ON THE UNIQUENESS AND MONOTONICITY OF SOLUTIONS OF FREE BOUNDARY PROBLEMS

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Abstract. For any $\Omega \subset \mathbb{R}^N$ smooth and bounded domain, we prove uniqueness of positive solutions of free boundary problems arising in plasma physics on Ω in a neat interval depending only by the best constant of the Sobolev embedding $H^1_0(\Omega) \hookrightarrow L^{2p}(\Omega)$, $p \in [1, \frac{N}{N-2})$ and show that the boundary density and a suitably defined energy share a universal monotonic behavior. At least to our knowledge, for $p > 1$, this is the first result about the uniqueness for a domain which is not a two-dimensional ball and in particular the very first result about the monotonicity of solutions, which seems to be new even for $p = 1$. The threshold, which is sharp for $p = 1$, yields a new condition which guarantees that there is no free boundary inside Ω. As a corollary, in the same range, we solve a long-standing open problem (dating back to the work of Berestycki-Brezis in 1980) about the uniqueness of variational solutions. Moreover, on a two-dimensional ball we describe the full branch of positive solutions, that is, we prove the monotonicity along the curve of positive solutions until the boundary density vanishes.

Keywords: Free boundary problem, bifurcation analysis, uniqueness, monotonicity.

Contents

1. **Introduction**

Letting $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be an open and bounded domain of class C^3, we are concerned with the free boundary problem

$$\begin{cases} -\Delta v = (v)_+^p \quad \text{in } \Omega \\ -\displaystyle\int_{\partial\Omega} \frac{\partial v}{\partial \nu} = I \\ v = \gamma \quad \text{on } \partial\Omega \end{cases} \qquad \mathbf{(F)}_I$$

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for the unknowns $\gamma \in \mathbb{R}$ and $v \in C^{2,r}(\overline{\Omega}\,)$, $r \in (0,1)$. Here $(v)_+$ is the positive part of v, ν is the exterior unit normal, $I > 0$ and $p \in [1, p_N)$ are fixed, with

$$p_N = \begin{cases} +\infty, \; N = 2 \\ \frac{N}{N-2}\,, \; N \geq 3. \end{cases}$$

Up to a rescaling, we will assume without loss of generality that $|\Omega| = 1$. Moreover, for $p > 1$, a scaling of the problem allows one to peak any positive constant to multiply $(v)_+^p$.

The study of $\mathbf{(F)}_I$ is motivated by Tokamak's plasma physics and we refer to the appendix in [42] for a short introduction and also to [24, 33] for a physical description of the problem. Initiated in [42, 43] with the model case $p = 1$, a systematic analysis of $\mathbf{(F)}_I$ has been undertaken in [10], where a more general problem is considered, the non-linearity $(v)_+^p$ being replaced by the positive part of any continuous and increasing function with growth of order at most p, for some $p \in [1, p_N)$. Indeed, the threshold p_N turns out to be a natural critical exponent for $\mathbf{(F)}_I$, see [10, 34] and the discussion in the sequel.
However, among many other things, it has been shown in [10] that for any $I > 0$ there exists at least one solution of $\mathbf{(F)}_I$. A lot of work has been done to understand solutions of $\mathbf{(F)}_I$ for $p \in (1, p_N]$, see [1, 3, 4, 6, 20, 28, 29, 31, 32, 35, 40, 44, 45, 46, 47], and in the model case $p = 1$, see [12, 13, 17, 18, 21, 36, 37, 38, 39], and the references quoted therein. Although we will not discuss this point here, a lot of work has been done in particular to understand the regularity (for solutions with $\gamma < 0$) of the free boundary $\partial\{x \in \Omega \,|\, v(x) > 0\}$, see [19, 26, 27] and references quoted therein.

However, the uniqueness results at hand about $\mathbf{(F)}_I$ seem to concern so far either the model case $p = 1$ and $N \geq 2$ ([21, 37, 43]) or the case of the ball for $N = 2$ and $p > 1$ ([6]), or either the case $N \geq 2$ where $(v)_+^p$ is replaced by the positive part of any continuous and increasing nonlinearity satisfying a suitable uniformly Lipschitz condition (whence at most of linear growth), see [10]. Otherwise, we are not aware of any uniqueness result for solutions of $\mathbf{(F)}_I$ neither for positive solutions, nor for the so-called variational solutions, see [10] and [3]. Actually, in dimension $N \geq 3$ and $p = \frac{N+2}{N-2}$ uniqueness fails for $\mathbf{(F)}_I$ on a ball for I small enough, see [4]. On the other hand, a non-uniqueness result for $\mathbf{(F)}_I$, with the nonlinearity rewrited in the form $\lambda(v)_+^p$, is obtained in [12, 13, 31, 38] for any $p \in [1, p_N)$ and λ sufficiently large. However, our main motivation comes from the fact that, at least to our knowledge, no results at all are available so far about the shape of branches of solutions (γ_I, v_I) of $\mathbf{(F)}_I$.

The main idea here is to identify the natural spectral setting together with the quantities which should share nice monotonicity properties (see also [7, 8, 9] where a different class of problems is considered). The first step is to move to a dual formulation ([10, 43]) of $\mathbf{(F)}_I$ via the constrained problem,

$$\begin{cases} -\Delta\psi = (\alpha + \lambda\psi)^p \quad \text{in } \Omega \\ \int\limits_{\Omega} (\alpha + \lambda\psi)^p = 1 \\ \psi > 0 \quad \text{in } \Omega, \quad \psi = 0 \quad \text{on } \partial\Omega \\ \alpha \geq 0 \end{cases} \qquad \mathbf{(P)}_\lambda$$

for the unknowns $\alpha \in \mathbb{R}$ and $\psi \in C^{2,r}_{0,+}(\overline{\Omega}\,)$. Here, $\lambda \geq 0$ and $p \in [1, p_N)$ are fixed and for $r \in (0,1)$ we set

$$C^{2,r}_0(\overline{\Omega}\,) = \{\psi \in C^{2,r}(\overline{\Omega}\,) \,:\, \psi = 0 \text{ on } \partial\Omega\},\; C^{2,r}_{0,+}(\overline{\Omega}\,) = \{\psi \in C^{2,r}_0(\overline{\Omega}\,) \,:\, \psi > 0 \text{ in } \Omega\}.$$

It is useful to define positive solutions as follows.

Definition. *We say that* (γ_I, v_I) *is a positive solution of* $(\mathbf{F})_I$ *if* $\gamma_I > 0$. *We say that* $(\alpha_\lambda, \psi_\lambda)$ *is a positive solution of* $(\mathbf{P})_\lambda$ *if* $\alpha_\lambda > 0$.

We remark that since $|\Omega| = 1$ and $\lambda \geq 0$ by assumption, then if $(\alpha_\lambda, \psi_\lambda)$ solves $(\mathbf{P})_\lambda$ we necessarily have,

$$\alpha_\lambda \leq 1,$$

and the equality holds if and only if $\lambda = 0$. We will frequently use this fact without further comments. Actually, if $\lambda = 0$, then $(\mathbf{P})_\lambda$ admits a unique solution $(\alpha_0, \psi_0) = (1, G[1])$ satisfying

$$\begin{cases} -\Delta\psi_0 = 1 \quad \text{in } \Omega \\ \\ \psi_0 = 0 \quad \text{on } \partial\Omega \end{cases}$$

where we define,

$$G[\rho](x) = \int_\Omega G_\Omega(x,y)\rho(y)\,dy, \; x \in \Omega.$$

Here G_Ω is the Green function of $-\Delta$ with Dirichlet boundary conditions on Ω. Obviously, to say that $(\alpha_\lambda, \psi_\lambda)$ is a solution of $(\mathbf{P})_\lambda$ is the same as to say that $\psi_\lambda = G[\rho_\lambda]$ and $\int_\Omega \rho_\lambda = 1$, where, here and in the rest of this work, we set

$$\rho_\lambda = (\alpha_\lambda + \lambda\psi_\lambda)^p.$$

Remark 1.1. *Let* q *be the conjugate index of* p, *that is*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

For any fixed $\lambda > 0$ *and* $p > 1$, $(\alpha_\lambda, \psi_\lambda)$ *is a positive solution of* $(\mathbf{P})_\lambda$ *if and only if, for* $I = I_\lambda = \lambda^q$, $(\gamma_I, v_I) = (\lambda^{\frac{1}{p-1}}\alpha_\lambda, \lambda^{\frac{1}{p-1}}(\alpha_\lambda + \lambda\psi_\lambda))$ *is a positive solution of* $(\mathbf{F})_I$. *Therefore, in particular, if* (γ_I, v_I) *solves* $(\mathbf{F})_I$ *then* $(\alpha_\lambda, \psi_\lambda) = (I^{-\frac{1}{p}}\gamma_I, I^{-1}(v_I - \gamma_I))$ *solves* $(\mathbf{P})_\lambda$ *and the identity* $I^{-\frac{1}{p}}v_I = \alpha_\lambda + \lambda\psi_\lambda$ *holds. This correspondence is singular for* $p = 1$ *and in this case we will stick with the formulation* $(\mathbf{P})_\lambda$. *Indeed, it is readily seen that for* $p = 1$, $(\mathbf{P})_\lambda$ *is equivalent to a more general problem than* $(\mathbf{F})_I$ *and positive solutions of* $(\mathbf{P})_\lambda$ *correspond to positive solutions of* $(\mathbf{F})_I$ *where the first equation is replaced by* $-\Delta v = \lambda(v)_+$.

All the statements below are concerned with $(\mathbf{P})_\lambda$, the corresponding statements about $(\mathbf{F})_I$ being immediately recovered via the latter remark. For the sake of clarity, we point out that if a map $\mathcal{M}$ from an interval $[a,b] \subset \mathbb{R}$ to a Banach space X is said to be real analytic, then it is understood that $\mathcal{M}$ can be extended in an open neighborhood of a and b where it admits a power series expansion in t, totally convergent in the X-norm.

For fixed $t \geq 1$ and $|\Omega| = 1$ we denote

$$\Lambda(\Omega, t) = \inf_{w \in H_0^1(\Omega), w \not\equiv 0} \frac{\int_\Omega |\nabla w|^2}{\left(\int_\Omega |w|^t\right)^{\frac{2}{t}}},$$

which provides the best constant in the Sobolev embedding $\|w\|_p \leq S_p(\Omega)\|\nabla w\|_2$, $S_p(\Omega) = \Lambda^{-2}(\Omega, p)$, $p \in [1, 2p_N)$. For $(\alpha_\lambda, \psi_\lambda)$ a solution of $(\mathbf{P})_\lambda$ we define the energy,

$$E_\lambda := \frac{1}{2}\int_\Omega |\nabla\psi_\lambda|^2 \equiv \frac{1}{2}\int_\Omega \rho_\lambda\psi_\lambda,$$

see Appendix A for more comments about the latter definition.

Set

$$\lambda^*(\Omega, p) = \sup\{\lambda > 0 \,:\, \alpha_\mu > 0 \text{ for any solution of } (\mathbf{P})_\mu, \forall\, \mu < \lambda\}.$$

It is not difficult to show (see Lemma B.1 in Appendix B) that the latter quantity is well defined. Finally, we denote by $\mathcal{G}(\Omega)$ the set of solutions of $\mathbf{(P)}_\lambda$ for $\lambda \in [0, \frac{1}{p}\Lambda(\Omega, 2p))$, $p \in [1, p_N)$ and let $B_r = \{x \in \mathbb{R}^N \,:\, |x| < r\}$ with volume $|B_r|$ and $\mathbb{D}_N$ be the N-dimensional ball of unit volume.

Theorem 1.2. *Let $p \in [1, p_N)$. Then $\lambda^*(\Omega, p) \geq \frac{1}{p}\Lambda(\Omega, 2p)$ and the equality holds if and only if $p = 1$. Moreover, we have:*

1. (Uniqueness): *for any $\lambda \in [0, \frac{1}{p}\Lambda(\Omega, 2p))$ there exists a unique solution $(\alpha_\lambda, \psi_\lambda)$ of $\mathbf{(P)}_\lambda$.*
2. (Monotonicity): *$\mathcal{G}(\Omega)$ is a real analytic simple curve of positive solutions $[0, \frac{1}{p}\Lambda(\Omega, 2p)) \ni \lambda \mapsto (\alpha_\lambda, \psi_\lambda)$ such that, for any $\lambda \in [0, \frac{1}{p}\Lambda(\Omega, 2p))$,*

$$\frac{d\alpha_\lambda}{d\lambda} < 0 \quad \text{and} \quad \frac{dE_\lambda}{d\lambda} > 0,$$

and

$$\alpha_\lambda = 1 + \mathrm{O}(\lambda), \;\; \psi_\lambda = \psi_0 + \mathrm{O}(\lambda), \;\; E_\lambda = E_0(\Omega) + \mathrm{O}(\lambda), \qquad \text{as } \lambda \to 0^+,$$

where,

$$E_0(\Omega) = \frac{1}{2}\int_\Omega \int_\Omega G_\Omega(x,y)\,dxdy \leq E_0(\mathbb{D}_N) = \frac{|B_1|^{-\frac{2}{N}}}{4(N+2)}.$$

In particular $\mathcal{G}(\Omega)$ can be extended continuously on $\lambda \in [0, \frac{1}{p}\Lambda(\Omega, 2p)]$ with $(\alpha_\lambda, \psi_\lambda) \to (\overline{\alpha}, \overline{\psi})$ as $\lambda \to \frac{1}{p}\Lambda(\Omega, 2p)^-$ and $\overline{\alpha} = 0$ if and only if $p = 1$.

The threshold $\frac{1}{p}\Lambda(\Omega, 2p)$ is sharp for $p = 1$ since in this case it is well known ([10, 37, 43]) that solutions are unique and positive if and only if $\lambda < \lambda^*(\Omega, 1) = \lambda^{(1)}(\Omega)$ while clearly $\Lambda(\Omega, 2) = \lambda^{(1)}(\Omega)$, where $\lambda^{(1)}(\Omega)$ is the first eigenvalue of $-\Delta$ on Ω with Dirichlet boundary conditions. The sharpness of the above result, for $p > 1$, will be discussed in a subsequent work. The bound $\lambda^*(\Omega, p) \geq \frac{1}{p}\Lambda(\Omega, 2p)$ is a non-linear generalization of the sufficient conditions, obtained in the sublinear case in [1, 4], which guarantees that there is no free boundary inside Ω. Clearly, $2E_0(\Omega)$ is just the torsional rigidity of Ω.
At least to our knowledge, for $p > 1$, this is the first result about the uniqueness of solutions for a domain which is not a two-dimensional ball. In particular, this is the very first result about the qualitative behavior of the branch of solutions and seems to be new even for $p = 1$, in which case, we actually describe the full branch of positive solutions, i.e. until the boundary density vanishes. Both the uniqueness and monotonicity of solutions hold for any smooth and bounded domain, in any dimension and for any subcritical exponent.
When $N = 2$ we derive also a sharp energy estimate for any solution of $\mathbf{(P)}_\lambda$, depending only on p, i.e. $E_\lambda \leq \frac{p+1}{16\pi}$ and the equality holds if and only if $\Omega = \mathbb{D}_2$ and $\alpha_\lambda = 0$, see Proposition 3.3.

Therefore, we succeed in the construction of a branch of positive solutions, that is, the case where there is no free boundary inside Ω, emanating from the unique positive solution (α_0, ψ_0) which share a universal (i.e. independent of Ω) monotonic profile, and prove that they are also the unique solutions of $\mathbf{(P)}_\lambda$ in $[0, \frac{1}{p}\Lambda(\Omega, 2p))$. The corresponding branch of solutions for $\mathbf{(F)}_I$ is real analytic in $(0, (\frac{1}{p}\Lambda(\Omega, 2p))^q)$, continuous in $(0, (\frac{1}{p}\Lambda(\Omega, 2p))^q]$ and tends to the trivial solution $(\gamma_0, v_0) \equiv (0, 0)$ as $I \to 0^+$. We remark that, while $\alpha_\lambda = I^{-\frac{1}{p}}\gamma_I$ is decreasing, γ_I need not be monotone and indeed it is not, at least on $\Omega = \mathbb{D}_2$, see Remark 1.5 below.

Theorem 1.2 is also relevant in the study of the so-called variational solutions of $\mathbf{(F)}_I$, see [10] and [3, 4]. A solution (γ_I, v_I) of $\mathbf{(F)}_I$ is a variational solution of $\mathbf{(F)}_I$ if it is a minimizer of the functional,

$$\Psi_I(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \frac{1}{p+1}\left(\int_\Omega (v)_+^{p+1}\right) + Iv(\partial\Omega), \tag{1.1}$$

on

$$\mathcal{H}_I = \left\{ v \in H \mid \int_\Omega (v)_+^p = I \right\},$$

where $I > 0$ and H is the space of $H^1(\Omega)$ functions whose boundary trace is constant. Analogously, variational solutions of $\mathbf{(P)}_\lambda$ are related to solutions of the dual variational principle **(VP)**, see Appendix A for more details. It has been shown in [43] that for $p = 1$ and any $\lambda > 0$ there exists at least one variational solution of $\mathbf{(P)}_\lambda$, while in [10] the authors proved that for any $p \in (1, p_N)$ and any $I > 0$ there exists at least one variational solution of $\mathbf{(F)}_I$. Actually, we know from [4, 6, 37] that the following holds true.

Theorem A. ([4, 6, 37])
1. *Let $p \in (1, p_N)$ and (γ_I, v_I) be a variational solution of $\mathbf{(F)}_I$. Then there exists $I^{**}(\Omega, p) \in (0, +\infty)$ such that $\gamma_I > 0$ if and only if $I \in (0, I^{**}(\Omega, p))$.*

2. *Let $p = 1$ and $(\alpha_\lambda, \psi_\lambda)$ be a any solution of $\mathbf{(P)}_\lambda$. Then there exists $\lambda^{**}(\Omega, 1) \in (0, +\infty)$ such that $\alpha_\lambda > 0$ if and only if $\lambda \in (0, \lambda^{**}(\Omega, 1))$. Moreover, $\lambda^{**}(\Omega, 1) = \lambda^{(1)}(\Omega)$.*

For $p > 1$ there exists a one to one correspondence ([10]) between variational solutions of $\mathbf{(F)}_I$ and variational solutions of $\mathbf{(P)}_\lambda$ and in particular we define,

$$\lambda^{**}(\Omega, p) = (I^{**}(\Omega, p))^{\frac{1}{q}}, \quad p > 1,$$

which shares the same property of $I^{**}(\Omega, p)$ in Theorem A, see Corollary A.1 in Appendix A. By the uniqueness results for the model case $p = 1$ ([21, 37, 43]), any solution is a variational solution and then in particular we have $\lambda^{**}(\Omega, 1) = \lambda^*(\Omega, 1)$. Similarly, the uniqueness property in [6] for $\Omega = \mathbb{D}_2$ yields that any solution is variational and $\lambda^{**}(\mathbb{D}_2, p) = \lambda^*(\mathbb{D}_2, p)$. Excluding these model cases, uniqueness was not known so far neither for variational solutions. Concerning this point we have the following,

Theorem 1.3. *For $p \in [1, p_N)$ and for any $\lambda \in [0, \frac{1}{p}\Lambda(\Omega, 2p))$ there exists a unique variational solution $(\alpha_\lambda, \psi_\lambda)$ of $\mathbf{(P)}_\lambda$. For $p \in (1, p_N)$ and for any $I \in (0, (\frac{1}{p}\Lambda(\Omega, 2p))^q)$ there exists a unique variational solution (γ_I, v_I) of $\mathbf{(F)}_I$. In particular the set of unique variational solutions of $\mathbf{(P)}_\lambda$ in $[0, \frac{1}{p}\Lambda(\Omega, 2p))$ coincides with $\mathcal{G}(\Omega)$ and the following inequalities hold,*

$$\lambda^{**}(\Omega, p) \geq \lambda^*(\Omega, p) \geq \frac{1}{p}\Lambda(\Omega, 2p) \geq \frac{1}{p}\Lambda(\mathbb{D}_N, 2p). \tag{1.2}$$

Observe that from Theorem 1.2 and (1.2) we have $\lambda^{**}(\Omega, p) \geq \lambda^*(\Omega, p) > \frac{1}{p}\Lambda(\Omega, 2p)$, for $p > 1$. On the other hand, the set of positive variational solutions is not empty for any $\lambda \in (\frac{1}{p}\Lambda(\Omega, 2p), \lambda^{**}(\Omega, p))$, see [3, 10]. The continuation of the curve of solutions $\mathcal{G}(\Omega)$, under generic assumptions, beyond $\frac{1}{p}\Lambda(\Omega, 2p)$, enjoying uniqueness and monotonicity properties, will be the topic of a future paper.

Concerning this point, for $\Omega = \mathbb{D}_2$ and for any $p \in [1, +\infty)$, we succeed in the description of the shape and monotonicity of the full branch of positive solutions. We stress that here only uniqueness was known so far ([6]). Let $\mathcal{G}^*(\mathbb{D}_2)$ be the set of unique solutions of $\mathbf{(P)}_\lambda$, then we have,

Theorem 1.4. *Let $p \in [1, +\infty)$. Then, $\mathcal{G}^*(\mathbb{D}_2)$ is a continuous curve, defined in $[0, \lambda^*(\mathbb{D}_2, p)]$, such that,*

$$\alpha_\lambda \text{ is strictly decreasing and } E_\lambda \text{ is strictly increasing in } (0, \lambda^*(\mathbb{D}_2, p)),$$

and

$$\alpha_{\lambda^*(\mathbb{D}_2, p)} = 0, \quad E_{\lambda^*(\mathbb{D}_2, p)} = \frac{p+1}{16\pi}.$$

Remark 1.5. *Actually, we prove something more about the regularity of $\alpha_\lambda, E_\lambda$, see section 5. Observe that, along $\mathcal{G}^*(\mathbb{D}_2)$, by the monotonicity of E_λ, we have $E_\lambda \geq E_0(\mathbb{D}_2) = \frac{1}{16\pi}$. On the other hand, by Proposition 3.3, we know $E_\lambda \leq \frac{p+1}{16\pi}$. Thus, along the branch, the solutions span the full energy range $[\frac{1}{16\pi}, \frac{p+1}{16\pi}]$. Besides, for $p > 1$, the parameter $\gamma_I = \lambda^{\frac{1}{p-1}}\alpha_\lambda$ varies continuously from $\gamma_0 = 0$ to $\gamma_{I^*(\mathbb{D}_2,p)} = 0$, whence in particular it is not monotone along the branch. For γ_I small enough there are at least two distinct pairs $\{I, v_I\}$ solving* $\mathbf{(F)}_I$.

The proof of Theorem 1.2 is based on the bifurcation analysis of the vectorial solutions $(\alpha_\lambda, \psi_\lambda)$ of the constrained problem $\mathbf{(P)}_\lambda$.
The crucial property of $\mathbf{(P)}_\lambda$ relies on its linearized operator, see the definition (2.8) of L_λ and the related eigenvalues equation (2.9) in section 2. The use of this operator is rather delicate since L_λ arises as the linearization of a constrained problem ($\int_\Omega \rho_\lambda = 1$) with respect to $(\alpha_\lambda, \psi_\lambda)$, which yields a non-local problem. As a consequence it is not true in general that its first eigenvalue, which we denote by $\sigma_1(\alpha_\lambda, \psi_\lambda)$, see (2.14), is simple and neither that if $\sigma_1(\alpha_\lambda, \psi_\lambda)$ is positive then the maximum principle holds. For example this is exactly what happens for $\lambda = 0$ on $\mathbb{D}_2$, where $\sigma_1(\alpha_0, \psi_0)$ can be evaluated explicitly (see [8]) and one finds that $\sigma_1(\alpha_0, \psi_0) = \lambda^{(2,0)}(\mathbb{D}_2) \simeq \pi(3,83)^2$ has three eigenfunctions, two of which indeed change sign. Here $\lambda^{(2,0)}(\Omega) > \lambda^{(1)}(\Omega)$ is the first non vanishing eigenvalue of $-\Delta$ on Ω on the space of $H^1(\Omega)$ vanishing mean functions with constant boundary trace.
In few words we argue as follows. We first prove that, if for a positive solution $(\alpha_\lambda, \psi_\lambda)$ with $\lambda \geq 0$ we have $0 \notin \sigma(L_\lambda)$, where $\sigma(L_\lambda)$ stands for the spectrum of L_λ, then the set of solutions of $\mathbf{(P)}_\lambda$ is locally a real analytic curve of positive solutions. In particular a real analytic curve of positive solutions exists around (α_0, ψ_0). A crucial information is that if $\sigma_1(\alpha_\lambda, \psi_\lambda)$ is positive, then $\frac{d\alpha_\lambda}{d\lambda} < 0$ and $\frac{dE_\lambda}{d\lambda} > 0$, see section 3. We evaluate the sign of the derivative of E_λ by a careful analysis of the Fourier modes of L_λ. At this point one can see that if for some $\overline{\lambda} > 0$ it holds $\sigma_1(\alpha_\lambda, \psi_\lambda) > 0$ for any solution $(\alpha_\lambda, \psi_\lambda)$ of $\mathbf{(P)}_\lambda$ with $\lambda \leq \overline{\lambda}$ and if $\overline{\alpha} > 0$ for a solution at $\overline{\lambda}$, then there exists one and only one solution of $\mathbf{(P)}_\lambda$ in $[0, \overline{\lambda}\,]$ which, by the monotonicity of α_λ, is also a positive solution, see Lemma 4.1. This interesting property seems to be the generalization to free boundary problems of similar properties for minimal solutions of semilinear elliptic equations with strictly positive, increasing and convex nonlinearities ([15, 25]). In particular, as a consequence of these facts, the proof of Theorem 1.2 is reduced to an a priori bound from below away from zero for $\sigma_1(\alpha_\lambda, \psi_\lambda)$ and α_λ.
The spectral estimate is obtained by using that $\sigma_1(\alpha_\lambda, \psi_\lambda)$ is strictly larger than the "standard" eigenvalue $\nu_{1,\lambda}$ corresponding to the problem without constraints, see (2.17). This is crucial as it rules out at once the natural strong competition between the value of the λ-threshold and the bound from below for α_λ. On the other side, the fact that $\lambda^*(\Omega, p) \geq \frac{1}{p}\Lambda(\Omega, p)$ follows essentially as consequence of the fact that if $(0, \psi_\lambda)$ solves $\mathbf{(P)}_\lambda$, then $\lambda\psi_\lambda$ is proportional to a positive strict subsolution of the corresponding linearized problem. The universal energy estimates for $N = 2$ is of independent interest, see Proposition 3.3.

Concerning Theorem 1.4 we remark that even on $\mathbb{D}_2$ it is not trivial to catch the shape of the full branch of positive solutions. Indeed, we have also to overcome the eventuality of singularities $\sigma_1(\alpha_\lambda, \psi_\lambda) = 0$ along the continuation of the curve $\mathcal{G}(\mathbb{D}_2)$. The non-local structure of L_λ makes the bifurcation analysis rather delicate. We handle this problem by a generalization of the Crandall-Rabinowitz ([15]) bending result suitable to describe solutions $(\alpha_\lambda, \psi_\lambda)$ of $\mathbf{(P)}_\lambda$ on any domain Ω near a singular point satisfying a suitable transversality condition. This result, although we will not state it explicitly, is of independent interest as it is just, via Remark 1.1, a Crandall-Rabinowitz bending-type result suitable to be applied to positive solutions of $\mathbf{(F)}_I$. We stress that the latter result would not work out if we would stick with the standard spectral analysis as it exploits the modified (constrained) spectral setting in its full strength. Besides its application to the case $\Omega = \mathbb{D}_2$, it will be exploited in a future work to obtain sufficient conditions to continue $\mathcal{G}(\Omega)$, under generic assumptions, until the boundary density vanishes.

The paper is organized as follows. In section 2 we set up the spectral and bifurcation analysis. Then, in section 3 we show the monotonicity of solutions while in section 4 we prove the uniqueness result. Finally, in section 5 we discuss the two-dimensional ball case. A brief discussion about variational solutions and uniqueness of solutions for λ small is postponed to Appendixes A and B, respectively.

2. **Spectral and bifurcation analysis**

In this section we develop the spectral and bifurcation analysis for solutions of $(\mathbf{P})_\lambda$ with $\lambda \geq 0$ and $p \in [1, p_N)$. Lemma 2.4 is the basic result which we will use to describe branches of solutions of $(\mathbf{P})_\lambda$ at regular points. We will also prove Lemma 2.6 and Theorem 2.5 which will be needed in section 5 for the two-dimensional ball case. In particular we will need there a generalization of the bending result of [15] for solutions of $(\mathbf{P})_\lambda$, see Proposition 2.7 below.

From now on, we will denote
$$\tau_\lambda = p\lambda.$$
Also, we write $\rho_{\lambda,\alpha} = \rho_{\lambda,\alpha}(\psi) = (\alpha + \lambda\psi)^p$ and $\rho_\lambda = (\alpha_\lambda + \lambda\psi_\lambda)^p$, and then in particular,
$$(\rho_{\lambda,\alpha}(\psi))^{\frac{1}{q}} = (\alpha + \lambda\psi)^{p-1} \text{ and } (\rho_\lambda)^{\frac{1}{q}} = (\alpha_\lambda + \lambda\psi_\lambda)^{p-1},$$
where it is understood that $(\rho_{\lambda,\alpha}(\psi))^{\frac{1}{q}} \equiv 1 \equiv (\rho_\lambda)^{\frac{1}{q}}$ for $p = 1$. Whenever $(\alpha_\lambda, \psi_\lambda)$ is a solution of $(\mathbf{P})_\lambda$ we denote,
$$< \eta >_\lambda = \frac{\int_\Omega (\rho_\lambda)^{\frac{1}{q}} \eta}{\int_\Omega (\rho_\lambda)^{\frac{1}{q}}} \quad \text{and} \quad [\eta]_\lambda = \eta - < \eta >_\lambda,$$
and define,
$$< \eta, \varphi >_\lambda := \frac{\int_\Omega (\rho_\lambda)^{\frac{1}{q}} \eta\varphi}{\int_\Omega (\rho_\lambda)^{\frac{1}{q}}} \quad \text{and} \quad \|\varphi\|_\lambda^2 := < \varphi, \varphi >_\lambda = < \varphi^2 >_\lambda = \frac{\int_\Omega (\rho_\lambda)^{\frac{1}{q}} \varphi^2}{\int_\Omega (\rho_\lambda)^{\frac{1}{q}}},$$
where $\{\eta, \varphi\} \subset L^2(\Omega)$. Clearly, for any solution ρ_λ is strictly positive in Ω. Therefore, it is easy to see that $< \cdot, \cdot >_\lambda$ defines a scalar product on $L^2(\Omega)$ whose norm is $\| \cdot \|_\lambda$. We will also adopt sometimes the useful shorthand notation,
$$m_\lambda = \int_\Omega (\rho_\lambda)^{\frac{1}{q}}.$$
Remark 2.1. *We will use the fact that,*
$$\int_\Omega (\rho_\lambda)^{\frac{1}{q}} [\eta]_\lambda^2 = \int_\Omega (\rho_\lambda)^{\frac{1}{q}} (\eta - < \eta >_\lambda)^2 \geq 0,$$
where the equality holds if and only if η is constant, whence in particular, in case $\eta \in H_0^1(\Omega)$, if and only if η vanishes identically. Also, since obviously $< [\eta]_\lambda >_\lambda = 0$, then,
$$< [\varphi]_\lambda, \eta >_\lambda = < \varphi, [\eta]_\lambda >_\lambda = < [\varphi]_\lambda [\eta]_\lambda >_\lambda, \ \forall \{\eta, \varphi\} \subset L^2(\Omega).$$
We will often use these properties when needed without further comments.

In the sequel we aim to describe possible branches of solutions of $(\mathbf{P})_\lambda$ around a positive solution, i.e. with $\alpha_\lambda > 0$. To this end, it is not difficult to construct an open subset A_Ω of the Banach space of triples $(\lambda, \alpha, \psi) \in \mathbb{R} \times \mathbb{R} \times C_0^{2,r}(\overline{\Omega})$ such that, on A_Ω, the density $\rho_{\lambda,\alpha} = \rho_{\lambda,\alpha}(\psi) = (\alpha + \lambda\psi)^p$ is well defined and
$$\alpha_\lambda + \lambda\psi_\lambda \geq \frac{\alpha_\lambda}{2} \quad \text{in } \overline{\Omega} \tag{2.1}$$

in a sufficiently small open neighborhood in A_Ω of any triple of the form $(\lambda,\alpha_\lambda,\psi_\lambda)$ whenever $(\alpha_\lambda,\psi_\lambda)$ is a positive solution of $\mathbf{(P)}_\lambda$.
At this point we introduce the map,

$$F : A_\Omega \to C^r(\overline{\Omega}\,), \quad F(\lambda,\alpha,\psi) := -\Delta\psi - \rho_{\lambda,\alpha}(\psi). \tag{2.2}$$

and

$$\Phi : A_\Omega \to \mathbb{R}\times C^r(\overline{\Omega}\,), \quad \Phi(\lambda,\alpha,\psi) = \begin{pmatrix} F(\lambda,\alpha,\psi) \\ \\ -1+\int_\Omega \rho_{\lambda,\alpha} \end{pmatrix}, \tag{2.3}$$

and, for a fixed $(\lambda,\alpha,\psi)\in A_\Omega$, its differential with respect to (α,ψ), that is the linear operator,

$$D_{\alpha,\psi}\Phi(\lambda,\alpha,\psi) : \mathbb{R}\times C^{2,r}_0(\overline{\Omega}\,) \to \mathbb{R}\times C^r(\overline{\Omega}\,),$$

which acts as follows,

$$D_{\alpha,\psi}\Phi(\lambda,\alpha,\psi)[s,\phi] = \begin{pmatrix} D_\psi F(\lambda,\alpha,\psi)[\phi] + d_\alpha F(\lambda,\alpha,\psi)[s] \\ \\ \int_\Omega \left(D_\psi \rho_{\lambda,\alpha}[\phi] + d_\alpha \rho_{\lambda,\alpha}[s]\right) \end{pmatrix},$$

where we have introduced the differential operators,

$$D_\psi F(\lambda,\alpha,\psi)[\phi] = -\Delta\phi - \tau_\lambda(\rho_{\lambda,\alpha})^{\frac{1}{q}}\phi, \quad \phi\in C^{2,r}_0(\overline{\Omega}\,), \tag{2.4}$$

$$D_\psi \rho_{\lambda,\alpha}[\phi] = \tau_\lambda(\rho_{\lambda,\alpha})^{\frac{1}{q}}\phi, \quad \phi\in C^{2,r}_0(\overline{\Omega}\,), \tag{2.5}$$

and

$$d_\alpha F(\lambda,\alpha,\psi)[s] = -p(\rho_{\lambda,\alpha})^{\frac{1}{q}}s, \quad s\in\mathbb{R}, \tag{2.6}$$

$$d_\alpha \rho_{\lambda,\alpha}[s] = p(\rho_{\lambda,\alpha})^{\frac{1}{q}}s, \quad s\in\mathbb{R}, \tag{2.7}$$

where we recall $\tau_\lambda = p\lambda$.
By the construction of A_Ω, see in particular (2.1), relying on known techniques about real analytic functions on Banach spaces ([11]), it is not difficult to show that $\Phi(\lambda,\alpha,\psi)$ is jointly real analytic in an open neighborhood of A_Ω around any triple of the form $(\lambda,\alpha_\lambda,\psi_\lambda)$ whenever $(\alpha_\lambda,\psi_\lambda)$ is a positive solution of $\mathbf{(P)}_\lambda$.

For fixed $\lambda\geq 0$ and $p\in[1,p_N)$, the pair $(\alpha_\lambda,\psi_\lambda)$ solves $\mathbf{(P)}_\lambda$ in the classical sense as defined in the introduction if and only if $\Phi(\lambda,\alpha_\lambda,\psi_\lambda)=(0,0)$, and we define the linear operator,

$$L_\lambda[\phi] = -\Delta\phi - \tau_\lambda(\rho_\lambda)^{\frac{1}{q}}[\phi]_\lambda. \tag{2.8}$$

We say that $\sigma=\sigma(\alpha_\lambda,\psi_\lambda)\in\mathbb{R}$ is an eigenvalue of L_λ if the equation,

$$-\Delta\phi - \tau_\lambda(\rho_\lambda)^{\frac{1}{q}}[\phi]_\lambda = \sigma(\rho_\lambda)^{\frac{1}{q}}[\phi]_\lambda, \tag{2.9}$$

admits a non-trivial weak solution $\phi\in H^1_0(\Omega)$. Let us define the Hilbert space,

$$Y_0 := \left\{\varphi\in\{L^2(\Omega), <\cdot,\cdot>_\lambda\} \,:<\varphi>_\lambda = 0\right\}, \tag{2.10}$$

and $T(f) := G[(\rho_\lambda)^{\frac{1}{q}}f]$, for $f\in L^2(\Omega)$. Since $T(Y_0)\subset W^{2,2}(\Omega)$, then the linear operator,

$$T_0 : Y_0\to Y_0,\; T_0(\varphi) = G[\tau_\lambda(\rho_\lambda)^{\frac{1}{q}}\varphi] - < G[\tau_\lambda(\rho_\lambda)^{\frac{1}{q}}\varphi] >_\lambda, \tag{2.11}$$

is compact. By a straightforward evaluation we see that T is also self-adjoint. As a consequence, standard results concerning the spectral decomposition of self-adjoint, compact, linear operators on Hilbert spaces show that Y_0 is the Hilbertian direct sum of the eigenfunctions of T_0, which can be represented as $\varphi_k=[\phi_k]_\lambda$, $k\in\mathbb{N}=\{1,2,\cdots\}$,

$$Y_0 = \overline{\mathrm{Span}\,\{[\phi_k]_\lambda,\; k\in\mathbb{N}\}},$$

for some $\phi_k \in H^1_0(\Omega)$, $k \in \mathbb{N} = \{1, 2, \cdots\}$. In fact, the eigenfunction φ_k, whose eigenvalue is $\mu_k \in \mathbb{R} \setminus \{0\}$, satisfies,

$$\mu_k \varphi_k = \left(G[\tau_\lambda (\rho_\lambda)^{\frac{1}{q}} \varphi_k] - < G[\tau_\lambda (\rho_\lambda)^{\frac{1}{q}} \varphi_k] >_\lambda \right).$$

In other words, by defining,

$$\phi_k := (\tau_\lambda + \sigma_k) G[(\rho_\lambda)^{\frac{1}{q}} \varphi_k],$$

it is easy to see that φ_k is an eigenfunction of T_0 with eigenvalue $\mu_k = \frac{\tau_\lambda}{\tau_\lambda + \sigma_k} \in \mathbb{R} \setminus \{0\}$ if and only if $\phi_k \in H^1_0(\Omega)$ and weakly solves,

$$-\Delta \phi_k = (\tau_\lambda + \sigma_k)(\rho_\lambda)^{\frac{1}{q}} [\phi_k]_\lambda \quad \text{in} \quad \Omega. \tag{2.12}$$

In particular we will use the fact that $\varphi_k = [\phi_k]_\lambda$ and

$$\phi_k = (\tau_\lambda + \sigma_k) G[(\rho_\lambda)^{\frac{1}{q}} [\phi_k]_\lambda], \quad k \in \mathbb{N} = \{1, 2, \cdots\}. \tag{2.13}$$

At this point, standard arguments in the calculus of variations show that,

$$\sigma_1 = \sigma_1(\alpha_\lambda, \psi_\lambda) = \inf_{\phi \in H^1_0(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla \phi|^2 - \tau_\lambda \int_\Omega (\rho_\lambda)^{\frac{1}{q}} [\phi]^2_\lambda}{\int_\Omega (\rho_\lambda)^{\frac{1}{q}} [\phi]^2_\lambda}. \tag{2.14}$$

The ratio in the right hand side of (2.14) is well defined because of Remark 2.1 and in particular we have,

$$\tau_\lambda + \sigma_1 > 0. \tag{2.15}$$

By the Fredholm alternative, if $0 \notin \{\sigma_j\}_{j \in \mathbb{N}}$, then $I - T_0$ is an isomorphism of Y_0 onto itself. Clearly, we can construct an orthonormal base of eigenfunctions satisfying,

$$< [\phi_i]_\lambda, [\phi_j]_\lambda >_\lambda = 0, \quad \forall i \neq j. \tag{2.16}$$

The following easy to verify inequality about the standard first eigenvalue $\nu_{1,\lambda}$ holds,

$$\nu_{1,\lambda} = \inf_{w \in H^1_0(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla w|^2 - \tau_\lambda \int_\Omega (\rho_\lambda)^{\frac{1}{q}} w^2}{\int_\Omega (\rho_\lambda)^{\frac{1}{q}} w^2} \leq \sigma_1(\alpha_\lambda, \psi_\lambda).$$

Actually, we will need the following slightly refined inequality which holds for any $p \in [1, p_N)$,

$$\sigma_1(\alpha_\lambda, \psi_\lambda) > \nu_{1,\lambda}. \tag{2.17}$$

Indeed we have,

$$\frac{\int_\Omega |\nabla \phi|^2 - \tau_\lambda \int_\Omega (\rho_\lambda)^{\frac{1}{q}} [\phi]^2_\lambda}{\int_\Omega (\rho_\lambda)^{\frac{1}{q}} [\phi]^2_\lambda} = \frac{\int_\Omega |\nabla \phi|^2}{\int_\Omega (\rho_\lambda)^{\frac{1}{q}} [\phi]^2_\lambda} - \tau_\lambda = \frac{1}{m_\lambda} \frac{\int_\Omega |\nabla \phi|^2}{< [\phi]^2_\lambda >_\lambda} - \tau_\lambda =$$

$$\frac{1}{m_\lambda} \frac{\int_\Omega |\nabla \phi|^2}{< \phi^2 >_\lambda - < \phi >^2_\lambda} - \tau_\lambda \geq \frac{1}{m_\lambda} \frac{\int_\Omega |\nabla \phi|^2}{< \phi^2 >_\lambda} - \tau_\lambda = \frac{\int_\Omega |\nabla \phi|^2}{\int_\Omega (\rho_\lambda)^{\frac{1}{q}} \phi^2} - \tau_\lambda = \frac{\int_\Omega |\nabla \phi|^2 - \tau_\lambda \int_\Omega (\rho_\lambda)^{\frac{1}{q}} \phi^2}{\int_\Omega (\rho_\lambda)^{\frac{1}{q}} \phi^2}$$

and for $\phi \in H^1_0(\Omega)$ the equality holds if and only if $< \phi >_\lambda = 0$. Therefore, if the equality $\nu_{1,\lambda} = \sigma_1(\alpha_\lambda, \psi_\lambda)$ would hold, then any eigenfunction ϕ_1 of $\nu_{1,\lambda}$ would satisfy $< \phi_1 >_\lambda = 0$, which is obviously impossible since ϕ_1 must have constant sign. Therefore, (2.17) holds as well.

Concerning $D_{\alpha,\psi}\Phi(\lambda, \alpha, \psi)$ we have,

Proposition 2.2. *For any positive solution $(\alpha_\lambda, \psi_\lambda)$ of* $(\mathbf{P})_\lambda$ *with $\lambda \geq 0$, the kernel of $D_{\alpha,\psi}\Phi(\lambda, \alpha_\lambda, \psi_\lambda)$ is empty if and only if the equation,*

$$-\Delta \phi - \tau_\lambda (\rho_\lambda)^{\frac{1}{q}} [\phi]_\lambda = 0, \quad \phi \in C^{2,r}_0(\overline{\Omega}) \tag{2.18}$$

admits only the trivial solution, or equivalently, if and only if 0 is not an eigenvalue of L_λ.

Proof. If $\phi \in H_0^1(\Omega)$ solves (2.18) and since Ω is of class C^3, then by standard elliptic regularity theory and a bootstrap argument we have $\phi \in C_0^{2,r}(\overline{\Omega})$. Therefore, in particular 0 is not an eigenvalue of L_λ if and only if (2.18) admits only the trivial solution.
Suppose first that there exists a non-vanishing pair $(s,\phi) \in \mathbb{R} \times C_0^{2,r}(\overline{\Omega})$ such that

$$D_{\alpha,\psi}\Phi(\lambda,\alpha_\lambda,\psi_\lambda)[s,\phi] = (0,0).$$

Then the equation $\int_\Omega \left(D_\psi \rho_{\lambda,\alpha}[\phi] + d_\alpha \rho_{\lambda,\alpha}[s] \right)\Big|_{(\alpha,\psi)=(\alpha_\lambda,\psi_\lambda)} = 0$ takes the form,

$$p\int_\Omega \left(\lambda(\rho_\lambda)^{\frac{1}{q}}\phi + (\rho_\lambda)^{\frac{1}{q}}s \right) = 0,$$

which is equivalent to $s = s_\lambda = -\lambda < \phi >_\lambda$. Substituting this relation into the first equation, $L_\lambda[\phi] = D_\psi F(\lambda,\alpha_\lambda,\psi_\lambda)[\phi] + d_\alpha F(\lambda,\alpha_\lambda,\psi_\lambda)[s_\lambda] = 0$, we conclude that ϕ is a non-trivial, classical solution of (2.18).
This shows one part of the claim, while on the other side, if a non-trivial, classical solution of (2.18) exists, then by arguing in the other way around, obviously we can find some $(s,\phi) \neq (0,0)$ such that $D_{\alpha,\psi}\Phi(\lambda,\alpha_\lambda,\psi_\lambda) = (0,0)$, as claimed. □

A relevant identity is satisfied by any eigenfunction which we summarize in the following,

Lemma 2.3. *Let* $(\alpha_\lambda,\psi_\lambda)$ *be a solution of* $\mathbf{(P)}_\lambda$ *and let* ϕ_k *be any eigenfunction of an eigenvalue* $\sigma_k = \sigma_k(\alpha_\lambda,\psi_\lambda)$*. Then the following identity holds,*

$$\frac{1}{m_\lambda} < \phi_k >_\lambda \equiv (\alpha_\lambda + \lambda < \psi_\lambda >_\lambda) < \phi_k >_\lambda = (\lambda(p-1) + \sigma_k) < \psi_\lambda[\phi_k]_\lambda >_\lambda . \tag{2.19}$$

Proof. The left hand side identity in (2.19) is an immediate consequence of the following identity,

$$(\alpha_\lambda + \lambda < \psi_\lambda >_\lambda) = \frac{1}{m_\lambda}\left(\int_\Omega (\rho_\lambda)^{\frac{1}{q}}(\alpha_\lambda + \lambda\psi_\lambda) \right) \equiv \frac{1}{m_\lambda}.$$

Therefore, we just need to prove the second equality. By assumption ϕ_k satisfies (2.12) which we multiply by ψ_λ and integrate by parts to obtain,

$$\int_\Omega \rho_\lambda \phi_k = (\tau_\lambda + \sigma_k)\int_\Omega (\rho_\lambda)^{\frac{1}{q}}\psi_\lambda[\phi_k]_\lambda.$$

Dividing by m_λ and since $\rho_\lambda = (\rho_\lambda)^{\frac{1}{q}}(\alpha_\lambda + \lambda\psi_\lambda)$ we find that,

$$\alpha_\lambda < \phi_k >_\lambda + \lambda < \psi_\lambda\phi_k >_\lambda = (\tau_\lambda + \sigma_k) < \psi_\lambda[\phi_k]_\lambda >_\lambda .$$

The conclusion immediately follows by observing that

$$< \psi_\lambda\phi_k >_\lambda = < \psi_\lambda[\phi_k] >_\lambda + < \psi_\lambda >_\lambda < \phi_k >_\lambda .$$

□

We state now the result needed to describe branches of solutions of $\mathbf{(P)}_\lambda$ at regular points.

Lemma 2.4. *Let* $(\alpha_{\lambda_0},\psi_{\lambda_0})$ *be a positive solution of* $\mathbf{(P)}_\lambda$ *with* $\lambda = \lambda_0 \geq 0$.
If 0 *is not an eigenvalue of* L_{λ_0}*, then:*
(i) $D_{\alpha,\psi}\Phi(\lambda_0,\alpha_{\lambda_0},\psi_{\lambda_0})$ *is an isomorphism;*
(ii) There exists an open neighborhood $\mathcal{U} \subset A_\Omega$ *of* $(\lambda_0,\alpha_{\lambda_0},\psi_{\lambda_0})$ *such that the set of solutions of* $\mathbf{(P)}_\lambda$ *in* $\mathcal{U}$ *is a real analytic curve of positive solutions* $J \ni \lambda \mapsto (\alpha_\lambda,\psi_\lambda) \in B$*, for suitable neighborhoods* J *of* λ_0 *and* B *of* $(\alpha_{\lambda_0},\psi_{\lambda_0})$ *in* $(0,+\infty) \times C_{0,+}^{2,r}(\overline{\Omega})$.
(iii) In particular if $(\alpha_{\lambda_0},\psi_{\lambda_0}) = (\alpha_0,\psi_0) = (1,G[1])$*, then* $(\alpha_\lambda,\psi_\lambda) = (1,\psi_0) + \mathrm{O}(\lambda)$ *as* $\lambda \to 0$.

Proof. For the sake of simplicity, in the rest of this proof we set $\lambda_0 = \lambda$.
By the construction of A_Ω, the map F as defined in (2.2) is jointly analytic in a suitable neighborhood of $(\lambda,\alpha_\lambda,\psi_\lambda)$. As a consequence, whenever (i) holds, then (ii) is an immediate consequence of the analytic implicit function theorem, see for example Theorem 4.5.4 in [11]. In particular

(iii) is a straightforward consequence of (ii). Therefore, we are just left with the proof of (i).

(i) Let $(t,f)\in\mathbb{R}\times C^r(\overline{\Omega}\,)$, then we have to prove that the equation,

$$D_{\alpha,\psi}\Phi(\lambda,\alpha_\lambda,\psi_\lambda)[s,\phi]=(t,f),$$

admits a unique solution $(s,\phi)\in\mathbb{R}\times C^{2,r}_0(\overline{\Omega}\,)$. In view of Proposition 2.2, it is enough to prove that it actually admits just one solution. The equation

$$\int_\Omega\left(D_\psi\rho_{\lambda,\alpha}[\phi]+d_\alpha\rho_{\lambda,\alpha}[s]\right)\bigg|_{(\alpha,\psi)=(\alpha_\lambda,\psi_\lambda)}=t$$

takes the form,

$$p\int_\Omega\left(\lambda(\rho_\lambda)^{\frac{1}{q}}\phi+(\rho_\lambda)^{\frac{1}{q}}s\right)=t,$$

which we solve as follows,

$$s=s_\lambda=\frac{t}{pm_\lambda}-\lambda<\phi>_\lambda.$$

Substituting this relation into the first equation, that is,

$$D_\psi F(\lambda,\alpha_\lambda,\psi_\lambda)[\phi]+d_\alpha F(\lambda,\alpha_\lambda,\psi_\lambda)[s_\lambda]=f,$$

by standard elliptic estimates we just need to find a solution $\phi_{f,t}\in C^0(\overline{\Omega}\,)$ of the equation,

$$\phi=G[\tau_\lambda(\rho_\lambda)^{\frac{1}{q}}[\phi]_\lambda]+G\,[f_t]\,,\qquad\text{where } f_t=f+t\frac{(\rho_\lambda)^{\frac{1}{q}}}{m_\lambda}. \tag{2.20}$$

Let $Y_{0,r}=Y_0\cap C^r(\overline{\Omega}\,)$ (see (2.10)) and $Y_{0,r}^\perp$ be the orthogonal complement of $Y_{0,r}$ in $C^r(\overline{\Omega}\,)$. Since $Y_{0,r}^\perp\oplus Y_{0,r}=C^r(\overline{\Omega}\,)$, then after projection on $Y_{0,r}^\perp$ and $Y_{0,r}$, we see that (2.20) is equivalent to the system,

$$\begin{cases}[\phi]_\lambda=[\;G[\tau_\lambda(\rho_\lambda)^{\frac{1}{q}}[\phi]_\lambda]+G[f_t]\;]_\lambda,\\[2ex] <\phi>_\lambda=<G[\tau_\lambda(\rho_\lambda)^{\frac{1}{q}}[\phi]_\lambda]+G[f_t]>_\lambda.\end{cases}$$

Since $0\notin\{\sigma_j(\alpha_\lambda,\psi_\lambda)\}_{j\in\mathbb{N}}$, then $I-T_0$ (see (2.11)) is an isomorphism of $Y_{0,r}$ onto itself. Therefore, the first equation, which has the form $(I-T_0)([\phi]_\lambda)=[G[f_t]]_\lambda$, has a unique solution $\varphi_{f,t}\in Y_{0,r}$. Let

$$\phi_{f,t}:=G[\tau_\lambda(\rho_\lambda)^{\frac{1}{q}}\varphi_{f,t}]+G[f_t]=\varphi_{f,t}+<G[\tau_\lambda(\rho_\lambda)^{\frac{1}{q}}\varphi_{f,t}]+G[f_t]>_\lambda.$$

By standard elliptic regularity theory we find that $\phi_{f,t}\in C^{2,r}_0(\overline{\Omega}\,)$, which in particular is the unique solution of both equations. □

The next result is about the transversality condition needed in section 5.

Theorem 2.5. *Let $(\alpha_\lambda,\psi_\lambda)$ be a positive solution of $\mathbf{(P)}_\lambda$. Suppose that any eigenfunction ϕ_k of a fixed vanishing eigenvalue $\sigma_k=\sigma_k(\alpha_\lambda,\psi_\lambda)=0$ satisfies $<\phi_k>_\lambda\neq 0$. Then σ_k is simple, that is, it admits only one eigenfunction.*
Let $\Omega\subset\mathbb{R}^2$ be symmetric and convex with respect to the coordinate directions x_i, $i=1,2$ and let $(\alpha_\lambda,\psi_\lambda)$ be a positive solution of $\mathbf{(P)}_\lambda$ with $\lambda>0$. Suppose that $\sigma_k=\sigma_k(\alpha_\lambda,\psi_\lambda)=0$ and let ϕ_k be any corresponding eigenfunction. Then:
(i) $<\phi_k>_\lambda\neq 0$;
(ii) $\sigma_k(\alpha_\lambda,\psi_\lambda)$ is simple, that is, it admits at most one eigenfunction.

Proof. If any eigenfunction ϕ_k of a vanishing eigenvalue $\sigma_k=0$ satisfies $<\phi_k>_\lambda\neq 0$, then there can be at most one such eigenfunction. Indeed, if there were more than one such eigenfunctions, say $\phi_{k,\ell}$, $\ell=1,2$, then, putting $a_\ell:=<\phi_{k,\ell}>_\lambda\neq 0$, $\ell=1,2$, we would find that $\phi=\phi_{k,1}-\frac{a_1}{a_2}\phi_{k,2}$ would be an eigenfunction of σ_k satisfying $<\phi>_\lambda=0$, which is a contradiction.

Concerning (i) we argue by contradiction and assume that $<\phi_k>_\lambda = 0$. In view of (2.12) and standard elliptic regularity theory ϕ_k is a classical solution of,

$$\begin{cases} -\Delta\phi_k = \tau_\lambda (\rho_\lambda)^{\frac{1}{q}}\phi_k \quad \text{in } \Omega, \\ \\ \phi_k = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2.21}$$

satisfying $m_\lambda <\phi_k>_\lambda = \int_\Omega (\rho_\lambda)^{\frac{1}{q}}\phi_k = 0$. However, by Theorem 3.1 in [16] (which relies on the symmetry and convexity properties of Ω) the nodal line of any solution of (2.21) cannot intersect the boundary. Here one is also using the fact that $\psi_\lambda \geq 0$ is a solution of $(\mathbf{P})_\lambda$ and that $\rho_\lambda = f_\lambda(\psi_\lambda)$, where $f_\lambda : [0,+\infty) \to [0,+\infty)$ is a C^1 function and $f_\lambda(0) \geq 0$. Therefore, ϕ_k has a fixed sign in a small enough neighborhood of the boundary and then, in particular, by the strong maximum principle we can assume without loss of generality that $\partial_\nu\phi_k = \frac{\partial\phi_k}{\partial\nu} < 0$ on $\partial\Omega$, where ν denotes the exterior unit normal. On the other side, by (2.21) and $<\phi_k>_\lambda = 0$, we see that $\int_{\partial\Omega}\partial_\nu\phi_k = 0$, which is the desired contradiction.
At this point (ii) follows immediately from (i) and the first part of the claim. □

The following result ensures that the first eigenvalue $\sigma_1(\alpha_\lambda,\psi_\lambda)$ of a positive variational solution must be non-negative.

Lemma 2.6. *Let $(\alpha_\lambda,\psi_\lambda)$ be a positive variational solution of $(\mathbf{P})_\lambda$. Then $\sigma_1(\alpha_\lambda,\psi_\lambda) \geq 0$.*

Proof. Since ρ_λ is a minimizer of the variational problem **(VP)** in Appendix A, then the Taylor formula shows that,

$$0 \leq J_\lambda(\rho_\lambda + \varepsilon f) - J_\lambda(\rho_\lambda) = \frac{\varepsilon^2}{2}\mathcal{D}^2 J_\lambda(\rho_\lambda)[f,f] + \mathrm{o}(\varepsilon^2),$$

for any $f \in C^2_0(\overline{\Omega})$ such that $\int_\Omega f = 0$. Therefore, for any such f, we find,

$$\mathcal{D}^2 J_\lambda(\rho_\lambda)[f,f] = \frac{1}{p}\int_\Omega (\rho_\lambda)^{\frac{1}{p}-1} f^2 - \lambda\int_\Omega f G[f] \geq 0,$$

which is well defined since $\rho_\lambda \geq \alpha_\lambda^p > 0$ on $\overline{\Omega}$. Clearly, for any $\varphi \in C^2_0(\overline{\Omega})$ such that $\int_\Omega (\rho_\lambda)^{\frac{1}{q}}\varphi = 0$ we can choose $f = (\rho_\lambda)^{\frac{1}{q}}\varphi$ and deduce that,

$$\int_\Omega (\rho_\lambda)^{\frac{1}{p}-1} f^2 - p\lambda\int_\Omega f G[f] = \int_\Omega (\rho_\lambda)^{\frac{1}{q}}\varphi^2 - p\lambda\int_\Omega (\rho_\lambda)^{\frac{1}{q}}\varphi\, G[(\rho_\lambda)^{\frac{1}{q}}\varphi] \geq 0.$$

Therefore, we have

$$\mathcal{A}(\phi) := \int_\Omega (\rho_\lambda)^{\frac{1}{q}}[\phi]^2_\lambda - \tau_\lambda\int_\Omega (\rho_\lambda)^{\frac{1}{q}}[\phi]_\lambda G[(\rho_\lambda)^{\frac{1}{q}}[\phi]_\lambda] \geq 0, \forall\, \phi \in C^2_0(\overline{\Omega}),$$

where we recall that $[\phi]_\lambda = \phi - <\phi>_\lambda$ and $\tau_\lambda = p\lambda$. On the other side, letting ϕ_1 be the first eigenfunction of (2.9) whose eigenvalue is σ_1, from (2.13) we see that,

$$0 \leq \frac{\mathcal{A}(\phi_1)}{m_\lambda} = <[\phi_1]^2_\lambda>_\lambda - \tau_\lambda <[\phi_1]_\lambda G[(\rho_\lambda)^{\frac{1}{q}}[\phi_1]_\lambda]>_\lambda =$$

$$<[\phi_1]^2_\lambda>_\lambda - \frac{\tau_\lambda}{\tau_\lambda+\sigma_1}<[\phi_1]_\lambda\phi_1>_\lambda = <[\phi_1]^2_\lambda>_\lambda \frac{\sigma_1}{\tau_\lambda+\sigma_1},$$

and then from (2.15) we immediately conclude that $\sigma_1 \geq 0$. □

Finally we obtain a generalization of the bending result of [15] for solutions of $(\mathbf{P})_\lambda$, which is an improvement of Lemma 2.4 in case of simple and vanishing eigenvalues satisfying a suitable transversality condition. We need here to exploit the modified spectral setting in its full strength.

Proposition 2.7. *Let $(\alpha_\lambda, \psi_\lambda)$ be a positive solution of $\mathbf{(P)}_\lambda$ with $\lambda > 0$ and suppose that the k-th eigenvalue $\sigma_k(\alpha_\lambda, \psi_\lambda) = 0$ is simple, that is, it admits only one eigenfunction, $\phi_k \in C^{2,r}_0(\overline{\Omega})$. If $< \phi_k >_\lambda \neq 0$, then there exists $\varepsilon > 0$, an open neighborhood $\mathcal{U}$ of $(\lambda, \alpha_\lambda, \psi_\lambda)$ in A_Ω and a real analytic curve $(-\varepsilon, \varepsilon) \ni s \mapsto (\lambda(s), \alpha(s), \psi(s))$ such that $(\lambda(0), \alpha(0), \psi(0)) = (\lambda, \alpha_\lambda, \psi_\lambda)$ and the set of solutions of $\mathbf{(P)}_\lambda$ in $\mathcal{U}$ has the form $(\lambda(s), \alpha(s), \psi(s))$, where $(\alpha(s), \psi(s))$ is a solution of $\mathbf{(P)}_\lambda$ for $\lambda = \lambda(s)$ for any $s \in (-\varepsilon, \varepsilon)$, with $\psi(s) = \psi_\lambda + s\phi_k + \xi(s)$, and*

$$< [\phi_k]_{\lambda(s)}, \xi(s) >_{\lambda(s)} = 0, \quad s \in (-\varepsilon, \varepsilon). \tag{2.22}$$

Moreover it holds,

$$\xi(0) \equiv 0 \equiv \xi'(0), \quad \alpha'(0) = -\lambda < \psi_\lambda >_\lambda, \quad \lambda'(0) = 0, \quad \psi'(0) = \phi_k, \tag{2.23}$$

and either $\lambda(s) = \lambda$ is constant in $(-\varepsilon, \varepsilon)$ or $\lambda'(s) \neq 0$, $\sigma_k(s) \neq 0$ in $(-\varepsilon, \varepsilon) \setminus \{0\}$, $\sigma_k(s)$ is simple in $(-\varepsilon, \varepsilon)$ and

$$< [\phi_k]_\lambda, \psi_\lambda >_\lambda \neq 0 \text{ and } < [\phi_k]_\lambda, \psi_\lambda >_\lambda \text{ has the same sign as } < \phi_k >_\lambda, \tag{2.24}$$

$$\frac{\sigma_k(s)}{\lambda'(s)} = \frac{p < [\phi_k]_\lambda, \psi_\lambda >_\lambda + \mathrm{o}(1)}{< [\phi_k]^2_\lambda >_\lambda + \mathrm{o}(1)}, \text{ as } s \to 0. \tag{2.25}$$

Proof. Since $< \phi_k >_\lambda \neq 0$ by assumption, then (2.24) is an immediate consequence of (2.19). Take $\varepsilon > 0$ small enough and $\delta_i = \delta_i(\varepsilon) > 0$, $i = 1, 2, 3$, such that

$$(\lambda + \mu, \alpha_\lambda + \beta, \psi_\lambda + s\phi_k + \xi) \subset A_\Omega, \ \forall\, (s, \mu, \beta, \xi) \in (-\varepsilon, \varepsilon) \times (-\delta_1(\varepsilon), \delta_1(\varepsilon)) \times (-\delta_2(\varepsilon), \delta_2(\varepsilon)) \times B^{\perp}_{k,\varepsilon},$$

where

$$B^{\perp}_{k,\varepsilon} = \left\{ \xi \in C^{2,r}_0(\overline{\Omega}) \, :< [\phi_k]_\lambda, \xi >_\lambda = 0, \ \|\xi\|_{C^{2,r}_0(\overline{\Omega})} < \delta_3(\varepsilon) \right\}.$$

Next, let us introduce the map,

$$\Phi_0 : (-\varepsilon, \varepsilon) \times (-\delta_1(\varepsilon), \delta_1(\varepsilon)) \times (-\delta_2(\varepsilon), \delta_2(\varepsilon)) \times B^{\perp}_{k,\varepsilon} \to \mathbb{R} \times C^r(\overline{\Omega}),$$

defined as follows,

$$\Phi_0(s, \mu, \beta, \xi) = \begin{pmatrix} -\Delta(\psi_\lambda + s\phi_k + \xi) - (\alpha_\lambda + \beta + (\lambda + \mu)(\psi_\lambda + s\phi_k + \xi))^p \\ \\ -1 + \int_\Omega (\alpha_\lambda + \beta + (\lambda + \mu)(\psi_\lambda + s\phi_k + \xi))^p \end{pmatrix}.$$

Clearly, with the notations introduced in (2.3), we have

$$\Phi_0(0, 0, 0, 0) = \Phi(\lambda, \alpha_\lambda, \psi_\lambda) = (0, 0).$$

Let us define,

$$X^{\perp}_k = \left\{ \phi \in C^{2,r}_0(\overline{\Omega}) \, :< [\phi_k]_\lambda, \phi >_\lambda = 0 \right\},$$

then the differential of Φ_0 with respect to (μ, β, ξ) at $(0, 0, 0, 0)$ acts on a triple $(s_\mu, s_\beta, \phi) \in \mathbb{R} \times \mathbb{R} \times X^{\perp}_k$ as follows,

$$D_{\mu,\beta,\xi}\Phi_0(0, 0, 0, 0)[s_\mu, s_\beta, \phi] = \begin{pmatrix} D_\psi F(\lambda, \alpha_\lambda, \psi_\lambda)[\phi] - p(\rho_\lambda)^{\frac{1}{q}} \psi_\lambda s_\mu - p(\rho_\lambda)^{\frac{1}{q}} s_\beta \\ \\ \tau_\lambda \int_\Omega (\rho_\lambda)^{\frac{1}{q}} \phi + p s_\mu \int_\Omega (\rho_\lambda)^{\frac{1}{q}} \psi_\lambda + p s_\beta \int_\Omega (\rho_\lambda)^{\frac{1}{q}} \end{pmatrix},$$

with $\tau_\lambda = p\lambda$ and $D_\psi F$ as in (2.4). The crux of the argument is to prove the following:

Lemma 2.8. *$D_{\mu,\beta,\xi}\Phi_0(0, 0, 0, 0)$ is an isomorphism of $\mathbb{R} \times \mathbb{R} \times X^{\perp}_k$ onto $\mathbb{R} \times C^r(\overline{\Omega})$.*

Proof. We will prove that for each $(t, f) \in \mathbb{R} \times C^r(\overline{\Omega})$, the vectorial equation

$$D_{\mu,\beta,\xi}\Phi_0(0, 0, 0, 0)[s_\mu, s_\beta, \phi] = \begin{pmatrix} f \\ \\ t \end{pmatrix},$$

admits a unique solution $(s_\mu, s_\beta, \phi) \in \mathbb{R} \times \mathbb{R} \times X_k^\perp$. From the second equation we deduce that,

$$s_\beta = -\frac{t}{pm_\lambda} - < \psi_\lambda >_\lambda s_\mu - \lambda < \phi >_\lambda,$$

where we recall that $m_\lambda = \int_\Omega (\rho_\lambda)^{\frac{1}{q}}$. Therefore, by substituting into the first equation, we find that the pair (s_μ, ϕ) solves the equation,

$$-\Delta\phi - \tau_\lambda (\rho_\lambda)^{\frac{1}{q}}[\phi]_\lambda - p(\rho_\lambda)^{\frac{1}{q}}[\psi_\lambda]_\lambda s_\mu = f + t\frac{(\rho_\lambda)^{\frac{1}{q}}}{m_\lambda}. \tag{2.26}$$

Let us write $C^r(\overline{\Omega}) = Y_k \oplus R$, where $Y_k = \mathrm{span}\{\phi_k\}$ and $R = \big\{f \in C^r(\overline{\Omega}) \,:\, \int_\Omega \phi_k f = 0\big\}$. Then, projecting (2.26) along Y_k, we find that,

$$-m_\lambda p < [\psi_\lambda]_\lambda, \phi_k > s_\mu = \int_\Omega \left(f + t\frac{(\rho_\lambda)^{\frac{1}{q}}}{m_\lambda} \right) \phi_k. \tag{2.27}$$

At this point it follows from (2.24) that (2.27) admits a unique solution s_μ. So we are left with showing that the projection of (2.26) onto R admits a unique solution as well.
First of all, let us observe that, since ϕ_k satisfies (2.12) with $\sigma_k = 0$, then the projection of (2.26) onto R takes the form,

$$-\Delta\phi - \tau_\lambda (\rho_\lambda)^{\frac{1}{q}}[\phi]_\lambda = g,$$

where

$$g = P_R \left(f + t\frac{(\rho_\lambda)^{\frac{1}{q}}}{m_\lambda} + p(\rho_\lambda)^{\frac{1}{q}}[\psi_\lambda]_\lambda s_\mu \right),$$

where P_R denotes the projection operator. Since $\alpha_\lambda > 0$ and $\lambda > 0$ by assumption, then $\rho_\lambda \geq \alpha_\lambda^p$ in $\overline{\Omega}$ and then, by standard elliptic estimates, it is enough to show that the equation,

$$\phi = \tau_\lambda G[(\rho_\lambda)^{\frac{1}{q}}[\phi]_\lambda] + G[(\rho_\lambda)^{\frac{1}{q}} g_1], \tag{2.28}$$

admits a unique solution $\phi \in Y_k^\perp := \{f \in C^r(\overline{\Omega}) \,:< [\phi_k]_\lambda, f >_\lambda = 0\}$, for any fixed $g_1 \in C^r(\overline{\Omega})$ satisfying $< g_1, \phi_k >_\lambda = 0$, where $< [\phi_k]_\lambda^2 >_\lambda = 1$. Next, let us decompose,

$$\phi = c_0 + c_k[\phi_k]_\lambda + \varphi, \quad g_1 = b_0 + b_k[\phi_k]_\lambda + g_2,$$

where $c_0 =< \phi >_\lambda$, $b_0 =< g_1 >_\lambda$, $c_k =< [\phi_k]_\lambda, \phi >_\lambda$, $b_k =< [\phi_k]_\lambda, g_1 >_\lambda$, $\{\varphi, g_2\} \in Y_{k,0} := Y_0 \cap Y_k^\perp$, $Y_0 = \{f \in C^r(\overline{\Omega}) \,:< f >_\lambda = 0\}$. It is easy to check, using (2.13), that we have $c_k = 0$, $b_k + b_0 < \phi_k >_\lambda = 0$ and

$$< G[(\rho_\lambda)^{\frac{1}{q}}\varphi], [\phi_k]_\lambda >_\lambda = 0 =< G[(\rho_\lambda)^{\frac{1}{q}} g_2], [\phi_k]_\lambda >_\lambda .$$

At this point, by using also (2.13), then a lengthy evaluation shows that (2.28) is equivalent to the system,

$$\begin{cases} c_0 = b_0 < G[(\rho_\lambda)^{\frac{1}{q}}] >_\lambda + \tau_\lambda < G[(\rho_\lambda)^{\frac{1}{q}}\varphi] >_\lambda + \frac{b_k}{\tau_\lambda} < \phi_k >_\lambda + < G[(\rho_\lambda)^{\frac{1}{q}} g_2] >_\lambda, \\ \varphi - \tau_\lambda [G[(\rho_\lambda)^{\frac{1}{q}}\varphi]]_\lambda = [G[(\rho_\lambda)^{\frac{1}{q}} g_2]]_\lambda + b_0 \Big([G[(\rho_\lambda)^{\frac{1}{q}}]]_\lambda - < [G[(\rho_\lambda)^{\frac{1}{q}}]]_\lambda, [\phi_k]_\lambda >_\lambda [\phi_k]_\lambda \Big), \\ b_0 < [G[(\rho_\lambda)^{\frac{1}{q}}]]_\lambda, [\phi_k]_\lambda >_\lambda + \frac{b_k}{\tau_\lambda} = 0, \end{cases}$$

with the constraint $b_k + b_0 < \phi_k >_\lambda = 0$. By using (2.13) once more we see that $< [G[(\rho_\lambda)^{\frac{1}{q}}]]_\lambda, [\phi_k]_\lambda >_\lambda = \frac{<\phi_k>_\lambda}{\tau_\lambda}$, whence the third equation is equivalent to $b_k + b_0 < \phi_k >_\lambda = 0$ and we are left with the first two equations. However the second equation takes the form,

$$(I - T_0)|_{Y_{k,0}}(\varphi) = f_2,$$

for a suitable $f_2 \in Y_{k,0}$, where T_0 is given in (2.11). Since $(I - T_0)|_{Y_{k,0}}(h) = 0$ has only the trivial solution, then by the Fredholm alternative it admits a unique solution, say φ_2. As a consequence φ_2 uniquely defines c_0 in the first equation, as claimed. $\square$

In view of Lemma 2.8, the existence of the real analytic map $(-\varepsilon, \varepsilon) \ni s \mapsto (\lambda(s), \alpha(s), \psi(s))$ satisfying the desired properties (including (2.22)) follows by the analytic implicit function theorem ([11]). Therefore, we are left with the proof of (2.23) and (2.25).
First of all, differentiating the constraint in $\mathbf{(P)}_\lambda$ with $\lambda(s) = \lambda + \mu(s)$, $\alpha(s) = \alpha_\lambda + \beta(s)$, $\psi(s) = \psi_\lambda + s\phi_k + \xi(s)$ we obtain,

$$p \int_\Omega (\rho_\lambda)^{\frac{1}{q}} \left(\beta'(0) + \mu'(0)\psi_\lambda + \lambda(\phi_k + \xi'(0)) \right) = 0,$$

which immediately implies that,

$$\beta'(0) = -\mu'(0) < \psi_\lambda >_\lambda - \lambda < \phi_k >_\lambda - \lambda < \xi'(0) >_\lambda . \tag{2.29}$$

Thus, by differentiating the equation in $\mathbf{(P)}_\lambda$ we obtain,

$$-\Delta(\phi_k + \xi'(0)) = p(\rho_\lambda)^{\frac{1}{q}} \left(\beta'(0) + \mu'(0)\psi_\lambda + \lambda(\phi_k + \xi'(0)) \right),$$

and so, by using (2.12) and (2.29), we conclude that,

$$-\Delta\xi'(0) = \tau_\lambda(\rho_\lambda)^{\frac{1}{q}}[\xi'(0)]_\lambda + p\mu'(0)(\rho_\lambda)^{\frac{1}{q}}[\psi_\lambda]_\lambda, \tag{2.30}$$

where, since $\xi(0) \equiv 0$, then differentiating (2.22), we also have $< [\phi_k]_\lambda, \xi'(0) >_\lambda = 0$. Therefore, multiplying (2.30) by ϕ_k and integrating by parts, we find that,

$$0 = m_\lambda \tau_\lambda < [\phi_k]_\lambda, \xi'(0) >_\lambda = \int_\Omega (-\Delta\phi_k)\xi'(0) =$$

$$\int_\Omega \phi_k(-\Delta\xi'(0)) = \int_\Omega \tau_\lambda(\rho_\lambda)^{\frac{1}{q}}[\xi'(0)]_\lambda\phi_k + p\mu'(0)\int_\Omega (\rho_\lambda)^{\frac{1}{q}}[\psi_\lambda]_\lambda\phi_k =$$

$$m_\lambda\tau_\lambda < [\phi_k]_\lambda, \xi'(0) >_\lambda + m_\lambda p\mu'(0) < [\phi_k]_\lambda, \psi_\lambda >_\lambda,$$

that is,

$$\mu'(0) < [\phi_k]_\lambda, \psi_\lambda >_\lambda = 0,$$

which, in view of (2.24), yields $\mu'(0) = 0$. Inserting this information back in (2.30), in view of $< [\phi_k]_\lambda, \xi'(0) >_\lambda = 0$, and since we assumed that σ_k is simple, then we conclude that $\xi'(0) \equiv 0$. Therefore, (2.29) shows that $\beta'(0) = -\lambda < \psi_\lambda >_\lambda$, which concludes the proof of (2.23).
Concerning (2.25), let us first observe that, since λ is real analytic in $(-\varepsilon, \varepsilon)$, then either $\lambda(s) = \lambda$ is constant or $\lambda'(s) \neq 0$ in $(-\varepsilon, \varepsilon) \setminus \{0\}$ for ε sufficiently small. Also, $\sigma_k(s) = \sigma_k(\alpha(s), \psi(s))$ is simple in $(-\varepsilon, \varepsilon)$ by the analytic perturbation theory of simple eigenvalues, see Proposition 4.5.8 in [11]. Clearly, in view of (2.24), we can also assume that $< [\psi(s)]_{\lambda(s)}, \phi_k(s) >_{\lambda(s)} \neq 0$ for any such s. Next, by arguing as in (2.30) we see that,

$$-\Delta\psi'(s) = p\lambda(s)\rho_{\lambda(s)}[\psi'(s)]_{\lambda(s)} + p\lambda'(s)\rho_{\lambda(s)}[\psi(s)]_{\lambda(s)}, \tag{2.31}$$

and so, multiplying (2.31) by $\phi_k(s)$, integrating by parts and using (2.12) with $\lambda = \lambda(s)$, then we conclude that,

$$\sigma_k(s) \int_\Omega \rho_{\lambda(s)}[\phi_k(s)]_{\lambda(s)}\psi'(s) = p\lambda'(s) \int_\Omega \rho_{\lambda(s)}[\phi_k(s)]_{\lambda(s)}\psi(s). \tag{2.32}$$

At this point we observe that, since $< [\psi(s)]_{\lambda(s)}, \phi_k(s) >_{\lambda(s)} \neq 0$, and $\int_\Omega \rho_{\lambda(s)}[\phi_k(s)]_{\lambda(s)}\psi'(s) = \mathrm{o}(1) + \int_\Omega [\phi_k]_\lambda^2$, as $s \to 0$, then, if $\lambda'(s) \neq 0$, we can write,

$$\frac{\sigma_k(s)}{\lambda'(s)} = \frac{p\int_\Omega \rho_{\lambda(s)}[\phi_k(s)]_{\lambda(s)}\psi(s) + \mathrm{o}(1)}{\int_\Omega \rho_{\lambda(s)}[\phi_k(s)]_{\lambda(s)}\psi'(s) + \mathrm{o}(1)},$$

for any s small enough, thus,

$$\frac{\sigma_k(s)}{\lambda'(s)} = \frac{p < [\phi_k]_\lambda, \psi_\lambda >_\lambda + o(1)}{< [\phi_k]_\lambda^2 >_\lambda + o(1)}, \text{ as } s \to 0,$$

which is (2.25). □

3. Monotonicity of solutions

In this section we are concerned with the monotonicity of the energy

$$2E_\lambda = \int_\Omega \rho_\lambda \psi_\lambda,$$

and of α_λ for positive solutions of $(\mathbf{P})_\lambda$ with $\sigma_1(\alpha_\lambda, \psi_\lambda) > 0$, see (2.14). The latter property is detected by making use of the spectral setting set up in the previous section. We recall that $m_\lambda = \int_\Omega (\rho_\lambda)^{\frac{1}{q}}$ and L_λ is the linearized operator defined in (2.8).

Proposition 3.1. *Let $(\alpha_{\lambda_0}, \psi_{\lambda_0})$ be a positive solution of $(\mathbf{P})_\lambda$ with $\lambda_0 \geq 0$ and suppose that 0 is not an eigenvalue of L_{λ_0}. Then, locally near λ_0, the map $\lambda \mapsto (\alpha_\lambda, \psi_\lambda)$ is a real analytic simple curve of positive solutions and*

$$2E_\lambda = \int_\Omega \rho_\lambda \psi_\lambda \quad \textit{and} \quad \eta_\lambda = \frac{d\psi_\lambda}{d\lambda},$$

are real analytic functions of λ with $\eta_\lambda \in C^2_0(\overline{\Omega})$. In particular we have:

$$\frac{dE_\lambda}{d\lambda} = \int_\Omega \rho_\lambda \eta_\lambda = m_\lambda \tau_\lambda < [\eta_\lambda]_\lambda, [\psi_\lambda]_\lambda >_\lambda + m_\lambda p \| [\psi_\lambda]_\lambda \|_\lambda^2. \tag{3.1}$$

Finally, if $\sigma_1 = \sigma_1(\alpha_\lambda, \psi_\lambda) > 0$, then,

$$< [\eta_\lambda]_\lambda, [\psi_\lambda]_\lambda >_\lambda \geq \frac{\sigma_1}{p} \| [\eta_\lambda]_\lambda \|_\lambda^2, \tag{3.2}$$

and in particular $\frac{dE_\lambda}{d\lambda} > 0$.

Proof. Clearly, $\frac{dE_\lambda}{d\lambda} > 0$ will follow immediately once we have (3.1), (3.2).
Since by assumption 0 is not an eigenvalue of L_λ, then we can apply Lemma 2.4. Therefore, locally around $(\alpha_{\lambda_0}, \psi_{\lambda_0})$, $(\alpha_\lambda, \psi_\lambda)$ is a real analytic function of λ and then, by standard elliptic estimates, we see that $\eta_\lambda \in C^2_0(\overline{\Omega})$ is a classical solution of,

$$-\Delta \eta_\lambda = \tau_\lambda (\rho_\lambda)^{\frac{1}{q}} \eta_\lambda + p(\rho_\lambda)^{\frac{1}{q}} \psi_\lambda + p(\rho_\lambda)^{\frac{1}{q}} \frac{d\alpha_\lambda}{d\lambda},$$

where $\frac{d\alpha_\lambda}{d\lambda}$ can be computed by the unit mass constraint in $(\mathbf{P})_\lambda$, that is

$$p \frac{d\alpha_\lambda}{d\lambda} = -\tau_\lambda < \eta_\lambda >_\lambda - p < \psi_\lambda >_\lambda .$$

Therefore, we conclude that η_λ is a solution of,

$$-\Delta \eta_\lambda = \tau_\lambda (\rho_\lambda)^{\frac{1}{q}} [\eta_\lambda]_\lambda + p(\rho_\lambda)^{\frac{1}{q}} [\psi_\lambda]_\lambda. \tag{3.3}$$

By using $(\mathbf{P})_\lambda$, we also have $E_\lambda = \frac{1}{2} \int_\Omega |\nabla \psi_\lambda|^2$, and in particular it holds,

$$\frac{d}{d\lambda} E_\lambda = \int_\Omega (\nabla \eta_\lambda, \nabla \psi_\lambda) = - \int_\Omega \eta_\lambda (\Delta \psi_\lambda) = \int_\Omega \rho_\lambda \eta_\lambda,$$

which proves the first equality sign in (3.1). At this point, by using $(\mathbf{P})_\lambda$ and (3.3), we see that,

$$\int_\Omega \rho_\lambda \eta_\lambda = \int_\Omega -(\Delta \psi_\lambda) \eta_\lambda = \int_\Omega -\psi_\lambda (\Delta \eta_\lambda) = m_\lambda \tau_\lambda < \psi_\lambda, [\eta_\lambda]_\lambda >_\lambda + m_\lambda p < \psi_\lambda, [\psi_\lambda]_\lambda >_\lambda, \tag{3.4}$$

which proves the second equality sign in (3.1).

For the last part of the statement, let

$$[\psi_\lambda]_\lambda = \sum_{j=1}^{+\infty} \xi_j [\phi_j]_\lambda, \quad [\eta_\lambda]_\lambda = \sum_{j=1}^{+\infty} \beta_j [\phi_j]_\lambda,$$

$$\xi_j = \int_\Omega (\rho_\lambda)^{\frac{1}{q}} [\phi_j]_\lambda [\psi_\lambda]_\lambda, \quad \beta_j = \int_\Omega (\rho_\lambda)^{\frac{1}{q}} [\phi_j]_\lambda [\eta_\lambda]_\lambda,$$

be the Fourier expansions of $[\psi_\lambda]_\lambda$ and $[\eta_\lambda]_\lambda$ in Y_0 (see (2.10)), with respect to the normalized projections $[\phi_j]_\lambda$, satisfying $\|[\phi_j]_\lambda\|_\lambda = 1$. After multiplying (3.3) by ϕ_j, using (2.9) and integrating by parts, we have,

$$\sigma_j \int_\Omega (\rho_\lambda)^{\frac{1}{q}} [\phi_j]_\lambda [\eta_\lambda]_\lambda = p \int_\Omega (\rho_\lambda)^{\frac{1}{q}} [\phi_j]_\lambda [\psi_\lambda]_\lambda, \text{ that is } \sigma_j \beta_j = p \xi_j, \tag{3.5}$$

where $\sigma_j = \sigma_j(\alpha_\lambda, \psi_\lambda)$. As a consequence, since $\sum\limits_{j=1}^{+\infty} (\beta_j)^2 =< [\eta_\lambda]_\lambda^2 >_\lambda$ and $\sum\limits_{j=1}^{+\infty} (\xi_j)^2 =< [\psi_\lambda]_\lambda^2 >_\lambda$, then we find that,

$$p < [\psi_\lambda]_\lambda, [\eta_\lambda]_\lambda >_\lambda = p \sum_{j=1}^{+\infty} \xi_j \beta_j = \sum_{j=1}^{+\infty} \sigma_j (\beta_j)^2 \geq \sigma_1 < [\eta_\lambda]_\lambda^2 >_\lambda, \tag{3.6}$$

which proves (3.2). □

Next we prove the monotonicity of α_λ whenever $\sigma_1(\alpha_\lambda, \psi_\lambda) > 0$.
For this and later purposes it is convenient to use the auxiliary function $u_\lambda = \lambda \psi_\lambda$ which satisfies

$$\begin{cases} -\Delta u_\lambda = \lambda\, (\alpha_\lambda + u_\lambda)^p \quad \text{in } \Omega \\ \int\limits_\Omega (\alpha_\lambda + u_\lambda)^p = 1 \\ u_\lambda > 0 \text{ in } \Omega, \quad u_\lambda = 0 \text{ on } \partial\Omega \\ \alpha_\lambda \geq 0 \end{cases} \tag{3.7}$$

where $\rho_\lambda = (\alpha_\lambda + u_\lambda)^p$, $(\rho_\lambda)^{\frac{1}{q}} = (\alpha_\lambda + u_\lambda)^{p-1}$.

Proposition 3.2. *Let $(\alpha_\lambda, \psi_\lambda)$ be a positive solution of* $\mathbf{(P)}_\lambda$ *(or either $(\alpha_\lambda, u_\lambda)$ a solution of (3.7), $\alpha_\lambda > 0$) with $\lambda \geq 0$. If $\sigma_1 = \sigma_1(\alpha_\lambda, \psi_\lambda) > 0$ then $\dfrac{d\alpha_\lambda}{d\lambda} < 0$.*

Proof. We recall that, by Lemma 2.4, if 0 is not an eigenvalue of L_λ, then ψ_λ and α_λ are locally real analytic functions of λ. Therefore, u_λ is also real analytic as a function of λ and $w_\lambda = \frac{du_\lambda}{d\lambda} \in C^2_0(\overline{\Omega})$ satisfies

$$\begin{cases} -\Delta w_\lambda = \tau_\lambda (\rho_\lambda)^{\frac{1}{q}} [w_\lambda]_\lambda + \rho_\lambda \quad \text{in } \Omega \\ w_\lambda = 0 \quad \text{on } \partial\Omega \end{cases} \tag{3.8}$$

where, since $\int_\Omega (\alpha_\lambda + u_\lambda)^p = 1$, then,

$$\frac{d\alpha_\lambda}{d\lambda} = - < w_\lambda >_\lambda . \tag{3.9}$$

We first prove the claim for $p > 1$ and $\lambda > 0$. Multiplying the equation in (3.8) by w_λ and integrating by parts we find that,

$$\alpha_\lambda \int_\Omega (\rho_\lambda)^{\frac{1}{q}} w_\lambda + \int_\Omega (\rho_\lambda)^{\frac{1}{q}} w_\lambda u_\lambda = \int_\Omega \rho_\lambda w_\lambda = \int_\Omega |\nabla w_\lambda|^2 - \tau_\lambda \int_\Omega (\rho_\lambda)^{\frac{1}{q}} [w_\lambda]_\lambda^2 \geq \tag{3.10}$$

$$\sigma_1(\alpha_\lambda,\psi_\lambda)\int_\Omega (\rho_\lambda)^{\frac{1}{q}}[w_\lambda]^2_\lambda.$$

On the other side, multiplying the equation in (3.7) by w_λ and integrating by parts we also find that,

$$\tau_\lambda \int_\Omega (\rho_\lambda)^{\frac{1}{q}}[w_\lambda]u_\lambda + \int_\Omega \rho_\lambda u_\lambda = \lambda\int_\Omega \rho_\lambda w_\lambda = \lambda\alpha_\lambda \int_\Omega (\rho_\lambda)^{\frac{1}{q}} w_\lambda + \lambda\int_\Omega (\rho_\lambda)^{\frac{1}{q}} w_\lambda u_\lambda, \tag{3.11}$$

which, after a straightforward evaluation yields,

$$\int_\Omega (\rho_\lambda)^{\frac{1}{q}} w_\lambda u_\lambda = \frac{1}{p-1}(p < u_\lambda >_\lambda + \alpha_\lambda)\int_\Omega (\rho_\lambda)^{\frac{1}{q}} w_\lambda - \frac{1}{\lambda(p-1)}\int_\Omega \rho_\lambda u_\lambda.$$

Substituting this expression of $\int_\Omega (\rho_\lambda)^{\frac{1}{q}} w_\lambda u_\lambda$ in (3.10) we obtain,

$$\tau_\lambda\left(\alpha_\lambda + < u_\lambda >_\lambda\right)\int_\Omega (\rho_\lambda)^{\frac{1}{q}} w_\lambda - \int_\Omega \rho_\lambda u_\lambda \geq \lambda(p-1)\sigma_1(\alpha_\lambda,\psi_\lambda)\int_\Omega (\rho_\lambda)^{\frac{1}{q}}[w_\lambda]^2_\lambda.$$

In other words we conclude that,

$$\tau_\lambda < w_\lambda >_\lambda = \frac{\tau_\lambda}{m_\lambda}\int_\Omega (\rho_\lambda)^{\frac{1}{q}} w_\lambda \geq \lambda(p-1)\sigma_1(\alpha_\lambda,\psi_\lambda)\int_\Omega (\rho_\lambda)^{\frac{1}{q}}[w_\lambda]^2_\lambda + \int_\Omega \rho_\lambda u_\lambda,$$

where we are using the identity

$$\alpha_\lambda + < u_\lambda >_\lambda = \frac{\int_\Omega (\rho_\lambda)^{\frac{1}{q}}(\alpha_\lambda + u_\lambda)}{m_\lambda} = \frac{\int_\Omega \rho_\lambda}{m_\lambda} = \frac{1}{m_\lambda}.$$

At this point, since $u_\lambda \geq 0$, $\frac{d\alpha_\lambda}{d\lambda} < 0$ immediately follows from (3.9).
For $p = 1$ and $\lambda > 0$ after a straightforward evaluation we deduce from (3.11) that $< w_\lambda >_\lambda = 2E_\lambda$ and then the conclusion follows from (3.9). For $p \geq 1$ and $\lambda = 0$ we have $w_\lambda = \psi_0$ and $< w_\lambda >_\lambda = 2E_0$ and the conclusion follows again from (3.9). □

We show next a sharp energy estimate for any solution of $(\mathbf{P})_\lambda$ with $N = 2$, depending only on the exponent p.

Proposition 3.3. *Let $N = 2$ and $p \in [1,+\infty)$. Let $(\alpha_\lambda,\psi_\lambda)$ be a solution of $(\mathbf{P})_\lambda$ (or equivalently $(\alpha_\lambda,u_\lambda)$ be a solution of (3.7)). Then it holds,*

$$2\lambda\left(\frac{p+1}{16\pi} - E_\lambda\right) \geq \alpha_\lambda(1-\alpha_\lambda^p), \tag{3.12}$$

where the equality holds if and only if $\Omega = \mathbb{D}_2$. In particular,

$$E_\lambda \leq \frac{p+1}{16\pi}, \tag{3.13}$$

and the equality holds if and only if $\Omega = \mathbb{D}_2$ and $\alpha_\lambda = 0$.

Proof. Let $\theta_\lambda = \|u_\lambda\|_{L^\infty(\Omega)}$ and set

$$\Omega(t) = \{x \in \Omega \,:\, u_\lambda > t\},\quad \Gamma(t) = \{x \in \Omega \,:\, u_\lambda = t\},\quad t \in [0,\theta_\lambda],$$

and

$$m(t) = \lambda \int\limits_{\Omega(t)} (\alpha_\lambda + u_\lambda)^p\,,\qquad \mu(t) = |\Omega(t)|,\qquad e(t) = \int\limits_{\Omega(t)} |\nabla u_\lambda|^2,$$

where $|\Omega(t)|$ is the area of $\Omega(t)$. If $\lambda = 0$ then (3.12) is trivially satisfied and (3.13) follows from (4.3). Hence, we consider now $\lambda > 0$. Since $|\Delta u_\lambda|$ is bounded below away from zero and since the boundary is smooth, then it is not difficult to see that actually $m(t)$ and $\mu(t)$ are continuous in $[0,\theta_\lambda]$ and piecewise smooth in $[0,\theta_\lambda]$, that is, of class C^1 with the exception of a finite number of points in $[0,\theta_\lambda]$. In particular the level sets have vanishing area $|\Gamma(t)|=0$ for any t and we will use the fact that,

$$m(0)=\lambda,\quad \mu(0)=1,\quad e(0)=\int\limits_{\Omega}|\nabla u_\lambda|^2\equiv 2\lambda^2 E_\lambda,$$

and

$$m(\theta_\lambda)=0,\quad \mu(\theta_\lambda)=0,\quad e(\theta_\lambda)=0.$$

By the co-area formula and the Sard Lemma we have,

$$-m^{'}(t)=\lambda\int\limits_{\Gamma(t)}\frac{(\alpha_\lambda+u_\lambda)^p}{|\nabla u_\lambda|}=\lambda\,(\alpha_\lambda+t)^p\int\limits_{\Gamma(t)}\frac{1}{|\nabla u_\lambda|}=\lambda\,(\alpha_\lambda+t)^p\,(-\mu^{'}(t)),\tag{3.14}$$

and

$$m(t)=-\int\limits_{\Omega(t)}\Delta u_\lambda=\int\limits_{\Gamma(t)}|\nabla u_\lambda|=-e^{'}(t),\tag{3.15}$$

for a.a. $t\in[0,\theta_\lambda]$. By the Schwarz inequality and the isoperimetric inequality we find that,

$$-m^{'}(t)m(t)=\lambda\int\limits_{\Gamma(t)}\frac{(\alpha_\lambda+u_\lambda)^p}{|\nabla u_\lambda|}\int\limits_{\Gamma(t)}|\nabla u_\lambda|=\lambda\,(\alpha_\lambda+t)^p\int\limits_{\Gamma(t)}\frac{1}{|\nabla u_\lambda|}\int\limits_{\Gamma(t)}|\nabla u_\lambda|\geq$$

$$\lambda\,(\alpha_\lambda+t)^p\,(|\Gamma(t)|_1)^2\geq\lambda\,(\alpha_\lambda+t)^p\,4\pi\mu(t),\ \text{for a.a. } t\in[0,\theta_\lambda],$$

where $|\Gamma(t)|_1$ denotes the length of $\Gamma(t)$. Therefore, we conclude that,

$$\frac{(m^2(t))^{'}}{8\pi}+\lambda\,(\alpha_\lambda+t)^p\,\mu(t)\leq 0,\ \text{for a.a. } t\in[0,\theta_\lambda].\tag{3.16}$$

By using the following identity,

$$(\alpha_\lambda+t)^p\,\mu(t)=\frac{1}{p+1}\left((\alpha_\lambda+t)^{p+1}\,\mu(t)\right)^{'}-\frac{1}{p+1}\,(\alpha_\lambda+t)^{p+1}\,\mu^{'}(t),\ \text{for a.a. } t\in[0,\theta_\lambda],$$

together with (3.16) and (3.14) we conclude that,

$$\left(\frac{m^2(t)}{8\pi}+\frac{\lambda}{p+1}\,(\alpha_\lambda+t)^{p+1}\,\mu(t)\right)^{'}-\frac{1}{p+1}\,(\alpha_\lambda+t)\,m^{'}(t)\leq 0,\ \text{for a.a. } t\in[0,\theta_\lambda].$$

Therefore, we see that,

$$-\frac{m^2(t)}{8\pi}-\frac{\lambda}{p+1}\,(\alpha_\lambda+t)^{p+1}\,\mu(t)+\frac{1}{p+1}\alpha_\lambda m(t)-\frac{1}{p+1}\int\limits_t^{\theta_\lambda}ds\,m^{'}(s)s\leq 0,\ \forall\, t\in[0,\theta_\lambda).$$

Clearly, by using (3.15), we have that,

$$-\int\limits_t^{\theta_\lambda}ds\,m^{'}(s)s=tm(t)+\int\limits_t^{\theta_\lambda}ds\,m(s)=tm(t)+e(t),$$

and we conclude that,

$$-\frac{m^2(t)}{8\pi}-\frac{\lambda}{p+1}\,(\alpha_\lambda+t)^{p+1}\,\mu(t)+\frac{1}{p+1}\,(\alpha_\lambda+t)\,m(t)+\frac{1}{p+1}e(t)\leq 0,\ \forall\, t\in[0,\theta_\lambda).\tag{3.17}$$

Evaluating (3.17) at $t=0$ we find that

$$-\frac{m^2(0)}{8\pi}-\frac{\lambda}{p+1}\alpha_\lambda^{p+1}\mu(0)+\frac{1}{p+1}\alpha_\lambda m(0)+\frac{1}{p+1}e(0)=$$

$$-\frac{\lambda^2}{8\pi}-\frac{\lambda}{p+1}\alpha_\lambda^{p+1}+\frac{\lambda}{p+1}\alpha_\lambda+\frac{2\lambda^2}{p+1}E_\lambda\leq 0,\tag{3.18}$$

which is (3.12). It is readily seen that the equality holds if and only if $\Gamma(t)$ is a disk for any t, whence if and only if u_λ is radial and $\Omega = \mathbb{D}_2$. The inequality (3.13) is a straightforward consequence of (3.12) and the fact that $\alpha_\lambda \leq 1$. Concerning the characterization of the equality sign in (3.13) we observe that if the equality holds, then necessarily $\alpha_\lambda(1-\alpha_\lambda^p) = 0$ and in particular the equality holds in (3.12). Therefore, if the equality holds in (3.13), then $\Omega = \mathbb{D}_2$ and either $\alpha_\lambda = 0$ or $\alpha_\lambda = 1$. But if $\Omega = \mathbb{D}_2$ and $\alpha_\lambda = 1$ then $E_\lambda = E_0(\mathbb{D}_2) = \frac{1}{8\pi}$ (Theorem 1.2) and then E_λ cannot be equal to $\frac{p+1}{16\pi}$ in this case. Therefore, if the equality holds in (3.13), then $\Omega = \mathbb{D}_2$ and $\alpha_\lambda = 0$. On the contrary, suppose that $\Omega = \mathbb{D}_2$ and $\alpha_\lambda = 0$. Then, since $\Omega = \mathbb{D}_2$, the equality holds in (3.12) and $\lambda(\frac{p+1}{16\pi} - E_\lambda) = 0$. But if $\lambda = 0$ then necessarily $\alpha_\lambda = 1$, and then α_λ cannot be zero in this case. Therefore, if $\Omega = \mathbb{D}_2$ and $\alpha_\lambda = 0$ then the equality holds in (3.13). This concludes the characterization of the equality sign in (3.13). □

4. **Uniqueness of solutions**

The following Lemma is the starting point of the proof of Theorem 1.2 and illustrates an interesting property of $\mathbf{(P)}_\lambda$.

Lemma 4.1. *Let $p \in [1, p_N)$.*
(i) If there exists $\overline{\lambda} > 0$ such that for any solution $(\alpha_\lambda, \psi_\lambda)$ with $\lambda \leq \overline{\lambda}$ it holds $\sigma_1(\alpha_\lambda, \psi_\lambda) > 0$, then $\mathbf{(P)}_\lambda$ has at most one positive solution for any $\lambda \leq \overline{\lambda}$.

(ii) If there exists $\overline{\lambda} > 0$ such that for any solution $(\alpha_\lambda, \psi_\lambda)$ with $\lambda \leq \overline{\lambda}$ it holds $\sigma_1(\alpha_\lambda, \psi_\lambda) > 0$ and if one of the following holds: either
(a) there exists a positive solution $(\overline{\alpha}, \overline{\psi})$ at $\lambda = \overline{\lambda}$, or
(b) if $(\alpha_\lambda, \psi_\lambda)$ is a solution with $\lambda \leq \overline{\lambda}$, then $\alpha_\lambda > 0$,
then $\mathbf{(P)}_\lambda$ has a unique solution for any $\lambda \leq \overline{\lambda}$. In particular the set of solutions in $[0, \overline{\lambda}]$ is a real analytic simple curve of positive solutions $\lambda \to (\alpha_\lambda, \psi_\lambda)$, $\lambda \in [0, \overline{\lambda}]$ and $\alpha_\lambda \geq \alpha_{\overline{\lambda}}$, for any $\lambda \leq \overline{\lambda}$.

Proof. By Lemma 3.4 in [10], for any $\overline{I} > 0$ we already have a uniform estimate for the $L^\infty(\Omega)$ norm of any solution of $\mathbf{(F)}_I$ but since the change of variables from (γ_I, v_I) is singular at $I = 0$ (see Remark 1.1), we cannot use those estimates directly for $\mathbf{(P)}_\lambda$.

Lemma 4.2. *Let $p \in [1, p_N)$. For any $\overline{\lambda} > 0$ there exists a positive constant $C_1 = C_1(r, \Omega, \overline{\lambda}, p, N)$ depending only on Ω, $\overline{\lambda}$, p, N and $r \in [0, 1)$ such that $\|\psi_\lambda\|_{C^{2,r}_0(\overline{\Omega})} \leq C_1$ for any solution $(\alpha_\lambda, \psi_\lambda)$ of $\mathbf{(P)}_\lambda$ with $\lambda \in [0, \overline{\lambda}]$.*

Proof. Since $\int_\Omega (\alpha_\lambda + \lambda\psi_\lambda)^p = 1$ it is well known ([30]) that for any $t \in [1, \frac{N}{N-1})$ there exists $C = C(t, N, \Omega)$ such that $\|\psi_\lambda\|_{W^{1,t}_0(\Omega)} \leq C(t, N, \Omega)$ for any solution of $\mathbf{(P)}_\lambda$. Thus, by the Sobolev inequalities, for any $1 \leq s < \frac{N}{N-2}$ we have $\|\psi_\lambda\|_{L^s(\Omega)} \leq C(s, N, \Omega)$, for some $C(s, N, \Omega)$. As a consequence, since $\alpha_\lambda \leq 1$ and $p < p_N$, then there exists some $m > 1$ depending on p and N such that $\|(\alpha_\lambda + \lambda\psi_\lambda)\|_{L^m(\Omega)} \leq C(p, N, \overline{\lambda}, s, \Omega)$, for any $\lambda \leq \overline{\lambda}$, for some $C(p, N, s, \overline{\lambda}, \Omega)$ and then, by standard elliptic estimates, we conclude that $\|\psi_\lambda\|_{W^{2,m}_0(\Omega)} \leq C_0(p, N, s, \overline{\lambda}, \Omega)$, for any $\lambda \leq \overline{\lambda}$, for some $C_0(p, N, s, \overline{\lambda}, \Omega)$. At this point, since Ω is of class C^3, the conclusion follows by standard elliptic estimates and a bootstrap argument. □

Proof of (i). We argue by contradiction and assume without loss of generality that a positive solution $(\overline{\alpha}, \overline{\psi})$ exists for $\lambda = \overline{\lambda}$. Since $\sigma_1(\alpha_\lambda, \psi_\lambda) > 0$ for any $\lambda \in [0, \overline{\lambda}]$ by assumption, then by Lemma 2.4 we deduce that there exists a small interval around $\overline{\lambda}$ where the set of solutions of $\mathbf{(P)}_\lambda$ is a real analytic curve of positive solutions. By Proposition 3.2 we deduce that α_λ is strictly decreasing and therefore in particular that $\alpha_\lambda \geq \overline{\alpha} > 0$ in the given left neighborhood. However we have $\sigma_1(\alpha_\lambda, \psi_\lambda) > 0$ for any $\lambda \in [0, \overline{\lambda}]$ and by Lemma 4.2, Lemma 2.4 and Proposition 3.2 we can continue this small curve of positive solutions without bifurcation points backward to the left as a curve of positive solutions defined in $\lambda \in (-\delta, \overline{\lambda}]$ for some small $\delta > 0$. At this point, if at any point in $(0, \overline{\lambda}]$ a positive solution would exist other than those on this curve, then it would generate in the same way another curve of positive solutions in $(-\delta, \overline{\lambda}]$. Obviously for both curves at $\lambda = 0$ we have $(\alpha_\lambda, \psi_\lambda) = (\alpha_0, \psi_0)$ which is the unique solution of $\mathbf{(P)}_\lambda$ for $\lambda = 0$. This is obviously impossible since then (α_0, ψ_0) would be a bifurcation point, in contradiction with Lemma 2.4.

Proof of (ii). If (a) holds we just know that there exists a positive solution $(\overline{\alpha}, \overline{\psi})$ at $\lambda = \overline{\lambda}$. Therefore, the same argument adopted in the first part shows that there is a real analytic curve of positive solutions emanating from $(\overline{\alpha}, \overline{\psi})$ and defined in $\lambda \in (-\delta, \overline{\lambda}]$ for some small $\delta > 0$. In particular, since $\sigma_1(\alpha_\lambda, \psi_\lambda) > 0$ for any $\lambda \in [0, \overline{\lambda}]$ then by Proposition 3.2 we deduce that $\alpha_\lambda \geq \overline{\alpha}$ along this curve. By the first part we have that there exists at most one positive solution in $[0, \overline{\lambda}]$ and we are done in this case.
If (b) holds we argue the other way around. By Lemma 2.4 we known that there exists $0 < \varepsilon < \overline{\lambda}$ small enough such that $\mathbf{(P)}_\lambda$ has at least one solution for any $\lambda \in [0, \varepsilon]$. Since by assumption $\sigma_1(\alpha_\lambda, \psi_\lambda) > 0$ for any $\lambda \in [0, \overline{\lambda}]$ and since (b) holds, then by Lemma 4.2 and Lemma 2.4 we can continue this curve forward to the right as a real analytic curve of positive solutions defined in $\lambda \in [0, \overline{\lambda}]$. In particular by Proposition 3.2 we deduce that $\alpha_\lambda \geq \alpha_{\overline{\lambda}}$ along this curve. By the first part we have that there exists at most one positive solution in $[0, \overline{\lambda}]$ and we are done in this case as well. □

By using Lemma 4.1, the proof of Theorem 1.2 is reduced to some uniform a priori estimates for $\sigma_1(\alpha_\lambda, \psi_\lambda)$ and α_λ, which we split in various Lemmas.

Proposition 4.3. *If* $(\alpha_\lambda, u_\lambda)$ *is a solution of* (3.7) (*or equivalently if* $(\alpha_\lambda, \psi_\lambda)$ *is a solution of* $\mathbf{(P)}_\lambda$) *and*

$$\lambda p \leq \Lambda(\Omega, 2p),$$

then,

$$\nu_{1,\lambda} = \inf_{w \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla w|^2 - \lambda p \int_\Omega (\rho_\lambda)^{\frac{1}{q}} w^2}{\int_\Omega (\rho_\lambda)^{\frac{1}{q}} w^2} \geq 0$$

Proof. Obviously we can assume that $\lambda > 0$. Let $w \in C_0^1(\overline{\Omega})$, $w \not\equiv 0$ then we have,

$$\frac{\int_\Omega |\nabla w|^2}{\int_\Omega (\rho_\lambda)^{\frac{1}{q}} w^2} \geq \frac{1}{\left(\int_\Omega (\alpha_\lambda + u_\lambda)^p\right)^{\frac{1}{q}}} \frac{\int_\Omega |\nabla w|^2}{\left(\int_\Omega w^{2p}\right)^{\frac{1}{p}}} = \frac{\int_\Omega |\nabla w|^2}{\left(\int_\Omega w^{2p}\right)^{\frac{1}{p}}} \geq \Lambda(\Omega, 2p)$$

which immediately implies that,

$$\nu_{1,\lambda} \geq \Lambda(\Omega, 2p) - \lambda p \geq 0, \; \forall\, \lambda p \leq \Lambda(\Omega, 2p).$$

□

Proposition 4.4. *For fixed* $p \in [1, p_N)$, *it holds* $\lambda^*(\Omega, p) \geq \frac{1}{p}\Lambda(\Omega, 2p)$ *and the equality holds if and only if* $p = 1$.

Proof. It is well known that for $p = 1$ ([43]) it holds $\lambda^*(\Omega, 1) = \lambda^{(1)}(\Omega)$, and since $\lambda^{(1)}(\Omega) \equiv \Lambda(\Omega, 2)$, then the equality holds for $p = 1$. Therefore, we will just prove that if $p > 1$ then $\lambda^*(\Omega, p)$ is well defined and satisfies $\lambda^*(\Omega, p) > \frac{1}{p}\Lambda(\Omega, 2p)$. To simplify the exposition let us denote $\lambda^* = \lambda^*(\Omega, p)$. By Lemma B.1 in Appendix B, λ^* is well defined. We argue by contradiction and suppose that $\lambda^* \leq \frac{1}{p}\Lambda(\Omega, 2p)$. It follows from Proposition 4.3, (2.17) and Lemma 4.1 that for any $\lambda < \lambda^*$ there exists a unique solution of $\mathbf{(P)}_\lambda$ and that these solutions form a real analytic simple curve of positive solutions. By Lemma 4.2 we can pass to the limit and obtain at least one solution (α_*, ψ_*) of $\mathbf{(P)}_\lambda$ for $\lambda = \lambda^*$. If $\alpha_* = 0$ then $u_* = \lambda^* \psi_*$ would be a solution of

$$\begin{cases} -\Delta u_* = \lambda^* (u_*)^p \quad \text{in } \Omega \\ \\ \displaystyle\int_\Omega (u_*)^p = 1 \\ \\ u_* > 0 \text{ in } \Omega, \quad u_* = 0 \text{ on } \partial\Omega. \end{cases} \tag{4.1}$$

Now the linearization of (4.1) where one just disregards the integral constraint takes the form

$$\begin{cases} -\Delta \phi = \lambda^* p (u_*)^{p-1} \phi \quad \text{in } \Omega \\ \\ \phi = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

and it is readily seen that $u_p = p^{-\frac{1}{p-1}} u_* < u_*$ is a positive solution of (4.2) with u_* replaced by u_p. Therefore, u_p is a positive strict subsolution of (4.2) and then we deduce by standard arguments that the first eigenvalue of (4.2) is negative. In particular we infer that

$$0 > \inf_{w \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla w|^2}{\int_\Omega (u_*)^{p-1} w^2} - \lambda^* p =$$

$$\inf_{w \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla w|^2 - \lambda^* p \int_\Omega (\rho_{\lambda^*})^{\frac{1}{q}} w^2}{\int_\Omega (\rho_{\lambda^*})^{\frac{1}{q}} w^2} = \nu_{1,\lambda^*}(u_*),$$

which contradicts Proposition 4.3. Therefore, we must have $\alpha_* > 0$ and in particular, by the monotonicity of α_λ (Proposition 3.2), we also have $\inf\limits_{[0,\lambda^*)} \alpha_\lambda > 0$. Using Proposition 4.3, (2.17) and Lemma 2.4 we see that there exists ϵ small enough such that we can continue the simple curve of unique positive solutions defined on $[0, \lambda^*]$ to a larger curve $\mathcal{G}_\epsilon$ defined in $[0, \lambda^* + \epsilon)$ such that in particular, by continuity, $\inf\limits_{(\alpha_\lambda, \psi_\lambda) \in \mathcal{G}_\epsilon} \alpha_\lambda > 0$. Therefore, by the definition of λ^*, either there exists another solution of $\mathbf{(P)}_\lambda$ for $\lambda = \lambda^*$ with $\alpha_\lambda = 0$ or there exists a sequence (α_n, ψ_n) of solutions of $\mathbf{(P)}_\lambda$ for $\lambda_n \to (\lambda^*)^+$ such that $\alpha_n \to 0$. In the latter case, by Lemma 4.2 we can pass to the limit and obtain at least one solution (α^*, ψ^*) of $\mathbf{(P)}_\lambda$ for $\lambda = \lambda^*$ with $\alpha^* = 0$. In particular we deduce that there exists at least another solution u^*, distinct from u_*, of (4.1). At this point the same argument adopted above shows that $\nu_{1,\lambda^*}(u^*) < 0$ which contradicts once more Proposition 4.3. □

At this point we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. By Proposition 4.4 we have $\lambda^*(\Omega, p) \geq \frac{1}{p}\Lambda(\Omega, 2p)$ where the equality holds if and only if $p = 1$. Whence $\alpha_\lambda > 0$ if $\lambda \in [0, \frac{1}{p}\Lambda(\Omega, 2p))$. Therefore, in view of (2.17) and Proposition 4.3, in particular we see that for any $\lambda \in [0, \frac{1}{p}\Lambda(\Omega, 2p))$ it holds $\sigma_1(\alpha_\lambda, \psi_\lambda) > 0$ and $\alpha_\lambda > 0$ for any solution of $\mathbf{(P)}_\lambda$. As a consequence by Lemma 4.1 and by Propositions 3.1 and 3.2, we deduce that for any $\lambda \in [0, \frac{1}{p}\Lambda(\Omega, 2p))$ there exists a unique solution of $\mathbf{(P)}_\lambda$ and that these solutions form a simple real analytic curve of positive solutions along which we have,

$$\frac{d\alpha_\lambda}{d\lambda} < 0, \quad \frac{dE_\lambda}{d\lambda} > 0, \quad \forall\, \lambda \in [0, \frac{1}{p}\Lambda(\Omega, 2p)).$$

Next, by using once more the fact that $\alpha_\lambda > 0$, we see that the right hand side of (3.12) is strictly positive in $[0, \frac{1}{p}\Lambda(\Omega, 2p))$ unless $\alpha_\lambda = 1$, which in turn implies $\lambda = 0$. On the other side by well known rearrangement estimates ([41]) we have,

$$E_0(\Omega) = \frac{1}{2}\int_\Omega \int_\Omega G_\Omega(x,y)\, dxdy \leq \frac{1}{2}\int_{\mathbb{D}_N} \int_{\mathbb{D}_N} G_{\mathbb{D}_N}(x,y)\, dxdy = \frac{N^{\frac{2}{N}}}{4(N+2)\omega_{N-1}^{\frac{2}{N}}}. \tag{4.3}$$

The fact that $\alpha_\lambda = 1 + \mathrm{O}(\lambda)$, $\psi_\lambda = \psi_0 + \mathrm{O}(\lambda)$, as $\lambda \to 0^+$ was already part of Lemma 2.4 and then, since $(\alpha_\lambda, \psi_\lambda)$ is a real analytic curve, it is easy to prove that, $E_\lambda = E_0(\Omega) + \mathrm{O}(\lambda)$, as $\lambda \to 0^+$.
By Lemma 4.2 we can pass to the limit as $\lambda \to (\frac{1}{p}\Lambda(\Omega, 2p))^-$ along $\mathcal{G}(\Omega)$ to obtain a solution $(\overline{\alpha}, \overline{\psi})$ of $\mathbf{(P)}_\lambda$ for $\lambda = \frac{1}{p}\Lambda(\Omega, 2p)$ and we deduce that $\mathcal{G}(\Omega)$ can be extended by continuity at $\lambda = \frac{1}{p}\Lambda(\Omega, 2p)$. If $p = 1$ then $\lambda^*(\Omega, 1) = \lambda^{(1)}(\Omega) \equiv \Lambda(\Omega, 2)$ and $\overline{\alpha} = 0$ ([43]). Otherwise, since $\lambda^*(\Omega, p) > \frac{1}{p}\Lambda(\Omega, 2p)$ for $p > 1$, we must have $\overline{\alpha} > 0$. □

The proof of Theorem 1.3 is based on the equivalence of the variational formulations of $\mathbf{(P)}_\lambda$ and $\mathbf{(F)}_I$, see Appendix A. Here

$$\lambda^{**}(\Omega, 1) = \lambda^{(1)}(\Omega) \quad \text{and} \quad \lambda^{**}(\Omega, p) = (I^{**}(\Omega, p))^{\frac{1}{q}}\,,\ p > 1, \tag{4.4}$$

denote the threshold for variational solutions of $\mathbf{(P)}_\lambda$, see Theorem A in the introduction and Corollary A.1 in Appendix A.

Proof of Theorem 1.3. By Theorem A in the introduction and Corollary A.1 in Appendix A we know that for variational solutions $\alpha_\lambda > 0$ if and only if $\lambda < \lambda^{**}(\Omega, p)$. Since $\lambda^*(\Omega, p)$ is by definition the threshold value for any solution, it cannot be larger than $\lambda^{**}(\Omega, p)$, which proves $\lambda^{**}(\Omega, p) \geq \lambda^*(\Omega, p)$ in (1.2). The fact that for $|\Omega| = 1$ it holds $\Lambda(\Omega, p) \geq \Lambda(\mathbb{D}, p)$ is well known ([14]) which together with the inequality $\lambda^*(\Omega, p) \geq \frac{1}{p}\Lambda(\Omega, p)$ of Theorem 1.2 concludes the proof of (1.2).
As a consequence of (1.2) we deduce that any variational solution with $\lambda < \frac{1}{p}\Lambda(\Omega, p)$ is a positive solution. Therefore, it follows from Theorem 1.2 that it is the unique positive solution in this range. In particular the set of variational solutions coincides with the set of positive solutions in this range. □

5. The case of the two-dimensional ball

We are concerned here with the case $\Omega = \mathbb{D}_2$ and $p \in [1, +\infty)$. By [6] we know that the solutions are unique and we denote with $\mathcal{G}^*(\mathbb{D}_2)$ the set of unique solutions of $\mathbf{(P)}_\lambda$ on $\mathbb{D}_2$. Recall also (4.4) and Corollary A.1 in Appendix A.

Proof of Theorem 1.4. For $p = 1$ we have by Theorem 1.2 that $\lambda^*(\mathbb{D}_2, 1) = \lambda^{(1)}(\mathbb{D}_2) \equiv \Lambda(\mathbb{D}_2, 2)$ and the set of solutions $\mathcal{G}^*(\mathbb{D}_2)$ is a real analytic curve in $[0, \lambda^*(\mathbb{D}_2, 1))$ which is continuous in $[0, \lambda^*(\mathbb{D}_2, 1)]$ and coincides with $\mathcal{G}(\mathbb{D}_2)$. Here we recall that $\mathcal{G}(\mathbb{D}_2)$ is the set of solutions in Theorem 1.2. In particular the unique ([10, 37, 43]) solution $(\overline{\alpha}, \overline{\psi})$ of $\mathbf{(P)}_\lambda$ with $\lambda = \lambda^*(\mathbb{D}_2, 1)$ satisfies $\overline{\alpha} = 0$ and the monotonicity properties of α_λ and E_λ of Theorem 1.2 hold in $[0, \lambda^*(\mathbb{D}_2, 1))$.

Therefore, we are left with the case $p > 1$ where we recall that, from Theorem 1.2, we have $\lambda^*(\mathbb{D}_2, p) > \frac{1}{p}\Lambda(\mathbb{D}_2, 2p)$. We will use the fact that, by the uniqueness of solutions ([6]), we have $\lambda^*(\mathbb{D}_2, p) = \lambda^{**}(\mathbb{D}_2, p)$ (see Corollary A.1) and to simplify the notations in the rest of this proof we set $\lambda^* = \lambda^*(\mathbb{D}_2, p)$.
By Proposition 4.3, by (2.17) and Lemma 2.4 and since $\lambda^* > \frac{1}{p}\Lambda(\mathbb{D}_2, 2p)$, we can continue the curve $\mathcal{G}(\mathbb{D}_2)$ in a right neighborhood of $\frac{1}{p}\Lambda(\mathbb{D}_2, 2p)$ as a larger simple curve of positive solutions with no bifurcation points $\mathcal{G}_\mu$, such that $\mathcal{G}(\mathbb{D}_2) \subset \mathcal{G}_\mu$, $\mu > \frac{1}{p}\Lambda(\mathbb{D}_2, 2p)$, and by continuity $\alpha_\lambda > 0$ and $\sigma_1(\alpha_\lambda, \psi_\lambda) > 0$ for any $\lambda < \mu$. Therefore, it is well defined,

$$\lambda_1 = \lambda_1(\mathbb{D}_2) := \sup\left\{\mu > \frac{1}{p}\Lambda(\mathbb{D}_2, 2p)\ :\ \alpha_\lambda > 0 \text{ and } \sigma_1(\alpha_\lambda, \psi_\lambda) > 0,\ \forall\,(\alpha_\lambda, \psi_\lambda) \in \mathcal{G}_\mu,\ \forall \lambda < \mu\right\}.$$

There are three possibilities: either
- $\lambda_1(\mathbb{D}_2) = +\infty$, or

- $\lambda_1(\mathbb{D}_2) < +\infty$ and either $\inf\limits_{\lambda\in[0,\lambda_1(\mathbb{D}_2))} \alpha_\lambda = 0$ or $\inf\limits_{\lambda\in[0,\lambda_1(\mathbb{D}_2))} \alpha_\lambda > 0$.

Since $\lambda^{**}(\mathbb{D}_2, p) < +\infty$ (see Appendix A), then $\lambda_1(\mathbb{D}_2) < +\infty$, which rules out the first possibility. By Propositions 3.1 and 3.2 $\frac{d\alpha_\lambda}{d\lambda} < 0$ and $\frac{dE_\lambda}{d\lambda} > 0$ continue to hold whenever $\alpha_\lambda > 0$ and $\sigma_1(\alpha_\lambda, \psi_\lambda) > 0$ are satisfied, whence for any $\lambda < \lambda_1(\mathbb{D}_2)$ as well. By (3.13) we have $E_\lambda \leq \frac{p+1}{16\pi}$ for any solution. Now if $\lambda_1(\mathbb{D}_2) < +\infty$ and $\inf\limits_{\lambda\in[0,\lambda_1(\mathbb{D}_2))} \alpha_\lambda = 0$ we have by definition that $\alpha_\lambda > 0$ and $\sigma_1(\alpha_\lambda, \psi_\lambda) > 0$ for any $\lambda < \lambda_1(\mathbb{D}_2)$, and then by Lemma 4.2, the uniqueness of solutions on $\mathbb{D}_2$ ([6]) and the monotonicity of α_λ (Proposition 3.2) it is not difficult to see that we can continue the curve up to $\lambda_1(\mathbb{D}_2)$ and that $\alpha_{\lambda_1(\mathbb{D}_2)} = 0$ holds. In particular $\lambda_1(\mathbb{D}_2) = \lambda^*$ and in this case we are done.

We are left with the discussion of the case, $\lambda_1(\mathbb{D}_2) < +\infty$ and $\inf\limits_{\lambda\in[0,\lambda_1(\mathbb{D}_2))} \alpha_\lambda > 0$.

First of all, by definition and by Lemmas 2.4 and 4.2, it is not difficult to see that in this case we have $\sigma_1(\alpha_1, \psi_1) = 0$, $\alpha_1 = \alpha_{\lambda_1(\mathbb{D}_2)} > 0$ and $\psi_1 = \psi_{\lambda_1(\mathbb{D}_2)}$. By Theorem 2.5 we see that on $\mathbb{D}_2$ the transversality condition, needed to apply Proposition 2.7, is always satisfied. Therefore, it follows from Proposition 2.7 that we can continue $\mathcal{G}_{\lambda_1(\mathbb{D}_2)}$ to a real analytic parametrization without bifurcation points,

$$\mathcal{G}^{(s_1+\delta_1)} = \{[0, s_1 + \delta_1) \ni s \mapsto (\lambda(s), \alpha(s), \psi(s))\},$$

where, for some $s_1 > 0$ and $\delta_1 > 0$, we have that for any $s \in [0, s_1 + \delta_1)$, $(\alpha(s), \psi(s))$ is a positive solution with $\lambda = \lambda(s)$ and $\lambda(s) = s$ for $s \leq s_1$. We claim that $\lambda(s)$ is monotonic increasing along the branch. Indeed, we recall from (2.24) in Proposition 2.7 that

$$< [\phi_1]_{\lambda_1(\mathbb{D}_2)}, \psi_1 >_{\lambda_1(\mathbb{D}_2)} \neq 0 \text{ and } < [\phi_1]_{\lambda_1(\mathbb{D}_2)}, \psi_1 >_{\lambda_1(\mathbb{D}_2)} \text{ has the same sign as } < \phi_1 >_{\lambda_1(\mathbb{D}_2)} .$$

Therefore, there is no loss of generality in assuming $< [\phi_1]_{\lambda_1(\mathbb{D}_2)}, \psi_1 >_{\lambda_1(\mathbb{D}_2)} > 0$. At this point we use (2.25) in Proposition 2.7, that is, putting $\sigma_1(s) = \sigma_1(\alpha(s), \psi(s))$,

$$\frac{\sigma_1(s)}{\lambda'(s)} = \frac{p < [\phi_1]_{\lambda_1(\mathbb{D}_2)}, \psi_1 >_{\lambda_1(\mathbb{D}_2)} + \mathrm{o}(1)}{< [\phi_1]^2_{\lambda_1(\mathbb{D}_2)} >_{\lambda_1(\mathbb{D}_2)} + \mathrm{o}(1)}, \text{ as } s \to 0,$$

and infer that $\sigma_1(s) \neq 0$ and has the same sign of $\lambda'(s)$ for s small enough in a deleted neighborhood of $s = 0$. Indeed the eigenvalues $\sigma_k(s) = \sigma_k(\alpha(s), \psi(s))$ of $L_{\lambda(s)}$ are real analytic functions of s ([11]) and then the level sets of a fixed $\sigma_k(s)$ cannot have accumulation points unless $\sigma_k(s)$ is constant on $(0, s_1 + \delta_1)$. However we can rule out the latter case since for $\lambda \leq \frac{1}{p}\Lambda(\Omega, 2p)$ we always have $\sigma_1(\alpha_\lambda, \psi_\lambda) > 0$ and then no $\sigma_n(s)$ can vanish identically.

At this point we use again that, by the uniqueness of solutions (Theorem A in the introduction) any solution of $\mathbf{(P)}_\lambda$ is also a variational solution (see Appendix A). Therefore, $(\alpha(s), \psi(s))$ is a positive variational solution and we infer from Lemma 2.6 that $\sigma_1(s) \geq 0$ for $s - s_1 > 0$ small enough. Since $\sigma_1(s) \neq 0$ for $s \neq 0$ small enough we deduce that $\sigma_1(s) > 0$ for $s - s_1 > 0$ small and then in particular that $\lambda'(s) > 0$. Therefore, $\lambda'(s) > 0$ for s in a small enough right neighborhood of s_1 and the curve $(\alpha_\lambda, \psi_\lambda)$ bends right. In particular this implies that, with the notations of Proposition 3.1, we can evaluate the following limit,

$$\lim_{\lambda\to\lambda_1(\mathbb{D}_2)} \frac{dE_\lambda}{d\lambda} = \lim_{\lambda\to\lambda_1(\mathbb{D}_2)} p\, m_\lambda \left(< [\psi_\lambda]^2_\lambda >_\lambda + \lambda < [\psi_\lambda]_\lambda, \eta_\lambda >_\lambda\right) =$$

$$\mathrm{O}(1) + p\, m_{\lambda_1(\mathbb{D}_2)} \lim_{\lambda\to\lambda_1(\mathbb{D}_2)} \lambda < [\psi_\lambda]_\lambda, \eta_\lambda >_\lambda = \mathrm{O}(1) + p\, m_{\lambda_1(\mathbb{D}_2)} \lim_{s\to 0} \frac{1}{\lambda'(s)} \lambda(s) < [\psi(s)]_{\lambda(s)}, \psi'(s) >_{\lambda(s)} =$$

$$\mathrm{O}(1) + p\, m_{\lambda_1(\mathbb{D}_2)} \lambda_1(\mathbb{D}_2) \lim_{s\to 0} \frac{1}{\lambda'(s)} \Big(< [\psi_1]_{\lambda_1(\mathbb{D}_2)}, \psi'(0) >_{\lambda_1(\mathbb{D}_2)} + \mathrm{o}(1)\Big) =$$

$$\mathrm{O}(1) + p\, m_{\lambda_1(\mathbb{D}_2)} \lambda_1(\mathbb{D}_2) \lim_{s\to 0} \frac{1}{\lambda'(s)} \left(< [\psi_1]_{\lambda_1(\mathbb{D}_2)}, \phi_1 >_{\lambda_1(\mathbb{D}_2)} + \mathrm{o}(1)\right) = +\infty. \tag{5.1}$$

Similarly, with the notations of Proposition 3.2, we have

$$\lim_{\lambda\to\lambda_1(\mathbb{D}_2)} \frac{d\alpha_\lambda}{d\lambda} = - \lim_{\lambda\to\lambda_1(\mathbb{D}_2)} (< \psi_\lambda >_\lambda + \lambda < \eta_\lambda >_\lambda) =$$

$$\mathrm{O}(1)-\lim_{\lambda\to\lambda_1(\mathbb{D}_2)}\lambda<\eta_\lambda>_\lambda=\mathrm{O}(1)-\lambda_1(\mathbb{D}_2)\lim_{s\to 0}\frac{1}{\lambda'(s)}<\psi'(s)>_{\lambda(s)}=$$

$$\mathrm{O}(1)-\lambda_1(\mathbb{D}_2)\lim_{s\to 0}\frac{1}{\lambda'(s)}\big(<\phi_1>_{\lambda_1(\mathbb{D}_2)}+\mathrm{o}(1)\big)=-\infty. \tag{5.2}$$

Summarizing we conclude that there exists $\overline{\lambda}_1$ such that we can continue $\mathcal{G}_{\lambda_1(\mathbb{D}_2)}$ to a larger branch $\mathcal{G}_{\overline{\lambda}_1}$ such that:
$(A1)_1$ for any $\lambda\in[0,\overline{\lambda}_1)$, $(\alpha_\lambda,\psi_\lambda)$ is a positive solution;
$(A1)_2$ $\overline{\lambda}_1>\lambda_1(\mathbb{D}_2)>\frac{1}{p}\Lambda(\mathbb{D}_2,2p)$;
$(A1)_3$ the inclusion $\{(\alpha_\lambda,\psi_\lambda),\,\lambda\in[0,\lambda_1(\mathbb{D}_2)]\}\subset\mathcal{G}_{\overline{\lambda}_1}$, holds,
$(A1)_4$ $\alpha_\lambda>0$, $\forall\,\lambda\in[0,\overline{\lambda}_1]$,
$(A1)_5$ $\sigma_1(\alpha_\lambda,\psi_\lambda)>0$, $\forall\,\lambda\in[0,\overline{\lambda}_1]\setminus\{\lambda_1(\mathbb{D}_2)\}$.
$(A1)_6$ $\frac{d\alpha_\lambda}{d\lambda}<0$ and $\frac{dE_\lambda}{d\lambda}>0$ for $\lambda\in[0,\overline{\lambda}_1]\setminus\{\lambda_1(\mathbb{D}_2)\}$ and $\frac{d\alpha_\lambda}{d\lambda}\to-\infty$, $\frac{dE_\lambda}{d\lambda}\to+\infty$ as $\lambda\to\lambda_1(\mathbb{D}_2)$.
Clearly, $(A1)_6$ follows from $(A1)_4$, $(A1)_5$, Propositions 3.1, 3.2 and (5.1), (5.2).

Therefore, we can argue by induction and for $k\geq 2$ define,

$$\lambda_k(\mathbb{D}_2):=\sup\left\{\mu>\lambda_{k-1}(\mathbb{D}_2)\,:\,\alpha_\lambda>0\text{ and }\sigma_1(\alpha_\lambda,\psi_\lambda)>0,\;\forall\,(\alpha_\lambda,\psi_\lambda)\in\mathcal{G}_\mu,\;\forall\lambda_{k-1}(\mathbb{D}_2)<\lambda<\mu\right\},$$

where as above we know that $\lambda_k(\mathbb{D}_2)<+\infty$. If there exists some $k\geq 2$ such that we also have $\inf\limits_{s\in(0,s_k)}\alpha(s)=0$, then by arguing as above we are done. Otherwise, the above procedure yields a sequence $\overline{\lambda}_k$ such that, for any $k\in\mathbb{N}$ we have, $\overline{\lambda}_k>\overline{\lambda}_{k-1}>\cdots>\overline{\lambda}_2>\overline{\lambda}_1$ and we can continue $\mathcal{G}_{\lambda_1(\mathbb{D}_2)}$ to a larger branch $\mathcal{G}^*(\mathbb{D}_2)$, such that, for any $k\in\mathbb{N}$, it holds:
$(Ak)_1$ for any $\lambda\in[0,\overline{\lambda}_k)$, $(\alpha_\lambda,\psi_\lambda)$ is a positive solution;
$(Ak)_2$ $\overline{\lambda}_k>\lambda_k(\mathbb{D})>\lambda_{k-1}(\mathbb{D}_2)>\cdots>\lambda_2(\mathbb{D}_2)>\lambda_1(\mathbb{D}_2)>\frac{1}{p}\Lambda(\mathbb{D}_2,2p)$;
$(Ak)_3$ the inclusion $\{(\alpha_\lambda,\psi_\lambda),\,\lambda\in[0,\lambda_k(\mathbb{D})]\}\subset\mathcal{G}^*(\mathbb{D})$, holds;
$(Ak)_4$ $\alpha_\lambda>0$, $\forall\,\lambda\in[0,\overline{\lambda}_k]$;
$(Ak)_5$ $\sigma_1(\alpha_\lambda,\psi_\lambda)>0$, $\forall\,\lambda\in[0,\overline{\lambda}_k]\setminus\{\lambda_k(\mathbb{D}_2)\}_{k\in\mathbb{N}}$;
$(Ak)_6$ $\frac{d\alpha_\lambda}{d\lambda}<0$ and $\frac{dE_\lambda}{d\lambda}>0$ for $\lambda\in[0,\overline{\lambda}_1]\setminus\{\lambda_k(\mathbb{D}_2)\}_{k\in\mathbb{N}}$ and $\frac{d\alpha_\lambda}{d\lambda}\to-\infty$, $\frac{dE_\lambda}{d\lambda}\to+\infty$ as $\lambda\to\lambda_k(\mathbb{D}_2)$.

At this point, since $\sup\limits_{k\in\mathbb{N}}\lambda_k(\mathbb{D}_2)<+\infty$, then $\lambda_k(\mathbb{D}_2)\to\overline{\lambda}$ as $k\to+\infty$. We claim that necessarily $\alpha_k=\alpha_{\lambda_k(\mathbb{D})}\to 0^+$. We argue by contradiction and assume that $\alpha_k\to\overline{\alpha}\in(0,1]$. Clearly, by Lemma 4.2 there exists $\overline{\psi}$ such that $(\overline{\alpha},\overline{\psi})$ is a positive solution. Let $\{\overline{\sigma}_n\}_{n\in\mathbb{N}}$ be the eigenvalues corresponding to $(\overline{\alpha},\overline{\psi})$. First of all we have that $0\in\sigma(L_{\overline{\lambda}})$, where $\sigma(L_{\overline{\lambda}})$ stands for the spectrum of $L_{\overline{\lambda}}$. Indeed, if this was not the case, then, since $\lambda(t_k)\to\overline{\lambda}$, by Lemma 2.4, by continuity and since the eigenvalues are isolated, we would have that there exists a fixed full neighborhood of 0 with empty intersection with $\sigma(L_{\lambda_k})$ for any k large enough. This is clearly impossible since by construction $0\in\sigma(L_{\lambda_k})$ for any k. As a consequence $\overline{\sigma}_1=0$. However, since $\overline{\alpha}\in(0,1]$ by assumption, then we can apply Proposition 2.7 and in particular conclude that $\overline{\sigma}_1(s)$ is a real analytic function of s ([11]) and then its zero level set cannot have accumulation points unless it vanishes identically. On one side for $\lambda\leq\frac{1}{p}\Lambda(\mathbb{D}_2,2p)$ we have $\sigma_1(\alpha_\lambda,\psi_\lambda)>0$ and then we deduce from $(A1)_2$ that no eigenvalues can vanish identically. Therefore, since the eigenvalues are isolated, by continuity we would have once more that there exists a fixed full neighborhood of $\overline{\sigma}_1=0$ with empty intersection with $\sigma(L_{\lambda_k})$ for any k large enough. This is clearly impossible since by construction $0\in\sigma(L_{\lambda_k})$ for any k. Therefore, $\overline{\alpha}=0$ and $\overline{\lambda}=\lambda^*$ in this case as well.
The energy identity follows by Proposition 3.3. □

Appendix A. Variational solutions

Problem $(\mathbf{P})_\lambda$ arises as the Euler-Lagrange equation of the constrained minimization principle $(\mathbf{VP})$ below for the plasma densities $\rho \in L^1(\Omega)$, which, for $p > 1$, is equivalent to the variational formulation of $(\mathbf{F})_I$. We shortly discuss here the variational solutions of $(\mathbf{F})_I$ and $(\mathbf{P})_\lambda$ and their equivalence and refer to [10] for a detailed discussion of this point. In this context α_λ is the Lagrange multiplier related to the "mass" constraint $\int_\Omega \rho_\lambda = 1$ while the Dirichlet energy is the density interaction energy,

$$\mathcal{E}(\rho) = \frac{1}{2}\int_\Omega \rho G[\rho],$$

which is easily seen to coincide with E_λ whenever $\psi_\lambda = G[\rho_\lambda]$, that is, $E_\lambda = \mathcal{E}(\rho_\lambda)$.
For any

$$\rho \in \mathcal{P}_\Omega := \left\{\rho \in L^{1+\frac{1}{p}}(\Omega)\,|\,\rho \geq 0 \text{ a.e. in } \Omega\right\},$$

and $\lambda \geq 0$, we define the free energy,

$$J_\lambda(\rho) = \tfrac{p}{p+1}\int_\Omega (\rho)^{1+\frac{1}{p}} - \frac{\lambda}{2}\int_\Omega \rho G[\rho]. \tag{A.1}$$

Let us consider the variational principle,

$$\mathcal{J}(\lambda) = \inf\left\{J_\lambda(\rho)\,:\,\rho \in \mathcal{P}_\Omega, \int_\Omega \rho = 1\right\} \qquad (\mathbf{VP})$$

It has been shown in [10, 43] that for each $\lambda > 0$ there exists at least one ρ_λ which solves the $(\mathbf{VP})$. In particular, ([10]) $\alpha = \alpha_\lambda \in \mathbb{R}$ arises as the Lagrange multiplier relative to the constraint $\int_\Omega \rho_\lambda = 1$ and if $\alpha_\lambda \geq 0$, then any minimizer ρ_λ yields a solution $(\alpha_\lambda, \psi_\lambda)$ of $(\mathbf{P})_\lambda$ where $\psi_\lambda = G[\rho_\lambda]$. Any such solution is called a variational solution of $(\mathbf{P})_\lambda$.

Solutions of $(\mathbf{F})_I$ are also found in [10] as minimizers of the functional $\Psi_I(v)$, $v \in \mathcal{H}_I$ as defined in (1.1). For fixed $I > 0$, a variational solution of $(\mathbf{F})_I$ is a solution of $(\mathbf{F})_I$ which is also a minimizer of Ψ_I on $\mathcal{H}_I$. It has been shown in [10] that, for $p \in (1, p_N)$, at least one variational solution exists for each $I > 0$. From Theorem A in the introduction there exists $I^{**}(\Omega, p) > 0$ such that if v is a variational solution of $(\mathbf{F})_I$ for some $\gamma = \gamma_I$, then $\gamma_I > 0$ if and only if $I \in (0, I^{**}(\Omega, p))$.
We will use a result in [10] p.421 which shows that, for $p > 1$, there is a one to one correspondence between variational solutions of $(\mathbf{F})_I$ and minimizers of

$$\inf\left\{J_1(\widetilde{\rho})\,:\,\widetilde{\rho} \in \mathcal{P}_\Omega, \int_\Omega \widetilde{\rho} = I\right\},$$

where J_λ has been defined in (A.1). We will also use the fact that the minimization of J_1 on $\mathcal{P}_\Omega \cap \{\int_\Omega \widetilde{\rho} = I\}$ is equivalent to $(\mathbf{VP})$. This is readily seen from $(\mathbf{VP})$ after the scaling $\rho = I^{-1}\widetilde{\rho}$ with $\lambda = I^{\frac{1}{q}}$ and in particular yields the Euler-Lagrange equation,

$$(\widetilde{\rho})^{\frac{1}{p}} = (\widetilde{\alpha} + G[\widetilde{\rho}])_+ \text{ in } \Omega. \tag{A.2}$$

Actually, it is not difficult to see that, if $\widetilde{\alpha} \geq 0$, then (A.2) is just the same as $(\mathbf{P})_\lambda$ with $\rho_\lambda = I^{-1}\widetilde{\rho}$, $\lambda = I^{\frac{1}{q}}$ and $\alpha_\lambda = I^{-\frac{1}{p}}\widetilde{\alpha}$.
From [10] p.421, we have: if $\widetilde{v}$ minimizes Ψ_I on $\mathcal{H}_I$, then $\widetilde{\rho} = (\widetilde{v})_+^p$ is a minimizer of J_1 on $\mathcal{P}_\Omega \cap \{\int_\Omega \widetilde{\rho} = I\}$. Therefore, if $(\gamma_I, \widetilde{v}_I)$ is a positive variational solution, then $\widetilde{\rho}_I = (\widetilde{v}_I)_+^p \equiv \widetilde{v}_I^p$ and in particular $\widetilde{\rho}_I\big|_{\partial\Omega} =: \gamma_I^p > 0$. Therefore, it follows from (A.2) that $\widetilde{\alpha}_I = (\widetilde{\rho}_I)^{\frac{1}{p}} = \gamma_I > 0$, and then scaling back we see that $(\alpha_\lambda, \psi_\lambda) = (I^{-\frac{1}{p}}\widetilde{\alpha}_I, G[I^{-1}\widetilde{\rho}_I])$ is a positive variational solution of $(\mathbf{VP})$.
On the other side, still from [10] p.421, we have: if $\widetilde{\rho}$ is a minimizer of J_1 on $\mathcal{P}_\Omega \cap \{\int_\Omega \widetilde{\rho} = I\}$, then there exists a unique $\gamma \in \mathbb{R}$ such that $\widetilde{v} = \gamma + G[\widetilde{\rho}] \in \mathcal{H}_I$ and $\widetilde{v}$ minimizes Ψ_I on $\mathcal{H}_I$. Now if $(\alpha_\lambda, \psi_\lambda)$ is a positive variational solution of $(\mathbf{P})_\lambda$ and $\rho_\lambda = (\alpha_\lambda + \lambda\psi_\lambda)^p$ then $\widetilde{\rho}_I = I\left(\rho_\lambda\right)_{\lambda = I^{\frac{1}{q}}}$

minimizes J_1 on $\mathcal{P}_\Omega \cap \{\int_\Omega \widetilde{\rho} = I\}$ and we infer that $\widetilde{v}_I = \gamma_I + G[\widetilde{\rho}_I] = \widetilde{\rho}_I^{\frac{1}{p}}$ is a minimizer of Ψ_I on $\mathcal{H}_I$ for a unique γ_I which satisfies $\gamma_I = \widetilde{\rho}_I^{\frac{1}{p}}\Big|_{\partial\Omega} = I^{\frac{1}{p}}\,(\alpha_\lambda)_{\lambda = I^{\frac{1}{q}}} > 0$. Therefore, we conclude that (γ_I, v_I) is positive variational solution of $\mathbf{(F)}_I$.

Summarizing, by using a duality argument introduced in [10], we have that, for $p > 1$, $(\alpha_\lambda, \psi_\lambda)$ is a positive variational solution of $\mathbf{(P)}_\lambda$ if and only if (γ_I, v_I) is a positive variational solution of $\mathbf{(F)}_I$. Therefore, as a corollary of Theorem A in the introduction, we conclude that

Corollary A.1. *Let $p \in [1, p_N)$. Then there exists $\lambda^{**}(\Omega, p) \in (0, +\infty)$ such that $(\alpha_\lambda, \psi_\lambda)$ is a positive variational solution of $\mathbf{(P)}_\lambda$ if and only if $\lambda \in (0, \lambda^{**}(\Omega, p))$. In particular, for $p > 1$, via Remark 1.1 we have $\lambda^{**}(\Omega, p) = (I^{**}(\Omega, p))^{\frac{1}{q}}$. Moreover, $\lambda^*(\mathbb{D}_2, p) = \lambda^{**}(\mathbb{D}_2, p)$.*

The identity $\lambda^*(\mathbb{D}_2, p) = \lambda^{**}(\mathbb{D}_2, p)$ is an immediate consequence of the uniqueness of solutions on $\mathbb{D}_2$ ([6]), which implies that any solution is variational.

Appendix B. Uniqueness of solutions for λ small

The following lemma is proved by a standard application of the contraction mapping principle and we prove it here just for reader's convenience.

Lemma B.1. *There exists $\overline{\lambda} > 0$ such that for any $\lambda \in [0, \overline{\lambda}]$ there exists one and only one solution $(\alpha_\lambda, \psi_\lambda)$ of $\mathbf{(P)}_\lambda$. Moreover, $\alpha_\lambda \geq \frac{1}{3}$ for any $\lambda \in [0, \overline{\lambda}]$.*

Proof. The proof is an immediate consequence of the following lemmas. Let us define

$$B_\infty = \left\{u \in L^\infty(\Omega) \,|\, \|u\|_{L^\infty(\Omega)} \leq C_1(r, \Omega),\, u \geq 0 \text{ a.e. in } \Omega\right\}$$

where $C_1(r, \Omega)$ is the constant in Lemma 4.2 concerning the uniform a priori bounds.

Lemma B.2. *Let $p \in [1, p_N)$. There exists $\overline{\lambda} > 0$ such that for any $\lambda \in [0, \overline{\lambda}]$ and for any $\alpha \in [0, 1]$ there exists one and only one solution $u = u_{\lambda,\alpha} \in C_0^{2,r}(\overline{\Omega})$ of the problem*

$$\begin{cases} -\Delta u = \lambda(\alpha + u)^p \quad \textit{in } \Omega \\ \\ u = 0 \quad \textit{on } \partial\Omega \\ \\ u \in B_\infty. \end{cases}$$

Moreover, for fixed $\lambda \in [0, \overline{\lambda}]$, the map $[0, 1] \ni \alpha \to u_\lambda[\alpha] = u_{\lambda,\alpha} \in B_\infty$ is continuous and $u_{0,\alpha} = 0$.

Proof. For $\overline{\lambda} > 0$ to be fixed later on and for fixed $\lambda \in [0, \overline{\lambda}]$ and $\alpha \in [0, 1]$ we define

$$T_{\lambda,\alpha}(u) = \lambda G[(\alpha + u)^p], \quad u \in B_\infty.$$

Clearly

$$\|T_{\lambda,\alpha}(u)\|_{L^\infty(\Omega)} \leq \lambda C_2(p, \Omega, \overline{\lambda} C_1(r, \Omega)),$$

whence we have $T_{\lambda,\alpha} : B_\infty \to B_\infty$ for any $\lambda \leq \frac{1}{C_2}$. Also,

$$\|T_{\lambda,\alpha}(u) - T_{\lambda,\alpha}(v)\|_{L^\infty(\Omega)} \leq \lambda p \|G[(\alpha + w)^{p-1}|u - v|]\|_{L^\infty(\Omega)} \leq$$
$$\lambda p \|G[(\alpha + w)^{p-1}]\|_{L^\infty(\Omega)} \|u - v\|_{L^\infty(\Omega)} \leq \lambda p \|G[(1 + |w|)^{p-1}]\|_{L^\infty(\Omega)} \|u - v\|_{L^\infty(\Omega)} \leq$$
$$\lambda C_3 \|u - v\|_{L^\infty(\Omega)}$$

where $w \in B_\infty$ satisfies $u \leq w \leq v$ and C_3 depends only by p, Ω and $\overline{\lambda} C_1(r, \Omega)$.
Therefore, we also have $\|T_{\lambda,\alpha}(u) - T_{\lambda,\alpha}(v)\|_{L^\infty(\Omega)} \leq \frac{1}{2}\|u - v\|_{L^\infty(\Omega)}$, for any $\lambda \leq \frac{1}{2C_3}$. As a consequence choosing $\overline{\lambda} \leq \min\{\frac{1}{C_2}, \frac{1}{2C_3}\}$, then $T_{\lambda,\alpha}$ is a contraction on B_∞. Whence, in particular, for any fixed $\alpha \in [0, 1]$, we have that for any $\lambda \in [0, \overline{\lambda}]$ there exists a unique solution of

$u = T_{\lambda,\alpha}(u)$. The existence and uniqueness claim follows since, by standard elliptic estimates, $u = u_{\lambda,\alpha} \in C_0^{2,r}(\overline{\Omega})$ solves the problem in the statement of the lemma if and only if $u \in B_\infty$ satisfies $u = T_{\lambda,\alpha}(u)$.
Concerning the continuity of $u_\lambda[\alpha] = u_{\lambda,\alpha}$ for $\alpha \in [0,1]$, we observe that if $\alpha_n \to \alpha$, then

$$\|u_\lambda[\alpha_n] - u_\lambda[\alpha]\|_{L^\infty(\Omega)} = \|T_{\lambda,\alpha_n}(u_{\lambda,\alpha_n}) - T_{\lambda,\alpha}(u_{\lambda,\alpha})\|_{L^\infty(\Omega)} \leq$$
$$\|T_{\lambda,\alpha_n}(u_{\lambda,\alpha_n}) - T_{\lambda,\alpha_n}(u_{\lambda,\alpha})\|_{L^\infty(\Omega)} + \|T_{\lambda,\alpha_n}(u_{\lambda,\alpha}) - T_{\lambda,\alpha}(u_{\lambda,\alpha})\|_{L^\infty(\Omega)} \leq$$
$$\lambda C_3\|u_\lambda[\alpha_n] - u_\lambda[\alpha]\|_{L^\infty(\Omega)} + p\lambda\|G[(s+u_{\lambda,\alpha})^{p-1}]\|_{L^\infty(\Omega)}|\alpha_n - \alpha| \leq$$
$$\frac{1}{2}\|u_\lambda[\alpha_n] - u_\lambda[\alpha]\|_{L^\infty(\Omega)} + \overline{\lambda}C_3|\alpha_n - \alpha|,$$

which readily implies the claim. Obviously $u_{0,\alpha} = T_{0,\alpha}[u_{0,\alpha}] = 0$. □

For fixed $\lambda \in [0,\overline{\lambda}\,]$ we consider the continuous map $[0,1] \ni \alpha \to u_\lambda[\alpha] \in B_\infty$ where $u_\lambda[\alpha] = u_{\lambda,\alpha}$. Then we have,

Lemma B.3. *By taking a smaller $\overline{\lambda}$ if necessary, for any fixed $\lambda \in [0,\overline{\lambda}\,]$ we have:*
(i) The map $u_\lambda[\alpha]$ is monotonic increasing,

$$u_\lambda[\alpha] \leq u_\lambda[\beta], \quad \forall\, 0 < \alpha < \beta \leq 1.$$

(ii) There exists a unique $\alpha_\lambda \in [\frac{1}{3}, 1]$ such that,

$$\int_\Omega (\alpha_\lambda + u_\lambda[\alpha_\lambda])^p = 1.$$

Proof. (i) If $\lambda = 0$ we have $u_{0,\alpha} = 0$ for any α and the conclusion is trivial. For any fixed $0 < \alpha < \beta \leq 1$ let $w = u_\lambda[\beta] - u_\lambda[\alpha] \in C_0^{2,r}(\overline{\Omega}\,) \cap B_\infty$, then

$$-\Delta w = \lambda(\beta + u_\lambda[\beta])^p - \lambda(\alpha + u_\lambda[\alpha])^p >$$
$$\lambda p(\alpha + u_\lambda[\alpha])^{p-1}(u_\lambda[\beta] - u_\lambda[\alpha]) = \lambda p(\alpha + u_\lambda[\alpha])^{p-1}w,$$

by the convexity of $f(t) = (\alpha + t)^p$ for $\alpha \in (0,1]$. Observe now that the first eigenvalue $\xi_{1,\lambda}$ of the linearized operator $-\Delta w + \lambda f^{'}(u_\lambda[\alpha])w$, $w \in H_0^1(\Omega)$ satisfies,

$$\xi_{1,\lambda} = \inf_{w \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla w|^2 - \lambda\int_\Omega f^{'}(u_\lambda[\alpha])|w|^2}{\int_\Omega |w|^2} = \tag{B.1}$$
$$\inf_{w \in H_0^1(\Omega)\setminus\{0\}} \left(\frac{\int_\Omega |\nabla w|^2}{\int_\Omega |w|^2} - \frac{\lambda\int_\Omega f^{'}(u_\lambda[\alpha])|w|^2}{\int_\Omega |w|^2} \right) \geq \lambda^{(1)}(\Omega) - \lambda p(1 + \overline{\lambda}C_1(r,\Omega))^{p-1} \geq \frac{1}{2}\lambda^{(1)}(\Omega),$$

where possibly we take a smaller $\overline{\lambda}$ to guarantee that $p\overline{\lambda}(1 + \overline{\lambda}C_1(r,\Omega))^{p-1} \leq \frac{1}{2}\lambda^{(1)}(\Omega)$. Here $\lambda^{(1)}(\Omega)$ is the first eigenvalue of $-\Delta$ on Ω with Dirichlet boundary conditions and $C_1(r,\Omega)$ is the constant appearing in Lemma B.2. In particular we conclude that $w = u_\lambda[\beta] - u_\lambda[\alpha] \geq 0$, as claimed.
(ii) For $\lambda = 0$ we have $u_{0,\alpha} = 0$ and then necessarily $\alpha_\lambda = 1$. For fixed $\lambda \in (0,\overline{\lambda}\,]$ and by Lemma B.2 and (i), the function

$$g(\alpha) = \int_\Omega (\alpha + u_\lambda[\alpha])^p, \quad \alpha \in (0,1],$$

is continuous and strictly increasing. Moreover, possibly taking $\overline{\lambda}$ small enough to guarantee that $(2C_1\overline{\lambda}\,)^p \leq \frac{1}{4}$, we have that,

$$\limsup_{\alpha\to 0} g(\alpha) \leq \limsup_{\alpha\to 0}\left(2^p\alpha^p + 2^p\int_\Omega (u_\lambda[\alpha])^p\right) \leq \frac{1}{4},$$

while, for any $\lambda \in (0,\overline{\lambda}\,]$, we also have,

$$g(1) = \int_\Omega (1 + u_\lambda[\alpha])^p > 1.$$

Therefore, for any $\lambda \in (0, \overline{\lambda})$, there exists one and only one α_λ such that $g(\alpha_\lambda) = 1$. On the other side we have,

$$1 = \int_\Omega (\alpha_\lambda + u_\lambda[\alpha_\lambda])^p \leq 2^p \alpha_\lambda^p + (2C_1\overline{\lambda})^p \leq 2^p\alpha_\lambda^p + \frac{1}{4},$$

whence $\alpha_\lambda \geq (\frac{3}{4})^{\frac{1}{p}}\frac{1}{2} > \frac{1}{3}$. □

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