, Thm. 1.4.2]. In addition, *(iv)* yields that Q_∞ maps $D(A^*)$ into $D(A)$, and on $D(A^*)$ we have $AQ_\infty = Q_\infty A^*$ $(= -Q/2$ by the Lyapunov equation). In particular, if 0 belongs to the resolvent set $\rho(A)$ we get an explicit formula for $Q_\infty = -\frac{1}{2}A^{-1}Q = -\frac{1}{2}Q(A^*)^{-1}$. About condition (v), we remark that $S_H(t)$ is self-adjoint and strongly continuous on H iff $Q^{-1/2}e^{tA}Q^{1/2}$ is self-adjoint and strongly continuous on X. Moreover, in the symmetric case not only $S_H(t)$ is strongly continuous, but there are M_1, $\beta > 0$ such that

$$\|S_H(t)\|_{\mathscr{L}(H)} \leq M_1 e^{-\beta t}, \quad t > 0. \tag{3.23}$$

See [26, Thm. 4.5]. Such estimate plays an important role in the asymptotic behavior of $T_p(t)$.

In the nonsymmetric case, $T_p(t)$ is not in general analytic, even in finite dimension: see the counterexamples in [21]. Necessary and sufficient conditions for analyticity were studied in the papers [42, 21, 23, 26, 31, 38]. In particular, [26] contains extensions and improvements of the previous ones, that are summarized in the next theorem.

Theorem 3.3 *The following conditions are equivalent:*

(i) $T_2(t)$ *is an analytic semigroup on* $L^2(X,\mu_\infty)$*;*

(ii) there exists $M > 0$ *such that* $|\langle Q_\infty A^*x, y\rangle| \leq M\langle Qx,x\rangle^{1/2}\langle Qy,y\rangle^{1/2}$*, for every* x, $y \in D(A^*)$*;*

(iii) $S_\infty(t)$ *is an analytic contraction semigroup*[2] *in* H_∞*.*

If in addition Q *has a bounded inverse, the above conditions are also equivalent to*

(iv) The operator AQ_∞ *has an extension belonging to* $\mathscr{L}(X)$*;*

(v) The operator $Q_\infty A^*$ *has an extension belonging to* $\mathscr{L}(X)$*.*

We refer to the conditions of Theorem 3.3 as "the analytic case". As in the symmetric case, if $T_2(t)$ is analytic on $L^2(X,\mu_\infty)$ then $T_p(t)$ is analytic on $L^p(X,\mu_\infty)$ for every $p \in (1,\infty)$, by a simple application of the Stein Interpolation Theorem (e.g., [29, Sect. 6.2]). Moreover, $T_p(t)$ is an analytic contraction semigroup and the optimal angle of analyticity θ_p has been determined in [31]; in [6] it has been proved that such angle coincides with the optimal angle for the bounded H^∞ calculus of $-L_p$. In addition, in the analytic case the semigroup e^{tA} maps H into itself, and the semigroup $S_H(t)$ is a strongly continuous, bounded analytic semigroup on H, see [34, Thm. 3.3]. For $p = 1$, $T_1(t)$ is not analytic, even in finite dimension. Characterizations of the domains $D(L_p)$ as suitable Sobolev spaces are known only in the analytic case.

The definition of the proper Sobolev spaces relies on the closability of the operator $\nabla_H : \mathcal{F}_0 \subset L^p(X,\mu_\infty) \to L^p(X,\mu_\infty;H)$, with $p \in [1,\infty)$. If $f \in \mathcal{F}_0$, $f(x) = \varphi(\langle x,x_1\rangle,\ldots,\langle x,x_n\rangle)$

[2] An analytic semigroup $T(t)$ on a real Banach space $\mathcal{X}$ is called "analytic contraction semigroup" if there exists a sector $\Sigma := \{z \neq 0 : \ |\arg z| < \theta\}$ with $\theta > 0$ such that the analytic extension $T(z)$ satisfies $\|T(z)\|_{\mathcal{L}(\mathcal{X}^{\mathbb{C}})} \leq 1$ for every $z \in \Sigma$. $\mathcal{X}^{\mathbb{C}}$ is the complexification of X.

with $\varphi \in C_b^2(\mathbb{R}^n)$ and $x_k \in D(A^*)$, we have $\nabla_H f(x) = \sum_{k=1}^n D_k\varphi(\langle x, x_1\rangle, \dots, \langle x, x_n\rangle)Qx_k$. Recalling (3.14), (1.3) and using the Lyapunov equation it is easy to see that for f, $g \in \mathcal{F}_0$ we have

$$\int_X (Lf\, g + f\, Lg)\mu_\infty(dx) = -\int_X \langle Q\nabla f, \nabla g\rangle \mu_\infty(dx) = -\int_X \langle \nabla_H f, \nabla_H g\rangle_H \mu_\infty(dx). \quad (3.24)$$

According to [24, Sect. 6], a sufficient condition for ∇_H be closable is that Q is one to one and the operator $W : H_\infty \to X$, $W(x) = Q^{1/2}Q_\infty^{-1/2}x$, is closable in X. Another sufficient condition, see [24, Cor. 5.6], is that e^{tA} maps H into itself and $S_H(t)$ is strongly continuous on H. So, in the analytic case (and, in particular, in the symmetric case) ∇_H is closable in $L^p(X, \mu_\infty)$ for every $p \in [1, \infty)$. See also [26, Prop. 8.3] and [24] for counterexamples to the closability of the gradient.

Whenever ∇_H is closable in $L^p(X, \mu_\infty)$, the Sobolev space $W_H^{1,p}(X, \mu_\infty)$ is defined as the domain of its closure (still called ∇_H), and it is a Banach space endowed with the graph norm

$$\|f\|^p_{W_H^{1,p}(X,\mu_\infty)} = \|f\|^p_{L^p(X,\mu_\infty)} + \int_X \|\nabla_H f(x)\|^p_H \,\mu_\infty(dx).$$

In particular, for $p = 2$ it is a Hilbert space with inner product $\langle f, g\rangle_{W_H^{1,2}(X,\mu_\infty)} = \langle f, g\rangle_{L^2(X,\mu_\infty)} + \langle \nabla_H f, \nabla_H g\rangle_{L^2(X,\mu_\infty;H)}$. In its turn, the operator $D_H^2 : \mathcal{F}_0 \subset L^p(X, \mu_\infty) \to L^p(X, \mu_\infty; \mathscr{L}_2(H))$ is closable, and $W_H^{2,p}(X, \mu_\infty)$ is defined as the domain of the closure (still called D_H^2), endowed with the graph norm

$$\|f\|^p_{W_H^{2,p}(X,\mu_\infty)} = \|f\|^p_{W_H^{1,p}(X,\mu_\infty)} + \int_X \|D_H^2 f(x)\|^p_{\mathscr{L}_2(H)} \,\mu_\infty(dx).$$

Another involved Sobolev-type space is the domain of the closure of $A_\infty^*\nabla_{H_\infty} : \mathcal{F}_0 \subset L^p(X, \mu_\infty) \to L^p(X, \mu_\infty; H_\infty)$ in $L^p(X, \mu_\infty)$, called $W^{1,p}_{AQ}(X, \mu_\infty)$ (we recall that A_∞ is the part of A in H_∞).

Using the notation in (3.13), for $f \in \mathcal{F}_0$ we have $\|\nabla_H f(x)\|_H = \|Q^{1/2}\nabla f(x)\|$, $\|D_H^2 f(x)\|^2_{\mathscr{L}_2(H)} = \mathrm{Tr}\,(QD^2 f(x))^2$, and $\|A_\infty^*\nabla_{H_\infty} f(x)\|^2_{H_\infty} = \langle A^*\nabla\varphi(x), Q_\infty A^*\nabla\varphi(x)\rangle$. In the symmetric case, using the Lyapunov equation we get $\|A_\infty^*\nabla_{H_\infty} f(x)\|^2_{H_\infty} = \langle \nabla\varphi(x), -AQ\nabla\varphi(x)\rangle/2$. In the case of the classical Ornstein-Uhlenbeck operator, we have $A = -I$, $Q_\infty = 2Q$, and the spaces $W_H^{1,p}(X, \mu_\infty) = W^{1,p}_{AQ}(X, \mu_\infty)$, $W_H^{2,p}(X, \mu_\infty)$ considered here coincide respectively with the spaces $W^{1,p}(X, \gamma)$, $W^{2,p}(X, \gamma)$ described in Section 1(iv), with $\gamma = \mathcal{N}_{0,2Q}$.

Before going on, we observe that the quadratic form

$$\mathcal{Q}(\varphi, \psi) := \frac{1}{2}\int_X \langle \nabla_H\varphi(x), \nabla_H\psi(x)\rangle_H \mu_\infty(dx), \quad \varphi,\ \psi \in W_H^{1,2}(X, \mu_\infty)$$

is closed, and in the symmetric case $-L_2$ is the operator associated with the form $\mathcal{Q}$ in $L^2(X, \mu_\infty)$, namely

$$D(L_2) = \{f \in W_H^{1,2}(X, \mu_\infty);\ \exists g \in L^2(X, \mu_\infty) \text{ s.t. } \mathcal{Q}(f, \psi) = \langle f, g\rangle_{L^2(X,\mu_\infty)}\}, \quad L_2 f = -g,$$

and therefore $D(-L_2)^{1/2} = W_H^{1,2}(X, \mu_\infty)$. Even in the nonsymmetric case, recalling that $\mathcal{F}_0$ is a core for L_2, formula (3.24) yields $D(L_2) \subset W_H^{1,2}(X, \mu_\infty)$ and (3.24) holds for any f, $g \in D(L_2)$. In particular, taking $f = g$ we get

$$\int_X Lf(x)\, f(x)\, \mu_\infty(dx) = -\frac{1}{2}\int_X \|\nabla_H f\|^2_H \,\mu_\infty(dx), \quad f \in D(L). \quad (3.25)$$

In the analytic case (see condition (ii) of Thm. 3.3) there is a sort of bounded extension of $Q_\infty A^*$ to H; more precisely, see [31], there exists an operator $B \in \mathscr{L}(H)$ such that $BQ_\infty x = Q_\infty A^* x$ for $x \in D(A^*)$, and that satisfies $B + B^* = -I$ in H by the Lyapunov equation. Moreover, $L_p f = \nabla_H^* B \nabla_H f$, for every f in the core $\mathcal{F}_0$. In the symmetric case we have $B = -I/2$, and this statement coincides with (2.6) for the classical Ornstein-Uhlenbeck operator.

The next theorem follows from [12, 13, 32, 34], and generalizes an earlier result of [14].

Theorem 3.4 *In the symmetric case for every $p \in (1, +\infty)$ we have $D(L_p) = W_H^{2,p}(X, \mu_\infty) \cap W_{AQ}^{1,p}(X, \mu_\infty)$, $D((-L_p)^{1/2}) = W_H^{1,p}(X, \mu_\infty)$, with equivalence of the respective norms.*

The next theorem follows from [33, 34]. We recall that in the analytic case e^{tA} maps H into itself, and $S_H(t) = e^{tA}_{|H} : H \to H$ is a strongly continuous semigroup. We denote by A_H its infinitesimal generator.

Theorem 3.5 *Let $1 < p < \infty$. In the analytic case, the following conditions are equivalent.*

(i) $D((-L_p)^{1/2}) = W_H^{1,p}(X, \mu_\infty)$, with equivalence of the respective norms;

(ii) the operator $-A_H$ admits bounded H^∞ functional calculus in H.

If such equivalent conditions are satisfied, we have $D(L_p) = W_H^{2,p}(X, \mu_\infty) \cap W_{AQ}^{1,p}(X, \mu_\infty)$, with equivalence of the respective norms.

Theorem 3.5 is a generalization of 3.4, since in the symmetric case (i) and (ii) are satisfied.

In [32] sufficient conditions were given in order that $D(L_p) \subset W_H^{2,p}(X, \mu_\infty)$ for $p \in (1, 2]$, even in the nonanalytic case.

Concerning summability improving, the following hypercontractivity result holds.

Theorem 3.6 *Fix $t > 0$ and let $1 \le p < q$ be such that*

$$q - 1 \le (p - 1) \|Q_\infty^{-1/2} e^{tA} Q_\infty^{1/2}\|_{\mathscr{L}(X)}^{-2}. \tag{3.26}$$

Then $T_p(t)(L^p(X, \mu)) \subset L^q(X, \mu)$, and $\|T_p(t) f\|_{L^q(X,\mu)} \le \|f\|_{L^p(X,\mu)}$ for every $f \in L^p(X, \mu)$.

The proof is in [22] and (in the case that Q_∞ is one to one) in [11]. Of course, the statement is meaningful only if (3.20) is satisfied. As we mentioned before, if (3.9) holds then (3.20) holds for every $t > 0$. Another simple example is the case that Q commutes with e^{tA} and (3.19) holds; then $Q_\infty^{-1/2} e^{tA} Q_\infty^{1/2} = e^{tA}$ and (3.20) is satisfied for large t if $M > 1$, for every $t > 0$ if $M = 1$, independently of the validity of (3.9). In particular, if $A = -\omega I$ with $\omega > 0$, (3.9) is not satisfied but (3.20) holds for every $t > 0$.

For the classical Ornstein-Uhlenbeck semigroup of Section 2 condition (3.26) coincides with the hypercontractivity property stated there.

It is well known, see [27, 18], that under appropriate assumptions the hypercontractivity of a semigroup is equivalent to the occurrence of a suitable logarithmic Sobolev inequality. But for general Ornstein-Uhlenbeck semigroups the assumptions of [27] are not necessarily satisfied, as shown in [22]. In the symmetric case, namely under the conditions of Theorem 3.2, they are satisfied, and by [13, Thm. 4.2] for every $\beta > 0$ the following conditions are equivalent.

(i) $\|Q^{-1/2}e^{tA}Q^{1/2}\|_{\mathscr{L}(X)} \le e^{-\beta t}$, for every $t > 0$;

(ii) $\|Q_\infty^{-1/2}e^{tA}Q_\infty^{1/2}\|_{\mathscr{L}(X)} \le e^{-\beta t}$, for every $t > 0$;

(iii) for every $f \in D(L_2)$ we have

$$\int_X |f(x)|^2 \log(|f(x)|)\mu_\infty(dx) \le \frac{2}{\beta}\langle -L_2 f, f\rangle_{L^2(X,\mu_\infty)} + \|f\|^2_{L^2(X,\mu_\infty)} \log(\|f\|_{L^2(X,\mu_\infty)}),$$

(iv) $T(t)$ is a contraction from $L^p(X,\mu_\infty)$ to $L^q(X,\mu_\infty)$ for every $t > 0$, $1 \le p \le q$ such that $q - 1 \le (p-1)e^{2\beta t}$.

In [22] it was remarked that if (3.20) holds for some $t > 0$, then there exist K, $\nu > 0$ such that

$$\left\|T_2(t)f - \int_X f(x)\mu_\infty(dx)\right\|_{L^2(X,\mu_\infty)} \le Ke^{-\nu t}\|f\|_{L^2(X,\mu_\infty)}, \quad t > 0,\ f \in L^2(X,\mu_\infty).$$

Notice that the operator $\Pi : L^2(X,\mu_\infty) \to L^2(X,\mu_\infty)$, $(\Pi f)(x) = \int_X f(x)\mu_\infty(dx)$ for a.e. $x \in X$, is just the orthogonal projection on the subspace of constant functions.

In general, exponential convergence of $T_2(t)f$ to Πf is related to the behavior of the semigroup $S_H(t)$. Indeed, if e^{tA} maps H into itself, for every $f \in C_b^1(X)$, $t > 0$ and $h \in H$ formula (3.8) yields

$$\frac{\partial T(t)f}{\partial h}(x) = \int_X \langle \nabla f(e^{tA}x + y), e^{tA}h\rangle_X \mu_t(dy) = \int_X \langle \nabla_H f(e^{tA}x + y), e^{tA}h\rangle_H \mu_t(dy),$$

and therefore, if $\|S_H(t)\|_{\mathscr{L}(H)} \le M_1 e^{-\beta t}$ for some M_1, $\beta > 0$, we argue as in Section 2 and we obtain

$$\begin{aligned}
|\langle \nabla_H T(t)f(x), h\rangle_H| = \Big|\frac{\partial T(t)f}{\partial h}(x)\Big| &\le M_1 e^{-\beta t}\|h\|_H \int_X \|\nabla_H f(e^{tA}x + y)\|_H\,\mu_t(dy) \\
&\le M_1 e^{-\beta t}\|h\|_H \Big(\int_X \|\nabla_H f(e^{tA}x + y)\|_H^2\,\mu_t(dy)\Big)^{1/2} \\
&= M_1 e^{-\beta t}\|h\|_H (T(t)\Big(\|\nabla_H f\|_H^2)(x)\Big)^{1/2}
\end{aligned}$$

namely,

$$\|\langle \nabla_H T(t)f(x)\|_H \le M_1 e^{-\beta t}\Big(T(t)(\|\nabla_H f\|^2)(x)\Big)^{1/2}, \quad t > 0,\ x \in X. \tag{3.27}$$

Squaring and integrating with respect to μ_∞ we get, for every $t > 0$,

$$\int_X \|\nabla_H T(t)\|_H^2\,d\mu_\infty \le M_1^2 e^{-2\beta t}\int_X T(t)(\|\nabla_H f\|_H^2)\,d\mu_\infty = M_1^2 e^{-2\beta t}\int_X \|\nabla_H f\|_H^2\,d\mu_\infty.$$

In the analytic case this estimate and (3.25) allow to obtain a Poincaré inequality,

$$\int_X |f - \Pi f|^2\,d\mu_\infty \le \frac{M_1^2}{2\beta}\int_X \|\nabla_H f\|_H^2\,d\mu_\infty, \quad f \in W_H^{1,2}(X,\mu_\infty) \tag{3.28}$$

by a classical method that seems to go back to [19] (the proof given in [15, Prop. 10.5.2] for a particular case works as well in general, using as main ingredients (3.25) and (3.27)).

By the invariance of μ_∞, $T_2(t)$ maps $L^2_0(X,\mu_\infty) := (I-\Pi)(L^2(X,\mu_\infty))$ into itself. Moreover, (3.28) and (3.25) yield $\langle L_2 f, f\rangle_{L^2(X,\mu_\infty)} \leq -(\beta/M_1^2)\|f\|^2_{L^2(X,\mu_\infty)}$ for every $f\in D(L_2)\cap L^2_0(X,\mu_\infty)$. By the general theory of semigroups (e.g. [45, Section IX.8]), $\|T_2(t)\|_{\mathscr{L}(L^2_0(X,\mu_\infty))} \leq e^{-\beta t/M_1^2}$ for $t>0$, and therefore

$$\|T_2(t)f - \Pi f\|_{L^2(X,\mu_\infty)} \leq e^{-\beta t/M_1^2}\|f\|_{L^2(X,\mu_\infty)}, \quad t>0,\ f\in L^2(X,\mu_\infty). \tag{3.29}$$

If in addition (3.20) holds for some $t>0$, the rate of convergence of $T_p(t)f$ to Πf is the same in all spaces $L^p(X,\mu_\infty)$, $1\leq p<\infty$. Indeed, if $p>2$ we fix $\tau>0$ such that $T(\tau)$ is a contraction from $L^2(X,\mu_\infty)$ to $L^p(X,\mu_\infty)$ (such a τ exists, since $Q_\infty^{-1/2}e^{tA}Q_\infty^{1/2}$ is a semigroup, and therefore if (3.20) holds for some $t>0$ then $\lim_{\tau\to\infty}\|Q_\infty^{-1/2}e^{\tau A}Q_\infty^{1/2}\|_{\mathscr{L}(X)} = 0$). For every $t\geq\tau$ and $f\in L^p(X,\mu_\infty)$ we have

$$\|T(t)f-\Pi f\|_{L^p(X,\mu_\infty)} = \|T(\tau)(T(t-\tau)f-\Pi f)\|_{L^p(X,\mu_\infty)} \leq \|T(t-\tau)f-\Pi f\|_{L^2(X,\mu_\infty)}$$

by Theorem 3.6, and using (3.29) we get

$$\|T(t)f-\Pi f\|_{L^p(X,\mu_\infty)} \leq e^{-\beta(t-\tau)/M_1^2}\|f\|_{L^2(X,\mu_\infty)} \leq e^{\beta\tau/M_1^2}e^{-\beta t/M_1^2}\|f\|_{L^p(X,\mu_\infty)}, \quad t\geq\tau.$$

Similarly, if $p<2$ we fix $\tau>0$ such that $T(\tau)$ is a contraction from $L^p(X,\mu_\infty)$ to $L^2(X,\mu_\infty)$. For every $t\geq\tau$ and $f\in L^p(X,\mu_\infty)$ we have

$$\|T(t)f-\Pi f\|_{L^p(X,\mu_\infty)} \leq \|T(t)f-\Pi f\|_{L^2(X,\mu_\infty)} = \|T(t-\tau)(T(\tau)f-\Pi(T(\tau)f))\|_{L^2(X,\mu_\infty)}$$

so that using (3.29) and then Theorem 3.6 we get

$$\|T(t)f-\Pi f\|_{L^p(X,\mu_\infty)} \leq e^{-\beta(t-\tau)/M_1^2}\|T(\tau)f\|_{L^2(X,\mu_\infty)} \leq e^{\beta\tau/M_1^2}e^{-\beta t/M_1^2}\|f\|_{L^p(X,\mu_\infty)}, \ t\geq\tau.$$

4 Ornstein-Uhlenbeck semigroups in Banach spaces

Many of the results of Section 3 have been extended to the case where X is a separable Banach space. In fact, the already mentioned papers [6, 26, 24, 31, 32, 33, 34, 40, 41, 42] deal with the Banach space case. A survey of the state of the art up to 2003 is in [26].

As in Section 3, $Q\in\mathscr{L}(X^*,X)$ is a symmetric positive operator, and $A: D(A)\subset X\to X$ is the infinitesimal generator of a strongly continuous semigroup e^{tA} on X. As in the Hilbert case, the basic assumption of this section is that for every $t>0$ the operator Q_t defined by (3.2) is the covariance of a Gaussian measure μ_t, and in this case the Ornstein-Uhlenbeck semigroup $T(t)$ is defined by (3.5).

If Q itself is a covariance and $A=-I$, $T(t)$ is the classical Ornstein-Uhlenbeck semigroup of Sect. 2, provided γ is the centered Gaussian measure on X with covariance $2Q$.

As in the Hilbert case, it is the transition semigroup of a stochastic differential equation in X, with a proper notion of mild solution, see [3, 44], and it is a contraction semigroup on $\mathcal{B}_b(X)$ that leaves invariant the spaces $C_b(X)$, $BUC(X)$, $C^\alpha_b(X)$, $C^k_b(X)$, $C^{\alpha+k}_b(X)$ for $\alpha\in(0,1)$, $k\in\mathbb{N}$.

The strong-Feller property of $T(t)$ is not easily recognizable as in the Hilbert case. Characterizations and sufficient conditions for $T(t)$ be strong-Feller are in [26, Sect. 6.1].

Concerning the behavior of $T(t)$ on $C_b(X)$, it is strongly continuous in the mixed topology, and the space $\mathcal{F}_0$ defined now by

$$\begin{aligned}\mathcal{F}_0 := \{f \in C_b(X) : f = \varphi(\langle \cdot, a_1\rangle, \dots, \langle \cdot, a_n\rangle);\ \varphi \in C_b^2(\mathbb{R}^n),\\ n \in \mathbb{N},\ a_i \in D(A^*),\ A^* Df(\cdot)(\cdot) \in C_b(X)\}\end{aligned} \tag{4.1}$$

is a core of the generator L of $T(t)$ in the mixed topology, by [26, Thm.6.6]. The domain of L is still given by (3.12), see [26, Section 6.1].

The spaces $H := H_Q$ and $H_t := H_{Q_t}$ introduced in Sect. 1(i) play the role of the spaces $Q^{1/2}(X)$, $Q_t^{1/2}(X)$ in the Hilbert case. We recall that H_t is the Cameron-Martin space of the measure μ_t.

As reported in Sect. 3 in the Hilbert space case, an important hypothesis to get smoothing properties if $T(t)$ aling H is that e^{tA} maps H into itself, and $S_H(t) := e^{tA}_{|H} : H \to H$ is a strongly continuous semigroup on H. Indeed, in this case e^{tA} maps H into H_t for every $t > 0$, and $\sup_{0<t<1} \|e^{tA}\|_{\mathscr{L}(H,H_t)} < \infty$, by [26, Thm. 3.5]. As a consequence, $T(t)$ is smoothing along H. See [34, Sect. 2] and [30] for representation formulae and estimates for any order H-derivatives of $T(t)f$ when $f \in C_b(X)$. Again, as in the Hilbert case, Schauder type theorems were proved in [30], that generalize the one stated in Sect. 2, and precisely for every $\alpha \in (0,1)$, $\lambda > 0$ and $f \in C_H^\alpha(X)$, $R(\lambda, L)f \in C_H^2(X)$ and $D_H^2 R(\lambda, L)f \in C_H^\alpha(X, \mathcal{L}^2(H))$. Notice that H is invariant under e^{tA} in the analytic case, see [34, Th. 3.3].

Concerning asymptotic behavior and existence of invariant measures, assumption (3.16) is generalized as follows.

$$\begin{cases} (i) & \forall f \in X^*\ \ \exists\, \text{weak} - \lim_{t\to\infty} Q_t f := Q_\infty f, \\ (ii) & Q_\infty \text{ is the covariance of a centered Gaussian measure } \mu_\infty. \end{cases} \tag{4.2}$$

Condition (i) is satisfied if (3.19) holds, in which case a representation formula similar to (3.17) holds, namely $Q_\infty f = \int_0^\infty e^{sA} Q e^{sA^*} f\, ds$ for every $f \in X^*$, where now the integral converges as a Pettis integral, see [26, Sect. 2]. As in the Hilbert case, if (i) holds the operator Q_∞ maps $D(A^*)$ into $D(A)$ and satisfies the Lyapunov equation (3.18); moreover (i) holds iff there exists a symmetric and positive operator $P \in \mathscr{L}(X^*, X)$ mapping $D(A^*)$ into $D(A)$ such that $PA^*f + APf = -Qf$ for every $f \in D(A^*)$, see [26, Sect. 4].

However, establishing whether a given symmetric positive operator is the covariance of a Gaussian measure is not as simple as in the Hilbert case. Necessary and sufficient conditions are in [44]. If (4.2) holds, denoting by $H_\infty := H_{Q_\infty}$ = the Cameron-Martin space of μ_∞ (as in the Hilbert case), several statements of the previous section are extendable to the Banach setting. In particular:

(a) e^{tA} maps H_∞ into itself, and $e^{tA}_{|H_\infty} : H_\infty \to H_\infty$ is a strongly continuous contraction semigroup, still denoted by $S_\infty(t)$. Moreover, for any $t > 0$ we have $H_\infty = H_t$ iff $\|S_\infty(t)\|_{\mathscr{L}(H_\infty)} < 1$.

(b) μ_∞ is an invariant measure of $T(t)$, and the arguments used in Sections 2 and 3 yield that $T(t)$ extends to a contraction C_0-semigroup $T_p(t)$ on $L^p(X, \mu_\infty)$, for every $p \in [1, +\infty)$.

(c) Conditions (i) and (iii) of Theorem 3.2 are still equivalent, see [26, Thm. 7.4]; if they hold $T_p(t)$ is an analytic contraction semigroup on $L^p(X, \mu_\infty)$ for every $p \in (1, \infty)$.

(d) Conditions (i), (ii), and (iii) of Theorem 3.3 are still equivalent, see [26, Sect. 8]; if they hold $T_p(t)$ is an analytic contraction semigroup on $L^p(X, \mu_\infty)$ for every $p \in (1, +\infty)$. The optimal angle of analyticity and the optimal angle for the bounded H^∞ calculus of $-L_p$ were determined in [31, 6], respectively, in the present Banach setting.

(e) Theorems 3.2 and 3.3 still hold, where the involved Sobolev spaces $W_H^{1,p}(X,\mu_\infty)$, $W_H^{2,p}(X,\mu_\infty)$, $W_{AQ}^{1,p}(X,\mu_\infty)$ are defined in a similar way to the Hilbert case. See [32, 33, 34].

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