SCUOLA NORMALE SUPERIORE

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GEOMETRY OF 1-CODIMENSIONAL MEASURES IN HEISENBERG GROUPS

DOCTORAL THESIS IN MATHEMATICS

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Breathe, breathe in the air
Don’t be afraid to care
Leave but don’t leave me
Look around and choose your own ground

For long you live and high you fly
And smiles you’ll give and tears you’ll cry
And all you touch and all you see
Is all your life will ever be

Run, rabbit, run
Dig that hole, forget the sun
And when at last the work is done
Don’t sit down, it’s time to dig another one

For long you live and high you fly
But only if you ride the tide
And balanced on the biggest wave
You race towards an early grave

Breathe, The Dark Side of the Moon

Contents

Notation

We add below a list of frequently used notations, together with the page of their first appearance:

$\lvert\cdot\rvert$	Euclidean norm,	1
$\lVert\cdot\rVert$	Koranyi norm,	9
$\lvert\lvert\lvert\cdot\rvert\rvert\rvert$	operatorial norm of matrices,	51
$\langle\cdot,\cdot\rangle$	scalar product in $\mathbb{R}^{2n}$,	9
$V(\cdot,\cdot)$	polarisation function of the Koranyi norm,	16
$\pi_H(\cdot)$	projection of $\mathbb{R}^{2n+1}$ onto the first $2n$ coordinates,	9
$\pi_T(\cdot)$	projection of $\mathbb{R}^{2n+1}$ onto the last coordinate,	9
x_H	shorthand for $\pi_H(x)$,	9
x_T	shorthand for $\pi_T(x)$,	9
τ_x	left translation by x,	9
D_λ	anisotropic dilations,	9
$\mathcal{V}$	centre of $\mathbb{H}^n$,	15
$U_r(x)$	open Euclidean ball of radius $r>0$ and centre x,	30
$B_r(x)$	open Koranyi ball of radious $r>0$ and centre x,	9
$\Theta_\alpha(\phi,x)$	α-dimensional density of the Radon measure ϕ at the point x,	10
$\phi_{x,r}$	dilated of a factor $r>0$ of the measure ϕ at the point $x\in\mathbb{H}^n$,	10
$\text{Tan}_\alpha(\phi,x)$	set of α-dimensional tangent measures to the measure ϕ at x,	10
$\mathcal{U}_{\mathbb{H}^n}(\alpha)$	set of α-uniform measures,	10
$\mathcal{M}$	set of Radon measures in $\mathbb{H}^n$,	10
$\text{supp}(\mu)$	support of the measure μ,	10
$\rightharpoonup$	weak convergence of measures,	10
$V(\mathfrak{n})$	the vertical hyperplane orthogonal to $\mathfrak{n}\in\mathbb{R}^{2n}$,	34
$\mathbb{K}(b,\mathcal{Q},\mathcal{T})$	the quadric $\langle b+\mathcal{Q}x,x\rangle+\mathcal{T}t=0$ where $(x,t)\in\mathbb{R}^{2n+1}$,	15
$\Sigma(f)$	characteristic set of the function $f:\mathbb{R}^{2n}\to\mathbb{R}$	36
$\Sigma(F)$	set where the horizontal gradient of the function $F:\mathbb{H}^n\to\mathbb{R}$ is null,	33
J	standard symplectic matrix,	9
$M(n,m)$	set of $n\times m$ matrices,	88
$\text{Sym}(2n)$	set of symmetric matrices on $\mathbb{R}^{2n}$,	15
$S(2n)$	subset of orthogonal matrices on $\mathbb{R}^{2n}$ inducing a linear isometry on $\mathbb{H}^n$,	9
$\mathcal{C}_c$	set of continuous functions with compact support,	10
$\mathcal{C}_c^{1,\alpha}(\Omega)$	set of $\mathcal{C}^{1,\alpha}$ functions with compact support contained in Ω,	88
$\mathcal{S}_E^\alpha$	α-dimensional spherical Hausdorff measure centred on the Borel set E,	12
$\mathcal{H}_{eu}^k$	Euclidean k-dimensional Hausdorff measure,	34
$\Gamma(\cdot)$	Gamma function,	13
$b_{k,s}^\mu$	k-th moment of the measure μ,	17

The symbol ϕ will always denote a measure with density and the symbols μ,ν uniform measures.

1. Introduction

In the Euclidean spaces the notion of rectifiability of a measure is linked to the metric by the celebrated:

Theorem 1.1 (Preiss, [37]). *Suppose $0 \leq m \leq n$ are integers, ϕ is a Radon measure on $\mathbb{R}^n$ and:*

$$0 < \Theta^m(\phi, x) := \lim_{r \to 0} \frac{\phi(U_r(x))}{r^m} < \infty \qquad \text{at } \phi\text{-almost every } x, \tag{1.1}$$

where $U_r(x)$ is the Euclidean ball of centre x and radius r. Then ϕ is m-rectifiable, i.e., ϕ-almost all of $\mathbb{R}^n$ can be covered by countably many m-dimensional Lipschitz submanifolds of $\mathbb{R}^n$.

The most difficult part of the proof of Theorem 1.1 is to show that the existence of the density, namely that (1.1) holds, implies that the measure ϕ has flat tangents, i.e.:

$$\text{Tan}(\phi, x) \subseteq \Theta^m(\phi, x)\{\mathcal{H}^m \llcorner V : V \text{ is an } m\text{-plane}\} \qquad \text{at } \phi\text{-almost every point.} \tag{1.2}$$

The fact that the inclusion (1.2) implies Theorem 1.1 is a consequence of the Marstrand-Mattila rectifiability criterion, see for instance Theorem 5.1 in [15]. The proof of such inclusion depends on the structure of the Euclidean ball and it is not known whether it is possible to extend it to a general finite dimensional Banach space. The only progress in this direction, to our knowledge, was done by A. Lorent, who proved that 2-locally uniform measures in ℓ^3_∞ are rectifiable, see Theorem 5 in [27]. As one should expect, although the assumption of local uniformity is far stronger than the mere existence of the density, already in this strengthened hypothesis, the proof is really intricate and exploits the particular shape of the ball.

In this thesis we investigate to what extent the local structure of 1-codimensional measures in $\mathbb{H}^n$ is affected by the regular behaviour of the measure of Koranyi balls. Although the Heisenberg group shares many similarities with the Euclidean spaces, it has Hausdorff dimension $2n + 2$ and it is a k-purely unrectifiable metric space for any $k \in \{n + 1, \ldots, 2n + 2\}$, i.e., for any compact set $K \subseteq \mathbb{R}^k$ and any Lipschitz function $f : K \to (\mathbb{H}^n, \|\cdot\|)$ we have:

$$\mathcal{H}^k_{\|\cdot\|}(f(K)) = 0,$$

where $\mathcal{H}^k_{\|\cdot\|}$ is Hausdorff measure associated to the Koranyi norm, see for instance Theorem 7.2 in [7] or Theorem 1.1 in [29]. This degeneracy of the structure of $\mathbb{H}^n$ poses a big obstacle to extending many Euclidean results and definitions to the context of the Heisenberg groups, or in bigger generality to Carnot groups. In particular it is not a priori clear what the correct notion of rectifiability should be, or even if there is one. In the paper [23] B. Franchi, R. Serapioni and F. Serra Cassano, introduced an intrinsic notion of rectifiability. A set $E \subseteq \mathbb{H}^n$ is said to be $(2n + 1)$-*intrinsic rectifiable* if for $\mathcal{S}^{2n+1}$-a.e. $x \in E$ there exists a $(2n + 1)$-dimensional homogeneous subgroup V_x such that:

$$\text{Tan}_{2n+1}(\mathcal{S}^{2n+1} \llcorner E, x) = \{\mathcal{S}^{2n+1} \llcorner V_x\},$$

and $\Theta^{2n+1}_*(\mathcal{S}^{2n+1} \llcorner V_x) > 0$. This definition makes the recovery of De Giorgi's rectifiability theorem of boundaries of finite perimeter sets possible in $\mathbb{H}^n$.

The main task of this thesis is to address the question of whether or not the definition of intrinsic rectifiability given by Franchi, Serapioni and Serra Cassano can be characterised by the metric in a similar way as rectifiability is in the Euclidean spaces. In other words we are interested in determining if a result in the spirit of Theorem 1.1 is available in $\mathbb{H}^n$, where *rectifiability* is replaced with this new *intrinsic rectifiability*.

The main result of the thesis is the proof of the analogue of the inclusion (1.2) in the Heisenberg groups, therefore getting closer to a *metric* justification of the notion of intrinsic rectifiability:

Theorem 1.2. *Suppose ϕ is a Radon measure in $\mathbb{H}^n$ such that:*

$$0 < \Theta^{2n+1}(\phi, x) := \lim_{r\to 0} \frac{\phi(B_r(x))}{r^{2n+1}} < \infty \qquad \text{for } \phi\text{-a.e. } x, \tag{1.3}$$

where $B_r(x)$ is the Koranyi ball. Then:

$$Tan_{2n+1}(\phi, x) \subseteq \Theta^{2n+1}(\phi, x)\mathfrak{M}(2n+1) \qquad \text{for } \phi\text{-a.e. } x,$$

where $\mathfrak{M}(2n+1)$ is the family of the Haar measures of $(2n+1)$-homogeneous subgroups of $\mathbb{H}^n$ which assign measure 1 to the unit ball.

This is the first example beyond the Euclidean spaces, where the mere existence of the density implies the flatness of the tangents. Theorem 1.2 leaves open the very interesting problem of determining whether or not a result in the spirit of the Marstrand-Mattila rectifiability criterion is available in the context of Heisenberg groups. An affirmative answer to such a question would imply, as in the Euclidean spaces, that ϕ is an intrinsic $(2n+1)$-rectifiable measure.

The study of the density problem in the Heisenberg groups was started in 2015 by V. Chousionis and J. Tyson in [14] where they proved that if ϕ is a Radon measure on $\mathbb{H}^n$ having α-density, i.e.:

$$0 < \Theta^{\alpha}(\phi, x) := \lim_{r\to 0} \frac{\phi(B_r(x))}{r^{\alpha}} < \infty \qquad \text{for } \phi\text{-a.e. } x, \tag{1.4}$$

then Marstrand theorem holds in $\mathbb{H}^n$ for the Koranyi norm, i.e., α is an integer in $\{0, \ldots, 2n+2\}$. This was done, very much as in the Euclidean spaces, by proving that (1.4) implies that ϕ-almost everywhere tangent measures to ϕ are α-uniform measures (see Definition 2.3) and that the support of such α-uniform measures are analytic manifolds. The same strategy with minor modifications works in general Carnot groups when endowed with a left invariant polynomial norm. The development of those ideas allowed Chousionis, Tyson and Magnani in [28] to characterise 1 and 2 uniform measures in $\mathbb{H}^1$ and to prove that vertically ruled 3-uniform measures are flat. As a byproduct of our analysis we complete the characterisation of uniform measures in $\mathbb{H}^1$, see Section 8.

We present here a survey of the strategy of the proof of Theorem 1.2, giving for each key step a brief discussion of the ideas involved. In Section 3, we prove that the support of a uniform measure μ is contained in a quadratic surface. This is a result in the spirit of Theorem 17.3 of [31] (cfr. also with Theorem 4.1 in [25]):

Theorem 1.3. *Let $\alpha \in \{1, \ldots, 2n+1\} \setminus \{2\}$, and suppose that μ is an α-uniform measure. Then there are $b \in \mathbb{R}^{2n}$, $\mathcal{T} \in \mathbb{R}$ and $\mathcal{Q} \in Sym(n)$ with $Tr(\mathcal{Q}) \neq 0$ such that:*

$$\mathrm{supp}(\mu) \subseteq \mathbb{K}(b, \mathcal{Q}, \mathcal{T}) := \{(x,t) \in \mathbb{R}^{2n+1} : \langle b, x\rangle + \langle x, \mathcal{Q}x\rangle + \mathcal{T}t = 0\}.$$

Despite the fact that we already know (thanks to Proposition 3.2 of [14]) that the supports of a uniform measure is an analytic variety, the algebraic simplicity of the quadrics containing the support in the 1-codimensional case will be a fundamental simplification in our computations.

The proof of Theorem 1.3 is based on an adaptation of the arguments of Section 3 of [37]. In particular we have extended Preiss's moments to this non-Euclidean context (the Heisenberg moments $b^{\mu}_{k,s}$ are introduced in Definition 3.8) in such a way that it is possible to prove (see Proposition 3.14) that for any $s > 0$ and any u is the support of a given uniform measure μ, we have:

$$\left|\sum_{k=1}^{4} b^{\mu}_{k,s}(u) - s\|u\|^4\right| \leq s^{\frac{5}{4}}\|u\|^5(2 + (s\|u\|^4)^2).$$

The left-hand side in the above expression is a polynomial of fourth degree in the coordinates of u, but with some work one can reduce (see Proposition 3.17) the above inequality to:

$$|\langle b(s), u_H\rangle + \langle \mathcal{Q}(s)[u_H], u_H\rangle + \mathcal{T}(s)u_T| \leq s^{\frac{1}{4}}\|u\|^3,$$

where u_H is the vector of the first $2n$ coordinates of u, u_T is the last coordinate of u and $b(s), \mathcal{Q}(s)$ and $\mathcal{T}(s)$ are introduced in Definition 3.16. From the above expression, sending s to 0 one gets the quadric containing $\mathrm{supp}(\mu)$. The most tricky part of Theorem 1.3 is to show that $\mathrm{Tr}(\mathcal{Q}) \neq 0$ and to the proof of this fact is devoted the entire Subsection 3.4.

When μ is a $(2n+1)$-uniform measure, one expects that the fact that Theorem 1.3 represents a strong information on the structure of $\mathrm{supp}(\mu)$. This idea is exploited in Section 4 where we prove:

Theorem 1.4. *The support of a $(2n+1)$-uniform measure μ is the closure of a union of connected components of $\mathbb{K}(b, \mathcal{Q}, \mathcal{T}) \setminus \Sigma$, where Σ is the set of those points where the tangent group to $\mathbb{K}(b, \mathcal{Q}, \mathcal{T})$ does not exists.*

The idea behind the proof of Theorem 1.4 is the following. Suppose $y \in \mathbb{K}(b, \mathcal{Q}, \mathcal{T}) \setminus \mathrm{supp}(\mu)$ and let z be a point with minimal Euclidean distance of y from $\mathrm{supp}(\mu)$. If $z \notin \Sigma$, thanks to Proposition 4.5, we know that $\mathrm{Tan}_{2n+1}(\mu, x) = \{\mathcal{S}^{2n+1}_V\}$ where V is the tangent group to $\mathbb{K}(b, \mathcal{Q}, \mathcal{T})$ at z and $\mathcal{S}^{2n+1}_V$ is its Haar measure. However, by means of careful computations (see Propositions 4.8 and 4.10) we show that the blowup of "the hole in the support" $B_{|y-z|}(y) \cap \mathbb{K}(b, \mathcal{Q}, \mathcal{T})$ is a non-empty open subset of V, which is in contradiction with the fact that the support the blowup of μ at z coincides with the whole V. This implies that the boundaries of holes of $\mathrm{supp}(\mu)$ inside $\mathbb{K}(b, \mathcal{Q}, \mathcal{T})$ must be contained in Σ and thus a standard connection argument proves Theorem 1.4.

Theorem 1.4 allows us get a better understanding of the behaviour of $(2n+1)$-uniform measures at infinity. In particular we prove that:

(i) if $\mathrm{Tan}_{2n+1}(\mu, \infty) \cap \mathfrak{M}(2n+1) \neq \emptyset$ then $\mu \in \mathfrak{M}(2n+1)$,

(ii) the set $\mathrm{Tan}_{2n+1}(\mu, \infty)$ is a singleton.

In the Euclidean space these properties arise from a careful analysis of the algebraic properties of moments. In our framework the structure of moments is much more complicated and therefore (i) and (ii) are proved by means of a geometric construction. Thanks to these two properties of $(2n+1)$-uniform measures, in Section 4, we prove the following:

Theorem 1.5. *Suppose there exists a functional $\mathscr{F} : \mathcal{M} \to \mathbb{R}$, continuous in the weak-$*$ convergence of measures, and a constant $\hbar = \hbar(\mathbb{H}^n) > 0$ such that:*

(i) if $\mu \in \mathfrak{M}(2n+1)$ then $\mathscr{F}(\mu) \leq \hbar/2$,

(ii) if μ is a $(2n+1)$-uniform cone (see Definition 5.1) and $\mathscr{F}(\mu) \leq \hbar$, then $\mu \in \mathfrak{M}(2n+1)$.

Then, for any ϕ Radon measure with $(2n+1)$-density and for ϕ-almost every x:

$$\mathit{Tan}_{2n+1}(\phi, x) \subseteq \Theta^{2n+1}(\phi, x)\mathfrak{M}(2n+1).$$

The proof of Theorem 1.5 follows closely its Euclidean counterpart, and it is a standard application of the very general principle that "a tangent to a tangent is a tangent" (see Proposition 2.5).

We are left to construct the functional $\mathscr{F}$ satisfying all the hypothesis of Theorem 1.5. Suppose φ is a smooth function with support contained in $B_2(0)$ such that $\varphi = 1$ on $B_1(0)$. We claim that the functional:

$$\mathscr{F}(\mu) := \min_{\mathfrak{m}\in\mathbb{S}^{2n-1}} \int \varphi(z)\langle \mathfrak{m}, z_H\rangle^2 d\mu(z),$$

satisfies all the hypothesis of Theorem 1.5 and therefore Theorem 1.2 follows. The fact that $\mathscr{F}$ is a continuous operator on Radon measures is easy to prove (see Proposition 8.1) and it is immediate to see that $\mathscr{F}$ is identically null on flat measures. The most challenging hypothesis to check, as in the Euclidean case, is the existence of $\hbar$.

Thanks to Theorem 1.3 there are two kinds of $(2n+1)$-uniform measures. The ones which are contained in a quadric for with $\mathcal{T} = 0$, that in the following are called *vertical*, and the ones with $\mathcal{T} \neq 0$, that we will call *horizontal*. The first step towards the verification of hypothesis (ii) of Theorem 1.5 is the following:

Theorem 1.6. *There exists a constant $\mathfrak{C}_3(n) > 0$ such that for any $\mathfrak{m} \in \mathbb{S}^{2n-1}$ and any horizontal $(2n+1)$-uniform cone μ we have:*

$$\int_{B_1(0)} \langle \mathfrak{m}, z_H\rangle^2 d\mu(z) \geq \mathfrak{C}_3(n).$$

The proof of Theorem 1.6 requires the entire the entire Section 6, but the arguments therein contained all rely on Theorem B.16, which is the main result of Appendix B. Since Theorem 1.6 requires so much work, we wish to discuss its proof more carefully, in order to help the reader keep in mind what the final goal of the Section 6 and Appendix B is.

If μ is a horizontal $(2n+1)$-uniform cone, we can find $\mathcal{D} \in \text{Sym}(2n) \setminus \{0\}$ such that $\text{supp}(\mu) \subseteq \mathbb{K}(0, \mathcal{D}, -1)$. In Theorem B.16, we prove that such $\mathcal{D}$ must satisfy the algebraic constraint (B.29) which implies that the operatorial norm $|||\mathcal{D}|||$ of $\mathcal{D}$ is bounded from above and below by universal positive constants $\mathfrak{C}_1(n)$ and $\mathfrak{C}_2(n)$, respectively (see Propositions 6.7 and 6.8) and thus Theorem 1.6 follows. We refer to the proof of Theorem 6.9 for further details.

While the bound from below easily follows from Theorem B.16, obtaining the bound from above is quite complicated. Suppose $\{\mu_i\}$ is a sequence of $(2n+1)$-uniform measures invariant under dilations and assume that $\text{supp}(\mu_i) \subseteq \mathbb{K}(0, \mathcal{D}_i, -1)$. If the sequence $|||\mathcal{D}_i|||$ diverges, then the limit points of the sequence $\{\mu_i\}$ can only be vertical $(2n+1)$-uniform cones. Defined $\mathcal{Q}$ to be one of the limit points of the sequence $\mathcal{D}_i/|||\mathcal{D}_i|||$, one can show that the algebraic constraints given by Theorem B.16 on $\mathcal{D}_i$ imply that for any $h \notin \text{Ker}(\mathcal{Q})$ we have:

$$2(\text{Tr}(\mathcal{Q}^2) - 2\langle \mathfrak{n}, \mathcal{Q}^2\mathfrak{n}\rangle + \langle \mathfrak{n}, \mathcal{Q}\mathfrak{n}\rangle^2) - (\text{Tr}(\mathcal{Q}) - \langle \mathfrak{n}, \mathcal{Q}\mathfrak{n}\rangle)^2 = 0,$$

where $\mathfrak{n} := \mathcal{Q}h/|\mathcal{Q}h|$. We refer to Proposition 6.3 for further details. By this key observation, via Proposition 6.5 we prove that the sequence $\{\mu_i\}$ can only have a flat measure as limit points. The fact that the limit must be flat together with the fact that all the eigenvalues of the $\mathcal{D}_i$ except one (see Proposition 6.6, which is again a consequence of Theorem B.16) must be bounded, implies that the assumption that such a sequence $\{\mu_i\}$ exists was absurd. Indeed the boundedness of all eigenvalues except one would prevent the limit of the μ_i's from being flat. See the proof of Proposition 6.7 for further details.

The above argument shows that the functional $\mathscr{F}$ disconnects horizontal $(2n+1)$-uniform cones and flat measures. The last piece of information we need to apply Theorem 1.5 is that $\mathscr{F}$ disconnects vertical non-flat $(2n+1)$-uniform cones from flat measures:

Theorem 1.7. *There exists a constant $\mathfrak{C}_{10}(n) > 0$ such that if μ is a vertical $(2n+1)$-uniform cone for which:*

$$\min_{\mathfrak{m}\in\mathbb{S}^{2n-1}} \int_{B_1(0)} \langle \mathfrak{m}, z_H\rangle^2 d\mu(z) \leq \mathfrak{C}_{10}(n),$$

then μ is flat.

The proof of the above theorem relies on Theorems 1.3, 1.4 and the representation formulas of Appendix A to get a very explicit and simple expression for the quadric containing $\mathrm{supp}(\mu)$ (see Proposition 7.2). Thanks to the structural similarities of these quadrics to their Euclidean counterparts we were able to rearrange Preiss's original disconnection argument to conclude the proof of Theorem 1.7 (see Theorem 7.7 and cfr. with the proof of Theorem 3.14 in [37]).

In Appendix C, we report some further results obtained during the PhD, which are however disconnected from the main topic of the thesis.

2. Preliminaries

In this preliminary section we recall many well known facts and introduce some notations. In case the proof of a Proposition is not present in literature, but the Euclidean argument applies verbatim, we will reduce ourselves to cite a reference where the Euclidean proof can be found.

2.1 The Heisenberg group $\mathbb{H}^n$

In this subsection we briefly recall some notations and very well known facts on the Heisenberg groups $\mathbb{H}^n$.Let $\pi_H : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n}$ be the projection onto the first $2n$ coordinates and $\pi_T : \mathbb{R}^{2n+1} \to \mathbb{R}$ be the projection onto the last one. The Lie groups $\mathbb{H}^n$ are the manifolds $\mathbb{R}^{2n+1}$ endowed with the product:

$$x * y := (x_H + y_H, x_T + y_T + 2\langle x_H, Jy_H\rangle),$$

where x_H and x_T are shorthands for $\pi_H(x)$ and $\pi_T(x)$ while J is the standard sympletic matrix on $\mathbb{R}^{2n}$:

$$J := \begin{pmatrix} 0 & \mathrm{id} \\ -\mathrm{id} & 0 \end{pmatrix}.$$

We metrize the group $(\mathbb{H}^n, *)$ with the *Koranyi distance* $d(\cdot,\cdot) : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{H}^n$ defined as:

$$d(x, y) := \left(|y_H - x_H|^4 + |y_T - x_T - 2\langle x_H, Jy_H\rangle|^2\right)^{\frac{1}{4}}.$$

Moreover we let $\|x\| := d(x, 0)$ be the so called *Koranyi norm* and $B_r(x) := \{z \in \mathbb{H}^n : d(z, x) \leq r\}$ the Koranyi ball. The geometry of $\mathbb{H}^n$ is quite rich and it is well known that $d(\cdot,\cdot)$ is left invariant, i.e., for any $z \in \mathbb{H}^n$ one has:

$$d(z * x, z * y) = d(x, y).$$

As a consequence, left translations $\tau_x(y) := x * y$ are isometries and we have that $d(x, y) = \|x^{-1} * y\| = \|y^{-1} * x\|$. Moreover, defined the anisotropic dilations $D_\lambda : \mathbb{H}^n \to \mathbb{H}^n$ as $D_\lambda(x) := (\lambda x_H, \lambda^2 x_T)$, we also have that $d(\cdot,\cdot)$ is homogeneous with respect to D_λ, i.e.:

$$d(D_\lambda(x), D_\lambda(y)) = \lambda d(x, y).$$

Besides the left translations, we have some other isometries of $(\mathbb{H}^n, d)$. Define:

$$\mathrm{S}(2n) := \{U \in O(2n) : U^T JU = J\} \cup \{U \in O(2n) : U^T JU = -J\}, \tag{2.1}$$

and let s be the function $s : \mathrm{S}(2n) \to \{-1, 1\}$ which satisfies $UJU = s(U)J$. It is easy to check that $\mathrm{S}(2n)$ is a group under multiplication and that $s(\cdot)$ is a homomorphism between $(\mathrm{S}(2n), \cdot)$ and $(\{-1, 1\}, \cdot)$. Furthermore we have:

Proposition 2.1. *Let $U \in S(2n)$. The map $\Xi_U : \mathbb{R}^{2n} \times \mathbb{R} \to \mathbb{R}^{2n} \times \mathbb{R}$ defined as:*

$$\Xi_U : (x,t) \mapsto (Ux, s(U)t), \tag{2.2}$$

is an isometry of $\mathbb{H}^n$.

For further references and a much more comprehensive account on the Heisenberg groups we refer to the monographs [12] and [33].

2.2 Measures with density and their blowups

We recall in this subsection some very well known facts about measures with density and their blowups.

Definition 2.2. A Radon measure ϕ on $\mathbb{H}^n$ is said to have α-density, for some $\alpha > 0$, if the limit:

$$\Theta^\alpha(\phi, x) := \lim_{r\to 0} \frac{\phi(B_r(x))}{r^\alpha},$$

exists finite and non-zero, for ϕ-almost every $x \in \mathbb{H}^n$.

Assume $\{\mu_k\}$ is a sequence of measures in $\mathcal{M}$. We say that $\{\mu_k\}$ converges to μ and we write $\mu_k \rightharpoonup \mu$, if:

$$\lim_{k\to\infty} \int f(x)d\mu_k(x) = \int f(x)d\mu(x) \qquad \text{for any } f \in \mathcal{C}_c(\mathbb{R}^n).$$

Since the paper is concerned with the study of the tangents to measures with α-density, we need a meaningful concept of tangent for a measure. Let ϕ be a Radon measure on $\mathbb{H}^n$ with α-density and denote by $\phi_{x,r}$ the measure that satisfies:

$$\phi_{x,r}(A) = \phi(x * D_r(A)), \tag{2.3}$$

for any Borel set $A \subseteq \mathbb{H}^n$. The set of tangent measures to ϕ at x is denoted by $\text{Tan}_\alpha(\phi, x)$ and it consists of the Radon measures μ for which there exists a sequence $r_i \to 0$ such that:

$$\frac{\phi_{x,r_i}}{r_i^\alpha} \rightharpoonup \mu.$$

The set $\text{Tan}_\alpha(\phi, x)$ is non-empty for ϕ-almost every $x \in \mathbb{H}^n$. Indeed, fix a point $x \in \mathbb{H}^n$ for which $\Theta^\alpha(\phi, x) < \infty$. Then for every $\rho > 0$:

$$r^{-\alpha}\phi_{x,r}(B_\rho(0)) = r^{-\alpha}\phi(B_{\rho r}(x)) \leq 2\Theta^\alpha(\phi, x)\rho^\alpha,$$

for sufficiently small r. Therefore the family of measures $r^{-\alpha}\phi_{x,r}$ is uniformly bounded on compact sets and compactness of measures yields the existence of a limit for a suitable subsequence.

Definition 2.3. We say that a Radon measure μ is an α-uniform measure if:

(i) $0 \in \text{supp}(\mu)$,

(ii) $\mu(B_r(x)) = r^\alpha$ for any $r > 0$ and any $x \in \text{supp}(\mu)$.

We will denote the set of α-uniform measures with the symbol $\mathcal{U}_{\mathbb{H}^n}(\alpha)$.

The following two propositions are of capital importance as Proposition 2.4 insures that the tangent measures to a measure ϕ with α-density are ϕ-almost everywhere α-uniform and Proposition 2.5 that for ϕ-almost every $x \in \mathbb{H}^n$, the tangent measures to any element of $\text{Tan}(\phi, x)$ are still in $\text{Tan}(\phi, x)$. The latter stability property is usually summarized in the effective but imprecise expression *tangent to tangents are tangents*.

Proposition 2.4. *Assume ϕ is a measure with α-density on $\mathbb{H}^n$. Then for ϕ almost every $x \in \mathbb{H}^n$ we have:*

$$\mathit{Tan}_\alpha(\phi, x) \subseteq \Theta^\alpha(\phi, x)\mathcal{U}_{\mathbb{H}^n}(\alpha).$$

Proof. The proof of this proposition follows almost without modifications the one given in the Euclidean case in Proposition 3.4 of [15]. □

Proposition 2.5. *Let ϕ be a Borel measures having α-density in $\mathbb{H}^n$. Then for ϕ-a.e. x if $\mu \in \mathit{Tan}_\alpha(\phi, x)$ we have:*

$$r^{-\alpha}\mu_{y,r} \in \mathit{Tan}_\alpha(\phi, x), \qquad \text{for every } y \in \operatorname{supp}(\mu) \text{ and } r > 0.$$

Proof. For a proof in generic metric groups see for instance Proposition 3.1 in [32]. □

Proposition 2.6. *Let ϕ be a Radon measure and $\mu \in \mathit{Tan}_\alpha(\phi, x)$ be such that $r_i^{-\alpha}\phi_{x,r_i} \rightharpoonup \mu$ for some $r_i \to 0$. If $y \in \operatorname{supp}(\mu)$, there exists a sequence $\{z_i\}_{i\in\mathbb{N}} \subseteq \operatorname{supp}(\phi)$ such that $D_{1/r_i}(x^{-1}z_i) \to y$.*

Proof. A simple argument by contradiction yields the claim, the proof follows verbatim its Euclidean analogue, Proposition 3.4 in [15]. □

If μ is an α-uniform measure, we can also define its blowups at infinity, or blowdowns. Such tangents at infinity are Radon measures ν for which there exists a sequence $\{R_i\} \to \infty$ such that:

$$R_i^{-\alpha}\phi_{0,R_i} \rightharpoonup \nu.$$

We will denote with $\mathrm{Tan}_\alpha(\mu, \infty)$, the set of tangent measures at infinity of μ. The following proposition is a strengthened version of Proposition 2.4 for uniform measures.

Proposition 2.7. *Assume μ is a α-uniform measure. Then for any $z \in \operatorname{supp}(\mu) \cup \{\infty\}$ we have:*

$$\emptyset \neq \mathit{Tan}_\alpha(\mu, z) \subseteq \mathcal{U}_{\mathbb{H}^n}(\alpha).$$

Proof. A straightforward adaptation of the proof of Lemma 3.5 in [15] yields the desired conclusion. □

The following is a compactness result for uniform measure and for their supports.

Lemma 2.8. *If $\{\mu\}_{i\in\mathbb{N}}$ is a sequence of α-uniform measures converging in the weak topology to some ν then:*

(i) *ν is an α-uniform measure,*

(ii) *if $y \in \operatorname{supp}(\nu)$ then there exists a sequence $\{y_i\} \subseteq \mathbb{H}^n$ such that $y_i \in \operatorname{supp}(\mu_i)$ and $y_i \to y$,*

(iii) *if there exists a sequence $\{y_i\} \subseteq \mathbb{H}^n$ such that $y_i \in \operatorname{supp}(\mu_i)$ and $y_i \to y$, then $y \in \operatorname{supp}(\nu)$.*

Proof. The proof of this lemma is an almost immediate adaptation of the proofs of Proposition 2.4 and Proposition 2.6. □

The following Theorem was proved by V. Chousionis, J.Tyson in [14]. They proved that if $\mathbb{H}^n$ is endowed with the Koranyi metric, the density problem reduces to integer exponents only, as in the Euclidean case:

Theorem 2.9. *The set $\mathcal{U}_{\mathbb{H}^n}(\alpha)$ is non-empty if and only if $\alpha \in \{0, 1, \dots, 2n+2\}$. In particular if ϕ is a measure with α-density in $\mathbb{H}^n$, then:*

$$\alpha \in \{0, \dots, 2n+2\}.$$

Remark 2.10. Note that $\mathcal{U}_{\mathbb{H}^n}(0) = \{\delta_0\}$. Moreover arguing as in Proposition 3.14 of [15], we can also deduce that $\mathcal{U}_{\mathbb{H}^n}(2n+2) = \{\mathcal{L}^{2n+1}\}$. From now on we will always assume $\alpha \in \{1, \dots, 2n+1\}$.

Remark 2.11. The stratification of zeros of holomorphic functions is the tool used by B. Kirchheim and D. Preiss in [24] and later by V. Chousionis and J.Tyson in [14] to prove Marstrand's theorem in the Euclidean spaces and in the Heisenberg groups, respectively. It is easy to see that the same argument yields Marstrand's theorem for any Carnot group endowed with a polynomial norm.

2.3 Basic properties of uniform measures in $\mathbb{H}^n$

Proposition 2.12. *Let $\Sigma : (\mathbb{H}^n, \|\cdot\|) \to (\mathbb{H}^n, \|\cdot\|)$ be a surjective isometry of $(\mathbb{H}^n, \|\cdot\|)$ into itself. If $\mu \in \mathcal{U}_{\mathbb{H}^n}(\alpha)$ and there exist $u \in \operatorname{supp}(\mu)$ such that $\Sigma(u) = 0$, then $\Sigma_\#(\phi) \in \mathcal{U}_{\mathbb{H}^n}(\alpha)$ and:*

$$\operatorname{supp}(\Sigma_\#(\phi)) = \Sigma(\operatorname{supp}(\phi)).$$

Proof. Since $\Sigma^{-1}(B_r(g)) = B_r(\Sigma^{-1}(g))$, for any $g \in \mathbb{H}^n$ and any $r > 0$ we have that:

$$\Sigma_\#\phi(B_r(g)) = \phi(\Sigma^{-1}(B_r(g))) = \phi(B_r(\Sigma^{-1}(g))).$$

If $g \in \Sigma(\operatorname{supp}(\phi))$, then there exists $h \in \operatorname{supp}(\phi)$ such that $g = \Sigma(h)$ and then:

$$\Sigma_\#\phi(B_r(g)) = \phi(B_r(h)) = r^\alpha.$$

In particular $\Sigma(\operatorname{supp}(\phi)) \subseteq \operatorname{supp}(\Sigma_\#(\phi))$. The other inclusion can be obtained similarly. □

Definition 2.13. Let $E \subseteq \mathbb{H}^n$ be a Borel set. For any $0 \leq \alpha \leq 2n+2$ and $\delta > 0$, define:

$$\mathscr{S}^\alpha_{\delta,E}(A) := \inf\left\{\sum_{j=1}^\infty r_j^s : A \cap E \subseteq \bigcup_{j=1}^\infty \operatorname{cl}(B_{r_j}(x_j)), r_j \leq \delta \text{ and } x_j \in E\right\},$$

whenever $\emptyset \neq A \subseteq \mathbb{H}^n$ is Borel and $\mathscr{S}^\alpha_{\delta,E}(\emptyset) = 0$. We define the E-centred spherical Hausdorff measure as:

$$\mathcal{S}^\alpha_E(A) := \sup_{B\subseteq A}\sup_{\delta>0}\mathscr{S}^\alpha_{\delta,E}(B).$$

In the case $E := \mathbb{H}^n$, our definition reduces to the standard spherical Hausdorff measure (see Definition 2.10.2(2) in [21]). In such a case we let $\mathcal{S}^\alpha := \mathcal{S}^\alpha_{\mathbb{H}^n}$.

Remark 2.14. Let $E \subseteq A$ be Borel subset of $\mathbb{H}^n$. It is easy to see that $\mathcal{S}^\alpha_A(S) \leq \mathcal{S}^\alpha_E(S) \leq 2^\alpha\mathcal{S}^\alpha_A(S)$ for any $S \subseteq E$ Borel. In particular the measures $\mathcal{S}^\alpha_A \llcorner E$ and $\mathcal{S}^\alpha_E$ are equivalent.

The following characterization of uniform measures easily follows from Federer's spherical differentiation theorems:

Proposition 2.15. *If μ is a α-uniform measure on $\mathbb{H}^n$, then $\mu = \mathcal{S}^\alpha_{\operatorname{supp}(\mu)}$.*

Proof. First of all note that since μ is uniform, $\mu(\partial B_r(x)) = 0$ for any $r \geq 0$ and $x \in \operatorname{supp}(\mu)$. If we let $\mathcal{F} := \{\operatorname{cl}(B_r(y)) : y \in E \text{ and } r > 0\}$, since μ is uniform, we also deduce that:

$$\lim_{r\to 0}\sup\left\{r^{-\alpha}\mu(\operatorname{cl}(B_r(y))) : x \in B_r(y) \in \mathcal{F}\right\} = 1. \tag{2.4}$$

The equality (2.4) together with Theorem 2.13 of [22], imply that whenever $V \subseteq \mathbb{H}^n$ is an open set, we have:

$$\mu(V) = \mathcal{S}^\alpha_{\operatorname{supp}(\mu)}(V).$$

In particular thanks to Lemma 1.9.4 of [10] the above identity implies that $\mu(E) = \mathcal{S}^\alpha_{\operatorname{supp}(\mu)}(E)$, for any Borel set E. □

Remark 2.16. The above proposition implicitly says that α-uniform measures are uniquely determined by their support.

Definition 2.17 (Radially symmetric functions)**.** We say that a function $\varphi : \mathbb{H}^n \to \mathbb{R}$ is radially symmetric if there exists a profile function $g : [0, \infty) \to \mathbb{R}$ such that $\varphi(z) = g(\|z\|)$.

Integrals of radially symmetric functions with respect to uniform measures are easy to compute and we have the following change of variable formula, which will be extensively used throughout the paper:

Proposition 2.18. *Let $\mu \in \mathcal{U}_{\mathbb{H}}(\alpha)$ and suppose $\varphi : \mathbb{H}^n \to \mathbb{R}$ is a radially symmetric non-negative function. Then for any $u \in \operatorname{supp}(\mu)$:*

$$\int_{\mathbb{H}^n} \varphi(u^{-1} * z) d\mu(z) = \alpha \int_0^\infty r^{\alpha-1} g(r) dr,$$

where g is the profile function associated to φ.

Proof. First one proves the formula for simple functions of the form:

$$\varphi(z) := \sum_{i=1}^k a_i \chi_{B_{r_i}(0)},$$

where $0 \leq a_i, r_i$ for any $i = 1, \ldots, k$. The result for a general φ follows by Beppo Levi convergence theorem. □

Proposition 2.19. *Let $p > 0$ and $\mu \in \mathcal{U}_{\mathbb{H}}(\alpha)$. Then:*

$$\int_{\mathbb{H}^n} \|z\|^p e^{-s\|z\|^4} d\mu(z) = \frac{\alpha}{4s^{\frac{\alpha+p}{4}}} \Gamma\left(\frac{\alpha+p}{4}\right).$$

Proof. The profile function associated to $\|z\|^p e^{-s\|z\|^4}$ is $r^p e^{-sr^4}$, thus by Proposition 2.18 we have that:

$$\begin{aligned}\int_{\mathbb{H}^n} \|z\|^p e^{-s\|z\|^4} d\mu(z) =& \alpha \int_0^\infty r^{\alpha-1} r^p e^{-sr^4} dr = \frac{\alpha}{s^{\frac{\alpha+p}{4}}} \int_0^\infty t^{\alpha+p-1} e^{-t^4} dt \\ =& \frac{\alpha}{4s^{\frac{\alpha+p}{4}}} \int_0^\infty x^{\frac{\alpha+p}{4}-1} e^{-x} dx = \frac{\alpha}{4s^{\frac{\alpha+p}{4}}} \Gamma\left(\frac{\alpha+p}{4}\right).\end{aligned}$$

□

3. Uniform measures have support contained in quadrics

First of all we introduce some notation:

Definition 3.1. Let $b \in \mathbb{R}^{2n}$, $\mathcal{Q} \in \text{Sym}(2n)$ and $\mathcal{T} \in \mathbb{R}$. We define $\mathbb{K}(b, \mathcal{Q}, \mathcal{T})$ to be the set of $(x, t) \in \mathbb{R}^{2n} \times \mathbb{R}$ for which:

$$\langle b, x\rangle + \langle x, \mathcal{Q}x\rangle + \mathcal{T}t = 0.$$

The main goal of this section is to prove the following:

Theorem 3.2. *Let $m \in \{1, \dots, 2n+1\} \setminus \{2\}$. For any $\mu \in \mathcal{U}_{\mathbb{H}^n}(m)$ there exist $b \in \mathbb{R}^{2n}$, $\mathcal{T} \in \mathbb{R}$ and $\mathcal{Q} \in$ Sym$(2n)$ with Tr$(\mathcal{Q}) \neq 0$ such that:*

$$\text{supp}(\mu) \subseteq \mathbb{K}(b, \mathcal{Q}, \mathcal{T}).$$

The proof of the above theorem is divided into 2 main steps. First we construct $b \in \mathbb{R}^{2n}$, $\mathcal{Q} \in \text{Sym}(2n)$, $\mathcal{T} \in \mathbb{R}$ for which:

$$\langle b, u_H\rangle + \langle u_H, \mathcal{Q}u_H\rangle + \mathcal{T}u_T = 0,$$

for any $u \in \text{supp}(\mu)$. Secondly we prove that $\text{Tr}(\mathcal{Q}) \neq 0$.

The first part of the above program follows closely Chapter 7 in [15] (for a more detailed explanation see the beginning of Subsection 3.1 below). The basic idea behind all these computations is that the identity $\mu(B_r(u)) = \mu(B_r(0))$ for any $u \in \text{supp}(\mu)$ and any $r > 0$ implies that:

$$\left| \fint_{B_r(u)} (r^4 - \|u^{-1} * z\|^4)^2 d\mu(z) - \fint_{B_r(0)} (r^4 - \|z\|^4)^2 d\mu(z) \right| \leq C\|u\|^5/r,$$

for any $u \in \mathbb{H}^n$ and any $r > 0$, from which it is not hard to build a quadric containing $\text{supp}(\mu)$ (for details see Subsection 3.3).

The second part (contained in Subsection 3.4) is devoted to prove that the quadric is non-degenerate. In the Euclidean case this is almost free, however in the sub-Riemannian context it requires some effort. In particular we are able to prove that if the support of μ is far away from the vertical axis $\mathcal{V} := \{x_H = 0\}$, then the quadric is non-degenerate. The reason for which we have to avoid the case $m = 2$ in Theorem 3.2 is that $\mathcal{S}^2_{\mathcal{V}}$ is a 2-uniform measure and indeed in that case the matrix $\mathcal{Q}$ of our construction is 0.

3.1 Moments in the Heisenberg group and their algebraic structure

One of the fundamental tools introduced by Preiss in [37] are moments of uniform measures. If μ is an m-uniform meausure in $\mathbb{R}^n$, for any $k \in \mathbb{N}$ and $s > 0$ he defines the k-th moment of μ:

$$b^{\mu}_{k,s}(u_1, \dots, u_k) := \frac{(2s)^{k+\frac{m}{2}}}{I(m)k!} \int_{\mathbb{R}^n} \prod_{i=1}^{k} \langle z, u_i\rangle e^{-s|z|^2} d\mu(z),$$

where $I(m) := \int_{\mathbb{R}^n} e^{-|z|^2} d\mu(z)$. Using these functions Preiss is able to prove the following expansion formula:

$$\left| \sum_{k=1}^{2q} b_{k,s}(u) - \sum_{k=1}^{q} \frac{s^k \|u\|^k}{k!} \right| \leq 5^{n+9}(s\|u\|^2), \tag{3.1}$$

for any $u \in \operatorname{supp}(\mu)$, which allows him to find algebraic equations for points in $\operatorname{supp}(\mu)$.

The problem we tackle in this subsection is to prove an analogue of the inequality (3.1) in a context where there is no scalar product inducing the metric. The strategy of our choice is to use a suitable polarization $V(\cdot,\cdot)$ of the Koranyi norm, which is a 4-th degree polynomial (see Proposition 3.4) for which a weak form of the Cauchy-Schwarz inequality holds (see Proposition 3.7). This will allow us to prove in the next subsection an inequality of the type (3.1) with our modified moments (see Proposition 3.14).

It is possible to prove an inequality of the type (3.1) even in Banach spaces with the suitable polarisation of the norm and with the proper definition of moments. The problem is that such an expansion would not yield many information on the structure of the support of measures as one really needs an explicit algebraic expression for the substitute of the scalar product in order to be able to push further the argument. In Carnot groups however one always have a smooth polynomial norm which can be used as in Heisenberg case, computations would be just much more complicated.

Definition 3.3 (Substitute for the scalar product)**.** Let $V : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{R}$ be the polarisation of the Koranyi norm, i.e.:

$$V(u,z) := \frac{\|u\|^4 + \|z\|^4 - \|u^{-1} * z\|^4}{2},$$

for any $u, z \in \mathbb{H}^n$.

Proposition 3.4. *The function $V(z,u)$ can be decomposed as:*

$$2V(u,z) = L(u,z) + Q(u,z) + T(u,z),$$

where:

(i) $L(u,z) := \langle u_H, 4|z_H|^2 z_H + 4z_T J z_H \rangle$,

(ii) $Q(u,z) := -4\langle z_H, u_H\rangle^2 - 2|z_H|^2|u_H|^2 - 4\langle Jz_H, u_H\rangle^2 + 2z_T u_T$,

(iii) $T(u,z) := \langle z_H, 4|u_H|^2 u_H + 4u_T J u_H \rangle$.

Remark 3.5. Note that $L(u,z), Q(u,z), T(u,z)$ are 1, 2, 3-intrinsic homogeneous in u, respectively, and moreover we have $L(z,u) = T(u,z)$.

Proof. Thanks to the definition of V and of $\|\cdot\|$, we have:

$$\begin{aligned} 2V(u,z) = \|u\|^4 + \|z\|^4 - \|u^{-1} * z\|^4 =& |u_H|^4 + |u_T|^2 + |z_H|^4 + |z_T|^2 - |z_H - u_H|^4 - |z_T - u_T - 2\langle u_H, Jz_H\rangle|^2 \\ =& -4\langle u_H, z_H\rangle^2 - 2|u_H|^2|z_H|^2 + 4|u_H|^2\langle u_H, z_H\rangle + 4|z_H|^2\langle u_H, z_H\rangle \\ & - 4\langle u_H, Jz_H\rangle^2 + 4z_T\langle u_H, Jz_H\rangle + 2u_T z_T - 4u_T\langle u_H, Jz_H\rangle. \end{aligned}$$

Recognising $L(u,z), Q(u,z), T(u,z)$ in the computation above proves the claim. □

Proposition 3.6. *For any $z, u \in \mathbb{H}^n$ the following estimates hold:*

(i) $|L(u,z)| \leq 4\|u\|\|z\|^3$,

(ii) $|Q(u,z)| \leq 12\|z\|^2\|u\|^2$,

(iii) $|T(u, z)| \leq 4\|z\|\|u\|^3$.

Proof. We start with proving the estimate for $L(u, z)$:

$$|L(u, z)| = |\langle u_H, 4|z_H|^2 z_H + 4z_T J z_H\rangle| \leq 4\|u\| \left||z_H|^2 z_H + z_T J z_H\right|.$$

Since z and Jz are orthogonal:

$$||z_H|^2 z_H + z_T J z_H|^2 = |z_H|^6 + z_T^2 |z_H|^2 = |z_H|^2 \|z\|^4.$$

From which we get:

$$|L(u, z)| \leq 4\|u\|\|z\|^2 |z_H|.$$

By Remark 3.5, we have that $L(z, u) = T(u, z)$ and hence point (iii) follows. At last we prove the bound for $Q(u, z)$:

$$|Q(u, z)| \leq 4\langle z_H, u_H\rangle^2 + 2|z_H|^2|u_H|^2 + 4\langle Jz_H, u_H\rangle^2 + 2|z_T||u_T| \leq 10|z_H|^2|u_H|^2 + 2|z_T||u_T| \leq 12\|z\|^2\|u\|^2.$$

□

Proposition 3.7 (Cauchy-Schwarz inequality for $V(\cdot, \cdot)$)**.** *For any $u, z \in \mathbb{H}^n$ the following holds:*

$$|V(u, z)| \leq 2\|u\|\|z\|(\|u\| + \|z\|)^2.$$

Proof. By the triangle inequality we have that $\|u^{-1} * z\| \geq |\|u\| - \|z\||$. Therefore:

$$\|u^{-1} * z\|^4 \geq |\|z\| - \|u\||^4 = \|u\|^4 - 4\|u\|^3\|z\| + 6\|u\|^2\|z\|^2 - 4\|u\|\|z\|^3 + \|z\|^4.$$

By the definition of V we conclude that:

$$4\|u\|^3\|z\| - 6\|u\|^2\|z\|^2 + 4\|u\|\|z\|^3 \geq \|u\|^4 + \|z\|^4 - \|u^{-1} * z\|^4 = 2V(u, z).$$

Collecting terms, we have:

$$2\|u\|\|z\|(\|u\| + \|z\|)^2 \geq V(u, z).$$

The bound from below for $V(u, z)$ is obtained similarly. □

The following definition extends from the Euclidean spaces to the Heisenberg group the notion of moment of a uniform measure given by Preiss in [37]:

Definition 3.8 (Preiss' moments)**.** For any $k \in \mathbb{N}$, $s > 0$ and any $u_1, \ldots, u_k \in \mathbb{H}^n$, we define:

$$b^{\mu}_{k,s}(u_1, \ldots, u_k) := \frac{s^{k+\frac{m}{4}}}{k! C(m)} \int_{\mathbb{H}^n} \prod_{i=1}^{k} 2V(u_i, z) e^{-s\|z\|^4} d\mu(z),$$

where $C(m) := \Gamma\left(\frac{m}{4} + 1\right)$. Moreover, if $u_1 = \ldots = u_k$, we let:

$$b^{\mu}_{k,s}(u) := b^{\mu}_{k,s}(u, \ldots, u).$$

Proposition 3.9. *For any $u \in \mathbb{H}^n$ the following estimate holds:*

$$|b^{\mu}_{k,s}(u)| \leq 16^k \frac{(\|u\| s^{\frac{1}{4}})^k}{k!} \frac{\Gamma(\frac{m+3k}{4})}{\Gamma\left(\frac{m}{4}\right)} ((\|u\| s^{\frac{1}{4}})^{2k} + 1).$$

Proof. Thanks to Proposition 3.7, we have the following preliminary estimate:

$$|b^\mu_{k,s}(u)| \leq s^{\frac{m}{4}}\frac{(2s)^k}{k!C(m)}\int_{\mathbb{H}^n}|V(u,z)|^k e^{-s\|z\|^4}d\mu(z) \leq s^{\frac{m}{4}}\frac{(2s)^k}{k!C(m)}\int_{\mathbb{H}^n}2^k\|u\|^k\|z\|^k(\|u\|+\|z\|)^{2k}e^{-s\|z\|^4}d\mu(z).$$

Moreover, Jensen inequality (used in the first line) and Proposition 2.19 (used in the third line to explicitly compute the integrals) imply that:

$$\begin{aligned}
|b^\mu_{k,s}(u)| \leq& 2^{3k}s^{\frac{m}{4}}\frac{(2s)^k}{k!C(m)}\int_{\mathbb{H}^n}\|u\|^k\|z\|^k(\|u\|^{2k}+\|z\|^{2k})e^{-s\|z\|^4}d\mu(z)\\
\leq& 2^{3k}s^{\frac{m}{4}}\frac{(2s)^k}{k!C(m)}\left(\int_{\mathbb{H}^n}\|u\|^{3k}\|z\|^k e^{-s\|z\|^4}d\mu(z)+\int_{\mathbb{H}^n}\|u\|^k\|z\|^{3k}e^{-s\|z\|^4}d\mu(z)\right)\\
=& 2^{4k}\frac{m}{4}\frac{\|u\|^k s^{\frac{k}{4}}}{k!C(m)}\left(\|u\|^{2k}s^{\frac{k}{2}}\Gamma\left(\frac{m+k}{4}\right)+\Gamma\left(\frac{m+3k}{4}\right)\right) \leq 16^k\frac{(\|u\|s^{\frac{1}{4}})^k}{k!}\frac{\Gamma\left(\frac{m+3k}{4}\right)}{\Gamma\left(\frac{m}{4}\right)}((\|u\|s^{\frac{1}{4}})^{2k}+1).
\end{aligned}$$

□

Definition 3.10. Let $\alpha \in \mathbb{N}^3 \setminus \{(0,0,0)\}$, $s>0$ and $u \in \mathbb{H}^n$. We define the functions $c_{\alpha,s} : \bigotimes_{i=0}^{|\alpha|}\mathbb{H}^n \to \mathbb{R}$ as:

$$c_{\alpha,s}(u) := \frac{1}{\alpha_1!\alpha_2!\alpha_3!}\frac{s^{|\alpha|+\frac{m}{4}}}{C(m)}\int_{\mathbb{H}^n}L(u,z)^{\alpha_1}Q(u,z)^{\alpha_2}T(u,z)^{\alpha_3}e^{-s\|z\|^4}d\mu(z),$$

where $|\alpha| := \alpha_1+\alpha_2+\alpha_3$. Moreover, for any $l \in \mathbb{N}$, we let:

$$A(l) := \{\alpha \in \mathbb{N}^3 \setminus \{(0,0,0)\} : \alpha_1+2\alpha_2+3\alpha_3 \leq l\}.$$

The moments $b^\mu_{k,s}$ can be expressed by means of the functions $c_{\alpha,s}$ defined above:

$$\begin{aligned}
b^\mu_{k,s}(u) =& \frac{s^{k+\frac{m}{4}}}{k!C(m)}\int_{\mathbb{H}^n}(2V(u,z))^k e^{-s\|z\|^4}d\mu(z) = \frac{s^{k+\frac{m}{4}}}{k!C(m)}\int_{\mathbb{H}^n}(L(u,z)+Q(u,z)+T(u,z))^k e^{-s\|z\|^4}d\mu(z)\\
=& \frac{s^{k+\frac{m}{4}}}{k!C(m)}\int_{\mathbb{H}^n}\sum_{|\alpha|=k}\frac{k!}{\alpha_1!\alpha_2!\alpha_3!}L(u,z)^{\alpha_1}Q(u,z)^{\alpha_2}T(u,z)^{\alpha_3}e^{-s\|z\|^4}d\mu(z)\\
=& \sum_{|\alpha|=k}\frac{1}{\alpha_1!\alpha_2!\alpha_3!}\frac{s^{k+\frac{m}{4}}}{C(m)}\int_{\mathbb{H}^n}L(u,z)^{\alpha_1}Q(u,z)^{\alpha_2}T(u,z)^{\alpha_3}e^{-s\|z\|^4}d\mu(z) = \sum_{|\alpha|=k}c_{\alpha,s}(u).
\end{aligned} \tag{3.2}$$

Proposition 3.11. *Let $\alpha \in \mathbb{N}^3 \setminus \{(0,0,0)\}$, $s>0$ and $u \in \mathbb{H}^n$. Then:*

$$|c_{\alpha,s}(u)| \leq D(\alpha)(s^{\frac{1}{4}}\|u\|)^{\alpha_1+2\alpha_2+3\alpha_3},$$

for some constant $D(\alpha)>0$.

Proof. Proposition 3.6 allows us to estimate the integrand in the definition of $c_{\alpha,s}$ in the following way (as it gives bounds on $|L(u,z)|$, $|Q(u,z)|$ and $|T(u,z)|$):

$$\begin{aligned}
|c_{\alpha,s}(u)| \leq& \frac{1}{\alpha_1!\alpha_2!\alpha_3!}\frac{s^{|\alpha|+\frac{m}{4}}}{C(m)}\int_{\mathbb{H}^n}|L(u,z)|^{\alpha_1}|Q(u,z)|^{\alpha_2}|T(u,z)|^{\alpha_3}e^{-s\|z\|^4}d\mu(z)\\
\leq& \frac{4^{\alpha_1+\alpha_3}12^{\alpha_2}}{\alpha_1!\alpha_2!\alpha_3!}\frac{s^{|\alpha|+\frac{m}{4}}}{C(m)}\|u\|^{\alpha_1+2\alpha_2+3\alpha_3}\int_{\mathbb{H}^n}\|z\|^{3\alpha_1+2\alpha_2+\alpha_3}e^{-s\|z\|^4}d\mu(z).
\end{aligned}$$

Moreover Proposition 2.19 yields an explicit value for the integrals in the last line of the above computations, therefore we get:

$$\begin{aligned}|c_{\alpha,s}(u)| \leq& \frac{4^{\alpha_1+\alpha_3}12^{\alpha_2}}{\alpha_1!\alpha_2!\alpha_3!}\frac{s^{|\alpha|+\frac{m}{4}}}{C(m)}\|u\|^{\alpha_1+2\alpha_2+3\alpha_3}\int_{\mathbb{H}^n}\|z\|^{3\alpha_1+2\alpha_2+\alpha_3}e^{-s\|z\|^4}d\mu(z)\\ =&\frac{4^{\alpha_1+\alpha_3}12^{\alpha_2}}{\alpha_1!\alpha_2!\alpha_3!}\frac{\Gamma\left(\frac{m+3\alpha_1+2\alpha_2+\alpha_3}{4}\right)}{\Gamma\left(\frac{m}{4}\right)}(\|u\|^4 s)^{\frac{\alpha_1+2\alpha_2+3\alpha_3}{4}}.\end{aligned}$$

With the choice: $D(\alpha) := \frac{4^{\alpha_1+\alpha_3}12^{\alpha_2}}{\alpha_1!\alpha_2!\alpha_3!}\frac{\Gamma\left(\frac{m+3\alpha_1+2\alpha_2+\alpha_3}{4}\right)}{\Gamma\left(\frac{m}{4}\right)}$, we get the desired conclusion. □

Proposition 3.12. *Assume μ is also invariant under dilations, i.e., for any $\lambda > 0$ we have $\frac{\mu_{0,\lambda}}{\lambda^m} = \mu$, where $\mu_{0,\lambda}$ was defined in* (2.3)*. Then:*

$$c_{\alpha,s}(u) = s^{\frac{\alpha_1+2\alpha_2+3\alpha_3}{4}}c_{\alpha,1}(u),$$

for any $s > 0$, $\alpha \in \mathbb{N}^3 \setminus \{(0,0,0)\}$ and any $u \in \mathbb{H}^n$.

Proof. For any $0 < \lambda$, we have that:

$$L(u, D_\lambda(z))^{\alpha_1}Q(u, D_\lambda(z))^{\alpha_2}T(u, D_\lambda(z))^{\alpha_3} = \lambda^{3\alpha_1+2\alpha_2+\alpha_3}L(u,z)^{\alpha_1}Q(u,z)^{\alpha_2}T(u,z)^{\alpha_3}.$$

Therefore, defining $\lambda := 1/s^{\frac{1}{4}}$, using the fact that $\mu_{0,\lambda}/\lambda^m = \mu$, we conclude that:

$$\begin{aligned}c_{\alpha,s}(u) =& \frac{1}{\alpha_1!\alpha_2!\alpha_3!}\frac{s^{|\alpha|+\frac{m}{4}}}{C(m)}\int_{\mathbb{H}^n}L(u,z)^{\alpha_1}Q(u,z)^{\alpha_2}T(u,z)^{\alpha_3}e^{-s\|z\|^4}d\mu(z)\\ =&\frac{1}{\alpha_1!\alpha_2!\alpha_3!}\frac{s^{\frac{\alpha_1+2\alpha_2+3\alpha_3}{4}}}{C(m)}\int_{\mathbb{H}^n}L(u,z)^{\alpha_1}Q(u,z)^{\alpha_2}T(u,z)^{\alpha_3}e^{-\|z\|^4}d\frac{\mu_{0,\lambda}(z)}{\lambda^m}.\end{aligned}$$

□

3.2 Expansion formulas for moments

This subsection is devoted to the proof of the expasion formula for the moments of uniform measures (3.3). Moreover in Proposition 3.15, we start to flesh out the complex algebra of the inequality (3.3), in order to build the desired quadric containing $\mathrm{supp}(\mu)$. We start with a technical lemma which will be required in the proof of Proposition 3.14:

Lemma 3.13. *For any $m, k \in \mathbb{N}$ we have the following estimate:*

$$\Gamma\left(\frac{3k+m}{4}\right) \leq 8^{\frac{m}{4}}\left(\frac{6k}{7}\right)^{\frac{3k}{4}}e^{-\frac{3}{4}k}\Gamma\left(\frac{m}{4}\right).$$

Proof. By definition of the Γ function we have:

$$\Gamma\left(\frac{3k+m}{4}\right) = \int_0^\infty t^{\frac{3k+m}{4}-1}e^{-t}dt \leq \|g\|_\infty\int_0^\infty t^{\frac{m}{4}-1}e^{-t/8}dt = 8^{\frac{m}{4}}\|g\|_\infty\Gamma\left(\frac{m}{4}\right),$$

where $g(t) := t^{\frac{3k}{4}}e^{-7t/8}$. The function g attains its maximum at $t_* := \frac{6k}{7}$ and thus:

$$\|g\|_\infty \leq \left(\frac{6k}{7}\right)^{\frac{3k}{4}}e^{-\frac{3}{4}k}.$$

This yields $\Gamma\left(\frac{3k+m}{4}\right) \leq 8^{\frac{m}{4}}\left(\frac{6k}{7}\right)^{\frac{3k}{4}}e^{-\frac{3}{4}k}\Gamma\left(\frac{m}{4}\right)$. □

The following proposition is the technical core of this section. As we already remarked, (3.3) will allow us to construct the algebraic surfaces containing $\mathrm{supp}(\mu)$. The proof follows closely its Euclidean analogue which can be found in Section 3.4 of [37] or in Lemma 7.6 of [15].

Proposition 3.14 (Expansion formula)**.** *There exists a constant $0 < G(m)$ such that for any $s > 0$, $q \in \mathbb{N}$ and $u \in \mathrm{supp}(\mu)$ we have:*

$$\left|\sum_{k=1}^{4q} b^{\mu}_{k,s}(u) - \sum_{k=1}^{q} \frac{s^k\|u\|^{4k}}{k!}\right| \leq G(m)(s\|u\|^4)^{q+\frac{1}{4}}(2+(s\|u\|^4)^{2q}). \tag{3.3}$$

Proof. First consider the case $s\|u\|^4 \geq 1$: triangle inequality and Proposition 3.9 (used to get the bound in the second line) imply that:

$$\begin{aligned}\left|\sum_{k=1}^{4q} b^{\mu}_{k,s}(u) - \sum_{k=1}^{q} \frac{s^k\|u\|^{4k}}{k!}\right| &\leq \left|\sum_{k=1}^{4q} b^{\mu}_{k,s}(u)\right| + \left|\sum_{k=1}^{q} \frac{s^k\|u\|^{4k}}{k!}\right| \\ &\leq \sum_{k=1}^{4q} 16^k \frac{(\|u\|s^{\frac{1}{4}})^k}{k!}\frac{\Gamma\left(\frac{m+3k}{4}\right)}{\Gamma\left(\frac{m}{4}\right)}((\|u\|s^{\frac{1}{4}})^{2k}+1) + \sum_{k=1}^{q}\frac{s^k\|u\|^{4k}}{k!} \\ &\leq (\|u\|^4 s)^{q+\frac{1}{4}}\Big(\big((\|u\|^4 s)^{2q}+1\big)E(m) + \sum_{k=1}^{q}\frac{1}{k!}\Big),\end{aligned}$$

where $E(m) := \sum_{k=1}^{4q} \frac{16^k}{k!}\frac{\Gamma\left(\frac{m+3k}{4}\right)}{\Gamma\left(\frac{m}{4}\right)}$. In order to prove the proposition in this case we are left to prove that $E(m)$ is finite. To do this, we use Lemma 3.13 to get an upper bound on $E(m)$:

$$E(m) = \sum_{k=1}^{\infty} \frac{16^k}{k!}\frac{\Gamma\left(\frac{m+3k}{4}\right)}{\Gamma\left(\frac{m}{4}\right)} \leq 8^{\frac{m}{4}} \sum_{k=1}^{\infty} \frac{16^k e^{-\frac{3}{4}k}}{k!}\left(\frac{6k}{7}\right)^{\frac{3k}{4}}.$$

The series on the right-hand side in the above inequality converges by the ratio testand thus by comparison $E(m)$ is also finite. Defined:

$$G(m) := \max\{E(m), e\},$$

we have that the proposition is proved in the case $s\|u\|^4 \geq 1$:

$$\left|\sum_{k=1}^{4q} b^{\mu}_{k,s}(u) - \sum_{k=1}^{q} \frac{s^k\|u\|^{4k}}{k!}\right| \leq G(m)(\|u\|^4 s)^{q+\frac{1}{4}}\left((\|u\|^4 s)^{2q}+2\right).$$

We have to prove the thesis in the case that $s\|u\|^4 < 1$. The well known identity $\sum_{k=0}^{\infty} \frac{s^k\|u\|^{4k}}{k!} = e^{s\|u\|^4}$ implies that:

$$\left|\sum_{k=0}^{q} \frac{s^k\|u\|^{4k}}{k!} - e^{s\|u\|^4}\right| = \left|\sum_{k=q+1}^{\infty} \frac{s^k\|u\|^{4k}}{k!}\right| \leq (s\|u\|^4)^{q+1}\sum_{k=q+1}^{\infty}\frac{1}{k!} \leq e(s\|u\|^4)^{q+1}.$$

For any fixed $s > 0$, we prove that for any $u \in \mathrm{supp}(\mu)$ such that $s\|u\|^4 < 1$, the series $\sum_{k=1}^{\infty} b_{k,s}(u)$ converges absolutely:

$$\sum_{k=1}^{\infty}|b_{k,s}(u)| \leq \sum_{k=1}^{\infty} 16^k \frac{(\|u\|s^{\frac{1}{4}})^k}{k!}\frac{\Gamma\left(\frac{m+3k}{4}\right)}{\Gamma\left(\frac{m}{4}\right)}((\|u\|s^{\frac{1}{4}})^{2k}+1) \leq 2\sum_{k=1}^{\infty}\frac{16^k}{k!}\frac{\Gamma\left(\frac{m+3k}{4}\right)}{\Gamma\left(\frac{m}{4}\right)} = 2E(m).$$

We can therefore estimate its tail:

$$\begin{aligned}\left|\sum_{k=1}^{\infty} b_{k,s}(u) - \sum_{k=1}^{4q} b_{k,s}(u)\right| \leq \left|\sum_{k=4q+1}^{\infty} b_{k,s}(u)\right| \leq \sum_{k=4q+1}^{\infty} 16^k \frac{(\|u\|s^{\frac{1}{4}})^k}{k!} \frac{\Gamma\left(\frac{m+3k}{4}\right)}{\Gamma\left(\frac{m}{4}\right)} ((\|u\|s^{\frac{1}{4}})^{2k}+1) \\ \leq (\|u\|^4 s)^{q+\frac{1}{4}}((\|u\|^4 s)^{2q+1}+1) \sum_{k=4q+1}^{\infty} \frac{16^k}{k!} \frac{\Gamma\left(\frac{m+3k}{4}\right)}{\Gamma\left(\frac{m}{4}\right)} \\ \leq (\|u\|^4 s)^{q+\frac{1}{4}}((\|u\|^4 s)^{2q+1}+1) E(m).\end{aligned} \tag{3.4}$$

The next step in the proof is to prove the following equality:

$$\sum_{k=1}^{\infty} b_{k,s}(u) = e^{s\|u\|^4} \tag{3.5}$$

for every $s > 0$ and any $u \in \operatorname{supp}(\mu)$ such that $s\|u\|^4 < 1$. Note that by definition:

$$\sum_{k=0}^{\infty} b_{k,s}(u) = \lim_{q\to\infty} \sum_{k=0}^{q} \frac{s^{\frac{m}{4}}}{k! C(m)} \int_{\mathbb{H}^n} (2sV(u,z))^k e^{-s\|z\|^4} d\mu(z).$$

We would like to exchange integral and the limit above using dominated convergence. To do so we first have to find a dominating function:

$$\left|\sum_{k=0}^{q} \frac{(2sV(u,z))^k}{k!} e^{-s\|z\|^4}\right| \leq e^{-s\|z\|^4} \sum_{k=0}^{q} \frac{(4s\|u\|\|z\|(\|u\|+\|z\|)^2)^k}{k!} = e^{-s\|z\|^4+4s\|u\|\|z\|(\|u\|+\|z\|)^2},$$

where in the first in the first inequality we applied Proposition 3.7. The function $f(\cdot) := e^{-s\|\cdot\|^4+4s\|u\|\|\cdot\|(\|u\|+\|\cdot\|)^2}$ is in $L^1(\mu)$ thanks to Proposition 2.18. Thus applying the dominated convergence theorem (pointwise convergence is obvious), we get:

$$\begin{aligned}\sum_{k=0}^{\infty} b_{k,s}(u) =& \frac{s^{\frac{m}{4}}}{C(m)} \int_{\mathbb{H}^n} \left(\sum_{k=0}^{\infty} \frac{(2sV(u,z))^k}{k!}\right) e^{-s\|z\|^4} d\mu(z) = \frac{s^{\frac{m}{4}}}{C(m)} \int_{\mathbb{H}^n} e^{2sV(u,z)} e^{-s\|z\|^4} d\mu(z) \\ =& \frac{s^{\frac{m}{4}}}{C(m)} e^{s\|u\|^4} \int_{\mathbb{H}^n} e^{-s\|z\|^4+2sV(u,z)-s\|z\|^4} d\mu(z).\end{aligned}$$

Thus by definition of $V(u,z)$, using Proposition 2.18 and Proposition 2.19 we get the desired equality (3.5).:

$$\sum_{k=0}^{\infty} b_{k,s}(u) = \frac{s^{\frac{m}{4}}}{C(m)} e^{s\|u\|^4} \int_{\mathbb{H}^n} e^{-s\|u^{-1}*z\|^4} d\mu(z) = e^{s\|u\|^4}.$$

Triangle inequality and the bound on the tail of the series $\sum_{k=0}^{\infty} b_{k,s}(u)$ (see equation (3.4)) conclude the proof:

$$\begin{aligned}\left|\sum_{k=1}^{4q} b_{k,s}(u) - \sum_{k=1}^{q} \frac{s^k\|u\|^4}{k!}\right| \leq& \left|\sum_{k=0}^{4q} b_{k,s}(u) - e^{s\|u\|^4}\right| + \left|e^{s\|u\|^4} - \sum_{k=0}^{q} \frac{s^k\|u\|^4}{k!}\right| \\ \leq& (\|u\|^4 s)^{q+\frac{1}{4}}((\|u\|^4 s)^{2q+1}+1)E(m) + e(s\|u\|^4)^{q+1} \\ \leq& G(m)(\|u\|^4 s)^{q+\frac{1}{4}}((\|u\|^4 s)^{2q}+2).\end{aligned}$$

□

Recall that in equation (3.2), we showed how $b_{k,s}$ are sum of functions $c_{\alpha,s}$:

$$b_{k,s}(u) = \sum_{|\alpha|=k} c_{\alpha,s}(u).$$

Proposition 3.11 implies that already for $q = 1$, in the left-hand side of inequality (3.3):

$$\left| \sum_{k=1}^{4} \sum_{|\alpha|=k} c_{\alpha,s}(u) - s\|u\|^4 \right| \leq G(m)(s\|u\|^4)^{\frac{5}{4}}(2 + (s\|u\|^4)^3),$$

there are a lot of terms bounded by $(s\|u\|)^{\frac{5}{4}}$. In the next proposition we get rid of those terms pushing them to the right-hand side.

Proposition 3.15. *For any $s > 0$ and any $u \in \operatorname{supp}(\mu)$ we have:*

$$\left| \sum_{\alpha \in A(4)} c_{\alpha,s}(u) - s\|u\|^4 \right| \leq (s\|u\|^4)^{\frac{5}{4}} B(s^{\frac{1}{4}}\|u\|), \tag{3.6}$$

where $B(\cdot)$ is a suitable polynomial whereas $c_{\alpha,s}(\cdot)$ and $A(4)$ where defined in Definition 3.10.

Proof. With the choice $q = 1$, the formula (3.3) turns into:

$$\left| \sum_{k=1}^{4} b_{k,s}(u) - s\|u\|^4 \right| \leq G(m)(s\|u\|^4)^{\frac{5}{4}}(2 + (s\|u\|^4)^3).$$

By the triangle inequality, we have:

$$\left| \sum_{\alpha \in A(4)} c_{\alpha,s}(u) - s\|u\|^4 \right| \leq \left| \sum_{k=1}^{4} \sum_{|\alpha|=k} c_{\alpha,s}(u) - s\|u\|^4 \right| + \left| \sum_{\substack{\alpha \notin A(4) \\ |\alpha| \leq 4}} c_{\alpha,s}(u) \right|.$$

By equation (3.2) we deduce that the first term in the right-hand side of the above inequality coincides with $\left| \sum_{k=1}^{4} b^{\mu}_{k,s}(u) - s\|u\|^4 \right|$ and thus we are just left to estimate the second one. Proposition 3.11 implies that:

$$\left| \sum_{\substack{\alpha \notin A(4) \\ |\alpha| \leq 4}} c_{\alpha,s}(u^{|\alpha|}) \right| \leq \left| \sum_{\substack{\alpha \notin A(4) \\ |\alpha| \leq 4}} D(\alpha)(s^{\frac{1}{4}}\|u\|)^{\alpha_1 + 2\alpha_2 + 3\alpha_3} \right|.$$

Therefore the claim holds true with the choice: $B(t) := G(m)t^3 + \sum_{\substack{\alpha \notin A(4) \\ |\alpha| \leq 4}} D(\alpha) t^{\alpha_1 + 2\alpha_2 + 3\alpha_3 - 5}$. □

3.3 Construction of the candidate quadric containing the support

Before describing the content of this subsection we give the following:

Definition 3.16. For any $s \in (0, \infty)$ we let:

(i) the horizontal barycentre of the measure μ at time s to be the vector in $\mathbb{R}^{2n}$:

$$b(s) := \frac{4s^{\frac{1}{2}+\frac{m}{4}}}{C(m)} \int (|z_H|^2 z_H + z_T J z_H) e^{-s\|z\|^4} d\mu(z),$$

(ii) the symmetric matrix $\mathcal{Q}(s)$ associated to the measure μ at time s to be the element of $\mathrm{Sym}(2n)$:

$$\begin{aligned}\mathcal{Q}(s) :=& -\frac{2s^{\frac{1}{2}+\frac{m}{4}}}{C(m)} \int_{\mathbb{H}^n} |z_H|^2 e^{-s\|z\|^4} d\mu(z)\mathrm{id}_{2n} - \frac{4s^{\frac{1}{2}+\frac{m}{4}}}{C(m)} \int (z_H \otimes z_H + Jz_H \otimes Jz_H) e^{-s\|z\|^4} d\mu(z)\\ &+ \frac{8s^{\frac{3}{2}+\frac{m}{4}}}{C(m)} \int (|z_H|^4 z_H \otimes z_H + z_T^2 Jz_H \otimes Jz_H) e^{-s\|z\|^4} d\mu(z)\\ &+ \frac{8s^{\frac{3}{2}+\frac{m}{4}}}{C(m)} \int_{\mathbb{H}^n} |z_H|^2 z_T (Jz_H \otimes z_H + z_H \otimes Jz_H) e^{-s\|z\|^4} d\mu(z),\end{aligned}$$

(iii) the vertical barycentre of the measure μ at time s to be the real number:

$$\mathcal{T}(s) := \frac{2s^{\frac{1}{2}+\frac{m}{4}}}{C(m)} \int z_T e^{-s\|z\|^4} d\mu(z).$$

The first half of this Subsection is devoted to the proof of Proposition 3.17, where from Proposition 3.15 we are able to further simplify the algebra of inequality (3.3) proving the existence of constant $0 < C$ such that:

$$|\langle b(s), u_H\rangle + \langle \mathcal{Q}(s)u_H, u_H\rangle + \mathcal{T}(s)u_T| \leq Cs^{\frac{1}{4}}\|u\|.$$

In the second half of this Subsection we prove that $b(\cdot)$, $\mathcal{Q}(\cdot)$ and $\mathcal{T}(\cdot)$ are bounded curves as s goes to 0 and therefore by compactness we can find $\overline{b}$, $\overline{\mathcal{Q}}$ and $\overline{\mathcal{T}}$ for which for any $u \in \mathrm{supp}(\mu)$ we have:

$$\langle \overline{b}, u_H\rangle + \langle u_H, \overline{\mathcal{Q}}u_H\rangle + \overline{\mathcal{T}}u_T = 0.$$

What is left to prove in Subsection 3.4 is that as $s \to 0$, we can find a limit $\overline{\mathcal{Q}}$ for which $\mathrm{Tr}(\overline{\mathcal{Q}}) \neq 0$.

Proposition 3.17. *For any $s > 0$ and any $u \in \mathrm{supp}(\mu)$ we have that the following inequality holds:*

$$|\langle b(s), u_H\rangle + \langle \mathcal{Q}(s)[u_H], u_H\rangle + \mathcal{T}(s)u_T| \leq s^{\frac{1}{4}}\|u\|^3 B'(s^{\frac{1}{4}}\|u\|),$$

where $B'(\cdot)$ is a suitable polynomial and $b(\cdot)$, $\mathcal{Q}(\cdot)$ and $\mathcal{T}(\cdot)$ where introduced in Definition 3.16.

Proof. First of all note that if $\alpha \notin A(2)$ (see Definition 3.10) then $\alpha_1 + 2\alpha_2 + 3\alpha_3 \geq 3$. Therefore Proposition 3.11 implies that:

$$\sum_{\substack{|\alpha|\leq 4\\ \alpha\notin A(2)}} |c_{\alpha,s}(u)| \leq (s^{\frac{1}{4}}\|u\|)^3 \sum_{\substack{|\alpha|\leq 4\\ \alpha\notin A(2)}} D(\alpha)(s^{\frac{1}{4}}\|u\|)^{\alpha_1+2\alpha_2+3\alpha_3-3} = (s^{\frac{1}{4}}\|u\|)^3 B''(s^{\frac{1}{4}}\|u\|).$$

where $B''(t) := \sum_{\substack{|\alpha|\leq 4\\ \alpha\notin A(2)}} D(\alpha)t^{\alpha_1+2\alpha_2+3\alpha_3-3}$. Hence Proposition 3.15 yields:

$$\begin{aligned}\left|\sum_{\alpha\in A(2)} c_{\alpha,s}(u)\right| \leq& \left|\sum_{\alpha\in A(4)} c_{\alpha,s}(u) - s\|u\|^4\right| + s\|u\|^4 + \sum_{\substack{|\alpha|\leq 4\\ \alpha\notin A(2)}} |c_{\alpha,s}(u)|\\ \leq& (s\|u\|^4)^{\frac{5}{4}} B(s^{\frac{1}{4}}\|u\|) + s\|u\|^4 + (s^{\frac{1}{4}}\|u\|)^3 B''(s^{\frac{1}{4}}\|u\|),\end{aligned}$$

where $B'(t) := t^2B(t) + t + B''(t)$. Since $A(2) = \{(1,0,0),(2,0,0),(0,1,0)\}$ we just have to prove that:

$$c_{(1,0,0),s}(u) + c_{(2,0,0),s}(u) + c_{(0,1,0),s}(u) = \langle b(s), u_H\rangle + \langle u_H, \mathcal{Q}(s)u_H\rangle + \mathcal{T}(s)u_T.$$

The expansion of $c_{(1,0,0),s}(u)$ yields:

$$\begin{aligned}c_{(1,0,0),s}(u) =& \frac{s^{1+\frac{m}{4}}}{C(m)}\int_{\mathbb{H}^n} L(u,z)e^{-s\|z\|^4}d\mu(z)\\ =&s^{\frac{1}{2}}\left\langle u_H, \frac{s^{\frac{1}{2}+\frac{m}{4}}}{C(m)}\int_{\mathbb{H}^n} 4|z_H|z_H + 4z_TJz_He^{-s\|z\|^4}d\mu(z)\right\rangle = s^{\frac{1}{2}}\langle u_H, b(s)\rangle,\end{aligned}$$

Expanding $c_{(2,0,0),s}(u)$ we get the first part of the quadric $\mathcal{Q}(s)$:

$$\begin{aligned}c_{(2,0,0),s}(u) =&\frac{1}{2}\frac{s^{2+\frac{m}{4}}}{C(m)}\int_{\mathbb{H}^n} L(u,z)^2e^{-s\|z\|^4}d\mu(z)\\ =&\frac{16s^{2+\frac{m}{4}}}{2C(m)}\int_{\mathbb{H}^n}\langle u_H, |z_H|^2z_H\rangle^2 + \langle u_H, z_TJz_H\rangle^2e^{-s\|z\|^4}d\mu(z)\\ &+\frac{32s^{2+\frac{m}{4}}}{2C(m)}\int_{\mathbb{H}^n}\langle u_H, |z_H|^2z_H\rangle\langle u_H, z_TJz_H\rangle e^{-s\|z\|^4}d\mu(z)\\ =&s^{\frac{1}{2}}\langle u_H, \mathcal{Q}_1(s)[u_H]\rangle + s^{\frac{1}{2}}\langle u_H, \mathcal{Q}_2(s)[u_H]\rangle,\end{aligned}$$

where:

$$\begin{aligned}\mathcal{Q}_1(s) :=&\frac{8s^{\frac{3}{2}+\frac{m}{4}}}{C(m)}\int_{\mathbb{H}^n}|z_H|^4z_H\otimes z_H + z_T^2Jz_H\otimes Jz_He^{-s\|z\|^4}d\mu(z),\\ \mathcal{Q}_2(s) :=&\frac{8s^{\frac{3}{2}+\frac{m}{4}}}{C(m)}\int_{\mathbb{H}^n}|z_H|^2z_T(z_H\otimes Jz_H + Jz_H\otimes z_H)e^{-s\|z\|^4}d\mu(z).\end{aligned}\tag{3.7}$$

At last $c_{(0,1,0),s}(u)$ contains the vertical barycentre and the second half of $\mathcal{Q}(s)$:

$$\begin{aligned}c_{(0,1,0),s}(u) =&\frac{s^{1+\frac{m}{4}}}{C(m)}\int_{\mathbb{H}^n} Q(u,z)e^{-s\|z\|^4}d\mu(z)\\ =&-\frac{s^{1+\frac{m}{4}}}{C(m)}\int_{\mathbb{H}^n}(4\langle z_H, u_H\rangle^2 + 4\langle Jz_H, u_H\rangle^2)e^{-s\|z\|^4}d\mu(z)\\ &+\frac{s^{1+\frac{m}{4}}}{C(m)}\int_{\mathbb{H}^n}(-2|z_H|^2|u_H|^2 + 2z_Tu_T)e^{-s\|z\|^4}d\mu(z).\end{aligned}$$

From which we deduce that:

$$c_{(0,1,0),s}(u) = -s^{\frac{1}{2}}\langle \mathcal{Q}_3(s)[u_H], u_H\rangle - s^{\frac{1}{2}}\langle \mathcal{Q}_4(s)[u_H], u_H\rangle + s^{\frac{1}{2}}\mathcal{T}(s)u_T,$$

where:

$$\begin{aligned}\mathcal{Q}_3(s) :=&\frac{4s^{\frac{1}{2}+\frac{m}{4}}}{C(m)}\int_{\mathbb{H}^n}(z_H\otimes z_H + Jz_H\otimes Jz_H)e^{-s\|z\|^4}d\mu(z),\\ \mathcal{Q}_4(s) :=&\frac{2s^{\frac{m+2}{4}}}{C(m)}\int_{\mathbb{H}^n}|z_H|^2e^{-s\|z\|^4}d\mu(z)\mathrm{id}_{2n}.\end{aligned}\tag{3.8}$$

Noticing that $\mathcal{Q}(s) = \mathcal{Q}_1(s) + \mathcal{Q}_2(s) - \mathcal{Q}_3(s) - \mathcal{Q}_4(s)\mathrm{id}$, the claim is proven. □

Remark 3.18. Define V to be the span of the horizontal projection of $\mathrm{supp}(\mu)$, i.e.:

$$V := \text{span}\{u_H : u \in \mathrm{supp}(\mu)\}.$$

Then with a small abuse of notation, we will always make the identification:

$$b(s) = \overline{B}(s) := \frac{4s^{\frac{1}{2}+\frac{m}{4}}}{C(m)} \int (|z_H|^2 z_H + z_T \pi_V(Jz_H)) e^{-s\|z\|^4} d\mu(z),$$

where the function $\pi_V : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is the orthogonal projection onto the subspace V. The reason for which we make this identification is that:

$$\langle b(s), u_H \rangle = \langle \overline{B}(s), u_H \rangle,$$

for any $u \in \mathrm{supp}(\mu)$ which explains why we take this freedom.

Proposition 3.19. *Both $\mathcal{Q}(s)$ and $\mathcal{T}(s)$ are bounded functions on $(0, \infty)$. To be precise:*

(i) Endowed Sym(n) with a norm $|\cdot|$ there exists a constant $0 < C_1$, such that $\sup_{s\in(0,\infty)}|Q(s)| \leq C_1$.

(ii) There exists a constant $0 < C_2$, such that $\sup_{s\in(0,\infty)}|\mathcal{T}(s)| \leq C_2$.

Remark 3.20. In particular the function $s \mapsto Tr(\mathcal{Q}(s))$ is bounded.

Proof. Proposition 3.11 implies that there exists a constant $0 < \tilde{G}$ for which:

$$\tilde{G}s^{\frac{1}{2}}\|u\|^2 \geq |c_{(2,0,0),s}(u) + c_{(0,1,0),s}(u)| = s^{\frac{1}{2}}|\langle u_H, \mathcal{Q}(s)u_H \rangle + \mathcal{T}(s)u_T| \geq s^{\frac{1}{2}} \left||\langle u_H, \mathcal{Q}(s)u_H \rangle| - |\mathcal{T}(s)||u_T|\right|.$$

Thus, it suffices to give a bound for $\mathcal{T}(s)$ and the other will follow:

$$|\langle u_H, \mathcal{Q}(s)[u_H] \rangle| \leq \left(\tilde{G} + \sup_{s\in[0,\infty)} |\mathcal{T}(s)| \right) \|u\|^2.$$

The estimate on the supremum norm of $\mathcal{T}(s)$ follows by its definition and by Proposition 2.19:

$$|\mathcal{T}(s)| \leq \frac{s^{\frac{1}{2}+\frac{m}{4}}}{C(m)} \int 2\|z_T\| e^{-s\|z\|^4} d\mu(z) \leq \frac{s^{\frac{1}{2}+\frac{m}{4}}}{C(m)} \int 2\|z\|^2 e^{-s\|z\|^4} d\mu(z) = 2\frac{\Gamma\left(\frac{m+2}{4}\right)}{\Gamma\left(\frac{m}{4}\right)}.$$

□

From the above proposition we deduce that for any sequence $\{s_j\}_{j\in\mathbb{N}}$ such that s_j tends to zero, by compactness we can extract a subsequence $\{s_{j_k}\}_{k\in\mathbb{N}}$, such that $\mathcal{Q}(s_{j_k})$ and $\mathcal{T}(s_{j_k})$ are converging to some $\tilde{\mathcal{Q}}$, $\tilde{\mathcal{T}}$. Therefore by Proposition 3.17:

$$0 \leq \lim_{k\to\infty} |\langle b(s_{j_k}), u_H \rangle + \langle \mathcal{Q}(s_{j_k})u_H, u_H \rangle + \mathcal{T}(s_{j_k})u_T| \leq \lim_{k\to\infty} s_{j_k}^{\frac{1}{4}} \|u\|^3 B'(s_{j_k}^{\frac{1}{4}}\|u\|) = 0.$$

This implies that for any $u \in \mathrm{supp}(\mu)$:

$$\lim_{k\to\infty} \langle b(s_{j_k}), u_H \rangle = -\langle \tilde{\mathcal{Q}}u_H, u_H \rangle - \tilde{\mathcal{T}}u_T. \tag{3.9}$$

Proposition 3.21. *There exists a $\overline{B} \in V$ such that $\lim_{k\to\infty} b(s_{j_k}) = \overline{B}$.*

Proof. First of all we note that $\langle b(s), v\rangle = 0$ for any $v \in V^{\perp}$, since $b(s_{j_k}) \in V$ for any $k \in \mathbb{N}$ (see Remark 3.18). Choose $\{u_1, \dots, u_l\} \subseteq \text{supp}(\mu)$, such that $(u_1)_H, \dots, (u_l)_H$ is a basis for V. Thus by equation (3.9), we have that:

$$\overline{B} = -\sum_{i=1}^{l} \left(\langle \tilde{\mathcal{Q}}(u_i)_H, (u_i)_H\rangle + \tilde{\mathcal{T}}(u_i)_T\right)(u_i)_H.$$

□

If the measure μ is invariant under dilations, finding a candidate (non-degenerate) quadric containing $\text{supp}(\mu)$ is quite easy:

Proposition 3.22. *If $\mu_{0,\lambda} = \mu$ for any $\lambda > 0$, then $b(s) = 0$ for any $s > 0$ and:*

$$\langle u_H, \mathcal{Q}(1)u_H\rangle + \mathcal{T}(1)u_T = 0,$$

for any $u \in \text{supp}(\mu)$.

Proof. In the proof of Proposition 3.17 we defined:

$$s^{\frac{1}{2}}\langle u_H, b(s)\rangle := c_{(1,0,0),s}(u),$$

and:

$$s^{\frac{1}{2}}(\langle u_H, \mathcal{Q}(s)u_H\rangle + \mathcal{T}(s)u_T) := c_{(2,0,0),s}(u) + c_{(0,1,0),s}(u).$$

Therefore Proposition 3.12 implies that:

$$s^{\frac{1}{2}}\langle u_H, b(s)\rangle = s^{\frac{1}{4}}\langle b(1), u_H\rangle.$$

and:

$$s^{\frac{1}{2}}(\langle u_H, \mathcal{Q}(s)u_H\rangle + \mathcal{T}(s)u_T) = s^{\frac{1}{2}}(\langle u_H, \mathcal{Q}(1)u_H\rangle + \mathcal{T}(1)u_T),$$

Therefore by Proposition 3.17 we have that:

$$\left|s^{-\frac{1}{4}}\langle b(1), u_H\rangle + \langle u_H, \mathcal{Q}(1)u_H\rangle + \mathcal{T}(1)u_T\rangle\right| \leq s^{\frac{1}{4}}\|u\|^3 B'(s^{\frac{1}{4}}\|u\|),$$

for any $u \in \text{supp}(\mu)$, any $s^{\frac{1}{4}}\|u\| \leq 1$. Sending s to 0 we deduce that $b(1) = 0$ and that:

$$\langle u_H, \mathcal{Q}(1)u_H\rangle + \mathcal{T}(1)u_T = 0.$$

□

Remark 3.23. Note that as we already mentioned, if $\mu = \mathcal{S}^2_{\mathcal{V}}$ where $\mathcal{V}$ is the vertical axis $\{z \in \mathbb{R}^{2n+1} : z_H = 0\}$, we have that μ is a 2-uniform measure but $\mathcal{Q}(1) = 0$. Therefore in this case our constructed quadric becomes trivial and thus not meaningful.

3.4 Non-degeneracy of the candidate quadric

The main result of this subsection can be stated as follows. Assume μ is a m-uniform measure in $\mathbb{H}^n$ for which:

$$\lim_{s\to 0} \text{Tr}(\mathcal{Q}(s)) = 0, \tag{3.10}$$

where $\mathcal{Q}(s)$ is the curve of symmetric matrices built in Proposition 3.17. Then:

$$\text{Tan}_m(\mu, \infty) = \{\mathcal{S}^2_{\mathcal{V}}\}.$$

It is not hard to show that the condition (3.10) is actually equivalent to:

$$b(s) \to 0, \qquad \mathcal{Q}(s) \to 0, \qquad \mathcal{T}(s) \to 0,$$

as $s \to 0$ and thus we are really characterizing uniform measures for which our construction fails. This part of the argument significantly parts ways with its Euclidean analogue (see Section 4 of [25]).

Proposition 3.24. *Let $\mathcal{Q}(s)$ be the matrix defined in Proposition 3.17. For any $s > 0$ the following equality holds:*

$$Tr(\mathcal{Q}(s)) = \frac{s^{\frac{m+2}{4}}}{C(m)} \int_{\mathbb{H}^n} |z_H|^2 (8s\|z\|^4 - (8+4n)) e^{-s\|z\|^4} d\mu(z).$$

Proof. Let $\{e_1, \ldots, e_{2n}\}$ be an orthonormal basis of $\mathbb{R}^{2n}$. Then:

$$\text{Tr}(\mathcal{Q}(s)) = \sum_{i=1}^{n} \langle e_i, \mathcal{Q}(s) e_i \rangle + \langle e_{i+n}, \mathcal{Q}(s) e_{i+n} \rangle.$$

Using the explicit expression for $\mathcal{Q}(s)$ we can compute both $\langle e_i, \mathcal{Q}(s) e_i \rangle$ for any $i = 1, \ldots, 2n$. If $1 \leq i \leq n$ we have:

$$\begin{aligned}\langle e_i, \mathcal{Q}(s) e_i \rangle = & -\frac{s^{\frac{1}{2}+\frac{m}{4}}}{C(m)} \int (4z_i^2 + 2|z_H|^2 + 4z_{i+n}^2) e^{-s\|z\|^4} d\mu(z) + \frac{s^{\frac{3}{2}+\frac{m}{4}}}{C(m)} \int (8|z_H|^4 z_i^2 + 8z_T^2 z_{i+n}^2) e^{-s\|z\|^4} d\mu(z) \\ & + \frac{s^{\frac{3}{2}+\frac{m}{4}}}{C(m)} \int_{\mathbb{H}^n} 8|z_H|^2 z_T (2z_{i+n} z_i) e^{-s\|z\|^4} d\mu(z).\end{aligned}$$

On the other hand if $n+1 \leq i \leq 2n$ another similar expression holds:

$$\begin{aligned}\langle e_{i+n}, \mathcal{Q}(s)[e_{i+n}] \rangle = & -\frac{s^{\frac{1}{2}+\frac{m}{4}}}{C(m)} \int (4z_{i+n}^2 + 2|z_H|^2 + 4z_i^2) e^{-s\|z\|^4} d\mu(z) \\ & + \frac{s^{\frac{3}{2}+\frac{m}{4}}}{C(m)} \int (8|z_H|^4 z_{i+n}^2 + 8z_T^2 z_i^2) e^{-s\|z\|^4} d\mu(z) \\ & - \frac{s^{\frac{3}{2}+\frac{m}{4}}}{C(m)} \int_{\mathbb{H}^n} 8|z_H|^2 z_T (2z_{i+n} z_i) e^{-s\|z\|^4} d\mu(z).\end{aligned}$$

Putting together the above computations with the definition of $\text{Tr}(\mathcal{Q}(s))$ we have:

$$\begin{aligned}\text{Tr}(\mathcal{Q}(s)) = & \sum_{i=1}^{n} -\frac{2s^{\frac{1}{2}+\frac{m}{4}}}{C(m)} \int (4z_{i+n}^2 + 2|z_H|^2 + 4z_i^2) e^{-s\|z\|^4} d\mu(z) \\ = & \frac{s^{\frac{m+2}{4}}}{C(m)} \int_{\mathbb{H}^n} |z_H|^2 (8s\|z\|^4 - (8+4n)) e^{-s\|z\|^4} d\mu(z).\end{aligned}$$

□

Proposition 3.25. *Let $f : (0, \infty) \to [0, \infty)$ be the function:*

$$f(s) := \int_{\mathbb{H}^n} |z_H|^2 e^{-s\|z\|^4} d\mu(z). \tag{3.11}$$

(i) If $\text{supp}(\mu) \not\subseteq \mathcal{V}$ one has $0 < f(s)$ for any $s > 0$,

(ii) *f is a smooth function and its derivatives are:*

$$f^{(i)}(s) = (-1)^i \int_{\mathbb{H}^n} \|z\|^{4i} |z_H|^2 e^{-s\|z\|^4} d\mu(z). \tag{3.12}$$

(iii) *$s^{\frac{m+2}{4}+i} f^{(i)}(\cdot)$ is a bounded function on $(0,\infty)$ for any $i \in \mathbb{N}$.*

Proof. Assume there exists $w \in \operatorname{supp}(\mu)$ such that $w_H \neq 0$. Then:

$$0 < \frac{|w_H|^2}{2} e^{-s\left(\frac{3\|w\|}{2}\right)^4} \mu(B_{\frac{|w_H|/2}{(}} w)) \leq \int_{\mathbb{H}^n} |z_H|^2 e^{-s\|z\|^4} d\mu(z) = f(s),$$

for any $s > 0$. The fact that f is smooth is proven showing that (3.12) holds and this is a standard application of the dominated convergence theorem. The last point is a direct consequence of Proposition 2.19 and the formula for $f^{(i)}$:

$$|s^{\frac{m+2}{4}+i} f^{(i)}(s)| \leq s^{\frac{m+2}{4}+i} \int_{\mathbb{H}^n} \|z\|^{4i+2} e^{-s\|z\|^4} d\mu(z) \leq \frac{m}{4} \Gamma\left(\frac{m+2}{4} + i\right).$$

□

Remark 3.26. Proposition 3.24 and Proposition 3.25 imply that:

(i) If $\operatorname{supp}(\mu) \not\subseteq \mathcal{V}$, we have $(-1)^i f^{(i)}(s) > 0$ for any $i \in \mathbb{N}$.

(ii) The expression of the trace can be rewritten as follows:

$$\operatorname{Tr}(\mathcal{Q}(s)) = \frac{s^{\frac{m+2}{4}}}{C(m)} \int_{\mathbb{H}^n} |z_H|^2 (8s\|z\|^4 - (8+4n)) e^{-s\|z\|^4} d\mu(z) = -8 \frac{s^{\frac{m+6}{4}}}{C(m)} f'(s) - (8+4n) \frac{s^{\frac{m+2}{4}}}{C(m)} f(s). \tag{3.13}$$

In particular this implies by Proposition 3.25 that $\operatorname{Tr}(\mathcal{Q}(s))$ is a smooth, bounded function on $(0,\infty)$.

Proposition 3.27. *The function f defined in* (3.11) *has the following representation by means of Tr$(\mathcal{Q}(\cdot))$:*

$$f(s) = -\frac{C(m)}{8s^{\frac{n+2}{2}}} \int_0^s \lambda^{\frac{2n-2-m}{4}} Tr(\mathcal{Q}(\lambda)) d\lambda,$$

for any $s > 0$.

Proof. Since f is smooth on $(0,\infty)$ by Proposition 3.25, the following equality holds true:

$$\frac{d}{ds}\left(s^{\frac{3}{2}+\frac{m}{4}} f(s)\right) = \left(\frac{3}{2} + \frac{m}{4}\right) s^{\frac{1}{2}+\frac{m}{4}} f(s) + s^{\frac{3}{2}+\frac{m}{4}} f'(s),$$

The expression for $\operatorname{Tr}(\mathcal{Q}(s))$ in terms of f and f' given in Remark 3.26 (ii) togheter with the above identity imply:

$$C(m)\operatorname{Tr}(\mathcal{Q}(s)) = -8s^{\frac{m+6}{4}} f'(s) - (8+4n)s^{\frac{m+2}{4}} f(s) = -8\frac{d}{ds}\left(s^{\frac{3}{2}+\frac{m}{4}} f(s)\right) + (4+2m-4n) s^{\frac{1}{2}+\frac{m}{4}} f(s). \tag{3.14}$$

Define now $g(s) := s^{\frac{3}{2}+\frac{m}{4}} f(s)$, and note that equation (3.14) becomes:

$$C(m)\operatorname{Tr}(\mathcal{Q}(s)) = -8g'(s) + (4+2m-4n) \frac{g(s)}{s}.$$

For any $\delta > 0$ the function $g(s)$ solves the following Cauchy problem:

$$\begin{cases} g'(s) + \frac{(2n-m-2)}{4s} g(s) = -\frac{C(m)}{8}\text{Tr}(\mathcal{Q}(s)), \\ g(\delta) = \delta^{\frac{3}{2}+\frac{m}{4}} f(\delta). \end{cases}$$

Such Cauchy Problem has an explicit unique solution on (δ, ∞) (as coeffcients are smooth, Lipschitz and the vector field is sublinear in g), which is:

$$h_\delta(s) = -\frac{C(m)}{8s^{\frac{2n-m-2}{4}}}\left[\int_\delta^s \lambda^{\frac{2n-m-2}{4}}\text{Tr}(\mathcal{Q}(\lambda))d\lambda + \delta^{\frac{2n+4}{4}} f(\delta)\right],$$

and coincides by uniqueness with g on (δ, ∞). Point (iii) of Proposition 3.25 implies that:

$$\left|\delta^{\frac{2n+4}{4}} f(\delta)\right| \leq C\delta^{\frac{2n+2-m}{4}},$$

for some $0 < C$. Moreover, since $m \leq 2n+1$ we have that $\delta^{\frac{2n+4}{4}} f(\delta) \to 0$ as $\delta \to 0$. This implies that for any fixed $s > 0$:

$$\begin{aligned} g(s) = \lim_{\delta\to 0} h_\delta(s) &= -\frac{C(m)}{8s^{\frac{2n-m-2}{4}}} \lim_{\delta\to 0}\left[\int_\delta^s \lambda^{\frac{2n-m-2}{4}}\text{Tr}(\mathcal{Q}(\lambda))d\lambda + \delta^{\frac{2n+4}{4}} f(\delta)\right] \\ &= -\frac{C(m)}{8s^{\frac{2n-m-2}{4}}}\int_0^s \lambda^{\frac{2n-m-2}{4}}\text{Tr}(\mathcal{Q}(\lambda))d\lambda, \end{aligned}$$

where the last equality comes from the fact that $|\cdot|^{\frac{2n-m-2}{4}}\text{Tr}(\mathcal{Q}(\cdot)) \in L^1([0,1])$ as $m \leq 2n+1$. □

Remark 3.28. Since $\text{Tr}(\mathcal{Q}(s))$ is bounded (see Remark 3.26(ii)), if:

$$\lim_{s\to 0} Tr(\mathcal{Q}(s)),$$

does not exists or exists non-zero, there is a sequence $\{s_j\}_{j\in\mathbb{N}}$ such that the trace of the matrices $\mathcal{Q}(s)$ converges to a non-zero value. Up to passing to a subsequence (for which $b(\cdot)$, $\mathcal{Q}(\cdot)$ and $\mathcal{T}(\cdot)$ converge) we can build $\overline{B}$, $\tilde{\mathcal{Q}}$ and $\tilde{\mathcal{T}}$ as in Proposition 3.24 for which $\text{Tr}(\mathcal{Q}) \neq 0$. This would imply that the quadric $\mathbb{K}(\bar{b}, \tilde{\mathcal{Q}}, \tilde{\mathcal{T}})$ would contain $\text{supp}(\mu)$ and would be non-degenerate. Therefore without loss of generality, in what follows we should always assume $\lim_{s\to 0} Tr(\mathcal{Q}(s)) = 0$.

Proposition 3.29. *Suppose that* $\lim_{s\to 0} Tr(\mathcal{Q}(s)) = 0$. *Then* $\lim_{s\to 0} s^{\frac{m+2}{4}} f(s) = 0$.

Proof. For any ϵ there exists a $\delta > 0$ such that for any $s \in (0,\delta)$ we have $|Tr(\mathcal{Q}(s))| \leq \epsilon$. In particular $Tr(\mathcal{Q}(s)) > -\epsilon$, and Proposition 3.27 implies that for $s \in (0,\delta)$:

$$f(s) = -\frac{C(m)}{8s^{\frac{n+2}{2}}}\int_0^s \lambda^{\frac{2n-2-m}{4}} Tr(\mathcal{Q}(\lambda))d\lambda < \frac{\epsilon C(m)}{8s^{\frac{n+2}{2}}}\int_0^s \lambda^{\frac{2n-2-m}{4}} d\lambda < \frac{\epsilon C(m)}{8s^{\frac{n+2}{2}}}\frac{s^{\frac{2n+2-m}{4}}}{\frac{2n+2-m}{4}} = \frac{\epsilon C(m)}{2(2n+2-m)s^{\frac{m+2}{4}}}.$$

Summing up, we proved that for any $s \in (0,\delta)$ we have $0 < s^{\frac{m+2}{4}} f(s) < \frac{\epsilon C(m)}{2(2n+2-m)}$. □

Proposition 3.30. *The following are equivalent:*

(i) $\lim_{s\to 0} s^{\frac{m+2}{4}} f(s) = 0$,

(ii) for any $\alpha > 0$ *there exists an* $R(\alpha) > 0$ *such that if* $R > R(\alpha)$, *then* $\text{supp}(\mu) \setminus B_R(0) \subseteq \{z : |z_H| \leq \alpha\|z\|\}$.

Proof. Suppose (ii) fails. Then there exists an $\alpha'>0$ such that for any $j\in\mathbb{N}$ there exists $y_j\in\mathrm{supp}(\mu)\setminus B_j(0)$ for which $|(y_j)_H|\geq\alpha'\|y_j\|$. We prove that along the sequence $s_j:=|(y_j)_H|^{-4}$, the function $s^{\frac{m+2}{4}}f(s)$ is bounded away from 0, which contradicts (i):

$$s_j^{\frac{m+2}{4}}\int_{\mathbb{H}^n}|z_H|^2e^{-s_j\|z\|^4}d\mu(z)\geq s_j^{\frac{m+2}{4}}\frac{|(y_j)_H|^2}{4}e^{-s_j\left(3\alpha'|(y_j)_H|/2\right)^4}\mu\left(B_{|(y_j)_H|/2}(y_j)\right)$$
$$=\frac{s_j^{\frac{m+2}{4}}}{2^{m+2}}e^{-s_j(3\alpha'|(y_j)_H|/2)^4}|(y_j)_H|^{m+2}\geq\frac{e^{-\frac{81(\alpha')^4}{16}}}{2^{m+2}},$$

where we used the fact that for any $z\in B_{|(y_j)_H|/2}(y_j)$ one has that $|(y_j)_H|/2\leq|z_H|$ and $\|z\|\leq3\alpha'|(y_j)_H|/2$. Viceversa suppose (ii) holds. This implies that for any $\alpha>0$:

$$s^{\frac{m+2}{4}}\int_{\mathbb{H}^n}|z_H|^2e^{-s\|z\|^4}d\mu(z)\leq s^{\frac{m+2}{4}}\int_{B_{R(\alpha)}(0)}\|z\|^2e^{-s\|z\|^4}d\mu(z)+s^{\frac{m+2}{4}}\alpha^2\int_{B^c_{R(\alpha)}(0)}\|z\|^2e^{-s\|z\|^4}d\mu(z),$$

The above computation, Proposition 2.18 and Proposition 2.19 imply that:

$$s^{\frac{m+2}{4}}\int_{\mathbb{H}^n}|z_H|^2e^{-s\|z\|^4}d\mu(z)\leq m\int_0^{s^{\frac{1}{4}}R(\alpha)}t^{m+1}e^{-t^4}dr+\alpha^2\frac{m}{4}\Gamma\left(\frac{m+2}{4}\right)$$

Therefore:

$$0\leq\limsup_{s\to0}s^{\frac{m+2}{4}}f(s)\leq\limsup_{s\to0}m\int_0^{s^{\frac{1}{4}}R(\alpha)}t^{m+1}e^{-t^4}dr+\alpha^2\frac{m}{4}\Gamma\left(\frac{m+2}{4}\right)\leq\alpha^2\frac{m}{4}\Gamma\left(\frac{m+2}{4}\right).$$

The arbitrariness of α concludes the proof. □

Proposition 3.31. *If* $\mathrm{supp}(\mu)\subseteq\mathcal{V}$, *then* $\mu=\mathcal{S}^2_{\mathcal{V}}$.

Proof. Since $\mathrm{supp}(\mu)\subseteq\mathcal{V}$, then $\mu(B_r(z))=\mu(B_r(z)\cap\mathcal{V})=r^m$, for any $z\in\mathrm{supp}(\mu)$ and any $r>0$. Note that:

$$B_r(z)\cap\mathcal{V}=\{(0,s)\in\mathbb{R}^{2n+1}:|s-z_T|<r^2\}=U_{r^2}(x)\cap\mathcal{V},$$

where as usual $U_{r^2}(z)$ denotes the Euclidean ball of radius r^2 and centre x. This implies that $\mu(U_r(z))=r^{\frac{m}{2}}$ and hence μ is a $m/2$-uniform measure with respect to Euclidean balls and which support is contained in the line $\mathcal{V}$. Mastrand theorem implies that $m/2$ must be an integer and since $\mathcal{V}$ is 1-dimensional, we deduce by differentiation that $m/2$ is either 0 or 1. As we escluded by hypoothesis $m=0$, we deduce by the classification of 1-uniform measures in $\mathbb{R}^n$ proved in [37] that $\mu=\frac{1}{2}\mathcal{H}^1_{eu}\llcorner\mathcal{V}$. Since $m=2$ and $\mathrm{supp}(\mu)=\mathcal{V}$, the result follows by Proposition 2.15. □

Proposition 3.32. *Suppose that for any* $\alpha>0$ *there exists an* $R(\alpha)>0$ *such that if* $R>R(\alpha)$, *then:*

$$\mathrm{supp}(\mu)\setminus B_R(0)\subseteq\{z:|z_H|\leq\alpha\|z\|\}.$$

Then $m=2$ *and* $\mathrm{Tan}_2(\mu,\infty)=\{\mathcal{S}^2_{\mathcal{V}}\}$.

Proof. Let $\eta\in\mathcal{C}^\infty_c(\mathbb{R}^{2n+1})$ such that $0\leq\eta\leq1$, $\eta(B_1(0))=1$ and $\eta(B_2(0)^c)=0$. For $r>0$ define:

$$\eta_R(z):=\eta\left(D_{1/R}(z)\right).$$

Let $\nu \in \mathrm{Tan}(\mu, \infty)$ and $\{\lambda_l\}_{l\in\mathbb{N}}$ be such that $\lambda_l \to \infty$ and $\frac{\mu_{0,\lambda_l}}{\lambda^m} \rightharpoonup \nu$. Then we have:

$$\int_{\mathbb{H}^n} \eta_R(z)|z_H|^2 e^{-\|z\|^2} d\nu(z) = \lim_{l\to\infty} \int_{\mathbb{H}^n} \eta_R(z)|z_H|^2 e^{-\|z\|^4} \frac{d\mu_{0,\lambda_l}(z)}{\lambda_l^m} \leq \lim_{l\to\infty} \int_{\mathbb{H}^n} \left|(D_{1/\lambda_l}(z))_H\right|^2 e^{-\|D_{1/\lambda_l}(z)\|^4} \frac{d\mu(z)}{\lambda_l^m}$$
$$= \lim_{l\to\infty} \frac{1}{\lambda_l^{m+2}} \int_{\mathbb{H}^n} |z_H|^2 e^{-\frac{\|z\|^4}{\lambda_l^4}} d\mu(z) = 0.$$

The last equality in the above computation is provided by Proposition 3.30 and our hypothesis on μ. The arbitrariness of $r > 0$ and dominated convergence theorem imply:

$$\int_{\mathbb{H}^n} |z_H|^2 e^{-\|z\|^2} d\nu(z) = 0.$$

Proposition 3.25 implies that $\mathrm{supp}(\nu) \subseteq \mathcal{V}$ and Proposition 3.31 implies that $\nu = \mathcal{S}^2_{\mathcal{V}}$. Therefore by Proposition 2.4 μ is a 2-uniform measure. ☐

As an immediate consequence we get:

Corollary 3.33. *Let $\mu \in \mathcal{U}_{\mathbb{H}^n}(m)$. If $\lim_{s\to 0} Tr(\mathcal{Q}(s)) = 0$, then $m = 2$ and:*

$$Tan_2(\mu, \infty) = \{\mathcal{S}^2_{\mathcal{V}}\}.$$

In particular for any $m \in \{1, \dots, 2n+1\} \setminus \{2\}$, there exist $b \in \mathbb{R}^{2n}$, $\mathcal{T} \in \mathbb{R}$ and $\mathcal{Q} \in Sym(n)$ with $Tr(\mathcal{Q}) \neq 0$ such that:

$$\mathrm{supp}(\mu) \subseteq \mathbb{K}(b, \mathcal{Q}, \mathcal{T}).$$

4. $(2n+1)$-uniform measures have no holes

Let μ be a $(2n+1)$-uniform measure (which should be considered fixed throughout the section) and $\mathbb{K}(b, \mathcal{Q}, \mathcal{T})$ be the non-degenerate quadric in which $\mathrm{supp}(\mu)$ is contained. The existence of such a quadric has been shown in Section 3. What is left to understand is whether $\mathrm{supp}(\mu)$ is some kind of very irregular set *inside* $\mathbb{K}(b, \mathcal{Q}, \mathcal{T})$ or if it has a better structure. To state our main result we need some notation. Let $F : \mathbb{R}^{2n+1} \to \mathbb{R}$ be the quadratic polynomial:

$$F(z) := \langle b, z_H \rangle + \langle z_H, \mathcal{Q} z_H \rangle + \mathcal{T} z_T, \tag{4.1}$$

whose zero-set is the quadric $\mathbb{K}(b, \mathcal{Q}, \mathcal{T})$. We define the set of singular points of $\mathbb{K}(b, \mathcal{Q}, \mathcal{T})$ as:

$$\Sigma(F) := \{x \in \mathbb{K}(b, \mathcal{Q}, \mathcal{T}) : b + 2(\mathcal{Q} - \mathcal{T} J) x_H = 0\}. \tag{4.2}$$

Usually $\Sigma(F)$ is called *characteristic set* if $\mathcal{T} \neq 0$ and *singular set* if $\mathcal{T} = 0$. We should not bother ourselves with such distinctions, and regar $\Sigma(F)$ as the set of points where $\mathbb{K}(b, \mathcal{Q}, \mathcal{T})$ behaves like a cone tip. The following theorem is the main result of this section and it should be regarded as an analogue of Proposition 4.3 in [25]. We show not only that $\mathrm{supp}(\mu)$ is not a fractal inside $\mathbb{K}(b, \mathcal{Q}, \mathcal{T})$ but also that it can be viewed as a quadratic surface with no holes:

Theorem 4.1. *The support of μ is (the closure of) a union of connected components of $\mathbb{K}(b, \mathcal{Q}, \mathcal{T}) \setminus \Sigma(F)$.*

We give now a short account of the content for each subsection. Subsection 4.1 is split in two parts. The main result of the first part is Proposition 4.5, where we show that everywhere outside $\Sigma(F)$, $(2n+1)$-uniform measures have flat blowups. Therefore, even though in principle μ may have holes, they are not inherited by tangents and therefore locally $\mathrm{supp}(\mu)$ behaves exactly as the whole surface $\mathbb{K}(b, \mathcal{Q}, \mathcal{T})$. The second part of Subsection 4.1 is devoted to the proof of Proposition 4.6, where we show that $\mathcal{T}$ is an invariant for μ: if $\mathcal{T} = 0$ then for any other quadric $\mathbb{K}(b', \mathcal{Q}', \mathcal{T}')$ containing $\mathrm{supp}(\mu)$ we have $\mathcal{T}' = 0$. This implies that there are two types of $(2n+1)$-uniform measures which are qualitatively different. If $\mathcal{T} = 0$, Theorem 4.1 implies that $\mathrm{supp}(\mu)$ is vertically ruled, and hence invariant by translations with elements of the centre, while if $\mathcal{T} \neq 0$ then $\mathrm{supp}(\mu)$ coincides with $\mathbb{K}(b, \mathcal{Q}, \mathcal{T})$, which is a t-graph, for a precise statement see Proposition 4.8.

Subsection 4.2 and Subsection 4.3 are completely devoted to the proof of Theorem 4.1 in the cases $\mathcal{T} \neq 0$ and $\mathcal{T} = 0$, respectively. The idea behind the proof is the same in both situations: pick a point $x \in \mathrm{supp}(\mu) \setminus \Sigma(F)$ for which for any $r > 0$ we have:

$$B_r(x) \cap \mathrm{supp}(\mu)^c \cap \mathbb{K}(b, \mathcal{Q}, \mathcal{T}) \neq \emptyset,$$

and show that the holes in the support pass to the blowup, contradicting the mentioned fact that at points of $\mathrm{supp}(\mu)$ outside $\Sigma(F)$ the tangent measure are planes.

4.1 Regularity of $(2n+1)$-uniform measures

The first part of this subsection is devoted to the study of the local properties of μ. First of all we introduce the definition of vertical hyperplane and flat measure:

Definition 4.2. For any $\mathfrak{n} \in \mathbb{R}^{2n}$ we define:

$$V(\mathfrak{n}) := \{z \in \mathbb{H}^n : \langle \mathfrak{n}, z_H \rangle = 0\},$$

and we say that $V(\mathfrak{n})$ is the *vertical hyperplane* orthogonal to $\mathfrak{n}$. A $(2n+1)$-uniform measure ν is said to be a *flat measure*, or simply *flat*, if:

$$\nu = \mathcal{S}^{2n+1}_{V(\mathfrak{n})},$$

for some $\mathfrak{n} \in \mathbb{R}^{2n}$.

Remark 4.3. The measure $\mathcal{S}^{2n+1}_{V(\mathfrak{n})}$ is invariant under translation by the elements of $V(\mathfrak{n})$. Let E be a Borel subset of $V(\mathfrak{n})$ and assume that the balls $\{B_i\}_{i\in\mathbb{N}}$, centred at points of $V(\mathfrak{n})$, are a countable cover of E. For any $v \in V(\mathfrak{n})$ the balls $\{v * B_i\}_{i\in\mathbb{N}}$ are still centred at points of $V(\mathfrak{n})$, are a countable cover of $v * E$ and:

$$\sum_{i\in\mathbb{N}} \mathrm{diam}(v * B_i) = \sum_{i\in\mathbb{N}} \mathrm{diam}(B_i),$$

since translations are isometries for the Koranyi metric. This in particular implies that $\mathcal{S}^{2n+1}_{V(\mathfrak{n})}(E) = \mathcal{S}^{2n+1}_{V(\mathfrak{n})}(v*E)$. The measure $\mathcal{S}^{2n+1}_{V(\mathfrak{n})}$ is also $(2n+1)$-uniform, indeed Proposition A.2 together with Corollary 7.6 in [23] imply that for any $r > 0$:

$$\mathcal{S}^{2n+1}_{V(\mathfrak{n})}(B_r(0)) = \frac{\mathcal{H}^{2n-2}_{eu}(B_r(x))}{\omega_{2n}} = r^{2n+1}, \tag{4.3}$$

where ω_{2n} is the volume of the unitary $2n$-dimensional Euclidean ball. In (4.3) the constant in the last equality does not coincide with the one of Corollary 7.6 in [23]. This is due to the fact that we are not using the same metric.

The following proposition shows that $(2n+1)$-uniform measure with support contained in a vertical hyperplane are flat. It is an adaptation of Remark 3.14 in [15].

Proposition 4.4. *Suppose that μ is a $(2n+1)$-uniform measure for which there exists an $\mathfrak{n} \in \mathbb{R}^{2n}$ for which $\mathrm{supp}(\mu) \subseteq V(\mathfrak{n})$. Then:*

$$\mu = \mathcal{S}^{2n+1}_{V(\mathfrak{n})}.$$

Proof. By Proposition 2.15 for any $x \in \mathrm{supp}(\mu)$ and any $r > 0$:

$$\mathcal{S}^{2n+1}_{\mathrm{supp}(\mu)}(B_r(x)) = \mu(B_r(x)) = r^{2n+1}.$$

Therefore, thanks to (4.3), for any $r > 0$ we have:

$$\mathcal{S}^{2n+1}_{\mathrm{supp}(\mu)}(B_r(0)) = \mathcal{S}^{2n+1}_{V(\mathfrak{n})}(B_r(0)).$$

Since $\mathrm{supp}(\mu)$ is closed in $V(\mathfrak{n})$, we deduce that $\mathrm{supp}(\mu) = V(\mathfrak{n})$: if a ball was missing somewhere the above equality would not be possible. $\square$

The following proposition shows that outside $\Sigma(F)$ tangents to μ are flat measures.

Proposition 4.5. *For any $x \in \mathrm{supp}(\mu) \setminus \Sigma(F)$ we have:*

$$Tan_{2n+1}(\mu, x) = \{\mathcal{S}^{2n+1}_{V(\mathfrak{n}(x))}\},$$

where $\mathfrak{n}(x) := b + 2(\mathcal{Q} - \mathcal{T}J)x_H$.

Proof. By Proposition 2.7 for any $x \in \mathrm{supp}(\mu)$ the set $\mathrm{Tan}_{2n+1}(\mu, x)$ is non-empty and it is contained in $\mathcal{U}_{\mathbb{H}^n}(2n+1)$. Pick any $\nu \in \mathrm{Tan}_{2n+1}(\mu, x)$ and recall that by definition of tangent, there exist $r_i \to 0$ such that $\mu_{x,r_i}/r_i^{2n+1} \rightharpoonup \nu$. Therefore Proposition 2.6 implies that for any $y \in \mathrm{supp}(\nu)$ there exists a sequence $\{x_i\} \subseteq \mathrm{supp}(\mu)$, for which $D_{1/r_i}(x^{-1}x_i) \to y$. Defined $y_i := D_{1/r_i}(x^{-1}x_i)$, we have that $x_i = xD_{r_i}(y_i)$ and thus for any $i \in \mathbb{N}$:

$$0 = \langle b, (x_i)_H\rangle + \langle (x_i)_H, \mathcal{Q}(x_i)_H\rangle + \mathcal{T}(x_i)_T = r_i\langle b, (y_i)_H\rangle + r_i\langle 2(\mathcal{Q} - \mathcal{T}J)x_H, (y_i)_H\rangle + r_i^2\langle (y_i)_H, \mathcal{Q}(y_i)_H\rangle.$$

Since $y_i \to y$, dividing by r_i and taking the limit as $i \to \infty$ we deduce:

$$0 = \langle b + 2(\mathcal{Q} - \mathcal{T}J)x_H, y_H\rangle.$$

This implies that for any $\nu \in \mathrm{Tan}_{2n+1}(\mu, x)$, we have $\mathrm{supp}(\nu) \subseteq V(\mathfrak{n}(x))$. The claim follows by Proposition 4.4. □

In the upcoming proposition we show that $\mathcal{U}_{\mathbb{H}^n}(2n+1)$ is split in two families that are characterized by the coefficient $\mathcal{T}$.

Proposition 4.6. *Suppose that there are $\overline{b} \in \mathbb{R}^{2n}$, $\overline{\mathcal{Q}} \in Sym(2n) \setminus \{0\}$ and $\overline{\mathcal{T}} \in \mathbb{R}$ such that:*

$$\mathrm{supp}(\mu) \subseteq \mathbb{K}(\overline{b}, \overline{\mathcal{Q}}, \overline{\mathcal{T}}).$$

Then $\mathcal{T} = 0$ if and only if $\overline{\mathcal{T}} = 0$.

Proof. Assume by contradiction that $\mathcal{T} \neq 0$ and $\overline{\mathcal{T}} = 0$. Since:

$$\mathrm{supp}(\mu) \subseteq S := \mathbb{K}(b, \mathcal{Q}, \mathcal{T}) \cap \mathbb{K}(\overline{b}, \overline{\mathcal{Q}}, 0),$$

by Proposition 2.15 and Remark 2.14, we have that:

$$\mu(B_r(0)) = \mathcal{S}^{2n+1}_{\mathrm{supp}(\mu)}(B_r(0)) \leq 2^{2n+1}\mathcal{S}^{2n+1}_{\mathbb{K}(b,\mathcal{Q},\mathcal{T})}(S \cap B_r(0)).$$

Note that the projection $\pi_H(S)$ has $\mathcal{L}^{2n}$-measure 0 in $\mathbb{R}^{2n}$. Therefore proposition A.5 implies that $\mathcal{S}^{2n+1}_{\mathbb{K}(b,\mathcal{Q},\mathcal{T})}(S \cap B_r(0)) = 0$, which contradicts the fact that $0 \in \mathrm{supp}(\mu)$. □

Definition 4.7. If there exist $b \in \mathbb{R}^{2n}$ and $\mathcal{Q} \in \mathrm{Sym}(2n) \setminus \{0\}$ such that:

$$\mathrm{supp}(\mu) \subseteq \mathbb{K}(b, \mathcal{Q}, 0),$$

then μ is said to be a *vertical uniform measure*. If such b and $\mathcal{Q}$ do not exist μ is said to be a *horizontal uniform measure*.

4.2 Structure of the support of horizontal uniform measures

In this subsection we prove Theorem 4.1 in case μ is a horizontal uniform measure and therefore throughout this subsection we assume $\mathcal{T} \neq 0$. Let $f : \mathbb{R}^{2n} \to \mathbb{R}$ be the smooth function:

$$f(h) := -\frac{\langle h, \mathcal{Q}h\rangle + \langle b, h\rangle}{\mathcal{T}}. \tag{4.4}$$

Since $\mathrm{supp}(\mu) \subseteq \mathrm{gr}(f)$, we deduce that:

$$\mathrm{supp}(\mu) \cap \Sigma(F) = \{(h, f(h)) \in \mathbb{R}^{2n+1} : h \in \Sigma(f)\},$$

where $\Sigma(f)$ is the set of characteristic points of f (see Subsection A in the Appendix for a more extensive explanation):

$$\Sigma(f) := \{x \in \mathbb{R}^{2n} : b + 2(\mathcal{Q} - \mathcal{T}J)h = 0\}. \tag{4.5}$$

Thanks to Proposition A.4, if $n > 1$ then $\Sigma(f)$ cannot disconnect $\mathbb{R}^{2n}$, as $\Sigma(f)$ is an affine space of dimension less then n. However if $n = 1$ there might be the case in which $\mathbb{R}^2 \setminus \Sigma(f)$ is split in two connected components, which we will denote in the following by C_i, with $i = 1, 2$. The following proposition is Theorem 4.1 in case μ is a horizontal measure. The idea behind the proof is to tuck cylinders inside holes of $\mathrm{supp}(\mu)$ in such a way that they are also tangent to $\mathrm{supp}(\mu)$ is some point outside $\Sigma(F)$ and to show with some careful computations that the tangents to μ at the point of tangency cannot be flat.

Proposition 4.8. *If $n > 1$ then $\mathrm{supp}(\mu) = \mathbb{K}(b, \mathcal{Q}, \mathcal{T})$. If $n = 1$ we have two cases:*

(i) if $\dim(\Sigma(f)) = 0$, then $\mathrm{supp}(\mu) = \mathbb{K}(b, \mathcal{Q}, \mathcal{T})$,

(ii) if $\dim(\Sigma(f)) = 1$, then either $\mathrm{supp}(\mu) = \mathbb{K}(b, \mathcal{Q}, \mathcal{T})$ or it coincides with the closure of the graph of $f|_{C_1}$ or $f|_{C_2}$.

Proof. Since $\mathrm{supp}(\mu)$ is a closed set and it is contained in $\mathbb{K}(b, \mathcal{Q}, \mathcal{T})$, then $S := \{p \in \mathbb{R}^{2n} : (p, f(p)) \in \mathrm{supp}(\mu)\}$ is closed in $\mathbb{R}^{2n}$. For any $y \in \mathbb{K}(b, \mathcal{Q}, \mathcal{T})$, there exists $z(y) \in S$ (possibly coinciding with y_H) such that:

$$|z(y) - y_H| = \mathrm{dist}_{\mathrm{eu}}(y_H, S).$$

As a consequence the (possibly empty) cylinder:

$$c_{|z(y)-y_H|}(y) := \{(x, t) \in \mathbb{R}^{2n} \times \mathbb{R} : |x - y_H| < |z(y) - y_H|\},$$

does not intersect $\mathrm{supp}(mu)$.

We claim that for any $y \in \mathbb{K}(b, \mathcal{Q}, \mathcal{T}) \setminus (\Sigma(F) \cup \mathrm{supp}(\mu))$ the tangency point $z(y)$ is contained in $\Sigma(f)$.

Let us prove the proposition assuming that the above claim holds. If $\Sigma(f) = \emptyset$, then $\mathrm{supp}(\mu) = \mathbb{K}(b, \mathcal{Q}, \mathcal{T})$ (as y cannot exist). Hence, without loss of generality we can assume $\Sigma(f) \neq \emptyset$.

If $n > 1$ then $\Omega := \mathbb{R}^{2n} \setminus \Sigma(f)$ is a connected open set and S is relatively closed inside it. Suppose that there exists a $p \in S \cap \Omega$ such that for any $r > 0$ there exists $q_r \in S^c \cap U_r(p)$. This would imply (for a sufficiently small $r > 0$):

$$\mathrm{dist}_{\mathrm{eu}}(S, q_r) < r < \mathrm{dist}_{\mathrm{eu}}(\Sigma(f), q_r),$$

contradicting the fact that if $z \in S$ satisfies $|z - p| = \mathrm{dist}_{\mathrm{eu}}(S, q_r)$ then $z \in \Sigma(f)$. Note that the same argument works in the remaining cases in which $n = 1$. If $\dim(\Sigma(f)) = 1$, where we apply the reasoning above to the connected components C_1 and C_2.

Let us now prove the claim. In order to ease notation in the following we define $\zeta := (z, f(z)) \in \mathrm{supp}(\mu)$. If by contradiction $z \notin \Sigma(f)$, then $\zeta \notin \Sigma(F)$ and thus Proposition 4.5 implies that:

$$\mathrm{Tan}_{2n+1}(\mu, \zeta) = \{\mathcal{S}^{2n+1} \llcorner V(\mathfrak{n}(\zeta))\},$$

where $\mathfrak{n}(\zeta) = -(b + 2(\mathcal{Q} - \mathcal{T}J)z)/\mathcal{T}$. Moreover Proposition 2.6 implies that if $w \in V(\mathfrak{n}(\zeta))$ and $r_i \to 0$, there exists a sequence $\{v_i\}_{i \in \mathbb{N}} \subseteq \mathrm{supp}(\mu)$ such that:

$$w_i := D_{1/r_i}(\zeta^{-1} v_i) \to w. \tag{4.6}$$

Since $v_i \notin c_{|z-y_H|}(y)$ by construction, we deduce that:

$$|z - y_H| \leq |(\zeta D_{r_i}(w_i))_H - y_H|,$$

which reduces to:

$$0 \leq 2\langle z - y_H, (w_i)_H \rangle + r_i |(w_i)_H|^2.$$

Taking the limit as $i \to \infty$ we deduce that:

$$\mathrm{supp}(\nu) \subseteq V(\mathfrak{n}(\zeta)) \cap \{w \in \mathbb{R}^{2n+1} : \langle z - y_H, w_H \rangle \geq 0\}.$$

If $z - y_H$ is not parallel to $\mathfrak{n}(\zeta)$ this would contradict Proposition 4.5 as $\mathrm{supp}(\nu)$ would be contained in a proper subset of $V(\mathfrak{n}(\zeta))$. This implies that there exists $\lambda \neq 0$ for which $z - y_H = \lambda \mathfrak{n}(\zeta)$. From (4.6) , we deduce that:

(α) $(v_i)_H = z + r_i w_H + R_i$, with $|R_i| = o(r_i)$,

(β) $r_i^2 w_T + o(r_i^2) = (v_i)_T - \zeta_T - 2\langle z, J(v_i)_H \rangle$.

Hence putting together (α) and (β), we get:

$$(v_i)_T = \zeta_T + 2r_i \langle z, Jw_H \rangle + 2\langle z, JR_i \rangle + r_i^2 w_T + o(r_i^2).$$

Since $\mathrm{supp}(\mu) \subseteq \mathrm{gr}(f)$, using the definition of f and (α), we deduce that:

$$\mathcal{T}(v_i)_T = \mathcal{T}\zeta_T - r_i \langle b + 2\mathcal{Q}z, w_H \rangle - \langle b + 2\mathcal{Q}z, R_i \rangle - r_i^2 \langle w_H, Qw_H \rangle + o(r_i^2).$$

The above computations, (β) and the fact that $w \in V(\mathfrak{n}(\zeta))$ imply:

$$\begin{aligned} r_i^2 w_T + o(r_i^2) =& r_i \langle \mathfrak{n}(\zeta), w_H \rangle + \langle \mathfrak{n}(\zeta), R_i \rangle - r_i^2 \frac{\langle w_H, \mathcal{Q}w_H \rangle}{\mathcal{T}} + o(r_i^2) \\ =& \langle \mathfrak{n}(\zeta), R_i \rangle - r_i^2 \frac{\langle w_H, \mathcal{Q}w_H \rangle}{\mathcal{T}} + o(r_i^2). \end{aligned}$$

Therefore recollecting terms the above equality to:

$$\langle \mathfrak{n}(\zeta), R_i \rangle = r_i^2 \left(\frac{\langle w_H, \mathcal{Q}w_H \rangle}{\mathcal{T}} + w_T \right) + o(r_i^2). \tag{4.7}$$

On the other hand, (α) and the definition of w_i imply that $(w_i)_H = w_i + \frac{R_i}{r_i}$. Since $w_i \notin c_{|z-y_H|}(y)$, we have:

$$0 \leq 2\langle z - y_H, (w_i)_H \rangle + r_i |(w_i)_H|^2 = 2\lambda \left\langle \mathfrak{n}(\zeta), \frac{R_i}{r_i} \right\rangle + r_i \left| w_H + \frac{R_i}{r_i} \right|^2,$$

where the last line comes from the fact that $w \in V(\mathfrak{n}(\zeta))$. Using (4.7) and dividing by r_i, we deduce:

$$\begin{aligned} 0 \leq& 2\lambda \left(\frac{\langle w_H, \mathcal{Q}w_H \rangle}{\mathcal{T}} + w_T \right) + \left| w_H + \frac{R_i}{r_i} \right|^2 + \frac{o(r_i)}{r_i} \\ =& 2\lambda \left(\frac{\langle w_H, \mathcal{Q}w_H \rangle}{\mathcal{T}} + w_T \right) + |w_H|^2 + \frac{o(r_i)}{r_i}. \end{aligned}$$

Sending $i \to \infty$, we get:

$$-\lambda w_T \leq \lambda \frac{\langle w_H, \mathcal{Q}w_H \rangle}{T} + |w_H|^2,$$

which constitutes a non-trivial bound on w_T and this contradicts Proposition 4.5. This implies that $z \in \Sigma(f)$. $\square$

4.3 Structure of the support of vertical uniform measures

In this subsection we prove Theorem 4.1 in case μ is a vertical uniform measure. First of all we need to establish the relation between the centre of an Euclidean ball tangent to $\mathrm{supp}(\mu)$ and the point of tangency.

Proposition 4.9. *Let $y \in \mathbb{K}(b, \mathcal{Q}, 0) \setminus \mathrm{supp}(\mu)$ and $\zeta \in \mathrm{supp}(\mu) \setminus \Sigma(F)$ be such that:*

$$|\zeta - y| = \mathrm{dist}_{\mathrm{eu}}(y, \mathrm{supp}(\mu)).$$

Then:

(i) $\zeta_T = y_T$,

(ii) there exists $\lambda \neq 0$ such that $\lambda(\zeta_H - y_H) = 2\mathcal{Q}\zeta_H + b$.

Proof. Since $\zeta \notin \Sigma(F)$, Proposition 4.5 implies that:

$$\mathrm{Tan}_{2n+1}(\mu, \zeta) = \{\mathcal{S}^{2n+1}_{V(2\mathcal{Q}\zeta_H+b)}\}.$$

By Proposition 2.6, for any $w \in V(2\mathcal{Q}\zeta_H + b)$ and any $r_i \to 0$, there exists a sequence $\{v_i\}_{i\in\mathbb{N}} \subseteq \mathrm{supp}(\mu)$ such that $w_i := D_{1/r_i}(\zeta^{-1}v_i) \to w$. Therefore writing this convergence componentwise, we have:

(α) $(v_i)_H = \zeta_H + r_i w_H + R_i$, with $|R_i| = o(r_i)$,

(β) $r_i^2 w_T + o(r_i^2) = (v_i)_T - \zeta_T - 2\langle \zeta_H, J(v_i)_H\rangle$.

Putting together conditions (α) and (β) we deduce that:

$$(v_i)_T = \zeta_T + 2r_i\langle \zeta_H, Jw_H\rangle + 2\langle \zeta_H, JR_i\rangle + r_i^2 w_T + o(r_i^2).$$

The fact that $v_i \notin U_{|\zeta-y|}(y)$ and the above expression for $(v_i)_T$ imply that:

$$\begin{aligned} 0 \leq & 2r_i\langle \zeta_H - y_H - 2(\zeta_T - y_T)J\zeta_H, w_H\rangle + 2\langle \zeta_H - y_H, R_i\rangle \\ & +|r_i w_H + R_i|^2 + 2(\zeta_T - y_T)(2\langle \zeta_H, JR_i\rangle + r_i^2 w_T + o(r_i^2)) \\ & +(2r_i\langle \zeta_H, Jw_H\rangle + 2\langle \zeta_H, JR_i\rangle + r_i^2 w_T + o(r_i^2))^2, \end{aligned} \tag{4.8}$$

for any $i \in \mathbb{N}$. Define $N := \zeta_H - y_H - 2(\zeta_T - y_T)J\zeta_H$. If $N = 0$, dividing the above inequality by r_i^2 and sending $i \to \infty$, we get:

$$0 \leq |w_H|^2 + 2(\zeta_T - y_T)w_T + 4\langle \zeta_H, Jw_i\rangle^2. \tag{4.9}$$

If $N = 0$ then $\zeta_T \neq y_T$, otherwise we would have that $\zeta = y$ and this is not possible by the choice of ζ and y. Therefore if $N = 0$, inequality (4.9) constitutes a non-trivial bound on w_T which is in contradiction with Proposition 4.5.

On the other hand, if $N \neq 0$, dividing by r_i inequality (4.8) and sending $i \to \infty$, we deduce that $0 \leq \langle N, w_H\rangle$. This implies that:

$$V(2\mathcal{Q}\zeta_H + b) \subseteq \{(x,t) : \langle N, x\rangle \geq 0\},$$

and therefore there exists $\lambda \neq 0$ such that $\lambda N = 2\mathcal{Q}\zeta_H + b$. Therefore (4.8):

$$\begin{aligned} 0 \leq & 2\langle N, R_i\rangle + |r_i w_H + R_i|^2 + 2(\zeta_T - y_T)(r_i^2 w_T + o(r_i^2)) \\ & +(2r_i\langle \zeta_H, Jw_H\rangle + 2\langle \zeta_H, JR_i\rangle + r_i^2 w_T + o(r_i^2))^2. \end{aligned} \tag{4.10}$$

The condition $v_i \in \mathbb{K}(b, \mathcal{Q}, 0)$ implies:

$$0 = \langle 2\mathcal{Q}\zeta_H + b, R_i\rangle + r_i^2\langle w_H, \mathcal{Q}w_H\rangle + 2r_i\langle w_H, QR_i\rangle + \langle R_i, \mathcal{Q}R_i\rangle,$$

which, together with the fact that $\lambda N = 2\mathcal{Q}\zeta_H + b$, yields:

$$-\lambda\langle N, R_i\rangle = r_i^2\langle w_H, \mathcal{Q}w_H\rangle + 2r_i\langle w_H, QR_i\rangle + \langle R_i, \mathcal{Q}R_i\rangle.$$

Using the above information, (4.10) becomes:

$$\begin{aligned}0 \leq -\frac{2}{\lambda}(r_i^2\langle w_H, \mathcal{Q}w_H\rangle &+ 2r_i\langle w_H, QR_i\rangle + \langle R_i, \mathcal{Q}R_i\rangle) + |r_i w_H + R_i|^2 + 2(\zeta_T - y_T)(r_i^2 w_T + o(r_i^2)) \\ &+ (2r_i\langle \zeta_H, Jw_H\rangle + 2\langle \zeta_H, JR_i\rangle + r_i^2 w_T + o(r_i^2))^2.\end{aligned}$$

Dividing the above inequality by r_i^2 and sending $i \to \infty$, we deduce that:

$$0 \leq -\frac{2}{\lambda}\langle w_H, \mathcal{Q}w_H\rangle + |w_H|^2 + 2(\zeta_T - y_T)w_T + \langle \zeta_H, Jw_H\rangle^2,$$

which if $\zeta_T \neq y_T$ isa non-trivial bound on w_T. This contradicts Proposition 4.5. Therefore $\zeta_T = y_T$ and thus:

$$\lambda(\zeta_H - y_H) = \lambda N = 2\mathcal{Q}\zeta_H + b.$$

□

In the following proposition we prove Theorem 4.1 in case μ is a vertical measure. The idea behind the proof is the following. Let ζ and y be as in the statement of Proposition 4.9. If $|\zeta_H - y_H|$ is small then the vector $\zeta_H - y_H$ roughly lies in the tangent space of $\mathbb{K}(b, \mathcal{Q}, 0)$ at ζ. Therefore (ii) of Proposition 4.9 implies that such a vector in the tangent space to $\mathbb{K}(b, \mathcal{Q}, 0)$ at ζ should be parallel to the normal to $\mathbb{K}(b, \mathcal{Q}, 0)$ at ζ, which is clearly not possible.

Proposition 4.10. *Assume C is a connected component of $\mathbb{K}(b, \mathcal{Q}, 0) \setminus \Sigma(F)$. Then:*

(i) either $\operatorname{supp}(\mu) \cap C = \emptyset$,

(ii) or $C \subseteq \operatorname{supp}(\mu)$.

Proof. If μ is flat there is nothing to prove. Therefore thanks to Proposition A.8 we can assume without loss of generality that:

$$\mathcal{S}^{2n+1}(\Sigma(F) \cap \mathbb{K}(b, \mathcal{Q}, 0)) = 0.$$

The set $C \cap \operatorname{supp}(\mu)$ is relatively closed in C, thus if it is also relatively open in C, by connectedness either $C \cap \operatorname{supp}(\mu) = \emptyset$ or $C \cap \operatorname{supp}(\mu) = C$. By contradiction suppose that this is not the case, and thus there exist $x \in \operatorname{supp}(\mu) \cap C$ and a $r_0 > 0$ such that:

(α) for any $0 < r < r_0$ there exists $y_r \in \operatorname{supp}(\mu)^c \cap C$,

(β) $\mathrm{cl}(U_{r_0}(x)) \cap \mathbb{K}(b, \mathcal{Q}, 0) = \mathrm{cl}(U_{r_0}(x)) \cap C$.

Thus, for any $0 < r < r_0$ there exists $\zeta_r \in \operatorname{supp}(\mu) \in B_r(x)$ such that:

$$|\zeta_r - y_r| = \mathrm{dist}_{\mathrm{eu}}(y_r, \operatorname{supp}(\mu)).$$

By Proposition 4.9, we deduce that for any $0 < r < r_0$:

(i) $(\zeta_r)_T = (y_r)_T$,

(ii) there exists $\lambda_r \neq 0$ such that:

$$\lambda_r((\zeta_r)_H - (y_r)_H) = 2\mathcal{Q}(\zeta_r)_H + b. \tag{4.11}$$

As $\zeta_r, y_r \in \mathbb{K}(b, \mathcal{Q}, 0)$ we have that:

$$0 = \langle (y_r)_H - (\zeta_r)_H, \mathcal{Q}[(y_r)_H - (\zeta_r)_H] \rangle + \langle 2\mathcal{Q}[(\zeta_r)_H] + b, \pi_H y_r - (\zeta_r)_H \rangle.$$

Therefore for a sufficiently small $r > 0$, equation (4.11) implies:

$$1 = \left| \left\langle \frac{(\zeta_r)_H - (y_r)_H}{|(\zeta_r)_H - (y_r)_H|}, \frac{2\mathcal{Q}(\zeta_r)_H + b}{|2\mathcal{Q}(\zeta_r)_H + b|} \right\rangle \right| = \frac{|\langle (y_r)_H - (\zeta_r)_H, \mathcal{Q}[(y_r)_H - (\zeta_r)_H] \rangle|}{|(y_r)_H - (\zeta_r)_H||2\mathcal{Q}(\zeta_r)_H + b|} \leq \frac{\|\mathcal{Q}\||(y_r)_H - (\zeta_r)_H|}{|2\mathcal{Q}(\zeta_r)_H + b|}.$$

However, since $|y_r - (\zeta_r)_H|$ converges to 0 and $2\mathcal{Q}(\zeta_r)_H + b$ converges to $2\mathcal{Q}x_H + b \neq 0$ as r tends to zero, we have a contradiction. □

5. Disconnectedness of $(2n+1)$-uniform cones implies rigidity of tangents

In this section we reduce the problem of establishing the flatness of blowups of measures with $(2n+1)$-density to the study of some properties of $(2n+1)$-uniform cones, that we introduce in the following:

Definition 5.1. An m-uniform measure μ on $\mathbb{H}^n$ is said to be an m-uniform cone if $\mu_{0,\lambda} = \mu$, for any $\lambda > 0$. We denote by $\mathcal{C}_{\mathbb{H}^n}(m)$ the set of m-uniform cones.

Such reduction consists in constructing a continuous functional on Radon measures which "disconnects" $(2n+1)$-flat measures to the non-flat $(2n+1)$-uniform cones in the following way:

Theorem 5.2. *Suppose that there exists a functional $\mathscr{F} : \mathcal{M} \to \mathbb{R}$, continuous in the weak-$*$ convergence of measures, and a constant $\hbar = \hbar(\mathbb{H}^n) > 0$ such that:*

(i) if $\mu \in \mathfrak{M}(2n+1)$ then $\mathscr{F}(\mu) \leq \hbar/2$,

(ii) if $\mu \in \mathcal{C}_{\mathbb{H}^n}(2n+1)$ and $\mathscr{F}(\mu) \leq \hbar$, then $\mu \in \mathfrak{M}(2n+1)$.

Then, for any ϕ Radon measure with $(2n+1)$-density and for ϕ-almost every x:

$$Tan_{2n+1}(\phi, x) \subseteq \Theta^{2n+1}(\phi, x)\mathfrak{M}(2n+1).$$

The proof of Theorem 5.2 relies on the following two properties of $(2n+1)$-uniform measures:

P.1 if $\mathrm{Tan}_{2n+1}(\mu, \infty) \cap \mathfrak{M}(2n+1) \neq \emptyset$ then $\mu \in \mathfrak{M}(2n+1)$,

P.2 the set $\mathrm{Tan}_{2n+1}(\mu, \infty)$ is a singleton.

In the Euclidean case, these properties are algebraic consequences of the development of moments. For instance the proof of **P.2** in $\mathbb{R}^n$ is quite immediate (see Theorem 3.6(2) of [37]), but it really relies on the fact that moments are symmetric multilinear functions. In $\mathbb{H}^n$ the structure of moments is much more complicated because they are not multilinear. This is the reason why we could prove these properties only in the codimension 1 case, where fairly strong structure results for $\mathrm{supp}(\mu)$ are available (see Section 4).

In Subsections 5.1 and 5.2 we will establish properties **P.1** and **P.2**, respectively, while in Subsection 5.3 we will prove Theorem 5.2.

5.1 Flatness at infinity implies flatness

In this section we prove **P.1**. As a first step, we show that if μ is a uniform measure whose support is contained in $\mathbb{K}(b, \mathcal{Q}, \mathcal{T})$, then any $\nu \in \mathrm{Tan}_{2n+1}(\mu, \infty)$ has support contained in $\mathbb{K}(0, \mathcal{Q}, \mathcal{T})$. This implies that if μ

has a flat tangent at infinity then $\mathbb{K}(0,\mathcal{Q},\mathcal{T})$ must contain a hyperplane. This is only possible when $\mathrm{rk}(\mathcal{Q})=1,2$ (see the proof of Theorem 5.10) and $\mathcal{T}=0$. In Proposition 5.6 we prove that if $\mathrm{rk}(\mathcal{Q})=1$, then μ must be flat, while in Proposition 5.9, we show that if $\mathrm{rk}(\mathcal{Q})=2$, then either μ is flat or it has a unique non-flat tangent at infitity.

Proposition 5.3. *Let μ be a $(2n+1)$-uniform measure for which $\mathrm{supp}(\mu)\subseteq\mathbb{K}(b,\mathcal{Q},\mathcal{T})$. Then for any $\nu\in Tan_{2n+1}(\mu,\infty)$ we have $\mathrm{supp}(\nu)\subseteq\mathbb{K}(0,\mathcal{Q},\mathcal{T})$.*

Proof. Proposition 2.8 implies for any $w\in\mathrm{supp}(\nu)$ there is are sequences $\{v_i\}_{i\in\mathbb{N}}\subseteq\mathrm{supp}(\mu)$ and $R_i\to\infty$ for which $w_i:=D_{1/R_i}(v_i)\to w$. The condition $v_i=D_{R_i}(w_i)\in\mathbb{K}(b,\mathcal{Q},\mathcal{T})$ reads:

$$R_i^2\langle (w_i)_H,\mathcal{Q}[(w_i)_H]\rangle+R_i\langle b,(w_i)_H\rangle+R_i^2\mathcal{T}(w_i)_T=0.$$

Dividing the above identity by R_i^2 and sending i to infinity, we get that:

$$\langle w_H,\mathcal{Q}[w_H]\rangle+\mathcal{T}w_T=0,$$

which implies that $\mathrm{supp}(\nu)\subseteq\mathbb{K}(0,\mathcal{Q},\mathcal{T})$. □

Proposition 5.4. *Let E,F be closed sets in $\mathbb{H}^n$ and suppose that $\mathcal{S}^{2n+1}_E$ is a $(2n+1)$-uniform measure and $\mathcal{S}^{2n+1}(E\cap F)=0$. The measure $\mathcal{S}^{2n+1}_{E\cup F}$ is $(2n+1)$-uniform measure if and only if $\mathcal{S}^{2n+1}(F)=0$.*

Proof. Let $x\in E$ be such that $\rho:=\mathrm{dist}(x,F)>0$ and fix $r>0$. For any $\delta>0$ we have:

$$\begin{aligned}\mathcal{S}^{2n+1}_{E\cup F}\llcorner E(B_r(x))=&\mathcal{S}^{2n+1}_{E\cup F}(B_r(x)\cap B(F,\delta)^c\cap E)+\mathcal{S}^{2n+1}_{E\cup F}(B_r(x)\cap B(F,\delta)\cap E)\\=&\mathcal{S}^{2n+1}_{E}(B_r(x)\cap B(F,\delta)^c)+\mathcal{S}^{2n+1}_{E\cup F}(B_r(x)\cap B(F,\delta)\cap E),\end{aligned}$$

where the last equality comes from the following observation. If $\{B_i\}_{i\in\mathbb{N}}$ is a covering of $B_r(x)\cap B(F,\delta)^c\cap E$ with balls of radii smaller than $\delta/2$ and centred at $E\cup F$, then the centres must be contained in E. Sending δ to 0, since F is closed, we deduce that:

$$\mathcal{S}^{2n+1}_{E\cup F}\llcorner E(B_r(x))=\mathcal{S}^{2n+1}_{E}(B_r(x)\cap F^c)+\mathcal{S}^{2n+1}_{E\cup F}(F\cap E)=\mathcal{S}^{2n+1}_{E}(B_r(x)),$$

where the last equality comes from the fact that $\mathcal{S}^{2n+1}\llcorner E$ and $\mathcal{S}^{2n+1}_E$ are mutually absolutely continuous and that by hypothesis $\mathcal{S}^{2n+1}(E\cap F)=0$.

Therefore for any $r<\rho/2$, we have $\mathcal{S}^{2n+1}_{E\cup F}(B_r(x))=\mathcal{S}^{2n+1}_{E}(B_r(x))=r^{2n+1}$, and on the other hand if $\mathcal{S}^{2n+1}(F)>0$ there exists a radius $r>\rho$ such that $\mathcal{S}^{2n+1}_{E\cup F}(B_r(x))>r^{2n+1}$. □

Corollary 5.5. *Suppose the quadric $\mathbb{K}(b,\mathcal{Q},\mathcal{T})$ is connected and that it supports a $(2n+1)$-uniform measure μ. If $\Sigma(F)\cap\mathbb{K}(b,\mathcal{Q},\mathcal{T})=\emptyset$, then $\mu=\mathcal{S}^{2n+1}_{\mathbb{K}(b,\mathcal{Q},\mathcal{T})}$.*

Proof. The corollary is an immediate consequence of Propositions 4.8(i) (for the case $\mathcal{T}\neq 0$) and 4.10 (for the case $\mathcal{T}=0$). □

We study here the case $\mathrm{rk}(\mathcal{Q})=1$, i.e., there exists a non-zero vector $\mathfrak{n}$ such that $\mathcal{Q}=\mathfrak{n}\otimes\mathfrak{n}$.

Proposition 5.6. *Let μ be a $(2n+1)$-uniform measure supported on $\mathbb{K}(b,\mathfrak{n}\otimes\mathfrak{n},0)$. Then μ is flat.*

Proof. By scaling we can assume that $\mathfrak{n}$ is unitary. If $b=0$ there is nothing to prove since Proposition 4.4 directly implies that $\mu\in\mathfrak{M}(2n+1)$. Therefore we can assume without loss of generality that $b\neq 0$. A consequence of Proposition 5.3 and the discussion of the case in which $b=0$, is that $\mathrm{Tan}(\mu,\infty)=\{\mathcal{S}^{2n+1}_{V(\mathfrak{n})}\}$.

There are two possibilities for b: either it is parallel to $\mathfrak{n}$ or it is not. We begin with the simpler case in which b is parallel to $\mathfrak{n}$. In such a case, there exists $\lambda \in \mathbb{R} \setminus \{0\}$ for which $b = \lambda\mathfrak{n}$ and:

$$\mathbb{K}(\lambda\mathfrak{n}, \mathfrak{n} \otimes \mathfrak{n}, 0) = V(\mathfrak{n}) \cup \Big(- \lambda\frac{\mathfrak{n}}{|\mathfrak{n}|} + V(\mathfrak{n})\Big).$$

The points w of the singular set $\Sigma(F)$ (see (4.2)) must satisfy the equation:

$$(\lambda + 2\langle w_H, \mathfrak{n}\rangle)\mathfrak{n} = 0,$$

which implies that $\Sigma(F) = \emptyset$. Since $\mathcal{S}^{2n+1}_{V(\mathfrak{n})}$ is $(2n+1)$-uniform, Proposition 5.4 together with Proposition 4.10, imply that $\mu = \mathcal{S}^{2n+1}_{V(\mathfrak{n})}$.

We are left to discuss the case in which b is not parallel to $\mathfrak{n}$. Since $\mathrm{Tan}_{2n+1}(\mu, \infty) = \{\mathcal{S}^{2n+1}_{V(\mathfrak{n})}\}$, Proposition 2.8 implies that for any $w \in V(\mathfrak{n})$ there exists a sequence $\{v_i\}_{i\in\mathbb{N}} \subseteq \mathrm{supp}(\mu)$ such that $D_{1/i}(v_i) \to w$. Let $u \in \mathbb{R}^{2n}$ be a unitary vector, orthogonal to $\mathfrak{n}$ and such that $\langle b, u\rangle > 0$. Moreover, let W be the the orthogonal in $\mathbb{R}^{2n}$ of the span of the vectors u and $\mathfrak{n}$ and denote by P_W the orthogonal projection on W. Recall that for every i the v_i's must satisfy the equation:

$$\langle b, (v_i)_H\rangle + \langle \mathfrak{n}, (v_i)_H\rangle^2 = 0,$$

which, decomposing v_i along u, $\mathfrak{n}$ and W, becomes:

$$\begin{aligned} 0 =&\langle b, \mathfrak{n}\rangle\langle (v_i)_H, \mathfrak{n}\rangle + \langle b, u\rangle\langle (v_i)_H, u\rangle + \langle b, P_W[(v_i)_H]\rangle + \langle \mathfrak{n}, (v_i)_H\rangle^2 \\ \geq&\langle b, \mathfrak{n}\rangle\langle (v_i)_H, \mathfrak{n}\rangle + \langle b, u\rangle\langle (v_i)_H, u\rangle + \langle b, P_W[(v_i)_H]\rangle, \end{aligned}$$

for any $i \in \mathbb{N}$. If we divide by i the above inequality and let $i \to \infty$, we get:

$$\langle b, u\rangle\langle w_H, u\rangle \leq -\langle b, P_W[w_H]\rangle, \tag{5.1}$$

since w_H is orthogonal to $\mathfrak{n}$. By the arbitrariness of $w \in V(\mathfrak{n})$, inequality (5.1) must be satisfied for any w_H orthogonal to $\mathfrak{n}$. Therefore, since (5.1) holds for both w_H and $-w_H$, then:

$$\langle b, u\rangle\langle w_H, u\rangle = -\langle b, P_W[w_H]\rangle.$$

However, the above identity cannot be satisfied for any w_H orthogonal to $\mathfrak{n}$, proving that $\mathbb{K}(b, \mathcal{Q}, 0)$ in this case cannot support a uniform measure. □

Proposition 5.7. *Let μ be a $(2n+1)$-uniform measure supported on $\mathbb{K}(b, \mathcal{Q}, 0)$. If $\mathcal{Q}$ is semidefinite, then rk$(\mathcal{Q}) = 1$.*

Proof. For any $\nu \in \mathrm{Tan}_{2n+1}(\mu, \infty)$, Proposition 5.3 implies that $\mathrm{supp}(\nu) \subseteq \mathbb{K}(0, \mathcal{Q}, 0)$. Suppose by contradiction that $\mathrm{rk}(\mathcal{Q}) \geq 2$, then:

$$\mathrm{supp}(\nu) \subseteq \bigcap_{i=1}^{\mathrm{rk}(\mathcal{Q})} V(\mathfrak{n}_i).$$

where $\mathfrak{n}_i$ are the eigenvectors relative to non-zero eigenvalues of $\mathcal{Q}$. This would imply by Proposition A.7 that $\mathcal{S}^{2n+1}(\mathrm{supp}(\nu)) = 0$, which is a contradiction. □

The following proposition will be useful in the rest of the section as it provides an efficient way to describe the structure of the support of tangent measures at infinity to those $(2n+1)$-uniform measures which are supported on graphs.

Proposition 5.8. *Let $\mathfrak{n} \in \mathbb{S}^{2n-1}$ and suppose that $g : \mathbb{R}^{2n-1} \to \mathbb{R}$ is a continuous function such that:*

$$\lim_{\lambda \to \infty} \frac{g(\lambda h)}{\lambda} = g^\infty\left(\frac{h}{|h|}\right)|h|, \tag{5.2}$$

for any $z \in \mathbb{R}^{2n-1}$ where $g^\infty : \mathbb{S}^{2n-2} \to \mathbb{R}$ is a continuous function. Moreover we define:

$$\Gamma := \{z + g(z)\mathfrak{n} : z \in \mathfrak{n}^\perp\} \qquad \text{and} \qquad \Gamma_\infty := \{z + g^\infty(z/|z|)|z|\mathfrak{n} : z \in \mathfrak{n}^\perp\}.$$

Let μ be a $(2n+1)$-uniform measure for which $\Gamma \times \mathbb{R} \subseteq \mathrm{supp}(\mu)$. Then for any $\nu \in Tan_{2n+1}(\mu, \infty)$ we have:

$$\Gamma_\infty \times \mathbb{R} \subseteq \mathrm{supp}(\nu).$$

Proof. Let $(w, \tau) \in \Gamma_\infty \times \mathbb{R}$ and define the curve $\gamma : [0, \infty) \to \Gamma \times \mathbb{R}$ as:

$$\gamma(t) := (tP_\mathfrak{n}(w) + g(tP_\mathfrak{n}(w))\mathfrak{n}, t^2\tau),$$

where $P_\mathfrak{n} : \mathbb{R}^{2n} \to \mathfrak{n}^\perp$ is the orthogonal projection on $\mathfrak{n}^\perp$. The curve γ is contained in $\mathrm{supp}(\mu)$ and

$$\lim_{t\to\infty} D_{1/t}(\gamma(t)) = \left(P_\mathfrak{n}(w) + g^\infty\left(\frac{P_\mathfrak{n}(w)}{|P_\mathfrak{n}(w)|}\right)|P_\mathfrak{n}(w)|\mathfrak{n}, \tau\right) = (w, \tau),$$

where the last equality comes from the fact that $w \in \Gamma_\infty$. Let $\nu \in \mathrm{Tan}_{2n+1}(\mu, \infty)$ and $R_i \to \infty$ be the sequence for which $R_i^{-(2n+1)}\mu_{0,R_i} \rightharpoonup \nu$. Then we have that:

$$D_{1/R_i}(\gamma(R_i)) \in \mathrm{supp}(R_i^{-(2n+1)}\mu_{0,R_i}).$$

Therefore (5.2) implies by Proposition 2.8 that $(w, \tau) \in \mathrm{supp}(\nu)$. By the arbitrariness of (w, τ) and of ν, we have that $\Gamma_\infty \times \mathbb{R} \subseteq \mathrm{supp}(\nu)$ for any $\nu \in \mathrm{Tan}_{2n+1}(\mu, \infty)$. □

The following proposition establishes both properties **P.1** and **P.2** in the case the quadric $\mathbb{K}(b, \mathcal{Q}, 0)$ with $\mathrm{rk}(\mathcal{Q}) = 2$.

Proposition 5.9. *Suppose μ is a $(2n+1)$-uniform measure supported on $\mathbb{K}(b, \mathcal{Q}, 0)$. If $rk(\mathcal{Q}) = 2$, then one of the following two mutually exclusive conditions holds:*

(i) $\mu \in \mathfrak{M}(2n+1)$,

(ii) $Tan_{2n+1}(\mu, \infty) = \{\nu\}$ *and ν is not flat.*

Proof. Since $\mathcal{Q}$ is symmetric, has rank 2 and has no sign (by Proposition 5.6), there are $e_1, e_2 \in \mathbb{R}^{2n}$ orthonormal vectors and $\lambda_1, \lambda_2 > 0$ for which $\mathcal{Q} = -\lambda_1^2 e_1 \otimes e_1 + \lambda_2^2 e_2 \otimes e_2$. We define $\mathfrak{n} := -\lambda_1 e_1 + \lambda_2 e_2$ and $\mathfrak{m} := \lambda_1 e_1 + \lambda_2 e_2$.

If $b = 0$ it is readily seen that $\mathbb{K}(0, \mathcal{Q}, 0) = V(\mathfrak{n}) \cup V(\mathfrak{m})$ and that the singular set $\Sigma(F)$ coincides with $V(\mathfrak{n}) \cap V(\mathfrak{m})$. In particular $\mathbb{K}(0, \mathcal{Q}, 0)$ is disconnected by $\Sigma(F)$ in four half planes which we denote by C_i, with $i = 1, \dots, 4$. We claim that $\mathrm{supp}(\mu)$ can coincide with the closure of the union of just two of the C_i's. First of all, Proposition 4.10 implies that $\mathrm{supp}(\mu)$ must coincide with the closure of the union of some of these half-planes and on the other hand Proposition 4.4 implies that $\mathrm{supp}(\mu)$ cannot coincide with the closure of just one half-plane. Moreover Proposition 5.4 shows that there cannot be more than 3 half-planes contained in the support of μ, and thus the only remaining possibility is that there are only two half-planes contained in $\mathrm{supp}(\mu)$. If these half-planes are contained in the same plane, then μ is flat, while if they are one contained in $V(\mathfrak{n})$ and one in $V(\mathfrak{m})$, then μ is not flat and its tangent at infinity is unique and coincide with μ itself (as μ is invariant under dilations).

If $b \neq 0$ we have two subcases. Either b is contained in the image of $\mathcal{Q}$ or it is not. First we discuss the simpler case in which $b \notin \text{span}(e_1, e_2)$ (note that this implies that $n > 1$). In this case $\mathbb{K}(b, \mathcal{Q}, 0)$ is a graph of a quadratic polynomial. Indeed, complete e_1, e_2 to an orthonormal basis $\{e_1, \ldots, e_{2n}\}$ of $\mathbb{R}^{2n}$ and assume without loss of generality that $\langle b, e_3\rangle \neq 0$. Then $\mathbb{K}(b, \mathcal{Q}, 0)$ is an e_3-graph, indeed if $h \in \mathbb{R}^{2n}$ satisfies $\langle h, b\rangle + \langle h, \mathcal{Q}h\rangle = 0$, then:

$$\langle h, e_3\rangle = -\frac{-\lambda_1^2\langle h, e_1\rangle^2 + \lambda_2^2\langle h, e_2\rangle^2 + \sum_{i\neq 3}\langle b, e_i\rangle\langle h, e_i\rangle}{\langle b, e_3\rangle}.$$

This implies that the quadric $\mathbb{K}(b, \mathcal{Q}, 0)$ is an e_3-graph and thus it is a connected set. Moreover, since the equation $b + 2\mathcal{Q}h = 0$ does not have solutions $h \in \mathbb{R}^{2n}$, the singular set $\Sigma(F)$ is empty. Therefore Proposition 5.5 implies that $\text{supp}(\mu) = \mathbb{K}(b, \mathcal{Q}, 0)$. This by Proposition 5.8 implies that $\text{supp}(\nu) = \mathbb{K}(0, \mathcal{Q}, 0)$ but this is not possible by the study of the case $b = 0$, and thus μ cannot be uniform.

Thus, we are left to study the case where $b \neq 0$ and $b = b_1e_1 + b_2e_2$ for some $b_1, b_2 \in \mathbb{R}$. For any $x \in \mathbb{K}(b, \mathcal{Q}, 0)$, once completed the squares we have that:

$$0 = -\left(\lambda_1\langle x, e_1\rangle - \frac{b_1}{2\lambda_1}\right)^2 + \left(\lambda_2\langle x, e_2\rangle + \frac{b_2}{2\lambda_2}\right)^2 + \frac{b_1^2}{4\lambda_1^2} - \frac{b_2^2}{4\lambda_2^2}, \tag{5.3}$$

For any $\overline{x} \in \Sigma(F)$ (see (4.2)), we have:

$$-2\lambda_1^2\langle \overline{x}_H, e_1\rangle e_1 + 2\lambda_2^2\langle \overline{x}_H, e_2\rangle e_2 + b_1e_1 + b_2e_2 = 0,$$

and in particular $b_1 = 2\lambda_1^2\langle \overline{x}_H, e_1\rangle$ and $b_2 = -2\lambda_2^2\langle \overline{x}_H, e_2\rangle$. This in particular implies by (5.3) that $\overline{x}$ cannot be contained in $\mathbb{K}(b, \mathcal{Q}, 0)$ if $b_1^2/4\lambda_1 - b_2^2/4\lambda_2 \neq 0$.

If $b_1^2/4\lambda_1 - b_2^2/4\lambda_2 > 0$ by the above discussion we deduce that $\Sigma(F) = \emptyset$. Thanks to the identity (5.3), the quadric $\mathbb{K}(b, \mathcal{Q}, 0)$ is easily seen to be the disjoint union of two e_1-graphs Γ_1 (which we assume contains 0) and Γ_2. The functions $g_1, g_2 : e_1^\perp \to e_1$ which define Γ_1 and Γ_2 respectively, satisfy the hypothesis of Proposition 5.8. Indeed:

$$\lim_{t\to\infty}\frac{g_{1,2}(th)}{t} = \pm\frac{\lambda_2|\langle h, e_2\rangle|}{\lambda_1}e_1 = g_{1,2}^\infty\left(\frac{h}{|h|}\right)|h|e_1.$$

Proposition 4.10 implies that Γ_1 must be contained in $\text{supp}(\mu)$ and therefore the graph of g_1^∞ is contained in the support of any tangent measure at infinity to μ by Proposition 5.8. Suppose now by contradiction that $\text{supp}(\mu)$ contains also Γ_2. Again by Proposition 5.8 we would have that the graph of g_2^∞ is contained in the support of any tangent measure at infinity to μ. However, since the union the graphs of g_1^∞ and g_2^∞ coincides with $\mathbb{K}(0, \mathcal{Q}, 0)$, by the discussion of the case $b = 0$, this is not possible. Therefore the support of any tangent measure ν at infinity to μ coincides with the graph of g_1^∞. Therefore by Proposition 2.15 $\text{Tan}_{2n+1}(\mu, \infty)$ is a singleton and its only element cannot be flat (as the graph of g_1^∞ is not a hyperplane). The case $b_1^2/4\lambda_1 - b_2^2/4\lambda_2 < 0$ is treated in the same way with the roles of e_1 and e_2 reversed.

If $b_1^2/4\lambda_1 - b_2^2/4\lambda_2 = 0$, the quadric $\mathbb{K}(b, \mathcal{Q}, 0)$ coincides with the solutions of the equation:

$$\left(\lambda_1x_1 - \frac{b_1}{2\lambda_1}\right)^2 = \left(\lambda_1x_2 + \frac{b_2}{2\lambda_2}\right)^2.$$

Let $\tau := (b_1/2\lambda_1^2)e_1 + (b_2/2\lambda_2^2)e_2$ and note that $\mathbb{K}(b, \mathcal{Q}, 0)$ coincides with $\tau * (V(\mathfrak{n}) \cup V(\mathfrak{m}))$. Since left translations are isometries of $\mathbb{H}^n$, the discussion of this case reduces to the one in which $b = 0$, concluding the proof of the proposition. □

Eventually, the following theorem concludes the proof of **P.1**.

Theorem 5.10. *If μ is a $(2n+1)$-uniform measure for which there exists $\mathfrak{n} \in \mathbb{S}^{2n-1}$ such that $\mathcal{S}^{2n+1}_{V(\mathfrak{n})} \in Tan_{2n+1}(\mu, \infty)$. Then $\mu = \mathcal{S}^{2n+1}_{V(\mathfrak{n})}$.*

Proof. If $\mathrm{supp}(\mu) \subseteq \mathbb{K}(b, \mathcal{Q}, \mathcal{T})$, then for any $\nu \in \mathrm{Tan}_{2n+1}(\mu, \infty)$ we have $\mathrm{supp}(\nu) \subseteq \mathbb{K}(0, \mathcal{Q}, \mathcal{T})$ by Proposition 5.3. In particular $V(\mathfrak{n}) \subseteq \mathbb{K}(0, \mathcal{Q}, \mathcal{T})$ and this implies that $\mathcal{T} = 0$. Complete $\mathfrak{n}$ to an orthonormal basis $\{\mathfrak{n}, e_2, \ldots, e_{2n}\}$ of $\mathbb{R}^{2n}$ and note that, since $V(\mathfrak{n}) \subseteq \mathbb{K}(0, \mathcal{Q}, 0)$, we have that:

$$\sum_{i=2}^{2n} \langle e_i, \mathcal{Q} e_i \rangle \langle w_H, e_i \rangle^2 + 2 \sum_{2 \le i < j \le 2n} \langle e_i, \mathcal{Q} e_j \rangle \langle w_H, e_i \rangle \langle w_H, e_j \rangle = 0,$$

for any $w \in V(\mathfrak{n})$ and thus $\langle e_i, \mathcal{Q} e_j \rangle = 0$ for any $2 \le i, j \le 2n$. This implies that for any $x \in \mathbb{H}^n$ we have:

$$\langle x_H, \mathcal{Q} x_H \rangle = \langle x_H, \mathfrak{n} \rangle \Big(\langle \mathfrak{n}, \mathcal{Q} \mathfrak{n} \rangle \langle x_H, \mathfrak{n} \rangle + 2 \sum_{i=2}^{2n} \langle \mathfrak{n}, \mathcal{Q} e_i \rangle \langle x_H, e_i \rangle \Big) = \langle x, \mathfrak{n} \rangle \langle \mathfrak{m}, x \rangle,$$

where $\mathfrak{m} := \langle \mathfrak{n}, \mathcal{Q} \mathfrak{n} \rangle \mathfrak{n} + 2 \sum_{i=2}^{2n} \langle \mathfrak{n}, \mathcal{Q} e_i \rangle e_i$. In particular $\mathrm{rk}(\mathcal{Q}) \le 2$ and if $\mathfrak{m}$ is parallel to $\mathfrak{n}$, Proposition 5.6 implies that μ is flat. On the other hand if $\mathfrak{m}$ is not parallel to $\mathfrak{n}$, since $\mathrm{rk}(Q) = 2$ and μ has a flat tangent at infinity, Proposition 5.9 implies that μ is flat. □

5.2 Uniqueness of the tangent at infinity

This subsection is devoted to prove **P.2**. The uniqueness of the tangents at infinity also implies that they are $(2n+1)$-uniform cones. Indeed, if for the sequence $R_i \to \infty$ we have $R_i^{-(2n+1)} \mu_{0,R_i} \rightharpoonup \nu$, for any $\lambda > 0$ we also have:

$$(\lambda R_i)^{-(2n+1)} \mu_{0,\lambda R_i} \rightharpoonup \lambda^{-(2n+1)} \nu_{0,\lambda}.$$

Therefore the uniqueness of the tangent implies that $\lambda^{-(2n+1)} \nu_{0,\lambda} = \nu$ for any $\lambda > 0$. Thus ν is a $(2n+1)$-uniform cone by Definition 5.1.

The idea behind the proof of the uniqueness of the tangent at infinity is the following. Let $\nu \in \mathrm{Tan}_{2n+1}(\mu, \infty)$ and fix a point $w \in \mathrm{supp}(\nu)$. If we can find a continuous curve $\gamma : [0, \infty) \to \mathbb{H}^n$ contained in $\mathrm{supp}(\mu)$ for which:

$$\lim_{t \to \infty} D_{1/t}(\gamma(t)) = w,$$

then $w \in \xi$ for any $\xi \in \mathrm{Tan}_{2n+1}(\mu, \infty)$ by Proposition 2.8(iii), since $D_{1/t}(\gamma(t)) \in t^{-(2n+1)} \mu_{0,t}$.

In the various cases, the curve γ will always be constructed inside the quadric $\mathbb{K}(b, \mathcal{Q}, \mathcal{T})$ supporting μ and its initial point $\gamma(0)$ will always be a point of $\mathrm{supp}(\mu)$. In order to make sure that the whole γ is contained in $\mathrm{supp}(\mu)$, we force γ to avoid the singular set $\Sigma(F)$, so that continuity implies that γ contained in just one connected component of $\mathbb{K}(b, \mathcal{Q}, 0) \setminus \Sigma(F)$. Since the staring point was contained in $\mathrm{supp}(\mu)$ by hypothesis, this implies that γ must be contained in $\mathrm{supp}(\mu)$ by Proposition 4.1.

Theorem 5.11. *Suppose μ is a $(2n+1)$-uniform measure. Then $\mathrm{Tan}_{2n+1}(\mu, \infty)$ is a singleton.*

Proof. Assume that $\mathrm{supp}(\mu) \subseteq \mathbb{K}(b, \mathcal{Q}, \mathcal{T})$ for some non-zero $\mathcal{Q} \in \mathrm{Sym}(2n)$. We can assume without loss of generality that $b \neq 0$. Indeed if $b = 0$, the singular set $\Sigma(F)$ and the quadric $\mathbb{K}(0, \mathcal{Q}, \mathcal{T})$ are dilation invariant. Therefore the measure μ by Proposition 4.1 is dilation invariant too. This implies that the tangent at infinity to μ coincides with μ itself and in particular $\mathrm{Tan}_{2n+1}(\mu, \infty)$ is a singleton. In the following the measure ν will be always considered an arbitrary element of $\mathrm{Tan}_{2n+1}(\mu, \infty)$. We also let $\{R_i\}$ be the sequence for which $R_i^{-2n-1} \mu_{0,R_i} \rightharpoonup \nu$ with $R_i \to \infty$.

First of all we study the simpler case in which $\mathcal{T} \neq 0$. For the reader's convenience we recall that f and $\Sigma(f)$ were introduced in the Section 4 in (4.4) and (4.5), respectively.

If $\mathcal{T} \neq 0$ and $\mathrm{supp}(\mu) = \mathbb{K}(b, \mathcal{Q}, \mathcal{T})$, for any $w \in \mathbb{K}(0, \mathcal{Q}, \mathcal{T})$, the curve:

$$t \mapsto (t w_H, -\langle t b + t^2 \mathcal{Q} w_H, w_H \rangle / \mathcal{T}) =: \gamma_w(t),$$

is contained in $\mathrm{supp}(\mu)$ and we have $\lim_{t\to\infty} D_{1/t}(\gamma_w(t)) = w$. This by the arbitrariness of w and Proposition 2.8 implies that $\mathbb{K}(0, \mathcal{Q}, \mathcal{T}) \subseteq \mathrm{supp}(\nu)$. On the other hand Proposition 5.3 implies the previous inclusion holds as an equality and thus Proposition 2.15 implies that $\mathrm{Tan}_{2n+1}(\mu, \infty)$ is a singleton.

If $\mathcal{T} \neq 0$ and $\mathrm{supp}(\mu) \subsetneq \mathbb{K}(b, \mathcal{Q}, \mathcal{T})$, Proposition 4.8 implies that $n = 1$ and that $\mathrm{supp}(\mu)$ is contained in the image under f (see (4.4)) of one of the two connected components in which $\mathbb{R}^2$ is splitted by $\Sigma(f)$ (see (4.5)). Let $p, v \in \mathbb{R}^2$ be such that $p + \mathrm{span}(v) = \Sigma(f)$ and $\mu = \mathcal{S}^3_{f(C)}$ where $C := p + H^+(v)$ and $H^+(v) := \{z \in \mathbb{R}^{2n} : \langle v, z\rangle \geq 0\}$. For any $w \in H^+(v)$ the curve:

$$t \mapsto (p + tw_H, -\langle p + tw_H, \mathcal{Q}(p + tw_H) + b\rangle/\mathcal{T}) =: \gamma_w(t),$$

is contained in $\mathrm{supp}(\mu)$ and $\lim_{t\to\infty} D_{1/t}(\gamma_w(t)) = w$. This implies by Proposition 2.8 that $\{(z, -\langle z, \mathcal{Q}z\rangle/\mathcal{T}) : z \in H^+(v)\} = \mathrm{supp}(\nu)$.

We are left to prove the thesis in the case in which $\mathcal{T} = 0$. In this case it is harder to build the curves we mentioned above because of the presence of the singular set $\Sigma(F)$. We will need to distinguish two cases in order to rule out this problem.

If $\mathcal{T} = 0$ and there exists $\mathfrak{n} \in \mathbb{R}^{2n}$ for which $\nu(V(\mathfrak{n})) > 0$, then $V(\mathfrak{n}) \subseteq \mathbb{K}(0, \mathcal{Q}, 0)$. Arguing as in the proof of Theorem 5.10, we deduce that either $\mathrm{rk}(\mathcal{Q}) = 1$ or $\mathrm{rk}(\mathcal{Q}) = 2$. Proposition 5.6 and Proposition 5.9 prove the thesis of the proposition in these cases. We can therefore assume without loss of generality that $\nu(V(\mathfrak{n})) = 0$ for any $\mathfrak{n} \in \mathbb{R}^{2n}$. This in particular implies that $\mathrm{rk}(\mathcal{Q}) \geq 3$ and by Proposition 5.7 we deduce that $\mathcal{Q}$ is not semidefinite. For the remainder of the proof we should consider $w \in \mathrm{supp}(\nu)$ fixed. By Proposition 2.8 we can also find a sequence $\{v_i\} \subseteq \mathrm{supp}(\mu)$ for which $D_{1/R_i}(v_i) \to w$. We can also assume that these $\{v_i\}$ are contained in the same connected component of $\mathbb{K}(b, \mathcal{Q}, 0) \setminus \Sigma(F)$.

At first assume that $\Sigma(F) \neq \emptyset$. Since $\nu(V(b)) = 0$, we can also assume without loss of generality that $\langle b, w_H\rangle \neq 0$. Let $\tilde{x} \in \Sigma(F)$ and define:

$$\gamma(t) := \big((v_i)_H + tw_H + t\theta(t)(\tilde{x})_H, (v_i)_T + t^2 w_T\big)$$

where $\theta(t) := \langle 2\mathcal{Q}[(v_i)_H] + b, w_H\rangle/\langle b, (v_i)_H + tw_H\rangle$. If i is big enough then $\langle b, (v_i)_H + tw_H\rangle \neq 0$ (since we have that $D_{1/R_i}(v_i) \to w$) and thus $\theta(t)$ is well defined for $t \geq 0$. First we check that $\lim_{t\to\infty} D_{1/t}(\gamma(t)) = w$. Indeed:

$$\lim_{t\to\infty} D_{1/t}(\gamma(t)) = \lim_{t\to\infty}\left(\frac{(v_i)_H + tw_H + t\theta(t)\tilde{x}}{t}, \frac{(v_i)_T + t^2 w_T}{t^2}\right) = \Big(w_H + \lim_{t\to\infty}\theta(t)(\tilde{x})_H, w_T\Big) = w,$$

since $\lim_{t\to\infty}\theta(t) = 0$. Secondly, we check that $\gamma(t) \in \mathbb{K}(b, \mathcal{Q}, 0)$ for any $t > 0$. Since $\tilde{x} \in \Sigma(F)$ we have:

$$2\mathcal{Q}(\tilde{x})_H + b = 0 \qquad \text{and} \qquad \langle(\tilde{x})_H, \mathcal{Q}(\tilde{x})_H + b\rangle = 0. \tag{5.4}$$

The identities in (5.4) imply:

$$0 = \langle(\tilde{x})_H, 2\mathcal{Q}(\tilde{x})_H + b\rangle - \langle(\tilde{x})_H, \mathcal{Q}(\tilde{x})_H + b\rangle = \langle(\tilde{x})_H, \mathcal{Q}(\tilde{x})_H\rangle = -\langle b, (\tilde{x})_H\rangle, \tag{5.5}$$

Thus (5.5) together with fact that $w \in \mathbb{K}(0, \mathcal{Q}, 0)$ imply:

$$\begin{aligned}\langle(\gamma(t))_H, b + \mathcal{Q}(\gamma(t))_H\rangle =& t\langle w_H, b\rangle + t\langle(v_i)_H, 2\mathcal{Q}w_H\rangle + t\theta(t)\langle(v_i)_H, 2\mathcal{Q}\tilde{x}\rangle + t^2\theta(t)\langle w_H, 2\mathcal{Q}\tilde{x}\rangle\\ =& t\langle w_H, b + 2\mathcal{Q}[(v_i)_H]\rangle - t\theta(t)\langle(v_i)_H + tw_H, b\rangle = 0,\end{aligned}$$

where the last equality comes from the definition of θ. Thanks to (5.4) and (5.5) we can also prove that γ does not intersect $\Sigma(F)$, indeed:

$$\langle 2\mathcal{Q}\gamma(t) + b, \tilde{x}\rangle = -\langle\gamma(t), b\rangle = -\langle(v_i)_H + tw_H, b\rangle \neq 0,$$

for any $t > 0$. This implies that for any $t \geq 0$, the curve γ is contained in the same connected component of $\mathbb{K}(0, \mathcal{Q}, 0) \setminus \Sigma(F)$ and since the initial point of γ is contained in $\mathrm{supp}(\mu)$, the whole curve γ by continuity is contained in the support of μ. and $\lim_{t\to\infty} D_{1/t}(\gamma(t)) = w$. Since $D_{1/t}(\gamma(t))$ is contained in the support of $t^{-(2n+1)}\mu_{0,t}$ and:

$$\lim_{t\to\infty} D_{1/t}(\gamma(t)) = w,$$

we deduce that $w \in \mathrm{supp}(\xi)$ for any $\xi \in \mathrm{Tan}_{2n+1}(\mu, \infty)$ by Proposition 2.8. Thanks to the arbitrariness of w and ξ we deduce that by Proposition 2.15 that the tangent at infinity is unique.

Finally suppose that $\Sigma(F) = \emptyset$. For any $x \in \mathbb{K}(b, \mathcal{Q}, 0)$ we have that:

$$0 = \lambda_1\langle x_H, e_1\rangle^2 + \langle b, e_1\rangle\langle x_H, e_1\rangle + \langle \mathcal{Q}[P_1(x_H)] + b, P_1(x_H)\rangle, \tag{5.6}$$

where e_1 is a unitary eigenvector of $\mathcal{Q}$ relative to a positive eigenvalue λ_1 and P_1 is the orthogonal projection on $e_1^\perp$. Since $\nu(V(e_1)) = 0$, we can assume without loss of generality that $\langle w_H, e_1\rangle > 0$, and since $w \in \mathbb{K}(0, \mathcal{Q}, 0)$ we have that:

$$\langle \mathcal{Q}[P_1(w_H)], P_1(w_H)\rangle = -\lambda_1\langle w_H, e_1\rangle^2 < 0. \tag{5.7}$$

Since $D_{1/R_i}(v_i) \to w$, defined $s(t) := P_1[(v_i)_H + tw_H]$ and provided i is sufficiently big, we have:

$$\langle \mathcal{Q}s(t) + b, s(t)\rangle < 0 \text{ for any } t \geq 0.$$

Therefore the curve:

$$\gamma_w(t) := \left(\frac{\sqrt{\langle b, e_1\rangle^2 - 4\lambda_1\langle \mathcal{Q}s(t) + b, s(t)\rangle} - \langle b, e_1\rangle}{2\lambda_1} e_1 + s(t), (v_i)_T + t^2 w_T\right),$$

is well defined for any $t \geq 0$. The component of γ_w along e_1 is by construction a solution to the equation:

$$\lambda_1\zeta^2 + \langle b, e_1\rangle\zeta + \langle \mathcal{Q}s(t) + b, s(t)\rangle = 0,$$

which by (5.6) implies that γ_w is contained in $\mathbb{K}(b, \mathcal{Q}, 0)$. Since γ_w is continuous, $\gamma_w(0) = v_i \in \mathrm{supp}(\mu)$ and $\Sigma(F) = \emptyset$, by Proposition 4.10 we deduce that $\gamma_w(t) \in \mathrm{supp}(\mu)$ for any $t \geq 0$. We are left to compute the limit $\lim_{t\to\infty} D_{1/t}(\gamma(t))$. In the case of the vertical component the computation is immediate, indeed $\lim_{t\to\infty}(\gamma_w(t))_T/t^2 = w_T$. The limit for the horizontal components is:

$$\begin{aligned}\lim_{t\to\infty}\frac{(\gamma_w(t))_H}{t} &= \lim_{t\to\infty}\frac{\sqrt{\langle b, e_1\rangle^2 - 4\lambda_1\langle \mathcal{Q}s(t) + b, s(t)\rangle} - \langle b, e_1\rangle}{2\lambda_1 t} e_1 + P_1[w_H]\\ &= \frac{\sqrt{-4\lambda_1\langle \mathcal{Q}w_H + b, w_H\rangle}}{2\lambda_1} e_1 + P_1[w_H] = w_H,\end{aligned}$$

where the last equality comes from the identity in (5.6) and the fact that $\langle w_H, e_1\rangle > 0$. Thus $\lim_{t\to\infty} D_{1/t}(\gamma(t)) = w$, which implies that $w \in \mathrm{supp}(\xi)$ for any $\xi \in \mathrm{Tan}_{2n+1}(\mu, \infty)$. The same argument we used in the previous case implies that the tangent at infinity is unique. □

5.3 Proof of Theorem 5.2

The following lemma insures that if ϕ is a measure with $(2n+1)$-density, at ϕ-almost every x there is a flat measure contained in $\mathrm{Tan}_{2n+1}(\mu, x)$.

Lemma 5.12. *If ϕ is a Radon measure with $(2n+1)$-density, then for ϕ almost every x there exists a $w \in \mathbb{R}^{2n}$ for which:*

$$\mathcal{S}^{2n+1}_{V(w)} \in \mathit{Tan}_{2n+1}(\phi, x).$$

Proof. Proposition 2.5 and Proposition 4.5 directly imply the claim. □

We are ready to finish the proof of Theorem 5.2. The argument we will use follows closely the proof of Proposition 6.10 of [15]. By contradiction, suppose there exists a point x such that:

(i) $\mathrm{Tan}_{2n+1}(\phi, x) \subseteq \Theta(\phi, x)\mathcal{U}_{\mathbb{H}^n}(2n+1)$,

(ii) there are $\zeta, \nu \in \mathrm{Tan}_{2n+1}(\phi, x)$ such that ν is flat and ζ is not flat,

(iii) Proposition 2.5 holds at x.

We can also assume without loss of generality that $\Theta(\phi, x) = 1$. Since ζ is not flat, its tangent at infinity χ cannot be flat otherwise Theorem 5.10 would imply that ζ is flat. In particular by the assumption on the functional $\mathscr{F}$ we have $\mathscr{F}(\chi) > \hbar$. Fix $r_k \to 0$ and $s_k \to 0$ such that:

$$\frac{\phi_{x,r_k}}{r_k^{2n+1}} \rightharpoonup \nu \qquad \text{and} \qquad \frac{\phi_{x,s_k}}{s_k^{2n+1}} \rightharpoonup \chi.$$

We can further suppose that $s_k < r_k$. Define for any $r > 0$ the function $f(r) := \mathscr{F}(r^{-(2n+1)}\phi_{x,r})$, and note that since F is continuous with respect to the weak-$*$ convergence of measures, f is continuous in r. Since ν is flat, then:

$$\lim_{r_k \to 0} f(r_k) = \mathscr{F}(\nu) \leq \hbar/2,$$

Thus for sufficiently small r_k we have $f(r_k) < \hbar$. On the other hand, since:

$$\lim_{s_k \to 0} f(s_k) = \mathscr{F}(\chi) > \hbar,$$

for suffciently small s_k we have that $f(s_k) > \hbar$. Fix $\sigma_k \in [s_k, r_k]$ such that $f(\sigma_k) = \hbar$ and $f(r) \leq \hbar$ for $r \in [\sigma_k, r_k]$. By compactness there exists a subsequence of $\{\sigma_k\}_{k\in\mathbb{N}}$, not relabeled, such that $\sigma_k^{-(2n+1)}\phi_{x,\sigma_k}$ converges weakly-$*$ to a measure $\xi \in \mathcal{U}_{\mathbb{H}^n}(2n+1)$. Clearly by continuity:

$$\mathscr{F}(\xi) = \lim_{\sigma_k \to 0} f(\sigma_k) = \hbar.$$

Note that $r_k/\sigma_k \to \infty$, otherwise if for some subsequence not relabeled, we had that r_k/σ_k converged to a constant C (larger than 1) we would conclude that $\frac{\xi_{0,C}}{C^{2n+1}} = \nu$ since:

$$\nu = \lim_{k\to\infty} \frac{\phi_{x,r_k}}{r_k^{2n+1}} = \lim_{k\to\infty} \left(\frac{\sigma_k}{r_k}\right)^{2n+1} \left(\frac{\phi_{x,\sigma_k}}{\sigma_k^{2n+1}}\right)_{0,r_k}.$$

In particular ξ would be flat, which is not possible as $F(\xi) = \hbar$. Note that for any given $R > 0$ we have:

$$(R\sigma_k)^{-(2n+1)}\phi_{x,R\sigma_k} \rightharpoonup R^{-(2n+1)}\xi_{0,R}.$$

This by continuity of $\mathscr{F}$ implies that:

$$\mathscr{F}(R^{-(2n+1)}\xi_{0,R}) = \lim_{k\to\infty} f(R\sigma_k).$$

Moreover, since $r_k/\sigma_k \to \infty$ we conclude that for any $R > 1$ we have that $R\sigma_k \in [\sigma_k, r_k]$ whenever k is large enough. This, by our choice of σ_k and r_k, implies that:

$$\mathscr{F}(R^{-(2n+1)}\xi_{0,R}) = \lim_{k\to\infty} f(R\sigma_k) \leq \hbar, \tag{5.8}$$

for every $R \geq 1$. Let ψ be the tangent measure at infinity to ξ, which by Theorem 5.11 is unique and it is a cone. Therefore, thanks to (5.8) we have that:

$$\mathscr{F}(\psi) = \lim_{R\to\infty} F(R^{-(2n+1)}\xi_{0,R}) \leq \hbar,$$

and in particular thanks to the assumptions on the functional $\mathscr{F}$, we have that $\psi \in \mathfrak{M}(2n+1)$. This is in contradiction with Theorem 5.10, which would imply that ξ is flat.

6. Limits of sequences of horizontal $(2n+1)$-uniform cones

The main result of Section 5, Theorem 5.2, implies that if we can prove that flat measures are quantitatively disconnected from the other $(2n+1)$-uniform cones, measures with $(2n+1)$-density have only flat tangents. A first step towards this direction is to show that horizontal $(2n+1)$-uniform cones are disconnected from $(2n+1)$-vertical cones.

Let $\{\mu_i\}_{i\in\mathbb{N}}$ be a sequence of horizontal $(2n+1)$-uniform cones supported on the quadrics $\mathbb{K}(0,\mathcal{D}_i,-1)$. Suppose these measures are weakly converging to a measure ν which by Proposition 2.8 is a $(2n+1)$-uniform cone. We define also the following sequence of matrices:

$$\mathcal{Q}_i := -\frac{\mathcal{D}_i}{\|\!|\mathcal{D}_i|\!\|},$$

where with the symbol $\|\!|A|\!\|$ we denote the operatorial norm of the matrix A. Furthermore we can assume, up to non-relabeled subsequences, that $-\mathcal{D}_i/\|\!|\mathcal{D}_i|\!\|$ converges to some matrix $\mathcal{Q}\in\text{Sym}(2n)$ with $\|\!|\mathcal{Q}|\!\|=1$.

The plan of the section is the following. First we prove that the measure ν is vertical if and only if the sequence of the norms $\|\!|\mathcal{D}_i|\!\|$ diverges. Secondly, we prove that the only possible vertical limits of sequences of horizontal $(2n+1)$-uniform cones are flat measures. This second step of the section is where the results of Appendix B come into play. Indeed Theorem B.16 applies to μ_i for any $i\in\mathbb{N}$ and it forces $\mathcal{D}_i$ to satisfy the following identity for any $z\in\mathbb{R}^{2n}$ for which $(\mathcal{D}_i+J)z\neq 0$:

$$\frac{\text{Tr}(\mathcal{D}_i^2)-2\langle\mathfrak{n}_i,\mathcal{D}_i^2\mathfrak{n}_i\rangle+\langle\mathfrak{n}_i,D_i\mathfrak{n}_i\rangle^2}{4(2n-1)}+\frac{n-1}{2n-1}-\frac{1}{4}+\frac{\langle\mathcal{D}_iJ\mathfrak{n}_i,\mathfrak{n}_i\rangle}{2n-1}-\frac{\big(\text{Tr}(\mathcal{D}_i)-\langle\mathfrak{n}_i,\mathcal{D}_i\mathfrak{n}_i\rangle\big)^2}{8(2n-1)}=0. \tag{6.1}$$

where $\mathfrak{n}_i(z):=\frac{(\mathcal{D}_i+J)z}{|(\mathcal{D}_i+J)z|}$ is the so called *horizontal normal* to the graph of $f(x)=\langle x,\mathcal{D}_ix\rangle$ at the point $(x,f(x))$. Since identity (6.1) holds for any $i\in\mathbb{N}$, we can deduce some constraints (see Proposition 6.3) for the limit matrix $\mathcal{Q}$. Using a similar argument we are able to prove in Proposition 6.6 that the sequence of the second biggest (in modulus) eigenvalue of $\mathcal{D}_i$ is bounded. Putting together these information, we are able to prove in Proposition 6.7 that there must exists a constant depending only on n which bounds the biggest eigenvalue of the matrices $\mathcal{D}_i$ associated quadrics $\mathbb{K}(0,\mathcal{D}_i,-1)$ support a horizontal $(2n+1)$-uniform measure.

The following proposition shows the equivalence between the geometric condition for ν to be vertical $(2n+1)$-uniform cone and the algebraic condition on the divergence of the sequence $\big\{\|\!|\mathcal{D}_i|\!\|\big\}_{i\in\mathbb{N}}$. This will be very useful in the forthcoming computations.

Proposition 6.1. *Let $\{\mu_i\}_{i\in\mathbb{N}}$ and ν as above. The following are equivalent:*

(i) ν *is supported on* $\mathbb{K}(0,\mathcal{Q},0)$ *where* $\mathcal{Q}$ *is the limit of* $\mathcal{D}_i/\|\!|\mathcal{D}_i|\!\|$ *as above,*

(ii) $\lim_{i\to\infty}\|\!|\mathcal{D}_i|\!\|=\infty$.

Proof. Proposition 2.8 implies that for any $y \in \mathrm{supp}(\nu)$ there exists a sequence $\{y_i\}_{i\in\mathbb{N}}$ such that $y_i \in \mathrm{supp}(\mu_i)$ and $y_i \to y$. Assume at first the sequence $|||\mathcal{D}_i|||$ is bounded. This implies that we can find a subsequence $\mathcal{D}_i$ (not relabeled) such that the matrices $\mathcal{D}_i$ converge to some $\mathcal{D} \in \mathrm{Sym}(2n)$. Thanks to our assumption on y_i we know that $(y_i)_T = \langle (y_i)_H, \mathcal{D}_i[(y_i)_H]\rangle$. Thus taking the limit as i to infinity, we get:

$$y_T = \langle y_H, \mathcal{D}y_H\rangle.$$

Therefore if $|||\mathcal{D}_i|||$ is bounded, we have that ν is supported on both the quadrics $\mathbb{K}(0,\mathcal{Q},0)$ and $\mathbb{K}(0,\mathcal{D},-1)$, however this is not possible thanks to Proposition 4.6 which implies that either $\mathcal{T}=0$ or $\mathcal{T}\neq 0$ for any quadric containing $\mathrm{supp}(\mu)$. Viceversa, if we suppose that $|||\mathcal{D}_i|||$ diverges, with some iterations of the triangle inequality, we deduce the follwoing bound:

$$|\langle y_H, \mathcal{Q}y_H\rangle| \leq \|\mathcal{Q}-\mathcal{Q}_i\||y_H|^2 + |y_H-(y_i)_H|(|y_H|+|(y_i)_H|) + \frac{|(y_i)_T|}{|||\mathcal{D}_i|||}. \tag{6.2}$$

The right-hand side of the above inequality goes to 0 as $i\to\infty$ since we can assume without loss of generality that $\|y_i\| \leq 2\|y\|$. Therefore for any $y \in \mathrm{supp}(\nu)$ we have that $\langle y_H, \mathcal{Q}y_H\rangle = 0$ and thus $\mathrm{supp}(\nu) \subseteq \mathbb{K}(0,\mathcal{Q},0)$. ☐

Remark 6.2. For later convenience we remark that up to considering an isometric copy of the sequence $\{\mu_i\}_{i\in\mathbb{N}}$, we can assume that the biggest eigenvalue of $\mathcal{Q}$ is 1. Without loss of generality we can assume that up to a non-relabeled subsequence, the biggest eigevalues have the same sign. Let $U \in \mathrm{S}(2n)$ and recall that the map Ξ_U introduced in Proposition 2.1 is a surjective isometry. In particular Propositions 2.1 and 2.12 imply that $(\Xi_U)_\#\mu$ is a $(2n+1)$-uniform measure. Moreover a routine computation shows that:

$$\mathrm{supp}((\Xi_U)_\#\mu) \subseteq \mathbb{K}(Ub, U\mathcal{Q}U^T, s(U)\mathcal{T}). \tag{6.3}$$

Suppose now that the biggest eigenvalue in modulus of $\mathcal{Q}$ is -1 and pick some $U \in \mathrm{S}(2n)$ for which $s(U)=-1$ (the function s was defined after (2.1)). Since $\mu_i \rightharpoonup \nu$ and $\mathrm{supp}(\mu_i) \subseteq \mathbb{K}(0,\mathcal{D}_i,-1)$, we have that:

$$(\Xi_U)_\#\mu_i \rightharpoonup (\Xi_U)_\#\nu \qquad \text{and} \qquad \mathrm{supp}((\Xi_U)_\#\mu_i) \subseteq \mathbb{K}(0,U\mathcal{D}_iU^T,1) = \mathbb{K}(0,-U\mathcal{D}_iU^T,-1) \text{ by (6.3)}$$

In particular, if $|||\mathcal{D}_i||| \to \infty$, Proposition 6.1 implies that $\mathrm{supp}((\Xi_U)_\#\nu) \subseteq \mathbb{K}(0,-U\mathcal{Q}U^T,0)$. Let v be an eigenvector of $\mathcal{Q}$ relative to the eigenvalue -1, and note that:

$$-U\mathcal{Q}U^T(Uv) = -U\mathcal{Q}v = Uv.$$

This argument also shows that if μ is a horizontal $(2n+1)$-uniform cone, we can always assume without loss of generality that $\mathrm{supp}(\mu) \subseteq \mathbb{K}(0,\mathcal{D},-1)$ where the biggest eigenvalue of $\mathcal{D}$ is positive.

From now on we shall always assume that $|||\mathcal{D}_i||| \to \infty$, or in other words that the limit measure ν is a vertical $(2n+1)$-uniform cone, and that 1 is an eigenvalue of $\mathcal{Q}$. In the following proposition we show how the quadric $\mathbb{K}(0,\mathcal{Q},0)$ "remembers" that the measure ν is the limit of a sequence of horizontal $(2n+1)$-uniform cones.

Proposition 6.3. *Let $\mathcal{Q}$ be the limit of the matrices $\mathcal{D}_i/|||\mathcal{D}_i|||$ as above. Then, for any $h \notin Ker(\mathcal{Q})$ we have:*

$$2(Tr(\mathcal{Q}^2) - 2\langle\mathfrak{n},\mathcal{Q}^2\mathfrak{n}\rangle + \langle\mathfrak{n},\mathcal{Q}\mathfrak{n}\rangle^2) - (Tr(\mathcal{Q}) - \langle\mathfrak{n},\mathcal{Q}\mathfrak{n}\rangle)^2 = 0, \tag{6.4}$$

where $\mathfrak{n} := \frac{\mathcal{Q}h}{|\mathcal{Q}h|}$.

Proof. Since $h \notin \ker(\mathcal{Q})$, there exists $N \in \mathbb{N}$ for which for any $i \geq N$ we have:

$$(\mathcal{D}_i + J)h \neq 0. \tag{6.5}$$

Indeed, if there was a (non-relabeled) subsequence of indeces for which $(\mathcal{D}_i + J)h = 0$, dividing the above equality by $|||\mathcal{D}_i|||$ and sending i to infinity, we would deduce that $\mathcal{Q}h = 0$. Note that defined $\mathfrak{n}_i(h) := (\mathcal{D}_i + J)h/|(\mathcal{D}_i + J)h|$, we have that:

$$\lim_{i\to\infty} \mathfrak{n}_i(h) = \frac{\mathcal{Q}h}{|\mathcal{Q}h|} = \mathfrak{n}.$$

Therefore Theorem B.16 implies that for any i for which (6.5) holds, we have:

$$\frac{\mathrm{Tr}(\mathcal{D}_i^2) - 2\langle \mathfrak{n}_i, \mathcal{D}_i^2\mathfrak{n}_i\rangle + \langle \mathfrak{n}_i, D_i\mathfrak{n}_i\rangle^2}{4(2n-1)} + \frac{n-1}{2n-1} - \frac{1}{4} + \frac{\langle \mathcal{D}_i J\mathfrak{n}_i, \mathfrak{n}_i\rangle}{2n-1} - \frac{(\mathrm{Tr}(\mathcal{D}_i) - \langle \mathfrak{n}_i, \mathcal{D}_i\mathfrak{n}_i\rangle)^2}{8(2n-1)} = 0. \tag{6.6}$$

As already remarked, equation (6.5) holds definitely, therefore dividing (6.6) by $|||\mathcal{D}_i|||$ and taking the limit as i goes to infinity, we obtain (6.4). □

The following corollary is an easy application of Proposition 6.3 in the case h is a non-zero eigenvector of $\mathcal{Q}$.

Corollary 6.4. *Let $\mathcal{Q}$ be the limit of the matrices $\mathcal{D}_i/|||\mathcal{D}_i|||$ as above. Then, every non-zero eigenvalue λ of $\mathcal{Q}$ satisfies the equation:*

$$-3\lambda^2 + 2Tr(\mathcal{Q})\lambda + (2Tr(\mathcal{Q}^2) - Tr(\mathcal{Q})^2) = 0. \tag{6.7}$$

In particular $\mathcal{Q}$ has at most two distinct non-zero eigenvalues.

Proof. For any $h \in \mathbb{R}^{2n}$ unitary eigenvector relative to the eigenvalue $\lambda \neq 0$, we have $\mathfrak{n}(h) = \frac{\mathcal{Q}h}{|\mathcal{Q}h|} = \mathrm{sgn}(\lambda)h$. Thanks to Proposition 6.3 we have that:

$$2(\mathrm{Tr}(\mathcal{Q}^2) - 2\lambda^2 + \lambda^2) - (\mathrm{Tr}(\mathcal{Q}) - \lambda)^2 = 0,$$

which collecting λ, proves the corollary. □

The following proposition shows that the non-zero eigenvalue of $\mathcal{Q}$ different from 1 does not exists. This implies that $\mathcal{Q}$ is semidefinite and ν is flat.

Proposition 6.5. *Suppose $\lim_{i\to\infty} |||\mathcal{D}_i||| = \infty$ and $\lim_{i\to\infty} \mathcal{D}_i/|||\mathcal{D}_i||| = \mathcal{Q}$. Then $rk(\mathcal{Q}) = 1$ and in particular the limit ν of the sequence $\{\mu_i\}$ is flat.*

Proof. Since 1 is an eigenvalue of $\mathcal{Q}$, Corollary 6.4 implies that 1 solves the equation (6.7). Therefore we have:

$$-3 + 2\mathrm{Tr}(\mathcal{Q}) + (2\mathrm{Tr}(\mathcal{Q}^2) - \mathrm{Tr}(\mathcal{Q})^2) = 0. \tag{6.8}$$

Using the above equality we deduce that (6.7) becomes:

$$-3\lambda^2 + 2\mathrm{Tr}(\mathcal{Q})\lambda + (3 - 2\mathrm{Tr}(\mathcal{Q})) = 0.$$

The above equation has the following solutions:

$$\lambda_1 = 1 \qquad \text{and} \qquad \lambda_2 = \frac{2\mathrm{Tr}(\mathcal{Q})}{3} - 1. \tag{6.9}$$

If $\lambda_2 \geq 0$, the matrix $\mathcal{Q}$ would be semi-definite. Since $\mathbb{K}(0, \mathcal{Q}, 0)$ supports a $(2n+1)$-uniform measure, Proposition 5.7 implies that $\mathrm{rk}(\mathcal{Q}) = 1$ and Proposition 5.6 implies that μ must be flat. Therefore we can assume without loss of generality that $\lambda_2 < 0$. Let now k_1 be the dimension of the eigenspace relative to the eigenvalue 1 and k_2 be the dimension of the eigenspace relative to the eigenvalue λ_2. By assumption $k_1 \geq 1$ and we can suppose without loss of generality that $k_2 \geq 1$, otherwise $\mathcal{Q}$ would be semidefinite and Proposition 5.6 and Proposition 5.7 imply that μ is flat. This implies, by (6.9) that:

$$\text{Tr}(\mathcal{Q}) = k_1 + \lambda_2 k_2 = k_1 + \frac{2\text{Tr}(\mathcal{Q})k_2}{3} - k_2.$$

The above equation together with (6.8), allows us to express $\text{Tr}(\mathcal{Q})$, $\text{Tr}(\mathcal{Q}^2)$ as functions of k_1, k_2:

$$\text{Tr}(\mathcal{Q}) = \frac{3(k_1 - k_2)}{3 - 2k_2} \qquad \text{and} \qquad \text{Tr}(\mathcal{Q}^2) = k_1 + \left(\frac{2k_1 - 3}{3 - 2k_2}\right)^2 k_2. \tag{6.10}$$

Substituting in (6.8) the above identities, we deduce that:

$$-3 + 2\frac{3(k_1 - k_2)}{3 - 2k_2} + 2k_1 + 2\left(\frac{2k_1 - 3}{3 - 2k_2}\right)^2 k_2 - \frac{9(k_1 - k_2)^2}{(3 - 2k_2)^2} = 0.$$

Recollecting terms, we can rearrange the above equality in the following fashion:

$$0 = k_1^2(8k_2 - 9) + k_1(8k_2^2 - 42k_2 + 36) + (-9k_2^2 + 36k_2 - 27),$$

which allow us to express k_1 in terms of k_2. Indeed the solutions of the above quadratic equation are:

$$k_1 = \frac{9k_2 - 9}{8k_2 - 9} \qquad \text{or} \qquad k_1 = 3 - k_2.$$

The only couple of natural numbers for which $k_1 = \frac{9k_2-9}{8k_2-9}$ holds is $(k_1, k_2) = (1, 0)$, which has been already taken in consideration. On the other hand, there are four couples of natural numbers for which $k_1 = 3 - k_2$, which are $(3, 0); (2, 1); (2, 1); (0, 3)$. However the only couples of natural numbers for which $\mathcal{Q}$ is not semi-defined are $(1, 2)$ and $(2, 1)$. In both these cases (6.10) implies that $\text{Tr}(\mathcal{Q}) = 3$ by (6.10) and therefore $\lambda_2 = 1$ by (6.9), which is in contradiction with the assumption that $\lambda_2 < 0$. □

We introduce here further notation. For any $i \in \mathbb{N}$ we let $\lambda_i(1), \ldots, \lambda_i(2n)$ to be the eigenvalues of $\mathcal{D}_i$. Such eigenvalues are ordered in the following way:

$$|||\mathcal{D}_i||| = \lambda_i(1) \geq |\lambda_i(2)| \geq \ldots \geq |\lambda_i(2n)|,$$

where we can assume without loss of generality that the biggest eigenvalue in modulus is positive thanks to Remark 6.2. We further let $\{e_i(1), \ldots, e_i(2n)\}$ be an orthonormal basis of $\mathbb{R}^{2n}$ for which $e_i(j)$ is an eigenvector relative to the eigenvalue $\lambda_i(j)$.

The following proposition will allow us to show in Proposition 6.7 that ν must be a horizontal $(2n+1)$-uniform cone. If this is not the case, we know by Proposition 6.5 that $\text{supp}(\nu)$ is a vertical hyperplane V. On the other hand, Proposition 6.6 implies that while $\lambda_1(j)$ is diverging, the other eigenvalues remain bounded. Therefore the limit of the quadrics $\mathbb{K}(0, \mathcal{D}_i, -1)$ must coincide with $\text{supp}(\mu)$ by Proposition 2.8 but on the other hand it must be a proper subset of V:

Proposition 6.6. *Let the sequence of matrices $\{\mathcal{D}_i\}$ be as above, i.e.:*

(i) $\mathbb{K}(0, \mathcal{D}_i, -1)$ *for any $i \in \mathbb{N}$ supports a $(2n+1)$-uniform cone μ_i which converges to some ν,*

(ii) $\lim_{i\to\infty} |||\mathcal{D}_i||| = \lim_{i\to\infty} \lambda_i(1) = \infty$.

Then the sequence $\{\lambda_i(2)\}_{i\in\mathbb{N}}$ is bounded.

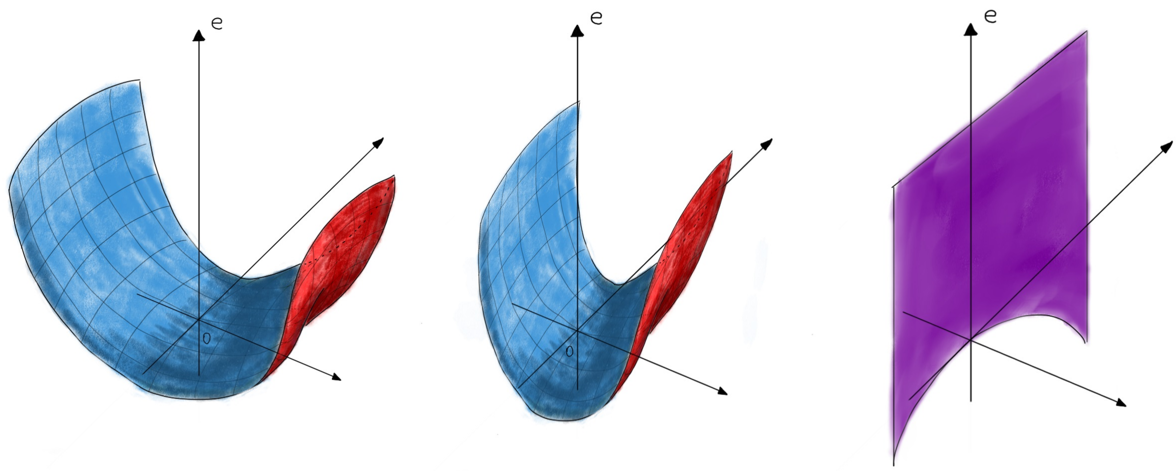

Figure 6.1: The image shows why a sequence of quadrics having only one eigenvalue diverging cannot converge to a full plane.

Proof. By contradiction assume that $|\lambda_i(2)| \to \infty$ and let $e_i := e_i(2)$ and $\lambda_i := \lambda_i(2)$. We want to apply Theorem B.16 to the points $h = e_i$. In order to do so we need to give an explicit expression to the quantities involved in (B.29) of Theorem B.16. First we compute the horizontal normal of $\mathbb{K}(0, \mathcal{D}_i, -1)$ at the point $(e_i, \langle e_i \mathcal{D}_i e_i \rangle)$:

$$\mathfrak{n}_i := \mathfrak{n}(e_i) := \frac{\mathcal{D}_i e_i + J e_i}{|\mathcal{D}_i e_i + J e_i|} = \frac{\lambda_i e_i + J e_i}{|\lambda_i e_i + J e_i|} = \frac{\lambda_i e_i + J e_i}{(1+\lambda_i^2)^{\frac{1}{2}}}$$

Secondly we compute the quantities $\langle \mathfrak{n}_i, \mathcal{D}_i \mathfrak{n}_i \rangle$, $\langle \mathfrak{n}_i, \mathcal{D}_i^2 \mathfrak{n}_i \rangle$ and $\langle \mathfrak{n}_i, \mathcal{D}_i J \mathfrak{n}_i \rangle$:

$$\begin{aligned} \langle \mathfrak{n}_i, \mathcal{D}_i^k \mathfrak{n}_i \rangle &= \left\langle \frac{\lambda_i e_i + J e_i}{(1+\lambda_i^2)^{\frac{1}{2}}}, D_i^k \left(\frac{\lambda_i e_i + J e_i}{(1+\lambda_i^2)^{\frac{1}{2}}} \right) \right\rangle = \lambda_i^k - \frac{\lambda_i^k + \langle e_i, J \mathcal{D}_i^k J e_i \rangle}{1+\lambda_i^2}, \\ \langle \mathfrak{n}_i, \mathcal{D}_i J \mathfrak{n}_i \rangle &= \left\langle \frac{\lambda_i e_i + J e_i}{(1+\lambda_i^2)^{\frac{1}{2}}}, D_i J \left(\frac{\lambda_i e_i + J e_i}{(1+\lambda_i^2)^{\frac{1}{2}}} \right) \right\rangle = \frac{\lambda_i^2 - \lambda_i \langle e_i, J \mathcal{D}_i J e_i \rangle}{1+\lambda_i^2}. \end{aligned} \tag{6.11}$$

In order to further simplify the notation, we define:

$$\begin{aligned} (I)_i :=& \mathrm{Tr}(\mathcal{D}_i^2) - 2\langle \mathfrak{n}_i, \mathcal{D}_i^2 \mathfrak{n}_i \rangle + \langle \mathfrak{n}_i, \mathcal{D}_i \mathfrak{n}_i \rangle^2, \\ (II)_i :=& \mathrm{Tr}(\mathcal{D}_i) - \langle \mathfrak{n}_i, \mathcal{D}_i \mathfrak{n}_i \rangle. \end{aligned}$$

With these notations, Theorem B.16 applied to $h = e_i$ turns into:

$$\frac{(I)_i}{4(2n-1)} + \frac{n-1}{2n-1} - \frac{1}{4} + \frac{\lambda_i^2 - \lambda_i \langle e_i, J \mathcal{D}_i J e_i \rangle}{1+\lambda_i^2} - \frac{(II)_i^2}{8(2n-1)} = 0. \tag{6.12}$$

We give now a more explicit description of both $(I)_i$ and $(II)_i$ using the expressions for $\langle \mathfrak{n}_i, \mathcal{D}_i \mathfrak{n}_i \rangle$, $\langle \mathfrak{n}_i, \mathcal{D}_i^2 \mathfrak{n}_i \rangle$

and $\langle \mathfrak{n}_i, \mathcal{D}_i J \mathfrak{n}_i\rangle$ we found in (6.11):

$$\begin{aligned}(I)_i =&\text{Tr}(\mathcal{D}_i^2) - 2\langle \mathfrak{n}_i, \mathcal{D}_i^2 \mathfrak{n}_i\rangle + \langle \mathfrak{n}_i, \mathcal{D}_i \mathfrak{n}_i\rangle^2 = \sum_{j=1}^{2n} \lambda_i(j)^2 - 2\lambda_i^2 - 2\frac{\lambda_i^2 + \langle e_i, J\mathcal{D}_i^2 J e_i\rangle}{1+\lambda_i^2} + \left(\lambda_i - \frac{\lambda_i + \langle e_i, J\mathcal{D}_i J e_i\rangle}{1+\lambda_i^2}\right)^2\\ =&\sum_{j\neq 2} \lambda_i(j)^2 - 2\frac{\lambda_i^2 + \langle e_i, J\mathcal{D}_i^2 J e_i\rangle}{1+\lambda_i^2} - 2\lambda_i \frac{\lambda_i + \langle e_i, J\mathcal{D}_i J e_i\rangle}{1+\lambda_i^2} + \frac{(\lambda_i + \langle e_i, J\mathcal{D}_i J e_i\rangle)^2}{(1+\lambda_i^2)^2},\\ (II)_i =&\text{Tr}(\mathcal{D}_i) - \langle \mathfrak{n}_i, \mathcal{D}_i \mathfrak{n}_i\rangle = \sum_{j\neq 2} \lambda_i(j) - \frac{\lambda_i + \langle e_i, J\mathcal{D}_i J e_i\rangle}{1+\lambda_i^2}.\end{aligned} \tag{6.13}$$

The absurd assumption that $\lambda_i \to \infty$ has the following consequences. First of all:

$$\lim_{i\to\infty} \frac{\langle e_i, J(\mathcal{D}_i/|||\mathcal{D}_i|||)^k J e_i\rangle}{(1+\lambda_i^2)} = 0, \tag{6.14}$$

for any $k \in \mathbb{N}$. Secondly, thanks to (6.13) and (6.14) we deduce that:

$$\lim_{i\to\infty} \frac{(I)_i}{|||\mathcal{D}_i|||^2} = 1 \qquad \text{and} \qquad \lim_{i\to\infty} \frac{(II)_i}{|||\mathcal{D}_i|||} = 1, \tag{6.15}$$

where we used the fact that by definition $\lambda_i(1) = |||\mathcal{D}_i|||$ and that $\lim_{i\to\infty} \lambda_i(j)/|||\mathcal{D}_i||| = 0$ for any $j \in \{2, \dots, 2n\}$ thanks to Proposition 6.5. Dividing by $|||\mathcal{D}_i|||^2$ the left-hand side of the identity (6.12), and sending $i \to \infty$ yields, thanks to (6.15):

$$\lim_{i\to\infty} \frac{(I)_i/|||\mathcal{D}_i|||^2}{4(2n-1)} - \frac{((II)_i/|||\mathcal{D}_i|||)^2}{8(2n-1)} + \frac{\left(\frac{n-1}{2n-1} - \frac{1}{4}\right)}{|||\mathcal{D}_i|||^2} + \frac{(\lambda_i/|||\mathcal{D}_i|||)^2 - (\lambda_i/|||\mathcal{D}_i|||)\langle e_i, J(\mathcal{D}_i/|||\mathcal{D}_i|||)J e_i\rangle}{1+\lambda_i^2} = \frac{1}{8(2n-1)},$$

which shows that the assumption $\lambda_i \to \infty$ is absurd thanks to fact that identity (6.12) holds for every $i \in \mathbb{N}$. □

Proposition 6.7. *There exists a constant $\mathfrak{C}_1(n) > 0$ such that if $\mathbb{K}(0, \mathcal{D}, -1)$ supports a $(2n+1)$-uniform measure, then $|||\mathcal{D}||| \leq \mathfrak{C}_1(n)$.*

Proof. By contradiction assume that there exists a sequence $\{\mu_i\}_{i\in\mathbb{N}}$ of $(2n+1)$-uniform measures such that $\mathbb{K}(0, \mathcal{D}_i, -1)$ supports μ_i and $|||\mathcal{D}_i||| \to \infty$. By the compactness of measures we can extract a converging subsequence and by Propositions 6.1 and 6.5 we deduce that the limit ν is flat. On the other hand Proposition 6.6 implies that with the exception of $\lambda_1(i)$, the eigenvalues are bounded in modulus by some constant $\mathfrak{C} > 0$ (which a priori depends on the sequence $\{\mu_i\}$). For any $y \in \text{supp}(\nu)$, Proposition 2.8 implies that we can find a sequence $\{y_i\} \subseteq \mathbb{H}^n$ such that $y_i \in \text{supp}(\mu_i)$ and $y_i \to y$. For such a sequence $\{y_i\}$, we have:

$$(y_i)_T =\langle (y_i)_H, \mathcal{D}_i (y_i)_H\rangle = \sum_{j=1}^{2n} \lambda_i(j)\langle e_i(j), (y_i)_H\rangle^2 \geq \sum_{j=2}^{2n} \lambda_i(j)\langle e_i(j), (y_i)_H\rangle^2 \geq -(2n-1)\mathfrak{C}|(y_i)_H|^2.$$

Sending i to infinity we deduce that $y_T \geq -(2n-1)\mathfrak{C}|y_H|^2$, which constitutes a non-trivial bound on y_T. This comes in contradiction with the fact that $\text{supp}(\nu)$ must contain a vertical hyperplane as ν is flat. □

Proposition 6.8. *There exists a constant $\mathfrak{C}_2(n) > 0$ such that if $\mathbb{K}(0, \mathcal{D}, -1)$ supports a $(2n+1)$-uniform measure, then $|||\mathcal{D}||| \geq \mathfrak{C}_2(n)$.*

Proof. Repeatedly applying the triangle inequality, and using the following trivial bounds:

$$\mathrm{Tr}(\mathcal{D}^2)\leq 2n|||\mathcal{D}|||^2,\qquad |\mathrm{Tr}(\mathcal{D})|\leq 2n|||\mathcal{D}|||,\qquad |\langle v,\mathcal{D}^k v\rangle|\leq |||\mathcal{D}|||^k|v|^2,\qquad |\langle v,DJv\rangle|\leq |||\mathcal{D}||||v|^2,$$

for any $v\in\mathbb{S}^{2n-1}$ we have:

$$\left|\frac{\mathrm{Tr}(\mathcal{D}^2)-2\langle v,\mathcal{D}^2v\rangle+\langle v,\mathcal{D}v\rangle^2}{4(2n-1)}+\frac{\langle \mathcal{D}Jv,v\rangle}{2n-1}-\frac{(\mathrm{Tr}(\mathcal{D})-\langle v,\mathcal{D}v\rangle)^2}{8(2n-1)}\right|\leq\frac{(4n^2+8n+7)|||\mathcal{D}|||^2+8|||\mathcal{D}|||}{8(2n-1)}.$$

Theorem B.16 implies that if $\mathbb{K}(0,\mathcal{D},-1)$ supports a $(2n+1)$-uniform measure, identity (B.29) must be satisfied for any $e\in\mathbb{S}^{2n-1}$ for which $(\mathcal{D}+J)e\neq 0$. In particular we have that:

$$\left|\frac{n-1}{2n-1}-\frac{1}{4}\right|\leq\frac{(4n^2+8n+7)|||\mathcal{D}|||^2+8|||\mathcal{D}|||}{8(2n-1)}. \tag{6.16}$$

With some algebraic computations, which we omit, we see that the positive solutions in $|||D|||$ to (6.16) are contained in $[1/\sqrt{4n^2+8n+7},\infty)$. □

Theorem 6.9. *There exists a constant $\mathfrak{C}_3(n)>0$ such that for any horizontal $\mathfrak{m}\in\mathbb{S}^{2n-1}$ and any $(2n+1)$-uniform cone μ we have:*

$$\int_{B_1(0)}\langle\mathfrak{m},z_H\rangle^2d\mu(z)\geq\mathfrak{C}_3(n).$$

Proof. We can suppose without loss of generality that $\mathrm{supp}(\mu)\subseteq\mathbb{K}(0,\mathcal{D},-1)$. Therefore Proposition 2.15 implies that $\mu=\mathcal{S}^{2n+1}_{\mathrm{supp}(\mu)}$ and by Proposition A.5 for any positive Borel function $h:\mathbb{H}^n\to\mathbb{R}$ we have:

$$\int h(z)\mu(z)=\int h(z)d\mathcal{S}^{2n+1}_{\mathrm{supp}(\mu)}(z)=\frac{1}{\mathfrak{c}_n}\int_{\pi_H(\mathrm{supp}(\mu))}h(z)|(\mathcal{D}+J)y|dy.$$

If $n>1$, Proposition 4.8 implies that $\pi_H(\mathrm{supp}(\mu))=\mathbb{R}^{2n}$, and thus:

$$\begin{aligned}\mathfrak{c}_n\int_{B_1(0)}\langle\mathfrak{m},z_H\rangle^2d\mu(z)=&\int_{|y|^4+\langle y,\mathcal{D}y\rangle^2\leq 1}\langle\mathfrak{m},y\rangle^2|(\mathcal{D}+J)y|dy\\ =&\int_{\mathbb{S}^{2n-1}}\langle\mathfrak{m},v\rangle^2|(\mathcal{D}+J)v|\int_0^{(1+\langle v,\mathcal{D}v\rangle^2)^{-1/4}}r^{2n+2}drd\sigma(v)\\ =&\frac{1}{(2n+3)}\int_{\mathbb{S}^{2n-1}}\frac{\langle\mathfrak{m},v\rangle^2|(\mathcal{D}+J)v|^2}{(1+\langle v,\mathcal{D}v\rangle^2)^{\frac{2n+3}{4}}}d\sigma(v),\end{aligned} \tag{6.17}$$

where $\sigma:=\mathcal{H}^{2n-1}_{eu}\llcorner\mathbb{S}^{2n-1}$ and in the second equality we performed the change of variables $y=rv$ in $\mathbb{R}^{2n}$. Since $\mathbb{K}(0,\mathcal{D},1)$ supports a uniform measure, Proposition 6.7 implies that $|||\mathcal{D}|||\leq\mathfrak{C}_1(n)$, and thus thanks to (6.17), we have:

$$\int_{B_1(0)}\langle\mathfrak{m},z_H\rangle^2d\mu(z)\geq\frac{\int_{\mathbb{S}^{2n-1}}\langle\mathfrak{m},v\rangle^2|(\mathcal{D}+J)v|d\sigma(v)}{(2n+3)\mathfrak{c}_n(1+\mathfrak{C}_1(n)^2)^{\frac{2n+3}{4}}}. \tag{6.18}$$

Suppose $e\in\mathbb{S}^{2n-1}$ is the vector at which $\mathcal{D}$ attains the operatorial norm, i.e., $|\mathcal{D}e|=|||\mathcal{D}|||$. Then e is an eigenvector of $\mathcal{D}$ and thus $|(\mathcal{D}+J)e|^2=|||D|||^2+1$. Therefore for any $u\in\mathbb{S}^{2n-1}\cap B_{1/8}(e)$, we have:

$$\begin{aligned}|(\mathcal{D}+J)u|^2\geq&|(\mathcal{D}+J)e|^2+2\langle(\mathcal{D}+J)e,(\mathcal{D}+J)(u-e)\rangle\geq(|||\mathcal{D}|||^2+1)-2|||\mathcal{D}+J|||^2|u-e|\\ \geq&(|||\mathcal{D}|||^2+1)(1-4|u-e|)\geq(\mathfrak{C}_2(n)^2+1)/2=:\mathfrak{C}_4(n),\end{aligned} \tag{6.19}$$

where in the second last inequality we used the Jensen inequality $\|\mathcal{D}+J\|^2 \leq 2(\|\mathcal{D}\|^2+1)$ and in the last one the bound on $\|\mathcal{D}\|$ yielded by Proposition 6.8. Putting together (6.17) and (6.19) we deduce that:

$$\int_{B_1(0)} \langle \mathfrak{m}, z_H\rangle^2 d\mu(z) \geq \frac{\mathfrak{C}_4(n)\int_{\mathbb{S}^{2n-1}\cap U_{1/8}(e)}\langle \mathfrak{m}, v\rangle^2 d\sigma(v)}{(2n+3)\mathfrak{c}_n(1+\mathfrak{C}_1(n)^2)^{\frac{2n+3}{4}}} \geq \frac{\mathfrak{C}_4(n)\mathfrak{C}_5(n)}{(2n+3)\mathfrak{c}_n(1+\mathfrak{C}_1(n)^2)^{\frac{2n+3}{4}}} =: \mathfrak{C}_3(n), \tag{6.20}$$

where $\mathfrak{C}_5(n) := \min_{\mathfrak{m}\in\mathbb{S}^{2n-1}}\int_{\mathbb{S}^{2n-1}\cap U_{1/8}(e)}\langle \mathfrak{m}, v\rangle^2 d\sigma(v)$ and as usual $U_{1/8}(e)$ denotes the Euclidean ball of radius $1/8$ and centre e in $\mathbb{R}^{2n}$.

The case in which $n=1$ and $\pi_H(\mathrm{supp}(\mu))$ is a half plane, the argument is the same. The only difference is that we have be careful to choose the vector e at which $\mathcal{D}$ attains the operatorial norm in $\pi_H(\mathrm{supp}(\mu))$. This choice of e together with the computations in (6.19) imply that $U_{1/8}(e) \subseteq \pi_H(\mathrm{supp}(\mu))$ and therefore (6.20) holds. $\square$

7. Disconnection of vertical non-flat cones and flat measures

This section is devoted to prove that non-flat vertical $(2n+1)$-uniform cones are quantitatively disconnected from flat measures. To be precise, we prove that there is a universal constant $\mathfrak{C}_{10}(n) > 0$ such that, if μ is a vertical $(2n+1)$-uniform cone and:

$$\min_{\mathfrak{m}\in\mathbb{S}^{2n-1}} \int_{B_1(0)} \langle \mathfrak{m}, z_H\rangle^2 d\mu(z) \leq \mathfrak{C}_{10}(n), \tag{7.1}$$

then μ is flat. The first step towards the proof is to use the study of the support of vertical $(2n+1)$-uniform cones carried on in Subsection 4.3 and the representation formulas of the perimeter given in Appendix A to obtain the following more explicit expression for the quadric containing μ (see Proposition 7.2). For any $w \in \mathrm{supp}(\mu)$, we have that:

$$|w_H|^2 - (2n-1) \fint \langle w_H, u\rangle^2 d\omega_\mu(u) = 0, \tag{7.2}$$

where $\omega_\mu := \mathcal{H}^{2n-2}_{eu} \llcorner \pi_H(\mathrm{supp}(\mu)) \cap \mathbb{S}^{2n-1}$. The existence of $\mathfrak{C}_{10}(n)$, as showed in Proposition 7.7, is a direct consequence of (7.2) by means of few algebraic manipulations.

The following technical lemma is a consequence of the coarea formula and the representation formulas for the intrinsic perimeter we proved in Appendix A.2.

Lemma 7.1. *Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^{2n} \to \mathbb{R}$ be positive Borel functions and μ a vertical $(2n+1)$-uniform cone. Defined $\omega_\mu := \mathcal{H}^{2n-2}_{eu} \llcorner \mathbb{S}^{2n-1} \cap \pi_H(\mathrm{supp}(\mu))$, we have that:*

(i)

$$\int_{B_1(0)} g(z_H) d\mu(z) = \frac{2}{\mathfrak{c}_n} \int_0^1 r^{2n-2}\sqrt{1-r^4} \int g(rv) d\omega_\mu(v) dr,$$

(ii)

$$\int f(z_T) g(z_H) d\mu(z) = \frac{1}{\mathfrak{c}_n} \int f(t)dt \cdot \int_0^\infty r^{2n-2} \int g(rv) d\omega_\mu(v) dr.$$

Proof. Suppose $\mathrm{supp}(\mu) \subseteq \mathbb{K}(0, \mathcal{D}, 0)$. As a first step we prove that for any positive Borel function $h : \mathbb{R}^{2n} \to \mathbb{R}$, defined $S := \pi_H(\mathrm{supp}(\mu))$, we have that:

$$\int h(y) d\mathcal{H}^{2n-1}_{eu} \llcorner S = \int_0^\infty r^{2n-2} \int h(rv) d\omega_\mu(v) dr, \tag{7.3}$$

where ω_μ is as above. To do so, for any $\delta > 0$ let $U(\mathrm{ker}(\mathcal{D}), \delta)$ be the open Euclidean neighbourhood of radius δ of the set $\mathrm{ker}(\mathcal{D})$ and define $\mathbb{K}_\delta := \pi_H(\mathbb{K}(0, \mathcal{D}, 0)) \setminus U(\mathrm{ker}(\mathcal{D}), \delta)$. The set $\mathbb{K}_\delta$ is a smooth submanifold of $\mathbb{R}^{2n}$

and since the function $u(x) := |x|$ is smooth on $\mathbb{R}^{2n} \setminus U(\ker(\mathcal{D}), \delta)$, we can apply the coarea formula in Remark 10.5 of Chapter 2 of [39] with the choice $g = \chi_{A\cap S}$, where A is any Borel set of $\mathbb{H}^n$. We therefore obtain:

$$\int_{A\cap S\cap \mathbb{K}_\delta} J^*_{\mathbb{K}_\delta} u(x) d\mathcal{H}^{2n-1}_{eu} = \int_{\mathbb{R}} \mathcal{H}^{2n-2}_{eu}(A\cap S\cap \mathbb{K}_\delta \cap r\mathbb{S}^{2n-1})dr,$$

where $J^*_{\mathbb{K}_\delta} u = |\nabla_T u|$ is the Jacobian of the tangetial gradient of u along $\mathbb{K}_\delta$. We claim that $J^*_{\mathbb{K}_\delta} u = 1$ on $\mathbb{K}_\delta$. To prove this, we note that the vector $x/|x|$ is contained in the tangent to $\mathbb{K}_\delta$ for any $x \in \mathbb{K}_\delta$. This is due to the fact that the curve $\gamma(s) := x + \frac{sx}{|x|}$ is contained in $\mathbb{K}_\delta$ provided $s \geq 0$. Complete $x/|x|$ to an othonormal basis of $\mathrm{Tan}(\mathbb{K}_\delta, x)$ with vectors $v_2, \ldots, v_{2n-1}$ and note that $\nabla_T u[v_i] = \langle \nabla u, v_i\rangle = 0$ for any $i = 2, \ldots, 2n-1$. This implies that $|\nabla_T u| = |\langle \nabla u(x), x/|x|\rangle| = 1$ and in particular:

$$\mathcal{H}^{2n-1}_{eu}(A\cap S\cap \mathbb{K}_\delta) = \int_{\mathbb{R}} \mathcal{H}^{2n-2}_{eu}(A\cap S\cap \mathbb{K}_\delta \cap r\mathbb{S}^{2n-1})dr,$$

Therefore sending δ to 0, by Beppo-Levi convergence theorem we get:

$$\begin{aligned}\mathcal{H}^{2n-1}_{eu}\llcorner S(A) = & \int_{\mathbb{R}} \mathcal{H}^{2n-2}_{eu}(A\cap S\cap r\mathbb{S}^{2n-1})dr = \int_{\mathbb{R}} r^{2n-2}\mathcal{H}^{2n-2}_{eu}\left(\frac{A}{r}\cap \mathbb{S}^{2n-1}\cap S\right)dr\\ = & \int_{\mathbb{R}} r^{2n-2}\int \chi_A(rv)d\mathcal{H}^{2n-2}_{eu}\llcorner \mathbb{S}^{2n-1}\cap S(v)dr.\end{aligned}$$

A standard approximation procedure with simple function proves the claim thanks to Beppo-Levi convergence theorem for a general positive measurable function.

We are ready to prove the identities (i) and (ii) of the statement. Thanks to Lemma A.10 we have that for any positive Borel function $G : \mathbb{H}^n \to \mathbb{R}$ we have:

$$\int G(z)d\mu(z) = \frac{1}{\mathfrak{c}_n}\int G(z_H, z_T)d\mathcal{H}^{2n-1}_{eu}\llcorner S(z_H)\otimes dz_T. \tag{7.4}$$

Therefore, in order to prove identity (i), we choose $G(z) := \chi_{B_1(0)}(z)g(z_H)$ and note that combining (7.3) and (7.4), we get:

$$\begin{aligned}\int_{B_1(0)} G(z)d\mu(z) = & \frac{1}{\mathfrak{c}_n}\int_{|y|^4+t^2\leq 1} g(y)d\mathcal{H}^{2n-1}_{eu}\llcorner S(y)\otimes \mathcal{L}^1(t) = \frac{2}{\mathfrak{c}_n}\int_{|y|\leq 1} g(y)\sqrt{1-|y|^4}d\mathcal{H}^{2n-1}_{eu}\llcorner S(y)\\ = & \frac{2}{\mathfrak{c}_n}\int_0^1 r^{2n-2}\sqrt{1-r^4}\int g(rv)d\omega_\mu(v)dr.\end{aligned}$$

Moreover, (ii) follows immediately with the choice $G(z) := f(z_T)g(z_H)$:

$$\begin{aligned}\int_{B_1(0)} G(z)d\mu(z) = & \frac{1}{\mathfrak{c}_n}\int f(t)g(y)d\mathcal{H}^{2n-1}_{eu}\llcorner S(y)\otimes dt = \frac{1}{\mathfrak{c}_n}\int f(t)dt\cdot \int g(y)d\mathcal{H}^{2n-1}_{eu}\llcorner S(y)\\ = & \frac{1}{\mathfrak{c}_n}\int f(t)dt\cdot \int_0^\infty r^{2n-2}\int g(rv)d\omega_\mu(v)dr,\end{aligned}$$

where in the first identity we applied (7.4), in the second Tonelli's theorem and in the last one (7.3). □

The following proposition is a refinement of Lemma 3.22 in case μ is a vertical $(2n+1)$-uniform cone. In such a case $\mathcal{T}(1) = 0$ and the integrals defining $\mathcal{Q}(1)$ ($\mathcal{T}$ and $\mathcal{Q}$ were introduced in Definition 3.16) can be explicitly computed thanks to Lemma 7.1.

Proposition 7.2. *Suppose μ is a vertical $(2n+1)$-uniform cone. For any $w \in \mathrm{supp}(\mu)$ we have:*

$$|w_H|^2 = (2n-1) \fint \langle w_H, u\rangle^2 d\omega_\mu(u), \tag{7.5}$$

where $\omega_\mu = \mathcal{H}^{2n-2}_{eu} \llcorner \mathbb{S}^{2n-1} \cap \pi_H(\mathrm{supp}(\mu))$.

Proof. Since μ is a cone, Proposition 3.22 implies that for any $w \in \mathrm{supp}(\mu)$ we have:

$$\langle w_H, \mathcal{Q}(1) w_H\rangle + \mathcal{T}(1) w_T = 0. \tag{7.6}$$

Moreover, since by assumption μ is a vertical $(2n+1)$-uniform cone, thanks to Proposition 4.6 we have $\mathcal{T}(1) = 0$. In order to prove the proposition, we are left to give an explicit expression for $\mathcal{Q}(1)$.

Let $\varphi : \mathbb{R}^{2n} \to \mathbb{R}$ be a k-homogeneous function and $i \in \mathbb{N}$. Lemma 7.1(ii) and the fact that $z_T^i \varphi(z) e^{-\|z\|^4} \in L^1(\mu)$ imply that:

$$\begin{aligned}\int z_T^i \varphi(z_H) e^{-\|z\|^4} d\mu(z) =& \frac{1}{\mathfrak{c}_n} \int t^i e^{-t^2} dt \cdot \int_0^\infty r^{2n-2} \int \varphi(rv) e^{-r^4} d\omega_\mu(v) dr \\ =& \frac{1}{\mathfrak{c}_n} \int t^i e^{-t^2} dt \cdot \int_0^\infty r^{2n+k-2} e^{-r^4} dr \cdot \int \varphi(v) d\omega_\mu(v)\end{aligned}$$

Thanks to the above identity, we deduce that if i is odd $\int z_T^i \varphi(z_H) e^{-\|z\|^4} d\mu(z) = 0$, while if i is even:

$$\int z_T^i \varphi(z_H) e^{-\|z\|^4} d\mu(z) = \frac{2\Gamma\left(\frac{2n-1+k}{4}\right)\Gamma\left(\frac{i+1}{2}\right)}{\mathfrak{c}_n} \int \varphi(v) d\omega_\mu(v). \tag{7.7}$$

In order to compute the matrix $\mathcal{Q}(1)$, it will be convenient to make use of the notation introduced in the proof of Proposition 3.17, where we defined the matrices $\mathcal{Q}_1, \mathcal{Q}_2$ and $\mathcal{Q}_3, \mathcal{Q}_4$ in (3.7) and (3.8) respectively. We define $\mathfrak{C}_6(n) := 2\Gamma\left(\frac{2n+1}{4}\right)\Gamma\left(\frac{1}{2}\right)/\mathfrak{c}_n C(2n+1)$, where the constant $C(2n+1)$ was introduced in Definition 3.8. Thanks to (7.7) and few algebraic calculations, we deduce that:

$$\begin{aligned}\frac{\mathcal{Q}_1(1)}{\mathfrak{C}_6(n)} =& \frac{8}{C(2n+1)} \int |z_H|^4 z_H \otimes z_H + z_T^2 J z_H \otimes J z_H e^{-\|z\|^4} d\mu(z) \\ =& 2(2n+1) \int v \otimes v d\omega_\mu(v) + 4 \int Jv \otimes Jv d\omega_\mu(v), \\ \frac{\mathcal{Q}_2(1)}{\mathfrak{C}_6(n)} =& \frac{8}{C(2n+1)} \int |z_H|^2 z_T (z_H \otimes J z_H + J z_H \otimes z_H) e^{-\|z\|^4} d\mu(z) = 0, \\ \frac{\mathcal{Q}_3(1)}{\mathfrak{C}_6(n)} =& \frac{2}{C(2n+1)} \int |z_H|^2 e^{-\|z\|^4} d\mu(z) \mathrm{id} = 2\omega_\mu(\mathbb{S}^{2n-1}) \mathrm{id}, \\ \frac{\mathcal{Q}_4(1)}{\mathfrak{C}_6(n)} =& \frac{4}{C(2n+1)} \int (z_H \otimes z_H + J z_H \otimes J z_H) e^{-\|z\|^4} d\mu(z) = 4 \int (v \otimes v + Jv \otimes Jv) d\omega_\mu(v).\end{aligned}$$

Summing up, since $\mathcal{Q}_1, \ldots, \mathcal{Q}_4$ were constructed in such a way that $\mathcal{Q}(1) = \mathcal{Q}_1(1) + \mathcal{Q}_2(1) - \mathcal{Q}_3(1) - \mathcal{Q}_4(1)$ (see the proof of Proposition 3.17), we deduce that:

$$\begin{aligned}\frac{\mathcal{Q}(1)}{\mathfrak{C}_6(n)\omega_\mu(\mathbb{S}^{2n-1})} =& 2(2n+1) \fint v \otimes v d\omega_\mu(v) + 4 \fint Jv \otimes Jv d\omega_\mu(v) - 2\mathrm{id} - 4 \fint (v \otimes v + Jv \otimes Jv) d\omega_\mu(v) \\ =& 2(2n-1) \fint u \otimes u d\omega_\mu(u) - 2\mathrm{id}.\end{aligned}$$

Therefore, thanks to (7.6), we deduce that for any $w \in \mathrm{supp}(\mu)$, we have:

$$0 = \langle w_H, \mathcal{Q}(1)w_H\rangle = 2\mathfrak{C}_6(n)\omega_\mu(\mathbb{S}^{2n-1})\Big((2n-1)\fint \langle v, w_H\rangle^2 d\omega_\mu(v) - |w_H|^2\Big).$$

□

Definition 7.3. For any vertical $(2n+1)$-uniform cone μ, define:

$$M := \fint u \otimes u d\omega_\mu(u),$$

where $\omega_\mu := \mathcal{H}^{2n-2}_{eu}\llcorner \pi_H(\mathrm{supp}(\mu)) \cap \mathbb{S}^{2n-1}$. Moreover, let $\alpha_1 \geq \ldots \geq \alpha_{2n} \geq 0$ be the eigenvalues of M and $e_1, \ldots, e_{2n}$ their relative eigenvectors.

Remark 7.4. The trace of the matrix M can be explicitly computed:

$$\mathrm{Tr}(M) = \sum_{i=1}^{2n} \alpha_i = \sum_{i=1}^{2n} \langle e_i, M e_i\rangle = \sum_{i=1}^{2n} \fint \langle u, e_i\rangle^2 d\omega_\mu(u) = 1.$$

The following proposition links the matrix M defined above to the functional $\min_{\mathfrak{m}\in\mathbb{S}^{2n-1}} \int_{B_1(0)} \langle \mathfrak{m}, z_H\rangle^2 d\mu(z)$, which will play a fundamental role in the proof of our main result.

Proposition 7.5. *There exists a constant $\mathfrak{C}_7(n) > 0$ for which for any vertical $(2n+1)$-uniform cone μ and any $\mathfrak{m} \in \mathbb{S}^{2n-1}$ we have:*

$$\int_{B_1(0)} \langle \mathfrak{m}, z_H\rangle^2 d\mu(z) = \mathfrak{C}_7(n)\langle \mathfrak{m}, M\mathfrak{m}\rangle.$$

Proof. Thanks to Lemma 7.1(i) we have that:

$$\int_{B_1(0)} \langle \mathfrak{m}, z_H\rangle^2 d\mu(z) = \frac{2}{\mathfrak{c}_n}\int_0^1 r^{2n}\sqrt{1-r^4}dr \int_{\mathbb{S}^{2n-1}} \langle \mathfrak{m}, v\rangle^2 d\omega_\mu(v) = \frac{2\mathfrak{C}_8(n)}{\mathfrak{c}_n}\int_{\mathbb{S}^{2n-1}} \langle \mathfrak{m}, v\rangle^2 d\omega_\mu(v),$$

where $\mathfrak{C}_8(n) := \int_0^1 r^{2n}\sqrt{1-r^4}dr$ and $\omega_\mu = \mathcal{H}^{2n-2}_{eu}\llcorner\mathbb{S}^{2n-1} \cap \pi_H(\mathrm{supp}(\mu))$. In order to prove the proposition, we are left to prove that $\omega_\mu(\mathbb{S}^{2n-1})$ is a constant depending only on n. Thanks to Lemma 7.1 implies that:

$$1 = \mu(B_1(0)) = \frac{2\omega_\mu(\mathbb{S}^{2n-1})}{\mathfrak{c}_n}\int_0^1 r^{2n-2}\sqrt{1-r^4}dr.$$

Therefore, defined $\mathfrak{C}_9 := \int_0^1 r^{2n-2}\sqrt{1-r^4}dr$ we have that $\omega_\mu(\mathbb{S}^{2n-1}) = \mathfrak{c}_n/2\mathfrak{C}_9(n)$ and in particular:

$$\int_{B_1(0)} \langle \mathfrak{m}, z_H\rangle^2 d\mu(z) = \frac{\mathfrak{C}_8(n)}{\mathfrak{C}_9(n)}\fint \langle \mathfrak{m}, v\rangle^2 d\omega_\mu(v) = \frac{\mathfrak{C}_8(n)}{\mathfrak{C}_9(n)}\langle \mathfrak{m}, M\mathfrak{m}\rangle.$$

□

An immediate consequence of Proposition 7.5 is that:

$$\min_{\mathfrak{m}\in\mathbb{S}^{2n-1}} \int_{B_1(0)} \langle \mathfrak{m}, z_H\rangle^2 d\mu(z) = \mathfrak{C}_7(n)\alpha_{2n} = \int_{B_1(0)} \langle e_{2n}, z_H\rangle^2 d\mu(z). \tag{7.8}$$

In particular we can link the value of $\int_{B_1(0)} \langle e_{2n}, z_H\rangle^2 d\mu(z)$ to the geometric structure of the measure μ thanks to the following:

Proposition 7.6. *Suppose μ is a vertical $(2n+1)$-uniform cone. For any $\delta > 0$ there exists an $\epsilon(\delta, n) > 0$ such that, if:*

$$\int_{B_1(0)} \langle e_{2n}, z_H \rangle^2 d\mu(z) \leq \epsilon(\delta, n),$$

then for any $x \in cl(U_1(0)) \cap e_{2n}^{\perp}$ there is $z \in \operatorname{supp}(\mu)$ for which $|z_H - e| \leq \delta$.

Proof. Assume by contradiction that there exists a $\delta > 0$, an infinitesimal sequence $\{\epsilon_i\}_{i\in\mathbb{N}}$ and a sequence of $(2n+1)$-uniform cones $\{\mu_i\}_{i\in\mathbb{N}}$ for which:

(α) defined $e_{2n}(i)$ be the minimum eigenvalue of the matrix $\int z_H \otimes z_H e^{-\|z\|^4} d\mu_i(z)$, we have:

$$\int_{B_1(0)} \langle z_H, e_{2n}(i) \rangle^2 d\mu_i(z) \leq \epsilon_i,$$

(β) there exists $x_i \in \operatorname{cl}(U_1(0)) \cap e_{2n}^{\perp}(i)$ for which $|z_H - x_i| \geq \delta$ for any $z \in \operatorname{supp}(\mu)$.

By compactness, up to non-relabeled subsequences, we can assume that $\mu_i \rightharpoonup \nu$, $e_{2n}(i) \to e$ and $x_i \to x$. Since $\langle \cdot, e_{2n}(i) \rangle^2 \chi_{B_1(0)}(\cdot)$ is uniformly converging to $\langle \cdot, e \rangle^2 \chi_{B_1(0)}(\cdot)$, we have:

$$\int_{B_1(0)} \langle z_H, e \rangle^2 d\nu(z) = 0.$$

This implies by Proposition 2.8 that ν is a $(2n+1)$-measure which support contained in $V(e)$. Therefore applying Proposition 5.6 we deduce that ν is flat.

On the other hand, since $\operatorname{supp}(\mu_i) \cap \operatorname{cl}(U_{\delta/2}(x_i)) \times \mathbb{R}e_{2n+1} = \emptyset$ for i sufficiently big, Proposition 2.8 imples that $\operatorname{supp}(\nu) \cap \operatorname{cl}(U_{\delta/2}(x)) \times \mathbb{R} = \emptyset$. This and the fact that $x \in e^{\perp}$ (see the assumptions on x_i in (β)) comes in contradiction with the fact that $\operatorname{supp}(\nu) = V(e)$. □

The following proposition shows that non-flat vertical $(2n+1)$-uniform cones are quantitatively disconnected from flat measures. The proof of this Therorem follows closely its Euclidean counterpart (see for instance Proposition 8.5 in [15]). The reason for this is the following.

Let μ be an m-uniform cone in $\mathbb{R}^n$. Then for any $w \in \operatorname{supp}(\mu)$ we have:

$$2\pi^{-m/2} \int \langle w, z \rangle^2 e^{-|z|^2} d\mu(z) = |w|^2,$$

see for instance identity (8.7) of Lemma 8.6 in [15]. The structure of the Euclidean quadric containing $\operatorname{supp}(\mu)$ has the same structure of the quadric in (7.5) which contains the support of vertical $(2n+1)$-uniform cones, and thus the same kind of algebraic computations work.

Theorem 7.7. *There exists a constant $\mathfrak{C}_{10}(n) > 0$ such that if μ is a vertical $(2n+1)$-uniform cone for which:*

$$\min_{\mathfrak{m} \in \mathbb{S}^{2n-1}} \int_{B_1(0)} \langle \mathfrak{m}, z_H \rangle^2 d\mu(z) \leq \mathfrak{C}_{10}(n),$$

then μ is flat.

Proof. Fix some $\delta > 0$ and suppose that $\min_{\mathfrak{m}\in\mathbb{S}^{2n-1}} \int_{B_1(0)} \langle \mathfrak{m}, z_H \rangle^2 d\mu(z) \leq \epsilon(\delta, n)/\mathfrak{C}_7(n)$. Identity (7.8) implies that $\int_{B_1(0)} \langle e_{2n}, z_H \rangle^2 d\mu(z) \leq \epsilon(\delta, n)$ and thus, thanks to Proposition 7.6, there exists a $z \in \operatorname{supp}(\mu)$ such that $|z_H - e_{2n-1}| \leq \delta$. Thanks to the order imposed on the α_i's, we have that $\alpha_i + (2n-2)\alpha_{2n-1} \leq \operatorname{Tr}(M) = 1$ for every $i \leq 2n-2$. This in particular implies that:

$$\alpha_i - \frac{1}{2n-1} \leq (2n-2)\left(\frac{1}{2n-1} - \alpha_{2n-1}\right), \text{ for every } i \leq 2n-2. \tag{7.9}$$

Since $z \in \mathrm{supp}(\mu)$, Proposition 7.2 implies (once (7.5) has been written in the notation of Definition 7.3) that:

$$0 = \sum_{i=1}^{2n} \left(\alpha_i - \frac{1}{2n-1} \right) \langle e_i, z_H \rangle^2. \tag{7.10}$$

We observe that since M is positive semidefinite and $\mathrm{Tr}(M) = 1$, we have that $\alpha_{2n} \leq 1/2n$. Therefore, putting together (7.9), (7.10) and the fact that $\alpha_{2n} \leq 1/2n$, we deduce that:

$$0 = \sum_{i=1}^{2n} \left(\alpha_i - \frac{1}{2n-1} \right) \langle e_i, z_H \rangle^2 \leq \sum_{i=1}^{2n-2} \left(\alpha_i - \frac{1}{2n-1} \right) \langle e_i, z_H \rangle^2 + \left(\alpha_{2n-1} - \frac{1}{2n-1} \right) \langle e_{2n-1}, z_H \rangle^2. \tag{7.11}$$

Summing up, inequalities (7.9) and (7.11) together with some algebraic manipulations imply:

$$\begin{aligned}
0 \leq &(2n-2)\Big(\frac{1}{2n-1} - \alpha_{2n-1}\Big) \sum_{i=1}^{2n-2} \langle e_i, z_H \rangle^2 - \Big(\frac{1}{2n-1} - \alpha_{2n-1}\Big)\langle e_{2n-1}, z_H \rangle^2 \\
= &(2n-2)\Big(\frac{1}{2n-1} - \alpha_{2n-1}\Big) \sum_{i=1}^{2n-2} \langle e_i, z_H \rangle^2 - \Big(\frac{1}{2n-1} - \alpha_{2n-1}\Big)(1 + \langle e_{2n-1}, z_H - e_{2n-1} \rangle)^2 \\
\leq &\Big(\frac{1}{2n-1} - \alpha_{2n-1}\Big)\Big((2n-2) \sum_{i=1}^{2n-2} |z_H - e_{2n-1}|^2 - (1 - |z_H - e_{2n-1}|)^2\Big) \\
\leq &\Big(\frac{1}{2n-1} - \alpha_{2n-1}\Big)((2n-2)^2\delta^2 - (1-\delta)^2),
\end{aligned} \tag{7.12}$$

where the last inequality comes from the choice of z. Therefore if δ is small enough, thanks to the inequality (7.12), we have that $\alpha_{2n-1} \geq 1/(2n-1)$. In this case, since M is positive semidefinite and $\mathrm{Tr}(M) = 1$, we have that:

$$\alpha_1 = \ldots = \alpha_{2n-1} = \frac{1}{2n-1} \qquad \text{and} \qquad \alpha_{2n} = 0.$$

This by the equation (7.8) implies that $\mathrm{supp}(\mu) \subseteq V(e_{2n})$ and thus by Proposition 5.6, μ must be flat. $\square$

8. Conclusions

In this section we complete the proof of Theorem 1.2. In order to conclude the proof we need to construct the continuous functional $\mathscr{F}$ on Radon measures which fulfills the hypothesis of Theorem 5.2.

Proposition 8.1. *Let η be a non-negative smooth function such that $\eta = 1$ on $B_1(0)$ and $\eta = 0$ on $B_2^c(0)$. Then the functional $\mathscr{F} : \mathcal{M} \to \mathbb{R}$ defined by:*

$$\mathscr{F}(\mu) := \min_{\mathfrak{m}\in\mathbb{S}^{2n-1}} \int \eta(z)\langle z_H, \mathfrak{m}\rangle^2 d\mu(z),$$

is continuous with respect to the weak convergence of measures.

Proof. Suppose that $\mu_i \rightharpoonup \nu$ and $\mathfrak{m}_i \in \mathbb{S}^{2n-1}$ are such that:

$$\int \eta(z)\langle z_H, \mathfrak{m}_i\rangle^2 d\mu_i(z) = \min_{\mathfrak{m}\in\mathcal{S}^{2n-1}} \int \eta(z)\langle z_H, \mathfrak{m}\rangle^2 d\mu_i(z).$$

Up to passing to a (non-relabeled) subsequence we can also suppose that $\mathfrak{m}_i$ converges to some $\mathfrak{m} \in \mathbb{S}^{2n-1}$. Thus the function $\eta(\cdot)\langle \pi_H(\cdot), \mathfrak{m}_i\rangle^2$ is uniformly converging to $\eta(\cdot)\langle \pi_H(\cdot), \mathfrak{m}\rangle^2$. Therefore we deduce that:

$$\lim_{i\to\infty} \int \eta(z)\langle z_H, \mathfrak{m}_i\rangle^2 d\mu_i(z) = \int \eta(z)\langle z_H, \mathfrak{m}\rangle^2 d\mu(z),$$

from which we infer that $\liminf_{i\to\infty} \mathscr{F}(\mu_i) \geq \mathscr{F}(\mu)$. On the other hand, let $\mathscr{F}(\nu) = \int \eta(z)\langle z_H, \overline{\mathfrak{m}}\rangle^2 d\mu(z)$ for some $\overline{\mathfrak{m}} \in \mathbb{S}^{2n-1}$. Since $\mu_i \rightharpoonup \nu$, we deduce that:

$$\lim_{i\to\infty} \int \eta(z)\langle z_H, \overline{\mathfrak{m}}\rangle^2 d\mu_i(z) = \int \eta(z)\langle z_H, \overline{\mathfrak{m}}\rangle^2 d\mu(z).$$

This implies that $\limsup_{i\to\infty} \mathscr{F}(\mu_i) \leq \mathscr{F}(\mu)$ and this concludes the proof. □

The following proposition shows that $\mathfrak{F}$ satisfies the hypothesis (ii) of Theorem 5.2.

Proposition 8.2. *There exists a constant $\hbar(n) > 0$ such that if μ is a $(2n+1)$-uniform cone and $\mathscr{F}(\mu) \leq \hbar(n)$, then μ is flat.*

Proof. Thanks to the definition of $\mathscr{F}$, we have that $\min_{\mathfrak{m}\in\mathbb{S}^{2n-1}} \int \langle z_H, \mathfrak{m}\rangle^2 d\mu(z) \leq \mathscr{F}(\mu)$. Therefore thanks to Theorems 6.9 and 7.7, if $\mathscr{F}(\mu) \leq \min\{\mathfrak{C}_3(n), \mathfrak{C}_{10}(n)\}/2 =: \hbar(n)$ the measure μ is flat. □

Eventually we are ready to prove our main result Theorem 1.2.

Theorem 8.3. *If ϕ is a measure with $(2n+1)$-density, then $Tan_{2n+1}(\phi, x) \subseteq \mathfrak{M}(2n+1)$ for ϕ-almost all $x \in \mathbb{H}^n$.*

Proof. Since $\mathscr{F}(\mu) = 0$, whenever μ is flat, Propositions 8.1 and 8.2 imply that we are in the hypothesis of Theorem 5.2. which proves the claim. □

A byproduct of our analysis is the conclusion of the classification of uniform measures in $\mathbb{H}^1$, which was systematically carried out in [13]. Our contribution is to prove that 3-uniform measures in $\mathbb{H}^1$ are flat. The final result reads:

Theorem 8.4. *In $\mathbb{H}^1$ we have the following complete classification of uniform measures:*

(i) If $\mu \in \mathcal{U}_{\mathbb{H}^1}(1)$, then $\mu = \mathcal{S}^1 \llcorner L$, where L is a horizontal line.

(ii) If $\mu \in \mathcal{U}_{\mathbb{H}^1}(2)$ then $\mu = \mathcal{S}^2 \llcorner \mathcal{V}$, where $\mathcal{V}$ is the vertical axis.

(iii) If $\mu \in \mathcal{U}_{\mathbb{H}^1}(3)$ then $\mu = \mathcal{S}^3 \llcorner W$, where W is a 2-dimensional vertical plane.

Proof. Points (i) and (ii) are Theorems 1.3 and 1.4 of [13]. Corollary 3.33 implies that if μ is supported on a quadric. Proposition 1.6 in [13] implies that such quadric cannot be a t-graph and therefore by Theorem 1.5 of the same paper we conclude the proof. □

A. Representations of sub-Riemannian spherical Hausdorff measure

In the following we will adopt the notations introduced in Section 4 and as usual we assume $b \in \mathbb{R}^{2n}$, $\mathcal{Q} \in \mathrm{Sym}(2n)$ and $\mathcal{T} \in \mathbb{R}$. The main goal of this section is to find a representation of the spherical Hausdorff measures $\mathcal{S}^{2n+1}_A$, where A is a Borel subset of the quadric $\mathbb{K}(b, \mathcal{Q}, \mathcal{T})$, in terms of the Euclidean Hausdorff measure $\mathcal{H}^{2n}_{eu}$. To do so, we will study separately the case where $\mathcal{T} \neq 0$ and the case $\mathcal{T} = 0$.

Before proceeding we need to review the definition and some well known facts about the horizontal perimeter measure in $\mathbb{H}^n$. For any $x \in \mathbb{H}^n$ and any $i \in \{1, \dots, n\}$ we define the vector fields:

$$X_i(x) := e_i - 2x_{i+n}e_{2n+1}, \qquad Y_i(x) := e_{i+n} + 2x_i e_{2n+1},$$

where $\{e_1, \dots, e_{2n+1}\}$ is the standard othonormal basis of $\mathbb{R}^{2n+1}$. We introduce here the *horizontal perimeter* of a set $E \subseteq \mathbb{H}^n$:

Definition A.1. The *horizontal perimeter* in an open set $\Omega \subseteq \mathbb{H}^n$ of a Lebesgue measurable set $E \subseteq \mathbb{H}^n$ is:

$$P(E, \Omega) := \sup\left\{ \int_E \mathrm{div}_{\mathbb{H}}\varphi dx : \varphi \in \mathcal{C}_c(\Omega, \mathbb{R}^{2n}),\ \|\varphi\|_\infty \leq 1 \right\},$$

where $\mathrm{div}_{\mathbb{H}}\varphi = \sum_{i=1}^n (X_i\varphi_i + Y_i\varphi_{n+i})$ and $\|\cdot\|_\infty$ is the usual supremum norm. If $P(E, \Omega) < \infty$ we say that E has finite horizontal perimeter in Ω.

Thanks to Riesz' representation theorem, since $\int_E \mathrm{div}_{\mathbb{H}}\varphi \leq P(E, \Omega)\|\varphi\|_\infty$, there is a positive Radon measure $|\partial E|_\Omega$ and a $|\partial E|_\Omega$-measurable function $\mathfrak{n}_H : \Omega \to \mathbb{R}^{2n}$ such that:

$$\int_E \mathrm{div}_{\mathbb{H}}\varphi dx = \int_\Omega \langle \varphi, \mathfrak{n}_H \rangle d|\partial E|_\Omega,$$

for any $\varphi \in \mathcal{C}_c(\Omega, \mathbb{R}^{2n})$. Denoted with $\mathfrak{n}$ the Euclidean unit normal to the quadric $\mathbb{K}(b, \mathcal{Q}, \mathcal{T})$ at x, we let $\mathfrak{n}_H(x)$ be the vector:

$$\mathfrak{n}_H(x) := (\langle X_1(x), \mathfrak{n}(x)\rangle, \dots, \langle X_n(x), \mathfrak{n}(x)\rangle, \langle Y_1(x), \mathfrak{n}(x)\rangle, \dots, \langle Y_{2n}(x), \mathfrak{n}(x)\rangle),$$

which is commonly known as the *horizontal normal* to $\mathbb{K}(b, \mathcal{Q}, \mathcal{T})$ at x. The vector $\mathfrak{n}_H(x)$ allows us to get the following useful:

Proposition A.2 (Monti, [33]). *Let $E \subseteq \mathbb{H}^n$ be a set with Euclidean Lipschitz boundary and Ω a fixed open set in $\mathbb{H}^n$. Then:*

$$|\partial E|_\Omega(A) = \int \chi_A |\mathfrak{n}_H| d\mathcal{H}^{2n} \llcorner \partial E \cap \Omega,$$

for any open set $A \subseteq \mathbb{H}^n$.

The above proposition together with Proposition 1.9.4 of [10] implies that the measures $|\partial E|_\Omega$ and $|\mathfrak{n}_H|\mathcal{H}^{2n}\llcorner\partial E\cap\Omega$ coincide on Borel sets.

Remark A.3. Note that if $\mathcal{T}=0$, then $\mathfrak{n}(x)$ is parallel to the vector $N(x):=(2\mathcal{Q}x+b,0)$. Moreover, since $\langle N(x),e_{2n+1}\rangle=0$, the projection of $N(x)$ on the plane spanned by $X_1,\dots,Y_{2n}$ coincides with $N(x)$ itself. Therefore in the case $\mathcal{T}=0$ we have that $\mathfrak{n}_H=\mathfrak{n}$.

A.1 Area on t-quadratic graphs

In this subsection we deal with the representation of spherical Hausdorff measure concentrated on subsets of horizontal quadrics. Therefore we can suppose without loss of generality that $\mathcal{T}=-1$ and let $f(x):=\langle b,x\rangle+\langle x,\mathcal{Q}x\rangle$ be the function defined in (4.4) and $\Sigma(f):=\{x\in\mathbb{R}^{2n}:2(\mathcal{Q}+J)x+b=0\}$ be the characteristic set of f, which was introduced in (4.5). Thanks to the simple algebraic context in which we are, the following characterisation of $\Sigma(f)$ is available:

Proposition A.4. *The set $\Sigma(f)$ is an affine plane in $\mathbb{R}^{2n}$ of dimension at most n.*

Proof. The proof is a simple argument by contradiction which yields the following stronger result. If A is a symmetric matrix and B is an invertible antisymmetric matrix then $\dim(\ker(A-B))\leq n$. □

It is worth noting that Proposition A.4 is a very simple case of the results obtained by Z. Balogh in [8].

Proposition A.5. *Let Ω be an open set in $\mathbb{R}^{2n}$ and define $\Omega':=\Omega\times\mathbb{R}$. For any positive Borel function $h:\mathbb{H}^n\to\mathbb{R}$ we have:*

$$\int h(z)d\mathcal{S}^{2n+1}_{\Omega'\cap\mathbb{K}(b,\mathcal{Q},\mathcal{T})}(z)=\frac{1}{\mathfrak{c}_n}\int_\Omega h(x,f(x))|b+2(\mathcal{D}+J)x|dx,$$

where $\mathfrak{c}_n=\mathcal{H}^{2n}_{eu}(B_1(0)\cap V(e))$ for any $e\in\mathbb{R}^{2n}$.

Proof. Let $E:=\{z\in\mathbb{R}^{2n+1}:z_T-f(z_H)\geq 0\}$ and note that by Proposition 3.2 of [33] we have that E is a set of intrinsic finite perimeter and:

$$|\partial E|_{\Omega'}(A)=\int_{\Omega\cap\pi_H(A)}|\nabla f(x)+2Jx|dx=\int_{\Omega\cap\pi_H(A)}|b+2(\mathcal{Q}+J)x|dx,$$

for any Borel set $A\subseteq\mathbb{H}^n$. It is immediate to see that the measure theoretic boundary of E (see for instance Definition 7.4 in [23]) coincides with $\mathbb{K}(b,\mathcal{Q},-1)$ and thus, thanks to Corollary 7.6 of [23], we deduce that:

$$|\partial E|_{\Omega'}=\mathfrak{c}_n\mathcal{S}^{2n+1}_{\Omega'\cap\mathbb{K}(b,\mathcal{Q},-1)}.\tag{A.1}$$

So far we proved that:

$$\mathcal{S}^{2n+1}_{\Omega'\cap\mathbb{K}(b,\mathcal{Q},-1)}(A)=\frac{1}{\mathfrak{c}_n}\int_\Omega\chi_A(x,f(x))|b+2(\mathcal{Q}+J)x|dx.$$

The standard approximation procedure of positive measurable functions together with Beppo-Levi convergence theorem concludes the proof. □

Remark A.6. The constant $\mathfrak{c}_n$ differs from the one of Corollary 7.6 of [23] since we defined the spherical Hausdorff in such a way that $\mathcal{S}^{2n+1}(B_1(0)\cap V(e))=1$. Furthermore we are using the Koranyi norm, while in [23] an equivalent, but different metric is used. We refer to the aformentioned article for further details.

A.2 Area on vertical quadric

In this subsection we deal with the representation of spherical Hausdorff measure concentrated on subsets of vertical quadrics. Therefore we let $F(x,t) := \langle b, x\rangle + \langle x, \mathcal{Q}x\rangle$ be the function defined in (4.1) which zeroes coincide with $\mathbb{K}(b,\mathcal{Q},0)$, and $\Sigma(F) : \{z \in \mathbb{K}(b,\mathcal{Q},0) : 2\mathcal{Q}x + b = 0\}$ be the set of singular points of $\mathbb{K}(b,\mathcal{Q},0)$, which was introduced in (4.2).

Proposition A.7. *Let a, b be two non-parallel vectors in $\mathbb{R}^{2n}$. Then $\mathcal{S}^{2n+1}(V(a)\cap V(b)) = 0$, where $V(\cdot)$ were introduced in Definition 4.2.*

Proof. The set $V(a)$ can be seen as the bounduary of the Lipschitz domain $\{x \in \mathbb{H}^n : \langle a, x_H\rangle \geq 0\}$. Therefore Proposition A.2 and the equivalence between the measures $\mathcal{S}_A^{2n+1}$ and $\mathcal{S}^{2n+1}\llcorner A$, implies that:

$$\mathcal{S}^{2n+1}(V(a)\cap V(b)) = \mathcal{H}^{2n}_{eu}(V(a)\cap V(b))/\mathfrak{c}_n = 0.$$

□

Proposition A.8. *One of the following two holds:*

(i) $b = 0$ *and* $\mathcal{Q} = a\otimes a$ *for some* $a \in \mathbb{R}^{2n}\setminus\{0\}$,

(ii) $\mathcal{S}^{2n+1}(\Sigma(F)) = 0$.

Proof. If $\dim(\ker(\mathcal{Q})) \leq 2n-2$, then $\Sigma(F)$ is contained in an affine subspace of dimension at most $2n-2$. Thus Proposition A.7 implies that $\mathcal{S}^{2n+1}(\Sigma(F)) = 0$. On the other hand, if $\dim(\ker(\mathcal{Q})) = 2n-1$, we have that $Q = a\otimes a$ for some $a \in \mathbb{R}^{2n}$ and the expression for F boils down to $F(x,t) = \langle a, x\rangle^2 + \langle b, x\rangle$. If $b \neq 0$ and it is not parallel to a, the equation $2\langle a, z_H\rangle a + b = 0$ can never be satisfied and thus $\Sigma(F) = \emptyset$. At last, if $a = \lambda b$, then $\mathbb{K}(b,\mathcal{Q},0) \subseteq V(a)\cup \lambda a + V(a)$, while $\Sigma(F) \subseteq \lambda a/2 + V(a)$, which implies $\Sigma(F) = \emptyset$. The only left out case is when $b = 0$ in which $\Sigma(F) = V(a)$, which proves the claim. □

Finally we can prove the representation formula for the spherical Hausdorff measure concentrated on vertical quadrics:

Proposition A.9. *Let Ω be an open set in $\mathbb{H}^n$. For any positive Borel function $h : \mathbb{H}^n \to \mathbb{R}$ we have:*

$$\int h(x)d\mathcal{S}^{2n+1}_{\mathbb{K}(b,\mathcal{Q},0)\cap\Omega}(x) = \frac{1}{\mathfrak{c}_n}\int h(x)d\mathcal{H}^{2n}_{eu}\llcorner(\mathbb{K}(b,\mathcal{Q},0)\cap\Omega)(x),$$

where $\mathfrak{c}_n$ is the constant introduced in Proposition A.5.

Proof. Let E be the open set $E := \{z \in \mathbb{R}^{2n+1} : F(z) < 0\}$ and note that $\partial E = \{F = 0\} = \mathbb{K}(b,\mathcal{Q},0)$. Thanks to Proposition A.2, since E has a Lipschitz boundary, we have that E is a set of intrinsic finite perimeter and:

$$|\partial E|_\Omega(A) = \int_{\partial E\cap\Omega} \chi_A|\mathfrak{n}_H|d\mathcal{H}^{2n}_{eu} = \mathcal{H}^{2n}_{eu}(\mathbb{K}(b,\mathcal{Q},0)\cap\Omega\cap A),$$

for any Borel set A of $\mathbb{H}^n$, since $|\mathfrak{n}_H| = 1$. It is easy to see that the measure theoretic boundary of E, see for instance Definition 7.4 of [23], coincides with $\mathbb{K}(b,\mathcal{Q},0)$. Therefore, thanks to Corollary 7.6 of [23] we have that:

$$|\partial E|_\Omega = \mathfrak{c}_n\mathcal{S}^{2n+1}_{\mathbb{K}(b,\mathcal{Q},0)\cap\Omega},$$

Summing up, we have that for any Borel set A in $\mathbb{H}^n$:

$$|\partial E|_\Omega(A) = \mathcal{H}^{2n}_{eu}(\mathbb{K}(b,\mathcal{Q},0)\cap\Omega\cap A)/\mathfrak{c}_n.$$

The usual approximation of positive measurable functions together with Beppo-Levi convergence theorem concludes the proof. □

Since the set $\mathrm{supp}(\mu) \setminus \Sigma(F)$ is relatively open set in $\mathbb{K}(b, \mathcal{Q}, 0)$ thanks to Proposition 4.10, we can find an open set Ω in $\mathbb{H}^n$ such that $\mathbb{K}(b, \mathcal{Q}, 0) \cap \Omega = \mathrm{supp}(\mu) \setminus \Sigma(F)$. Therefore Proposition A.9 implies that:

$$\mathcal{S}^{2n+1}_{\mathrm{supp}(\mu)\setminus\Sigma(F)} = \frac{\mathcal{H}^{2n}_{eu} \llcorner \mathrm{supp}(\mu) \setminus \Sigma(F)}{\mathfrak{c}_n}.$$

Note the above identity in the case in which $b = 0$ and $\mathcal{Q} = a \otimes a$, can be rewritten as $\mathcal{S}^{2n+1}_{\mathrm{supp}(\mu)} = \mathcal{H}^{2n}_{eu} \llcorner \mathrm{supp}(\mu)/\mathfrak{c}_n$ since the Euclidean normal is well defined on the whole $V(a)$ despite the fact that $\Sigma(F) = V(a)$. Thanks to Proposition A.8 we can assume that $\mathcal{S}^{2n+1}(\Sigma(F)) = 0$ and thus:

$$\mathcal{S}^{2n+1}_{\mathrm{supp}(\mu)} = \frac{\mathcal{H}^{2n}_{eu} \llcorner \mathrm{supp}(\mu)}{\mathfrak{c}_n}. \tag{A.2}$$

Lemma A.10. *If μ is a vertical $(2n + 1)$-uniform cone then:*

$$\mu = \frac{1}{\mathfrak{c}_n} \mathcal{H}^{2n-1}_{eu} \llcorner \pi_H(\mathrm{supp}(\mu)) \otimes \mathcal{H}^1_{eu} \llcorner \mathbb{R}e_{2n+1}.$$

Proof. First of all, thanks to the identity (A.2) used in the first and second equality respectively, we deduce that:

$$\mu = \mathcal{S}^{2n+1}_{\mathrm{supp}(\mu)} = \frac{1}{\mathfrak{c}_n} \mathcal{H}^{2n}_{eu} \llcorner \mathrm{supp}(\mu).$$

Since by Proposition 4.10 we have that $\mathrm{supp}(\mu) = \pi_H(\mathrm{supp}(\mu)) \times \mathbb{R}e_{2n+1}$, Proposition 3.2.23 of [21] implies:

$$\mathcal{H}^{2n}_{eu} \llcorner \mathrm{supp}(\mu) = \mathcal{H}^{2n-1}_{eu} \llcorner \pi_H(\mathrm{supp}(\mu)) \otimes \mathcal{H}^1_{eu} \llcorner \mathbb{R}e_{2n+1}.$$

□

B. Taylor expansion of area on quadratic t-cones

Before giving a short account on the content of this appendix, we introduce some notation. Let $\mathcal{D} \in \text{Sym}(2n) \setminus \{0\}$ and define the function $f : \mathbb{R}^{2n} \to \mathbb{R}$ as:

$$f(h) := \langle h, \mathcal{D}h \rangle.$$

Furthermore, we let $|\partial\mathbb{K}|$ be the intrinsic perimeter measure of the epigraph of f in $\mathbb{H}^n$, which is well defined since f is a smooth function, see Proposition A.2. Throughout this entire section, we fix a point $x \notin \Sigma(f)$ (recall that $\Sigma(f)$ was introduced in (4.5)), and we let $\mathcal{X} := (x, f(x))$.

The main goal of this section is to determine an asymptotic expansion of $|\partial\mathbb{K}|(B_r(\mathcal{X}))$ for r small. More precisely, written:

$$|\partial\mathbb{K}|(B_r(\mathcal{X})) = \mathfrak{c}(\mathcal{X})r^{2n+1} + \mathfrak{d}(\mathcal{X})r^{2n+2} + \mathfrak{e}(\mathcal{X})r^{2n+3} + O(r^{2n+4}),$$

we want to find an expression for the coefficients $\mathfrak{c}, \mathfrak{d}, \mathfrak{e}$ in terms of x, $\mathcal{D}$ and n. The coefficient $\mathfrak{c}$ will be quite easy to study and we will show that it is a constant depending only on n. On the other hand the coefficients $\mathfrak{d}$ and $\mathfrak{e}$ will need much more work and they will play a fundamental role in the study of the geometric properties of 1-codimensional uniform measures carried on in Section 6.

Definition B.1. Let $\mathcal{D}$, $\mathcal{X}$ and f be as above. We denote as:

$$\mathfrak{n} := \frac{(\mathcal{D} + J)x}{|(\mathcal{D} + J)x|},$$

the horizontal normal at $\mathcal{X}$ to $\text{gr}(f)$ and moreover we let $c := 2|(\mathcal{D} + J)x|$.

The following proposition gives a first characterisation of the shape of the intersection between $B_r(\mathcal{X})$ with $\text{gr}(f)$. In particular we construct a function G at the point x whose sublevel sets are the horizontal projection of $B_r(\mathcal{X}) \cap \text{gr}(f)$.

Proposition B.2. *Following the notations introduced above, we have:*

$$\pi_H(B_r(\mathcal{X}) \cap gr(f)) = x + \left\{ w \in \mathbb{R}^{2n} : G(w) \leq r^4 \right\},$$

where $G(w) := |w|^4 + |c\langle \mathfrak{n}, w \rangle + \langle w, \mathcal{D}w \rangle|^2$.

Proof. By definition of f and of the Koranyi norm, we have:

$$B_r(\mathcal{X}) := \{z \in \mathbb{R}^{2n+1} : |z_H - x|^4 + |z_T - f(x) - 2\langle x, Jz_H \rangle|^2 \leq r^4\}.$$

Therefore, the intersection of $B_r(\mathcal{X})$ with $\mathrm{gr}(f)$ is:

$$\begin{aligned} B_r(\mathcal{X}) \cap \mathrm{gr}(f) =&\{(y, f(y)) \in \mathbb{R}^{2n+1} : |x-y|^4 + |f(x) - f(y) + 2\langle x, Jy\rangle|^2 \leq r^4\} \\ =&x + \{(w, f(w)) \in \mathbb{R}^{2n+1} : |w|^4 + |f(x) - f(x+w) + 2\langle x, Jw\rangle|^2 \leq r^4\}, \end{aligned}$$

where in the last line we have performed the change of variable $y = x + w$. By definition of f, we have:

$$f(x) - f(x+w) = \langle x, \mathcal{D}x\rangle - \langle x+w, \mathcal{D}(x+w)\rangle = -2\langle x, \mathcal{D}w\rangle - \langle w, \mathcal{D}w\rangle.$$

In particular, this implies that:

$$\begin{aligned} \pi_H(B_r(\mathcal{X}) \cap \mathrm{gr}(f)) =&\{w \in \mathbb{R}^{2n} : |w|^4 + |-2\langle x, \mathcal{D}w\rangle - \langle w, \mathcal{D}w\rangle + 2\langle x, Jw\rangle|^2 \leq r^4\} \\ =&\{w \in \mathbb{R}^{2n} : |w|^4 + |c\langle \mathfrak{n}, w\rangle + \langle w, \mathcal{D}w\rangle|^2 \leq r^4\}. \end{aligned} \tag{B.1}$$

Identity (B.1) and the definition of G conclude the proof. ☐

The following proposition introduces a special set of polar coordinates, which are going to be very useful in the study of the intersection $B_r(\mathcal{X}) \cap \mathrm{gr}(f)$ when r is small.

Proposition B.3. *For any $w \in \mathbb{R}^{2n} \setminus x + \mathrm{span}(\mathfrak{n})$ there exists a unique triple $(\vartheta, \rho, v) \in \mathscr{C} := [-\frac{\pi}{2}, \frac{\pi}{2}) \times (0, \infty) \times \mathbb{S}^{2n-1} \cap \mathfrak{n}^\perp$ such that:*

$$w = x + \frac{\sin\vartheta}{c}\rho^2\mathfrak{n} + \cos\vartheta\rho v =: x + \mathcal{P}(\vartheta, \rho, v). \tag{B.2}$$

Proof. Any $w \in \mathbb{R}^{2n} \setminus x + \mathrm{span}(\mathfrak{n})$ can be uniquely written as $w = x + \lambda n + u$ for some $\lambda \in \mathbb{R}$ and $u \in \mathfrak{n}^\perp \neq 0$. Defined $\rho := \sqrt{\big(\|u\|^2 + \sqrt{\|u\|^4 + 4\lambda^2 c^2}\big)/2}$, we have that $\rho \neq 0$ and:

$$\left(\frac{c\lambda}{\rho^2}\right)^2 + \left(\frac{\|u\|}{\rho}\right)^2 = 1, \tag{B.3}$$

since ρ^2 solves the equation $\zeta^2 - \|u\|^2\zeta - c^2\lambda^2 = 0$. By (B.3), there is a unique $\vartheta \in [-\pi/2, \pi/2)$ for which $\sin\vartheta = c\lambda/\rho^2$ and $\cos\vartheta = \|u\|/\rho$. Eventually, if we let $v := u/\|u\|$, thanks to the definition of θ, we have:

$$w = x + \lambda n + \|u\|v = x + \frac{\sin\vartheta}{c}\rho^2\mathfrak{n} + \cos\vartheta\rho v.$$

☐

In order to simplify the notations in the forthcoming propositions, we define:

$$\alpha_{\mathfrak{n}} := \langle \mathfrak{n}, \mathcal{D}\mathfrak{n}\rangle, \qquad \beta_{\mathfrak{n}}(v) := \langle v, \mathcal{D}\mathfrak{n}\rangle, \qquad \gamma(v) := \langle v, \mathcal{D}v\rangle \qquad \text{for } v \in \mathbb{R}^{2n}. \tag{B.4}$$

In the following proposition we give an explicit expression of G in the new polar coordinates $\mathcal{P}(\vartheta, \rho, v)$ introduced in Proposition B.3.

Proposition B.4. *Let us define the function $H : \mathscr{C} \to \mathbb{R}$ as:*

$$H(\vartheta, \rho, v) := G(\mathcal{P}(\vartheta, \rho, v)),$$

where G was introduced in Proposition B.2 and both $\mathscr{C}$ and $\mathcal{P}(\vartheta, \rho, v)$ were defined in Proposition B.3. Then H has the following explicit expression:

$$H(\vartheta, \rho, v) := A(\vartheta, v)\rho^4 + \frac{\overline{B}(\vartheta, v)}{c}\rho^5 + \frac{\overline{C}(\vartheta, v)}{c^2}\rho^6 + \frac{\overline{D}(\vartheta, v)}{c^3}\rho^7 + \frac{\overline{E}(\vartheta, \rho)}{c^4}\rho^8,$$

where:

(i) $A(\vartheta, v) := (\cos\vartheta^4 + (\cos^2\vartheta\gamma(v) + \sin\vartheta)^2)$,

(ii) $\overline{B}(\vartheta, v) := cB(\vartheta, v) := 4\sin\vartheta\cos\vartheta\beta_{\mathfrak{n}}(v)(\cos^2\vartheta\gamma(v) + \sin\vartheta)$,

(iii) $\overline{C}(\vartheta, v) := c^2C(\vartheta, v) := \sin^2\vartheta(\cos^2\vartheta(2 + 4\beta_{\mathfrak{n}}(v)^2 + 2\gamma(v)\alpha_{\mathfrak{n}}) + 2\sin\vartheta\alpha_{\mathfrak{n}})$,

(iv) $\overline{D}(\vartheta, v) := c^3D(\vartheta, v) := 4\alpha_{\mathfrak{n}}\beta_{\mathfrak{n}}(v)\sin\vartheta^3\cos\vartheta$,

(v) $\overline{E}(\vartheta) := c^4E(\vartheta) := (1 + \alpha_{\mathfrak{n}}^2)\sin^4\vartheta$,

and (ϑ, v) *varies in* $[-\pi/2, \pi/2) \times \mathbb{S}^{2n-1} \cap \mathfrak{n}^{\perp}$.

Proof. For any $x + w \in \mathbb{R}^{2n} \setminus x + \text{span}(\mathfrak{n})$, by Proposition B.3 we can find a unique $(\vartheta, \rho, v) \in \mathscr{C}$ such that $w = \mathcal{P}(\vartheta, \rho, v)$. Note that the following identities hold:

$$\begin{aligned}
|w|^4 &= |\mathcal{P}(\vartheta, \rho, v|^4 = \frac{\sin\vartheta^4}{c^4}\rho^8 + 2\frac{\sin\vartheta^2\cos^2\vartheta}{c^2}\rho^6 + \cos^4\vartheta\rho^4,\\
c\langle\mathfrak{n}, w\rangle &= c\langle\mathfrak{n}, \mathcal{P}(\vartheta, \rho, v)\rangle = \left\langle c\mathfrak{n}, \frac{\sin\vartheta}{c}\rho^2\mathfrak{n} + \cos\vartheta\rho v\right\rangle = \sin\vartheta\rho^2,\\
\langle w, \mathcal{D}w\rangle &= \langle\mathcal{P}(\vartheta, \rho, v), \mathcal{D}[\mathcal{P}(\vartheta, \rho, v)]\rangle = \frac{\sin\vartheta^2}{c^2}\rho^4\alpha_{\mathfrak{n}} + 2\frac{\sin\vartheta\cos\vartheta}{c}\rho^3\beta_{\mathfrak{n}}(v) + \cos^2\vartheta\rho^2\gamma(v).
\end{aligned} \tag{B.5}$$

Therefore, thanks to the definition of G and H and the three identities in (B.5), we have:

$$\begin{aligned}
H(\rho, \vartheta, v) =&G(\mathcal{P}(\vartheta, \rho, v)) = |\mathcal{P}(\vartheta, \rho, v)|^4 + |c\langle\mathfrak{n}, \mathcal{P}(\vartheta, \rho, v)\rangle + \langle\mathcal{P}(\vartheta, \rho, v), \mathcal{D}[\mathcal{P}(\vartheta, \rho, v)]\rangle|^2\\
=&\frac{\sin\vartheta^4}{c^4}\rho^8 + 2\frac{\sin\vartheta^2}{c^2}\cos^2\vartheta\rho^6 + \cos^4\vartheta\rho^4 + (\cos^2\vartheta\gamma(v) + \sin\vartheta)^2\rho^4 + 4\frac{\sin^2\vartheta\cos^2\vartheta}{c^2}\beta_{\mathfrak{n}}(v)^2\rho^6\\
&+\frac{\sin^4\vartheta}{c^4}\alpha_{\mathfrak{n}}^2\rho^8 + 4\frac{\sin\vartheta\cos\vartheta}{c}(\cos^2\vartheta\gamma(v) + \sin\vartheta)\beta_{\mathfrak{n}}(v)\rho^5 + 2(\cos^2\vartheta\gamma(v) + \sin\vartheta)\frac{\sin^2\vartheta}{c^2}\alpha_{\mathfrak{n}}\rho^6\\
&+4\frac{\sin\vartheta^3\cos\vartheta}{c^3}\alpha_{\mathfrak{n}}\beta_{\mathfrak{n}}(v)\rho^7.
\end{aligned}$$

The claim follows recollecting the various powers of ρ. □

We summarize in the following lemma some algebraic properties of the functions $A, \ldots, E$ introduced in Proposition B.4, as they will be very useful in the forthcoming computations.

Lemma B.5. *Consider* $(\vartheta, v) \in [-\pi/2, \pi/2) \times \mathbb{S}^{2n} \cap n^{\perp}$ *fixed. Then:*

(i) $A(\vartheta, v) = A(\vartheta, -v)$ *and* $C(\vartheta, v) = C(\vartheta, -v)$,

(ii) $B(\vartheta, v) = -B(\vartheta, -v)$ *and* $D(\vartheta, v) = -D(\vartheta, -v)$,

(iii) $B(\vartheta, v) = -B(-\vartheta, v)$ *and* $D(\vartheta, v) = -D(-\vartheta, v)$,

(iv) E *does not depend on* v,

(v) A *is bounded away from* 0, *i.e.:*

$$\omega := \min_{[-\pi/2, \pi/2) \times \mathbb{S}^{2n} \cap n^{\perp}} A(\vartheta, v) > 0.$$

Proof. The first four points are direct consequence of the definition of A, B, C, D, E, and of $\alpha_{\mathfrak{n}}, \beta_{\mathfrak{n}}(v), \gamma(v)$ (see (B.4)). We are left to prove the last point. Since $A(\pi/2, v) = A(-\pi/2, v)$, the function A has minimum in $(-\pi/2, \pi/2) \times \mathbb{S}^{2n} \cap n^{\perp}$. Suppose such minimum is 0 and it is attained at (ϑ, v). This would imply that $0 = \cos\vartheta^4 + (\cos^2\vartheta\gamma(v) + \sin\vartheta)^2$, but this is not possible as it would force $\sin\vartheta = \cos\vartheta = 0$. □

The following proposition allows us to determine precisely the shape of the set $\pi_H(B_r(\mathcal{X})\cap \mathrm{gr}(f))$ when r is small:

Proposition B.6. *There exists an* $0<\mathfrak{r}_1(\mathcal{X})=\mathfrak{r}_1$ *such that for any* $0<r<\mathfrak{r}_1$, *if* $\rho(r)$ *is a solution to the equation:*

$$H(\rho(r),\vartheta,v)=r^4, \tag{B.6}$$

then:

$$\rho(r)=P_{\vartheta,v}(r)+O(r^4):=\frac{r}{A^{\frac{1}{4}}}-\frac{Br^2}{4A^{\frac{3}{2}}}+\left(\frac{7}{32}\frac{B^2}{A^{\frac{11}{4}}}-\frac{C}{4A^{\frac{7}{4}}}\right)r^3+O(r^4), \tag{B.7}$$

and the remainder $O(r^4)$ *is independent on* v *and* ϑ.

Proof. By Proposition B.4, equation (B.6) turns into:

$$A\rho(r)^4+B\rho(r)^5+C\rho(r)^6+D\rho(r)^7+E\rho(r)^8=r^4, \tag{B.8}$$

where we dropped the dependence on v and ϑ from A, B, C, D and E to simplify the notation.

We claim that there are $\mathfrak{r}_1>0$ and $\mathfrak{c}_1>0$, independent on ϑ and v, such that $\rho(r)\leq \mathfrak{c}_1 r$ for any $0<r<\mathfrak{r}_1$. Suppose by contradiction there exists a sequence (r_i,ϑ_i,v_i) such that $G(\rho(r_i),\vartheta_i,v_i)=r_i^4$ for any $i\in\mathbb{N}$, $\{r_i\}_{i\in\mathbb{N}}$ is infinitesimal and $\rho(r_i)>2r_i/\omega^{1/4}$, where ω was introduced in Proposition B.4(v). By (B.8), we have that for any $i\in\mathbb{N}$:

$$\begin{aligned}1=&A\left(\frac{\rho(r_i)}{r_i}\right)^4+B\left(\frac{\rho(r_i)}{r_i}\right)^5r_i+C\left(\frac{\rho(r_i)}{r_i}\right)^6r_i^2+D\left(\frac{\rho(r_i)}{r_i}\right)^7r_i^3+E\left(\frac{\rho(r_i)}{r_i}\right)^8r_i^4\\ >&A\left(\frac{2}{\omega^{1/4}}\right)^4+B\left(\frac{2}{\omega^{1/4}}\right)^5r_i+C\left(\frac{2}{\omega^{1/4}}\right)^6r_i^2+D\left(\frac{2}{\omega^{1/4}}\right)^7r_i^3+E\left(\frac{2}{\omega^{1/4}}\right)^8r_i^4.\end{aligned} \tag{B.9}$$

Define the constant:

$$M:=\left(1+\frac{2}{\omega^{1/4}}\right)^8\max_{[-\pi/2,\pi/2]\times\mathbb{S}^{2n}\cap n^\perp}|B|+|C|+|D|+|E|,$$

and note that provided $r_i<\min(1,1/M)$, the inequality (B.9) implies:

$$1>A\left(\frac{2}{\omega^{1/4}}\right)^4-Mr_i>A\left(\frac{2}{\omega^{1/4}}\right)^4-1.$$

This is not possible, since $A/\omega\geq 1$ and thus the claim is proved. Therefore thanks to (B.8), together with the uniform bound on ρ proved above, we deduce that:

$$\rho(r)=\frac{r}{A^{\frac{1}{4}}}+R_1(r), \tag{B.10}$$

where $|R_1(r)|\leq \mathfrak{c}_2r^2$ for any $0<r<\mathfrak{r}_1$ and for some constant $\mathfrak{c}_2>0$ independent on ϑ and v. Plugging the above expression for $\rho(r)$ inside (B.9), we can find a more explicit expression for R_1. Indeed, with some algebraic computations, which we omit, we get:

$$r^4=r^4+4A^{\frac{1}{4}}r^3R_1(r)+\frac{Br^5}{A^{\frac{5}{4}}}+R_2(r),$$

where $|R_2|\leq \mathfrak{c}_3r^6$ for any $0<r<\mathfrak{r}_1$ and for some constant $\mathfrak{c}_3$ independent on ϑ and v, from which we deduce:

$$R_1(r)=-\frac{Br^2}{4A^{\frac{3}{2}}}-\frac{R_2(r)}{4A^{\frac{1}{4}}r^3}.$$

Substituting the above identity in (B.10), we have:

$$\rho(r) = \frac{r}{A^{\frac{1}{4}}} - \frac{Br^2}{4A^{\frac{3}{2}}} + R_3(r), \tag{B.11}$$

where $R_3(r) := R_2(r)/4A^{\frac{1}{4}}r^3$. In particular $|R_3(r)| \leq \mathfrak{c}_3/4\omega^{\frac{1}{4}}r^3$ for any $0 < r < r_0$. Once again, substituting the newfound expression for $\rho(r)$ given by (B.11) in (B.9), we get the following expression for R_3:

$$R_3(r) := \left(\frac{7}{32}\frac{B^2}{A^{\frac{11}{4}}} - \frac{C}{4A^{\frac{7}{4}}}\right)r^3 + R_4(r),$$

where $|R_4(r)| \leq \mathfrak{c}_4 r^4$ for any $0 < r < \mathfrak{r}_1$ and for some constant $\mathfrak{c}_4 > 0$ independent on ϑ and v. □

Proposition B.6 has the following almost immeiate consequence:

Corollary B.7. *For any $r > 0$ and any $\delta \in \mathbb{R}$ sufficiently small, define:*

$$\mathcal{B}_{r,\delta} := \mathcal{P}\left(\left\{(\rho,\vartheta,v) \in \mathscr{C} : \rho \leq P_{\vartheta,v}(r) + \delta r^3\right\}\right).$$

There exists an $\epsilon_0(\mathcal{X}) = \epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$, there exists $0 < \mathfrak{r}_2(\epsilon) = \mathfrak{r}_2$ such that for any $0 < r < \mathfrak{r}_2$, we have:

$$\mathcal{B}_{r,-\epsilon} \subseteq \pi_H(B_r(\mathcal{X}) \cap gr(f)) \subseteq \mathcal{B}_{r,\epsilon}.$$

Proof. Proposition B.2 and the definition of $\mathcal{P}$ (recall that the function $\mathcal{P}$ was defined in B.2) imply:

$$\pi_H(B_r(\mathcal{X}) \cap \mathrm{gr}(f)) = x + \mathcal{P}\left(\left\{(\rho,\vartheta,v) \in \mathscr{C} : H(\rho,\vartheta,v) \leq r^4\right\}\right). \tag{B.12}$$

The function $\rho \mapsto H(\rho,\vartheta,v)$ is a polynomial of 8th degree in ρ, and thus the equation:

$$H(\rho,\vartheta,v) = r^4, \tag{B.13}$$

has at most 8 solutions in ρ. Assume $\rho_1 < \ldots < \rho_k$ are the positive distinct solutions of the above equation, where $k \in \{1,\ldots,8\}$. If $\rho > \rho_k$ then $H(\rho,\vartheta,v) > r^4$ and on the other hand, since $H(0,\vartheta,v) = 0$, if $0 \leq \rho < \rho_1$ then $G(\rho,\vartheta,v) < r^4$. This implies that:

$$\mathcal{P}\left(\{(\rho,\vartheta,v) \in \mathscr{C} : \rho \leq \rho_1\}\right) \subseteq \pi_H(B_r(\mathcal{X}) \cap \mathrm{gr}(f)) \subseteq \mathcal{P}\left(\{(\rho,\vartheta,v) \in \mathscr{C} : \rho \leq \rho_k\}\right).$$

Proposition B.6 concludes the proof since ρ_1 and ρ_k coincide up to an error of order r^4. □

The following technical lemma will be needed in the computations of Proposition B.13, and it is a Taylor expansion formula for the sub-Riemmanian area element at a non-characteristic point of a horizontal quadric.

Lemma B.8. *For any $(\rho,\vartheta,v) \in \mathscr{C}$ we have:*

$$2|(\mathcal{D} + J)[x + \mathcal{P}(\rho,\vartheta,v)]| = c + \mathcal{A}\rho + \mathcal{B}\rho^2 + R_1(\rho),$$

where:

(i) $\mathcal{A} := \mathcal{A}(\vartheta,v) := 2\cos\vartheta(\beta_{\mathfrak{n}}(v) + \langle Jv, n\rangle)$,

(ii) $\overline{\mathcal{B}} = c\mathcal{B} := c\mathcal{B}(\vartheta,v) := 2\left(\alpha_{\mathfrak{n}}\sin\vartheta + |P_{\mathfrak{n}}[(D+J)v]|\cos^2\vartheta\right)$*, and $P_{\mathfrak{n}}$ is the orthogonal projection on $\mathfrak{n}^\perp$.*

(iii) $|R_1(\rho)| \leq \mathfrak{c}_5\rho^3$ *for any $0 < \rho < \mathfrak{r}_3$ and for some constant $\mathfrak{c}_5 > 0$, independent on ϑ and v.*

Proof. In order to simplify the notation we let $M := 2(\mathcal{D}+J)$. First of all we want to find an explicit expression for the vector $M[x+\mathcal{P}(\rho,\vartheta,v)]$.

$$M[x+\mathcal{P}(\rho,\vartheta,v)] = M\left[x+\frac{\sin\vartheta}{c}\rho^2\mathfrak{n}+\cos\vartheta\rho v\right] = c\mathfrak{n}+\frac{2\sin\vartheta}{c}\rho^2 M\mathfrak{n}+2\cos\vartheta\rho Mv$$

Secondly, we compute the squared norm of the vector $M[x+\mathcal{P}(\rho,\vartheta,v)]$:

$$\begin{aligned}|M[x+\mathcal{P}(\rho,\vartheta,v)]|^2 =& c^2+4c\cos\vartheta\rho\langle Mv,n\rangle+4\sin\vartheta\rho^2\langle M\mathfrak{n},\mathfrak{n}\rangle+4\cos^2\theta\rho^2|Mv|^2\\ &+\frac{8\sin\vartheta\cos\theta}{c}\rho^3\langle M\mathfrak{n},Mv\rangle+\frac{4\sin^2\vartheta}{c^2}\rho^4|M\mathfrak{n}|^2.\end{aligned}$$

Note that by definition of $\alpha_{\mathfrak{n}}$ and $\beta_{\mathfrak{n}}(v)$, we have that $\langle Mv,\mathfrak{n}\rangle=\beta_{\mathfrak{n}}(v)+\langle Jv,\mathfrak{n}\rangle$ and thus $4c\cos\vartheta\langle Mv,\mathfrak{n}\rangle = 2c\mathcal{A}$. Moreover the fact that $\langle M\mathfrak{n},\mathfrak{n}\rangle=\alpha_{\mathfrak{n}}$ and that $|Mv|^2-\langle Mv,\mathfrak{n}\rangle^2 = |P_{\mathfrak{n}}(Mv)|^2$ imply, by means of few algebrai computation:

$$4\sin\vartheta\langle M\mathfrak{n},\mathfrak{n}\rangle+4\cos^2\vartheta|Mv|^2 = 2c\mathcal{B}+\mathcal{A}^2.$$

Therefore, with some algebraic computation that we omit, we can show that:

$$\begin{aligned}|M[x+\mathcal{P}(\rho,\vartheta,v)]|-c-\mathcal{A}\rho-\mathcal{B}\rho^2 =& \frac{|M[x+\mathcal{P}(\rho,\vartheta,v)]|^2-(c+\mathcal{A}\rho+\mathcal{B}\rho^2)^2}{|M[x+\mathcal{P}(\rho,\vartheta,v)]|+c+\mathcal{A}\rho+\mathcal{B}\rho^2}\\ =& \frac{\left(\frac{8\sin\vartheta\cos\theta}{c}\langle M\mathfrak{n},Mv\rangle-2\mathcal{A}\mathcal{B}\right)\rho^3+\left(\frac{4\sin^2\vartheta}{c^2}|M\mathfrak{n}|^2-\mathcal{B}^2\right)\rho^4}{|M[x+\mathcal{P}(\rho,\vartheta,v)]|+c+\mathcal{A}\rho+\mathcal{B}\rho^2}.\end{aligned}$$

Therefore there are a sufficiently small $\mathfrak{r}_3>0$ and a constant $\mathfrak{c}_5$ that can be chosen independent on ϑ and v, for which:

$$\big||M[x+\mathcal{P}(\rho,\vartheta,v)]|-c-\mathcal{A}\rho-\mathcal{B}\rho^2\big|\le\mathfrak{c}_5\rho^3, \qquad \text{for any } 0<\rho<\mathfrak{r}_3$$

□

Remark B.9. For any $(\vartheta,v)\in[-\pi/2,\pi/2]\times\mathbb{S}^{2n-1}\cap n^\perp$, we have that:

(i) $\mathcal{A}(\vartheta,v)=-\mathcal{A}(\vartheta,-v)$,

(ii) $\mathcal{B}(\vartheta,v)=\mathcal{B}(\vartheta,-v)$.

Proposition B.10. *The functions $\mathcal{A}(\cdot,\cdot)$ and $\mathcal{B}(\cdot,\cdot)$ defined in the statement of Proposition B.8 have the following symmetries. For any $0<r<dist(x,\Sigma(f))$, we have that:*

$$|\partial\mathbb{K}|(B_r(\mathcal{X}))=\int_{\mathbb{S}^{2n-1}\cap\mathfrak{n}^\perp}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{\{H(\rho,\vartheta,v)\le r^4\}}\Xi(\rho,\vartheta,v)d\rho d\vartheta d\sigma(v),$$

where:

(i) $\Xi(\rho,\vartheta,v):=\frac{2\rho^{2n}}{c}\cos^{2n-2}\vartheta(1+\sin^2\vartheta)\big|(D+J)[x+\mathcal{P}(\rho,\vartheta,v)]\big|$,

(ii) $\sigma:=\mathcal{H}^{2n-2}_{eu}\llcorner\mathbb{S}^{2n-1}\cap\mathfrak{n}^\perp$.

Proof. Proposition 3.2 in [33] implies that

$$|\partial\mathbb{K}|(B_r(\mathcal{X})) = \int_{U_r} |\nabla f(w) + 2Jw| dw = 2\int_{U_r} |(D+J)w| dw. \tag{B.14}$$

We want to perform in the right-hand side of the above equation, the following change of variables:

$$w = x + \frac{\sin\vartheta\rho^2}{c}\mathfrak{n} + \cos\vartheta\rho v = x + \mathcal{P}(\rho,\vartheta,v), \text{ where } (\rho,\vartheta,v)\in\mathscr{C}. \tag{B.15}$$

If $n > 1$, we can parametrize the sphere $\mathbb{S}^{2n-1}\cap\mathfrak{n}^\perp$ with the usual polar coordinates and therefore we let $v = v(\psi)$ where $\psi \in \Psi := [-\pi,\pi) \times [\pi/2,\pi,2]^{2n-3}$. The Jacobian determinant (obtained using the Laplace formula, we omit the computations) of the change of variables (B.15) is:

$$\left|\det\frac{\partial w(\rho,\vartheta,v)}{\partial\rho\partial\vartheta\partial\psi}\right| = \frac{\rho^{2n}}{c}(1+\sin^2\vartheta)\cos^{2n-2}\vartheta\left|\left(v(\psi),\frac{\partial v(\psi)}{\partial\psi}\right)\right|.$$

Therefore (B.14) becomes:

$$\begin{aligned}|\partial\mathbb{K}|(B_r(\mathcal{X})) =& 2\int_\Psi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int \chi_{U_r}(x+\mathcal{P}(\rho,\vartheta,\psi))\frac{\rho^{2n}}{c}(1+\sin^2\vartheta)\cos^{2n-2}\vartheta\left|\left(v(\psi),\frac{\partial v(\psi)}{\partial\psi}\right)\right| d\rho d\vartheta d\psi\\ =& \int_\Psi\left|\left(v(\psi),\frac{\partial v(\psi)}{\partial\psi}\right)\right|\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int\chi_{U_r}(x+\mathcal{P}(\rho,\vartheta,\psi))\Xi(\rho,\vartheta,v(\psi))d\rho d\vartheta d\psi\\ =& \int_{\mathbb{S}^{2n-1}\cap\mathfrak{n}^\perp}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int\chi_{U_r}(x+\mathcal{P}(\rho,\vartheta,\psi))\Xi(\rho,\vartheta,v)d\rho d\vartheta d\sigma(v)\\ =& \int_{\mathbb{S}^{2n-1}\cap\mathfrak{n}^\perp}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{\{H(\rho,\vartheta,v)\le r^4\}}\Xi(\rho,\vartheta,v)d\rho d\vartheta d\sigma(v),\end{aligned}$$

where the last line comes from the identity (B.12). If $n = 1$, the computation is simpler since $\mathbb{S}^{2n-1}\cap\mathfrak{n}^\perp = \{\pm J\mathfrak{n}\}$. ☐

The following two lemmas will allow us to compute some integrals in Propositions B.13 and B.15.

Lemma B.11. *For any $k\in\mathbb{N}$, any $\alpha > \frac{k+1}{2}$ we have:*

$$\int_{-\infty}^{\infty}\frac{x^k}{(1+x^2)^\alpha}dx = \begin{cases}0 & \text{if } k \text{ is odd,}\\ \frac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\alpha-\frac{k+1}{2}\right)}{\Gamma(\alpha)} & \text{if } k \text{ is even.}\end{cases}$$

Proof. If k is odd, then $\int x^k/(1+x^2)^\alpha = 0$. If on the other hand k is even, the change of variable $t = 1/(1+x^2)^\alpha$ implies:

$$\int_{-\infty}^{\infty}\frac{x^k}{(1+x^2)^\alpha}dx = \int_0^1(1-t)^{\frac{k+1}{2}-1}t^{(\alpha-\frac{k+1}{2})-1}dt = \beta\left(\frac{k+1}{2},\alpha-\frac{k+1}{2}\right) = \frac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\alpha-\frac{k+1}{2}\right)}{\Gamma(\alpha)},$$

where $\beta(\cdot,\cdot)$ is the Euler's beta function. The last equality follows from a well known property of β, see for instance Theorem 12.41 in [41]. ☐

Lemma B.12. *Suppose $f:\mathbb{R}\to\mathbb{R}$ is a measurable function such that $f(x)/(1+x^2)^\alpha\in L^1(\mathbb{R})$ and let $\mathfrak{d}(\theta) := \cos^{2n-2}\vartheta(\cos^2\vartheta+2\sin^2\vartheta)$. Then the following identity holds:*

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\mathfrak{d}(\vartheta)\frac{\cos^{4\alpha-2n-1}\vartheta f\left(\frac{\sin\vartheta}{\cos^2\vartheta}\right)}{A^\alpha}d\vartheta = \int_{-\infty}^{\infty}\frac{f(x)}{\left(1+\left(x+\gamma(v)\right)^2\right)^\alpha}dx,$$

where $\gamma(v)$ was defined in (B.4).

Proof. Thanks to the definition of A, we have that:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathfrak{d}(\vartheta)\frac{\cos^{4\alpha-2n-1}(\vartheta)f\left(\frac{\sin\vartheta}{\cos^2\vartheta}\right)}{A^\alpha}d\vartheta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathfrak{d}(\vartheta)\frac{\cos^{4\alpha-2n-1}\vartheta f\left(\frac{\sin\vartheta}{\cos^2\vartheta}\right)}{\left(\cos^4\vartheta+(\cos^2\vartheta+\gamma(v)\sin\vartheta)^2\right)^\alpha}d\vartheta$$
$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^2\vartheta+2\sin^2\vartheta}{\cos^3\vartheta}\frac{f\left(\frac{\sin\vartheta}{\cos^2\vartheta}\right)}{\left(1+(\frac{\sin\vartheta}{\cos^2\vartheta}+\gamma(v))^2\right)^\alpha}d\vartheta = \int_{-\infty}^{\infty}\frac{f(x)}{\left(1+\left(x+\gamma(v)\right)^2\right)^\alpha}dx,$$

where the last equality is obtained with the change of variable $x=\sin\vartheta/\cos^2\vartheta$. □

Proposition B.13 is the technical core of this appendix. It gives a first description of the structure of the coefficients $\mathfrak{c}$, $\mathfrak{d}$ and $\mathfrak{e}$.

Proposition B.13. *For any $\epsilon>0$ there exists $\mathfrak{r}_4=\mathfrak{r}_4(\epsilon)>0$ such that for any $0<r<\mathfrak{r}_4$:*

$$|\partial\mathbb{K}|(B_r(\mathcal{X}))=\mathfrak{c}_n r^{2n+1}+\mathfrak{e}(\mathcal{X})r^{2n+3}+\epsilon R_2(r), \tag{B.16}$$

where defined $\mathbb{S}(\mathfrak{n}):=\mathbb{S}^{2n-1}\cap\mathfrak{n}^\perp$, we have:

(i) $\mathfrak{c}(\mathcal{X})=\mathfrak{c}_n:=\frac{\sqrt{\pi}\Gamma\left(\frac{2n-1}{4}\right)\sigma(\mathbb{S}(\mathfrak{n}))}{(2n+1)\Gamma\left(\frac{2n+1}{4}\right)}$,

(ii)

$$\mathfrak{e}(\mathcal{X})=\int_{\mathbb{S}(\mathfrak{n})}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\mathfrak{d}(\vartheta)\left(\left(\frac{7}{32}+\frac{n}{16}\right)\frac{\overline{B}^2}{A^2}-\frac{\overline{C}}{4A}-\frac{\mathcal{A}\overline{B}}{4A}+\frac{\overline{\mathcal{B}}}{(2n+3)}\right)}{c^2A^{\frac{2n+3}{4}}}d\vartheta d\sigma.$$

(iii) $|R_2(r)|\leq\mathfrak{C}_6(n)r^{2n+3}$ *for any $0<r<\mathfrak{r}_4$ and for some constant $\mathfrak{C}_6(n)$ depending only on $\mathcal{X}$.*

Proof. Thanks to Lemma B.8 and Proposition B.10, we deduce that:

$$|\partial\mathbb{K}|(B_r(\mathcal{X}))=\int\chi_{U_r}(x+\mathcal{P}(\rho,\vartheta,\psi))\Xi(\rho,\vartheta,v)d\rho d\vartheta d\sigma(v)$$
$$=\int_{\mathbb{S}(\mathfrak{n})}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{\{H(\rho,\vartheta,v)\leq r^4\}}\mathfrak{d}(\vartheta)\left(\rho^{2n}+\frac{\mathcal{A}}{c}\rho^{2n+1}+\frac{\mathcal{B}}{c}\rho^{2n+2}+\frac{R(\rho)\rho^{2n}}{c}\right)d\rho d\vartheta d\sigma(v).$$

We now proceed giving estimates of each term in the last line of the above identity. In the proof of Proposition B.6 we showed that $\{H(\rho,\vartheta,v)\leq r^4\}\subseteq\{\rho\leq\mathfrak{c}_1 r\}$ whenever $0<r<\mathfrak{r}_1$ and $\mathfrak{c}_1$ was a constant independent on ϑ and v. Therefore if $r\leq\mathfrak{r}_1$:

$$\int_{\{H(\rho,\vartheta,v)\leq r^4\}}R(\rho)\rho^{2n}d\rho\leq\int_0^{\mathfrak{c}_1 r}R(\rho)\rho^{2n}d\rho\leq\mathfrak{c}_5\int_0^{\mathfrak{c}_1 r}\rho^{2n+3}d\rho\leq\frac{\mathfrak{c}_5 c_1^{2n+4}r^{2n+4}}{(2n+4)}, \tag{B.17}$$

where the second inequality comes from Lemma B.8 provided $\mathfrak{c}_1 r\leq\mathfrak{r}_3$. Moreover, by Proposition B.7 for any $\epsilon>0$ there exists $\mathfrak{r}_2>0$ for which for any $0<r<\mathfrak{r}_2$ we have:

$$\begin{aligned}\left|\int_{\{H(\rho,\vartheta,v)\leq r^4\}}\rho^j d\rho-\int_0^{\mathcal{P}_{\vartheta,v}(r)}\rho^j d\rho\right|\leq&\int_0^{P_{\vartheta,v}(r)+\epsilon r^3}\rho^j d\rho-\int_0^{P_{\vartheta,v}(r)-\epsilon r^3}\rho^j d\rho\\ =&\frac{(P_{\vartheta,v}(r)+\epsilon r^3)^{j+1}-(P_{\vartheta,v}(r)-\epsilon r^3)^{j+1}}{j+1}\leq\frac{2^{j+1}\epsilon P_{\vartheta,v}(r)^j r^3}{j+1}.\end{aligned} \tag{B.18}$$

Thanks to the bounds (B.17) and (B.18), for any $0 < r < \min\{\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_3\}$ we have:

$$|R_1^{\vartheta,v}(r)| := \left| \int_{\{H(\rho,\vartheta,v)\le r^4\}} \left(\rho^{2n} + \frac{\mathcal{A}}{c}\rho^{2n+1} + \frac{\mathcal{B}}{c}\rho^{2n+2} + \frac{R(\rho)\rho^{2n}}{c}\right) d\rho - \int_0^{P_{\vartheta,v}(r)} \left(\rho^{2n} + \frac{\mathcal{A}}{c}\rho^{2n+1} + \frac{\mathcal{B}}{c}\rho^{2n+2}\right) d\rho \right|$$
$$\le \epsilon\frac{2^{2n+1}P_{\vartheta,v}(r)^{2n}r^3}{2n+1} + \epsilon\frac{2^{2n+2}\|\mathcal{A}\|_\infty P_{\vartheta,v}(r)^{2n+1}r^3}{(2n+2)c} + \epsilon\frac{2^{2n+3}\|\mathcal{B}\|_\infty P_{\vartheta,v}(r)^{2n+2}r^3}{(2n+3)c} + \frac{\mathfrak{c}_5\mathfrak{c}_1^{2n+4}r^{2n+4}}{(2n+4)}, \tag{B.19}$$

where $\|\mathcal{A}\|_\infty$ and $\|\mathcal{B}\|_\infty$ are the maximum of the functions $\mathcal{A}$ and $\mathcal{B}$ on $[-\pi/2, \pi/2] \times \mathbb{S}(\mathfrak{n})$. Thanks to the definition of $P_{\vartheta,v}$, which was given in (B.7), we can find $\mathfrak{r}_5 > 0$ depending only on ϵ and $\mathcal{X}$, such that for any $0 < r < \mathfrak{r}_5$ we have $|R_1^{\vartheta,v}| \le \epsilon\mathfrak{C}_7(n)r^{2n+3}$, where the constant $\mathfrak{C}_7(n)$ depends only on $\mathcal{X}$. Finally, there exists an $\mathfrak{r}_6 > 0$ such that and a constant $\mathfrak{C}_8(n) > 0$ depending only on $\mathcal{X}$ such that:

$$\begin{aligned}|\Delta_1^{\vartheta,v}(r)| &:= \left| P_{\vartheta,v}(r)^{2n+1} - \frac{r^{2n+1}}{A^{\frac{2n+1}{4}}} + \frac{(2n+1)Br^{2n+2}}{4A^{\frac{2n+6}{4}}} - \frac{2n+1}{A^{\frac{2n+3}{4}}}\left((2n+7)\frac{B^2}{32A^2} - \frac{C}{4A}\right) r^{2n+3} \right| \le \mathfrak{C}_8(n)r^{2n+4},\\
|\Delta_2^{\vartheta,v}(r)| &:= \left| P_{\vartheta,v}(r)^{2n+2} - \frac{r^{2n+2}}{A^{\frac{2n+2}{4}}} + \frac{(2n+2)Br^{2n+3}}{4A^{\frac{2n+7}{4}}} \right| \le \mathfrak{C}_8(n)r^{2n+4},\\
|\Delta_3^{\vartheta,v}(r)| &:= \left| P_{\vartheta,v}(r)^{2n+3} - \frac{r^{2n+3}}{A^{\frac{2n+3}{4}}} \right| \le \mathfrak{C}_8(n)r^{2n+4}.\end{aligned} \tag{B.20}$$

We omit the computations proving the three bounds above since they can easily obtained by explicitly computing $P_{\vartheta,v}(r)^j$ and truncating the expression to the desired order of r. Therefore, (B.19) and (B.20) imply that for any $0 < r < \min(\mathfrak{r}_5, \mathfrak{r}_6)$ we have:

$$\begin{aligned}\int_{\{H(\rho,\vartheta,v)\le r^4\}} \left(\rho^{2n} + \frac{\mathcal{A}}{c}\rho^{2n+1} + \frac{\mathcal{B}}{c}\rho^{2n+2} + \frac{R(\rho)\rho^{2n}}{c}\right) d\rho &= \int_0^{P_{\vartheta,v}(r)} \left(\rho^{2n} + \frac{\mathcal{A}}{c}\rho^{2n+1} + \frac{\mathcal{B}}{c}\rho^{2n+2}\right) d\rho + R_1^{\vartheta,v}(r)\\
&= \frac{P_{\vartheta,v}(r)^{2n+1}}{2n+1} + \frac{\mathcal{A}P_{\vartheta,v}(r)^{2n+2}}{(2n+2)c} + \frac{\mathcal{B}P_{\vartheta,v}(r)^{2n+3}}{(2n+3)c} + R_1^{\vartheta,v}(r)\\
= \frac{r^{2n+1}}{(2n+1)A^{\frac{2n+1}{4}}} + \left(\frac{\mathcal{A}}{(2n+2)c} - \frac{B}{4A}\right)\frac{r^{2n+2}}{A^{\frac{2n+2}{4}}} &+ \left((2n+7)\frac{\overline{B}^2}{32A^2} - \frac{\overline{C}}{4A} - \frac{\mathcal{A}\overline{B}}{4A} + \frac{\overline{\mathcal{B}}}{(2n+3)}\right)\frac{r^{2n+3}}{A^{\frac{2n+3}{4}}} + R_2^{\vartheta,v}(r),\end{aligned}$$

where $R_2^{\vartheta,v}(r) := R_1^{\vartheta,v}(r) + \Delta_1^{\vartheta,v}(r) + \Delta_2^{\vartheta,v}(r) + \Delta_3^{\vartheta,v}(r)$ and the functions $\overline{B}, \ldots, \overline{E}$ and $\mathcal{A}, \overline{\mathcal{B}}$ where introduced in the statement of Proposition B.4 and in Lemma B.8 respectively. Thanks to the definitions of $R^{\vartheta,v}$ and $\Delta_i^{\vartheta,v}$, for any $\epsilon > 0$ there exists $\mathfrak{r}_7 := \mathfrak{r}_7(\epsilon) > 0$ such that for any $0 < r < \mathfrak{r}_7$, we have $|R_2^{\vartheta,v}(r)| \le \epsilon\mathfrak{C}_9(n)r^{2n+3}$, where $\mathfrak{C}_9(n)$ is a constant which depends only on ϵ and $\mathcal{X}$. Therefore, using what we have deduced so far we get:

$$\begin{aligned}|\partial\mathbb{K}|(B_r(x)) =& \frac{r^{2n+1}}{(2n+1)}\int_{\mathbb{S}(\mathfrak{n})}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathfrak{d}(\vartheta)}{A^{\frac{2n+1}{4}}} d\vartheta d\sigma + r^{2n+2}\int_{\mathbb{S}(\mathfrak{n})}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathfrak{d}(\vartheta)}{A^{\frac{2n+2}{4}}}\left(\frac{\mathcal{A}}{(2n+2)c} - \frac{B}{4A}\right) d\vartheta d\sigma\\
&+ r^{2n+3}\int_{\mathbb{S}(\mathfrak{n})}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathfrak{d}(\vartheta)}{A^{\frac{2n+3}{4}}}\left(\frac{(2n+7)\overline{B}^2}{32A^2} - \frac{\overline{C}}{4A} - \frac{\mathcal{A}\overline{B}}{4A} + \frac{\overline{\mathcal{B}}}{(2n+3)}\right) d\vartheta d\sigma + R_3(r),\end{aligned}$$

where $R_3(r) := \int_{\mathbb{S}(\mathfrak{n})}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} R_2^{\vartheta,v}(r) d\vartheta d\sigma$ and for any $0 < r < \mathfrak{r}_7$ we have $|R_3(r)| \le \epsilon\mathfrak{c}_{10}r^{2n+3}$ for some constant $\mathfrak{c}_{10}$ depending only on $\mathcal{X}$. Moreover, since B and $\mathcal{A}$ are odd functions on $\mathbb{S}(\mathfrak{n})$ (see Proposition B.5 and Remark

B.9) we deduce that:

$$\int_{\mathbb{S}(\mathfrak{n})} \frac{\mathcal{A}}{A^{\frac{2n+2}{4}}} d\sigma = 0, \qquad \int_{\mathbb{S}(\mathfrak{n})} \frac{B}{4A^{\frac{2n+6}{4}}} d\sigma = 0.$$

This, by Fubini implies that:

$$\begin{aligned}|\partial\mathbb{K}|(B_r(x)) =& \frac{r^{2n+1}}{(2n+1)} \int_{\mathbb{S}(\mathfrak{n})} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathfrak{d}(\vartheta)}{A^{\frac{2n+1}{4}}} d\vartheta d\sigma \\ &+ r^{2n+3} \int_{\mathbb{S}(\mathfrak{n})} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathfrak{d}(\vartheta)}{A^{\frac{2n+3}{4}}} \left(\frac{(2n+7)\overline{B}^2}{32A^2} - \frac{\overline{C}}{4A} - \frac{\mathcal{A}\overline{B}}{4A} + \frac{\overline{\mathcal{B}}}{(2n+3)} \right) d\vartheta d\sigma + R_3(r),\end{aligned}$$

We are left to prove that $\int_{\mathbb{S}(\mathfrak{n})} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathfrak{d}(\vartheta)}{A^{\frac{2n+1}{4}}} d\vartheta d\sigma$ is a constant depending only on n. This is done by explicit computation:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathfrak{d}(\vartheta)}{A^{\frac{2n+1}{4}}} d\vartheta = \int_{-\infty}^{\infty} \frac{dx}{(1+(x+\gamma(v))^2)^{\frac{2n+1}{4}}} = \frac{\sqrt{\pi}\Gamma\left(\frac{2n-1}{4}\right)}{\Gamma\left(\frac{2n+1}{4}\right)},$$

where we used Lemma B.12 in the first equality and Lemma B.11 in the second . □

Remark B.14. The constant $\mathfrak{c}_n$ is the same constant appearing in Propositions A.5 and A.9. Although we could deduce from Proposition A.5 that the leading term in the expansion (B.16) is constant and its value, we nevertheless decided to carry out the explicit computation (the last line in the proof of Proposition B.13) of the coefficient $\mathfrak{c}(\mathcal{X})$.

In the previous propositions we gave a first characterisation of the coefficient of the Taylor developments of the perimeter of quadratic surfaces. The coefficient relative r^{2n+1} has been proved to be a constant depending only on n and the one of r^{2n+2} has been proved null. In the following proposition we investigate more carefully the structure of the coefficient relative to r^{2n+3}:

Proposition B.15. *In the notation of the previous propositions we have:*

$$\mathfrak{e}(\mathcal{X}) = \frac{1}{4}\frac{Tr(\mathcal{D}^2) - 2\langle\mathfrak{n}, \mathcal{D}^2\mathfrak{n}\rangle + \langle\mathfrak{n}, \mathcal{D}\mathfrak{n}\rangle^2}{2n-1} + \frac{n-1}{2n-1} - \frac{1}{4} + \frac{\langle\mathcal{D}J\mathfrak{n}, \mathfrak{n}\rangle}{2n-1} - \frac{1}{8}\frac{(Tr(\mathcal{D}) - \langle\mathfrak{n}, \mathcal{D}\mathfrak{n}\rangle)^2}{2n-1}.$$

Proof. Thanks to Proposition B.13, we deduce that:

$$\mathfrak{e}(\mathcal{X}) = \int_{\mathbb{S}(\mathfrak{n})} \Bigg(\underbrace{\frac{2n+7}{32} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathfrak{d}(\vartheta)\overline{B}^2}{A^{\frac{2n+11}{4}}} d\vartheta}_{\text{(I)}} - \underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathfrak{d}(\vartheta)\overline{C}}{4A^{\frac{2n+7}{4}}} d\vartheta}_{\text{(II)}} - \underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathfrak{d}(\vartheta)\mathcal{A}\overline{B}}{4A^{\frac{2n+7}{4}}} d\vartheta}_{\text{(III)}} + \underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathfrak{d}(\vartheta)\overline{\mathcal{B}}}{(2n+3)A^{\frac{2n+3}{4}}} d\vartheta}_{\text{(IV)}} \Bigg) d\sigma. \tag{B.21}$$

Now we study each term separately. Let us start studying (I). Since $\overline{B}^2 = \cos^{10}\vartheta\Big(16\beta_{\mathfrak{n}}(v)^2(\frac{\sin\vartheta}{\cos^2\vartheta})^2(\frac{\sin\vartheta}{\cos^2\vartheta} + \gamma(v))\Big)$, Lemmas B.11 and B.12 imply:

$$\begin{aligned}\frac{2n+7}{32}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathfrak{d}(\vartheta)\overline{B}^2}{A^{\frac{2n+11}{4}}} d\vartheta &= \frac{2n+7}{32}\int_{-\infty}^{\infty} \frac{16\beta_{\mathfrak{n}}(v)^2 x^2(\gamma(v)+x)^2}{(1+(\gamma(v)+x)^2)^{\frac{2n+11}{4}}} d\vartheta = \frac{(2n+7)\beta_{\mathfrak{n}}(v)^2}{2}\int_{-\infty}^{\infty} \frac{x^2(x-\gamma(v))^2}{(1+x^2)^{\frac{2n+11}{4}}} dx \\ =& \frac{(2n+7)\beta_{\mathfrak{n}}(v)^2}{2}\left(\int_{-\infty}^{\infty} \frac{x^4}{(1+x^2)^{\frac{2n+11}{4}}} dx + \gamma(v)^2\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^{\frac{2n+11}{4}}} dx\right) = 2\mathcal{C}_n\beta_{\mathfrak{n}}(v)^2\left(\frac{3}{4} + \frac{(2n+1)\gamma(v)^2}{8}\right),\end{aligned} \tag{B.22}$$

where $\mathcal{C}_n := \frac{\sqrt{\pi}\Gamma\left(\frac{2n+1}{4}\right)}{\frac{2n+3}{4}\Gamma\left(\frac{2n+3}{4}\right)}$.

We turn now our attention to (II). Since $\overline{C} = \cos^6\vartheta(\frac{\sin\vartheta}{\cos^2\vartheta})^2\Big((2+4\beta_\mathfrak{n}(v)^2+2\gamma(v)\alpha_\mathfrak{n})+2\alpha_\mathfrak{n}\frac{\sin\vartheta}{\cos^2\vartheta}\Big)$, Lemmas B.11 and B.12 imply:

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\mathfrak{d}(\vartheta)\overline{C}}{4A^{\frac{2n+7}{4}}}d\vartheta =&\frac{1}{2}\int_{-\infty}^{\infty}\frac{x^2((1+2\beta_\mathfrak{n}(v)^2+\gamma(v)\alpha_\mathfrak{n})+\alpha_\mathfrak{n}x)}{(1+(x+\gamma(v))^2)^{\frac{2n+7}{4}}}dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{(x-\gamma(v))^2((1+2\beta_\mathfrak{n}(v)^2)+\alpha_\mathfrak{n}x)}{(1+x^2)^{\frac{2n+7}{4}}}dx\\ =&\frac{(1+2\beta_\mathfrak{n}(v)^2-2\alpha_\mathfrak{n}\gamma(v))}{2}\int_{-\infty}^{\infty}\frac{x^2}{(1+x^2)^{\frac{2n+7}{4}}}dx+\frac{(1+2\beta_\mathfrak{n}(v)^2)\gamma(v)^2}{2}\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^{\frac{2n+7}{4}}}\\ =&\mathcal{C}_n\frac{(2n+1)(1+2\beta_\mathfrak{n}^2(v))\gamma(v)^2+2+4\beta_\mathfrak{n}(v)^2-4\gamma(v)\alpha_\mathfrak{n}}{8}.\end{aligned} \tag{B.23}$$

Since $\mathcal{A}\overline{B} = 8\cos^6\vartheta(\beta_\mathfrak{n}(v)+\langle Jv,\mathfrak{n}\rangle)\beta_\mathfrak{n}(v)\frac{\sin\vartheta}{\cos^2\vartheta}\Big(\gamma(v)+\frac{\sin\vartheta}{\cos^2\vartheta}\Big)$, Lemmas B.11 and B.12 imply:

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\mathfrak{d}(\vartheta)\mathcal{A}\overline{B}}{4A^{\frac{2n+7}{4}}}d\vartheta =& 2(\beta_\mathfrak{n}(v)+\langle Jv,\mathfrak{n}\rangle)\beta_\mathfrak{n}(v)\int_{-\infty}^{\infty}\frac{x(\gamma(v)+x)}{(1+(\gamma(v)+x)^2)^{\frac{2n+7}{4}}}d\vartheta\\ =&2(\beta_\mathfrak{n}(v)+\langle Jv,\mathfrak{n}\rangle)\beta_\mathfrak{n}(v)\int_{-\infty}^{\infty}\frac{x(x-\gamma(v))}{(1+x^2)^{\frac{2n+7}{4}}}dx = \mathcal{C}_n(\beta_\mathfrak{n}(v)+\langle Jv,\mathfrak{n}\rangle)\beta_\mathfrak{n}(v),\end{aligned} \tag{B.24}$$

which concludes the discussion of the integral (III). Finally, we are left with the discussion of (IV). Thanks to the fact that $\overline{\mathcal{B}} = 2\cos^2\vartheta\Big(\alpha_\mathfrak{n}\frac{\sin\vartheta}{\cos^2\vartheta}+|P_\mathfrak{n}(\mathcal{D}+J)v|^2\Big)$ Lemmas B.11 and B.12, imply that:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\mathfrak{d}(\vartheta)}{A^{\frac{2n+3}{4}}}\frac{\overline{\mathcal{B}}}{(2n+3)}d\vartheta =\frac{2}{2n+3}\int_{-\infty}^{\infty}\frac{\alpha_\mathfrak{n}x+|P_\mathfrak{n}(\mathcal{D}+J)v|^2}{(1+(x+\gamma(v)))^{\frac{2n+3}{4}}}dx = \mathcal{C}_n\frac{-\alpha_\mathfrak{n}\gamma(v)+|P_\mathfrak{n}(\mathcal{D}+J)v|^2}{2}. \tag{B.25}$$

Plugging the identities (B.22), (B.23), (B.24), (B.25) into (B.21), we get:

$$\begin{aligned}\frac{\mathfrak{e}(\mathcal{X})}{\mathcal{C}_n\sigma(\mathbb{S}(\mathfrak{n}))} =& ⨍_{\mathbb{S}(\mathfrak{n})}\left[\beta_\mathfrak{n}(v)^2\left(\frac{3}{2}+\frac{2n+1}{4}\gamma(v)^2\right)-\frac{(2n+1)(1+2\beta_\mathfrak{n}^2(v))\gamma(v)^2+2+4\beta_\mathfrak{n}(v)^2-4\gamma(v)\alpha_\mathfrak{n}}{8}\right]d\sigma(v)\\ &+⨍_{\mathbb{S}(\mathfrak{n})}\left[-(\beta_\mathfrak{n}(v)+\langle Jv,\mathfrak{n}\rangle)\beta_\mathfrak{n}(v)+\frac{-\alpha_\mathfrak{n}\gamma(v)+|P_\mathfrak{n}(\mathcal{D}+J)v|^2}{2}\right]d\sigma(v)\\ =&-\frac{2n+1}{8}\underbrace{⨍_{\mathbb{S}(\mathfrak{n})}\gamma(v)^2d\sigma}_{\text{(V)}}-\frac{1}{4}-\underbrace{⨍_{\mathbb{S}(\mathfrak{n})}\beta_\mathfrak{n}(v)\langle Jv,n\rangle d\sigma}_{\text{(VI)}}+\underbrace{⨍_{\mathbb{S}(\mathfrak{n})}\frac{|P_\mathfrak{n}(\mathcal{D}+J)v|^2}{2}d\sigma}_{\text{(VII)}}.\end{aligned}$$

Eventually, in order to make the expression for $\mathfrak{e}(\mathcal{X})$ explicit, we need to compute the integrals (V), (VI) and (VII). To do so, we let $\mathcal{E} := \{m_1,\dots,m_{2n}\}$ be an orthonormal basis of $\mathbb{R}^{2n}$ such that $m_1 = \mathfrak{n}$ and:

$$Jm_i = \begin{cases} m_{n+i} & \text{if } i\in\{1,\dots,n\},\\ -m_{i-n} & \text{if } i\in\{n+1,\dots,2n\}.\end{cases} \tag{B.26}$$

With respect to the basis $\mathcal{E}$, the points $v\in\mathbb{S}(\mathfrak{n})$ are written as $v=\sum_{i=2}^{2n}v_im_i$ where $v_i := \langle v,m_i\rangle$. This is due to the fact that $v \in n^\perp$ by definition of $\mathbb{S}(\mathfrak{n})$. With these notations, the integral (V) becomes:

$$\begin{aligned}
\fint_{\mathbb{S}(\mathfrak{n})} \gamma(v)^2 d\sigma(v) =& \fint_{\mathbb{S}(\mathfrak{n})} \left(\sum_{i,j=2}^{2n} \langle m_i, \mathcal{D} m_j \rangle v_i v_j \right)^2 d\sigma(v) \\
=& \sum_{i,j,k,l=2}^{2n} \langle m_i, \mathcal{D} m_j \rangle \langle m_k, \mathcal{D} m_l \rangle \fint_{\mathbb{S}(\mathfrak{n})} v_i v_j v_k v_l d\sigma(v) \\
=& \sum_{i=2}^{2n} \langle m_i, \mathcal{D} m_i \rangle^2 \fint_{\mathbb{S}(\mathfrak{n})} v_i^4 d\sigma(v) + \sum_{\substack{2 \le i,j \le 2n \\ i \ne j}} \langle m_i, \mathcal{D} m_i \rangle \langle m_j, \mathcal{D} m_j \rangle \fint_{\mathbb{S}(\mathfrak{n})} v_i^2 v_j^2 d\sigma(v) \\
&+ 2 \sum_{\substack{2 \le i,j \le 2n \\ i \ne j}} \langle m_i, \mathcal{D} m_j \rangle^2 \fint_{\mathbb{S}(\mathfrak{n})} v_i^2 v_j^2 d\sigma(v),
\end{aligned}$$

where the last equality comes from the fact that integrals of odd functions on $\mathbb{S}(\mathfrak{n})$ are null. By direct computation or using formulas stated at the beginning of section $2c$ in [25], we have that:

$$\begin{aligned}
\fint_{\mathbb{S}(\mathfrak{n})} \gamma(v)^2 d\sigma(v) =& \frac{3}{4n^2-1} \sum_{i=2}^{2n} \langle m_i, \mathcal{D} m_i \rangle^2 + \frac{1}{4n^2-1} \sum_{i \ne k=2}^{2n} \langle m_i, \mathcal{D} m_i \rangle \langle m_k, \mathcal{D} m_k \rangle + \frac{2}{4n^2-1} \sum_{i \ne j=2}^{2n} \langle m_i, \mathcal{D} m_j \rangle^2 \\
=& \frac{2}{4n^2-1} \sum_{i,j=2}^{2n} \langle m_i, \mathcal{D} m_j \rangle^2 + \frac{1}{4n^2-1} \left(\sum_{i=2}^{2n} \langle m_i, \mathcal{D} m_i \rangle \right)^2 .
\end{aligned}$$

Since the matrix $\mathcal{D}$ is symmetric, we have that $\mathrm{Tr}(\mathcal{D}^2) = \sum_{i,j=1}^{2n} \langle m_i, \mathcal{D} m_j \rangle^2$. Thanks to this identity we deduce that:

$$\begin{aligned}
\fint_{\mathbb{S}(\mathfrak{n})} \gamma(v)^2 d\sigma(v) =& \frac{2\mathrm{Tr}(\mathcal{D}^2) - 4 \sum_{i=2}^{2n} \langle m_i, \mathcal{D}\mathfrak{n} \rangle^2 - 2 \langle \mathfrak{n}, \mathcal{D}\mathfrak{n} \rangle^2 + \left(\mathrm{Tr}(\mathcal{D}) - \langle \mathfrak{n}, \mathcal{D}\mathfrak{n} \rangle \right)^2}{4n^2-1} \\
=& \frac{2\mathrm{Tr}(\mathcal{D}^2) - 4 \langle \mathfrak{n}, \mathcal{D}^2 \mathfrak{n} \rangle + 2 \langle \mathfrak{n}, \mathcal{D}\mathfrak{n} \rangle^2 + \left(\mathrm{Tr}(\mathcal{D}) - \langle \mathfrak{n}, \mathcal{D}\mathfrak{n} \rangle \right)^2}{(2n-1)(2n+1)},
\end{aligned}$$

where the last identity comes from the fact that $|\mathcal{D}\mathfrak{n}|^2 = \langle \mathfrak{n}, \mathcal{D}^2 \mathfrak{n} \rangle$ and some algebraic manipulations. The computation of the integral (VI) is much easier. Indeed:

$$\fint_{\mathbb{S}(\mathfrak{n})} \langle Jv, \mathfrak{n} \rangle \beta_{\mathfrak{n}}(v) d\sigma(v) = \sum_{i,j=0}^{2n-2} \langle J m_i, \mathfrak{n} \rangle \langle \mathcal{D} m_j, \mathfrak{n} \rangle \fint_{\mathbb{S}(\mathfrak{n})} v_i v_j d\sigma(v) = -\frac{\langle \mathcal{D} J \mathfrak{n}, \mathfrak{n} \rangle}{2n-1},$$

where the last equality comes from the fact that $\langle J m_j, \mathfrak{n} \rangle \ne 0$ if and only if $j = n+1$, by the choice of the basis $\mathcal{E}$, and that:

$$\fint_{\mathbb{S}(\mathfrak{n})} v_i v_j d\sigma(v) = \begin{cases} 0 & \text{if } i \ne j, \\ \frac{1}{2n-1} & \text{if } i = j. \end{cases} \tag{B.27}$$

We are left to study the integral (VII). Since $P_{\mathfrak{n}}$ is the orthogonal projection on $\mathfrak{n}^\perp$, we have that:

$$|P_{\mathfrak{n}}(Mv)|^2 = \sum_{i=2}^{2n} \langle m_i, Mv \rangle^2 = \sum_{i,j,k=2}^{2n} v_j v_k \langle m_i, M m_j \rangle \langle m_i, M m_k \rangle.$$

Furthermore, thanks to (B.27), we deduce that:

$$\fint_{\mathbb{S}(\mathfrak{n})}|P_{\mathfrak{n}}((\mathcal{D}+J)v)|^2d\sigma(v)=\sum_{i,j,k=2}^{2n}\langle m_i,(\mathcal{D}+J)m_j\rangle\langle m_i,(\mathcal{D}+J)m_k\rangle\fint_{\mathbb{S}(\mathfrak{n})}v_jv_kd\sigma(v)$$
$$=\sum_{i,j=2}^{2n}\langle m_i,(\mathcal{D}+J)m_j\rangle^2\fint_{\mathbb{S}(\mathfrak{n})}v_j^2d\sigma(v)=\frac{1}{2n-1}\sum_{i,j=2}^{2n}\langle m_i,(\mathcal{D}+J)m_j\rangle^2.$$

We wish now to make $\sum_{i,j=2}^{2n}\langle m_i,(\mathcal{D}+J)m_j\rangle^2$ more explicit. To do so note that by definition of $\mathcal{E}$, we have:

$$\langle m_i,Jm_j\rangle=\begin{cases}1 & \text{if } i\in\{1,\dots,n\},\\ -1 & \text{if } i\in\{n+1,\dots,2n\},\\ 0 & \text{otherwise.}\end{cases}\tag{B.28}$$

The identities in (B.28) imply that $\sum_{i,j=2}^{2n}\langle m_i,Jm_j\rangle^2=2n-2$ and since $\mathcal{D}$ is symmetric, we also have that:

$$\sum_{i,j=2}^{2n}\langle m_i,\mathcal{D}m_j\rangle\langle m_i,Jm_j\rangle=0.$$

Summing up what we have proved up to this point, we get:

$$\fint_{\mathbb{S}(\mathfrak{n})}|P_{\mathfrak{n}}((\mathcal{D}+J)v)|^2d\sigma(v)=\frac{1}{2n-1}\sum_{i,j=2}^{2n}\langle m_i,\mathcal{D}m_j\rangle^2+2\langle m_i,\mathcal{D}m_j\rangle\langle m_i,Jm_j\rangle+\langle m_i,Jm_j\rangle^2$$
$$=\frac{\sum_{i,j=2}^{2n}\langle m_i,\mathcal{D}m_j\rangle^2+2n-2}{2n-1}=\frac{\mathrm{Tr}(\mathcal{D}^2)-2\langle\mathfrak{n},\mathcal{D}^2\mathfrak{n}\rangle+\langle\mathfrak{n},\mathcal{D}\mathfrak{n}\rangle^2}{2n-1}+\frac{2n-2}{2n-1},$$

where the last equality as in the discussion of the case (V) comes from the identity $\mathrm{Tr}(\mathcal{D}^2)=\sum_{i,j=1}^{2n}\langle m_i,\mathcal{D}m_j\rangle^2$ and a few algebraic manipulations. Finally putting togheter the expressions of the integrals (V), (VI) and (VII), we get:

$$\frac{c^2\mathfrak{e}(\mathcal{X})}{\mathcal{C}_n\sigma(\mathbb{S}(\mathfrak{n}))}=-\frac{2n+1}{8}\left(\frac{2\mathrm{Tr}(\mathcal{D}^2)-4\langle\mathfrak{n},\mathcal{D}^2\mathfrak{n}\rangle+2\langle\mathfrak{n},\mathcal{D}\mathfrak{n}\rangle^2+(\mathrm{Tr}(\mathcal{D})-\langle\mathfrak{n},\mathcal{D}[\mathfrak{n}]\rangle)^2}{(2n-1)(2n+1)}\right)-\frac{1}{4}+\frac{\langle\mathcal{D}J\mathfrak{n},\mathfrak{n}\rangle}{2n-1}$$
$$+\frac{1}{2}\left(\frac{\mathrm{Tr}(\mathcal{D}^2)-2\langle\mathfrak{n},\mathcal{D}^2\mathfrak{n}\rangle+\langle\mathfrak{n},\mathcal{D}\mathfrak{n}\rangle^2}{2n-1}+\frac{2n-2}{2n-1}\right)$$
$$=\frac{1}{4}\frac{\mathrm{Tr}(\mathcal{D}^2)-2\langle\mathfrak{n},\mathcal{D}^2\mathfrak{n}\rangle+\langle\mathfrak{n},\mathcal{D}\mathfrak{n}\rangle^2}{2n-1}+\frac{n-1}{2n-1}-\frac{1}{4}+\frac{\langle\mathcal{D}J\mathfrak{n},\mathfrak{n}\rangle}{2n-1}-\frac{(\mathrm{Tr}(\mathcal{D})-\langle\mathfrak{n},\mathcal{D}\mathfrak{n}\rangle)^2}{8(2n-1)},$$

where the last identity is obtained from the previous ones with few algebraic computations. □

Theorem B.16. *Assume μ is a $(2n+1)$-uniform measure supported on $\mathbb{K}(0,\mathcal{D},-1)$. For any $h\in\mathbb{R}^{2n}\setminus\Sigma(f)$ (see (4.5)) we have:*

$$0=\frac{Tr(\mathcal{D}^2)-2\langle\mathfrak{n}(h),\mathcal{D}^2\mathfrak{n}(h)\rangle+\langle\mathfrak{n}(h),\mathcal{D}\mathfrak{n}(h)\rangle^2}{4(2n-1)}+\frac{n-1}{2n-1}-\frac{1}{4}+\frac{\langle\mathcal{D}J\mathfrak{n}(h),\mathfrak{n}(h)\rangle}{2n-1}-\frac{(Tr(\mathcal{D})-\langle\mathfrak{n}(h),\mathcal{D}\mathfrak{n}(h)\rangle)^2}{8(2n-1)},\tag{B.29}$$

where $\mathfrak{n}(h):=\frac{(\mathcal{D}+J)h}{|\mathcal{D}+J)h|}$.

Proof. Suppose that $\mathcal{X} := (h, f(h)) \in \operatorname{supp}(\mu)$. Then Proposition 4.8 implies that there exists $\mathfrak{r}_8 > 0$ such that:

$$B_r(\mathcal{X}) \cap \mathbb{K}(0, \mathcal{D}, 1) = B_r(\mathcal{X}) \cap \operatorname{supp}(\mu),$$

for any $0 < r < \mathfrak{r}_8$. Therefore Propositions 2.15, A.5 and B.13 imply that:

$$r^{2n+1} = \mu(B_r(\mathcal{X})) = \mathcal{S}^{2n+1}_{\operatorname{supp}(\mu)}(B_r(\mathcal{X})) = \frac{|\partial\mathbb{K}|(B_r(\mathcal{X}))}{\mathfrak{c}_n} = r^{2n+1} + \frac{\mathfrak{e}(\mathcal{Z})}{\mathfrak{c}_n} r^{2n+3} + \frac{R_2(r)}{\mathfrak{c}_n},$$

whenever $0 < r < \min(\mathfrak{r}_4, \mathfrak{r}_8)$. In particular we deduce that $\mathfrak{e}(\mathcal{X}) = 0$, since $|R_2(r)| \leq \mathfrak{C}_6(n) r^{2n+3}$ for any $0 < r < \mathfrak{r}_4$ and the constant $\mathfrak{C}_6(n)$ depends only on $\mathcal{X}$.

If $n > 1$ or $\dim(\Sigma(f)) = 0$, Proposition 4.8 and Proposition B.15 prove the claim since $\pi_H(\operatorname{supp}(\mu)) = \mathbb{R}^{2n}$. On the other hand if $n = 1$ and $\dim(\Sigma(f)) = 1$, Proposition 4.8 implies that one of the two connected components of $\mathbb{R}^2 \setminus \Sigma(f)$ (which are halfspaces which bonduaty pass through 0) is contained in $\pi_H(\operatorname{supp}(\mu))$ and thus equation (B.29) holds for any z contained in such connected component. However, the structure of the coefficient $\mathfrak{e}(\mathcal{X})$ and the fact that $\mathfrak{n}(-h) = -\mathfrak{n}(h)$ imply that equation (B.29) holds on $\mathbb{R}^2 \setminus \Sigma(f)$. ☐

C. Further results obtained during the PhD

C.1 Non-differentiability sets of typical Lipschitz functions

The characterisation of the non-differentiability sets of real valued Lipschitz functions on the real line goes back to Zahorski who proved in [42] that:

Theorem C.1. *For any $G_{\delta\sigma}$ subset of the real line of Lebesgue measure zero E, there exists a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ which is non-differentiable everywhere on E and differentiable everywhere on $\mathbb{R} \setminus E$.*

Viceversa, given a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$, the set of points at which f is non-differentiable is a $G_{\delta\sigma}$ Lebesgue-null set.

At this point is quite natural to ask whether any similar characterisation is available for Lipschtiz maps $f : \mathbb{R}^n \to \mathbb{R}^m$. As it turns out already when the domain is $\mathbb{R}^2$, the answer is much more complicated and only partially known. Indeed M. Doré and O. Maleva in [19] and [20] constructed a compact set with Hausdorff dimension 1 in the $\mathbb{R}^n$ on which every Lipschitz function has a differentiability point (the first Lebesgue-null set with this property was first constructed by D. Preiss in [35]).

In order to solve the problem, one could hope to use the intuitive idea that the typical Lipschitz functions have the worst differentiability behaviour, and thus the problem may be solved by means of the Baire Cathegory Theorem on a suitable space of Lipschitz functions. This approach was attempted in 1995 by D. Preiss and J. Tišer in [36] where they showed that:

Theorem C.2 (Preiss, Tišer). *Let $E \Subset (0, 1)$ be an analytic set. The following are equivalent:*

(i) E is contained in an F_σ subset of $[0, 1]$ with Lebesgue measure zero.

(ii) The set S of those 1-Lipschitz functions differentiable at no point of E is residual in $\mathrm{Lip}_1([0, 1], \mathbb{R})$, the space of 1-Lipschitz functions on $[0, 1]$ with values in $\mathbb{R}$ endowed with the supremum norm.

Theorem C.2 is both good and bad news. On the one hand it shows that if E is covered by countably many closed Lebesgue-null sets, then the Baire Cathegory Theorem produces non-differentiable functions on E. On the other, if E does not satisfy this topological condition (for istance if $E \subseteq [0, 1]$ is residual and Lebesgue-null), they proved that the typical Lipschitz function *has* a point of differentiability in E, showing that this topological approach cannot tell the full story, in view of Zahorski's theorem.

There are many possible generalisations of the above result in higher dimensions. The one in which we are interested is the following: is it possible to build a map from $\mathbb{R}^n$ to $\mathbb{R}^n$ which is non-differentiable in any direction of a given purely unrectifiable Borel set E by means of the Baire Category Theorem? The answer to this question, depends on the topological properties of the set E:

Theorem C.3. *Let $E \Subset (0, 1)^n$ be an analytic set and let $n \leq m$. Then the follwing are equivalent:*

(i) E is contained in a countable union of closed purely unrectifiable sets,

(ii) *the set S of those 1-Lipschitz functions which are non-differentiable in every direction at every point of E is residual in $(\mathrm{Lip}_1([0,1]^n, \mathbb{R}^m), \|\cdot\|_\infty)$, the space of 1-Lipschitz functions on $[0,1]^n$ with values in $\mathbb{R}^m$ endowed with the supremum norm.*

As in the one dimensional case, the proof of (ii)⇒(i) shows that if (i) does not hold, then the typical Lipschitz function *has* a differentiability point in E. This gives an intuitive justification to why even the construction of fully non-differentiable functions on non-compact purely unrectifiable sets in [2] is so intricate.

C.1.1 Scheme of the proof

The proof of the implication $(i) \Rightarrow (ii)$ of Theorem C.3 heavily relies on techniques introduced in [5] and in [2]. Fix $\epsilon > 0$, $e \in \mathbb{S}^{2n-1}$ and a closed purely unrectifiable set E. It is possible (see Lemma 4.12 in [5]) to build for any $e \in \mathbb{S}^{n-1}$ a 1-Lipschitz function g_e such that:

(α) $\|g_e\|_\infty \leq \epsilon$,

(β) $|Dg_e(x) - e| < \epsilon$ for any $x \in E$.

Using these functions it is not hard to construct maps $G : \mathbb{R}^n \to \mathbb{R}^n$ with small supremum norms and such that $DG(x) \approx \mathrm{id}_n$ for any $x \in E$. Pick any smooth function f, and let $u \in \mathbb{S}^{n-1}$, $v \in \mathbb{S}^{m-1}$ and define:

$$\tilde{f}(x) := f(x) - Df(x)[G(x)] + g_u(x)v.$$

The function $\tilde{f}$ is close to f in the supremum topology, and on E its derivative along the direction v is near to u. In the end this construction (with some fine tuning) and the density of smooth functions in $\mathrm{Lip}_1([0,1]^n, \mathbb{R}^m)$ prove the implication (i)⇒(ii).

To explain the proof of the implication (ii)⇒(i) we need to introduce the Banach-Mazur game first. Let E be a set which cannot be covered by countably many compact purely unrectifiable sets and define the family of 1-Lipschitz functions:

$$B := \{f \in \mathrm{Lip}_1([0,1]^n, \mathbb{R}^m) : \text{ there are } x \in E,\ e \in \mathbb{R}^n \text{ s.t. } f(x+te) \text{ is differentiable at } t = 0\}.$$

Consider the following game with two players. Player (I) chooses an open set $U_1 \subseteq \mathrm{Lip}_1([0,1]^n, \mathbb{R}^m)$; then Player (II) chooses an open set $V_1 \subseteq U_1$; then Player (I) chooses an open set $U_2 \subseteq V_1$ and so on. Player (II) wins if $\bigcap_i V_i \subseteq B$, otherwise Player I wins. If we can build a winning strategy for Player (II), Theorem 6.1 of [34] implies that the set B is residual in $\mathrm{Lip}_1([0,1]^n, \mathbb{R}^m)$.

The proof that V_k can be chosen in such a fashion that Player (II) wins is based on the following two observations:

(α') Theorem 2 of [40] says that E is residual in a closed set F having any portion of positive width,

(β') if two continuous piece-wise congruent mappings (which where introduced in [11]) are close in the supremum norm, then the set where their directional derivative along $e \in \mathbb{S}^{n-1}$ are not close, has small width with respect to the cone of axis e (of a suitable amplitude).

Player II at each turn chooses piece-wise congruent mappings f_k, sets M_k and directions e_k (converging to some e) such that the sets V_k are (small enough) balls centred at f_k. The turn of Player II starts by arbitrarily picking a piece-wise congruent mapping f_k in U_k. The direction e_k is chosen close to e_{k-1} in such a way that the width of M_{k-1} along a cone of axis e_k (of sufficiently small amplitude) is positive. Eventually Player II must deal with the construction of M_k. Since $E \cap F$ is residual in F, we can find a sequence of relatively open sets E_k in F such that $E \cap F \subseteq \bigcap E_k$. Let G_k be the set given by point (β') which enjoys the two following properties:

(α'') the complement of G_k has a complement with very small width,

(β'') on G_k the function f_k and f_{k-1} have close derivatives along the direction e_k.

Player II defines M_k to be a non-empty relatively open set in F, compactly contained in $E_{k-1} \cap M_{k-1} \cap G_k$. Moreover point ($\alpha'$) insures that we can always find such an M_k having positive width with respect to a cone with axis e_k.

The functions f_k are uniformly converging to some $f \in \mathrm{Lip}_1([0,1]^n, \mathbb{R}^m)$ and the sets M_k are constructed in such a way that their intersection is non-empty (thanks to the finite intersection property of compact sets), it is contained in $E \cap F$ and point (β'') implies that f is differentiable along e at any point of $\bigcap M_k$.

Related results

The problem of the characterisation of non-differentiability sets of Lipschitz functions between Euclidean spaces has quite a long history, originally motivated by the attempt to prove a Rademacher-type theorems on Banach spaces (see for istance the monograph [26]). The paper [35] by D. Preiss could be arguably considered the first fundamental contribution to the theory, where among other things, he constructs a Lebesgue-null set in $\mathbb{R}^2$ on which every Lipschitz function has a differentiability point, showing that Rademacher's Theorem does not tell the full story. In 2005 G. Alberti, M. Csörnyei and D. Preiss announced in [4] and [3] a geometric characterisation of non-differentiability sets of Lipschitz functions and the proof that any Lebesgue-null set in $\mathbb{R}^2$ is contained in a non-differentiability set of some Lipschitz function $f : \mathbb{R}^2 \to \mathbb{R}^2$. On the other hand, more recently D. Preiss and G. Speight proved in [38] that for any $m < n$ there exists a Lebesgue-null set $\mathcal{N} \subseteq \mathbb{R}^n$ for which every Lipschitz map $f : \mathbb{R}^n \to \mathbb{R}^m$ has a point of differentiability on $\mathcal{N}$.

On the measure-theoretic side, in 2015 G. Alberti and A. Marchese proved in [5] that the Rademacher Theorem can be extended to finite mass Borel measures (when the definition of differentiability is suitably weakened) and in 2016 G. De Philippis and F. Rindler showed in [16] that if every Lipschitz function is differentiable μ-a.e. in the standard sense then μ is absolutely continuous with respect to Lebesgue.

C.2 Contact sets of non-involutive distributions and non-rectifiability in Carnot-Carathéodory spaces

This piece of work is a collaboration with G. Alberti and A. Massaccesi. Before stating the results, we introduce some notations. Suppose V is a distribution of k-planes of class $\mathcal{C}$ on the open set $\Omega \subseteq \mathbb{R}^n$ and let $S \subseteq \Omega$ be a k-dimensional manifold of class $\mathcal{C}$.

We say that $x \in S$ is a *tangency point* of S with respect to V if and only if $\mathrm{Tan}(S, z) = V(z)$. The set of such points is called the *tangency set* of S with respect to V and denoted by $\tau(S, V)$.

We say that V is *involutive* at a point $x \in \Omega$ if for every couple of vector fields X, X' of class $\mathcal{C}$ which are tangent to V the commutator $[X, X'](x)$ belongs to $V(x)$. We say that V is involutive if it is involutive at every point of Ω. The collection of all points x where V is not involutive is called the *non-involutivity set* of V and denoted by $N(V)$.

We say that *Frobenius theorem holds* for the k-rectifiable set $\mathcal{S}$ and the non-involutive distribution of k-planes V if $\mathcal{H}^k(\Sigma \cap N(V)) = 0$. It is a classical fact that for $\mathcal{C}^2$ manifolds the Frobenius theorem holds for any non-involutive distribution, and it has been proved by Z. Balogh in [9] the same holds for $\mathcal{C}^{1,1}$ manifolds.

C.2.1 Description of the results

Lusin-type theorems and failing of Frobenius theorem

In the first part of the paper we focus on constructing a k-dimensional $\mathcal{C}$ surface tangent to a continuous distribution of k-planes in a set of positive $\mathcal{H}^k$-measure:

Proposition C.4. *If V is a continuous distribution of k-planes of class $\mathcal{C}$ in $\mathbb{R}^n$, there exists a surface S of dimension k and of class $\cap_{0<\alpha<1} C^{1,\alpha}$ such that:*

$$\mathcal{H}^k(\tau(V,S)) > 0.$$

This was already achieved by Z. Balogh in Theorem 8.2 of [9] assuming some symmetry on the distribution V. As in [9], the main tool to construct such a surface is a Lusin-type theorem. The version stated below represents a slight improvement on the ones that can be found in [1] and [30]:

Theorem C.5 (Lusin-type theorem)**.** *Suppose m, n are integers and let μ be a Radon measure on $\mathbb{R}^n$ and Ω be an open set with $\mu(\Omega) < \infty$. Then, for every bounded Borel map $\mathcal{M} : \Omega \times \mathbb{R}^m \to M(n,m)$ and every $\epsilon, \eta > 0$ there exists a compact set K and a function $u \in \bigcap_{0<\alpha<1} \mathcal{C}^{1,\alpha}_c(\Omega)$ with $\|u\|_\infty \leq \eta$ such that:*

(i) $\mu(\Omega \setminus K) \leq \epsilon$,

(ii) $Du(x) = M(x, u(x))$ *for every* $x \in K$.

With this we have shown that despite the distribution V can be non-involutive (without any further geometric assumption), we can notheless construct a quite regular surface tangent to it on a set of positive, showing that Frobenius theorem cannot hold if the regularity drops below $C^{1,1}$.

Recovery of Frobenius theorem for tangency sets with good boundary

At first glance one may think that the story is concluded and there is no hope of recovering Frobenius's theorem for surfaces with regularity below $\mathcal{C}^{1,1}$. This is not quite correct. Indeed, as already remarked in [6], combining Corollary 4.1 of [17] and Corollary 1.1 of [18], one deduces that the contact set of a $\mathcal{C}^1$ non-involutive distribution V of k-planes with a $\mathcal{C}^1$, k-dimensional surface S is $\mathcal{H}^k$-null provided $V\mathcal{H}^k \llcorner \tau(S,V)$ is a normal current. This corresponds to the intuitive idea that the tangency set $\tau(S,V)$ has a "lot of holes", and thus it cannot have a "good boundary". Our original contribution is to show that if the boundary of $\tau(S,V)$ has some kind of fractional regularity, the tangency set must be small, allowing us to recover the Frobenius Theorem for the the couple S and V:

Theorem C.6. *Suppose V and S are as above. Let $T := V\mathcal{H}^k \llcorner \tau(S,V)$ and assume that ∂T the boundary acts on $\mathcal{C}^\alpha$-forms for some $\alpha \in [0,1)$, i.e.:*

$$|\langle \partial T, \omega \rangle| \leq C\|\omega\|_\alpha = \|\omega\|_\infty + \sup_{y\neq x} \frac{|\omega(y)-\omega(x)|}{|y-x|}^\alpha. \tag{C.1}$$

for any smooth, compactly supported $(k-1)$-form ω. Then $\mathcal{H}^k(\tau(S,V)) = 0$.

The bound (C.1) is implied by ∂T being supported on a Borel set having box-dimension bigger than α. The condition on the box-dimension seems to be quite precise, indeed we are able to construct a current whose boundary has Hausdorff dimension α for which (C.1) fails.

Pure unrectifiability of Carnot-Carathéodory spaces

A distribution of class $\mathcal{C}^1$ of k-planes V on $\mathbb{R}^n$ is said to have the Hörmander property if at any point x of Ω, there exists $N(x) \in \mathbb{N}$ such that the elementary commutators of $X_1, \dots, X_k$ of length at most $N(x)$ span $\mathbb{R}^n$ at x. We say that an absolutely continuous curve $\gamma : [0,1] \to \mathbb{R}^n$ is said *horizontal* if for $\mathcal{L}^1$-almost every $t \in [0,1]$ one has:

$$\gamma'(t) \in V(\gamma(t)).$$

For any $x, y \in \Omega$, we define *Carnot-Carathéodory distance* as:

$$d_V(x,y) := \inf\{\ell(\gamma) : \gamma \text{ is a horizontal curve, } \gamma(0) = x, \gamma(1) = y\}.$$

where ℓ is the Euclidean length of γ. The metric space $(\mathbb{R}^n, d_V)$ is said to be a Carnot-Carathéodory space. The main result of this third section is the following:

Theorem C.7. *The metric space $(\mathbb{R}^n, d_V)$ is m-purely unrectifiable for any $m \geq k$.*

To our knowledge, this statement was known only in the case of Carnot groups, see [7] and [29]. One may wonder why in a work concerned with contact sets between surfaces and distribution of planes, we prove such a result. The point is that given a Lipschitz function $f : K \Subset \mathbb{R}^k \to (\mathbb{R}^n, d_V)$ the set $f(K)$ is contained in a C^2 k-rectifiable set such that $\text{Tan}(f(K), x) = V(x)$ for $\mathcal{H}^k_{eu}$-almost every $x \in f(K)$. Therefore Lipschitz images are countable union of contact sets between C^2 surfaces and the non-involutive distribution V. Thanks to the above discussion we know that $f(K)$ is $\mathcal{H}^k_{eu}$-null. The leap from the $\mathcal{H}^k_{eu}$-nullness to the $\mathcal{H}^k_{d_V}$-nullness is somewhat delicate. Thanks to the fact that f is Lipschtz it is possible to show that for any euclidean ball $U(x, r)$ centred at x and with radius r, we have that $f(K) \cap U(x, r) \subseteq B(x, Cr)$, where C is a constant depending only on K and $B(x, r)$ is the Carnot-Carathéodory ball. This easily implies the claim.

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