EXISTENCE OF MINIMIZERS FOR THE SDRI MODEL IN 2D: WETTING AND DEWETTING REGIME WITH MISMATCH STRAIN

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Abstract. The model introduced in [45] in the framework of the theory on Stress-Driven Rearrangement Instabilities (SDRI) [3, 43] for the morphology of crystalline materials under stress is considered. As in [45] and in agreement with the models in [50, 55], a mismatch strain, rather than a Dirichlet condition as in [16], is included into the analysis to represent the lattice mismatch between the crystal and possible adjacent (supporting) materials. The existence of solutions is established in dimension two in the absence of graph-like assumptions and of the restriction to a finite number m of connected components for the free boundary of the region occupied by the crystalline material, thus extending previous results for epitaxially strained thin films and material cavities [6, 34, 35, 45]. Due to the lack of compactness and lower semicontinuity for the sequences of m-minimizers, i.e., minimizers among configurations with at most m connected boundary components, a minimizing candidate is directly constructed, and then shown to be a minimizer by means of uniform density estimates and the convergence of m-minimizers' energies to the energy infimum as $m \to \infty$. Finally, regularity properties for the morphology satisfied by every minimizer are established.

1. Introduction

In this paper we establish existence and regularity properties for the solutions of the variational model for Stress-Driven Rearrangement Instabilities (SDRI) [3, 23, 43] that was introduced in [45]. Under the name of SDRI are included all those material morphologies, such as boundary irregularities, cracks, filaments, and surface patterns, which a crystalline material may exhibit in the presence of external forces, such as in particular the chemical bonding forces with adjacent materials. In order to release the induced stresses, atoms rearrange from the material optimal crystalline order and instabilities may develop.

The main advancement provided by the results in this manuscript with respect to [45] is the absence of the unphysical restriction on the number of connected components for the boundary of the region occupied by the crystalline material, by also avoiding graph-like assumptions for such boundaries assumed for the specific settings of epitaxially strained thin films in [6, 16, 34] and material voids in [35]. In particular, with respect to [16] we include into the analysis the *dewetting regime*, i.e., the presence of other fixed materials with possibly different boundary surface tensions, even if by only treating the two dimensional case, and we establish regularity results for the crystalline morphologies and instabilities satisfied by every minimizer. Furthermore, our strategy stems from the approach used in [22] for the *Mumford-Shah* functional, and hence differs from the method introduced in [16], which instead is based on allowing displacements to attain *a limit value* ∞ on

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sets with positive measure (and on technically assigning a zero cost to the elastic-energy contribution related to those sets).

The SDRI model of [45] is a variational model introduced in the framework of the SDRI theory initiated in the seminal papers of [3] and [43], and on the basis of the subsequent analytical descriptions provided in the context of epitaxially strained thin films [6, 24, 25, 34], crystal cavities [8, 35], capillarity droplets [26, 30], fractures [7, 11, 17, 19, 36], and boundary debonding and delamination [4, 49]. All such settings are included and can be treated simultaneously in the SDRI model [45] (see Section 2.5). In agreement with [3, 43] since SDRI morphologies relate to the boundary of crystalline materials and depend on the bulk rearrangements, the energy $\mathcal{F}$ characterizing the SDRI model displays both an *elastic bulk energy* and a *surface energy* denoted by $\mathcal{W}$ and $\mathcal{S}$, respectively. More precisely, the energy $\mathcal{F}$ is defined as

$$\mathcal{F}(A,u) := \mathcal{S}(A,u) + \mathcal{W}(A,u) \tag{1.1}$$

for any admissible *configurational pair* (A,u) consisting of a set A that represents the region occupied by the crystalline material in a fixed *container* $\Omega\subset\mathbb{R}^d$ for $d\in\mathbb{N}$, i.e.,

$$A\in\mathcal{A} := \{A\subset\overline{\Omega}:\ A \text{ is } \mathcal{L}^2\text{-measurable and } \partial A \text{ is } \mathcal{H}^1\text{-rectifiable}\},$$

and of a displacement function u of the bulk materials (with respect to the optimal crystal arrangement) given by

$$u\in GSBD^2(\mathrm{Int}(A\cup S\cup\Sigma);\mathbb{R}^d)\cap H^1_{\mathrm{loc}}(\mathrm{Int}(A)\cup S;\mathbb{R}^d),$$

where $S\subset\mathbb{R}^d\setminus\Omega$ is the region occupied by a fixed material, which we denote *substrate* in analogy with the thin-film setting and we consider possibly different from the material in the container, and

$$\Sigma := \partial S\cap\partial\Omega$$

represents the *contact surface* between the container Ω and the substrate S. In the following we refer to $\mathcal{C}$ as the *configurational space* and to each configuration $(A,u)\in\mathcal{C}$ as a *free crystal* with A and u as the *free-crystal region* and the *free-crystal displacement*, respectively (see Figure 1).

The bulk elastic energy $\mathcal{W}$ in (1.1) is defined in [45] by

$$\mathcal{W}(A,u) = \int_{A\cup S} W(z, e(u)-M_0)\,dz,$$

where the *elastic density* W is given by

$$W(z,M) := \mathbb{C}(z)M : M \tag{1.2}$$

for any $z\in\Omega\cup S$ and any $(d\times d)$-symmetric matrix $M\in\mathbb{M}^{d\times d}_{\mathrm{sym}}$, and for a positive-definite elasticity tensor $\mathbb{C}$, and attains its minimum value zero for every z at a fixed strain $M_0\in M\in\mathbb{M}^{d\times d}_{\mathrm{sym}}$ in the following referred to as *mismatch strain.* The inclusion in (1.2) of a mismatch strain M_0 defined by

$$M_0 := \begin{cases} e(u_0) & \text{in } \Omega,\\ 0 & \text{in } S,\end{cases} \tag{1.3}$$

for a fixed $u_0\in H^1(\mathbb{R}^d;\mathbb{R}^d)$, together with the fact that both M_0 and $\mathbb{C}$ are let free of jumping across Σ, allows to model the presence of two different materials in the substrate and in the free crystals, and in particular to take into account the *lattice mismatch* between their optimal crystalline lattices that is crucial, e.g., in the setting of heteroepitaxy [24, 25].

The surface energy $\mathcal{S}$ in (1.1) is defined as

$$\mathcal{S}(A,u) = \int_{\partial A}\psi(z,u,\nu)\,d\mathcal{H}^{d-1},$$

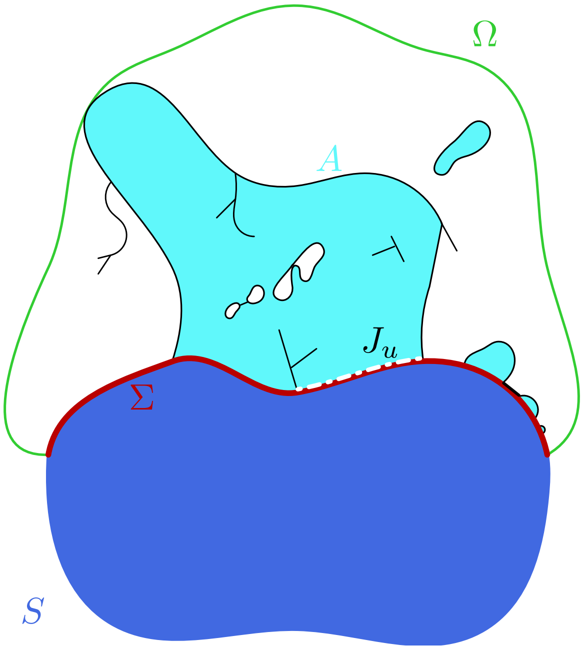

FIGURE 1. An admissible free-crystal region A is displayed in light blue in the container Ω, while the substrate S is represented in dark blue. The boundary of A (with the cracks) is depicted in black, the container boundary in green, the contact surface Σ in red (thicker line) while the free-crystal delamination region J_u with a white dashed line.

where the *surface tension* ψ is given by

$$\psi(z,u,\nu) := \begin{cases} \varphi(z,\nu_A(z)) & z \in \Omega \cap \partial^* A, \\ 2\varphi(z,\nu_A(z)) & z \in \Omega \cap (A^{(1)} \cup A^{(0)}) \cap \partial A, \\ \varphi(z,\nu_S(z)) + \beta(z) & z \in \Sigma \cap A^{(0)} \cap \partial A, \\ \beta(z) & z \in \Sigma \cap \partial^* A \setminus J_u, \\ \varphi(z,\nu_S(z)) & z \in J_u, \end{cases} \tag{1.4}$$

with $\varphi \in C(\overline{\Omega} \times \mathbb{R}^d; [0,+\infty))$ being a Finsler norm such that $c_1|\xi| \le \varphi(x,\xi) \le c_2|\xi|$ for some $c_1, c_2 > 0$ and representing the *anisotropy* of the free-crystal material, β denoting the *relative adhesion coefficient* on Σ such that, as for capillarity problems [26, 30],

$$|\beta(z)| \le \varphi(z,\nu_S(z))$$

for every $z \in \Sigma$, ν coinciding with the exterior normal on the reduced boundary $\partial^* A$, and $A^{(\delta)}$ denoting the set of points of A with density $\delta \in [0,1]$.

The anisotropic form of ψ in (1.4) distinguishes various portions of the free-crystal topological boundary ∂A: the *free boundary* $\partial^* A \cap \Omega$, the family of *internal cracks* $A^{(1)} \cap \Omega \cap \partial A$, the family of *external filaments* $A^{(0)} \cap \Omega \cap \partial A$, the *delaminated region* J_u, i.e., the portion on the contact surface Σ where there is no bonding between the free crystal and the substrate (even if they are adjacent), the *adhesion area* where the free-crystal displacement is continuous through Σ, i.e., $\Sigma \cap \partial^* A \setminus J_u$, and the *wetting layer* represented by the filaments on Σ, i.e., $\Sigma \cap A^{(0)}$. In particular, ψ weights the different portions of ∂A in relation to the active chemical bondings present at each portion, i.e., φ when there is no extra chemical bonding, such as at the free profile and at the delaminated region, and β at the adhesion contact area with the substrate, while both the cracks and at external filaments are counted 2φ and the wetting layer sees the contribution of both ψ and β.

We consider the case $d = 2$ as in [45], with the fixed sets Ω and S being bounded Lipschitz open connected sets such that Σ is a Lipschitz 1-manifold. For $d \geq 3$ results are available for the isotropic Griffith model with L^p-fidelity term (of the type (2.19)) in [11] and with Dirichlet conditions for the displacements at the boundary in [12]. Moreover, a similar energy as the SDRI energy introduced in [45] was subsequently found in [16] as a relaxation formula separately for thin films and material voids, for the different setting with a Dirichlet condition imposed at $\partial\Omega$, and in the wetting regime, i.e., the case where free crystals are expected to cover the substrate. Unfortunately the strategy employed in [16] is not implementable in our setting, where rather than prescribing a Dirichlet condition as in [16], the mismatch strain (1.3) (which depends on the substrate region S) is considered in the elastic energy in analogy with the models in [55] and [50, Section 4.2.2] (see also the mathematical treatments [24, 25, 34, 47]).

In fact, the existence results in [16] are achieved by working (in the proofs) with displacements in a larger space than the classical framework of small displacements of linearized elasticity, namely the space $GSBD^p_\infty$ for $p > 1$ that includes displacements attaining a limit value ∞ in a set of finite perimeter (on which their strain $e(u)$ is defined to be zero [16, Page 1055]). Such a method works well with a Dirichlet condition that keeps the displacements anchored, while in our setting it would be always convenient for the displacements in $GSBD^p_\infty$ of the minimizing sequences to escape to infinity, as this would result with the definition of the energy in [16] in the minimum (zero) value of the elastic energy for the limiting free-crystal region. A treatment for $d \geq 3$ of the model under consideration in this paper with mismatch strain (and without Dirichlet conditions) is under preparation [46] by implementing the ideas in this manuscript together with the ones in [45], but without the need of Golab's Theorem (and without employing the space $GSBD^p_\infty$ for the displacements).

Therefore, we must proceed differently here and we rely on the results of [45] for $d = 2$. We begin by observing that, as shown in [45], the specific weights of (1.4) are crucial to obtain the lower semicontinuity of the energy $\mathcal{F}$ under the constraint on a fixed number $m \in \mathbb{N}$ of boundary connected components for the free-crystal regions, which represented an extension of the more restrictive graph condition assumed in [34] for the particular setting of epitaxially strained thin films and the starshapedness condition in [35] for material cavities. More precisely, by considering the subfamily $\mathcal{C}_m$ of configurations with free crystals presenting at most $m \in \mathbb{N}$ boundary connected components, namely

$$\mathcal{C}_m := \Big\{(A, u) \in \mathcal{C} : \ \partial A \text{ has at most } m \text{ connected components}\Big\},$$

in [45, Theorem 2.8] it is shown that

$$\liminf_{k\to\infty} \mathcal{F}(A_k, u_k) \geq \mathcal{F}(A, u)$$

for every sequence $\{(A_k, u_k)\} \subset \mathcal{C}_m$ converging in a properly chosen topology $\tau_{\mathcal{C}}$ to a configuration $(A, u) \in \mathcal{C}_m$. In particular, the convergence with respect to $\tau_{\mathcal{C}}$ prescribes that $\mathcal{H}^1(\partial A_k)$ are equibounded, $\mathrm{sdist}(\cdot, \partial A_k) \to \mathrm{sdist}(\cdot, \partial A)$ locally uniformly in $\mathbb{R}^2$ with *sdist* representing the *signed distance* function (recall definition at (2.2)), and $u_n \to u$ a.e. in $\mathrm{Int}(A) \cup S$. We notice that the restriction to the subfamily $\mathcal{C}_m$ was needed in [45] to establish not only the lower semicontinuity, but also the compactness with respect to $\tau_{\mathcal{C}}$, which indeed fails in $\mathcal{C}$ (see Remark 2.3), so that by means of the *direct method* of the calculus of variations, the existence of minimizers $(A_m, u_m) \in \mathcal{C}_m$ of $\mathcal{F}$ among all configurations in $\mathcal{C}_m$ followed in [45, Theorem 2.6].

The aim of the investigation contained in this paper is to recover the full generality avoiding any extra hypothesis on the admissible free-crystal regions. This is achieved by retrieving compactness with respect to the free-crystal regions at least for any sequence of m-minimizers $(A_m, u_m) \in \mathcal{C}_m$, and by combining the strategies of [22] and [45]. More precisely, the use in [45] of the Golab-type Theorem [40] is avoided for the compactness of the free-crystal regions by adapting to our setting the classical *density-estimate* arguments first introduced for surface energies and the Mumford-Shah functional (see, e.g., [2, 28, 52]), and then extended to the Griffith functional [12, 19], which in turns allow us also to establish some regularity results. Moreover, in our setting there is the extra difficulty with respect to [22] that the compactness and lower semicontinuity along sequences of m-minimizers (with respect to the topology used to find such m-minimizers through the direct method) are both missing. We overcome this issue, by directly constructing a minimizing candidate, proving that it belongs to the class

$$\widetilde{\mathcal{A}} := \Big\{ A \subset \overline{\Omega} : \ A \text{ is } \mathcal{L}^2\text{-measurable and } \mathcal{H}^1(\partial A) < +\infty \Big\},$$

and establishing a "lower-semicontinuity inequality" (see (1.7) below) along the selected sequence of m-minimizers (A_m, u_m) (see Subsection 1.1 for more details).

Since $\mathcal{A} \subset \widetilde{\mathcal{A}}$, for proving such lower-semicontinuity property we introduce an auxiliary energy $\widetilde{\mathcal{F}}$ defined in the larger family $\widetilde{\mathcal{C}}$ of configurations (A, u) for which $A \in \widetilde{\mathcal{A}}$, i.e.,

$$\widetilde{\mathcal{F}}(A, u) := \widetilde{\mathcal{S}}(A, u) + \mathcal{W}(A, u),$$

with auxiliary surface energy $\widetilde{\mathcal{S}}$ defined as

$$\widetilde{\mathcal{S}}(A, u) = \int_{\partial A} \widetilde{\psi}(z, u, \nu) d\mathcal{H}^{d-1},$$

where the surface tension $\widetilde{\psi}$ is given by

$$\widetilde{\psi}(z, u, \nu) := \begin{cases} \varphi(z, \nu_A(z)) & z \in \Omega \cap \partial^* A, \\ 2\varphi(z, \nu_A(z)) & z \in S_u^A, \\ \beta(z) & z \in \Sigma \cap \partial^* A \setminus J_u, \\ \varphi(z, \nu_S(z)) & z \in J_u \end{cases}$$

for S_u^A denoting the jump set of u along the $\mathcal{H}^1$-rectifiable portion of the cracks (see (2.6) for the precise definition).

The results of this paper are twofold: The existence results contained in Theorem 2.6 and the regularity properties of Theorem 2.7. More precisely, in Theorem 2.6 we prove the existence of a minimum configuration of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ among all configurations in $\mathcal{C}$ and $\widetilde{\mathcal{C}}$, respectively, with free-crystal region satisfying a volume constraint, i.e., we solve the minimum problems

$$\inf_{(A,u) \in \mathcal{C},\ |A| = \mathtt{v}} \mathcal{F}(A, u) \tag{1.5}$$

and

$$\inf_{(A,u) \in \widetilde{\mathcal{C}},\ |A| = \mathtt{v}} \widetilde{\mathcal{F}}(A, u) \tag{1.6}$$

for a fixed volume parameter $\mathtt{v} \in (0, |\Omega|)$ or, if $S = \emptyset$, $\mathtt{v} = |\Omega|$. Furthermore, the minimum problems (1.5) and (1.6) are proven to be equivalent to the *unconstraint minimum problems* consisting in minimizing *volume-penalized versions* $\mathcal{F}^\lambda$ and $\widetilde{\mathcal{F}}^\lambda$ of the functionals $\mathcal{F}$ and $\widetilde{\mathcal{F}}$, for a *penalization constant* $\lambda > 0$ provided that $\lambda \geq \lambda_1$ for some uniform constant $\lambda_1 > 0$.

In Theorem 2.7 regularity properties shared by all solutions of (1.5) and (1.6) are found. Notice that we cannot directly apply the arguments of [34, 35] based on the *external sphere condition* considered in [15] because of the absence of graph and star-shapedness assumptions on the admissible free-crystal regions. As a byproduct of Theorem 2.6 and Proposition 5.1 given a configuration (A, u) minimizing (1.5) resp. (1.6), we can construct a configuration $(A', u) \in \mathcal{C}$ which minimizes both minimum problems (1.5) and (1.6) such that A' is an open set with cracks coinciding in Ω with the jump set of the corresponding minimizing free-crystal displacement u, and boundary $\partial A'$ satisfying uniform upper and lower density estimates. Furthermore, we also observe that, any connected component E of A' that does not intersect $\Sigma \setminus J_u$ (up to $\mathcal{H}^1$-negligible sets), must have a sufficiently large area, i.e.,

$$|E| \geq (c_1\sqrt{4\pi}/\lambda_1)^2,$$

and must satisfy $u = u_0$ in E up to adding a rigid displacement.

1.1. **Paper organization and detail of the proofs.** The paper is organized in 5 sections. In Section 2 we introduce the mathematical setting, recall the SDRI model from [45], and carefully state the main results of the paper.

In Section 3 we prove the upper and lower density estimates for the local decay of the energy $\mathcal{F}$ on any sequence of m-minimizers $(A_m, u_m) \in \mathcal{C}_m$ (see Theorem 3.1) by considering a local version of $\mathcal{F}^\lambda$ (see (2.9)), adapting arguments of [2, 12, 19] to our setting with displacements paired with free-crystal regions, and paying extra care to the fact that $\mathbb{C}$ is possibly not constant (but in $L^\infty(\Omega \cup S) \cap C^0(\Omega)$).

In Section 4 we prove compactness and lower-semicontinuity properties for a sequence of m-minimizers. We begin by establishing in Proposition 4.1 the compactness for a sequence of m-minimizers $\{(A_m, u_m)\}$ with free-crystal regions A_m not containing isolated points of such free-crystal regions to a limiting set of finite perimeter $A \subset \Omega$ by means of both the Blaschke-type selection principle [45, Proposition 3.1] and the density estimates established in Section 3. Then, in Proposition 4.3, we further extend the (already generalized) Golab-type Theorem [40, Theorem 4.2] to a priori not-connected $\mathcal{H}^1$-measurable (not necessarily $\mathcal{H}^1$-rectifiable) sets satisfying uniform density estimates (see [22] for the isotropic case). The compactness of the displacements in $\{(A_m, u_m)\}$ is then proved in Propositions 4.4 by carefully constructing the limiting displacement u in view of the property that for every connected component E_i of A the set in which displacements u_m diverge is either the whole component E_i or $\emptyset$, which follows from [45, Theorem 3.7]. Finally, in Proposition 4.6 we establish the lower-semicontinuity property

$$\liminf_{h\to\infty} \mathcal{F}(A_{m_h}, u_{m_h}) \geq \widetilde{\mathcal{F}}(A, u), \tag{1.7}$$

by treating separately the elastic and the surface energy. For the latter we employ a *blow-up method* differently performed for each portion of the ∂A where $\widetilde{\psi}$ is supported. In particular extra care is needed for the jump set J_u and jump set along cracks S_u^A (since there is no bound on the number of connected components), where we need to extend some ideas from [45, Proposition 4.1].

In Section 5 we prove the main results of the manuscript, i.e., the existence and regularity results that are contained in Theorems 2.6 and 2.7, respectively. In order to prove Theorem 2.6 we first establish in Proposition 5.1 the equalities

$$\inf_{(B,v)\in\widetilde{\mathcal{C}},\ |B|=\mathtt{v}} \widetilde{\mathcal{F}}(B,v) = \inf_{(B,v)\in\mathcal{C},\ |B|=\mathtt{v}} \mathcal{F}(B,v) = \lim_{m\to\infty} \inf_{(B,v)\in\mathcal{C}_m,\ |B|=\mathtt{v}} \mathcal{F}(B,v). \tag{1.8}$$

(recall that the second equality follows from [45, Theorem 2.6]) by using similar arguments previously used in [45, Theorem 2.6]. In particular, (1.7) and (1.8) imply that the configuration $(A, u) \in \widetilde{\mathcal{C}}$ is a minimizer of $\widetilde{\mathcal{F}}$ in $\widetilde{\mathcal{C}}$. In Theorem 5.3 we establish the uniform density estimates for the jump set S_u^A of u along cracks for a minimizer (A, u) of $\widetilde{\mathcal{F}}$. In particular, S_u^A is then essentially closed, and using this fact in Proposition 5.4 we construct a configuration $(A', u) \in \mathcal{C}$, which minimizes boths $\widetilde{\mathcal{F}}$ and $\mathcal{F}$, starting from a minimizer (A, u) of $\widetilde{\mathcal{F}}$ in $\widetilde{\mathcal{C}}$. Moreover, (A', u) solves both (1.5) and (1.6) and satisfies the properties stated in Theorem 2.7. Theorem 2.7 is then a direct consequence of Proposition 5.4, comparison arguments, the isoperimetric inequality in $\mathbb{R}^2$, and the equivalence of the constrained minimum problems and the unconstrained penalized minimum problem related to the energies $\mathcal{F}^\lambda$ and $\widetilde{\mathcal{F}}^\lambda$.

We conclude the manuscript with Appendix A that contains some subsidiary results recalled for Reader's convenience since very relevant in the arguments used throughout the paper.

2. Mathematical setting

In this section we recall the SDRI model from [45], collect all the definitions and hypotheses and state the main results of the paper. Since our model is two-dimensional, unless otherwise stated, all sets we consider are subsets of $\mathbb{R}^2$. We choose the standard basis $\{\mathbf{e_1} = (1,0), \mathbf{e_2} = (0,1)\}$ in $\mathbb{R}^2$ and denote the coordinates of $x \in \mathbb{R}^2$ with respect to this basis by (x_1, x_2). We denote by $\mathrm{Int}(A)$ the interior of $A \subset \mathbb{R}^2$. Given a Lebesgue measurable set E, we denote by χ_E its characteristic function and by $|E|$ its Lebesgue measure. The set

$$E^{(\alpha)} := \Big\{x \in \mathbb{R}^2 : \lim_{r\to 0} \frac{|E \cap B_r(x)|}{|B_r(x)|} = \alpha\Big\}, \qquad \alpha \in [0,1],$$

where $B_r(x)$ denotes the ball in $\mathbb{R}^2$ centered at x of radius $r > 0$, is called the set of points of density α of E. Clearly, $E^{(\alpha)} \subset \partial E$ for any $\alpha \in (0,1)$, where

$$\partial E := \{x \in \mathbb{R}^2 : \ B_r(x) \cap E \neq \emptyset \text{ and } B_r(x) \setminus E \neq \emptyset \text{ for any } r > 0\}$$

is the topological boundary. The set $E^{(1)}$ is the *Lebesgue set* of E and $|E^{(1)} \Delta E| = 0$. We denote by $\partial^* E$ the *reduced* boundary of a set E of finite perimeter [2, 41], i.e.,

$$\partial^* E := \Big\{x \in \mathbb{R}^2 : \ \exists \nu_E(x) := -\lim_{r\to 0} \frac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))}, \quad |\nu_E(x)| = 1\Big\}.$$

The vector $\nu_E(x)$ is called the *generalized outer normal* to E.

Remark 2.1. If E is a set of finite perimeter, then

- $\overline{\partial^* E} = \partial E^{(1)}$ (see e.g., [52, Eq. 15.3]);
- $\partial^* E \subseteq E^{(1/2)}$ and $\mathcal{H}^1(E^{(1/2)} \setminus \partial^* E) = 0$ (see e.g., [52, Theorem 16.2]);
- $P(E,B) = \mathcal{H}^1(B \cap \partial^* E) = \mathcal{H}^1(B \cap E^{(1/2)})$ for any Borel set E;

where $P(E,B)$ and $\mathcal{H}^1$ denote the *perimeter* of E in B and the 1-dimensional Hausdorff measure, respectively.

An $\mathcal{H}^1$-measurable set K is called $\mathcal{H}^1$*-rectifiable* if $\mathcal{H}^1(K) < \infty$ and there exist countably many Lipschitz functions $f_i : \mathbb{R} \to \mathbb{R}^2$ such that

$$\mathcal{H}^1\Big(K \setminus \bigcup_{i\geq 1} f_i(\mathbb{R})\Big) = 0 \tag{2.1}$$

(see e.g., [2, Definition 2.57]). Notice that one can assume in (2.1) that the functions f_i are C^1, since Lipschitz functions are a.e. differentiable. By the Besicovitch-Marstrand-Mattila Theorem ([2, Theorem 2.63] a Borel set $K \subset \mathbb{R}^2$ with $\mathcal{H}^1(K) < +\infty$ is $\mathcal{H}^1$-rectifiable if and only if $\theta^*(K,x) = \theta_*(K,x) = 1$ for $\mathcal{H}^1$-a.e. $x \in K$, where

$$\theta^*(K,x) := \limsup_{r\to 0^+} \frac{\mathcal{H}^1(B_r(x) \cap K)}{2r} \quad \text{and} \quad \theta_*(K,x) := \liminf_{r\to 0^+} \frac{\mathcal{H}^1(B_r(x) \cap K)}{2r}.$$

In particular, any $\mathcal{H}^1$-rectifiable set K admits a approximate tangent line at $\mathcal{H}^1$-a.e. $x \in K$, see e.g., [52, Remark 10.3]. When $\theta_*(K,x) = \theta^*(K,x) = 1$, we write for simplicity $\theta(K,x) = 1$. A Borel set $K \subset \mathbb{R}^2$ with $\mathcal{H}^1(K) < +\infty$ is said *purely unrectifiable* if $\mathcal{H}^1(K \cap \Gamma) = 0$ for every 1-dimensional Lipschitz graph $\Gamma \subset \mathbb{R}^2$ (see e.g., [2, Definition 2.64]).

Moreover, by [29, Theorem 5.7] , if $K \subset \mathbb{R}^2$ is an arbitrary Borel set with $\mathcal{H}^1(E) < +\infty$, then there exist Borel subsets K^r and K^u of K such that $K = K^r \cup K^u$, K^r is $\mathcal{H}^1$-rectifiable and K^u is purely unrectifiable, and such a decomposition is unique up to a $\mathcal{H}^1$-negligible set. More precisely, if $K = L^r \cup L^u$ with $\mathcal{H}^1$-rectifiable L^r and purely unrectifiable L^u, then $\mathcal{H}^1(K^r \Delta L^r) = \mathcal{H}^1(K^u \Delta L^u) = 0$. In what follows we call K^r and K^u the rectifiable and purely unrectifiable parts of K, respectively. When $A \subset \mathbb{R}^2$ with $\mathcal{H}^1(\partial A) < +\infty$, we denote by $\partial^r A$ and $\partial^u A$ the $\mathcal{H}^1$-rectifiable and purely unrectifiable parts of ∂A, respectively.

The notation $\mathrm{dist}(\cdot, E)$ stands for the distance function from the set $E \subset \mathbb{R}^2$ with the convention that $\mathrm{dist}(\cdot, \emptyset) \equiv +\infty$. Given a set $A \subset \mathbb{R}^2$, we consider also signed distance function from ∂A, negative inside, defined as

$$\mathrm{sdist}(x, \partial A) := \begin{cases} \mathrm{dist}(x, A) & \text{if } x \in \mathbb{R}^2 \setminus A, \\ -\mathrm{dist}(x, \mathbb{R}^2 \setminus A) & \text{if } x \in A. \end{cases} \tag{2.2}$$

Remark 2.2. The following assertions are equivalent:

(a) $\mathrm{sdist}(x, \partial E_k) \to \mathrm{sdist}(x, \partial E)$ locally uniformly in $\mathbb{R}^2$;

(b) $E_k \xrightarrow{K} \overline{E}$ and $\mathbb{R}^2 \setminus E_k \xrightarrow{K} \mathbb{R}^2 \setminus \mathrm{Int}(E)$, where K denotes the Kuratowski convergence of sets [20].

Moreover, either assumption implies $\partial E_k \xrightarrow{K} \partial E$.

Given $r > 0$, $\nu \in \mathbb{S}^1$ and $x \in \mathbb{R}^2$ we denote by $Q_{r,\nu}(x)$ the square of sidelength $2r$ centered at x whose sides are either parallel or perpendicular to ν. When $\nu = \mathbf{e_2}$ or $\nu = \mathbf{e_1}$, we drop the dependence on ν and write $Q_r(x)$. If in addition $x = 0$, we write just Q_r. We also set

$$I_r := [-r, r] \times \{0\},\ Q_r^+ = \{x \in Q_r : \ x \cdot \mathbf{e}_2 > 0\}, \text{ and } Q_r^- = \{x \in Q_r : \ x \cdot \mathbf{e}_2 < 0\}. \tag{2.3}$$

Given $x \in \mathbb{R}^2$ and $r > 0$, the blow-up map $\sigma_{x,r}$ is defined as

$$\sigma_{x,r}(y) = \frac{y - x}{r}. \tag{2.4}$$

The blow-up of $K \subset \mathbb{R}^2$ is defined as $\sigma_{x,r}(K)$.

Given an open set $U \subset \mathbb{R}^2$ and a metric space X we denote by $\mathrm{Lip}(U; X)$ the family of all Lipschitz functions $\psi : U \to X$. We denote by $\mathrm{Lip}(\psi)$ the Lipschitz constant of $\psi \in \mathrm{Lip}(U; X)$. Furthermore, $GSBD(U; \mathbb{R}^2)$ denotes the collection of all *generalized special functions of bounded deformation* (see [14, 21] for their definition and properties). Given $u \in GSBD(U; \mathbb{R}^2)$ we denote with $e(u) \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$ the *approximate symmetric gradient* of u, for which

$$\operatorname*{ap\,lim}_{y \to x} \frac{[u(y) - u(x) - e(u)(x)(y - x)] \cdot (y - x)}{|y - x|^2} = 0$$

holds for a.e. $x \in U$ by [21, Theorem 9.1], and with J_u the *jump set* of u, which is $\mathcal{H}^1$-rectifiable by [21, Theorem 6.2]. Let us also define

$$GSBD^2(U, \mathbb{R}^2) := \{u \in GSBD(U; \mathbb{R}^2) : \ e(u) \in L^2(U; \mathbb{M}^{2\times 2}_{\mathrm{sym}})\}.$$

Given a $\mathcal{H}^1$-rectifiable set $M \subset \overline{U}$, we consider a normal vector ν_M to its approximate tangent line and we denote by u_M^+ and u_M^- the approximate limits of $u \in GSBD^2(U; \mathbb{R}^2)$ with respect to ν_M, i.e.,

$$u_M^+(x) := \operatorname*{ap\,lim}_{\substack{(y-x)\cdot\nu_M > 0,\\ y \in U}} u(y) \quad \text{and} \quad u_M^-(x) := \operatorname*{ap\,lim}_{\substack{(y-x)\cdot\nu_M < 0\\ y \in U}} u(y) \tag{2.5}$$

for every $x \in M$ whenever they exist (see [21, Definition 2.4]). We refer to u_M^+ and u_M^- as the *two-sided traces* of u at M and we notice that they are uniquely determined up to a permutation when changing the sign of ν_M. If $U = \mathrm{Int}(A)$ for some measurable set A with $\mathcal{H}^1(\partial A) < +\infty$ and $M := \partial^r A$, we use the simplified notations $u_{\partial A}^\pm$ on $A^{(1)} \cap \partial^r A$, and $\mathrm{tr}_A u := u^+_{\partial A}$ on $\partial^* A$, where on $\partial^* A$ we always choose ν_M in (2.5) as the generalized outer unit normal to A. Moreover, we define

$$S_u^A := \{x \in A^{(1)} \cap \partial^r A : \, u^+_{\partial A}(x) \neq u^-_{\partial A}(x)\}. \tag{2.6}$$

Note that S_u^A is $\mathcal{H}^1$-rectifiable. We refer to S_u^A the jump set of u along the cracks of A.

A linear function $a : \mathbb{R}^2 \to \mathbb{R}^2$, defined as $Ax = Mx + b$, where M is 2×2-matrix and $b \in \mathbb{R}^2$, is an (infinitesimal) rigid displacement if $M = -M^T$.

2.1. **The SDRI model.** Given two nonempty bounded Lipschitz connected open sets $\Omega \subset \mathbb{R}^2$ and $S \subset \mathbb{R}^2 \setminus \Omega$ such that $\overline{\Omega} \cap \overline{S} \neq \emptyset$ and the set $\Sigma := \partial S \cap \partial \Omega$ is a Lipschitz 1-manifold, we define the family of admissible regions for the *free crystal* and the space of *admissible configurations* by

$$\mathcal{A} := \{A \subset \overline{\Omega} : \; A \text{ is } \mathcal{L}^2\text{-measurable and } \partial A \text{ is } \mathcal{H}^1\text{-rectifiable}\}$$

and

$$\begin{aligned}\mathcal{C} := \big\{(A,u) : \, & A \in \mathcal{A}, \\ & u \in GSBD^2(\mathrm{Int}(A \cup S \cup \Sigma); \mathbb{R}^2) \cap H^1_{\mathrm{loc}}(\mathrm{Int}(A) \cup S; \mathbb{R}^2)\big\},\end{aligned}$$

respectively. By Proposition A.1 any $A \in \mathcal{A}$ has finite perimeter. Furthermore, $J_u \subset \Sigma \cap \overline{\partial^* A}$ since $u \in H^1_{\mathrm{loc}}(\mathrm{Int}(A) \cup S; \mathbb{R}^2)$.

The *energy* of admissible configurations is given by $\mathcal{F} : \mathcal{C} \to [-\infty, +\infty]$,

$$\mathcal{F} := \mathcal{S} + \mathcal{W}, \tag{2.7}$$

where $\mathcal{S}$ and $\mathcal{W}$ are the surface and elastic energies of the configuration, respectively. The surface energy of $(A,u) \in \mathcal{C}$ is defined as

$$\begin{aligned}\mathcal{S}(A,u) := & \int_{\Omega \cap \partial^* A} \varphi(x, \nu_A(x)) d\mathcal{H}^1(x) \\ & + \int_{\Omega \cap (A^{(1)} \cup A^{(0)}) \cap \partial A} \big(\varphi(x, \nu_A(x)) + \varphi(x, -\nu_A(x))\big) d\mathcal{H}^1(x) \\ & + \int_{\Sigma \cap A^{(0)} \cap \partial A} \big(\varphi(x, \nu_\Sigma(x)) + \beta(x)\big) d\mathcal{H}^1(x) \\ & + \int_{\Sigma \cap \partial^* A \setminus J_u} \beta(x) d\mathcal{H}^1(x) + \int_{J_u} \varphi(x, -\nu_\Sigma(x)) \, d\mathcal{H}^1(x),\end{aligned} \tag{2.8}$$

where $\varphi : \overline{\Omega} \times \mathbb{S}^1 \to [0, +\infty)$ and $\beta : \Sigma \to \mathbb{R}$ are Borel functions denoting the *anisotropy* of crystal and the *relative adhesion* coefficient of the substrate, respectively, and $\nu_\Sigma := \nu_S$. In the following we refer to the first term in (2.8) as the *free-boundary energy*, to the second as the *energy of internal cracks and external filaments*, to the third as the *wetting-layer energy*, to the fourth as the *contact energy*, and to the last as the *delamination energy*. In applications instead of $\varphi(x, \cdot)$ it is more convenient to use its positively one-homogeneous extension $|\xi|\varphi(x, \xi/|\xi|)$. With a slight abuse of notation we denote this extension also by φ.

The elastic energy of $(A,u) \in \mathcal{C}$ is defined as

$$\mathcal{W}(A,u) := \int_{A \cup S} W(x, e(u(x)) - M_0(x)) dx,$$

where the elastic density W is determined as the quadratic form

$$W(x, M) := \mathbb{C}(x) M : M,$$

by the so-called *stress-tensor*, a measurable function $x \in \Omega \cup S \to \mathbb{C}(x)$, where $\mathbb{C}(x)$ is a nonnegative fourth-order tensor in the Hilbert space $\mathbb{M}^{2\times 2}_{\mathrm{sym}}$ of all 2×2-symmetric matrices with the natural inner product

$$M : N = \sum_{i,j=1}^{2} M_{ij} N_{ij}$$

for $M = (M_{ij})_{i,j=1}^2$, $N = (N_{ij})_{i,j=1}^2 \in \mathbb{M}^{2\times 2}_{\rm sym}$.

The *mismatch strain* $x \in \Omega \cup S \mapsto M_0(x) \in \mathbb{M}^{2\times 2}_{\rm sym}$ is given by

$$M_0 := \begin{cases} e(u_0) & \text{in } \Omega, \\ 0 & \text{in } S, \end{cases}$$

for a fixed $u_0 \in H^1(\mathbb{R}^2;\mathbb{R}^2)$.

Given $m \in \mathbb{N}$, let $\mathcal{A}_m$ be a collection of all $A \in \mathcal{A}$ such that ∂A has at most m connected components and let

$$\mathcal{C}_m := \Big\{(A,u) \in \mathcal{C} :\ A \in \mathcal{A}_m\Big\}$$

to be the set of constrained admissible configurations. For simplicity, we assume that $\mathcal{C}_\infty = \mathcal{C}$.

Remark 2.3. The reason to introduce $\mathcal{C}_m$ is that $\mathcal{C}_m$ is both closed under $\tau_{\mathcal{C}}$-convergence (see [45, Definition 2.5]) and $\mathcal{F}$ is lower semicontinuous with respect to $\tau_{\mathcal{C}}$ in $\mathcal{C}_m$ (see [45, Theorems 2.7 and 2.8]). Such two properties do not apply instead to $\mathcal{C}$ as the following examples show.

We begin by recalling that a sequence $\{(A_k,u_k)\} \subset \mathcal{C}$ is said to $\tau_{\mathcal{C}}$-converge to $(A,u) \subset \mathcal{C}$ and we denote by $(A_k,u_k) \xrightarrow{\tau_{\mathcal{C}}} (A,u)$, if

- $\sup\limits_{k\ge1} \mathcal{H}^1(\partial A_k) < \infty$,
- $\mathrm{sdist}(\cdot,\partial A_k) \to \mathrm{sdist}(\cdot,\partial A)$ locally uniformly in $\mathbb{R}^2$ as $k\to\infty$,
- $u_k \to u$ a.e. in $\mathrm{Int}(A) \cup S$.

Let $X := \{x_n\}$ be a countable dense set in Ω and $A \in \mathcal{A}$ such that $|A| = \mathtt{v} \in (0,|\Omega|]$. Then the sets $A_k := A \setminus \{x_1,\dots,x_k\} \in \mathcal{A}$, $k \in \mathbb{N}$, are such that $|A_k| = \mathtt{v} \in (0,|\Omega|)$, $\mathcal{H}^1(\partial A_k) = \mathcal{H}^1(\partial A)$, and $(A_k,0) \xrightarrow{\tau_{\mathcal{C}}} (A\setminus X,0)$ as $k\to\infty$, but $A\setminus X \notin \mathcal{A}$ since $\partial(A\setminus X) = \overline{A}$. Therefore, compactness with respect to $\tau_{\mathcal{C}}$ fails in $\mathcal{C}$.

Furthermore, let $\Gamma \subset A$ be a segment such that $\mathcal{H}^1(\Gamma) > 0$, $B := A \setminus \Gamma$, $B_k := A \setminus (\Gamma \cap \{x_1,\dots,x_k\})$ for every $k \in \mathbb{N}$, and assume that X is dense in Γ. We notice that $\{(B_k,0)\} \subset \mathcal{C}$, $(B,0) \in \mathcal{C}$, $|B_k| = |B| = |A|$, $(B_k,0) \xrightarrow{\tau_{\mathcal{C}}} (B,0)$ as $k\to\infty$. However,

$$\mathcal{F}(B_k,0) = \mathcal{F}(A,0) < \mathcal{F}(A\setminus\Gamma,0) = \mathcal{F}(B,0).$$

Therefore, lower semicontinuity of $\mathcal{F}$ with respect to $\tau_{\mathcal{C}}$ fails in $\mathcal{C}$.

2.2. **Localized energies.** In this section we introduce the notion of quasi minimizers of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ in Ω and the localized version $\mathcal{F}(\cdot;O) : \mathcal{C}_m \to \mathbb{R}$ of $\mathcal{F}$ for open sets $O \subset \Omega$ and for $m \in \mathbb{N}\cup\{\infty\}$ with the convention $\mathcal{C}_\infty := \mathcal{C}$. We define

$$\mathcal{F}(A,u;O) := \mathcal{S}(A;O) + \mathcal{W}(A,u;O), \tag{2.9}$$

where

$$\mathcal{S}(A;O) := \int_{O\cap\partial^* A} \varphi(y,\nu_A)\,d\mathcal{H}^1 + 2\int_{O\cap(A^{(1)}\cup A^{(0)})\cap\partial A} \varphi(y,\nu_A)\,d\mathcal{H}^1$$

and

$$\mathcal{W}(A,u;O) = \int_{O\cap A} \mathbb{C}(y)e(u) : e(u)\,dy,$$

are the localized versions of the surface and elastic energies, respectively. Since we define the localized energy $\mathcal{F}(\cdot;O)$ only for open subsets O of Ω, the localized surface energy

$\mathcal{S}(\cdot;O)$ does not depend on u and the localized elastic energy $\mathcal{W}(\cdot;O)$ can be defined without u_0; see also Remark 2.5 below.

Definition 2.4. Given $\Lambda \geq 0$ and $m \in \mathbb{N} \cup \{\infty\}$, the configuration $(A,u) \in \mathcal{C}_m$ is a local (Λ, m)-*minimizer* of $\mathcal{F} : \mathcal{C}_m \to \mathbb{R}$ in O if

$$\mathcal{F}(A,u;O) \leq \mathcal{F}(B,v;O) + \Lambda|A\Delta B|$$

whenever $(B,v) \in \mathcal{C}_m$ with $A\Delta B \subset\subset O$ and $\operatorname{supp}(u-v) \subset\subset O$. Furthermore, we define

$$\Phi(A,u;O) := \inf\Big\{\mathcal{F}(B,v;O) : (B,v) \in \mathcal{C}_m, \\ B\Delta A \subset\subset O,\ \operatorname{supp}(u-v) \subset\subset O\Big\} \tag{2.10}$$

and

$$\Psi(A,u;O) := \mathcal{F}(A,u;O) - \Phi(A,u;O) \tag{2.11}$$

for every $(A,u) \in \mathcal{C}_m$ and every open set $O \subset\subset \Omega$.

Remark 2.5. By [45, Theorem 2.6] (see also (3.1) below) for any minimizer (A,u) of $\mathcal{F}$ in $\mathcal{C}_m$, the configuration $(A, u-u_0)$ is a (λ_0, m)-minimizer of $\mathcal{F}(\cdot,\cdot;\Omega)$. Indeed, since (A,u) is a minimizer of $\mathcal{F}^{\lambda_0}$ in $\mathcal{C}_m$, the function $\widehat{u} := u - u_0$ minimizes $\mathcal{C}_m \ni (B,v) \mapsto \widehat{\mathcal{F}}^{\lambda_0}(B,v) := \mathcal{F}^{\lambda_0}(B, v+u_0)$. Hence, for any open set $O \subset \Omega$ and $(B,v) \in \mathcal{C}_m$ with $A\Delta B \subset\subset O$ and $\operatorname{supp}(u-u_0-v) \subset\subset O$ we have $\widehat{\mathcal{F}}^{\lambda_0}(A, u-\widehat{u}_0) \leq \widehat{\mathcal{F}}^{\lambda_0}(B,v)$ so that

$$\mathcal{F}(A,u-u_0;O) \leq \mathcal{F}(B,v;O) + \lambda_0\big||A|-|B|\big| \leq \mathcal{F}(B,v;O) + \lambda_0\big|A\Delta B\big|.$$

Similarly, if (A,u) is a minimizer of $\widetilde{\mathcal{F}}$ in $\widetilde{\mathcal{C}}$, the configuration $(A,u-u_0)$ is a λ_0-minimizer of $\widetilde{\mathcal{F}}(\cdot;O)$.

2.3. **Auxiliary model.** We also introduce a *weak* formulation of the SRDI model defined in Section 2.1 for which the more general family $\widetilde{\mathcal{C}}$ of admissible configurations, given by

$$\widetilde{\mathcal{C}} := \big\{(A,u) : A \in \widetilde{\mathcal{A}}, \\ u \in GSBD^2(\operatorname{Int}(A \cup S \cup \Sigma);\mathbb{R}^2) \cap H^1_{\mathrm{loc}}(\operatorname{Int}(A) \cup S;\mathbb{R}^2)\big\},$$

is considered, where

$$\widetilde{\mathcal{A}} := \Big\{A \subset \overline{\Omega} :\ A \text{ is } \mathcal{L}^2\text{-measurable and } \mathcal{H}^1(\partial A) < +\infty\Big\}.$$

The auxiliary energy $\widetilde{\mathcal{F}} : \widetilde{\mathcal{C}} \to \mathbb{R}$ is defined as

$$\widetilde{\mathcal{F}} := \widetilde{\mathcal{S}} + \mathcal{W},$$

where

$$\widetilde{\mathcal{S}}(A,u) := \int_{\Omega\cap\partial^* A} \varphi(x,\nu_A(x))\,d\mathcal{H}^1(x) \\ + \int_{S_u^A} \big(\varphi(x,\nu_A(x)) + \varphi(x,-\nu_A(x))\big)\,d\mathcal{H}^1(x) \\ + \int_{\Sigma\cap\partial^* A\setminus J_u} \beta(x)\,d\mathcal{H}^1(x) + \int_{J_u} \varphi(x,-\nu_\Sigma(x))\,d\mathcal{H}^1(x), \tag{2.12}$$

where $S_u^A \subset \Omega$ by definition (2.6).

2.4. **Main results.** We begin by stating the hypotheses which will be assumed throughout the paper:

(H1) $\varphi \in C(\overline{\Omega} \times \mathbb{R}^2)$ and is a Finsler norm, i.e., there exist $c_2 \ge c_1 > 0$ such that for every $x \in \overline{\Omega}$, $\varphi(x, \cdot)$ is a norm in $\mathbb{R}^2$ satisfying
$$c_1|\xi| \le \varphi(x,\xi) \le c_2|\xi| \tag{2.13}$$
for any $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^2$;

(H2) $\beta \in L^\infty(\Sigma)$ and satisfies
$$-\varphi(x,\nu_\Sigma(x)) \le \beta(x) \le \varphi(x,\nu_\Sigma(x)) \tag{2.14}$$
for $\mathcal{H}^1$-a.e. $x \in \Sigma$;

(H3) $\mathbb{C} \in L^\infty(\Omega \cup S) \cap C^0(\overline{\Omega})$ and there exists $c_4 \ge c_3 > 0$ such that
$$2c_3\, M : M \le \mathbb{C}(x)M : M \le 2c_4\, M : M \tag{2.15}$$
for any $x \in \Omega \cup S$ and $M \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$;

(H4) Either $\mathtt{v} \in (0, |\Omega|)$ or $S = \emptyset$.

Given $\mathcal{G} \in \{\mathcal{F}, \widetilde{\mathcal{F}}\}$, we use the notation:
$$\mathcal{X}_{\mathcal{G}} := \begin{cases} \mathcal{C} & \text{if } \mathcal{G} = \mathcal{F}, \\ \widetilde{\mathcal{C}} & \text{if } \mathcal{G} = \widetilde{\mathcal{F}}. \end{cases}$$

The first result is the *existence* of solutions without constraint on the number of free-crystal boundary components.

Theorem 2.6 (**Existence**). *Assume* (H1)-(H4). *Let $\mathcal{G} \in \{\mathcal{F}, \widetilde{\mathcal{F}}\}$. Then the minimum problem*
$$\inf_{(B,v)\in\mathcal{X}_{\mathcal{G}},\ |B|=\mathtt{v}} \mathcal{G}(B,v) \tag{2.16}$$
admits a solution. Moreover, there exists $\lambda_1 > 0$ such that $(A,u) \in \mathcal{X}_{\mathcal{G}}$ is a solution of (2.16) *if and only if it solves*
$$\inf_{(B,v)\in\mathcal{X}_{\mathcal{G}}} \mathcal{G}^\lambda(B,v)$$
for every $\lambda \ge \lambda_1$, where
$$\mathcal{G}^\lambda(B,v) := \mathcal{G}(B,v) + \lambda\big||B| - \mathtt{v}\big|; \tag{2.17}$$

For simplicity we call the solutions of (2.16) *global minimizers.*

The second result is a *partial regularity* of the free-crystal boundaries. We recall that the definition of S_u^A is provided in (2.6).

Theorem 2.7 (**Properties of global minimizers**). *Assume* (H1)-(H4). *Let $\mathcal{G} \in \{\mathcal{F}, \widetilde{\mathcal{F}}\}$ and $(A,u) \in \mathcal{X}_{\mathcal{G}}$ be a solution of* (2.16)*. Define*
$$A' := \mathrm{Int}(A^{(1)}) \setminus \overline{\Gamma}, \tag{2.18}$$
where $\overline{\Gamma}$ is the closure of $\{x \in S_u^A : \theta_*(S_u^A, x) > 0\}$, and, with a slight abuse of notation, consider u as defined in $A' \cup S$ (and so, also on the $\mathcal{L}^2$-negligible set $A' \setminus \mathrm{Int}(A)$). Then:

(1) A' is open, $\theta_*(S_u^{A'}, x) > 0$ for all $x \in S_u^{A'}$, $|A'\Delta A| = 0$, $\mathcal{H}^1(\partial A \Delta \partial A') = 0$, $\mathcal{H}^1(S_u \Delta S_u^{A'}) = 0$, $(A', u) \in \mathcal{C}$, and
$$\mathcal{G}(A,u) = \mathcal{F}(A',u) = \inf_{(B,v)\in\mathcal{C},\, |B|=\mathtt{v}} \mathcal{F}(B,v) = \inf_{(B,v)\in\widetilde{\mathcal{C}},\, |B|=\mathtt{v}} \widetilde{\mathcal{F}}(B,v);$$

(2) for any $x \in \Omega$ and $r \in (0, \min\{1, \mathrm{dist}(x, \partial\Omega)\})$,

$$\frac{\mathcal{H}^1(Q_r(x) \cap \partial A')}{r} \leq \frac{16c_2 + 4\lambda_1}{c_1};$$

(3) there exist $\varsigma_0 = \varsigma_0(c_3, c_4) \in (0,1)$ and $R_0 = R_0(c_1, c_2, c_3, c_4, \lambda_1) > 0$, where $\lambda_1 > 0$ is given in Theorem 2.6, with the following property: if $x \in \Omega \cap \partial A'$, then

$$\frac{\mathcal{H}^1(Q_r(x) \cap \partial A')}{r} \geq \varsigma_0$$

for any square $Q_r(x) \subset\subset \Omega$ with $r \in (0, R_0)$.

(4) $\overline{A'^{(1)} \cap \partial A'} = \overline{S_u^{A'}}$ and

$$\mathcal{H}^1(\overline{S_u^{A'}} \setminus S_u^{A'}) = 0,$$

hence cracks essentially coincide with the jump set for the displacement u;

(5) If $E \subset A'$ is any connected component of A' with $\mathcal{H}^1(\partial E \cap \Sigma \setminus J_u) = 0$, then $|E| \geq (c_1\sqrt{4\pi}/\lambda_1)^2$ and $u = u_0 + a$ in E, where a is a rigid displacement.

In what follows we refer to the estimates in (2) and (3) as the (uniform) *upper and lower density estimate*, respectively. Note that by assertion (1), the assertions (3) and (5) directly hold also for solutions (A, u) of (2.16).

2.5. **Examples.** We recall from [45] that the SDRI energy (2.7) coincides with the functionals of the following free-boundary problems considered in the Literature when restricted to the corresponding subfamilies of admissible configurations in $\mathcal{C}$:

(a) *Epitaxially strained thin films, e.g.,* [6, 24, 25, 34, 39, 47]: $\Omega := (a,b) \times (0, +\infty)$, $S := (a,b) \times (-\infty, 0)$ for some $a < b$, free crystals in the subfamily

$$\mathcal{A}_{\mathrm{subgraph}} := \{A \subset \Omega : \exists h \in BV(\Sigma; [0,\infty)) \text{ and l.s.c. such that } A = A_h\} \subset \mathcal{A}_1,$$

where $A_h := \{(x^1, x^2) : 0 < x^2 < h(x^1)\}$, and admissible configurations in the subspace

$$\mathcal{C}_{\mathrm{subgraph}} := \{(A,u) : A \in \mathcal{A}_{\mathrm{subgraph}}, u \in H^1_{\mathrm{loc}}(\mathrm{Int}(A \cup S \cup \Sigma); \mathbb{R}^2)\} \subset \mathcal{C}_1$$

(see also [5, 42]);

(b) *Crystal cavities, e.g.,* [35, 38, 54, 56]: $\Omega \subset \mathbb{R}^2$ smooth set containing the origin, $S := \mathbb{R}^2 \setminus \Omega$, free crystals in the subfamily

$$\mathcal{A}_{\mathrm{starshaped}} := \{A \subset \Omega : \text{open and } \Omega \setminus A \text{ starshaped w.r.t. } (0,0)\} \subset \mathcal{A}_1,$$

and the space of admissible configurations

$$\mathcal{C}_{\mathrm{starshaped}} := \{(A,u) : A \in \mathcal{A}_{\mathrm{starshaped}}, u \in H^1_{\mathrm{loc}}(\mathrm{Int}(A \cup S \cup \Sigma); \mathbb{R}^2)\} \subset \mathcal{C}_1;$$

(b) *Capillarity droplets, e.g.,* [9, 26, 30]: $\Omega \subset \mathbb{R}^2$ is a bounded Lipschitz open set (or a cylinder), admissible configurations in the collection

$$\mathcal{C}_{\mathrm{capillarity}} := \{(A, u_0) : A \in \mathcal{A}\} \subset \mathcal{C} \quad \text{or} \quad \widetilde{\mathcal{C}}_{\mathrm{capillarity}} := \{(A, u_0) : A \in \widetilde{\mathcal{A}}\} \subset \widetilde{\mathcal{C}};$$

(d) *Griffith fracture model, e.g.,* [7, 11, 12, 17, 19, 36, 37]: $S = \Sigma = \emptyset$, $E_0 \equiv 0$, and the space of configurations

$$\mathcal{C}_{\mathrm{Griffith}} := \{(\Omega \setminus K, u) : K \text{ closed, } \mathcal{H}^1\text{-rectifiable, } u \in H^1_{\mathrm{loc}}(\Omega \setminus K; \mathbb{R}^2)\} \subset \mathcal{C};$$

(e) *Mumford-Shah model, e.g.,* [2, 22, 51]: $S = \Sigma = \emptyset$, $E_0 = 0$, $\mathbb{C}$ is such that the elastic energy $\mathcal{W}$ reduces to the Dirichlet energy, and the space of configurations

$$\mathcal{C}_{\mathrm{Mumfard-Shah}} := \{(\Omega \setminus K, u) \in \mathcal{C}_{\mathrm{Griffith}} : u = (u_1, 0)\} \subset \mathcal{C};$$

(f) *Boundary delaminations, e.g.,* [4, 31, 44, 48, 49, 57]: the SDRI model includes also the setting of debonding and edge delamination in composites [57]. The focus is here on the 2-dimensional film and substrate vertical section, while in [4, 48, 49] a reduced model for the horizontal interface between the film and the substrate is derived.

For the cases (a) and (b), the existence results for the SDRI model in $\mathcal{C}_{\rm subgraph}$ and $\mathcal{C}_{\rm starshaped}$ can be found for example in [45, Theorem 2.9 and Remark 2.10]. For (c), the same statements of Theorems 2.6 and 2.7 hold with $\mathcal{X}_\mathcal{G} := \mathcal{C}_{\rm capillarity}$ if $\mathcal{G} = \mathcal{F}$ or $\mathcal{X}_\mathcal{G} := \widetilde{\mathcal{C}}_{\rm capillarity}$ if $\mathcal{G} = \widetilde{\mathcal{F}}$ (note that S_u and Γ are empty in this case). For (d)-(f), we postpone the analysis to future investigations since some modifications in the proofs is needed to include *boundary Dirichlet conditions* or *fidelity terms* of type

$$\kappa \int_{\Omega\setminus K} |u-g|^p dx \tag{2.19}$$

for $p\in(1,\infty)$, $\kappa>0$, and $g\in L^\infty(\Omega)$, which are generally considered (and needed) in these mechanical applications.

3. Decay estimates for m-minimizers

In this section we always assume (H4). We recall that by [45, Theorem 2.6] under the hypotheses (H1)-(H3) both the volume-contrained minimum problem

$$\inf_{(A,u)\in\mathcal{C}_m,\ |A|=\mathtt{v}} \mathcal{F}(A,u),$$

and the unconstrained minimum problem

$$\inf_{(A,u)\in\mathcal{C}_m} \mathcal{F}^\lambda(A,u)$$

admit a solution for any $m\in\mathbb{N}$. Moreover, by [45, Theorem 2.6] there exists $\lambda_0>0$ such that

$$\inf_{(A,u)\in\mathcal{C},\ |A|=\mathtt{v}} \mathcal{F}(A,u) = \inf_{(A,u)\in\mathcal{C}} \mathcal{F}^\lambda(A,u) = \lim_{m\to\infty} \inf_{(A,u)\in\mathcal{C}_m,\ |A|=\mathtt{v}} \mathcal{F}(A,u) \tag{3.1}$$

for every $\lambda\ge\lambda_0$.

The main results of this section are the following density estimates for the quasi-minimizers of $\mathcal{F}$ in $\mathcal{C}_m$ with $m\in\mathbb{N}\cup\{\infty\}$.

Theorem 3.1 (**Density estimates for** (Λ,m)**-minimizers**). *There exist* $\varsigma_* = \varsigma_*(c_3,c_4)\in(0,1)$ *and* $R_* = R_*(c_1,c_2,c_3,c_4,\lambda_0)>0$, *where* c_i *are given by* (2.13) *and* (2.15), *with the following property. Let* $(A,u)\in\mathcal{C}_m$ *be a* (Λ,m)*-minimizer of* $\mathcal{F}(\cdot,\cdot;\Omega)$ *in* $\mathcal{C}_m$ *for some* $m\in\mathbb{N}\cup\{\infty\}$. *Then for any* $x\in\Omega$ *and* $r\in(0,\mathrm{dist}(x,\partial\Omega))$,

$$\frac{\mathcal{H}^1(Q_r(x)\cap\partial A)}{r} \le \frac{16c_2+4\Lambda}{c_1}. \tag{3.2}$$

Moreover, if $x\in\Omega$ *belongs to the closure of the set* $\{y\in\Omega\cap\partial A:\ \theta_*(\partial A,y)>0\}$, *then*

$$\frac{\mathcal{H}^1(Q_r(x)\cap\partial A)}{r} \ge \varsigma_* \tag{3.3}$$

for any square $Q_r(x)\subset\subset\Omega$ *with* $r\in(0,R_*)$.

To prove Theorem 3.1 we start with the following adaptation of [11, Theorem 3] to our setting (of set-function pairs).

Lemma 3.2. *There exist $\eta \in (0, 1/32)$ and $c_0 > 0$ with the following property: For any $m \in \mathbb{N} \cup \{\infty\}$, any admissible $(A, u) \in \mathcal{C}_m$, and any square $Q_R(x_0) \subset \Omega$ of sidelength $2R > 0$ with*

$$\delta := \Big(\frac{\mathcal{H}^1(Q_R(x_0) \cap \partial^r A)}{R}\Big)^{1/2} < \eta \tag{3.4}$$

there exist $v \in GSBD^2(\mathrm{Int}(\Omega \cup S \cup \Sigma); \mathbb{R}^2)$, $B \in \mathcal{A}$ with $(B, v\big|_B) \in \mathcal{C}_m$, $R' \in (R(1-\sqrt{\delta}), R)$ and a Lebesgue measurable set $\omega \subset\subset Q_R(x_0)$ such that

(1) $v \in C^\infty(Q_{R(1-\sqrt{\delta})}(x_0))$, $A \Delta B \subset\subset Q_{R'}(x_0) \setminus Q_{R(1-\sqrt{\delta})}(x_0)$ *and* $\mathrm{supp}\,(\widetilde{u} - v) \subset\subset Q_R(x_0)$, *where*
$$\widetilde{u} := u\chi_{Q_R(x_0) \cap A} + \xi \chi_{Q_R(x_0) \setminus A}, \tag{3.5}$$
where $\xi \in Q_R$ is chosen such that $Q_R \cap \partial^* A \subset J_{\widetilde{u}}$;

(2) $\mathcal{H}^1(\partial B \setminus \partial A) \le c_0 \sqrt{\delta}\, \mathcal{H}^1([Q_R(x_0) \setminus Q_{R(1-\sqrt{\delta})}(x_0)] \cap \partial A)$;

(3) $|\omega| \le c_0 \delta\, \mathcal{H}^1(Q_R(x_0) \cap \partial A)$ *and*
$$\int_{Q_R(x_0)\setminus\omega} |v - \widetilde{u}|^2 dx \le c_0 \delta^2 R^2 \int_{Q_R(x_0)} |e(\widetilde{u})|^2 dx;$$

(4) *for any $\psi \in \mathrm{Lip}(Q_R; [0,1])$ and elasticity tensor $\mathbb{C} \in L^\infty(Q_R)$ with*
$$d_1 M : M \le \mathbb{C}(x) M : M \le d_2 M : M, \qquad (x, M) \in Q_R \times \mathbb{M}^{2\times 2}_{\mathrm{sym}}, \tag{3.6}$$
there exist $d_3 := d_3(c_0, d_1, d_2) > 0$ and $s := s(c_0, d_1, d_2) \in (0, 1/2)$ such that
$$\int_{Q_R(x_0)} \psi \mathbb{C}(x) e(v) : e(v) dx \le \int_{Q_R(x_0)\cap A} \psi \mathbb{C}(x) e(u) : e(u) dx + d_3\, \delta^s\, (1 + R\,\mathrm{Lip}(\psi)) \int_{Q_R(x_0)\cap A} |e(u)|^2 dx.$$

The proof of Lemma 3.2 is an adaptation of the arguments of [11, Theorem 3] to our situation of functional depending on set-function pairs with extra care paid for the constraint on the number of boundary connected components. The idea is to treat the boundary of each admissible region as a jump of a properly defined displacement. In particular, we choose such displacement of the type (3.5), where ξ is selected as in the construction used in the proof of [45, Lemma 3.10]. We also notice that the constants η and $c := c_0/(1 + \sqrt{2}/24) > 0$ are given by [11, Theorem 3].

Proof of Lemma 3.2. By translating and rescaling if necessary, we assume that $x_0 = 0$ and $R = 1$. Notice that since $\mathcal{H}^1(Q_1 \cap \partial A) < +\infty$, by Proposition A.2 there exists $\xi \in (0,1)^2$ such that the set

$$\{x \in Q_1 \cap \partial^* A : \mathrm{tr}_A(u) \text{ exists and is equal to } \xi\}$$

is $\mathcal{H}^1$-negligible. By [41, Theorem 4.4] up to a $\mathcal{H}^1$-negligible set we can cover $Q_1 \cap \partial^* A$ with C^1-maps so that by [21, Theorem 5.2] $\mathrm{tr}_A(u)$ exists $\mathcal{H}^1$-a.e. on $Q_1 \cap \partial^* A$.

Let

$$\widetilde{u} := u\chi_{Q_1 \cap A} + \xi \chi_{Q_1 \setminus A}.$$

Note that $\widetilde{u} \in GSBD^2(Q_1; \mathbb{R}^2)$ and by the choice of ξ and by [21, Definition 2.4] $Q_1 \cap \partial^* A \subset J_{\widetilde{u}}$. In addition, by possibly adding to $\widetilde{u}$ a function in $SBD^2(Q_1; \mathbb{R}^2) \cap W^{1,\infty}(Q_1 \setminus \partial A; \mathbb{R}^2)$ with small $W^{1,\infty}(Q_1 \setminus \partial A; \mathbb{R}^2)$ norm, jump on the set $Q_1 \cap \partial^r A$, and supported near $Q_1 \cap \partial A$, we can assume without loss of generality that $Q_1 \cap J_{\widetilde{u}} \supset Q_1 \cap \partial^r A$ up to a $\mathcal{H}^1$-negligible set*. Notice that

$$\delta := \mathcal{H}^1(Q_1 \cap \partial^r A)^{1/2} = \mathcal{H}^1(Q_1 \cap J_{\widetilde{u}})^{1/2}$$

*A similar argument was used in [13, p. 1359, above Eq. 4.19] and set $N := [1/\delta]$ so that $(-N\delta, N\delta)^2 \subset Q_1$. For $i := 0, 1, \dots, N-1$ let $Q^i := (-(N-i)\delta, (N-i)\delta)^2$ and $C^i := Q^i \setminus Q^{i+1}$ (assuming $C^{N-1} := Q^{N-1}$). Up to a slight translation of Q^i we assume that $\mathcal{H}^1(\partial A \cap \partial Q^i) = 0$ for all i. By [11, Lemma 3.3] we find $i_0 \ge 1$ such that

$$\begin{cases} \int_{C^{i_0} \cup C^{i_0+1}} |e(\widetilde{u})|^2 dx \le 8\sqrt{\delta} \int_{Q_1 \setminus Q_{1-\sqrt{\delta}}} |e(\widetilde{u})|^2 dx, \\ \mathcal{H}^1(\partial A \cap (C^{i_0} \cup C^{i_0+1})) \le 8\sqrt{\delta}\, \mathcal{H}^1(\partial A \cap (Q_1 \setminus Q_{1-\delta})). \end{cases}$$

We partition Q^{i_0+1} into pairwise disjoint squares with sidelength δ and divide the slice C^{i_0} into dyadic slices

$$G_j := (-(N - i_0 - 2^{-j})\delta, (N - i_0 - 2^{-j})\delta)^2 \setminus (-(N - i_0 - 2^{-j+1})\delta, (N - i_0 - 2^{-j+1})\delta)^2,$$

then we partition each slice G_j into pairwise disjoint squares $Q_{j,l}$ of sidelength $2^{-j}\delta$ whose sides are parallel to the coordinate axis. Let $\mathcal{V}_0$ be the collection of all squares of sidelength δ that cover the central square Q^{i_0+1} and let $\mathcal{V}$ be the union of $\mathcal{V}_0$ and of the collection of all $Q_{j,l}$. Following [11] we differentiate between "good" and "bad" squares in $\mathcal{V}$. A square $Q \in \mathcal{V}$ is "good" if

$$\mathcal{H}^1(Q''' \cap \partial A) \le \eta \delta_Q, \tag{3.7}$$

where Q''' is the square with the same center as Q and dilated by $7/6$, and $\delta_Q := \delta$ if $Q \in \mathcal{V}_0$ and $\delta_Q := 2^{-j}\delta$ if $Q \subset G_j$. A square Q is "bad" if it does not satisfy (3.7). By (3.4) $\delta^2 = \mathcal{H}^1(Q_1 \cap \partial A) < \eta\delta$, hence, by definition, all squares in $\mathcal{V}_0$ are good and by [11, Eq. 12] the sum of the perimeters of all bad squares satisfies

$$\sum_{Q \text{ bad}} \mathcal{H}^1(\partial^* Q) \le \widetilde{c}_0 \sqrt{\delta}\, \mathcal{H}^1((Q_1 \setminus Q_{1-\sqrt{\delta}}) \cap \partial A) \tag{3.8}$$

for some $\widetilde{c}_0 > 0$. Since $\delta < \eta$, by [11, Theorem 3] there exist $\widetilde{v} \in GSBD^2(Q_1; \mathbb{R}^2)$, $r \in (1 - \sqrt{\delta}, 1)$ and a Lebesgue measurable set $\widetilde{\omega} \subset\subset Q_r$ such that

(a1) $\widetilde{v} \in C^\infty(Q_{1-\sqrt{\delta}})$, $\widetilde{u} = \widetilde{v}$ in $Q_1 \setminus Q_r$ and $\mathcal{H}^1(J_{\widetilde{u}} \cap \partial Q_r) = \mathcal{H}^1(J_{\widetilde{v}} \cap \partial Q_r) = 0$;
(a2) $\mathcal{H}^1(J_{\widetilde{v}} \setminus J_{\widetilde{u}}) \le \widetilde{c}_0 \sqrt{\delta}\, \mathcal{H}^1((Q_1 \setminus Q_{1-\sqrt{\delta}}) \cap J_{\widetilde{u}})$;
(a3) $|\widetilde{\omega}| \le \widetilde{c}_0 \delta \mathcal{H}^1(Q_r \cap \partial A)$ and

$$\int_{Q_1 \setminus \widetilde{\omega}} |\widetilde{v} - \widetilde{u}|^2 dx \le \widetilde{c}_0 \delta^2 \int_{Q_1} |e(\widetilde{u})|^2 dx;$$

(a4) for any $\psi \in \mathrm{Lip}(Q_1; [0,1])$ and elasticity tensor $\mathbb{C} \in L^\infty(Q_1)$ satisfying (3.6) there exists $d_3 := d_3(\widetilde{c}_0, d_1, d_2) > 0$ such that

$$\int_{Q_1} \psi \mathbb{C}(x) e(\widetilde{v}) : e(\widetilde{v}) dx \le \int_{Q_1} \psi \mathbb{C}(x) e(\widetilde{u}) : e(\widetilde{u}) dx + d_3 \delta^s (1 + \mathrm{Lip}(\psi)) \int_{Q_1} |e(\widetilde{u})|^2 dx$$

with $s \in (0,1)$ depending only on $\widetilde{c}_0$, d_1 and d_2;
(a5) $J_{\widetilde{v}} \subset \partial^* D \cup (J_{\widetilde{u}} \setminus Q^{i_0+1})$ and $J_{\widetilde{v}} \setminus J_{\widetilde{u}} \subset \partial^* D$, where D is the union of all bad squares.

Note that for proving (a4) in [11] a *mollifying argument* is used (together with the fact that $\mathbb{C}$ is assumed to be constant in [11]). As in our setting $\mathbb{C}$ is in general not constant, we revised such argument (see [11, Eq. 23]), by using the fact that the energy

$$w \in GSBD^2(O) \mapsto \int_O \mathbb{C} e(w) : e(w) dx$$

is quadratic with respect to the $e(w)$ and hence, we have convexity and we can employ *Cauchy-Schwartz inequality* for positive semidefinite bilinear forms to obtain

$$\begin{aligned}&\int_O \mathbb{C}(x)e(\widetilde{v}) : e(\widetilde{v})dx \le \int_O \mathbb{C}(x)e(\widetilde{u}) : e(\widetilde{u})dx + 2\int_O \mathbb{C}(x)e(\widetilde{v}) : [e(\widetilde{v}) - e(\widetilde{u})]dx \\ &\le \int_O \mathbb{C}(x)e(\widetilde{u}) : e(\widetilde{u})dx + 2\Big[\int_O \mathbb{C}(x)e(\widetilde{v}) : e(\widetilde{v})dx\Big]^{1/2} \times \\ &\qquad\qquad \times \Big[\int_O \mathbb{C}(x)[e(\widetilde{v}) - e(\widetilde{u})] : [e(\widetilde{v}) - e(\widetilde{u})]dx\Big]^{1/2}\end{aligned}$$

for any open set $O \subset Q_1$. Since the inequality $a^2 \le b^2 + 2ac$, where $a, b, c \ge 0$, implies* $a \le b + 2c$, we get

$$\begin{aligned}\Big[\int_O \mathbb{C}(x)e(\widetilde{v}) : e(\widetilde{v})dx\Big]^{1/2} \le & \Big[\int_O \mathbb{C}(x)e(\widetilde{u}) : e(\widetilde{u})dx\Big]^{1/2} \\ & + 2\Big[\int_O \mathbb{C}(x)[e(\widetilde{v}) - e(\widetilde{u})] : [e(\widetilde{v}) - e(\widetilde{u})]dx\Big]^{1/2} \\ \le & (1 + c\delta^s)\Big[\int_O \mathbb{C}(x)e(\widetilde{u}) : e(\widetilde{u})dx\Big]^{1/2}\end{aligned}$$

so that Eq.23 of [11] holds also in our setting.

Let $\mathcal{V}_i$ be the family of all bad squares Q intersecting $\mathrm{Int}(A)$ and $D_i := \bigcup_{Q\in\mathcal{V}_i} Q$. For every $Q \in \mathcal{V}_i$ we define I_Q as the segment of smallest length connecting $(Q''' \cap \partial A) \setminus \overline{Q}$ to ∂Q with the convention that $I_Q = \emptyset$ if $(Q''' \cap \partial A) \setminus \overline{Q} = \emptyset$ or $Q \cap \mathrm{Int}(\Omega \setminus A) \ne \emptyset$. By the definition of Q''' and Q, $\mathcal{H}^1(I_Q) \le \frac{\sqrt{2}}{24}\mathcal{H}^1(\partial Q)$.

Let

$$B := \Big[\big(A \setminus \overline{D_i}\big) \cup \partial D_i\Big] \setminus \bigcup_{Q\in\mathcal{V}_i} I_Q$$

and

$$v := \widetilde{v}\chi_{Q_1} + \widetilde{u}\chi_{(\Omega\cup S)\setminus Q_1}.$$

We claim that B, v and $\widetilde{\omega}$ satisfy the assertions of the lemma.

Indeed, from (a4) applied with $\psi \equiv 1$ and $\mathbb{C} = I$ it follows that $v \in GSBD^2(\mathrm{Int}(B);\mathbb{R}^2)$. Moreover, by (a5) $v \in H^1_{\mathrm{loc}}(\mathrm{Int}(B);\mathbb{R}^2)$, thus, $(B, v) \in \mathcal{C}$. Let us show that if $A \in \mathcal{A}_m$ for some $m \in \mathbb{N}$, then $B \in \mathcal{A}_m$. Indeed, by the construction of B, for each bad square Q, the dilated square Q''' contains inside "large" portions of the boundary ∂A. Now if ∂A intersects $\overline{Q}$, then $I_Q = \emptyset$ and the modification $[A \setminus \overline{Q}] \cup \partial Q \setminus I_Q$ does not increase the number of boundary components. Otherwise, if ∂A does not increase $\overline{Q}$, so that it intersects only $Q''' \setminus \overline{Q}$, then adding a small segment I_Q to connect $\partial A \cap [Q''' \setminus \overline{Q}]$ to $\overline{Q}$ again does not increase the number of boundary components of $[A \setminus \overline{Q}] \cup \partial Q \setminus I_Q$. Now, from the disjointness of the cubes $Q \in \mathcal{V}_i$ it follows that $B \in \mathcal{C}_m$. Therefore, if $(A, u) \in \mathcal{C}_m$ for some $m \in \mathbb{N}$, then $(B, v\big|_B) \in \mathcal{C}_m$.

By (a1) it follows that $v \in C^\infty(Q_{1-\sqrt{\delta}})$. Moreover, by the definition of B, $A\Delta B \subset\subset Q_{r_h} \setminus Q_{1-\sqrt{\delta}}$ for some $r_h \in (1-\sqrt{\delta}, 1)$ such that $D_i \subset Q_{r_h}$. Also, by (a1) $\mathrm{supp}\,(\widetilde{u} - \widetilde{v}) \subset\subset Q_1$ so that $\mathrm{supp}\,(\widetilde{u} - v) \subset\subset Q_1$, and (1) follows.

*Note that $a \le b + 2c$ follows from $a^2 \le b^2 + 2ac$ as it yields $(a - 2c)^2 \le a(a - 2c) \le b^2$.

Moreover, by the definition of B, I_Q and (3.8)

$$\mathcal{H}^1(\partial B\setminus\partial A)\le\sum_{Q\in\mathcal{V}_i}P(Q)+\sum_{Q\in\mathcal{V}_i}\mathcal{H}^1(I_Q)$$
$$\le\Big(1+\frac{\sqrt{2}}{24}\Big)\sum_{Q\in\mathcal{V}_i}P(Q)\le c_0\sqrt{\delta}\mathcal{H}^1((Q_1\setminus Q_{1-\sqrt{\delta}})\cap\partial A),$$

where $c_0:=\widetilde{c}_0(1+\sqrt{2}/24)$, and (2) follows.

Next, by (a3) $|\omega|\le c_0\delta\mathcal{H}^1(Q_1\cap\partial A)$, and

$$\int_{Q_1\setminus\omega}|v(x)-\widetilde{u}(x)|^2dx=\int_{Q_1\setminus\widetilde{\omega}}|\widetilde{v}(y)-\widetilde{u}(y)|^2dy\le\widetilde{c}_0\delta^2\int_{Q_1}|e(\widetilde{u})|^2dy$$
$$\le c_0\delta^2\int_{Q_1}|e(\widetilde{u})|^2(y)dy.$$

Finally, by (a4) and the definition of v (i.e., $v=\widetilde{v}$ in Q_1) for any $\psi\in\mathrm{Lip}(Q_1)$ and $\mathbb{C}\in L^\infty(Q_1)$ satisfying (3.6) we have

$$\int_{Q_1}\psi(x)\mathbb{C}(x)e(v):e(v)dx=\int_{Q_1}\psi(x)\mathbb{C}(x)e(\widetilde{v}):e(\widetilde{v})dx$$
$$\le\int_{Q_1}\psi(x)\mathbb{C}(x)e(\widetilde{u}):e(\widetilde{u})dx+d_3\,(1+\mathrm{Lip}(\psi))\int_{Q_1}|e(\widetilde{u})|^2dx$$
$$=\int_{Q_1\cap A}\psi(x)\mathbb{C}(x)e(u):e(u)dx+d_3\delta^s\,(1+\mathrm{Lip}(\psi))\int_{Q_1\cap A}|e(u)|^2dx,$$

since $\widetilde{u}$ is constant in $Q_1\setminus A$. Hence, (4) follows. □

The following proposition is a generalization to our setting of [11, Theorem 4] established for the Griffith model.

Proposition 3.3. *Let $Q_R(x_0)\subset\Omega$ be a square of side length $2R>0$. Consider sequences $\{m_h\}\subset\mathbb{N}\cup\{\infty\}$, Finsler norms $\{\varphi_h\}$ and ellipticity tensors $\{\mathbb{C}_h\}$ such that $\{\mathbb{C}_h\}$ is equicontinuous in $\overline{Q_R(x_0)}$ and there exist $d_3,d_4,d_5>0$ with*

$$d_3M:M\le\mathbb{C}_h(x)M:M\le d_4M:M\quad\text{for all } (x,M)\in\overline{Q_R(x_0)}\times\mathbb{M}^{2\times2}_{\mathrm{sym}},\tag{3.9}$$

and

$$d_5\sup_{(x,\nu)\in Q_R\times\mathbb{S}^1}\varphi_h(x,\nu)\le\inf_{(x,\nu)\in Q_R\times\mathbb{S}^1}\varphi_h(x,\nu),\tag{3.10}$$

and define $\mathcal{F}_h$ and Ψ_h in $\mathcal{C}_{m_h}$ as in (2.9) and (2.11), respectively, with φ_h, $\mathbb{C}_h$ and m_h in places of φ, $\mathbb{C}$ and m. Let $\{(A_h,u_h)\}\subset\mathcal{C}_{m_h}$ be such that

$$\lim_{h\to\infty}\Psi_h(A_h,u_h;Q_R(x_0))=0,\tag{3.11}$$

$$\lim_{h\to\infty}\mathcal{H}^1(Q_R(x_0)\cap\partial A_h)=0,\tag{3.12}$$

$$\sup_{h\ge1}\mathcal{F}_h(A_h,u_h;Q_R(x_0))=:M<\infty.\tag{3.13}$$

Then there exist $u\in H^1(Q_R(x_0))$, an elasticity tensor $\mathbb{C}\in C^0(\overline{Q_R(x_0)};\mathbb{M}^{2\times2}_{\mathrm{sym}})$, sequences $\{\xi_j\}\subset(0,1)^2$ of vectors and $\{a_j\}$ of rigid displacements and subsequences $\{(A_{h_j},u_{h_j})\}$, $\{\varphi_{h_j}\}$ and $\{\mathbb{C}_{h_j}\}$ such that

(a) $\mathbb{C}_{h_j}\to\mathbb{C}$ uniformly in $\overline{Q_R(x_0)}$ and $w_j:=u_{h_j}\chi_{Q_R(x_0)\cap A_{h_j}}+\xi_j\chi_{Q_R(x_0)\setminus A_{h_j}}-a_j\to u$ pointwise a.e. in $Q_R(x_0)$, and $e(w_j)\rightharpoonup e(u)$ in $L^2(Q_R(x_0))$ as $j\to\infty$;

(b) for all $v \in u + H^1_0(Q_R(x_0))$

$$\int_{Q_R(x_0)} \mathbb{C}(y)e(u) : e(u)\,dy \le \int_{Q_R(x_0)} \mathbb{C}(y)e(v) : e(v)\,dy; \tag{3.14}$$

(c) for any $r \in (0, R]$

$$\lim_{j\to\infty} \mathcal{F}_h(A_{h_j}, u_{h_j}; Q_r(x_0)) = \int_{Q_r(x_0)} \mathbb{C}(x)e(u) : e(u)\,dx. \tag{3.15}$$

Proof. Without loss of generality, we suppose $R = 1$ and $x_0 = 0$. Let

$$c_{1,h} := \inf_{(x,\nu)\in Q_1\times\mathbb{S}^1} \varphi_h(x, \nu), \qquad c_{2,h} := \sup_{(x,\nu)\in Q_1\times\mathbb{S}^1} \varphi_h(x, \nu); \tag{3.16}$$

by (3.10) we have $d_5 c_{2,h} \le c_{1,h}$. Since $\sup_h \mathcal{H}^1(Q_1 \cap \partial A_h) < \infty$, by Proposition A.2 for every $h \ge 1$ there exists $\xi_h \in (0, 1)^2$ such that

$$\mathcal{H}^1(\{y \in Q_1 \cap \partial A_h :\ \mathrm{tr}_{A_h}(u_h) \text{ exists and equals to } \xi_h \text{ at } y\}) = 0.$$

Therefore

$$\widetilde{u}_h := \begin{cases} u_h & \text{in } Q_1 \cap A_h, \\ \xi_h & \text{in } Q_1 \setminus A_h \end{cases} \tag{3.17}$$

belongs to $GSBD^2(Q_1; \mathbb{R}^2)$ with $J_{\widetilde{u}_h} \subset Q_1 \cap \partial A_h$ and

$$\lim_{h\to\infty} \mathcal{H}^1(J_{\widetilde{u}_h}) = 0 \tag{3.18}$$

in view of (3.12). Further we suppose $\mathcal{H}^1(J_{\widetilde{u}_h}) < 1/4$ for any $h \ge 1$.

By [10, Proposition 2] and (3.9), there exist a constant c (depending only on d_3) and sequences $\{\widetilde{\omega}_h\}$ of a Lebesgue measurable subsets of Q_1 with $|\widetilde{\omega}_h| \le c\mathcal{H}^1(Q_1 \cap \partial A_h)$ and $\{a_h\}$ of rigid motions such that

$$\int_{Q_1\setminus\widetilde{\omega}_h} |\widetilde{u}_h - a_h|^2 dx \le c \int_{Q_1} \mathbb{C}_h(x)e(\widetilde{u}_h) : e(\widetilde{u}_h)dx. \tag{3.19}$$

By (3.9) and (3.13), there exists $u \in L^2(Q_1)$ such that up to a subsequence $(\widetilde{u}_h - a_h)\chi_{Q_1\setminus\widetilde{\omega}_h} \rightharpoonup u$ weakly in $L^2(Q_1)$. Furthermore from (3.9) and (3.13) we obtain

$$\sup_{h\ge1} \int_{Q_1} |e(\widetilde{u}_h - a_h)|^2\,dx + \mathcal{H}^1(J_{\widetilde{u}_h}) < \infty,$$

and hence, by [14, Theorem 1.1] there exist a subsequence still denoted by $\{\widetilde{u}_h - a_h\}$ for which the set

$$E := \{y \in Q_1 :\ \lim_{h\to\infty} |\widetilde{u}_h(y) - a_h(y)| \to \infty\}$$

has finite perimeter and $\widetilde{u} \in GSBD^2(Q_1 \setminus E; \mathbb{R}^2)$ with $\widetilde{u} = 0$ in E such that

$$\begin{gathered} \widetilde{u}_h - a_h \to \widetilde{u} \qquad \text{a.e. in } Q_1 \setminus E \\ e(\widetilde{u}_h - a_h) \rightharpoonup e(\widetilde{u}) \qquad \text{in } L^2(Q_1 \setminus E; \mathbb{M}^{2\times2}_{\mathrm{sym}}), \\ \mathcal{H}^1((Q_1 \setminus \partial^*E) \cap J_{\widetilde{u}}) + \mathcal{H}^1(Q_1 \cap \partial^*E) = \mathcal{H}^1(J_{\widetilde{u}} \cup \partial^*E) \le \liminf_{h\to+\infty} \mathcal{H}^1(J_{\widetilde{u}_h}) = 0. \end{gathered} \tag{3.20}$$

In particular, $P(E, Q_1) = 0$ so that by the relative isoperimetric inequality either $|E| = |Q_1|$ or $|E| = 0$. By the definition of E, (3.12), the uniform $L^2(Q_1)$-boundedness of $\{(\widetilde{u}_h - a_h)\chi_{Q_1\setminus\widetilde{\omega}_h}\}$ which is a consequence of (3.19) and (3.13), and Fatou's Lemma it follows that $|E| = 0$. Hence, from (3.20) we get $\widetilde{u}_h - a_h \to \widetilde{u}$ a.e. in Q_1 and $e(\widetilde{u}_h - a_h) \rightharpoonup e(\widetilde{u})$ in $L^2(Q_1; \mathbb{M}^{2\times2}_{\mathrm{sym}})$, and all relations in (3.20) hold in Q_1 and $\widetilde{u} = u$ a.e. in Q_1. In particular, since $\mathcal{H}^1(J_u) = 0$, by Proposition A.3 we have that $u \in H^1(Q_1; \mathbb{R}^2)$. In view of the fact that our elastic energy is invariant under rigid deformations, we suppose $a_h = 0$ for any $h \ge 1$.

Next we prove (3.14). Let $v \in H^1(Q_1;\mathbb{R}^2)$ be such that $\operatorname{supp}(u-v) \subset\subset Q_r$ for some $r \in (0,1)$. Let $\psi \in C^1_c(Q_r;[0,1])$ be a cut-off function with $\{0<\psi<1\} \subset \{u=v\} \cap Q_{r'}$ and $\operatorname{supp}(u-v) \subseteq \{\psi \equiv 1\} \subseteq Q_{r''}$ for some $r''<r'<r$. By (3.18) and Lemma 3.2 applied with (A_h,u_h) and Q_r there exist $\widetilde{v}_h \in GSBD^2(\mathrm{Int}(\Omega\cup S\cup\Sigma);\mathbb{R}^2)$, $B_h \in \mathcal{A}_{m_h}$ with $(B_h,\widetilde{v}_h\big|_{B_h}) \in \mathcal{C}_{m_h}$, $r_h \in (r(1-\sqrt{\delta_h}),r)$ and a Lebesgue measurable set $\omega_h \subset\subset Q_r$ such that

(a1) $\widetilde{v}_h \in C^\infty(Q_{r(1-\sqrt{\delta_h})})$, $A_h \Delta B_h \subset\subset Q_{r_h} \setminus Q_{r(1-\sqrt{\delta_h})}$ and $\operatorname{supp}(\widetilde{u}_h - \widetilde{v}_h) \subset\subset Q_r$;

(a2) $\mathcal{H}^1(\partial B_h \setminus \partial A_h) \le c_0 \sqrt{\delta_h}\, \mathcal{H}^1([Q_r \setminus Q_{r(1-\sqrt{\delta_h})}] \cap \partial A_h)$;

(a3) $|\omega_h| \le c_0\delta_h\, \mathcal{H}^1(Q_r\cap\partial A_h)$ and

$$\int_{Q_r\setminus\omega_h} |\widetilde{v}_h - \widetilde{u}_h|^2 dx \le c_0\delta_h^2 r^2 \int_{Q_r\cap A_h} |e(u_h)|^2 dx;$$

(a4) for any $\eta \in \mathrm{Lip}(Q_r;[0,1])$

$$\begin{aligned}\int_{Q_r} \eta\mathbb{C}_h e(\widetilde{v}_h) : e(\widetilde{v}_h)dx \le & \int_{Q_r\cap A_h} \eta\mathbb{C}_h e(u_h) : e(u_h)dx \\ & + d_3\,\delta_h^s\,(1+r\,\mathrm{Lip}(\eta)) \int_{Q_r\cap A_h} |e(u_h)|^2 dx,\end{aligned} \tag{3.21}$$

where $\delta_h := r^{-1/2}\mathcal{H}^1(Q_r\cap\partial A_h)^{1/2} \to 0$, and d_3 and s are constants. We assume that h is large enough so that $r_h > r'$. Set

$$v_h := (1-\psi)\widetilde{v}_h + \psi v.$$

We observe that $\operatorname{supp}(u_h - v_h\big|_{B_h}) \subset\subset Q_r$: by (a1) and the definition of ψ, there exists $r_0 \in (r_h,r)$ such that $A_h \setminus Q_{r_0} = B_h \setminus Q_{r_0}$ and $\widetilde{u}_h = \widetilde{v}_h = v_h$ in $Q_r \setminus Q_{r_0}$ and hence, $u_h\big|_{Q_r\cap A_h\setminus Q_{r_0}} = \widetilde{u}_h\big|_{Q_r\cap A_h\setminus Q_{r_0}} = v_h\big|_{Q_r\cap B_h\setminus Q_{r_0}}$. Thus, (B_h,v_h) is an admissible configuration in (2.10) and from (3.11) and the definition of deviation it follows that

$$\mathcal{F}_h(A_h,u_h;Q_1) \le \mathcal{F}_h(B_h,v_h;Q_1) + o(1), \tag{3.22}$$

where $o(1)\to 0$ as $h\to\infty$. We observe that

$$\begin{aligned}\mathcal{S}_h(B_h;Q_1) - \mathcal{S}_h(A_h;Q_1) \le & \mathcal{S}_h(B_h;Q_r\setminus\overline{Q_{r(1-\sqrt{\delta_h})}}) - \mathcal{S}_h(A_h;Q_r\setminus\overline{Q_{r(1-\sqrt{\delta_h})}}) \\ \le & \int_{(\partial^*B_h\setminus\partial^*A_h)\cap Q_r\setminus\overline{Q_{r(1-\sqrt{\delta_h})}}} \varphi(x,\nu_{B_h})d\mathcal{H}^1 \\ & + 2\int_{(Q_r\setminus\overline{Q_{r(1-\sqrt{\delta_h})}})\cap(B_h^{(1)}\cup B_h^{(0)})\cap(\partial B_h\setminus\partial A_h)} \varphi(x,\nu_{A_h})d\mathcal{H}^1 \\ \le & 2c_{2,h}\mathcal{H}^1(\partial B_h\setminus\partial A_h) \le 2c_0c_{2,h}\sqrt{\delta_h}\mathcal{H}^1([Q_r\setminus Q_{r(1-\sqrt{\delta_h})}]\cap\partial A_h) \\ \le & \frac{2c_0\sqrt{\delta_h}}{d_5}\mathcal{S}_h(A_h;Q_1) = o(1)\end{aligned}$$

as $h\to+\infty$, where we used in the first inequality (a1), in the second the definition and nonnegativity of $\mathcal{S}_h$, in the third (3.16), in the fourth (a2) in the last again (3.16) and the definition of $\mathcal{S}_h$, and finally in the equality we used (3.13). Thus, (3.22) is rewritten as

$$\mathcal{W}_h(A_h,u_h;Q_1) \le \mathcal{W}_h(B_h,v_h;Q_1) + o(1). \tag{3.23}$$

Note that by (a1), (a3), (3.18), (3.20) and Fatou's Lemma, $\widetilde{v}_h\chi_{Q_r\setminus\omega_h} \to u$ a.e. in Q_r and by (a3) $\chi_{Q_r\setminus\omega_h} \to 1$ a.e. in Q_r. Therefore, for a.e. $x\in Q_r$ there exists $h_x \ge 1$ such that $\chi_{Q_r\setminus\omega_h}(x) = 1$ for every $h > h_x$ and $\widetilde{v}_h(x) = \widetilde{v}_h(x)\chi_{Q_r\setminus\omega_h}(x) \to u(x)$. So

$$\widetilde{v}_h \to u \text{ a.e. in } Q_r. \tag{3.24}$$

We claim that $\widetilde{v}_h \to u$ strongly in $L^2_{\rm loc}(Q_r)$. To see this we fix $\rho \in (0,r)$, and, since $\delta_h \to 0$ by (a1), there exists $h_\rho \geq 1$ such that $\widetilde{v}_h \in H^1(Q_\rho)$ for every $h > h_\rho$. From (3.9), (3.13) and (3.21) as well as the Korn-Poincaré inequality

$$\sup_{h>h_\rho} \|\widetilde{v}_h - b_h\|_{H^1(Q_\rho)} < \infty$$

for some sequence $\{b_h\}$ of rigid displacements. On the one hand, by Rellich-Kondrachov Theorem there exist $z \in H^1(Q_\rho;\mathbb{R}^2)$ and not relabelled subsequence such that $\widetilde{v}_h - b_h \to z$ in $L^2(Q_\rho;\mathbb{R}^2)$ and a.e. in Q_ρ. On the other hand, by (3.24) $b_h = \widetilde{v}_h - (\widetilde{v}_h - b_h)$ converges to $b := u - z$ a.e. in Q_ρ. Since b_h is a rigid displacement, so is b and hence $b_h \to b$ uniformly in Q_ρ. Therefore,

$$\limsup_{h\to\infty} \|\widetilde{v}_h - u\|_{L^2(Q_\rho)} \leq \limsup_{h\to\infty} \|\widetilde{v}_h - b_h - z\|_{L^2(Q_\rho)} + \limsup_{h\to\infty} \|b_h - b\|_{L^2(Q_\rho)} = 0,$$

and the claim follows.

Since $u = v$ out of $\{\psi = 1\}$, the claim implies $\widetilde{v}_h \to v$ strongly in $L^2(\{0 < \psi < 1\})$, and hence,

$$\lim_{h\to\infty} \int_{Q_r} |\nabla\psi \odot (v - \widetilde{v}_h)\big|_{A_h}|^2 \leq \liminf_{h\to\infty} \int_{\{0<\psi<1\}} |\nabla\psi \odot (v - \widetilde{v}_h)|^2 = 0, \tag{3.25}$$

where $X \odot Y = (X \otimes Y + Y \otimes X)/2$, Thus, by the definition of v_h and the equality

$$e(v_h) = (1-\psi)e(\widetilde{v}_h) + \psi e(v) + \nabla\psi \odot (v - \widetilde{v}_h),$$

we estimate

$$\begin{aligned}
&\int_{Q_r} \mathbb{C}_h e(v_h) : e(v_h)dx \\
&= \int_{Q_r} (1-\psi)^2 \mathbb{C}_h e(\widetilde{v}_h) : e(\widetilde{v}_h)dx + \int_{Q_r} \psi^2 \mathbb{C}_h e(v) : e(v)dx \\
&\quad + \int_{Q_r} \mathbb{C}_h(\nabla\psi \odot (v - \widetilde{v}_h)) : (\nabla\psi \odot (v - \widetilde{v}_h))dx \\
&\quad + 2\int_{Q_r} (1-\psi)\mathbb{C}_h e(\widetilde{v}_h) : (\nabla\psi \odot (v - \widetilde{v}_h))dx \\
&\quad + 2\int_{Q_r} \psi \mathbb{C}_h e(v) : (\nabla\psi \odot (v - \widetilde{v}_h))dx \\
&= \int_{Q_r} (1-\psi)^2 \mathbb{C}_h e(\widetilde{v}_h) : e(\widetilde{v}_h)dx + \int_{Q_r} \psi^2 \mathbb{C}_h e(v) : e(v)dx + o(1) \\
&\leq \int_{Q_r\cap A_h} (1-\psi)^2 \mathbb{C}_h e(u_h) : e(u_h)dx + \int_{Q_r} \psi^2 \mathbb{C}_h e(v) : e(v)dx + o(1),
\end{aligned} \tag{3.26}$$

where in the second equality we use (3.13), (3.21) with $\eta \equiv 1$, (3.25), (3.9) and the Hölder inequality, while in the last inequality we use (3.21) with $\eta = (1-\psi)^2$ and (3.17). Now (3.23), (3.26) and (3.17) imply

$$\int_{Q_r} (2\psi - \psi^2)\mathbb{C}_h e(\widetilde{u}_h) : e(\widetilde{u}_h)dx \leq \int_{Q_r} \psi^2 \mathbb{C}_h e(v) : e(v)dx + o(1). \tag{3.27}$$

Since $\{\mathbb{C}_h\}$ is equibounded (see (3.9)) and equicontinuous, by the Arzela-Ascoli Theorem, there exist a (not relabelled) subsequence and an elasticity tensor $\mathbb{C} \in C^0(Q_1;\mathbb{M}^{2\times 2}_{\rm sym})$ such that $\mathbb{C}_h \to \mathbb{C}$ uniformly in Q_1. Hence, letting $h \to \infty$ in (3.27) and using the convexity of the elastic energy and (3.20), we obtain

$$\int_{Q_r} (2\psi - \psi^2)\mathbb{C}(y)e(u) : e(u)dy \leq \int_{Q_r} \psi^2 \mathbb{C}(y)e(v) : e(v)\,dy. \tag{3.28}$$

By the choice of ψ, (3.28) implies

$$\int_{Q_{r''}} \mathbb{C}(y)e(u):e(u)dy \le \int_{Q_r} \mathbb{C}(y)e(v):e(v)\,dy. \tag{3.29}$$

Since r'' is arbitrary, letting $r'' \nearrow r$ we deduce that (3.29) holds also with $r''=r$. Since $\operatorname{supp}(u-v) \subset\subset Q_r$, this implies (3.14).

It remains to prove (3.15). If we take $v=u$ in (3.27) and use $0\le\psi\le1$ and $\psi=1$ in $Q_{r''}$ we get

$$\begin{aligned}\int_{Q_{r''}} \mathbb{C}e(u):e(u)dx &\le \liminf_{h\to\infty}\int_{Q_{r''}} \mathbb{C}_h e(\widetilde u_h):e(\widetilde u_h)dx\\ &\le \limsup_{h\to\infty}\int_{Q_{r''}} \mathbb{C}_h e(\widetilde u_h):e(\widetilde u_h)dx \le \int_{Q_r}\mathbb{C}e(u):e(u)dx.\end{aligned}$$

Since r'' is arbitrary, letting $r''\nearrow r$ we deduce

$$\lim_{h\to\infty}\int_{Q_r} \mathbb{C}_h e(\widetilde u_h):e(\widetilde u_h)dx = \int_{Q_r}\mathbb{C}e(u):e(u)dx. \tag{3.30}$$

Now we prove that

$$\lim_{h\to\infty} \mathcal{S}_h(A_h;Q_r)=0 \tag{3.31}$$

for any $r\in(0,1)$. By (3.12), we can find $h_r>0$ such that

$$\mathcal{H}^1(Q_1\cap\partial A_h) < (1-r)/5 \tag{3.32}$$

for any $h>h_r$, and hence there is no connected component of ∂A_h intersecting both ∂Q_r and ∂Q_1. Also by the relative isoperimetric inequality, passing to further subsequence we suppose that either

$$\lim_{h\to\infty}|Q_1\cap A_h|=0 \tag{3.33}$$

or

$$\lim_{h\to\infty}|Q_1\setminus A_h|=0. \tag{3.34}$$

First assume that (3.33) holds. Let $E_h\subset A_h$ be the set consisting of all connected components of $\overline{A_h}$ not intersecting ∂Q_1. Then, $(A_h\setminus E_h, u_h\big|_{A_h\setminus E_h})$ is an admissible configuration in (2.10), thus,

$$\mathcal{F}_h(A_h,u_h;Q_1)\le \Phi_h(A_h,u_h;Q_1)+o(1)\le \mathcal{F}_h(A_h\setminus E_h,u_h;Q_1)+o(1), \tag{3.35}$$

where in the first inequality we use (3.11) and in the second we use the definition of Φ_h. Hence,

$$\begin{aligned}\mathcal{S}(A_h;Q_r)\le&\mathcal{S}(E_h;Q_1)=\mathcal{S}_h(A_h;Q_1)-\mathcal{S}_h(A_h\setminus E_h;Q_1)\\ \le&\mathcal{F}_h(A_h;Q_1)-\mathcal{F}_h(A_h\setminus E_h;Q_1)\le o(1),\end{aligned}$$

where we used in the first inequality the definition of E_h, which entitles that $U_r\cap\partial A_h\subset\partial E_h$, in the equality the disjointness of $\overline{A_h\setminus E_h}$ and $\overline{E_h}$ which follows by (3.32), and in the second inequality the nonnegativity of the elastic energy and in the third (3.35). Hence, (3.31) follows.

Now assume that (3.34) holds and let $\delta_h := r^{-1/2}\sqrt{\mathcal{H}^1(Q_r\cap\partial A_h)}\to0$. Fix any $\rho\in(0,r)$. By (3.12), we can find $h_{r,\rho}>0$ such that $\delta_h<\min\{1-r,r-\rho\}/5$ for any $h>h_{r,\rho}$. Since $A_h\in\mathcal{A}_{m_h}$, no connected component of ∂A_h intersects both ∂Q_r and ∂Q_ρ. Let $F_h\subset Q_1\setminus A_h$ be the union of all connected components of $\overline{Q_1\setminus A_h}$ lying strictly inside Q_1 (so F_h is a union of "holes" and $\partial F_h\subset\partial A_h$). Let $\psi\in C_c^1(Q_r;[0,1])$ be a cut-off function with $\{0<\psi<1\}\subset Q_{r'}$ and $\{\psi\equiv1\}\subseteq Q_{r''}$ for some $r''<r'<r$. Set $A_h' := A_h\cup\overline{F_h}$. Applying Lemma 3.2 with $(A_h',\widetilde u_h\big|_{A_h'})$, Q_r and $m=m_h$ we find

$\widetilde{v}_h' \in GSBD^2(\mathrm{Int}(\Omega\cup S\cup\Sigma);\mathbb{R}^2)$, $B_h' \in \mathcal{A}_{m_h}$ with $(B_h', \widetilde{v}_h'|_{B_h}) \in \mathcal{C}_{m_h}$, $r_h \in (r(1-\sqrt{\delta_h}), r)$ and a Lebesgue measurable set $\omega_h' \subset\subset Q_r$ such that

(b1) $\widetilde{v}_h' \in C^\infty(Q_{r(1-\sqrt{\delta_h})})$, $A_h'\Delta B_h' \subset\subset Q_{r_h}\setminus Q_{r(1-\sqrt{\delta_h})}$ and $\mathrm{supp}\,(\widetilde{u}_h - \widetilde{v}_h') \subset\subset Q_r$;

(b2) $\mathcal{H}^1(\partial B_h'\setminus\partial A_h') \le c_0\sqrt{\delta_h}\,\mathcal{H}^1([Q_r\setminus Q_{r(1-\sqrt{\delta_h})}]\cap\partial A_h')$;

(b3) $|\omega_h'| \le c_0\delta_h\,\mathcal{H}^1(Q_r\cap\partial A_h')$ and

$$\int_{Q_r\setminus\omega_h'} |\widetilde{v}_h' - \widetilde{u}_h|^2dx \le c_0\delta_h^2 r^2\int_{Q_r\cap A_h'}|e(u_h)|^2dx;$$

(b4) for any $\eta\in\mathrm{Lip}(Q_r;[0,1])$

$$\int_{Q_r}\eta\mathbb{C}e(\widetilde{v}_h'):e(\widetilde{v}_h')dx \le \int_{Q_r\cap A_h}\eta\mathbb{C}e(u_h):e(u_h)dx + d_3\,\delta_h^s\,(1+r\mathrm{Lip}(\eta))\int_{Q_r\cap A_h}|e(u_h)|^2dx,$$

where d_3 and s are constants. Set

$$v_h' := (1-\psi)\widetilde{v}_h' + \psi u.$$

By the definition of A_h' and (b1) $(B_h', v_h'|_{B_h'})$ is an admissible configuration for $\Phi_h(A_h,u_h;Q_1)$ in (2.10). Thus from (3.11) and (3.34)

$$\mathcal{F}_h(A_h,u_h;Q_1) \le \mathcal{F}_h(B_h', v_h'|_{B_h'};Q_1) + o(1). \tag{3.36}$$

Now as in the proof of (3.27)

$$\begin{aligned}
&\mathcal{W}_h(B_h', v_h'|_{B_h'};Q_1) - \mathcal{W}_h(A_h,u_h;Q_1)\\
&\le \int_{Q_r}\psi^2\mathbb{C}_he(u):e(u)dx - \int_{Q_r}(2\psi-\psi^2)\mathbb{C}_he(\widetilde{u}_h):e(\widetilde{u}_h)dx + o(1)\\
&\le \int_{Q_r}\mathbb{C}_he(u):e(u)dx - \int_{Q_{r''}}\mathbb{C}_he(\widetilde{u}_h):e(\widetilde{u}_h)dx + o(1).
\end{aligned}\tag{3.37}$$

Moreover,

$$\begin{aligned}
\mathcal{S}_h(B_h';Q_1) - \mathcal{S}_h(A_h;Q_1) &= \Big(\mathcal{S}_h(B_h';Q_1) - \mathcal{S}_h(A_h';Q_1)\Big) + \Big(\mathcal{S}_h(A_h';Q_1) - \mathcal{S}_h(A_h;Q_1)\Big)\\
&\le \mathcal{S}_h(B_h';Q_r\setminus Q_{r(1-\sqrt{\delta_h})}) - \mathcal{S}_h(A_h;Q_\rho) \le 2c_{2,h}\mathcal{H}^1(\partial B_h'\setminus\partial A_h') - \mathcal{S}_h(A_h;Q_\rho)\\
&\le 2c_0c_{2,h}\sqrt{\delta_h}\mathcal{H}^1([Q_r\setminus Q_{r(1-\sqrt{\delta_h})}]\cap\partial A_h) - \mathcal{S}_h(A_h;Q_\rho)\\
&\le \frac{2c_0\sqrt{\delta_h}}{d_5}\mathcal{S}_h(A_h;Q_1) - \mathcal{S}_h(A_h;Q_\rho) = o(1) - \mathcal{S}_h(A_h;Q_\rho),
\end{aligned}\tag{3.38}$$

where we used in the first inequality (b1) and the definition of A_h', in the second and in the last inequalities the definition of $\mathcal{S}_h$, (3.16) and (3.10), in the third inequality (b2), and in the last equality (3.13) and that $\delta_h\to 0$ by (3.12). Hence, (3.36), (3.37) and (3.38) imply

$$\mathcal{S}_h(A_h;Q_\rho) + \int_{Q_{r''}}\mathbb{C}_he(\widetilde{u}_h):e(\widetilde{u}_h)dx \le \int_{Q_r}\mathbb{C}_he(u):e(u)dx + o(1).$$

Thus, letting $h\to\infty$ and using (3.30) we get

$$\limsup_{h\to\infty}\mathcal{S}_h(A_h;Q_\rho) + \int_{Q_{r''}}\mathbb{C}e(u):e(u)dx \le \int_{Q_r}\mathbb{C}e(u):e(u)dx.$$

Now letting $r''\to r$ we get

$$\limsup_{h\to\infty}\mathcal{S}_h(A_h;Q_\rho) = 0. \tag{3.39}$$

Observe that the function $B \mapsto \mathcal{S}_h(A_h; B)$ defined for Borel sets $B \subset Q_1$ extends to a bounded nonnegative Radon measure μ_h in Q_1. Since (3.39) holds for any $\rho \in (0, r)$, μ_h converges to 0 in the weak* sense, and thus (3.31) follows. $\square$

Recall that by [18, Proposition 3.4] if the elasticity tensor $\mathbb{C}$ is constant, then for any $\gamma \in (0, 2)$ there exists $c_\gamma := c_\gamma(c_3, c_4) > 0$ such that for every local minimizer $(\Omega, u) \in \mathcal{C}$ of $\mathcal{F}(\cdot; O)$, u is analytic in O and for any square $Q_R(x) \subset\subset O$ and $r \in (0, R)$,

$$\int_{Q_r(x)} \mathbb{C}e(u) : e(u)\, dx \le c_\gamma \left(\frac{r}{R}\right)^{2-\gamma} \int_{Q_R(x)} \mathbb{C}e(u) : e(u)\, dx. \tag{3.40}$$

Given $\gamma \in (0, 1)$ let

$$\tau_0 = \tau_0(\gamma, c_3, c_4) := \min\{1, \tfrac{1}{2} c_\gamma^{-\frac{1}{4-2\gamma}}\},$$

where c_γ is the constant appearing in (3.40). Using Proposition 3.3 and repeating similar arguments of [12, 19] we get the following decay property of the functional $\mathcal{F}$.

Proposition 3.4. *For any $\tau \in (0, \tau_0)$ there exist $\varsigma = \varsigma(\tau) \in (0, 1)$ and $\vartheta := \vartheta(\tau) \in (0, 1)$ with the following property: If there exist $m \in \mathbb{N} \cup \{\infty\}$, $(A, u) \in \mathcal{C}_m$ and a square $Q_\rho(x) \subset\subset \Omega$ such that*

$$\mathcal{H}^1(Q_\rho(x) \cap \partial A) \le 2\varsigma\rho \quad \textit{and} \quad \mathcal{F}(A, u; Q_\rho(x)) \le (1 + \vartheta)\Phi(A, u; Q_\rho(x)),$$

then

$$\mathcal{F}(A, u; Q_{\tau\rho}(x)) \le \tau^{2-\gamma} \mathcal{F}(A, u; Q_\rho(x)).$$

Proof. We argue by contradiction. Assume that there exists $\tau \in (0, \tau_0)$ such that for all $\varsigma, \vartheta \in (0, 1)$ we can find $m := m(\varsigma, \vartheta) \in \mathbb{N} \cup \{\infty\}$, $(A, u) := (A(\varsigma, \vartheta), u(\varsigma, \vartheta)) \in \mathcal{C}_m$ and $Q_\rho(x) \subset\subset \Omega$ with $\rho := \rho(\varsigma, \vartheta)$ and $x := x(\varsigma, \vartheta)$ satisfying

$$\mathcal{H}^1(Q_\rho(x) \cap \partial A) \le 2\varsigma\rho \quad \text{and} \quad \mathcal{F}(A, u; Q_\rho(x)) \le (1 + \vartheta)\Phi(A, u; Q_\rho(x)), \tag{3.41}$$

but

$$\mathcal{F}(A, u; Q_{\tau\rho}(x)) > \tau^{2-\gamma} \mathcal{F}(A, u; Q_\rho(x)). \tag{3.42}$$

Let us choose any positive real numbers $\varsigma_h, \vartheta_h \to 0$, and denote for simplicity $m_h := m(\varsigma_h, \vartheta_h)$, $(A_h, u_h) = (A(\varsigma_h, \vartheta_h), u(\varsigma_h, \vartheta_h))$, $\rho_h := \rho(\varsigma_h, \vartheta_h)$, $x_h = x(\varsigma_h, \vartheta_h)$. By (3.41) and (3.42),

$$\mathcal{H}^1(Q_{\rho_h}(x_h) \cap \partial A_h) \le 2\varsigma_h \rho_h, \tag{3.43}$$

$$\mathcal{F}(A_h, u_h; Q_{\rho_h}(x_h)) \le (1 + \vartheta_h)\Phi(A_h, u_h; Q_{\rho_h}(x_h)), \tag{3.44}$$

but

$$\mathcal{F}(A_h, u_h; Q_{\tau\rho_h}(x_h)) > \tau^{2-\gamma} \mathcal{F}(A_h, u_h; Q_{\rho_h}(x_h)) \tag{3.45}$$

for any h. Note that $\mathcal{F}(A_h, u_h; Q_{\rho_h}(x_h)) > 0$. Let us define the rescaled energy $\mathcal{F}_h(\cdot; Q_1) : \mathcal{C}_{m_h} \to \mathbb{R}$ as in (2.9) with

$$\varphi_h(y, \nu) := \frac{\rho_h \varphi(x_h + \rho_h y, \nu)}{\mathcal{F}(A_h, u_h; Q_{\rho_h}(x_h))}$$

in place of $\varphi(y, \nu)$ and

$$\mathbb{C}_h(y) := \mathbb{C}(x_h + \rho_h y)$$

in place of $\mathbb{C}(y)$, for $y \in Q_1$. We notice that

$$\mathcal{F}_h(E_h, v_h; Q_1) = 1 \tag{3.46}$$

for

$$E_h := \sigma_{x_h, \rho_h}(A_h)$$

(see definition of blow-up map $\sigma_{x,r}$ at (2.4)) and

$$v_h(y) := \frac{u_h(x_h + \rho_h y)}{\sqrt{\mathcal{F}(A_h, u_h; B_{\rho_h}(x_h))}}.$$

By (3.43) we obtain

$$\mathcal{H}^1(Q_1 \cap \partial E_h) < 2\varsigma_h$$

while (3.44) and (3.46) entails

$$\Psi_h(E_h, v_h; Q_1) \le \vartheta_h \Phi_h(E_h, v_h; Q_1) \le \vartheta_h \mathcal{F}_h(E_h, v_h; Q_1) = \vartheta_h,$$

where Φ_h and Ψ_h are defined as in (2.10) and (2.11) (again with φ_h and $\mathbb{C}_h$ in places of φ and $\mathbb{C}$, respectively). By (2.15) $\{\mathbb{C}_h\}$ is equibounded. Since Ω is bounded, there exists $x_0 \in \overline{\Omega}$ such that, up to extracting a subsequence, $x_h \to x_0$ as $h \to +\infty$. As $\rho_h \to 0$, one has $x_h + \rho_h y \to x_0$ for every $y \in \overline{Q_1}$. Thus $\{\mathbb{C}_h\}$ is also equicontinuous and $\mathbb{C}_h \to \mathbb{C}_0 := \mathbb{C}(x_0)$ uniformly in $\overline{Q_1}$. In view of (3.43), (3.44) and (3.46), we can apply Proposition 3.3 to find $v \in H^1(Q_1; \mathbb{R}^2)$, vectors $\xi_h \in (0,1)^2$, and infinitesimal rigid displacements a_h such that, up to a subsequence,

$$w_h := v_h \chi_{Q_1 \cap E_h} + \xi_h \chi_{Q_1 \setminus E_h} - a_h \to v$$

pointwise a.e. in Q_1, $e(w_h) \rightharpoonup e(v)$ in $L^2(Q_1)$ as $h \to +\infty$, and

$$\lim_{h\to+\infty} \mathcal{F}_h(E_h, w_h; Q_r) = \lim_{h\to+\infty} \mathcal{F}_h(E_h, v_h; Q_r) = \int_{Q_r} \mathbb{C}_0(x) e(v) : e(v)dx \tag{3.47}$$

for any $r \in (0,1]$. In particular, from (3.47) and (3.45) it follows that

$$\begin{aligned}\int_{Q_\tau} \mathbb{C}_0(x)e(v) : e(v)dx &= \lim_{h\to+\infty} \mathcal{F}(E_h, v_h; Q_\tau) \\ &\ge \lim_{h\to+\infty} \tau^{2-\gamma} \mathcal{F}(E_h, v_h; Q_1) = \tau^{2-\gamma} \int_{Q_1} \mathbb{C}_0(x)e(v) : e(v)dx.\end{aligned}$$

Since $\mathbb{C}_0$ is constant, applying (3.40) with $r := \tau$ and $R := 1$ we get

$$\begin{aligned}c_\gamma \tau^{2-\gamma} \int_{Q_1} \mathbb{C}_0(x)e(v) : e(v)dx &\ge \int_{Q_\tau} \mathbb{C}_0(x)e(v) : e(v)dx \\ &\ge \tau^{\gamma-2} \int_{Q_1} \mathbb{C}_0(x)e(v) : e(v)dx.\end{aligned}$$

Now recalling that $\mathcal{F}_h(E_h, v_h; Q_1) = 1$, by (3.47) we get $\int_{Q_1} \mathbb{C}_0(x)e(v) : e(v)dx = 1$, thus, $\tau^{2-\gamma} \ge c_\gamma^{-1/2} > \tau_0^{2-\gamma}$, a contradiction. □

By employing the arguments of [53, Section 4.3] and using Proposition 3.4 we establish the following lower bound for $\mathcal{F}$.

Proposition 3.5. *Given $\tau \in (0,\tau_0)$, let $\varsigma := \varsigma(\tau) \in (0,1)$ and $\vartheta := \vartheta(\tau) \in (0,1)$ be as in Proposition 3.4. Let $(A,u) \in \mathcal{C}_m$ be a (Λ, m)-minimizer of $\mathcal{F}$ in $Q_{r_0}(x_0)$ for some $m \in \mathbb{N} \cup \{\infty\}$ and $r_0 > 0$, and let*

$$J_A^* := \{y \in Q_{r_0}(x_0) \cap \partial A : \ \theta_*(\partial A, y) > 0\}.$$

Then,

$$\mathcal{F}(A, u; Q_\rho(x)) \ge 2c_1\varsigma\rho \tag{3.48}$$

for every $x \in \overline{J_A^*}$ and for every square $Q_\rho(x) \subset Q_{r_0}(x_0)$ with $\rho \in (0, R_0)$, where

$$R_0 := R_0(r_0, \Lambda, c_1, \tau) := \min\Big\{r_0, \frac{\sqrt{\pi}\, c_1 \vartheta}{\Lambda(2+\vartheta)}\Big\}.$$

Proof. Fix $m \in \mathbb{N}\cup\{\infty\}$. Note that for any $(C,w),(D,v) \in \mathcal{C}_m$ and $O \subset \Omega$ with $C\Delta D \subset\subset O$

$$
\begin{aligned}
\sqrt{4\pi}\,|C\Delta D|^{1/2} \le & \mathcal{H}^1(\partial^*(C\Delta D)) \le \mathcal{H}^1(O\cap \partial^* C) + \mathcal{H}^1(O\cap\partial^* D) \\
\le & \frac{\mathcal{S}(C,O)+\mathcal{S}(D,O)}{c_1} \le \frac{\mathcal{F}(C,w;O)+\mathcal{F}(D,v;O)}{c_1},
\end{aligned}
\tag{3.49}
$$

where in the first inequality we used the isoperimetric inequality, in the second $\partial^*(C\Delta D) \subset O\cap(\partial^* C\cup\partial^* D)$, in the third (2.13) and the definition of $\mathcal{S}(\cdot;O)$ and in the last the nonnegativity of $\mathcal{W}(\cdot;O)$. Thus, from the (Λ,m)-minimality of (A,u) in $Q_{r_0}(x_0)$ we deduce that

$$
\begin{aligned}
\mathcal{F}(A,u;Q_r(x)) \le & \mathcal{F}(B,v;Q_r(x)) + \Lambda|A\Delta B|^{\frac12}|A\Delta B|^{\frac12} \\
\le & \mathcal{F}(B,v;Q_r(x)) + \frac{\Lambda r}{\sqrt{\pi}\,c_1}\Big(\mathcal{F}(A,u;Q_r(x))+\mathcal{F}(B,v;Q_r(x))\Big)
\end{aligned}
\tag{3.50}
$$

for any $Q_r(x)\subset Q_{r_0}(x_0)$ and $(B,v)\in\mathcal{C}_m$ with $A\Delta B\subset\subset Q_r(x)$ and $\mathrm{supp}\,(u-v)\subset\subset Q_r(x)$, where in the last inequality we used (3.49) and the inequality $|A\Delta B| \le |Q_r| = 4r^2$. Let $r>0$ be small enough so that $\frac{\Lambda r}{\sqrt{\pi}\,c_1} \le \frac{\vartheta}{2+\vartheta}$, where $\vartheta := \vartheta(\tau)\in(0,1)$ is given by Proposition 3.4. From (3.50) we obtain

$$
\mathcal{F}(A,u;Q_r(x)) \le (1+\vartheta)\mathcal{F}(B,v;Q_r(x)),
$$

which by the arbitrariness of (B,v) is equivalent to

$$
\mathcal{F}(A,u;Q_r(x)) \le (1+\vartheta)\Phi(A,u;Q_r(x)). \tag{3.51}
$$

Now we prove (3.48). Let $x\in J_A^*$. For simplicity we suppose that $x=0$. Assume by contradiction that for such $m\in\mathbb{N}\cup\{\infty\}$, $(A,u)\in\mathcal{C}_m$ and for some $Q_\rho\subset\subset Q_{r_0}(x_0)$ with $\rho\in(0,R_0)$ we have

$$
\mathcal{F}(A,u;Q_\rho) < 2c_1\varsigma\rho.
$$

Then by the nonnegativity of the elastic energy and (2.13),

$$
2c_1\varsigma\rho > \mathcal{F}(A,u;Q_\rho) \ge \int_{Q_\rho\cap\partial A}\varphi(x,\nu_A)d\mathcal{H}^1 \ge c_1\mathcal{H}^1(Q_\rho\cap\partial A)
$$

so that

$$
\mathcal{H}^1(Q_\rho\cap\partial A) < 2\varsigma\rho. \tag{3.52}
$$

By (3.52) and (3.51) we can apply Proposition 3.4 and obtain that

$$
\mathcal{F}(A,u;Q_{\tau\rho}) \le \tau^{2-\gamma}\mathcal{F}(A,u;Q_\rho) \le 2c_1\varsigma\tau^{2-\gamma}\rho
$$

Hence,

$$
\mathcal{H}^1(Q_{\tau\rho}\cap\partial A) \le 2\varsigma\tau^{2-\gamma}\rho < 2\varsigma\tau\rho,
$$

where we used $\gamma,\tau\in(0,1)$, and by induction

$$
\mathcal{H}^1(Q_{\tau^n\rho}\cap\partial A) \le 2\varsigma\tau^{(2-\gamma)n}\rho < 2\varsigma\tau^n\rho, \quad n\in\mathbb{N}.
$$

However, by the choice of x

$$
0 < \theta_*(\partial A,x) = \liminf_{n\to+\infty}\frac{\mathcal{H}^1(Q_{\tau^n\rho}\cap\partial A)}{2\tau^n\rho} \le \lim_{n\to+\infty}\frac{2c_1\varsigma\tau^{(1-\gamma)n}}{2c_1} = 0,
$$

a contradiction. This contradiction implies (3.48) for $x\in J_A^*$.

Now consider any $x\in Q_{r_0}(x_0)\cap\overline{J_A^*}$ and $\rho\in(0,R_0)$ with $Q_\rho(x)\subset Q_{r_0}(x_0)$, and let us choose a sequence $\{Q_{\rho_k}(x_k)\}$ of squares with $x_k\in J_A^*$ and $\rho_1\le\rho_2\le\ldots\le\rho$ such that

$$
Q_{\rho_1}(x_1)\subseteq Q_{\rho_2}(x_2)\subseteq \ldots Q_\rho(x) \qquad \text{and} \qquad Q_\rho(x) = \bigcup_k Q_{\rho_k}(x_k).
$$

Notice that $x_k \to x$ and $\rho_k \to \rho$. By De Giorgi-Letta Theorem [2, Theorem 1.53], both maps

$$O \mapsto \int_{O\cap\partial^* A} \varphi(x,\nu_A)d\mathcal{H}^1 + 2\int_{O\cap(A^{(0)}\cup A^{(1)})\cap\partial A} \varphi(x,\nu_A)d\mathcal{H}^1$$

and

$$O \mapsto \int_{O\cap A} \mathbb{C}(y)e(u):e(u)dy,$$

defined at open sets $O \subset\subset \Omega$, uniquely extend to positive Borel measures μ_1 and μ_2 in Ω. Therefore, from the continuity of μ_1 and μ_2 (see e.g. [2, Remark 1.3]) and the validity of (3.48) with x_k and ρ_k it follows that

$$\begin{aligned}\mathcal{F}(A,u;Q_\rho(x)) =&\mu_1(Q_\rho(x)) + \mu_2(Q_\rho(x)) = \lim_{k\to+\infty}[\mu_1(Q_\rho(x_k)) + \mu_2(Q_{\rho_k}(x_k))]\\ &= \lim_{k\to+\infty}\mathcal{F}(A,u;Q_{\rho_k}(x_k)) \geq \lim_{k\to+\infty}(2c_2\varsigma\rho_k) = 2c_2\varsigma\rho.\end{aligned}$$

□

Now we are ready to prove (3.2) and (3.3).

Proof of Theorem 3.1. Let $m \in \mathbb{N}\cup\{\infty\}$ and (A,u) be a (Λ,m)-minimizer of $\mathcal{F}(\cdot,\cdot;\Omega)$. We begin by establishing (3.2). Let $x \in \Omega$, $r \in (0,\min\{1,\operatorname{dist}(x,\partial\Omega)\})$, and $Q_r := Q_r(x)$. Since (3.2) is trivial if $Q_r \cap \partial A = \emptyset$, then we assume that $Q_r\cap\partial A \neq \emptyset$ and so $E := (A\setminus\overline{Q_r})\cup\partial Q_r \in \mathcal{A}_m$. By the (Λ,m)-minimality of (A,u)

$$\mathcal{F}(A,u;Q_r) \leq \mathcal{F}(E,u;Q_r) + \Lambda|Q_r|.$$

Hence, by the nonnegativity $\mathcal{W}(A\cap Q_r,u;Q_r)$

$$\int_{Q_r\cap\partial A}\varphi(x,\nu_A)d\mathcal{H}^1 \leq 2\int_{\partial Q_r}\varphi(x,\nu_{Q_r})d\mathcal{H}^1 + 4\Lambda r^2$$

and hence (2.13) entails (3.2). In particular, since $E\Delta A \subset\subset Q_\rho$ for every $\rho \in (r,\operatorname{dist}(x,\partial\Omega))$, we also have

$$\begin{aligned}\mathcal{F}(A,u;Q_\rho) \leq&\mathcal{F}(E,u;Q_\rho) + \Lambda|Q_r| = \mathcal{F}(E,u;Q_\rho\setminus\overline{Q_r}) + \mathcal{S}(E,u;\overline{Q_r}) + 4\Lambda r^2\\ \leq&\mathcal{F}(E,u;Q_\rho\setminus\overline{Q_r}) + 2\int_{\partial Q_r}\varphi(x,\nu_{Q_r})d\mathcal{H}^1 + 4\Lambda r^2\\ \leq&\mathcal{F}(E,u;Q_\rho\setminus\overline{Q_r}) + 16c_2 r + 4\Lambda r^2\end{aligned}$$

and hence, letting $\rho\searrow r$ and using $r\leq 1$ we get

$$\mathcal{F}(A,u;\overline{Q_r}) \leq (16c_2+4\Lambda)r. \tag{3.53}$$

Now assuming that x belongs to the closure of the set $\{y\in\Omega\cap\partial A:\ \theta_*(\partial A,y)>0\}$, we prove (3.3). For $\tau_o := \tau_0/2$, let $\varsigma_o = \varsigma(\tau_o) \in (0,1)$ and $R_o = R_0(1,\Lambda,c_1,\tau_o) > 0$ be as in Proposition 3.5. Then by (3.48),

$$\mathcal{F}(A,u;Q_{\kappa r}) \geq 2c_1\varsigma_o\kappa r \tag{3.54}$$

for $\kappa\in(0,1]$ and for any square $Q_r\subset\Omega$ with $r\in(0,R_o)$. We consider $\varsigma_* := \varsigma(\tau_*)$, $\vartheta_* := \vartheta(\tau^*)$, and $R_* := \min\{R(1,\Lambda,c_1,\tau_*),R_o\}$ as given by Proposition 3.4 for $\tau_* := \min\{\frac{\tau_0}{2},\big(\frac{c_1\varsigma_o}{16c_2+4\Lambda}\big)^{\frac{1}{1-\gamma}}\}$. By contradiction, if $\mathcal{H}^1(Q_r\cap\partial A) < \varsigma_* r$, then by applying (3.51) with $\kappa = \tau_*$ we obtain

$$\mathcal{F}(A,u;Q_r) \leq (1+\vartheta_*)\Phi(A,u;Q_r).$$

Then by Proposition 3.4,

$$\mathcal{F}(A,u;Q_{\tau_* r}) \leq \tau_*^{2-\gamma}\mathcal{F}(A,u;Q_r)$$

so that by (3.54) and (3.53)

$$\tau_*^{1-\gamma} \geq \frac{2c_1\varsigma_o}{16c_2 + 4\Lambda},$$

which is a contradiction. □

4. Compactness and lower-semicontinuity properties

For the convenience of the reader, we divide the prove into several propositions. We start by showing the compactness of free crystal regions of the sequence of constrained minimizers $\{(A_m, u_m)\}$.

Proposition 4.1. *Assume that either* $\mathtt{v} \in (0, |\Omega|)$ *or* $S = \emptyset$. *There exist* $m_h \nearrow +\infty$, $(A_{m_h}, u_{m_h}) \in \mathcal{C}_{m_h}$ *and* $A \in \widetilde{\mathcal{A}}$ *such that*

(a) *for any* $h \in \mathbb{N}$, (A_{m_h}, u_{m_h}) *is a minimizer of* $\mathcal{F}$ *in* $\mathcal{C}_{m_h}$ *with* $|A| = \mathtt{v}$ *such that* ∂A_{m_h} *does not contain isolated points;*

(b) $\mathrm{sdist}(\cdot, \partial A_{m_h}) \to \mathrm{sdist}(\cdot, \partial A)$ *locally uniformly in* $\mathbb{R}^2$ *as* $h \to \infty$;

(c) *for any* $x \in \Omega \cap \partial A$ *and* $r \in (0, \min\{R_*, \mathrm{dist}(x, \partial\Omega)\})$

$$\frac{c_1\varsigma_*}{8\pi c_2} \leq \frac{\mathcal{H}^1(Q_r(x) \cap \partial A)}{2r} \leq \frac{2\pi c_2}{c_1\varsigma_*}, \tag{4.1}$$

where $\varsigma_* := \varsigma_*(c_3, c_4) \in (0,1)$ *and* $R_* := R_*(c_1, c_2, c_3, c_4) > 0$ *are given in Theorem 3.1.*

Proof. By [45, Theorem 2.6] there exists a minimizer $(A_m, u_m) \in \mathcal{C}_m$ for every $m \in \mathbb{N}$. Without loss of generality we assume that ∂A_m does not contain isolated points. In fact, if ∂A_m has a isolated point x in $A_m^{(0)}$, then $A_m \backslash \{x\} \in \mathcal{A}_m$ and $\mathcal{F}(A_m, u_m) = \mathcal{F}(A_m \backslash \{x\}, u_m)$. Analogously, if ∂A_m has an isolated point in $A_m^{(1)}$, then there exists $r > 0$ such that $B_r(x) \cap \partial A_m = \{x\}$ (and $B_r(x) \subset A_m \cup \{x\} \in \mathcal{C}_m$). In view of Proposition A.3 the function u_m, arbitrarily extended to x belongs to $H^1_{\mathrm{loc}}(B_r(x))$, hence, the configuration $(A_m \cup \{x\}, u_m) \in \mathcal{C}_m$ and satisfies $\mathcal{F}(A_m, u_m) = \mathcal{F}(A_m \cup \{x\}, u_m)$.

In view of Remark 2.5 $(A_m, u_m - u_0)$ is a (λ_0, m)-minimizer of $\mathcal{F}(\cdot, \cdot; \Omega)$. Moreover, since ∂A_m does not contain isolated points $\theta_*(\partial A_m, x) > 0$ for any $x \in \partial A_m$, hence by Theorem 3.1 the density estimates (3.2) and (3.3) hold for all $x \in \Omega \cap \partial A_m$.

By [45, Proposition 3.1], there exist $A \subset \Omega$ and a subsequence $\{(A_{m_h}, u_{m_h})\}$ such that $\mathrm{sdist}(\cdot, \partial A_{m_h}) \to \mathrm{sdist}(\cdot, \partial A)$ as $h \to \infty$. Consider the sequence $\mu_h := \mathcal{H}^1 \llcorner \partial A_{m_h}$ of positive Radon measures. By Theorem 3.1

$$\frac{\varsigma_*}{2} \leq \frac{\mu_h(Q_r(x))}{2r} \leq \frac{2\pi c_2}{c_1} \tag{4.2}$$

for every $x \in \Omega \cap \partial A_{m_h}$ and $Q_r(x) \subset\subset \Omega$ with $r \in (0, R_*)$. By (2.13), (2.14) and (3.1),

$$\begin{aligned}\mu_h(\mathbb{R}^2) = \mathcal{H}^1(\partial A_{m_h}) \leq & \mathcal{H}^1(\partial\Omega) + \frac{\mathcal{F}(A_{m_h}, u_{m_h}) + 2c_2\mathcal{H}^1(\Sigma)}{c_1} \\ \leq & \mathcal{H}^1(\partial\Omega) + \frac{\mathcal{F}(A_1, u_1) + 2c_2\mathcal{H}^1(\Sigma)}{c_1},\end{aligned}$$

hence, by compactness, there exist a not relabelled subsequence and a positive Radon measure μ in $\mathbb{R}^2$ such that $\mu_h \rightharpoonup^* \mu$ as $h \to \infty$. We claim that

$$\overline{\Omega \cap \partial A} \subseteq \mathrm{supp}\,\mu \subseteq \partial A.$$

Indeed, let $x \in \Omega \cap \partial A$ and $r \in (0, \min\{\mathrm{dist}(x, \partial\Omega), R_*\})$. By the sdist-convergence, there exists $x_h \in Q_r(x) \cap \partial A_{m_h}$ with $x_h \to x$ such that $Q_{r/2}(x_h) \subset Q_r(x)$ and hence, by the weak* convergence and (4.2),

$$\mu(\overline{Q_r(x)}) \geq \limsup_{h\to\infty} \mu_h(\overline{Q_r(x)}) \geq \limsup_{h\to\infty} \mu_h(Q_{r/2}(x_h)) \geq \varsigma_* r.$$

This implies $x \in \operatorname{supp}\mu$. Conversely, if, by contradiction, there exists $x \in \operatorname{supp}\mu \setminus \partial A$, then we can find $r > 0$ such that $Q_r(x) \cap \partial A = \emptyset$. From the sdist-convergence it follows that $Q_{r/2}(x) \cap \partial A_{m_h} = \emptyset$ for h large enough, and hence,

$$0 < \mu(Q_{r/2}(x)) \leq \liminf_{h\to\infty} \mu_h(Q_{r/2}(x)) = 0,$$

which is a contradiction.

From (4.2) it follows that

$$\frac{\varsigma_*}{2} \leq \frac{\mu(Q_r(x))}{2r} \leq \frac{2\pi c_2}{c_1} \tag{4.3}$$

for any $x \in \Omega\cap\operatorname{supp}\mu$ any $r \in (0, R_*)$ with $Q_r(x) \subset\subset \Omega$. Indeed, let $x \in \Omega\cap\operatorname{supp}\mu$ and let $R(x) := \min\{R_*, \mathrm{dist}(x, \partial\Omega)\}$. Then by the weak* convergence $\mu_h(Q_r(x)) \to \mu(Q_r(x)) = 0$ for a.e. $r \in (0, R(x))$. In particular, (4.3) holds for a.e. $r \in (0, R(x))$. Since μ is a Radon measure, (4.3) extends to all $r \in (0, R(x))$ by the left-continuity of the map $r \mapsto \mu(Q_r(x))$.

From (4.3) and [2, Theorem 2.56] it follows that

$$\varsigma_* \mathcal{H}^1 \llcorner (\Omega \cap \operatorname{supp}\mu) \leq \mu \llcorner \Omega \leq \frac{4\pi c_2}{c_1} \mathcal{H}^1 \llcorner (\Omega \cap \operatorname{supp}\mu). \tag{4.4}$$

Thus, $\mu\llcorner\Omega$ is absolutely continuous with respect to $\mathcal{H}^1\llcorner(\Omega\cap\operatorname{supp}\mu)$ and $\mathcal{H}^1(\operatorname{supp}\mu) < \infty$. By (4.4),

$$\mathcal{H}^1(\partial A) \leq \mathcal{H}^1(\Omega \cap \partial A) + \mathcal{H}^1(\partial\Omega \cap \partial A) \leq \frac{1}{\varsigma_*}\mu(\Omega) + \mathcal{H}^1(\partial\Omega) < \infty.$$

Finally let us prove (4.1). Fix any $x \in \Omega \cap \partial A$ and let $R(x) := \min\{R_*, \mathrm{dist}(x, \partial\Omega)\}$. Then by (4.4)

$$\frac{\varsigma_* \mathcal{H}^1(Q_r(x) \cap \partial A)}{2r} \leq \frac{\mu(Q_r(x))}{2r} \leq \frac{4\pi c_2}{c_1} \frac{\mathcal{H}^1(Q_r(x) \cap \partial A)}{2r}.$$

This and (4.3) imply

$$\frac{\varsigma_* \mathcal{H}^1(Q_r(x) \cap \partial A)}{2r} \leq \frac{2\pi c_2}{c_1} \quad \text{and} \quad \frac{\varsigma_*}{2} \leq \frac{4\pi c_2}{c_1} \frac{\mathcal{H}^1(Q_r(x) \cap \partial A)}{2r},$$

and hence, (4.1) follows. □

We notice that by Proposition A.1 the limit set A in Proposition 4.1 is of finite perimeter. However, a priori, by the arguments of Proposition 4.1, its topological boundary ∂A does not need to be $\mathcal{H}^1$-rectifiable, and so in $\mathcal{A}$. This issue is overcome by introducing the extended class $\widetilde{\mathcal{A}}$ and the auxiliary model $\widetilde{F}$ in Section 2.3.

Corollary 4.2. *Let $\{A_{m_h}\}$ and A be as in Proposition 4.1. Then $A_{m_h} \to A$ in $L^1(\mathbb{R}^2)$ as $h \to \infty$.*

Proof. Since $\mathcal{H}^1(\partial A) < \infty$ and $A_{m_h} \xrightarrow{K} \overline{A}$ as $h \to \infty$, one has $\chi_{A_{m_h}}(x) \to \chi_A(x)$ as $h \to \infty$ for a.e. $x \in \mathbb{R}^2$. Now Corollary 4.2 follows from the Dominated Convergence Theorem. □

The following result generalizes [40, Theorem 4.2] since it applies to set Γ a priori not connected and even not necessarily $\mathcal{H}^1$-rectifiable), but satisfying uniform density estimates. Recall that we denote by Γ^r and Γ^u the $\mathcal{H}^1$-rectifiable and purely unrectifiable parts of a Borel 1-set Γ.

Proposition 4.3. *Let $\Gamma \subset \mathbb{R}^2$ be a Borel set such that $\mathcal{H}^1(\Gamma) < +\infty$ and for some $r_0, c, C > 0$ and for all $x \in \Gamma$*

$$c \le \frac{\mathcal{H}^1(Q_r(x))}{2r} \le C, \qquad r \in (0, r_0). \tag{4.5}$$

Then for any $R > 0$ and a.e. $x \in \Gamma^r$ one has

$$\overline{Q_{R,\nu_\Gamma(x)}(x)} \cap \sigma_{x,\rho}(\Gamma) \overset{K}{\to} \overline{Q_{R,\nu_\Gamma(x)}(x)} \cap T_x \tag{4.6}$$

and

$$\mathcal{H}^1 \llcorner (\sigma_{x,\rho}(\Gamma)) \overset{*}{\rightharpoonup} \mathcal{H}^1 \llcorner T_x \tag{4.7}$$

as $\rho \to 0$, where $\sigma_{x,r}$ is the blow-up map defined in (2.4) and T_x is the generalized tangent line to Γ at x. Moreover, for any $\mathcal{H}^1$-measurable $\Gamma' \subset \Gamma$ and $\mathcal{H}^1$-a.e. $x \in [\Gamma']^r$ the relations (4.6) and (4.7) hold with Γ' in place of Γ.

Proof. By [33, Theorem 3.3], Γ^r (and hence $[\Gamma']^r$) has a approximate tangent line at $\mathcal{H}^1$-a.e. x, therefore, (4.7) follows from [2, Remark 2.80]. To prove (4.6) with Γ choose $x \in \Gamma$ such that $\theta(\Gamma, x) = 1$ and T_x exists. Without loss of generality we assume that $x = 0$ and $\nu_\Gamma(x) = \mathbf{e_2}$ is the unit normal to T_x. First we prove

$$\sigma_{0,r}(\Gamma) \overset{K}{\to} T_0 \tag{4.8}$$

as $r \searrow 0$. Indeed, let $\mu_r := \mathcal{H}^1 \llcorner (\sigma_{0,r}(\Gamma))$ and $\mu_0 := \mathcal{H}^1 \llcorner T_0$. Given $r > 0$, since $\mu_r(Q_\rho(x)) = \frac{\mathcal{H}^1(Q_{\rho r}(rx))}{r}$, by (4.5) for all $x \in \sigma_{0,r}(\Gamma)$ and $\rho \in (0, r_0/r)$ one has

$$c \le \frac{\mu_r(Q_\rho(x))}{2\rho} \le C. \tag{4.9}$$

Let $r_k \searrow 0$ be any sequence. By compactness of sets in the Kuratowski convergence, passing to a further not relabelled subsequence if necessary we suppose that

$$\sigma_{0,r_k}(\Gamma) \overset{K}{\to} L \tag{4.10}$$

for some closed set $L \subset \mathbb{R}^2$ as $k \to \infty$. We claim that $L = T_0$. If there exists $x \in T_0 \setminus L$, then for some $\rho > 0$, $Q_\rho(x) \cap L = \emptyset$. By (4.10), $Q_{\rho/2}(x) \cap \sigma_{0,r_k}(\Gamma) = \emptyset$ for all large k so that $\mu_{r_k}(Q_{\rho/2}(x)) = 0$. Then by (4.7)

$$0 = \lim_{k\to\infty} \mu_{r_k}(Q_{\rho/2}(x)) \ge \mu_0(Q_{\rho/2}(x)) \ge \rho,$$

a contradiction. If there exists $x \in L \setminus T_0$, then for some $Q_\rho(x) \cap T_0 = \emptyset$ for some $\rho > 0$ and there exists a sequence $x_k \in \sigma_{0,r_k}(\Gamma)$ such that $x_k \to x$. Then $Q_{\rho/2}(x_k) \subset Q_\rho(x)$ for all large k so that by (4.7) and (4.9),

$$0 = \mu_0(\overline{Q_\rho(x)}) \ge \limsup_{k\to\infty} \mu_{r_k}(\overline{Q_\rho(x)}) \ge \limsup_{k\to\infty} \mu_{r_k}(Q_{\rho/2}(x_k)) \ge c\rho,$$

a contradiction. Thus, $L = T_0$. Since the sequence $r_k \searrow 0$ is arbitrary, (4.8) follows. Now (4.6) is obvious.

To prove the assertion for Γ', fix any $x \in \Gamma'$ such that $\theta(\Gamma, x) = \theta(\Gamma', x) = 1$ and both generalized tangents T_x^Γ and $T_x^{\Gamma'}$ of Γ and Γ' exist. Note that $T_x^\Gamma = T_x^{\Gamma'} =: T_x$. For shortness, assume that $x = 0$ and $\nu_\Gamma(x) = \mathbf{e_2}$. Since in general Γ' does not satisfy the uniform density estimates of type (4.5), we cannot argue as above.

Let $r_k \searrow 0$ be arbitrary sequence such that $\sigma_{0,r_k}(\Gamma') \to L$ for some closed set $L \subset \mathbb{R}^2$. Then for every $x \in L$ there exists a sequence $x_k \in \sigma_{0,r_k}(\Gamma')$ such that $x_k \to x$. Since $\Gamma' \subset \Gamma$ and by (4.8) $\sigma_{0,r_k}(\Gamma) \overset{K}{\to} T_0$, we have $x_k \in \sigma_{0,r_k}(\Gamma)$ and $x_k \to x \in T_0$. Thus, $L \subset T_0$. To prove the converse inclusion, assume that there exists $x \in T_0 \setminus L$. Since L is closed there exists $r>0$ such that $B_{2r}(x)\cap L=\emptyset$. As we mentioned in the beginning of the proof, for $\mu_k := \mathcal{H}^1 \llcorner (\sigma_{0,r_k}(\Gamma'))$ we have $\mu_k \xrightarrow{*} \mathcal{H}^1(T_0)$. In particular, for every $\rho\in(0,r)$

$$\lim_{k\to+\infty} \mu_k(B_\rho(x)) = \mathcal{H}^1(B_\rho(x)\cap T_0) = 2\rho.$$

Hence, $B_\rho(x)\cap\sigma_{0,r_k}(\Gamma')\neq\emptyset$ for each such ρ and thus, taking a sequence $\rho_n\to 0$ and using a diagonal argument we obtain a sequence $x_n\in\sigma_{0,r_{k_n}}(\Gamma')$ converging to x. So $x\in L$, a contradiction.

Since $r_k\to 0$ is arbitrary, one has $\sigma_{0,r}(\Gamma')\xrightarrow{K} T_0$ as $r\to 0$. □

Next we turn to the compactness of displacements of the sequence of constrained minimizers $\{(A_m,u_m)\}$.

Proposition 4.4. *Let A_{m_h} and A be as in Proposition 4.1. Let $\{E_i\}_{i\in\mathbb{N}}$ be the family of all connected components of* $\mathrm{Int}(A)$. *There exist a further (not relabelled) subsequence of $\{(A_{m_h},u_h)\}$, a sequence $\{a_h\}$ of rigid displacements, a subset N of $\mathbb{N}$, a function $v_0\in H^1(S)$ and a family $\{v_i\in GSBD^2(\mathrm{Int}(E_i))\cap H^1_{\rm loc}(\mathrm{Int}(E_i)\cup S)\}_{i\in N}$ such that*

$$|u_{m_h}+a_h|\to+\infty$$

a.e. in $\bigcup_{i\in\mathbb{N}\setminus N}E_i$,

$$u_{m_h}+a_h \rightharpoonup v_0\chi_S + \sum_{i\in N} v_i\chi_{E_i}$$

weakly in $H^1_{\rm loc}((\cup_{i\in N}E_i)\cup S)$ (and hence a.e. in $(\cup_{i\in N}E_i)\cup S$),

$$e(u_{m_h}) \to e(v_0)\chi_S + \sum_{i\in N} e(v_i)\chi_{E_i}$$

weakly in $L^2_{\rm loc}((\cup_{i\in N}E_i)\cup S)$.

The main difference of our compactness result from [14, Theorem 1.1] is not only that in our setting we have the set-function coupling, but also we need to select those components of limiting free crytal region where the displacements diverge and those in which they don't. This first requires to actually prove that the behavior is consistent inside each component of the limiting free-crystal region, which is achieved using [45, Proposition 3.7].

Proof. Since S is connected and Lipschitz, by the Korn-Poincaré inequality and the Rellich-Kondrachov Theorem there exists a further not relabelled subsequence $\{u_{m_h}\}$, a sequence $\{a_h\}$ of infinitesimal rigid displacements and $v_0\in H^1(S;\mathbb{R}^2)$ such that $u_{m_h}+a_h\to v_0$ weakly in $H^1(S;\mathbb{R}^2)$ and a.e. in S.

We define the set $N\subset\mathbb{N}$ as follows: For each $i\in\mathbb{N}$ fix some ball $B_i\subset\subset E_i$. Since $A_{m_h}\xrightarrow{K} A$, there exists $h_i^0>0$ such that $B_i\subset\subset A_{m_h}$ for all $h>h_i^0$. By (2.15) and (3.1)

$$\sup_{h>h_i^0}\int_{B_i}|e(u_{m_h}+a_h)|^2dx \le \frac{1}{2c_3}\sup_{h>h_i^0}\int_{A_{m_h}\cup S}\mathbb{C}(x)e(u_{m_h}):e(u_{m_h})dx<+\infty,$$

and thus, by [45, Proposition 3.7] either $|u_{m_h}+a_h|\to+\infty$ a.e. in B_i or up to a subsequence, $u_{m_h}+a_h$ converges a.e. in B_i. By a diagonal argument, we choose a further not relabelled subsequence $\{u_{m_h}\}$ and the subset N of indices $i\in\mathbb{N}$ such that for every $i\in N$ the sequence $w_h:=u_{m_h}+a_h\to v_i$ converges a.e. in B_i as $h\to+\infty$.

We claim that for every $i\in N$ there exists $v_i\in H^1_{\rm loc}(E_i;\mathbb{R}^2)\cap GSBD^2(E_i;\mathbb{R}^2)$ such that $w_h\to v_i$ weakly in $H^1_{\rm loc}(E_i;\mathbb{R}^2)$ and a.e. in E_i as $h\to\infty$. To prove the claim we fix $i\in N$ and let $D\subset\subset E_i$ be an arbitrary connected open set containing B_i. Since $\mathrm{sdist}(\cdot,\partial A_{m_h})\to$

$\mathrm{sdist}(\cdot,\partial A)$ locally uniformly in $\mathbb{R}^2$, there exists $h_D>0$ such that $D\subset\subset \mathrm{Int}(A_{m_h})$ for all $h>h_D$. Note that $w_h\in H^1(D)$ and

$$\sup_{h>h_D}\int_D |e(w_h)|^2dx \le C := \frac{1}{2c_3}\sup_{h>h_D}\int_{A_{m_h}\cup S}\mathbb{C}(x)e(u_{m_h}):e(u_{m_h})dx<+\infty, \tag{4.11}$$

where in the first inequality we used (2.15) and in the second (3.1). Since w_h has finite limit a.e. in $B_i\subset D$, by [45, Proposition 3.7] there exists $v_i^D\in H^1_{\mathrm{loc}}(D)\cap GSBD^2(D)$ and a subsequence $\{w_h^D\}$ of $\{w_h\}$ such that $w_h^D\to v_i^D$ weakly in H^1_{loc} and a.e. in D. Now choosing a sequence $D_1\subset\subset D_2\subset\subset\ldots\subset\subset E_i$ of connected open sets such that $B_i\subset D_1$ and $E_i=\cup_j D_j$ and using a diagonal argument we choose a (not relablled) subsequence $\{w_h\}$ and $v_i\in H^1_{\mathrm{loc}}(E_i)\cap GSBD^2_{\mathrm{loc}}(E_i)$ such that $w_h\to v_i$ weakly in $H^1_{\mathrm{loc}}(E_i)$ and a.e. in E_i. In particular, $e(w_h)\to e(v_i)$ weakly in $L^2_{\mathrm{loc}}(E_i)$ and hence, by convexity and (4.11)

$$\int_{D_j}|e(v_i)|^2dx\le \liminf_{h\to+\infty}\int_{D_j}|e(w_h)|^2dx\le C.$$

Hence, letting $j\to\infty$ we get $v_i\in GSBD^2(E_i)$.

Let us now show that by the choice of N, for every $j\in\mathbb{N}\setminus N$ one has $|u_{m_h}+a_h|\to+\infty$ a.e. in E_j as $h\to+\infty$. Indeed, by definition, if $i\notin N$, then $|u_{m_h}+a_h|\to+\infty$ a.e. in $B_i\subset\subset E_i$. Let $D\subset\subset E_i$ be any connected open set containing B_i. As in (4.11) we can show $\|e(u_{m_h}+a_h)\|^2_{L^2(D)}$ is uniformly bounded for all sufficiently large h, and therefore, by [45, Proposition 4.7] $|u_{m_h}+a_h|\to+\infty$ a.e. in D.

Finally, since $u_{m_h}+a_h\to u$ weakly in $H^1_{\mathrm{loc}}((\cup_{i\in N}E_i)\cup S)$, it follows that $e(u_{m_h})=e(u_{m_h}+a_h)\to e(u)$ weakly in $L^2_{\mathrm{loc}}((\cup_{i\in N}E_i)\cup S)$. □

Proposition 4.4 allows us to define a “limit” displacement.

Proposition 4.5. *Let $\{(A_{m_h},u_{m_h})\}$, $\{a_h\}$, A, N and $\{v_i\}_{i\in N\cup\{0\}}$ satisfy the assertion of Proposition 4.4 and let*

$$u:=v_0\chi_S+\sum_{i\in N}v_i\chi_{E_i}+\sum_{j\in\mathbb{N}\setminus N}u_0\chi_{E_j},$$

where u_0 is the displacement defining the mismatch strain M_0. Then

$$\liminf_{h\to\infty}\mathcal{W}(A_{m_h},u_{m_h})\ge\mathcal{W}(A,u). \tag{4.12}$$

Proof. Fix arbitrary open set $D\subset\subset\mathrm{Int}(A)\cup S$. By Proposition 4.4 $u_{m_h}+a_h\to u$ weakly in $L^2(D\cap[(\cup_{i\in N}E_i)\cup S])$, hence, by the convexity of the elastic energy

$$\begin{aligned}\liminf_{h\to\infty}\mathcal{W}(A_{m_h},u_{m_h}) &= \liminf_{h\to\infty}\int_{A_{m_h}\cup S}W(x,e(u_{m_h})-M_0)dx\\ &\ge\liminf_{h\to\infty}\Big(\int_{D\cap S}W(x,e(u_{m_h})-M_0)dx+\sum_{j\in N}\int_{D\cap E_i}W(x,e(u_{m_h})-M_0)dx\Big)\\ &\ge\int_{D\cap S}W(x,e(u)-M_0)dx+\sum_{i\in N}\int_{D\cap E_i}W(x,e(u)-M_0)dx,\end{aligned}$$

where we recall that $M_0=e(u_0)$. Since $e(u)-M_0=0$ a.e. in $\cup_{j\in\mathbb{N}\setminus N}E_j$, this inequality can also be rewritten as

$$\liminf_{h\to\infty}\mathcal{W}(A_{m_h},u_{m_h})\ge\int_{D\cap(A\cup S)}W(x,e(u)-M_0)dx.$$

Now letting $D\nearrow\mathrm{Int}(A)\cup S$ and using $|A\setminus\mathrm{Int}(A)|\le|\partial A|=0$ we get (4.12). □

Now we establish the following “lower semicontinuity” of $\mathcal{F}(A_m,u_m)$.

Proposition 4.6. *Let $\{(A_{m_h}, u_{m_h})\}$, A and u be as in Proposition 4.5. Then* $(\mathrm{Int}(A), u) \in \widetilde{\mathcal{C}}$ *and*

$$\liminf_{h\to\infty} \mathcal{S}(A_{m_h}, u_{m_h}) \geq \widetilde{\mathcal{S}}(\mathrm{Int}(A), u), \tag{4.13}$$

where $\widetilde{\mathcal{S}}$ is defined in (2.12).

We postpone the proof of this proposition after the following auxiliary lemma, needed to treat the delamination and jumps along the cracks.

Lemma 4.7. *Recall the definition of the sets I_r and $Q_r^{\pm}$ from (2.3). Let ϕ be any norm in $\mathbb{R}^2$. Let $\{D_k\}$ and $\{m_k\}$ be sequences of subsets of Q_4 and of natural numbers, respectively, satisfying*

(a) the number of connected components ∂D_k lying strictly inside Q_4 does not exceed m_k;

(b) $\mathrm{sdist}(\cdot, \partial D_k) \to -\mathrm{dist}(\cdot, I_4)$ *uniformly in Q_4 and*

$$\sup_k \mathcal{H}^1(Q_1 \cap \partial D_k) < +\infty;$$

(c) there exists a sequence $\{w_k\} \subset GSBD^2(Q_4)$ such that $J_{w_k} \subset Q_1 \cap \partial D_k$ and

$$\sup_k \int_{Q_1} |e(w_k)|^2 dx < +\infty;$$

(d) there exist $\xi^{\pm} \in \mathbb{R}^2$ such that

$$w_k \to w_0 := \xi^+ \chi_{Q_1^-} + \xi^- \chi_{Q_1^+ \setminus U_1^\infty} \quad a.e. \text{ in } Q_1 \setminus U_1^\infty$$

and

$$|w_k| \to +\infty \quad a.e. \text{ in } U_1^\infty,$$

where U_1^∞ is either $\emptyset$ or Q_1^+.

Then there exists a subsequence $\{k_h\} \subset \mathbb{N}$ such that for any $\delta \in (0,1)$ we can find $h_\delta > 0$ for which

$$\int_{Q_1 \cap \partial^* D_{k_h}} \phi(\nu_{D_{k_h}}) d\mathcal{H}^1 + 2 \int_{Q_1 \cap D_{k_h}^{(1)} \cap \partial D_{k_h}} \phi(\nu_{D_{k_h}}) d\mathcal{H}^1 \geq 2 \int_{I_1} \phi(\mathbf{e_2}) d\mathcal{H}^1 - \delta \tag{4.14}$$

for all $h > h_\delta$.

Before the proof of Lemma 4.7 we recall some notations and results from [14]. Given $\xi \in \mathbb{R}^2 \setminus \{0\}$, let $\Pi_\xi := \{y \in \mathbb{R}^2 : \, y \cdot \xi = 0\}$. For every set $B \subset \mathbb{R}^2$ and for every $y \in \Pi_\xi$ we define

$$B_y^\xi := \{t \in \mathbb{R} : \, y + t\xi \in B\}.$$

Moreover, for every $u : B \to \mathbb{R}^2$ we define $\widehat{u}_y^\xi : B_y^\xi \to \mathbb{R}$ by

$$\widehat{u}_y^\xi(t) := u(y + t\xi) \cdot \xi.$$

When $u \in GSBD^2(Q_1)$, then $\widehat{u}_y^\xi \in SBV^2_{\mathrm{loc}}([Q_1]_y^\xi)$ for $\mathcal{H}^1$-a.e. $\pi_\xi(Q_1)$ and for all $\xi \in \mathbb{R}^2 \setminus \{0\}$. In this case we define

$$I_y^\xi(u) := \int_{[Q_1]_y^\xi} |(\dot{u})_y^\xi|^2 dt,$$

where $(\dot{u})_y^\xi$ is the density of the absolutely continuity part of $D\widehat{u}_y^\xi$ and also

$$II_y^\xi(u) := |D(\tau(u \cdot \xi)_y^\xi)|([Q]_1^\xi),$$

where $\tau(t) := \arctan(t)$. Recall that

$$\int_{\Pi_\xi} I_y^\xi(u)\mathcal{H}^1(y) + \int_{\Pi_\xi} II_y^\xi(u)\mathcal{H}^1(y) \le \int_{Q_1} |e(u)|dx + \int_{Q_1} |e(u)|^2 dx + \mathcal{H}^1(J_u)$$

(see e.g. [14, Eq. 3.8 and 3.9]).

Proof. The proof is similar to [45, Lemma 4.7]. Since ϕ is even,

$$\phi(\xi) = \sup_{\eta\in\mathbb{R}^2,\, \phi^o(\eta)=1} |\xi\cdot\eta|, \quad \xi \in \mathbb{R}^2,$$

where ϕ^o is the dual norm of ϕ. By the compactness of $B^{\phi^o} := \{\eta \in \mathbb{R}^2 : \phi^o(\eta) = 1\}$, for any countable set $\{\eta_i\}$ dense in B^{ϕ^o} and for any $\mathcal{H}^1$-rectifiable set $K \subset \mathbb{R}^2$

$$\phi(\nu_K(x)) = \sup_{i\ge1} |\nu_K(x)\cdot\eta_i| \quad \text{for } \mathcal{H}^1\text{-a.e. } x \in K.$$

Hence, by [27, Lemma 6] for any open set $U \subset \mathbb{R}^2$

$$\int_{U\cap K} \phi(\nu_K)d\mathcal{H}^1 = \sup_k \sup \Big\{ \sum_{i=1}^k \int_{A_i\cap K} |\nu_K\cdot\eta_i| d\mathcal{H}^1 : A_i \subset\subset U \text{ open and pairwise disjoint}\Big\}.$$

Moreover, by the area formula for any Borel set B

$$\int_{B\cap K} |\nu_K\cdot\xi| d\mathcal{H}^1 = |\xi| \int_{\pi_\xi(B)} \mathcal{H}^0(K\cap B_y^\xi)d\mathcal{H}^1(y),$$

where $\pi_\xi(z) = z - \big(z\cdot\frac{\xi}{|\xi|}\big)\frac{\xi}{|\xi|}$ and given $y \in \pi_\xi(B)$, $B_y^\xi = \pi_\xi^{-1}(y)\cap B$.

Step 1: There exists an at most countable set $\varUpsilon \subset B^{\phi^o}$ such that

$$\lim_{k\to+\infty} \mathcal{H}^1(\pi_\xi(I_1)\setminus \pi_\xi(J_{w_k})) = 0 \tag{4.15}$$

for any $\xi \in B^{\phi^o}\setminus\varUpsilon$.

Indeed, let $\varUpsilon$ be the set of all $\xi \in B^{\phi^o}$ for which there exists $y \in \pi_\xi(I_1)$ such that $\mathcal{H}^1(\pi_\xi^{-1}(y)\cap\partial D_k) > 0$. By assumption (b) and Proposition A.2 the set $\varUpsilon$ is at most countable. Let $\{w_{k_l}\}$ be arbitrary not relabelled subsequence of $\{w_k\}$. In view of [14, Eq. 3.23] (applied with $A = U_1^\infty$) for any $\xi \in B^{\phi^o}\setminus\varUpsilon$, $\epsilon > 0$ and for $\mathcal{H}^1$-a.e. $y \in \pi_\xi(Q_1)$ there exists a further subsequence $w_{k_{l_h}}$ (possibly depending on ξ, ϵ and y)

$$\mathcal{H}^0(J_{[\widehat{w}_0]_y^\xi}\cap[Q_1\setminus U_1^\infty]_y^\xi) + \mathcal{H}^0([\partial U_1^\infty]_y^\xi) \le \liminf_{h\to+\infty}\Big[\mathcal{H}^1(J_{[w_{k_{l_h}}]_y^\xi}) + \epsilon(I_y^\xi(w_{k_{l_h}}) + II_y^\xi w_{k_{l_h}})\Big]. \tag{4.16}$$

By the definition of w_0 and U_1^∞, the left-side of (4.16) is equal to 1 for $\mathcal{H}^1$-a.e. $y \in \pi_\xi(I_1)$. Theorefore, for such y and for sufficiently small $\epsilon > 0$ we have $\liminf_{h\to+\infty}\mathcal{H}^1(J_{[w_{k_{l_h}}]_y^\xi}) \ge 1$. Hence, for $\mathcal{H}^1$-a.e. $y \in \pi_\xi(I_1)$ the line $\pi_\xi^{-1}(y)$ intersects $J_{w_{k_{l_h}}}$ for all h and (4.15) follows.

Note that by [45, Proposition 4.6]

$$\liminf_{k\to+\infty}\int_{Q_1\cap J_{w_k}} \phi(\nu_{J_{w_k}})d\mathcal{H}^1 \ge \int_{I_1}\phi(\mathbf{e_2})d\mathcal{H}^1. \tag{4.17}$$

Step 2: Now we improve (4.17) by including coefficient 2 on the right-hand side of the inequality in the presence of a small error term.

We proceed in three substeps. We redefine the displacement w_k in the convex envelope V_k^i of each connected component K_k^i of ∂D_k in such a way that ∂V_k^i become jump sets with the left-hand side of (4.14) lowered up to a small error.

Substep 2.1: First we identify $\{V_k^i\}$.

Fix any $\delta\in(0,1)$. By (b) there exists $k_\delta^1>0$ such that $([-2,2]\times[-2,-\delta])\cup([-2,2]\times[\delta,2])\subset \mathrm{Int}(D_k)$ for any $k\ge k_\delta^1$. Let $F_k:=Q_1\cap D_k$. Note that $\partial F_k\subset (Q_1\cap\partial D_k)\cup(\{\pm1\}\times[-\delta,\delta])$ and since $D_k\in\mathcal{A}_{m_k}$, the number of connected components $\{L_k^j\}_{j\ge1}$ of ∂F_k does not exceed m_k. Note that $F_k\subset[-1,1]\times[-\delta,\delta]$ and

$$
\begin{aligned}
\alpha_k &:= \int_{Q_1\cap\partial^* E_k}\phi(\nu_{E_k})d\mathcal{H}^1 + 2\int_{Q_1\cap E_k^{(1)}\cap\partial E_k}\phi(\nu_{E_k})d\mathcal{H}^1\\
&\ge \int_{Q_2\cap\partial^* F_k}\phi(\nu_{F_k})d\mathcal{H}^1 + 2\int_{Q_2\cap F_k^{(1)}\cap\partial F_k}\phi(\nu_{F_k})d\mathcal{H}^1 - 4\delta\\
&= \sum_{j\ge1}\Big[\int_{Q_2\cap\partial^* F_k\cap L_k^j}\phi(\nu_{F_k})d\mathcal{H}^1 + 2\int_{Q_2\cap F_k^{(1)}\cap\partial F_k\cap L_k^j}\phi(\nu_{F_k})d\mathcal{H}^1\Big] - 4\delta := \alpha_k'.
\end{aligned}
\tag{4.18}
$$

Next repeating the same arguments of Step 1 in the proof of [45, Lemma 4.7] we can find a family $\{V_k^i\}_i$ of at most countably many pairwise disjoint closed convex sets with non-empty interior such that for each L_k^j there exists a unique V_i with $L_k^j\subset V_k^i$ and

$$
\alpha_k' \ge \sum_{i\ge1}\int_{\partial V_k^i}\phi(\nu_{V_k^i})d\mathcal{H}^1 - 6\delta \tag{4.19}
$$

see e.g. [45, Eq. 4.34]

Substep 2.2: Now we replace w_k with another function v_k associated to V_k^i. Fix $\xi_0\in\mathbb{R}^2$ such that the jump set of the function

$$
v_k := w_k\chi_{Q_1\setminus\cup_i V_k^i} + \xi_0\chi_{\cup_i V_k^i}
$$

coincide with $\cup_i\partial V_k^i$ (up to a $\mathcal{H}^1$-negligible set).

By assumption (b) $\cup_i\partial V_k^i\overset{K}{\to} I_1$ as $k\to+\infty$. Moreover, as in Step 1 we can find a countable set $\varUpsilon'\subset B^{\phi^o}$ such that by assumption (b) and (4.15)

$$
\begin{aligned}
\limsup_{k\to+\infty}\mathcal{H}^1(\pi_\xi(I_1)\setminus\pi_\xi(\cup_i\partial V_k^i)) &\le \limsup_{k\to+\infty}\mathcal{H}^1(\pi_\xi(I_1)\setminus\pi_\xi(\partial D_k))\\
&\le \lim_{k\to+\infty}\mathcal{H}^1(\pi_\xi(I_1)\setminus\pi_\xi(J_{w_k}))=0
\end{aligned}
$$

for all $\xi\in B^{\phi^o}\setminus(\varUpsilon\cup\varUpsilon')$. Moreover, by assumption (d) $v_k\to w_0$ a.e. in $Q_1\setminus U_1^\infty$ and $|v_k|\to+\infty$ a.e. in U_1^∞.

Substep 2.3: By convexity of each V_k^i we observe that

$$
\liminf_{k\to+\infty}\mathcal{H}^0(\pi_\xi^{-1}(y)\cap J_{v_k})\ge 2 = 2\mathcal{H}^0(J_{[\widehat{w}_0]_y^\xi}\cap[Q_1\setminus U_1^\infty]_y^\xi).
$$

for all $\xi\in B^{\phi^o}\setminus(\varUpsilon\cup\varUpsilon')$ and $\mathcal{H}^1$-a.e. $y\in\pi_\xi$. Thus, by repeating the arguments of Step 1 in the proof of [45, Proposition 4.6] we get

$$
\liminf_{k\to+\infty}\int_{\cup_i\partial V_k^i}\phi(\nu_{\cup_i V_k^i})d\mathcal{H}^1 = \liminf_{k\to+\infty}\int_{J_{v_k}}\phi(\nu_{J_{v_k}})d\mathcal{H}^1 \ge 2\int_{I_1}\phi(\mathbf{e_2})d\mathcal{H}^1,
$$

which together with (4.18) and (4.19) implies the assertion of the lemma. □

Now we are ready to prove (4.13).

Proof of Proposition 4.6. For shortness, let

$$
G:=\mathrm{Int}(A).
$$

We define

$$
\widetilde{u}_h := (u_{m_h}+a_h)\chi_{A_{m_h}} + \eta\chi_{\Omega\setminus A_{m_h}}
$$

and

$$\widetilde{u} := u\chi_G + \eta\chi_{\Omega\setminus G}$$

for $\eta \in (0,1)^2$ such that $\Omega \cap \partial^* A_{m_h} \subset J_{\widetilde{u}_h}$ and $\Omega \cap \partial^* G \subset J_{\widetilde{u}}$ up to an $\mathcal{H}^1$-negligible set. Such η exists by Proposition A.2 in view of the estimate

$$\mathcal{H}^1(\partial A_{m_h}) \le \frac{\mathcal{S}(A_{m_h}, u_{m_h})}{c_1} + \frac{2c_2\mathcal{H}^1(\partial\Omega)}{c_1} \le \frac{\mathcal{F}(A_1, u_1)}{c_1} + \frac{2c_2\mathcal{H}^1(\partial\Omega)}{c_1},$$

which holds for every $h \ge 1$. Notice that $\widetilde{u}_h \in GSBD^2(\mathrm{Int}(\Omega \cup S \cup \Sigma)) \cap H^1_{\mathrm{loc}}((\Omega \cup S) \setminus \partial A_{m_h})$, $\widetilde{u} \in GSBD^2(\mathrm{Int}(\Omega \cup S \cup \Sigma)) \cap H^1_{\mathrm{loc}}((\Omega \cup S) \setminus \partial G)$, $J_{\widetilde{u}} \subset (\Omega \cap \partial G) \cup (\Sigma \cap J_u)$ and

$$\mathcal{H}^1(J_{\widetilde{u}_h}) + \int_\Omega |e(\widetilde{u}_h)|^2 dx \le \mathcal{F}(A_{m_h}, u_{m_h}) + \mathcal{H}^1(\Sigma) \le M := \mathcal{F}(A_1, u_1) + \mathcal{H}^1(\Sigma) < \infty \quad (4.20)$$

for every $h \ge 1$. Moreover, by Proposition 4.4, the definitions of u, $\widetilde{u}_h$ and $\widetilde{u}$,

$$\widetilde{u}_h \to \widetilde{u} \quad \text{a.e. in } [S \cup (\Omega \setminus G)] \cup \bigcup_{i\in N} E_i \quad (4.21)$$

and

$$|\widetilde{u}_h| \to +\infty \quad \text{a.e. in } \bigcup_{j\in\mathbb{N}\setminus N} E_j, \quad (4.22)$$

where $\{E_i\}$ and N are provided by Proposition 4.4.

We recall that a priori ∂A, and hence ∂G, does not need to be $\mathcal{H}^1$-rectifiable. Therefore, by [21, Theorem 6.2] $J_{\widetilde{u}} \subset (\Omega \cap \partial^r G) \cup (\Sigma \cap J_u)$, where we recall that $\partial^r G$ is $\mathcal{H}^1$-rectifiable part of ∂G.

To prove (4.13) we use similar arguments as in [45, Proposition 4.1]. Let $g \in L^\infty(\Sigma \times \{0,1\})$ be such that

$$g(x,s) := \varphi(x, \nu_\Sigma(x)) + s\beta(x)$$

for which we know by (2.14) that $g \ge 0$ and

$$|g(x,1) - g(x,0)| \le \varphi(x, \nu_\Sigma(x)) \qquad \text{for a.e. } x \in \Sigma. \quad (4.23)$$

Let μ_h be the sequence of positive Radon measures defined at Borel sets $B \subset \mathbb{R}^2$ as

$$\begin{aligned}\mu_h(B) := &\int_{B\cap\Omega\cap\partial^* A_{m_h}} \varphi(x, \nu_{A_{m_h}}) d\mathcal{H}^1 + 2\int_{B\cap\Omega\cap(A^{(1)}_{m_h}\cup A^{(0)}_{m_h})\cap\partial A_{m_h}} \varphi(x, \nu_{A_{m_h}}) d\mathcal{H}^1 \\ &+ \int_{B\cap\Sigma\cap A^{(0)}_{m_h}\cap\partial A_{m_h}} [\varphi(x,\nu_\Sigma) + g(x,1)]\, d\mathcal{H}^1(x) + \int_{B\cap\Sigma\setminus\partial A_{m_h}} g(x,0) d\mathcal{H}^1 \\ &+ \int_{B\cap\Sigma\cap\partial^* A_{m_h}\setminus J_{u_{m_h}}} g(x,1) d\mathcal{H}^1 + \int_{B\cap\Sigma\cap J_{u_{m_h}}} [g(x,0) + \varphi(x,\nu_\Sigma)]\, d\mathcal{H}^1\end{aligned}$$

and let μ be the positive measure defined at Borel sets $B \subset \mathbb{R}^2$ as

$$\begin{aligned}\mu(B) := &\int_{B\cap\Omega\cap\partial^* G} \varphi(x, \nu_A) d\mathcal{H}^1 + 2\int_{B\cap\Omega\cap G^{(1)}\cap\partial G\cap J_{\widetilde{u}}} \varphi(x, \nu_G) d\mathcal{H}^1 \\ + &\int_{B\cap\Sigma\setminus\partial G} g(x,0) d\mathcal{H}^1 + \int_{B\cap\Sigma\cap\partial^* G\setminus J_u} g(x,1) d\mathcal{H}^1 + \int_{B\cap\Sigma\cap J_{\widetilde{u}}} [g(x,0) + \varphi(x,\nu_\Sigma)]\, d\mathcal{H}^1.\end{aligned}$$

Since $S^A_{\widetilde{u}} := G^{(1)} \cap \partial G \cap J_{\widetilde{u}}$ and $\Sigma \cap J_{\widetilde{u}} = \Sigma \cap J_u$, we have

$$\mu_h(\mathbb{R}^2) = \mathcal{S}(A_{m_h}, u_{m_h}) + \int_\Sigma \varphi(x, \nu_\Sigma) d\mathcal{H}^1$$

and

$$\mu(\mathbb{R}^2) = \widetilde{\mathcal{S}}(G, u) + \int_\Sigma \varphi(x, \nu_\Sigma) d\mathcal{H}^1.$$

Hence, to establish (4.13) it suffices to prove

$$\liminf_{h\to\infty} \mu_h(\mathbb{R}^2) \ge \mu(\mathbb{R}^2). \tag{4.24}$$

Since $\sup_h \mu_h(\mathbb{R}^2) < +\infty$, by compactness, there exists a positive Radon measure μ_0 in $\mathbb{R}^2$ such that (up to a subsequence) $\mu_h \rightharpoonup^* \mu_0$ as $h \to \infty$. We show

$$\mu_0 \ge \mu \tag{4.25}$$

and we observe that (4.24) immediately follows from (4.25). To establish (4.25) it suffices to prove

$$\frac{d\mu_0}{d\mathcal{H}^1 \llcorner (\Omega \cap \partial^* G)}(x) \ge \varphi(x, \nu_G(x)) \qquad \text{for a.e. } x \in \Omega \cap \partial^* G, \tag{4.26a}$$

$$\frac{d\mu_0}{d\mathcal{H}^1 \llcorner (\Sigma \cap \partial^* G)}(x) \ge g(x, 1) \qquad \text{for a.e. } x \in \Sigma \cap \partial^* G, \tag{4.26b}$$

$$\frac{d\mu_0}{d\mathcal{H}^1 \llcorner (\Sigma \setminus \partial G)}(x) \ge \varphi(x, \nu_\Sigma(x)) \qquad \text{for a.e. } x \in \Sigma \setminus \partial G, \tag{4.26c}$$

$$\frac{d\mu_0}{d\mathcal{H}^1 \llcorner S^A_{\widetilde{u}}}(x) \ge 2\varphi(x, \nu_G(x)) \qquad \text{for a.e. } x \in S^A_{\widetilde{u}}, \tag{4.26d}$$

$$\frac{d\mu_0}{d\mathcal{H}^1 \llcorner (\Sigma \cap J_{\widetilde{u}})}(x) \ge 2\varphi(x, \nu_\Sigma(x)) \qquad \text{for a.e. } x \in \Sigma \cap J_{\widetilde{u}} \tag{4.26e}$$

since $g(x, 0) = \varphi(x, \nu_\Sigma)$.

The proof of the estimates (4.26a)-(4.26e) follows from similar arguments used in [45, Proposition 4.1] with special care needed for (4.26d). In fact for (4.26d) we cannot employ the strategy used for [45, Eq. 4.40c] that was hinged on the uniform bound on the number of boundary components, which here we do not have. We instead adapt the arguments employed in [45, Eq. 4.40g] by using Lemma 4.7.

Next we detail the proofs of (4.26a)-(4.26e).

Proof of (4.26a). Note that $A = G$ up to a negligible set. By Corollary 4.2 $A_{m_h} \to A$ in $L^1(\mathbb{R}^2)$, thus, the proof of (4.26a) can be done following the arguments of [45, Eq. 4.40a] using Reshetnyak lower semicontinuity Theorem [2, Theorem 2.38].

Proof of (4.26b). Since $A_{m_h} \to G$ in $L^1(\mathbb{R}^2)$, we have $D\chi_{A_{m_h}} \rightharpoonup^* D\chi_G$. Thus, the proof of (4.26b) directly follows from [1, Lemma 3.8] (see also the proof of [45, Eq. 4.40d]).

Proof of (4.26c). Let $x_0 \in \Sigma \setminus \partial G$ and let $r_0 := \mathrm{dist}(x_0, \partial G) > 0$. Since $\mathbb{R}^2 \setminus \mathrm{Int}(A_{m_h}) \xrightarrow{K} \mathbb{R}^2 \setminus \mathrm{Int}(A) = \mathbb{R}^2 \setminus G$, there exists $r_1 \in (0, r_0)$ such that $B_r(x_0) \cap \mathrm{Int}(A_{m_h})$ for any $r \in (0, r_1)$. Therefore, for any $r \in (0, r_1)$

$$\begin{aligned}
\mu_h(\overline{B_r(x_0)}) &= \int_{\overline{B_r(x_0)} \cap \Sigma \cap A^{(0)}_{m_h} \cap \partial A_{m_h}} \big[\varphi(x, \nu_\Sigma) + g(x, 1)\big] d\mathcal{H}^1(x) + \int_{\overline{B_r(x_0)} \cap \Sigma \setminus \partial A_{m_h}} g(x, 0) d\mathcal{H}^1 \\
&\ge \int_{\overline{B_r(x_0)} \cap \Sigma \cap A^{(0)}_{m_h} \cap \partial A_{m_h}} g(x, 0) d\mathcal{H}^1(x) + \int_{\overline{B_r(x_0)} \cap \Sigma \setminus \partial A_{m_h}} g(x, 0) d\mathcal{H}^1 \\
&= \int_{\overline{B_r(x_0)} \cap \Sigma} g(x, 0) d\mathcal{H}^1,
\end{aligned}$$

where in the inequality we used (4.23). Thus, taking *limsup* as $h \to +\infty$ and using $\mu_h \rightharpoonup^* \mu_0$ we get

$$\mu_0(\overline{B_r(x_0)}) \ge \int_{B_r(x_0) \cap \Sigma} g(x, 0) d\mathcal{H}^1.$$

Now (4.26c) follows from the Besicovitch Differentiation Theorem.

Proofs of (4.26d) *and* (4.26e). We establish

$$\frac{d\mu_0}{d\mathcal{H}^1 \llcorner K} \geq 2\phi(x,\nu_K) \quad \text{for } \mathcal{H}^1\text{-a.e. } x \in K, \tag{4.27}$$

where

$$K = S^A_{\widetilde{u}} \cup (\Sigma \cap J_{\widetilde{u}}).$$

Let $x \in K$ be such that $\theta(K,x) = 1$. Then either $x \in S_{\widetilde{u}} \subset G^{(1)} \cap \partial^r G$ or $x \in \Sigma \cap J_{\widetilde{u}}$. By setting $E_0 := S$ and recalling that $\mathrm{Int}(A) = \cup_{i\in\mathbb{N}} E_i$, in view of Proposition 4.4 we have one of the following:

- (b1) there exists $i_0 \in N$ such that $x \in E^{(1)}_{i_0} \cap \partial^r E_{i_0}$, $\theta(E^{(1)}_{i_0} \cap \partial^r E_{i_0}, x) = 1$ and $u_{m_h} + a_{m_h} \to u$ a.e. in E_{i_0};
- (b2) there exist $i_0 \in N \cup \{0\}$ and $j_0 \in \mathbb{N} \setminus N$ such that $x \in \partial^* E_{i_0} \cap \partial^* E_{j_0}$ and $u_{m_h} + a_{m_h} \to u$ a.e. in E_{i_0} and $|u_{m_h} + a_{m_h}| \to \infty$ a.e. in E_{j_0};
- (b3) there exist $i_1, i_2 \in N \cup \{0\}$ with $i_1 \neq i_2$ such that $x \in \partial^* E_{i_1} \cap \partial^* E_{i_2}$ and $u_{m_h} + a_{m_h} \to u$ a.e. in $E_{i_1} \cup E_{i_2}$.

Let L denote the set among $E^{(1)}_{i_0} \cap \partial^r E_{i_0}$, $\partial^* E_{i_0} \cap \partial^* E_{j_0}$ and $\partial^* E_{i_1} \cap \partial^* E_{i_2}$ containing x. Without loss of generality we assume that $x \in Y \subset L$, where Y is defined as the set of points $y \in L \subset \partial A$ satisfying:

- (c1) $\theta(\partial G, y) = \theta(\partial A, y) = \theta(L, y) = 1$ and $\nu_G(y) = \nu_A(y) = \nu_L(y)$ exists. If $y \in \Sigma$, then additionally, $\theta(\Sigma, x) = 1$ and ν_Σ also exists;
- (c2) as $\rho \to 0$ the sets $\overline{Q_{R,\nu_L}(y)} \cap \sigma_{\rho,x}(\partial A)$, $\overline{Q_{R,\nu_L}(x)} \cap \sigma_{\rho,y}(\partial G)$ and $\overline{Q_{R,\nu_L}(y)} \cap \sigma_{\rho,y}(L)$ converge $\overline{Q_{R,\nu_L}(y)} \cap T_y$ in the Kuratowski sense, where $R > 0$ and T_y is the generalized tangent line to ∂A at y;
- (c3) one-sided traces $\widetilde{u}^+(y)$ and $\widetilde{u}^-(y)$ of $\widetilde{u}$ w.r.t. L exist and are not equal;
- (c4) $\frac{d\mu_0}{d\mathcal{H}^1 \llcorner K}(y)$ exists and is finite.

In fact, $\mathcal{H}^1(L \setminus Y) = 0$ since for (c1) we notice that $Y \subset L \subset \partial^r A$ and $\partial^r A$ is $\mathcal{H}^1$-rectifiable, for (c2) we use Proposition 4.3 by observing that the points of Σ and $\Omega \cap \partial A$ satisfy uniform density estimates in view of the Lipchitzianity of Σ and Proposition 4.1, respectively, for (c3) we use [21, Definition 2.4] and the existence of traces of GBD-functions along C^1-manifolds [21, Theorem 6.2] and the fact that being a jump set of $\widetilde{u}$, the set K (and also L) can be covered by at most countably many one-dimensional C^1-graphs (up to a $\mathcal{H}^1$-negligible set), and finally for (c4) we use Besicovitch Differentiation Theorem.

Without loss of generality, we assume $x = 0$, $\nu_K(x) = \mathbf{e_2}$, T_x is the x_1-axis and $\mathbf{e_2}$ is the outer normal of E_{i_0}.

Let $4r_0 := \mathrm{dist}(0, \partial\Omega)$ if $0 \in \Omega$ and $4r_0 := \mathrm{dist}(0, \partial\Sigma)$ if $0 \in \Sigma$; since Σ is Lipschitz, it consists of at most countably many open connected components in $\partial\Omega$, and hence, $r_0 > 0$. By weak convergence,

$$\lim_{h\to\infty} \mu_h(\overline{Q_r}) = \mu_0(Q_r) \tag{4.28}$$

for a.e. $r \in (0, r_0)$. By assumption (b3), [21, Definition 2.4] and [21, Remark 2.2] separately applied to $Q_1^+ := Q_1 \cap \{x_2 > 0\}$ and $Q_1 \setminus Q_1^+$ we have

$$\lim_{r\to 0} \int_{Q_1} |\tau(\widetilde{u}(rx)) - \tau(u_0(x))| dx = 0, \tag{4.29}$$

where

$$u_0 := \widetilde{u}^+(0)\chi_{Q_1^+} + \widetilde{u}^-(0)\chi_{Q_1 \setminus Q_1^+}$$

and

$$\tau(z) = (\arctan z_1, \arctan z_2), \quad z = (z_1, z_2) \in \mathbb{R}^2.$$

For every $r \in (0, r_0)$ let

$$U_r^\infty := \{x \in Q_1 : \liminf_{h\to\infty} |\widetilde{u}_h(rx)| = +\infty\}.$$

Unlike the proof of [45, Eq. 4.40g], (4.22) implies that U_r^∞ can have positive measure. By (4.21) and the Dominated Convergence Theorem

$$\lim_{h\to\infty} \int_{Q_1 \setminus U_r^\infty} |\tau(\widetilde{u}_h(rx)) - \tau(\widetilde{u}(rx))| dx = 0. \tag{4.30}$$

By (c2) applied with $R = 8$, Proposition 4.3 and (c1)-(c3)

$$Q_8 \cap \sigma_r(\partial A) \xrightarrow{K} I_8 \qquad \text{and} \qquad \mathcal{H}^1 \llcorner (Q_8 \cap \sigma_r(\partial A)) \overset{*}{\rightharpoonup} \mathcal{H}^1 \llcorner I_8,$$

$$Q_8 \cap \sigma_r(L) \xrightarrow{K} I_8 \quad \text{and} \quad \mathcal{H}^1 \llcorner (Q_8 \cap \sigma_r(L))) \overset{*}{\rightharpoonup} \mathcal{H}^1 \llcorner I_8$$

as $r \to 0$. Hence, by [45, Proposition A.5]

$$\mathrm{sdist}(\cdot, \sigma_r(\partial A)) \to -\mathrm{dist}(\cdot, T_0), \tag{4.31a}$$

$$\mathrm{sdist}(\cdot, \sigma_r(\partial E_{i_0})) \to -\mathrm{dist}(\cdot, T_0), \tag{4.31b}$$

$$\mathrm{sdist}(\cdot, \sigma_r(\partial [E_{i_0} \cup E_{j_0}])) \to -\mathrm{dist}(\cdot, T_0), \tag{4.31c}$$

$$\mathrm{sdist}(\cdot, \sigma_r(\partial [E_{i_1} \cup E_{i_2}])) \to -\mathrm{dist}(\cdot, T_0) \tag{4.31d}$$

locally uniformly in $\overline{Q_4}$ as $r \to 0$. Let

$$U_0^\infty = \begin{cases} \emptyset & \text{in cases (c1) and (c3)}, \\ Q_1^+ & \text{in case (c2)}. \end{cases}$$

By the definitions of E_{i_0}, E_{j_0}, E_{i_1} and E_{i_2} and (4.31b)-(4.31d)

$$\lim_{r\to 0} |U_r^\infty \Delta U_0^\infty| = 0. \tag{4.32}$$

Step 1: We choose sequences $h_k \nearrow \infty$ and $r_k \searrow 0$ as follows. By (4.28), (4.29), (4.31a) and (4.32) for any $k \in \mathbb{N}$ there exists $r_k \in (0, \frac{1}{k})$ such that (4.28) holds with $r = r_k$ and

$$\|\mathrm{sdist}(\cdot, \sigma_{r_k}(\partial A)) + \mathrm{dist}(\cdot, T_0)\|_{L^\infty(Q_4)} < \frac{1}{k^2}, \tag{4.33a}$$

$$\int_{Q_1} |\tau(\widetilde{u}(r_k x)) - \tau(u_0(x))| dx < \frac{1}{k^2}, \tag{4.33b}$$

$$|U_{r_k}^\infty \Delta U_0^\infty| < \frac{1}{k^2}. \tag{4.33c}$$

Given $k \geq 1$ and r_k, since A_{m_h} *sdist*-converges to A and the function τ is bounded, by (4.30), (4.33c) and (4.28) we can choose h_k such that

$$\frac{1}{h_k r_k} < \frac{1}{k}, \tag{4.34a}$$

$$\|\mathrm{sdist}(\cdot, \sigma_{r_k}(\partial A_{m_{h_k}})) - \mathrm{sdist}(\cdot, \sigma_{r_k}(\partial A))\|_{L^\infty(Q_4)} < \frac{1}{k}, \tag{4.34b}$$

$$\int_{Q_1 \setminus U_0^\infty} |\tau(\widetilde{u}_{h_k}(r_k x)) - \tau(\widetilde{u}(r_k x))| dx < \frac{1}{k}, \tag{4.34c}$$

$$\mu_{h_k}(\overline{Q_{r_k}}) \leq \mu_0(Q_{r_k}) + r_k^2. \tag{4.34d}$$

Notice that by (4.34a), $h_k \to \infty$ as $k \to \infty$.

Let

$$D_k := \sigma_{r_k}(A_{m_{h_k}} \cup S)$$

and

$$w_k(x) := \widetilde{u}_{h_k}(r_k x), \quad x \in Q_1.$$

Then the number of connected components of ∂D_k lying strictly inside Q_4 does not exceed m_{h_k}, and $w_k \in GSBD^2(Q_1)$ with $J_{w_k} \subset Q_1 \cap \partial D_k$. By (4.34b) and (4.33a),

$$\mathrm{sdist}(\cdot, \partial D_k) \to -\mathrm{dist}(\cdot, T_0) \quad \text{uniformly in } Q_4 \text{ as } k \to \infty.$$

Moreover, by (4.33b) and (4.34c) $w_k \to u_0$ a.e. in $Q_1 \setminus U_1^\infty$ and $|w_k| \to +\infty$ a.e. in U_1^∞. By the finiteness of

$$\frac{d\mu_0}{\mathcal{H}^1 \llcorner L}(0) = \lim_{k\to\infty} \frac{\mu_0(Q_{r_k})}{2r_k}$$

and (4.34d)

$$\mathcal{H}^1(Q_1 \cap \partial D_k) = \frac{\mathcal{H}^1(Q_{r_k} \cap \partial A_{m_{h_k}})}{r_k} \le \frac{\mu_{h_k}(\overline{Q_{r_k}})}{c_1 r_k} \le C := \frac{2}{c_1} \frac{d\mu_0}{\mathcal{H}^1 \llcorner L}(0) + 1 \tag{4.35}$$

for all large k. Moreover, by changing variables as $x = r_k y$ and using (4.20) we get

$$\int_{Q_1} |e(w_k)|^2 dx = \int_{Q_{r_k}} |e(\widetilde{u}_k)|^2 dy \le M$$

for all k; note that the first equality holds only in dimension two.

Fix $\delta \in (0,1)$. Since φ is uniformly continuous, there exists $k_\delta^0 > 0$ such that

$$|\varphi(x,\nu) - \varphi(0,\nu)| < \delta, \qquad x \in \overline{Q_{r_k}},\ \nu \in \mathbb{S}^1.$$

Therefore, by the definitions of D_k and μ_h, the nonnegativity of g as well as (4.35)

$$\frac{\mu_{h_k}(\overline{Q_{r_k}})}{r_k} \ge \int_{Q_1 \cap \partial^* D_k} \phi(\nu_{D_k}) d\mathcal{H}^1 + 2\int_{Q_1 \cap D_k^{(1)} \cap \partial D_k} \phi(\nu_{D_k}) d\mathcal{H}^1 - 2Cc_2\delta, \tag{4.36}$$

where

$$\phi(\nu) = \varphi(0,\nu).$$

By Lemma 4.7 applied with sequences $\{D_k\}$ and $\{m_{h_k}\}$ we find $k_\delta^2 > k_\delta^1$ such that

$$\int_{Q_1 \cap \partial^* D_k} \phi(\nu_{D_k}) d\mathcal{H}^1 + 2\int_{Q_1 \cap D_k^{(1)} \cap \partial D_k} \phi(\nu_{D_k}) d\mathcal{H}^1 \ge 2\int_{I_1} \phi(\mathbf{e_2}) d\mathcal{H}^1 - \delta.$$

Thus, by (4.36) and (4.34d) we get

$$\frac{\mu_0(Q_{r_k})}{2r_k} + \frac{r_k}{2} \ge \int_{I_1} \phi(\mathbf{e_2}) d\mathcal{H}^1 - \frac{2Cc_1 + 1}{2}\delta$$

for all $k > k_\delta^2$. Now letting first $k \to +\infty$ and then $\delta \to 0$ we get (4.27). □

5. Proof of the main results

The aim of this section is to prove theorems of Section 2.4. We start by showing that the volume-constraint infima of $\mathcal{F}$ in $\mathcal{C}$ and of $\widetilde{\mathcal{F}}$ in $\widetilde{\mathcal{C}}$ in fact coincide.

Proposition 5.1. *Assume hypotheses (H1)-(H3) and let* $\mathtt{v} \in (0, |\Omega|)$ *or* $S = \emptyset$. *Then*

$$\inf_{(A,u)\in\mathcal{C},\,|A|=\mathtt{v}} \mathcal{F}(A,u) \;=\; \inf_{(A,u)\in\widetilde{\mathcal{C}},\,|A|=\mathtt{v}} \widetilde{\mathcal{F}}(A,u) = \inf_{(A,u)\in\widetilde{\mathcal{C}}} \widetilde{\mathcal{F}}^\lambda(A,u) \tag{5.1}$$

for any $\lambda \ge \lambda_0$, *where* λ_0 *is given by* [45, Theorem 2.6] *and* $\widetilde{\mathcal{F}}^\lambda$ *is given by* (5.17).

Proof. We repeat similar arguments to [45, Section 5]. Note that for any $\lambda > 0$

$$\inf_{(A,u)\in\mathcal{C},\,|A|=\mathtt{v}} \mathcal{F}(A,u) \;\ge\; \inf_{(A,u)\in\widetilde{\mathcal{C}},\,|A|=\mathtt{v}} \widetilde{\mathcal{F}}(A,u) \ge \inf_{(A,u)\in\widetilde{\mathcal{C}}} \widetilde{\mathcal{F}}^\lambda(A,u). \tag{5.2}$$

Further we fix any $\lambda \ge \lambda_0$. Recall that from [45] for such λ

$$\inf_{(A,u)\in\mathcal{C},\,|A|=\mathtt{v}} \mathcal{F}(A,u) = \lim_{m\to+\infty} \min_{(A,u)\in\mathcal{C}_m,\,|A|=\mathtt{v}} \mathcal{F}(A,u) = \lim_{m\to+\infty} \min_{(A,u)\in\mathcal{C}_m} \mathcal{F}^\lambda(A,u),$$

where $\mathcal{F}^\lambda$ is given by (2.17). Thus, in view of (5.2) to prove (5.1) it is enough to establish that for $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ and $(A_\epsilon, u_\epsilon) \in \mathcal{C}_{n_\epsilon}$ such that

$$\inf_{(A,u)\in\widetilde{\mathcal{C}}} \widetilde{\mathcal{F}}^\lambda(A,u) + \epsilon > \mathcal{F}^\lambda(A_\epsilon, u_\epsilon). \tag{5.3}$$

To prove the existence of n_ϵ and $(A_\epsilon, u_\epsilon) \in \mathcal{C}_{n_\epsilon}$, we repeat essentially the same arguments of the proof of [45, Eq. 5.4]. For the convenience of the reader we give the detailed proof. Given $\epsilon > 0$ let $(B_1, v_1) \in \widetilde{\mathcal{C}}$ be such that

$$\inf_{(A,u)\in\widetilde{\mathcal{C}}} \mathcal{F}^\lambda(A,u) > \mathcal{F}^\lambda(B_1, v_1) - \epsilon. \tag{5.4}$$

Since $|B_1| = |\mathrm{Int}(B_1)|$ and $\mathcal{F}^\lambda(B_1, v_1) \ge \mathcal{F}^\lambda(\mathrm{Int}(B_1), v_1)$, we may assume that $B_1 = \mathrm{Int}(B_1)$, i.e., B_1 is open.

Step 1: First we remove the jump set J_{v_1} of v_1 on Σ making a hole in Ω. Recall that by our choice, ν_Σ is always directed towards Ω. Since Σ is Lipschitz, by the regularity of $\mathcal{H}^1 \llcorner \Sigma$, there exists a relatively open set $\Sigma' \subset \Sigma$ such that $\mathcal{H}^1(J_{v_1} \setminus \Sigma') = 0$ and $\mathcal{H}^1(\Sigma' \setminus J_{v_1}) < \frac{\epsilon}{c_2}$.

Let $r \in (0, \frac{\epsilon}{\lambda \mathcal{H}^1(\Sigma)})$ be such that

$$|\varphi(x,\nu) - \varphi(y,\nu)| < \frac{\epsilon}{\mathcal{H}^1(\Sigma)} \tag{5.5}$$

whenever $|x-y| < 4r$. Since Σ is Lipschitz, by Vitaly Covering Lemma we can find an at most countable family $\{Q_{r_j,\nu_\Sigma(x_j)}(x_j)\}_{j\ge1}$ of disjoint open squares such that $x_j \in \Sigma$, $r_j \in (0,r)$, $\Sigma \cap Q_{r,\nu_\Sigma(x_j)}(x_j)$ is a graph in $\nu_\Sigma(x_j)$-direction, Σ crosses two opposite sides of each $Q_{r,\nu_\Sigma(x_j)}(x_j)$ parallel to $\nu_\Sigma(x_j)$ and

$$\mathcal{H}^1\Big(\Sigma' \setminus \bigcup_j \overline{Q_{r_j,\nu_\Sigma(x_j)}(x_j)}\Big) = 0. \tag{5.6}$$

Note that $\sum\limits_j r_j < \mathcal{H}^1(\Sigma)$. For each j define

$$\Sigma_j := (\Sigma \cap \overline{Q_{r,\nu_\Sigma(x_j)}(x_j)}) + \rho_j \nu_\Sigma(x_j),$$

where $\rho_j \in (0, r_j)$ is such that Σ_j still connects two vertical sides of $Q_{r,\nu_\Sigma(x_j)}(x_j)$ and $\sum_j \rho_j < \frac{\epsilon}{2c_2}$. Let U_j be the open set whose boundaries are Σ_j, $\Sigma \cap \overline{Q_{r,\nu_\Sigma(x_j)}(x_j)}$ and two vertical sides of $Q_{r,\nu_\Sigma(x_j)}(x_j)$. Note that $\{U_j\}_j$ is a countable family of pairwise disjoint open sets.

Let $B_2 := B_1 \setminus \overline{\cup_j U_j}$ and $v_2 := v_1\Big|_{B_2\cup S}$. Then using the localized version of $\mathcal{S}$ we get

$$\mathcal{S}(B_2, v_2) \le \mathcal{S}(B_1, v_1 - u_0; \Omega \setminus \overline{\cup_j U_j}) + \sum_j \Big(\int_{\Sigma_j} \varphi(x, \nu_\Sigma(x)) d\mathcal{H}^1 + 2c_2\rho_j \Big). \tag{5.7}$$

By (5.5) and the definition of Σ_j

$$\int_{\Sigma_j} \varphi(x,\nu_\Sigma(x)) d\mathcal{H}^1 \le \int_{\Sigma\cap Q_{r,\nu_\Sigma(x_j)}(x_j)} \varphi(y, \nu_\Sigma(y)) d\mathcal{H}^1 + \frac{\epsilon \mathcal{H}^1(\Sigma \cap Q_{r,\nu_\Sigma(x_j)}(x_j))}{\mathcal{H}^1(\Sigma)}.$$

Thus summing this inequality in j and using pairwise disjointness of $Q_{r,\nu_\Sigma(x_j)}(x_j)$ and (5.6) we get

$$\sum_j \int_{\Sigma_j} \varphi(x,\nu_\Sigma(x)) d\mathcal{H}^1 \le \int_{\Sigma'} \varphi(y,\nu_\Sigma(y)) d\mathcal{H}^1 + \frac{\epsilon \mathcal{H}^1(\Sigma')}{\mathcal{H}^1(\Sigma)}.$$

Using the definition of Σ' we obtain

$$\sum_j \int_{\Sigma_j} \varphi(x, \nu_\Sigma(x))d\mathcal{H}^1 \le \int_{J_{v_1}} \varphi(y, \nu_\Sigma(y))d\mathcal{H}^1 + 2\epsilon.$$

Inserting this in (5.7) and using the inequality $\sum_j \rho_j < \frac{\epsilon}{2c_2}$ we get

$$\mathcal{S}(B_2, v_2) \le \mathcal{S}(B_1, v_1 - u_0; \Omega \setminus \overline{\cup_j U_j}) + \int_{J_{v_1}} \varphi(y, \nu_\Sigma(y))d\mathcal{H}^1 + 3\epsilon \le \mathcal{S}(B_1, v_1) + 3\epsilon.$$

Then by the nonnegativity of the elastic energy, for (B_2, v_2) we get

$$\widetilde{\mathcal{F}}(B_2, v_2) \le \widetilde{\mathcal{F}}(B_1, v_1) + 3\epsilon.$$

Notice that by our construction $\Sigma \cap J_{v_2}$ is $\mathcal{H}^1$-negligible, hence by Proposition A.3 $v_2 \in H^1_{\rm loc}(\mathrm{Int}(B_2 \cup S \cup \Sigma))$.

Finally we estimate the volume contribution of B_2. Since $U_j \subset Q_{r,\nu_\Sigma(x_j)}(x_j)$ and $r_j \le r < \frac{\epsilon}{\lambda \mathcal{H}^1(\Sigma)}$, using $\sum_j r_j < \mathcal{H}^1(\Sigma)$ we get

$$|B_1 \setminus B_2| \le \sum_j |U_j| \le \sum_j r_j^2 \le r \sum_j r_j < \frac{\epsilon}{\lambda}.$$

Therefore,

$$\widetilde{\mathcal{F}}^\lambda(B_1, v_1) \ge \widetilde{\mathcal{F}}^\lambda(B_2, v_2) - 4\epsilon. \tag{5.8}$$

Step 2: Let $\{E_i\}_{i\ge1}$ be all open connected components of B_2 (recall that B_2 is open). We remove all sufficiently small connected components of B_1. Using the localized versions of $\mathcal{S}$ and $\mathcal{W}$ we have

$$\mathcal{W}(B_2, v_2 - u_0; \Omega) = \sum_{i\ge1} \mathcal{W}(E_i, v_2 - u_0; \Omega).$$

Since $\partial E_i \cap \partial E_j \subset B_2^{(1)} \cap \partial B_2$ and $\varphi(x, \cdot)$ is even,

$$\mathcal{S}(B_2, v_2; \Omega) = \sum_{i\ge1} \mathcal{S}(E_i, v_2; \Omega).$$

Hence, there exists $N_1 \in \mathbb{N}$ such that the set $B_3 := \cup_{i=1}^{N_1} E_i$ satisfies

$$\mathcal{S}(B_2, v_2; \Omega) + \mathcal{W}(B_2, v_2 - u_0; \Omega) + \epsilon > \mathcal{S}(B_3, v_2; \Omega) + \mathcal{W}(B_3, v_2 - u_0; \Omega),$$
$$0 \le |B_2| - |B_3| < \frac{\epsilon}{\lambda}.$$

Thus,

$$\mathcal{F}^\lambda(B_2, v_2) > \mathcal{F}^\lambda(B_3, v_3) - 2\epsilon, \tag{5.9}$$

where $v_3 := v_2\Big|_{B_3}$.

Step 3: Let $\{F_j\}_{j\ge1}$ be all connected components of $\Omega \setminus \overline{B_3}$ such that $\partial F_j \subset \partial B_3$ (hence, F_i are holes in B_3). We fill in all sufficiently small holes. Since $\mathcal{S}(B_3, v) < +\infty$, there exists $N_2 \ge 1$ such that

$$\sum_{i>N_2} \mathcal{S}(F_i, v_3; \Omega) + \sum_{i>N_2} \mathcal{W}(F_i, v_3 - u_0; \Omega) < \epsilon, \quad \sum_{i>N_2} |F_i| < \frac{\epsilon}{\lambda}.$$

Then the set $B_4 := B_3 \cup (\cup_{i>N_2} F_i)$ and the function $v_4 := v_3 \chi_{B_2 \cup S} + u_0 \chi_{\cup_{i>N_2} F_i}$ satisfies

$$\mathcal{F}^\lambda(B_3, v_3) > \mathcal{F}^\lambda(B_4, v_4) - 2\epsilon. \tag{5.10}$$

By construction, $\overline{\partial^* B_4}$ has at most $N_1 + N_2$ connected components.

Step 5: Finally we construct $(A_\epsilon, u_\epsilon) \in \mathcal{C}_{n_\epsilon}$ satisfying (5.3) for some $n_\epsilon \in \mathbb{N}$. Let $B_5 := \mathrm{Int}(\overline{B_4})$. Since B_4 can have finitely many "substantial" holes $B_5 \cap \partial B_4 = \emptyset$. In particular, if we extend v_4 arbitrarily to the set $B_4^{(1)}\cap\partial B_4$ and denote the extension by v_5, then $v_5\in GSBD^2(\mathrm{Int}(B_5\cup S\cup\Sigma))$ and $J_{v_5}=S_{v_4}^{B_4}$ up to a $\mathcal{H}^1$-negligible set, where S_u^A is defined in (2.6). Since $v_5=v_4$ a.e. in B_5, by (5.4)-(5.10)

$$\int_{B_5\cup S}\mathbb{C}(x)e(v_5):e(v_5)=\mathcal{W}(B_4,v_4)\le\widetilde{\mathcal{F}}(B_4,v_4)+c_2\mathcal{H}^1(\Sigma)\le C+9\epsilon,$$

where $C:=\max\{1,\inf_{\widetilde{\mathcal{C}}}\widetilde{\mathcal{F}}\}$ is independent of ϵ.

By [13, Theorem 1.1] there exists $u_\epsilon\in SBV^2(\mathrm{Int}(B_5\cup S\cup\Sigma))\cap L^\infty(\mathrm{Int}(B_5\cup S\cup\Sigma))$ such that J_{u_ϵ} is contained in a union Γ of finitely many closed connected pieces of C^1-curves in $\mathrm{Int}(B_5\cup S\cup\Sigma)$, $u_\epsilon\in W^{1,\infty}(\mathrm{Int}(B_5\cup S\cup\Sigma))$ and

$$\int_{B_5\cup S}|e(u_\epsilon)-e(v_5)|^2dx\le\frac{\epsilon}{4(C+11\epsilon)(\|\mathbb{C}\|_\infty+1)}\tag{5.11}$$

and

$$\mathcal{H}^1(J_{u_\epsilon}\Delta J_{v_5})<\frac{\epsilon}{2c_2}.\tag{5.12}$$

Since $J_{v_5}\subset B_5$, we can assume that the squares $\{Q_j\}_{j\ge1}$ of Vitali cover in [13, Eq. 4.3a] satisfies $Q_j\subset\subset B_5$. Therefore, we may assume that $\Gamma\subset\overline{B_5}$. Since $\mathcal{H}^1\llcorner\Gamma$ is regular, we may extract finitely many intervals of Γ whose union Γ' still covers J_{u_ϵ} and satisfies $\mathcal{H}^1(\Gamma'\setminus J_{u_\epsilon})<\frac{\epsilon}{2c_2}$. Now we define

$$A_\epsilon:=B_5\setminus\overline{\Gamma'}.$$

Recall that both $\Sigma\cap J_{v_5}$ and $\Sigma\cap J_{e-\epsilon}$ are $\mathcal{H}^1$-negligible. By the definition of B_5 and Γ', there exists $n_\epsilon\in\mathbb{N}$ such that $(A_\epsilon,u_\epsilon)\in\mathcal{C}_{n_\epsilon}$. By the definition of $\widetilde{\mathcal{S}}$, B_5 and v_5 as well as by (5.12) we have

$$\begin{aligned}\widetilde{\mathcal{S}}(B_4,v_4)=&\int_{\Omega\cap\partial^*B_5}\varphi(x,\nu_{B_5})d\mathcal{H}^1+2\int_{B_5\cap J_{v_5}}\varphi(x,\nu_{J_{v_5}})d\mathcal{H}^1+\int_{\Sigma\cap\partial^*B_5}\beta d\mathcal{H}^1\\ \ge&\int_{\Omega\cap\partial^*B_5}\varphi(x,\nu_{B_5})d\mathcal{H}^1+2\int_{B_5\cap J_{u_\epsilon}}\varphi(x,\nu_{J_{u_\epsilon}})d\mathcal{H}^1+\int_{\Sigma\cap\partial^*B_5}\beta d\mathcal{H}^1-\epsilon.\end{aligned}$$

Thus, by the definition of A_ϵ and Γ'

$$\begin{aligned}\widetilde{\mathcal{S}}(B_4,v_4)\ge&\int_{\Omega\cap\partial^*A_\epsilon}\varphi(x,\nu_{A_\epsilon})d\mathcal{H}^1+2\int_{A_\epsilon^{(1)}\cap\Gamma'}\varphi(x,\nu_{\Gamma'})d\mathcal{H}^1+\int_{\Sigma\cap\partial^*A_\epsilon}\beta d\mathcal{H}^1-2\epsilon\\ =&\mathcal{S}(A_\epsilon,u_\epsilon)-2\epsilon.\end{aligned}\tag{5.13}$$

Moreover, using the relations $|A_\epsilon\Delta B_4|=0$ and $v_4=v_5$ a.e. in B_5 and Cauchy-Schwartz inequality for nonnegative symmetric forms we obtain

$$\begin{aligned}\mathcal{W}(A_\epsilon,u_\epsilon)\le&\mathcal{W}(B_4,v_4)+2\int_{B_5\cup S}\mathbb{C}(x)e(u_\epsilon):(e(u_\epsilon)-e(v_5))\\ \le&\mathcal{W}(B_4,v_4)+2\Big(\int_{B_5\cup S}\mathbb{C}(x)e(u_\epsilon):e(u_\epsilon)dx\Big)^{1/2}\times\\ &\times\Big(\int_{B_5\cup S}\mathbb{C}(x)(e(u_\epsilon)-e(v_5)):(e(u_\epsilon)-e(v_5))dx\Big)^{1/2}.\end{aligned}\tag{5.14}$$

Similarly,

$$\begin{aligned}&\int_{B_4\cup S}\mathbb{C}(x)e(u_\epsilon):e(u_\epsilon)dx\\ \le&\mathcal{W}(B_4,v_4)+2\Big(\mathcal{W}(B_4,v_4)\Big)^{1/2}\Big(\int_{B_5\cup S}\mathbb{C}(x)(e(u_\epsilon)-e(v_5)):(e(u_\epsilon)-e(v_5))dx\Big)^{1/2}\\ \le&(C+9\epsilon)+2\sqrt{(C+9\epsilon)\|\mathbb{C}\|_\infty}\,\|e(u_\epsilon)-e(v_5)\|_{L^2}\le C+10\epsilon,\end{aligned}$$

where in the last inequality we used (5.11). Therefore, by (5.14) and again by (5.11)

$$\mathcal{W}(A_\epsilon,u_\epsilon) \le \mathcal{W}(B_4,v_4) + 2\sqrt{(C+10\epsilon)\|\mathbb{C}\|_\infty}\ \|e(u_\epsilon)-e(v_5)\|_{L^2} \le \mathcal{W}(B_4,v_4)+\epsilon. \tag{5.15}$$

Now combining (5.13) and (5.15) as well as using $|B_5| = |A_\epsilon|$ we get

$$\widetilde{\mathcal{F}}^\lambda(B_4,v_4) \ge \mathcal{F}^\lambda(A_\epsilon,u_\epsilon) - 3\epsilon. \tag{5.16}$$

Since $(A_\epsilon,u_\epsilon)\in\mathcal{C}_{n_\epsilon}$, by (5.4), (5.8), (5.9), (5.10) and (5.16) we get

$$\inf_{(A,u)\in\widetilde{\mathcal{C}}} \widetilde{\mathcal{F}}(A,u) + 12\epsilon \ge \mathcal{F}(A_\epsilon,u_\epsilon),$$

and (5.3) follows. □

Proposition 5.1 implies that the configuration (A,u) given by Proposition 4.6 is a volume-constraint minimizer of $\widetilde{\mathcal{F}}$ in $\widetilde{\mathcal{C}}$.

Proposition 5.2. *Let $(A,u)\in\widetilde{\mathcal{C}}$ be given by Proposition 4.6. Then $(\mathrm{Int}(A),u)$ is a minimizer of $\widetilde{\mathcal{F}}$ in $\widetilde{\mathcal{C}}$ under the volume constraint $|A| = \mathtt{v}$. Moreover, let λ_0 be as in Proposition 5.1 and let $(\widetilde{A},\widetilde{u})\in\widetilde{\mathcal{C}}$ be any volume-constraint minimizer of $\widetilde{\mathcal{F}}$. Then $(\widetilde{A},\widetilde{u})$ is a minimizer of $\widetilde{\mathcal{F}}^\lambda$ for all $\lambda\ge\lambda_0$, where*

$$\widetilde{\mathcal{F}}^\lambda(B,v) := \widetilde{\mathcal{F}}(B,v) + \lambda\big||B|-\mathtt{v}\big|, \quad (B,v)\in\widetilde{\mathcal{C}}, \quad \lambda>0. \tag{5.17}$$

Proof. Note that since $|\mathrm{Int}(A)\Delta A| = 0$ and $(\mathrm{Int}(A),u)\in\widetilde{\mathcal{C}}$, by Propositions 4.5, 4.6 and 5.1

$$\widetilde{\mathcal{F}}(\mathrm{Int}(A),u) = \inf_{(B,v)\in\mathcal{C}} \mathcal{F}(B,v) = \inf_{(B,v)\in\widetilde{\mathcal{C}}} \widetilde{\mathcal{F}}(B,v) = \inf_{(B,v)\in\mathcal{C}} \widetilde{\mathcal{F}}^\lambda(B,v)$$

for all $\lambda\ge\lambda_0$. Thus, $(\mathrm{Int}(A),u)$ is a minimizer of both $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}^{\lambda_0}$. The same is true for every minimizer (B,v) of $\widetilde{\mathcal{F}}$. □

Theorem 5.3 (Density estimates for minimizers of $\widetilde{\mathcal{F}}^\lambda$). *Given $\lambda>0$, let $(A,u)\in\widetilde{\mathcal{C}}$ be any minimizer of $\widetilde{\mathcal{F}}^\lambda(\cdot,\cdot)$ in $\widetilde{\mathcal{C}}$ and let $\xi\in\mathbb{R}^2$ be such that for the function*

$$\widetilde{u} := u\chi_{A\cup S} + \xi\chi_{\Omega\setminus A}$$

one has $\Omega\cap\partial^*A\subset J_{\widetilde{u}}$. Then for any $x\in\Omega$ and $r\in(0,\mathrm{dist}(x,\partial\Omega))$,

$$\frac{\mathcal{H}^1(Q_r(x)\cap J_{\widetilde{u}})}{r} \le \frac{16c_2+4\lambda}{c_1}. \tag{5.18}$$

Moreover, there exist $\varsigma^* = \varsigma^*\in(0,1)$ and $R^*>0$ not depending on (A,u) with the following property. If $x\in\Omega$ belongs to the closure $J_{\widetilde{u}}^c$ of the set $\{y\in\Omega\cap J_{\widetilde{u}}:\ \theta_*(J_{\widetilde{u}},y)>0\}$, then

$$\frac{\mathcal{H}^1(Q_r(x)\cap J_{\widetilde{u}})}{r} \ge \varsigma^* \tag{5.19}$$

for any square $Q_r(x)\subset\subset\Omega$ with $r\in(0,\min\{R^*,\mathrm{dist}(x,\partial\Omega)\})$, and if $x\in\Omega$ belongs to the closure S_u^c of $\{x\in S_u^A:\ \theta_*(S_u^A,x)>0\}$, then

$$\frac{\mathcal{H}^1(Q_r(x)\cap S_u^A)}{r} \ge \varsigma^* \tag{5.20}$$

for any $r\in(0,\min\{R^*,\mathrm{dist}(x,\overline{\partial^*A})\}$. In particular,

$$\mathcal{H}^1(\Omega\cap(J_{\widetilde{u}}^c\setminus J_{\widetilde{u}}) = \mathcal{H}^1(\mathrm{Int}(A^{(1)})\cap(S_u^c\setminus S_u^A) = 0. \tag{5.21}$$

Proof of Theorem 5.3. As in Remark 2.5 (A,u) is a minimizer of $\widetilde{\mathcal{F}}^\lambda$ if and only if $(A,u+u_0)$ minimizes the $\widehat{\widetilde{\mathcal{F}}^\lambda}(\cdot) := \widetilde{\mathcal{F}}^\lambda(\cdot - u_0)$. Thus, we can introduce the following localized version of $\widetilde{\mathcal{F}}$ in open subsets O of Ω which does not see the substrate:

$$\widetilde{\mathcal{F}}(B,v;O) := \widetilde{\mathcal{S}}(B;O) + \mathcal{W}(B,v;O)$$

where

$$\widetilde{\mathcal{S}}(B,v;O) := \int_{O\cap\partial^* B}\varphi(y,\nu_B)d\mathcal{H}^1 + 2\int_{O\cap B^{(1)}\cap\partial B\cap S_v}\varphi(y,\nu_B)d\mathcal{H}^1,$$

the $\mathcal{W}(\cdot;O)$ is given as in (2.9) and S_v^A is defined as in (2.6). Then the minimality of (A,u) implies that $(A,u+u_0)$ is a quasi-minimizer of $\widetilde{\mathcal{F}}(\cdot;O)$ in O, namely,

$$\widetilde{\mathcal{F}}(A,u+u_0;O) \le \widetilde{\mathcal{F}}(B,v;O) + \lambda_0|A\Delta B|$$

whenever $(B,v)\in\widetilde{\mathcal{C}}$ with $A\Delta B\subset\subset O$ and $\operatorname{supp}(u+u_0-v)\subset\subset O$. Now the proof of the existence of ς^* and R^* satisfying (5.18) and (5.19) runs along the same lines of the proof of Theorem 3.1 for $m=\infty$, therefore, we do not repeat it here. Note that ς^* and R^* depend only on c_i and λ.

Let $A_\circ := \operatorname{Int}(A^{(1)})$. We claim that

$$\partial A_\circ = \overline{\partial^* A}.$$

Indeed, note that $A^{(1)}\setminus A_\circ \subset \partial A^{(1)} = \overline{\partial^* A}$, where in the equality we used $\overline{\partial^* A} = \overline{\partial^* A^{(1)}} = \partial A^{(1)}$ see e.g., [52, Eq. 15.3]. Thus, $A_\circ$ is also equivalent to A, and hence, $\partial^* A_\circ = \partial^* A = \partial^* A^{(1)}$. In particular, $\partial A^{(1)} = \overline{\partial^* A_\circ} \subset \partial A_\circ$. On the other hand, assume that there exists $x\in\partial A_\circ\setminus\partial A^{(1)}$. Since $\partial A^{(1)}$ is closed, there exists $r>0$ such that $\overline{B_r(x)}\cap\partial A^{(1)}=\emptyset$. Hence, either $B_r(x)\subset\operatorname{Int}(A^{(1)}) = A_\circ$ or $\overline{B_r(x)}\cap\overline{A^{(1)}}=\emptyset$. Since $A_\circ$ is open and $x\in\partial A_\circ$, the inclusion $B_r(x)\subset A_\circ$ is not possible. On the other hand, since $\overline{A_\circ}\subset\overline{A^{(1)}}$ and $x\in\partial A_\circ$, the relation $\overline{B_r(x)}\cap\overline{A^{(1)}}=\emptyset$ is also not possible. Thus, $\partial A_\circ\subseteq\partial A^{(1)}$.

To prove (5.20) we fix $\Omega'\subset\subset\Omega$. We claim that $\widetilde{u}\big|_{A_\circ}$ is a minimizer of Griffith functional $\mathcal{G}: GSBD^2(\operatorname{Int}(A_\circ\cup S\cup\Sigma))\to\mathbb{R}$,

$$\mathcal{G}(v) := \int_{A_\circ\cap J_v}\varphi(x,\nu_{J_v})d\mathcal{H}^1 + \int_{A_\circ}\mathbb{C}(x)e(v):e(v)dx$$

with Dirichlet boundary condition $v=\widetilde{u}=u$ in $A_\circ\setminus\Omega'$. Indeed, for every $v\in GSBD^2(A_\circ)$ with $\widetilde{u}=v$ in $A_\circ\setminus\Omega'$ we define $B:=A_\circ\setminus\overline{J_v}$. Then $(B,v)\in\widetilde{\mathcal{C}}$ and by the minimality of (A,u)

$$\mathcal{G}(u)-\mathcal{G}(v) = \widetilde{\mathcal{F}}(A,u)-\widetilde{\mathcal{F}}(B,v)\le 0.$$

Since $S_v^B = J_{\widetilde{u}\big|_{A_\circ}}$ up to a $\mathcal{H}^1$-negligible set, (5.20) directly follows from the density estimates for the jump set of Griffith minimizers (see e.g. [12]) with possibly smaller $\varsigma^*\in(0,1)$ and $R^*>0$.

Finally, we prove (5.21) only for S_u^A, the other being similar. Let $\Gamma := \{x\in S_u^A : \theta_*(S_u^A,x)>0\}$. Note that $S_u^c=\overline{\Gamma}$.

We claim that

$$\mathcal{H}^1(A_\circ\cap(\overline{\Gamma}\setminus\Gamma)) = 0. \tag{5.22}$$

Indeed, let $\mu := \mathcal{H}^1\llcorner\Gamma$. Then $\mu(\overline{\Gamma}\setminus\Gamma)=0$. By the regularity of μ, for every $\epsilon>0$ there exists an open set $U\subset\mathbb{R}^2$ such that $L:=A_\circ\cap(\overline{\Gamma}\setminus\Gamma)\subset U$ and $\mu(U)=\mathcal{H}^1(U\cap\Gamma)<\epsilon$. Note that $\overline{\Gamma}\subset\overline{\{y\in\Omega\cap J_{\widetilde{u}}:\,\theta_*(J_{\widetilde{u}},x)>0\}}$, where $\widetilde{u}$ is given by Theorem 5.3. Hence, for (5.19) holds for all points of $\overline{\Gamma}$. By the definition of the closure, and Vitali Covering Lemma we can find at most countable pairwise disjoint family $\{\overline{B_{r_i}(x_i)}\}_i$ of closed balls $\overline{B_{r_i}(x_i)}$ with $x_i\in A_\circ\cap\Gamma$, $r_i\le\min\{R^*,\epsilon,\operatorname{dist}(x,\partial\overline{A})\}$ such that $A_\circ\cap(\overline{\Gamma}\setminus\Gamma)\subset\cup_i\overline{B_{5r_i}(x_i)}$. Without loss of generality we may assume that $B_{r_i}(x_i) \subset U$. Since $Q_{r_i/\sqrt{2}}(x_i) \subset B_{r_i}(x_i) \subset Q_{r_i}(x_i)$, from the definition of Hausdorff premeasure, (5.19) and disjointness of $\{B_{r_i}(x_i)\}$ as well as the choice of U we obtain

$$\begin{aligned}\mathcal{H}_{10\epsilon}(A_\circ \cap (\overline{\Gamma} \setminus \Gamma)) \le & \sum_{i\ge1} 2\pi(5r_i) \le \frac{10\pi\sqrt{2}}{\varsigma^*} \sum_{i\le1} \mathcal{H}^1(Q_{r_i/\sqrt{2}}(x_i) \cap \Gamma) \\ = & \frac{10\pi\sqrt{2}}{\varsigma^*} \mathcal{H}^1\big(\cup_i Q_{r_i/\sqrt{2}}(x_i) \cap \Gamma\big) \le \frac{10\pi\sqrt{2}}{\varsigma^*} \mathcal{H}^1(U \cap \Gamma) < \frac{10\pi\sqrt{2}\epsilon}{\varsigma^*}.\end{aligned}$$

Now letting $\epsilon \to 0$ we get (5.22). □

In the following proposition we construct a "regular" minimizer of $\mathcal{F}$ starting from a minimizer of $\widetilde{\mathcal{F}}$ in $\widetilde{\mathcal{C}}$.

Proposition 5.4. *Given $\lambda > 0$, let $(A, u) \in \widetilde{\mathcal{C}}$ be any minimizer of $\widetilde{\mathcal{F}}^\lambda$. Define*

$$A' := \mathrm{Int}(A^{(1)}) \setminus \overline{\Gamma},$$

where $\Gamma := \{x \in S_u^A : \theta_*(S_u^A, x) > 0\}$, and, with a slight abuse of notation, consider u as defined in $A' \cup S$ (and so, also on the $\mathcal{L}^2$-negligible set $A' \setminus \mathrm{Int}(A)$). Then $(A', u) \in \mathcal{C}$ is such that $\widetilde{\mathcal{F}}(A, u) = \mathcal{F}(A', u)$ and satisfy the following assertions:

(1) A' is open, $\theta_*(S_u^{A'}, x) > 0$ for all $x \in S_u^{A'}$, $|A\Delta A'| = 0$ and $u\chi_{A\cup S} = u\chi_{A'\cup S}$ a.e. in $\Omega \cup S$.
(2) The closure of $A'^{(1)} \cap \partial A'$ coincide with $\overline{S_u^{A'}}$ and $\mathcal{H}^1(\overline{S_u^{A'}} \setminus S_u^{A'}) = 0$.
(3) Let $\varsigma*$ and R^* be given by Theorem 5.3. Then

$$\frac{\mathcal{H}^1(Q_r(x) \cap \partial A')}{r} \le \frac{16c_2 + 4\lambda_0}{c_1} \tag{5.23}$$

for every square $Q_r(x) \subset \Omega$ and

$$\frac{\mathcal{H}^1(Q_r(x) \cap \partial A')}{r} \ge \varsigma* \tag{5.24}$$

for every $Q_r(x) \subset \Omega$ with for any $x \in \partial A'$ and $r \in (0, R^*)$.

Proof. Note that by definition A' is open and $|A'\Delta A| = 0$. Moreover, $S_u^{A'} \subset \Gamma$, and by (5.20) all points of $\Omega \cap \overline{\Gamma}$ satisfy uniform lower density estimates, hence, $\theta_*(S_u^{A'}, x) > 0$ for any $x \in S_u^{A'}$.

We claim that $A' \in \mathcal{A}$. Indeed, let $\widetilde{u}$ be given as in Theorem 5.3. By definition

$$\Omega \cap J_{\widetilde{u}}^c = \Omega \cap \partial A' \quad \text{and} \quad \partial A' \subset J_{\widetilde{u}}^c \cup \Sigma, \tag{5.25}$$

where $J_{\widetilde{u}}^c$ is the closure of the set $\{x \in J_{\widetilde{u}} : \theta_*(J_{\widetilde{u}}, x) > 0\}$. Since $J_{\widetilde{u}}$ is $\mathcal{H}^1$-rectifiable, so is $J_{\widetilde{u}}^c$ in view of (5.21). Therefore, $\partial A'$ is $\mathcal{H}^1$-rectifiable, i.e., $A' \in \mathcal{A}$. Note that by construction $\mathcal{H}^1(A' \cap J_{\widetilde{u}}) = 0$ hence, by Proposition A.3 $\widetilde{u} \in H^1_{\mathrm{loc}}(A')$ and, since $u = \widetilde{u}$ a.e. in A' it follows that $u \in H^1_{\mathrm{loc}}(A')$.

Since $|A\Delta A'| = 0$ and $u = u$ a.e. in A', it follows that

$$\mathcal{W}(A, u) = \mathcal{W}(A', u).$$

Moreover, by the definition of Γ and S_u^A,

$$|\mathcal{S}(A', u) - \widetilde{\mathcal{S}}(A, u)| = \int_{\mathrm{Int}(A^{(1)})\cap(\overline{\Gamma}\setminus S_u^A)} \varphi(x, \nu_\Gamma) d\mathcal{H}^1 \le c_2\mathcal{H}^1(\mathrm{Int}(A^{(1)}) \cap (\overline{\Gamma} \setminus \Gamma)) = 0,$$

where in the last equality we used (5.21). Finally, (5.23) and (5.24) follows from (5.25) and density estimates of Theorem 5.3. □

Now we are ready to prove the existence of global minimizers of $\mathcal{F}$.

Proof of Theorem 2.6. First we prove the assertion for $\mathcal{G} = \mathcal{F}$.

Let $(A_m, u_m) \in \mathcal{C}_m$ be a minimizer of $\mathcal{F}$ satisfying the volume constraint $|A_m| = \mathtt{v}$ and let (A_{m_h}, u_{m_h}), and A and u be as in Proposition 4.6. By (3.1), (4.13) and (4.12) we have

$$\inf_{(B,v)\in\mathcal{C},\,|B|=\mathtt{v}} \mathcal{F}(B,v) = \lim_{h\to+\infty} \mathcal{F}(A_{m_h}, u_{m_h}) \geq \widetilde{\mathcal{F}}(\mathrm{Int}(A), u).$$

Since $|\mathrm{Int}(A)| = \mathtt{v}$, by Propositions 5.1 and 5.2

$$\inf_{(B,v)\in\mathcal{C},\,|B|=\mathtt{v}} \mathcal{F}(B,v) = \inf_{(B,v)\in\widetilde{\mathcal{C}},\,|B|=\mathtt{v}} \widetilde{\mathcal{F}}^{\lambda_0}(B,v) = \widetilde{\mathcal{F}}^{\lambda_0}(\mathrm{Int}(A),u) = \widetilde{\mathcal{F}}(\mathrm{Int}(A),u), \tag{5.26}$$

hence, $(\mathrm{Int}(A), u)$ is a minimizer of $\widetilde{\mathcal{F}}^{\lambda_0}$ in $\widetilde{\mathcal{C}}$. Then by Proposition 5.4 there exists $(A', u) \in \mathcal{C}$ such that

$$\widetilde{\mathcal{F}}(\mathrm{Int}(A), u) = \mathcal{F}(A', u),$$

and hence, in view of (5.26), (A', u) is a solution to (2.16).

The proof of the second assertion (i.e., the existence of λ_1 for which the set of minimizers in $\mathcal{C}$ of both $\mathcal{F}$ and $\mathcal{F}^\lambda$ coincide for all $\lambda \geq \lambda_1$) can be done using the first one and also following the arguments of [32, Theorem 1.1] and [45, Proposition A.6]. Without loss of generality we assume that $\lambda_1 \geq \lambda_0$, where λ_0 is given by Proposition 5.1.

Now we prove Theorem 2.6 for $\mathcal{G} = \widetilde{\mathcal{F}}$. We have already shown above that the configuration $(\mathrm{Int}(A), u)$ given by Proposition 4.6 solves the minimum problem (2.16) with $\mathcal{G} = \widetilde{\mathcal{F}}$. In view of (5.1) every volume-constraint minimizer of $\widetilde{\mathcal{F}}$ also minimizer of $\widetilde{\mathcal{F}}^\lambda$ for all $\lambda \geq \lambda_1$. To prove the converse assertion, we fix any minimizer $(A, u) \in \widetilde{\mathcal{C}}$ of $\widetilde{\mathcal{F}}^\lambda$ for $\lambda \geq \lambda_1$. By Proposition 5.4 there exists $(A', u) \in \mathcal{C}$ such that $|A'| = |A|$ and $\mathcal{F}(A', u) = \widetilde{\mathcal{F}}(A, u)$. By the first part of the proof and (5.1) we know that

$$\inf_{(B,v)\in\mathcal{C}} \mathcal{F}^\lambda(B,v) = \inf_{(B,v)\in\mathcal{C},|B|=\mathtt{v}} \mathcal{F}(B,v) = \inf_{(B,v)\in\widetilde{\mathcal{C}}} \widetilde{\mathcal{F}}^\lambda(B,v) = \mathcal{F}^\lambda(A',u).$$

Hence, (A', u) is the minimizer of $\mathcal{F}^\lambda$. Since $\lambda \geq \lambda_1$ according to the first part of the proof, $|A'| = \mathtt{v}$. Hence, $|A| = \mathtt{v}$ and (A, u) minimizer of (2.16). □

We are ready now to study the properties of the minimizers of $\mathcal{F}$ in $\mathcal{C}$ provided by Theorem 2.6.

Proof of Theorem 2.7. First we properties (1)-(4) the assertion for $\mathcal{G} = \widetilde{\mathcal{F}}$.

Consider any solution $(A, u) \in \widetilde{\mathcal{C}}$ of (2.16). By Proposition 5.4 there exists a $(A', u) \in \mathcal{C}$ with A' defined as in (2.18), such that the properties (1)-(4) hold except the conditions $\mathcal{H}^1(\partial A \Delta \partial A') = 0$ and $\mathcal{H}^1(S_u^A \Delta S_u^{A'}) = 0$ of (1). To prove these two equations it is enough to observe that

$$0 = |\mathcal{F}(A,u) - \mathcal{F}(A',u)| = 2\int_{A^{(1)}\cap(\partial A\Delta\partial A')} \varphi(x,\nu_A)\,d\mathcal{H}^1$$

and

$$0 = |\widetilde{\mathcal{F}}(A,u) - \widetilde{\mathcal{F}}(A',u)| = 2\int_{A^{(1)}\cap(S_u^A\Delta S_u^{A'})} \varphi(x,\nu_A)\,d\mathcal{H}^1.$$

Now we assume that $\mathcal{G} = \mathcal{F}$ and let $(A, u) \in \mathcal{C}$ be a solution to (2.16). Since $(A, u) \in \widetilde{\mathcal{C}}$, by Proposition 5.1

$$\inf_{(B,v)\in\widetilde{\mathcal{C}},|B|=\mathtt{v}} \widetilde{\mathcal{F}}(B,v) = \mathcal{F}(A,u) \geq \widetilde{\mathcal{F}}(A,u).$$

Therefore, (A, u) is also a volume-constraint minimizer of $\widetilde{\mathcal{F}}$. Thus, applying first part of the proof we establish that $(A', u) \in \mathcal{C}$ satisfies (1)-(4).

Finally, notice that if $E \subset A'$ is a connected component of A' with $\mathcal{H}^1(\partial E \cap \Sigma \setminus J_u) = 0$, then for (A', v) with $v = u\chi_{(A\cup S)\setminus E} + (u_0 + a)\chi_E$, where a is any rigid displacement, we have

$$\mathcal{S}(A', u) \geq \mathcal{S}(A', v)$$

and

$$\mathcal{W}(A', u) \geq \mathcal{W}(A', v), \tag{5.27}$$

where in (5.27) equality holds if and only of $u = u_0 + a$ in E. Therefore, by the minimality of (A', u) it follows that $u = u_0 + a$ in E. It remains to prove

$$|E| \geq 4\pi \left(\frac{c_1}{\lambda_0}\right)^2. \tag{5.28}$$

Consider the competitor $(A' \setminus E, u) \in \mathcal{C}$. By minimality and Theorem 2.6, $\mathcal{F}^{\lambda_1}(A', u) \leq \mathcal{F}^{\lambda_1}(A' \setminus E, u)$, so that by (5.27) and the additivity of the surface energy, $\mathcal{S}(E, u) \leq \lambda_1 |E|$. Then by (2.13) and the isoperimetric inequality in $\mathbb{R}^2$

$$\lambda_1 |E| \geq c_1 \mathcal{H}^1(\partial E) \geq c_1 \sqrt{4\pi} |E|^{1/2}.$$

Hence, (5.28) follows. □

Appendix A.

We include in this section auxiliary results used in the paper for the convenience of the Reader. We begin by a property satisfied by the free-crystal regions in $\mathcal{A}$ and $\widetilde{\mathcal{A}}$.

Proposition A.1. *Let $A \subset \mathbb{R}^2$ be a bounded $\mathcal{L}^2$-measurable set with $\mathcal{H}^1(\partial A) < +\infty$. Then A is a set of finite perimeter in $\mathbb{R}^2$.*

Proof. Since $A \Delta \mathrm{Int}(\overline{A}) \subset \overline{A} \setminus \mathrm{Int}(A) = \partial A$, we have $|A \Delta \mathrm{Int}(\overline{A})| \leq |\partial A| = 0$, and hence, it suffices to prove that the open set $E := \mathrm{Int}(\overline{A})$ has finite perimeter in $\mathbb{R}^2$. Note that by construction, $\partial E \subset \partial A$ and $\mathcal{H}^1(\partial E) \leq \mathcal{H}^1(\partial A) < +\infty$.

We divide the proof of $E \in BV(\mathbb{R}^2, \{0,1\})$ into three steps.

Step 1. We claim that if E is simply connected, then $E \in BV(\mathbb{R}^2; \{0,1\})$. Indeed, in this case ∂E is a connected compact set with $\mathcal{H}^1(\partial E) \leq \mathcal{H}^1(\partial A) < +\infty$ and by [33, Lemma 3.12] it contains a closed curve Γ enclosing $\overline{E}$. Since $\mathcal{H}^1(\Gamma) < +\infty$, it is rectifiable in the sense of [33, Section 3.2]: its length $\mathcal{H}^1(\Gamma)$ is well-approximated by the length of closed polygonal curves π_k whose vertices lie on Γ, i.e., $\mathcal{H}^1(\pi_k) \to \mathcal{H}^1(\Gamma)$. Let E_k be the set enclosed by π_k and observe that $\pi_k \xrightarrow{K} \Gamma$. Since E_k are Lipschitz sets, they are sets of finite perimeter and $P(E_k) = \mathcal{H}^1(\pi_k) \leq \mathcal{H}^1(\Gamma) + 1$ for large k. Since E is open, for every $x \in E$ there exists a ball $B_r(x) \subset E$ and by the Kuratowski convergence of π_k to Γ, it follows that $B_r(x) \subset E_k$ for large k, and hence $\chi_{E_k}(x) = \chi_E(x) = 1$. Similarly, $\chi_{E_k}(x) = \chi_E(x) = 0$ for every $x \in \mathbb{R}^2 \setminus \overline{E}$ provided k is large enough. Therefore, $\chi_{E_k} \to \chi_E$ a.e. in $\mathbb{R}^2$ and hence, $E_k \to E$ in $L^1(\mathbb{R}^2)$. Now by the L^1-lower semicontinuity of perimeter (see [52, Proposition 12.15]), E is a set of finite perimeter.

Step 2. We claim that if E is connected, then $E \in BV(\mathbb{R}^2; \{0,1\})$. Indeed, let E' be the smallest simply connected open set containing E (basically, E' is contructed by filling in all "holes" in E) and let

$$F := E' \setminus \overline{E}$$

be the union of all holes. Since $\partial E' \subset \partial E$ and $\mathcal{H}^1(\partial E) \leq \mathcal{H}^1(\partial A) < +\infty$, by Step 1 $E' \in BV(\mathbb{R}^2; \{0,1\})$. Observing $E = E' \setminus \overline{F}$, to conclude this step it is enough to prove that F has finite perimeter. Since every open set in $\mathbb{R}^2$ is a union of at most countably many connected components[†], we have $F = \cup_j F_j$, where $\{F_j\}$ are open, connected and $F_i \cap F_j = \emptyset$ for $i \neq j$. Since E is connected, each F_j is simply connected, and hence, by Step 1 $F_j \in BV(\mathbb{R}^2; \{0,1\})$. Moreover, the set $\partial F_i \cap \partial F_j$, $i \neq j$, can have at most one point. Indeed, otherwise, by the definition of F and the connectedness of E we could find a curve $\gamma \subset \partial F_i \cap \partial F_j \cap \partial E$ with $\mathcal{H}^1(\gamma) > 0$, which contradicts the equality $E = \mathrm{Int}(\overline{E})$. Therefore, observing $\partial F = \bigcup \partial F_j \subset \partial E$, we obtain

$$\sum_j P(F_j) \leq \sum_j \mathcal{H}^1(\partial F_j) = \mathcal{H}^1\Big(\bigcup_j \partial F_j\Big) = \mathcal{H}^1(\partial F) \leq \mathcal{H}^1(\partial E) < +\infty.$$

Thus, $F = \bigcup_j F_j$ has finite perimeter in $\mathbb{R}^2$.

Step 3. Now we prove that $E \in BV(\mathbb{R}^2; \{0,1\})$ (without assuming any extra connectedness assumption). Let $\{E_j\}$ be the family of connected components of E. Since $\mathcal{H}^1(\partial E_j) \leq \mathcal{H}^1(\partial E) < +\infty$, by Step 2 $E_j \in BV(\mathbb{R}^2; \{0,1\})$. Therefore, since $\partial E = \cup_j \partial E_j$ we obtain that

$$\sum_j P(E_j) \leq \sum_j \mathcal{H}^1(\partial E_j) \leq \mathcal{H}^1\Big(\bigcup_j \partial E_j\Big) + \sum_{i<j} \mathcal{H}^1(\partial E_i \cap \partial E_j) \leq 2\mathcal{H}^1\Big(\bigcup_j \partial E_j\Big) = 2\mathcal{H}^1(\partial E),$$

and hence, by the finiteness of $\mathcal{H}^1(\partial E)$, the set $E = \cup_j E_j$ has finite perimeter in $\mathbb{R}^2$. □

The following proposition, which is based on [52, Proposition 2.16], is used throughout the paper.

Proposition A.2. *Let $K \subset \mathbb{R}^2$ be such that $\mathcal{H}^1(K) < +\infty$ and let $\{E_t\}_{t \in \Upsilon}$ be a family of sets parametrized by $t \in \Upsilon$ such that*

$$\mathcal{H}^1(K \cap E_t \cap E_s) = 0 \tag{A.1}$$

and $\mathcal{H}^1(K \cap E_t) > 0$. Then Υ is at most countable.

Proof. The proof runs along the lines of the proof of [52, Proposition 2.16]. For $j \in \mathbb{N}$ let $\Upsilon_j \subset \Upsilon$ be the set of all $t \in \Upsilon$ such that $\mathcal{H}^1(K \cap E_t) > \frac{1}{j}$. Then by (A.1) Υ_j cannot contain more than $j\mathcal{H}^1(K)$ elements. Since $\Upsilon = \cup_j \Upsilon_j$, the set Υ is at most countable. □

We finally state a regularity property of $GSBD$ functions with $\mathcal{H}^{d-1}$-negligible jump.

Proposition A.3. *Let $U \subset \mathbb{R}^d$ be a connected bounded open set and $u \in GSBD^2(U)$ be such that $\mathcal{H}^{d-1}(J_u) = 0$. Then $u \in H^1_{\mathrm{loc}}(U)$.*

Proof. Indeed, for $r > 0$ let $Q := x_0 + (-r, r)^d \subset U$ be any cube centered at $x \in U$ and let $0 < \theta'' < \theta' < 1$. For shortness, write $Q' := x_0 + (-\theta' r, \theta' r)^d$ and $Q'' := x_0 + (-\theta'' r, \theta'' r)^d$. By [11, Proposition 3.1 (1)] (see also [10, Theorem 1.1]) there exists a $\mathcal{L}^2$-measurable set $\omega \subset Q'$ and a rigid displacement $a : \mathbb{R}^d \to \mathbb{R}^d$ such that $|\omega| \leq c_* r \mathcal{H}^{d-1}(J_u) = 0$ and

$$\int_{Q'} |u - a|^{\frac{2d}{d-1}} dx = \int_{Q' \setminus \omega} |u - a|^{\frac{2d}{d-1}} dx \leq c_* r^2 \Big(\int_Q |e(u)|^2\Big)^{\frac{d}{d-1}},$$

where c_* depends only on d. Hence, $u \in L^{\frac{2d}{d-1}}_{\mathrm{loc}}(Q)$. Next, fix any mollifier $\rho_1 \in C^\infty(B_r(0))$ with $\rho_\epsilon \in C_c^\infty(B_{(\theta' - \theta'')\epsilon})$, where $\rho_\epsilon(x) := \rho_1(x/\epsilon)$, $\epsilon \in (0, r)$. By [11, Proposition 3.1] there exists $\overline{p} > 0$ depending on n and ϵ such that

$$\int_{Q''} |e(u * \rho_\epsilon) - e(u) * \rho_\epsilon|^2 dx \leq c\Big(\frac{\mathcal{H}^{d-1}(J_u)}{r^{d-1}}\Big)^{\overline{p}} \int_Q |e(u)|^2 dx = 0,$$

[†]This property easily follows by fact that we can always choose in each connected component a different point with rational coordinates

where c depends on n, ρ_1 and ϵ. Hence,

$$e(u * \rho_\epsilon) = e(u) * \rho_\epsilon \quad \text{a.e. in } Q''. \tag{A.2}$$

Recall that $u * \rho_\epsilon \in C^\infty(Q'')$. Since $e(u) \in L^2(Q)$, $e(u) * \rho_\epsilon \in C^\infty(Q'') \cap L^2(Q'')$ in particular, $e(u * \rho_\epsilon) \in C^\infty(Q'') \cap L^2(Q'')$. By Poincaré-Korn inequality $u * \rho_\epsilon \in H^1(Q'')$. Since $e(u) * \rho_\epsilon \to e(u)$ in $L^2(Q'')$ as $\epsilon \to 0$, in view of (A.2) there exists $\epsilon_0 > 0$ such that

$$\|e(u * \rho_\epsilon)\|_{L^2(Q'')} \le \|e(u)\|_{L^2(Q'')} + 1 \quad \text{for all } \epsilon \in (0, \epsilon_0).$$

Moreover, by Poincaré-Korn inequality for any $\epsilon \in (0, \epsilon_0)$ there exists a rigid displacement a_ϵ such that

$$\|u * \rho_\epsilon - a_\epsilon\|_{H^1(Q'')} \le C\|e(u * \rho_\epsilon)\|_{L^2(Q'')} \le C(\|e(u)\|_{L^2(Q'')} + 1),$$

where C is the Poincaré-Korn constant for a cube. Thus, the family $\{u * \rho_\epsilon\}_\epsilon$ is uniformly bounded in $H^1(Q'')$. Since $u * \rho_\epsilon \to u$ in $L^2(Q'')$, there exists a rigid displacement a such that $a_\epsilon \to a$ in $L^2(Q'')$. Then $u * \rho_\epsilon - a_\epsilon$ weakly converges to $u - a$ in $H^1(Q'')$, i.e., $u - a \in H^1(Q'')$. Since a is linear and θ'' is arbitrary, $u \in H^1_{\mathrm{loc}}(Q)$. Now covering U with finitely many cubes of edgelength $2r$ we get $u \in H^1_{\mathrm{loc}}(U)$. □

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