EQUIVALENCE OF SOLUTIONS OF EIKONAL EQUATION IN METRIC SPACES

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ABSTRACT. In this paper we prove the equivalence between some known notions of solutions to the eikonal equation and more general analogs of the Hamilton-Jacobi equations in complete and rectifiably connected metric spaces. The notions considered are that of curve-based viscosity solutions, slope-based viscosity solutions, and Monge solutions. By using the induced intrinsic (path) metric, we reduce the metric space to a length space and show the equivalence of these solutions to the associated Dirichlet boundary problem. Without utilizing the boundary data, we also localize our argument and directly prove the equivalence for the definitions of solutions. Regularity of solutions related to the Euclidean semi-concavity is discussed as well.

1. Introduction

1.1. **Background and motivation.** In this paper, we are concerned with first order Hamilton-Jacobi equations in metric spaces. The Hamilton-Jacobi equations in the Euclidean spaces are widely applied in various fields such as optimal control, geometric optics, computer vision, image processing. It is well known that the notion of viscosity solutions provides a nice framework for the well-posedness of first order fully nonlinear equations; we refer to [14, 6] for comprehensive introduction.

In seeking to further develop various fields such as optimal transport [5, 38], mean field games [13], topological networks [37, 28, 1, 26, 27] etc., the Hamilton-Jacobi equations in a general metric space $(\mathbf{X}, d)$ have recently attracted great attention, see for example [21, 22]. Typical forms of the equations include

$$H(x, u, |\nabla u|) = 0 \quad \text{in } \Omega, \tag{1.1}$$

and its time-dependent version

$$\partial_t u + H(x, t, u, |\nabla u|) = 0 \quad \text{in } (0, \infty) \times \mathbf{X} \tag{1.2}$$

with necessary boundary or initial value conditions. Here $\Omega \subsetneq \mathbf{X}$ is an open set and $H : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function called the Hamiltonian of the Hamilton-Jacobi equation. While ∂_t denotes the time differentiation, $|\nabla u|$ stands for a generalized notion of the gradient norm of u in metric spaces.

In this paper we also pay particular attention to the so-called eikonal equation

$$|\nabla u|(x) = f(x) \quad \text{in } \Omega. \tag{1.3}$$

Here $f : \Omega \to [0, \infty)$ is a given continuous function satisfying

$$\inf_{\Omega} f > 0.$$

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The eikonal equation in the Euclidean space has important applications in various fields such as geometric optics, electromagnetic theory and image processing [29, 32].

Several new notions of viscosity solutions have recently been proposed in the general metric setting. We refer the reader to Section 2 for a review with precise definitions and basic properties of solutions, and to the relevant papers we mention below.

Using rectifiable curves in the space $\mathbf{X}$, Giga, Hamamuki and Nakayasu [22] discussed a notion of metric viscosity solutions to (1.3) and established well-posedness under the Dirichlet condition

$$u = \zeta \quad \text{on } \partial\Omega, \tag{1.4}$$

where ζ is a given bounded continuous function on $\partial\Omega$ satisfying an appropriate regularity assumption to be discussed later. In the sequel, this type of solutions will be called curve-based solutions (or c-solutions for short). The definition of c-solutions (Definition 2.1) essentially relies on optimal control interpretations along rectifiable curves and requires very little structure of the space. The same approach is used in [33] to study the evolution problem (1.2) with the Hamiltonian $H(x,)$ convex in v. Moreover, unique viscosity solutions of the eikonal equation in the sense of [22] are also constructed on fractals like the Sierpinski gasket [10].

On the other hand, when $(\mathbf{X}, d)$ is a complete geodesic space, by interpreting $|\nabla u|$ as the local slope of a locally Lipschitz function u, Ambrosio and Feng [4] provide a different viscosity approach to (1.2) for a class of convex Hamiltonians. This was extended to the class of potentially nonconvex Hamiltonians H by Gangbo and Święch [20, 21], who proposed a generalized notion of viscosity solutions via appropriate test classes and proved uniqueness and existence of the solutions to more general Hamilton-Jacobi equations in length spaces. Stability and convexity of such solutions are studied respectively in [34] and in [30]. Since this definition of solutions is based on the local slope, we shall call them slope-based solutions (or s-solutions for short) below. See the precise definition in Definition 2.3.

Since either approach above provides a generalized viscosity solution theory for first order nonlinear equations, especially the uniqueness and existence results, it is natural to expect that they actually agree with each other in the wider setting of metric spaces. This motivates us to explore the relations between both types of viscosity solutions and to understand further connections to other possible approaches.

1.2. **Equivalence of solutions to the Dirichlet problem.** In the first part of this work (Section 3), we compare c- and s-solutions of the eikonal equation and show their equivalence when continuous solutions to the Dirichlet problem exist. To this end, we assume that $(\mathbf{X}, d)$ is complete and rectifably connected; namely, for any $x, y \in \mathbf{X}$, there exists a rectifiable curve in $\mathbf{X}$ joining x and y.

Our key idea is to use the induced intrinsic metric of the space $\mathbf{X}$, which is given by

$$\tilde{d}(x, y) = \inf\{\ell(\xi) : \xi \text{ is a rectifiable curve connecting } x \text{ and } y\} \tag{1.5}$$

for $x, y \in \mathbf{X}$, where $\ell(\xi)$ denotes the length of the curve ξ with respect to the original metric d. It is then easily seen that $(\mathbf{X}, \tilde{d})$ is a length space. Since this change of metric preserves the property of c-solutions, we can directly compare the notion of c-solutions in $(\mathbf{X}, d)$ with s-solutions in $(\mathbf{X}, \tilde{d})$. In order to preserve the completeness of the metric space, we assume throughout the paper that

$$\tilde{d} \to 0 \text{ as } d \to 0. \tag{1.6}$$

Before introducing our main results, we emphasize that in the sequel, by "local" we mean the property in question holds in sufficiently small open balls with respect to the intrinsic metric of the space.

Our first main result is as follows.

Theorem 1.1 (Equivalence between c- and s-solutions). *Let* $(\mathbf{X}, d)$ *be a complete rectifiably connected metric space and* $\tilde{d}$ *be the intrinsic metric given by* (1.5). *Assume that* $(\mathbf{X}, d)$ *also satisfies* (1.6). *Suppose that* $\Omega \subsetneq \mathbf{X}$ *is an open set bounded with respect to the metric* $\tilde{d}$. *Assume that* $f \in C(\Omega)$ *satisfies* $\inf_\Omega f > 0$. *If* u *is a c-solution of* (1.3) *with respect to* d*, then* u *is a locally Lipschitz s-solution of* (1.3) *with respect to* $\tilde{d}$. *In addition, if* f *is uniformly continuous in* Ω*,* ζ *is uniformly continuous on* $\partial\Omega$*, and there exists a modulus of continuity* σ *such that*

$$|\zeta(y) - u(x)| \leq \sigma(\tilde{d}(x, y)) \quad \text{for all } x \in \Omega \text{ and } y \in \partial\Omega, \tag{1.7}$$

then u *is the unique c-solution and s-solution of* (1.3) *satisfying* (1.7).

In the second result of the theorem above, we can replace the assumptions of uniform continuity of u and f by their continuity if $(\mathbf{X}, d)$ is additionally assumed to be proper, that is, any bounded closed subset of $\mathbf{X}$ is compact.

As the metric d was replaced with the metric $\tilde{d}$, the proof of the first statement in Theorem 1.1 is more or less analogous to the classical arguments to show the relation between a viscosity solution and its optimal control interpretation (cf. [6]).

The c-solutions and f considered in [22] are in general only continuous along curves, and therefore might not be continuous with respect to either the metric d or the metric $\tilde{d}$; see [22, Example 4.9]. Here we assume the stronger requirement of continuity with respect to the metric d. Then (1.7) in the second result of Theorem 1.1 guarantees that the c-solution u is uniformly continuous up to $\partial\Omega$. Recall that the comparison principle (and thus the uniqueness) for s-solutions [21] needs such an assumption. More precisely, it was shown in [21, Theorem 5.3] that any s-subsolution u and any s-supersolution v satisfy $u \leq v$ in $\overline{\Omega}$ whenever there exists a modulus σ such that

$$u(x) \leq \zeta(y) + \sigma(\tilde{d}(x, y)), \quad v(x) \geq \zeta(y) - \sigma(\tilde{d}(x, y)) \tag{1.8}$$

for all $x \in \Omega$ and $y \in \partial\Omega$. One thus needs to use (1.7) to validate the comparison principle and conclude the uniqueness and equivalence. In Section 3.3, for the sake of completeness, sufficient conditions for (1.7) are discussed.

1.3. **Local equivalence of solutions.** Our result in Theorem 1.1 states that c- and s-solutions of (1.3) coincide. However, we show the equivalence in presence of the Dirichlet boundary condition so as to use the comparison principle. One may wonder about a more direct proof of the equivalence without using the boundary condition.

The second part of this work is devoted to answering this question. Our method is based on localization of our arguments in the first part. To this end, we introduce a local version of the notion of c-solutions, which we call local c-solutions, by restricting the definition in a small metric ball centered at each point in Ω; see Definition 2.2. We also include a third notion, called Monge solutions, in our discussion. We compare locally the notions of s-, local c-, and Monge s olutions of (1.3) in a complete length space.

The Monge solution is known to be an alternative notion of solutions to Hamilton-Jacobi equations in Euclidean spaces [35, 7]. In a complete length space $(\mathbf{X}, d)$, our generalized definition of Monge solutions to (1.3) is quite simple; it only requires a locally Lipschitz function u to satisfy

$$|\nabla^- u|(x) = f(x) \quad \text{for } \textit{every } x \in \Omega,$$

where $|\nabla^- u|(x)$, given by

$$|\nabla^- u|(x) = \limsup_{y \to x} \frac{\max\{u(x) - u(y), 0\}}{d(x, y)},$$

denotes the sub-slope of u at x; see also the definition in (2.8). One advantage of this notion is that it does not involve any viscosity tests and the comparison principle can be easily established. We show that locally uniformly continuous c-, s- and Monge solutions of the eikonal equation are actually equivalent under a weaker positivity assumption on f. A more precise statement is given below.

Theorem 1.2 (Local equivalence between solutions of eikonal equation)**.** *Let $(\mathbf{X}, d)$ be a complete length space and $\Omega \subset \mathbf{X}$ be an open set. Assume that f is locally uniformly continuous and $f > 0$ in Ω. Let $u \in C(\Omega)$. Then the following statements are equivalent:*

(a) *u is a local c-solution of* (1.3)*;*
(b) *u is a locally uniformly continuous s-solution of* (1.3)*;*
(c) *u is a Monge solution of* (1.3)*.*

In addition, if any of (a)–(c) holds, then u is locally Lipschitz with

$$|\nabla u|(x) = |\nabla^- u|(x) = f(x) \quad \textit{for all } x \in \Omega. \tag{1.9}$$

We actually prove more:

- The notions of all locally uniformly continuous subsolutions are equivalent and locally Lipschitz (Proposition 4.5, Proposition 4.6);
- The notions of locally Lipschitz s-supersolutions and Monge supersolutions are equivalent (Proposition 4.7(i));
- Any locally Lipschitz local c-supersolution is a Monge supersolution (Proposition 4.7(ii));
- Any Monge solution is a local c-solution (Proposition 4.8).

We however do not know whether an s- or Monge supersolution needs to be a local c-supersolution.

In proving Theorem 1.2, it turns out that the local Lipschitz continuity of solutions is an important ingredient. Note that the local Lipschitz continuity holds for Monge solutions by definition, and it can be easily deduced for c-subsolutions as well because the space $(\mathbf{X}, \tilde{d})$ is a length space, as shown in Lemma 3.3. In contrast, the Lipschitz regularity of s-solutions of (1.3) is less straightforward. Our proof requires the assumption on the local uniform continuity of s-solutions and f due to possible lack of compactness for general length spaces. As stated in Corollary 4.10, we can remove such an assumption if $(\mathbf{X}, d)$ is proper (and therefore the length space space X is a geodesic space because of the generalized Hopf-Rinow theorem in metric spaces [23, 9], which can be viewed as an immediate consequence of the Arzela-Ascoli theorem). The notions of continuity and uniform continuity in a compact set are clearly equivalent.

In this work we will assume that $(\mathbf{X}, d)$ satisfies the hypotheses of Theorem 1.1. It is worth stressing that in this section $(\mathbf{X}, d)$ is assumed to be a length space only for simplicity from the point of view of c-solutions. The s-solutions and Monge solutions are not defined for the case that $\mathbf{X}$ is not a length space. For metric spaces that are not length spaces, the slopes in the definition of s-solutions and Monge solutions require us to replace the metric d with $\tilde{d}$.

It is not difficult to extend our discussion on the equivalence to the general equation (1.1) under a monotonicity assumption on $p \to H(x, r, p)$. When $p \to H(x, r, p)$ is increasing, we can simply apply an implicit-function-type argument to locally reduce the problem to the eikonal equation.

Besides, as in (1.9) in Theorem 1.2, in this general case for any solution u we can obtain the continuity of $|\nabla u|$ as well as the following property:

$$|\nabla u|(x) = |\nabla^{-} u|(x) \quad \text{for all } x \in \Omega; \tag{1.10}$$

in other words, the solution itself actually lie in the test class for s-subsolutions proposed in [20, 21]. This type of properties also appears in the study of time-dependent Hamilton-Jacobi equations on metric spaces (cf. [31]). Our analysis reveals that (1.10) resembles the semi-concavity in the Euclidean space. In fact, in the Euclidean space, (1.10) implies the existence of a C^1 test function everywhere from above, which is a typical property of semi-concave functions. We expect more applications of the regularity property (1.10), since in general metric spaces defining convex functions is not trivial at all. It would be interesting to see further properties on such regular solutions in relation to the structure of PDEs and the geometric property.

The rest of the paper is organized as follows. In Section 2, we review the definitions and properties of c- and s-solutions of Hamilton-Jacobi equations in metric spaces. Section 3 is devoted to the proof of Theorem 1.1. In Section 4 we propose the notion of Monge solutions and prove Theorem 1.2.

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2. Metric viscosity solutions of Hamilton-Jacobi equations

In this section we review the two notions of viscosity solutions to Hamilton-Jacobi equations in metric spaces, mentioned in the first section. We focus on the stationary equation (1.1) and particularly the eikonal equation (1.3).

In the setting of general metric spaces, one needs to have an analog of $|\nabla u|$. To do so, let us recall the definitions of viscosity solutions in [22] and [21].

2.1. Curve-based solutions.

Given an interval $I = [a, b] \subset \mathbb{R}$ and a continuous function (curve) $\xi : I \to \Omega$, we define the length of γ by

$$\ell(\xi) := \sup_{a=t_0<t_1<\cdots<t_k=b} \sum_{j=0}^{k-1} d(\xi(t_j), \xi(t_{j+1})).$$

Note that if $\ell(\xi) < \infty$, then the real-valued function $s_\xi : [a, b] \to [0, \ell(\gamma)]$ defined by

$$s_\xi(t) := \ell(\xi|_{[a,t]})$$

is a monotone increasing function, and hence it is differentiable at almost every $t \in [a, b]$. We denote s'_ξ by $|\xi'|$, which can be equivalently defined by

$$|\xi'|(t) = \lim_{\tau \to 0} \frac{d(\xi(t+\tau), \xi(t))}{|\tau|}$$

for almost every $t \in (a, b)$. Note that if ξ is an absolutely continuous curve (and so s_ξ is absolutely continuous real-valued function), then

$$s_\xi(t) = \int_a^t |\xi'|(\tau)\, d\tau$$

for all $t \in [a, b]$.

For any interval $I \subset \mathbb{R}$, we say an absolutely continuous curve $\xi : I \to \mathbf{X}$ is admissible if

$$|\xi'| \leq 1 \quad \text{a.e. in } I.$$

Note that these are 1-Lipschitz curves. Let $\mathcal{A}(I, \mathbf{X})$ denote the set of all admissible curves in $\mathbf{X}$ defined on I; without loss of generality, we only consider intervals I for which $0 \in I$. For any $x \in \mathbf{X}$, we write $\xi \in \mathcal{A}_x(I, \mathbf{X})$ if $\xi(0) = x$. We also need to define the exit time and entrance time of a curve ξ:

$$\begin{aligned} T^+_\Omega[\xi] &:= \inf\{t \in I : t \geq 0, \xi(t) \notin \Omega\}; \\ T^-_\Omega[\xi] &:= \sup\{t \in I : t \leq 0, \xi(t) \notin \Omega\}. \end{aligned} \tag{2.1}$$

Since any absolutely continuous curve is rectifiable and one can always reparametrize a rectifiable curve by its arc length (cf. [24, Theorem 3.2]), hereafter we do not distinguish the difference between an absolutely continuous curve and a rectifiable (or Lipschitz) curve.

Definition 2.1 (Definition 2.1 in [22])**.** An upper semicontinuous (USC) function u in Ω is called a curve-based viscosity subsolution or c-subsolution of (1.3) if for any $x \in \Omega$ and $\xi \in \mathcal{A}_x(\mathbb{R}, \Omega)$, we have

$$\left|\phi'(0)\right| \leq f(x) \tag{2.2}$$

whenever $\phi \in C^1(\mathbb{R})$ such that $t \mapsto u(\xi(t)) - \phi(t)$, $t \in \xi^{-1}(\Omega)$, attains a local maximum at $t = 0$.

A lower semicontinuous (LSC) function u in Ω is called a curve-based viscosity supersolution or c-supersolution of (1.3) if for any $\varepsilon > 0$ and $x \in \Omega$, there exists $\xi \in \mathcal{A}_x(\mathbb{R}, \mathbf{X})$ and $w \in LSC(T^-, T^+)$ with $-\infty < T^\pm = T^\pm_\Omega[\xi] < \infty$ such that

$$w(0) = u(x), \quad w \geq u \circ \xi - \varepsilon, \tag{2.3}$$

and

$$\left|\phi'(t_0)\right| \geq f(\xi(t_0)) - \varepsilon \tag{2.4}$$

whenever $\phi \in C^1(\mathbb{R})$ such that $w(t) - \phi(t)$ attains a minimum at $t = t_0 \in (T^-, T^+)$. A function $u \in C(\Omega)$ is said to be a curve-based viscosity solution or c-solution if it is both a c-subsolution and a c-supersolution of (1.3).

In the definition of supersolutions, in general we cannot merely replace w with $u \circ \xi$. Suppose that $\mathbf{X}$ is not a geodesic space but a length space. When $f \equiv 1$, as we expect that the distance function $u = d(\cdot, x_0)$ is still a solution for any $x_0 \in \mathbf{X}$, the supersolution property for $u \circ \xi$ without approximation would imply that ξ is a geodesic, which is a contradiction.

The regularity of u above can be relaxed, since we only need its semicontinuity along each curve ξ. In fact, one can require a c-subsolution (resp., c-supersolution) to be merely arcwise upper (resp., lower) semicontinuous; consult [22] for details. However, in order to obtain our main results in this paper, we need to impose the conventional continuity of u rather than the arcwise continuity.

The notions of c-subsolutions and c-supersolutions with respect to the metric d and the intrinsic metric $\tilde{d}$ given in (1.5) are equivalent. Note that ξ is a curve with respect to d if and only if it is a curve with respect to $\tilde{d}$ because of our assumption (1.6). It can also be seen that the speed $|\xi'|$ of the curve remains the same in both metrics; see Lemma 3.1. Hence, the class of admissible curves in Definition 2.1 does not depend on the choice between d and $\tilde{d}$.

Although there seems to be no requirement on the metric space in the definition above, it is implicitly assumed in the definition of the c-supersolution that each point $x \in \Omega$ can be connected to the boundary $\partial\Omega$ by a curve of finite length.

Uniqueness of c-solutions of (1.3) with boundary data (1.4) is shown by proving a comparison principle [22, Theorem 3.1]. The existence of solutions in $C(\overline{\Omega})$, on the other hand, is based on an optimal control interpretation; in particular, it is shown in [22, Theorem 4.2 and Theorem 4.5] that

$$u(x) = \inf_{\xi \in C_x} \left\{ \int_0^{T_\Omega^+[\xi]} f(\xi(s))\, ds + \zeta\left(\xi(T_\Omega^+[\xi])\right) \right\} \tag{2.5}$$

is a c-solution of (1.3) and (1.4) provided that for each $x \in \overline{\Omega}$ we have

$$C_x := \left\{\xi \in \mathcal{A}_x([0,\infty), \mathbf{X}) \,:\, T_\Omega^+[\xi] \in (0,\infty)\right\} \neq \emptyset$$

and ζ satisfies a boundary regularity, see (3.17) below.

Note in the definition of c-supersolution, for each (x,ε) the conditions (2.3) and (2.4) for (ξ, w) are satisfied for all $t \in (T^-, T^+)$. We localize this definition as follows. Recall the notions of $T_{\mathcal{O}}^{\pm}[\xi]$ for open sets $\mathcal{O} \subset \mathbf{X}$ from (2.1).

Definition 2.2 (Local curve-based solutions)**.** A function $u \in LSC(\Omega)$ is said to be a local c-supersolution if for each $x \in \Omega$ there exists $r > 0$ with $B_r^d(x) \subset \Omega$, and for each $\varepsilon > 0$ we can find a curve $\xi_\varepsilon \in \mathcal{A}_x(\mathbb{R}, \mathbf{X})$ with $\xi_\varepsilon(0) = x$ and a function $w \in LSC(t_r^-, t_r^+)$ with $t_r^- := T^-_{B_r^d(x)}[\xi_\varepsilon]$ and $t_r^+ := T^+_{B_r^d(x)}[\xi_\varepsilon]$ such that

$$w(0) = u(x), \quad w(t) \geq u \circ \xi_\varepsilon(t) - \varepsilon \ \text{ for all } t \in (t_r^-, t_r^+),$$

and

$$\left|\phi'(t_0)\right| \geq f(\xi_\varepsilon(t_0)) - \varepsilon$$

whenever $\phi \in C^1(\mathbb{R})$ such that $w(t) - \phi(t)$ attains a minimum at $t = t_0 \in (t_r^-, t_r^+)$. A function $u \in C(\Omega)$ is said to be a local c-solution if it is both a c-subsolution and a local c-supersolution of (1.3).

In the definition of local c-supersolutions given above, the ball $B_r^d(x)$ is taken with respect to d. If $(\mathbf{X}, d)$ is a complete rectifiably connected metric space such that the intrinsic metric $\tilde{d}$ defined in (1.5) satisfies the consistency condition (1.6), then it is equivalent to use metric balls $B_r(x)$ with respect to $\tilde{d}$. This definition is studied mainly in Section 4, where we assume $(\mathbf{X}, d)$ to be a length space. For length spaces balls with respect to d and balls with respect to $\tilde{d}$ are the same, that is, $B_r^d(x) = B_r(x)$.

A c-supersolution (resp., c-solution) is clearly a local c-supersolution (resp., local c-solution), but it is not clear to us whether the reverse is also true in general. The notion of c-subsolutions is already a localized one, and we therefore do not have to define "local c-subsolutions" separately.

2.2. **Slope-based solutions.** We next discuss the definition proposed in [20], which relies more on the property of geodesic or length metric. We denote by $\mathrm{Lip}_{loc}(\Omega)$ the set of locally Lipschitz continuous functions on an open subset Ω of a complete length space $(\mathbf{X}, d)$. For $u \in \mathrm{Lip}_{loc}(\Omega)$ and for $x \in \Omega$, we define the local slope of u to be

$$|\nabla u|(x) := \limsup_{y \to x} \frac{|u(y) - u(x)|}{d(x,y)}. \tag{2.6}$$

Let

$$\begin{aligned} \overline{\mathcal{C}}(\Omega) &:= \{u \in \mathrm{Lip}_{loc}(\Omega) \,:\, |\nabla^+ u| = |\nabla u| \text{ and } |\nabla u| \text{ is continuous in } \Omega\}, \\ \underline{\mathcal{C}}(\Omega) &:= \{u \in \mathrm{Lip}_{loc}(\Omega) \,:\, |\nabla^- u| = |\nabla u| \text{ and } |\nabla u| \text{ is continuous in } \Omega\}, \end{aligned} \tag{2.7}$$

where, for each $x \in \mathbf{X}$,

$$|\nabla^{\pm} u|(x) := \limsup_{y \to x} \frac{[u(y) - u(x)]_{\pm}}{d(x,y)} \tag{2.8}$$

with $[a]_+ := \max\{a, 0\}$ and $[a]_- := -\min\{a, 0\}$ for any $a \in \mathbb{R}$. In this work we call $|\nabla^+ u|$ and $|\nabla^- u|$ the (local) super- and sub-slopes of u respectively; they are also named super- and sub-gradient norms in the literature (cf. [31]).

Concerning the test classes $\overline{\mathcal{C}}(\Omega)$ and $\underline{\mathcal{C}}(\Omega)$, it is known from [20, Lemma 7.2] and [21, Lemma 2.3] that in a length space X, $Ad(\cdot, x_0)^2$ belongs to $\underline{\mathcal{C}}(\Omega)$ for for any $x_0 \in \mathbf{X}$ and $A > 0$; moreover, $Ad(\cdot, x_0)^2$ belongs to $\overline{\mathcal{C}}(\Omega)$ for any $x_0 \in \mathbf{X}$ and $A < 0$.

Now we recall from [21] the definition of s-solutions of a general class of Hamilton-Jacobi equations.

Definition 2.3 (Definition 2.6 in [21])**.** An USC (resp., LSC) function u in an open set $\Omega \subset \mathbf{X}$ is called a slope-based viscosity subsolution (resp., slope-based viscosity supersolution) or s-subsolution (resp., s-supersolution) of (1.1) if

$$H_{|\nabla \psi_2|^*(x)}(x, u(x), |\nabla \psi_1|(x)) \le 0 \tag{2.9}$$

$$\left(\text{resp.,} \quad H^{|\nabla \psi_2|^*(x)}(x, u(x), |\nabla \psi_1|(x)) \ge 0\right) \tag{2.10}$$

holds for any $\psi_1 \in \underline{\mathcal{C}}(\Omega)$ (resp., $\psi_1 \in \overline{\mathcal{C}}(\Omega)$) and $\psi_2 \in \mathrm{Lip}_{loc}(\Omega)$ such that $u - \psi_1 - \psi_2$ attains a local maximum (resp., minimum) at a point $x \in \Omega$, where, for any $(x, \rho, p) \in \Omega \times \mathbb{R} \times \mathbb{R}$ and $a > 0$,

$$H_a(x, \rho, p) = \inf_{|q-p| \le a} H(x, \rho, q), \quad H^a(x, \rho, p) = \sup_{|q-p| \le a} H(x, \rho, q) \quad \text{for } a \ge 0,$$

and $|\nabla \psi_2|^*(x) = \limsup_{y \to x} |\nabla \psi_2|(y)$. We say that $u \in C(\Omega)$ is an s-solution of (1.1) if it is both an s-subsolution and an s-supersolution of (1.1).

In the case of (1.3), we can define subsolutions (resp., supersolutions) by replacing (2.9) (resp., (2.10)) with

$$|\nabla \psi_1|(x) \le f(x) + |\nabla \psi_2|^*(x) \tag{2.11}$$

$$(\text{resp.,} \quad |\nabla \psi_1|(x) \ge f(x) - |\nabla \psi_2|^*(x)). \tag{2.12}$$

When $\mathbf{X} = \mathbb{R}^N$, it is not difficult to see that $C^1(\Omega) \subset \overline{\mathcal{C}}(\Omega) \cap \underline{\mathcal{C}}(\Omega)$ for any open set $\Omega \subset \mathbf{X}$. Hence, s-subsolutions, s-supersolutions and s-solutions of (1.1) in this case reduce to conventional viscosity subsolutions, supersolution and solutions respectively.

Concerning the test functions in a general geodesic or length space $(\mathbf{X}, d)$, it is known that, for any $k \geq 0$, $x_0 \in \mathbf{X}$, the function $x \mapsto k\varphi(d(x, x_0))$ (resp., $x \mapsto -k\varphi(d(x, x_0))$) belongs to the class $\underline{\mathcal{C}}(\Omega)$ (resp., $\overline{\mathcal{C}}(\Omega)$) provided that $\varphi \in C^1([0, \infty))$ satisfies $\varphi'(0) = 0$ and $\varphi' \geq 0$; see details in [20, Lemma 7.2] and [21, Lemma 2.3].

Comparison principles for s-solutions are given in [21, Theorem 5.3] for the eikonal equation and in [21, Theorem 5.1, Theorem 5.3] for more general Hamilton-Jacobi equations.

3. Equivalence between curve- and metric-based solutions

We give a proof of Theorem 1.1 in this section. Let us begin with some elementary results on the space $\mathbf{X}$ with the intrinsic metric $\tilde{d}$. Then we show that any c-subsolutions and c-supersolutions are respectively s-subsolutions and s-supersolutions.

3.1. Metric change.

Let $(\mathbf{X}, d)$ be a complete rectifiably connected metric space. We compare the notions of viscosity solutions to (1.3) provided respectively in Definition 2.1 and in Definition 2.3. The key to our argument is to use (1.5) to connect the geometric setting of two notions.

It is clear that

$$d(x, y) \leq \tilde{d}(x, y) \quad \text{for all } x, y \in \mathbf{X}. \tag{3.1}$$

Therefore bounded sets in $(\mathbf{X}, \tilde{d})$ are bounded in $(\mathbf{X}, d)$. Under the assumption of rectifiable connectedness of (X, d), we also see that $\tilde{d}(x, y) < \infty$ for any $x, y \in \mathbf{X}$ and $\tilde{d}$ is a metric on $\mathbf{X}$. Moreover, by (3.1), it is clear that open sets in $(\mathbf{X}, d)$ are also open in $(\mathbf{X}, \tilde{d})$.

The metric $\tilde{d}$ is also used in [22] to study the continuity and stability of c-solutions. In the rest of this work, $B_r(x)$ denotes the open ball centered at $x \in \mathbf{X}$ with radius $r > 0$ with respect to the intrinsic metric $\tilde{d}$.

The induced intrinsic structure leads us to the following elementary fact that $(\mathbf{X}, \tilde{d})$ is a length space. One can find this classical result in [8, Proposition 2.3.12] and [36, Corollary 2.1.12] for instance. See also [17] for the arc-length parametrization of rectifiable curves with respect to d and with respect to $\tilde{d}$.

Lemma 3.1 (Length space under intrinsic metric)**.** *Assume that $(\mathbf{X}, d)$ is a complete rectifiably connected metric space. Let $\tilde{d}$ be the intrinsic metric of a metric space $(\mathbf{X}, d)$ as defined in (1.5). Then $(\mathbf{X}, \tilde{d})$ is a length space. Moreover, for any rectifiable curve ξ, $\xi(s) : I \to \mathbf{X}$ is a parametrization with respect to d if and only if it is a parametrization with respect to $\tilde{d}$ and the speed $|\xi'|$ in both metrics coincide.*

We remark that a similar intrinsic metric is constructed in [15, 16, 17] involving a given measure on the space. Ours is standard and simpler, since measures do not play a role in the current work.

The completeness of the metric space is needed to properly define s-solutions under the metric $\tilde{d}$. Note that $(\mathbf{X}, \tilde{d})$ is complete if $(\mathbf{X}, d)$ is complete, since $d \leq \tilde{d}$ holds and $(\mathbf{X}, d)$ satisfies the condition (1.6).

3.2. **Equivalence between solutions of the Dirichlet problem.** Let us start proving Theorem 1.1. We first prove that any c-subsolution is an s-subsolution. We need the following characterization of c-subsolutions given in [22].

Proposition 3.2 (Proposition 2.6 in [22]). *Assume that $f \in C(\Omega)$ with $f \geq 0$ in Ω. Let u be upper semicontinuous in Ω. Then the following statements are equivalent:*

(1) *u is a c-subsolution of (1.3) in (Ω, d).*

(2) *The inequality*

$$u(\xi(t_1)) \leq \int_{t_1}^{t_2} f(\xi(r))\, dr + u(\xi(t_2)) \tag{3.2}$$

for all $\xi \in \mathcal{A}(\mathbb{R}, \Omega)$ and $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$.

Such characterization enables us to deduce the local Lipschitz continuity of c-subsolutions with respect to the metric $\tilde{d}$.

Lemma 3.3 (Local Lipschitz continuity of c-subsolutions). *Let $(\mathbf{X}, d)$ be a complete rectifiably connected metric space and $\tilde{d}$ be the intrinsic metric given by (1.5). Assume that (1.6) holds. Let $\Omega \subsetneq \mathbf{X}$ be an open set. Assume that $f \in C(\Omega)$ and $f \geq 0$ in Ω. If u is upper semicontinuous in Ω and is a c-subsolution of (1.3) in (Ω, d), then $u \in \mathrm{Lip}_{loc}(\Omega)$. In particular, for any $x_0 \in \Omega$ and $r > 0$ such that $B_{2r}(x_0) \subset \Omega$ and f is bounded in $B_{2r}(x_0)$, u satisfies*

$$|u(x) - u(y)| \leq \tilde{d}(x, y) \sup_{B_{2r}(x_0)} f \quad \text{for all } x, y \in B_r(x_0). \tag{3.3}$$

Proof. Fix arbitrarily $x_0 \in \Omega$ and $r > 0$ small such that $B_{2r}(x_0) \subset \Omega$ (with respect to $\tilde{d}$) and f is bounded on $B_{2r}(x_0)$. For any $x, y \in B_r(x_0)$ and any $0 < \varepsilon < 2r - \tilde{d}(x, y)$, there exists an arc-length parametrized rectifiable curve ξ_0 joining x and y satisfying

$$\ell(\xi_0) \leq \tilde{d}(x, y) + \varepsilon < 2r.$$

It follows that $\xi_0 \subset B_{2r} \subset \Omega$. Applying the characterization of c-subsolutions in Proposition 3.2(ii) with $\xi = \xi_0$, we have

$$u(x) - u(y) \leq (\tilde{d}(x, y) + \varepsilon) \sup_{B_{2r}(x_0)} f.$$

Passing to the limit $\varepsilon \to 0$, we get

$$u(x) - u(y) \leq \tilde{d}(x, y) \sup_{B_{2r}(x_0)} f.$$

Exchanging the roles of x and y, we thus obtain (3.3). □

We next continue to use Proposition 3.2 to show that c-subsolutions of (1.3) are s-subsolutions with respect to the metric $\tilde{d}$.

Proposition 3.4 (Implication of subsolution property). *Let $(\mathbf{X}, d)$ be a complete rectifiably connected metric space and $\tilde{d}$ be the intrinsic metric given by (1.5). Assume that (1.6) holds. Let $\Omega \subsetneq \mathbf{X}$ be an open set. Assume that $f \in C(\Omega)$ and $f \geq 0$ in Ω. If u is upper semicontinuous in Ω and is a c-subsolution of (1.3) in (Ω, d), then u is an s-subsolution of (1.3) in $(\Omega, \tilde{d})$.*

Proof. Since $(\mathbf{X}, \tilde{d})$ is a length space, our notation $\mathrm{Lip}_{loc}(\Omega)$ now denotes the set of all locally Lipschitz functions on Ω with respect to the intrinsic metric $\tilde{d}$. Note that if u is upper semicontinuous with respect to the metric d, then it is upper semicontinous with respect to $\tilde{d}$, since $\tilde{d} \to 0$ if and only if $d \to 0$ due to (1.5) and (1.6).

Fix $x_0 \in \Omega$ arbitrarily. Assume that there exists $\psi_1 \in \underline{\mathcal{C}}(\Omega)$ and $\psi_2 \in \mathrm{Lip}_{loc}(\Omega)$ such that $u - \psi_1 - \psi_2$ attains a local maximum at a point x_0. So there is some $r_0 > 0$ with $B_{2r_0}(x_0) \subset \Omega$ such that

$$u(x) - u(x_0) \leq (\psi_1 + \psi_2)(x) - (\psi_1 + \psi_2)(x_0)$$

for all $x \in B_{r_0}(x_0)$.

Moreover, for any fixed $\varepsilon \in (0, 1)$, by the continuity of f, we can make $0 < r_0$ smaller so that

$$|f(x) - f(x_0)| \leq \varepsilon \tag{3.4}$$

if $x \in B_{r_0}(x_0)$. We fix such $r_0 > 0$ (and keep in mind that r_0 now also depends on ε).

For any $r \in (0, r_0/2)$ and any $x \in \Omega$ with $0 < \tilde{d}(x, x_0) < r$, there exists an arc-length parametrized curve ξ in Ω such that $\xi(0) = x_0$ and $\xi(t) = x$, where

$$t = \ell(\xi) \leq \tilde{d}(x, x_0) + \varepsilon \tilde{d}(x, x_0). \tag{3.5}$$

Applying Proposition 3.2 for such a curve with $t_1 = 0, t_2 = t$, we get

$$u(x_0) \leq \int_0^t f(\xi(s))\, ds + u(x),$$

and therefore,

$$(\psi_1 + \psi_2)(x_0) - (\psi_1 + \psi_2)(x) \leq \int_0^t f(\xi(s))\, ds.$$

Dividing the inequality above by $\tilde{d}(x, x_0)$, we get

$$\frac{\psi_1(x_0) - \psi_1(x)}{\tilde{d}(x, x_0)} \leq \frac{1}{\tilde{d}(x, x_0)} \int_0^t f(\xi(s))\, ds + \frac{\psi_2(x) - \psi_2(x_0)}{\tilde{d}(x, x_0)}. \tag{3.6}$$

Since $\varepsilon < 1$ and $r < r_0/2$, we have

$$\tilde{d}(\xi(s), x_0) \leq t \leq r + \varepsilon r < r_0$$

for all $s \in [0, t]$, by (3.4). Therefore

$$f(\xi(s)) \leq f(x_0) + \varepsilon$$

for all $s \in [0, t]$. Hence (3.6) yields

$$\frac{\psi_1(x_0) - \psi_1(x)}{\tilde{d}(x, x_0)} \leq \frac{t}{\tilde{d}(x, x_0)}(f(x_0) + \varepsilon) + \frac{\psi_2(x) - \psi_2(x_0)}{\tilde{d}(x, x_0)}.$$

Using (3.5) and recalling that the choice of r_0 depends on ε, we thus have

$$\frac{\psi_1(x_0) - \psi_1(x)}{\tilde{d}(x, x_0)} \leq (1 + \varepsilon)(f(x_0) + \varepsilon) + \frac{\psi_2(x) - \psi_2(x_0)}{\tilde{d}(x, x_0)} \tag{3.7}$$

for all $\varepsilon > 0$ and all $x \in \Omega$ with $x \in B_{r_0}(x_0)$.

Since $\psi_1 \in \underline{\mathcal{C}}(\Omega)$, there exists a sequence of points $x_n \in \Omega$ such that, as $n \to \infty$, we have $x_n \to x_0$ and

$$\frac{\psi_1(x_0) - \psi_1(x_n)}{\tilde{d}(x_n, x_0)} \to |\nabla^- \psi_1|(x_0) = |\nabla \psi_1|(x_0).$$

Adopting (3.7) with $x = x_n$ and sending $n \to \infty$ and then $\varepsilon \to 0$, we end up with the desired inequality (2.11) at $x = x_0$. $\square$

We next show that any c-supersolution is an s-supersolution. We again use a result presented in [22].

Proposition 3.5 (Proposition 2.8 in [22])**.** *Assume that $f \in C(\Omega)$ with $f \geq 0$. Assume $\inf_\Omega f > 0$. Let u be a lower semicontinuous c-supersolution of (1.3). Then for any $\varepsilon > 0$ and $x_0 \in \Omega$, there exists $\xi_\varepsilon \in \mathcal{A}_{x_0}([0,\infty),\Omega)$ satisfying $T = T^+_\Omega[\xi_\varepsilon] < \infty$ and*

$$u(x_0) \geq \int_0^t f(\xi_\varepsilon(s))\, ds + u(\xi_\varepsilon(t)) - \varepsilon(1+t) \tag{3.8}$$

for all $0 \leq t \leq T$.

Remark 3.6. If u is a local c-supersolution instead, then for each $x_0 \in \Omega$ there is a sufficiently small $r > 0$ such that for each $\varepsilon > 0$ we can find a choice $\xi_\varepsilon \in \mathcal{A}_{x_0}([0,\infty), B_r(x_0))$ such that for all $0 \leq t \leq T^+_{B_r(x_0)}[\xi_\varepsilon]$, (3.8) holds. This is seen by directly adapting the proof of [22, Proposition 2.8].

Proposition 3.7 (Implication of supersolution property)**.** *Let $(\mathbf{X}, d)$ be a complete rectifiably connected metric space and $\tilde{d}$ be the intrinsic metric given by (1.5). Assume that (1.6) holds. Let $\Omega \subsetneq \mathbf{X}$ be an open set. Assume that $f \in C(\Omega)$ with $\inf_\Omega f > 0$. If u is a lower semicontinuous c-supersolution of (1.3) in (Ω, d), then u is an s-supersolution of (1.3) in $(\Omega, \tilde{d})$.*

Proof. Since u is lower semicontinuous with respect to the metric d, it is easily seen that u is also lower semicontinuous with respect to $\tilde{d}$, thanks to (1.5) and (1.6).

Fix $x_0 \in \Omega$ arbitrarily. Assume that there exist $\psi_1 \in \overline{\mathcal{C}}(\Omega)$ and $\psi_2 \in \mathrm{Lip}_{loc}(\Omega)$ such that $u - \psi_1 - \psi_2$ attains a local minimum at a point x_0. We thus have $r_0 > 0$ such that $B_{2r_0}(x_0) \subset \Omega$ and

$$u(y) - u(x_0) \geq (\psi_1 + \psi_2)(y) - (\psi_1 + \psi_2)(x_0)$$

for all $y \in B_{r_0}(x_0)$.

Applying Proposition 3.5, for any $\varepsilon > 0$ satisfying $\sqrt{\varepsilon} < \min\{r, \tilde{d}(x_0, \partial\Omega)\}$, we can find $\xi \in \mathcal{A}_{x_0}([0,\infty),\Omega)$ such that (3.8) holds for all $0 \leq t \leq T^+_\Omega[\xi]$. Since $T^+_\Omega[\xi] \geq \tilde{d}(x_0, \partial\Omega) > 0$, we can take $t = \sqrt{\varepsilon}$ and $x_\varepsilon = \xi(\sqrt{\varepsilon})$ in (3.8) to get

$$(\psi_1 + \psi_2)(x_\varepsilon) - (\psi_1 + \psi_2)(x_0) \leq -\int_0^{\sqrt{\varepsilon}} f(\xi(s))\, ds + \varepsilon(1 + \sqrt{\varepsilon}).$$

Dividing this relation by $\sqrt{\varepsilon}$, we get

$$\frac{1}{\sqrt{\varepsilon}} \int_0^{\sqrt{\varepsilon}} f(\xi(s))\, ds + \frac{\psi_2(x_\varepsilon) - \psi_2(x_0)}{\sqrt{\varepsilon}} - \sqrt{\varepsilon}(1 + \sqrt{\varepsilon}) \leq \frac{\psi_1(x_0) - \psi_1(x_\varepsilon)}{\sqrt{\varepsilon}}. \tag{3.9}$$

Noticing that

$$\tilde{d}\,(x_\varepsilon, x_0) \leq \sqrt{\varepsilon},$$

we deduce that

$$\frac{\psi_1(x_\varepsilon) - \psi_1(x_0)}{\sqrt{\varepsilon}} \leq \frac{|\psi_1(x_\varepsilon) - \psi_1(x_0)|}{\sqrt{\varepsilon}} \leq \frac{|\psi_1(x_\varepsilon) - \psi_1(x_0)|}{\tilde{d}\,(x_\varepsilon, x_0)}$$

and

$$\frac{\psi_2(x_\varepsilon) - \psi_2(x_0)}{\sqrt{\varepsilon}} \geq -\frac{|\psi_2(x_\varepsilon) - \psi_2(x_0)|}{\sqrt{\varepsilon}} \geq -\frac{|\psi_2(x_\varepsilon) - \psi_2(x_0)|}{\tilde{d}\,(x_\varepsilon, x_0)}.$$

Hence, combining the above inequalities together implies that

$$\frac{|\psi_1(x_\varepsilon) - \psi_1(x_0)|}{\tilde{d}\,(x_\varepsilon, x_0)} \geq \frac{1}{\sqrt{\varepsilon}} \int_0^{\sqrt{\varepsilon}} f(\xi(s))\, ds - \frac{|\psi_2(x_\varepsilon) - \psi_2(x_0)|}{\tilde{d}\,(x_\varepsilon, x_0)} - \sqrt{\varepsilon}(1 + \sqrt{\varepsilon}).$$

Letting $\varepsilon \to 0$, we are led to (2.12) with $x = x_0$ as desired. □

Remark 3.8. By Remark 3.6, it is not difficult to see that if u is only a local c-supersolution, then for any $x_0 \in \Omega$ the same result as in Proposition 3.7 holds in $B_r(x_0)$ with $r > 0$ small. In fact, the proof will be the same except that Ω should be replaced by $B_r(x_0)$.

We now prove Theorem 1.1.

Proof of Theorem 1.1. In view of Lemma 3.3, we know that any c-solution of (1.3) is locally Lipschitz with respect to $\tilde{d}$. Now by Proposition 3.4 and Proposition 3.7 we see that any c-solution u of (1.3) is a locally Lipschitz s-solution. If u satisfies (1.7) and f is uniformly continuous with $\inf_\Omega f > 0$, then in the bounded domain Ω we can apply the comparison principle for s-solutions (cf. [21, Theorem 5.3]) to show that u must be the only s- and c-solution of the Dirichlet problem. □

3.3. **Boundary value.** In light of the second part of Theorem 1.1, the importance of the condition (1.7) is clear. We now give sufficient conditions for (1.7), which is also important for the existence of c-solutions. Recall that ζ is a continuous function on $\partial\Omega$, playing the role of the Dirichlet boundary data in (1.4).

Proposition 3.9 (Boundary consistency)**.** *Assume that* $(\mathbf{X}, d)$ *is a complete rectifiably connected metric space and* $\tilde{d}$ *be the induced intrinsic metric given by* (1.5)*. Assume that* (1.6) *holds. Let* $\Omega \subsetneq \mathbf{X}$ *be an open set. Suppose that* $f \in C(\overline{\Omega})$ *is bounded and* $f \geq 0$ *in* $\overline{\Omega}$*. Let* u *be given by* (2.5) *with* $\zeta \in C(\partial\Omega)$ *given.*

(1) *If there exists* $L > 0$ *such that* ζ *is* L*-Lipschitz on* $\partial\Omega$ *with respect to the metric* $\tilde{d}$*, then*

$$u(x) - \zeta(y) \leq \tilde{d}(x, y) \max\left\{L,\ \sup_{\overline{\Omega}} f\right\} \quad \text{for all } x \in \Omega \text{ and } y \in \partial\Omega. \tag{3.10}$$

(2) *If* ζ *satisfies a stronger condition:*

$$|\zeta(x) - \zeta(y)| \leq \tilde{d}(x, y) \inf_{\overline{\Omega}} f \quad \text{for every } x, y \in \partial\Omega, \tag{3.11}$$

then

$$|u(x) - \zeta(y)| \leq \tilde{d}(x, y) \sup_{\overline{\Omega}} f \quad \text{for all } x \in \Omega \text{ and } y \in \partial\Omega. \tag{3.12}$$

Proof. For simplicity of notation, denote

$$m := \inf_{\overline{\Omega}} f, \quad M := \sup_{\overline{\Omega}} f.$$

Fix $x \in \Omega$ and $y \in \partial\Omega$. Then for any $\varepsilon > 0$, there exists an arc-length parametrized curve $\xi \in \mathcal{A}_x(\mathbb{R}, \mathbf{X})$ such that $\xi(t) = y$ and

$$\tilde{d}(x, y) \leq t \leq \tilde{d}(x, y) + \varepsilon.$$

This curve may not stay in Ω, but there exists $z = \xi(t_1) \in \partial\Omega$, where

$$t_1 := T^+_\Omega[\xi] = \inf\{s : \xi(s) \in \partial\Omega\}.$$

Since we have

$$t - t_1 \geq \tilde{d}(y, z) \quad \text{and} \quad t \leq \tilde{d}(x, z) + \tilde{d}(y, z) + \varepsilon,$$

it follows that $t_1 \leq \tilde{d}(x, z) + \varepsilon$ and therefore

$$\tilde{d}(x, z) + \tilde{d}(z, y) \leq t \leq \tilde{d}(x, y) + \varepsilon. \tag{3.13}$$

Now in view of (2.5), we have

$$u(x) \leq \zeta(z) + \int_0^{t_1} f(\xi(s))\, ds \leq \zeta(z) + M(\tilde{d}(x, z) + \varepsilon). \tag{3.14}$$

Thanks to the L-Lipschitz continuity of ζ, we have

$$\zeta(z) \leq \zeta(y) + L\tilde{d}(y, z). \tag{3.15}$$

Applying (3.13) and (3.15) in (3.14), we thus get

$$u(x) \leq \zeta(y) + L\tilde{d}(y, z) + M\tilde{d}(x, z) + M\varepsilon \leq \zeta(y) + \max\{L, M\}\tilde{d}(x, y) + 2\max\{L, M\}\varepsilon. \tag{3.16}$$

Since the above holds for all $\varepsilon > 0$, we have (3.10) immediately.

In order to show (3.12), we need the stronger condition (3.11), which means that ζ is m-Lipschitz on $\partial\Omega$ with respect to $\tilde{d}$. By (2.5), for any $\varepsilon > 0$, there exists $y_\varepsilon \in \partial\Omega$ and a curve $\xi_\varepsilon \in \mathcal{A}_x(\mathbb{R}, \overline{\Omega})$ such that with $t_\varepsilon > 0$ chosen so that $\xi_\varepsilon(t_\varepsilon) = y_\varepsilon$, we have

$$u(x) \geq \zeta(y_\varepsilon) + \int_0^{t_\varepsilon} f(\xi(s))\, ds - \varepsilon \geq \zeta(y_\varepsilon) + m\tilde{d}(x, y_\varepsilon) - \varepsilon.$$

Using (3.11), we have

$$u(x) \geq \zeta(y) - m\tilde{d}(y, y_\varepsilon) + m\tilde{d}(x, y_\varepsilon) - \varepsilon \geq \zeta(y) - m\tilde{d}(x, y) - \varepsilon.$$

Letting $\varepsilon \to 0$, we obtain

$$u(x) - \zeta(y) \geq -m\tilde{d}(x, y) \geq -M\tilde{d}(x, y),$$

which, combined with (3.10) with $L = m$, completes the proof. □

The condition (3.11) gives a quite restrictive constraint on the oscillation of the boundary data ζ. A weaker condition than (3.11) is that

$$\begin{cases} \zeta(x) - \zeta(y) \leq \displaystyle\int_0^t f(\xi(s))|\xi'|(s)\, ds \\ \text{for every } x, y \in \partial\Omega \text{ and every rectifiable curve } \xi : [0, t] \to \overline{\Omega} \text{ with } \xi(0) = x \text{ and } \xi(t) = y. \end{cases} \tag{3.17}$$

This condition is employed in [22] to guarantee the existence of continuous solutions to the Dirichlet problem. We can use (3.17) instead of (3.11) to obtain a continuity result weaker than (3.12) provided that Ω enjoys a better regularity like the so-called quasiconvexity.

Proposition 3.10 (Boundary consistency under domain quasiconvexity)**.** *Let $(\mathbf{X}, d)$ be a complete rectifiably connected metric space and $\tilde{d}$ be given by* (1.5)*. We suppose that* (1.6) *holds. Let $\Omega \subsetneq \mathbf{X}$ be an open set. Assume that $f \in C(\overline{\Omega})$ is bounded and $f \geq 0$ in $\overline{\Omega}$. Assume in addition that $\overline{\Omega}$ is σ_Ω–convex in $(\mathbf{X}, \tilde{d})$ with respect to a modulus of continuity σ_Ω, i.e., for any $x, y \in \overline{\Omega}$, there exist a rectifiable curve ξ in $\overline{\Omega}$ joining x and y and satisfying $\ell(\xi) \leq \sigma_\Omega(\tilde{d}(x, y))$. Let ζ satisfy* (3.17) *and u be given by* (2.5)*. Then* (1.7) *holds with $\sigma(t) = 2\sigma_\Omega(t)\sup_{\overline{\Omega}} f$ for $t \geq 0$.*

Proof. Let us still take $M = \sup_{\overline{\Omega}} f$ for simplicity of notation. Fix $x \in \Omega$ and $y \in \partial\Omega$. Using the same argument as in the proof of Proposition 3.9, we can easily prove that

$$u(x) - \zeta(y) \le 2M\sigma_\Omega(\tilde{d}(x, y)).$$

Indeed, since the quasiconvexity of Ω and (3.17) yield

$$|\zeta(y_1) - \zeta(y_2)| \le M\sigma_\Omega(\tilde{d}(y_1, y_2))$$

for any $y_1, y_2 \in \partial\Omega$, we only need to respectively substitute the terms $m\tilde{d}(y, z)$ and $m\tilde{d}(x, y)$ in (3.15) and (3.16) with $M\sigma_\Omega(\tilde{d}(y, z))$ and $M\sigma_\Omega(\tilde{d}(x, y))$.

Let us now show that

$$u(x) - \zeta(y) \ge -\sigma(\tilde{d}(x, y)). \tag{3.18}$$

In fact, for any $\varepsilon > 0$, we can use (2.5) again to find $y_\varepsilon \in \partial\Omega$ and an arc-length parametrized curve $\xi_1 \in \mathcal{A}_x([0, t_1))$ with $t_1 > 0$ such that $\tilde{d}(x, y_\varepsilon) \ge t_1 - \varepsilon$ and

$$u(x) \ge \zeta(y_\varepsilon) + \int_0^{t_1} f(\xi_1(s))\, ds - \varepsilon. \tag{3.19}$$

Note that there exists another arc-length parametrized curve $\xi_2 \in \mathcal{A}_x([0, t_2))$ such that $\xi_2(0) = x$, $\xi_2(t_2) = y$ and $\sigma_\Omega(\tilde{d}(x, y)) \ge t_2$. We thus can join ξ_1 and ξ_2 by taking

$$\xi(s) = \begin{cases} \xi_1(t_1 - s) & \text{if } s \in [0, t_1], \\ \xi_2(s - t_1) & \text{if } s \in (t_1, t_1 + t_2]. \end{cases}$$

Adopting (3.11), we have

$$\zeta(y) \le \zeta(y_\varepsilon) + \int_0^{t_1+t_2} f(\xi(s))\, ds,$$

which, combined with (3.19), implies that

$$\begin{aligned} u(x) &\ge \zeta(y) + \int_0^{t_1} f(\xi_1(s))\, ds - \int_0^{t_1+t_2} f(\xi(s))\, ds - \varepsilon = \zeta(y) - \int_0^{t_2} f(\xi_2(s))\, ds - \varepsilon \\ &\ge \zeta(y) - Mt_2 - \varepsilon \ge \zeta(y) - M\sigma_\Omega(\tilde{d}(x, y)) - \varepsilon. \end{aligned}$$

We conclude the proof of (3.18) by letting $\varepsilon \to 0$. □

3.4. **Slope-based solutions in general metric spaces.** Motivated by our above results, we generalize the definition of viscosity solutions proposed in [21]. In particular, by utilizing the induced intrinsic metric $\tilde{d}$ given by (1.5), we can now study the general equation (1.1) in a general metric space $(\mathbf{X}, d)$ that is not necessarily a length space but a complete rectifiably connected metric space satisfying (1.6).

We first extend the notion of pointwise local slope for any locally Lipschitz function u by continuing to use the notation $|\nabla u|$:

$$|\nabla u|(x) := \limsup_{y \to x} \frac{|u(y) - u(x)|}{\tilde{d}(x, y)}, \quad \text{for any } x \in \mathbf{X}. \tag{3.20}$$

Such an idea has already been formally stated in [22, Equation (2.3)]. If (X, d) is a length space, then $\tilde{d} = d$ and (3.20) agrees with (2.6).

Analogously, if $(\mathbf{X}, d)$ is a complete rectifiably connected metric space satisfying (1.6), we can define upper and lower slopes by taking, for $u \in \mathrm{Lip}_{loc}(\mathbf{X})$,

$$|\nabla^{\pm} u|(x) := \limsup_{y \to x} \frac{[u(y) - u(x)]_{\pm}}{\tilde{d}(x, y)}, \quad \text{for any } x \in \mathbf{X}. \tag{3.21}$$

These are again consistent with (2.8) if (X, d) is a length space.

By using the definition in (2.7), we can provide the test classes $\overline{\mathcal{C}}(\Omega)$ and $\underline{\mathcal{C}}(\Omega)$ but on an open set Ω of a more general metric space.

As a result, $u \in \overline{\mathcal{C}}(\Omega)$ (resp., $u \in \underline{\mathcal{C}}(\Omega)$) if and only if for any $x \in \Omega$,

$$\limsup_{y\to x} \frac{u(y)-u(x)}{\tilde{d}(x,y)} = |\nabla u|(x) \quad \left(\text{resp., } \limsup_{y\to x} \frac{u(x)-u(y)}{\tilde{d}(x,y)} = |\nabla u|(x)\right).$$

Now Definition 2.3 can be used to define viscosity solutions of (1.1) in a complete rectifiably connected metric space as long as we replace d by $\tilde{d}$.

We conclude this section by remarking that our approach above is based on a pointwise version of the notion of upper gradients, which, as an important substitute of the Euclidean gradients, has recently attracted a great deal of attention in the study of Sobolev spaces in metric measure spaces; see for instance [24, 25] for introduction on this topic.

4. Monge Solutions and Local Equivalence

In this section we aim to show the equivalence of c- and s-solutions of the eikonal equation without relying on the boundary condition. Our discussion involves a third notion of solutions to Hamilton-Jacobi equations in general metric spaces, which generalizes the so-called Monge solution of the eikonal equation in the Euclidean space studied in [35, 7] etc.

Thanks to our remarks in Section 3.4, it is sufficient to set up the problem in a complete length space $(\mathbf{X}, d)$. Our results in this section can be applied to more general rectifiably connected spaces by taking the induced intrinsic metric $\tilde{d}$ as in (1.5). Throughout this section, we shall always focus our attention on the complete length space $(\mathbf{X}, d)$.

4.1. Definition and uniqueness of Monge solutions.

Let us begin with the definition of Monge solutions for the general Hamilton-Jacobi equation in a complete length space.

Definition 4.1 (Definition of Monge solutions)**.** A function $u \in \mathrm{Lip}_{loc}(\Omega)$ is called a Monge subsolution (resp., Monge supersolution) of (1.1) if, at any $x \in \Omega$,

$$H\left(x, u(x), |\nabla^- u|(x)\right) \leq 0 \quad \left(\text{resp., } H\left(x, u(x), |\nabla^- u|(x)\right) \geq 0\right).$$

A function $u \in \mathrm{Lip}_{loc}(\Omega)$ is said to be a Monge solution if u is both a Monge subsolution and a Monge supersolution, i.e., u satisfies

$$H\left(x, u(x), |\nabla^- u|(x)\right) = 0 \tag{4.1}$$

at any $x \in \Omega$.

In the case of (1.3), the definition of Monge subsolutions (resp., supersolutions) reduces to

$$|\nabla^- u|(x) \leq f(x) \quad (\text{resp., } |\nabla^- u|(x) \geq f(x)) \tag{4.2}$$

for all $x \in \Omega$. Then u is a Monge solution of (1.3) if

$$|\nabla^- u| = f \quad \text{in } \Omega. \tag{4.3}$$

The notion of Monge solutions of (1.3) in the Euclidean space is studied in [35], where the right hand side f is allowed to be more generally lower semicontinuous in $\overline{\Omega}$. Such a notion, still in the Euclidean space, was later generalized in [7] to handle general Hamilton-Jacobi equations with discontinuities. The definitions of Monge solutions in [35, 7] require the optical length function, which can be regarded as a Lagrangian structure for H. In contrast, our definition of Monge solutions does not rely on optical length functions. We only consider continuous H in this work and the discontinuous case will be discussed in our forthcoming paper [12].

One advantage of using the Monge solutions is that its uniqueness can be easily obtained. In what follows, we give a comparison principle for Monge solutions of the eikonal equation.

Theorem 4.2 (Comparison principle for Monge solutions of eikonal equation)**.** *Let* $(\mathbf{X}, d)$ *be a complete length space and* $\Omega \subsetneq \mathbf{X}$ *be a bounded open set in* $(\mathbf{X}, d)$. *Assume that* $f \in C(\Omega)$ *is bounded and satisfies* $\inf_\Omega f > 0$. *Let* $u \in C(\overline{\Omega}) \cap \mathrm{Lip}_{loc}(\Omega)$ *be a bounded Monge subsolution and* $v \in C(\overline{\Omega}) \cap \mathrm{Lip}_{loc}(\Omega)$ *be a bounded Monge supersolution of* (1.3). *If*

$$\lim_{\delta \to 0} \sup \left\{ u(x) - v(x) : x \in \overline{\Omega},\ d(x, \partial\Omega) \leq \delta \right\} \leq 0, \tag{4.4}$$

then $u \leq v$ *in* $\overline{\Omega}$. *Here* $d(x, \partial\Omega)$ *is given by* $\inf_{y \in \partial\Omega} d(x, y)$.

Proof. Since u and v are bounded, we may assume that $u, v \geq 0$ by adding a positive constant to them. It suffices to show that $\lambda u \leq v$ in Ω for all $\lambda \in (0, 1)$. Assume by contradiction that there exists $\lambda \in (0, 1)$ such that $\sup_\Omega(\lambda u - v) > 2\mu$ for some $\mu > 0$. By (4.4), we may take $\delta > 0$ small such that

$$\lambda u(x) - v(x) \leq u(x) - v(x) \leq \mu$$

for all $x \in \overline{\Omega} \setminus \Omega_\delta$, where we denote $\Omega_r = \{x : \Omega : d(x, \partial\Omega) > r\}$ for $r > 0$. We choose $\varepsilon \in (0, \delta/2)$ such that

$$\sup_\Omega (\lambda u - v) > 2\mu + \varepsilon^2$$

and

$$\varepsilon < (1 - \lambda) \inf_{\Omega_{\delta/2}} f. \tag{4.5}$$

We have such an $\varepsilon > 0$ because $\inf_\Omega f > 0$. Thus there exists $x_0 \in \Omega$ such that $\lambda u(x_0) - v(x_0) \geq \sup_\Omega(\lambda u - v) - \varepsilon^2 > 2\mu$ and therefore $x_0 \in \Omega_\delta$.

By Ekeland's variational principle (cf. [18, Theorem 1.1], [19, Theorem 1]), there exists $x_\varepsilon \in B_\varepsilon(x_0) \subset \Omega_{\delta/2}$ such that

$$\lambda u(x_\varepsilon) - v(x_\varepsilon) \geq \lambda u(x_0) - v(x_0)$$

and $x \mapsto \lambda u(x) - v(x) - \varepsilon d(x_\varepsilon, x)$ attains a local maximum in Ω at $x = x_\varepsilon$. It follows that

$$v(x_\varepsilon) - v(x) \leq \lambda u(x_\varepsilon) - \lambda u(x) + \varepsilon d(x_\varepsilon, x) \tag{4.6}$$

for all $x \in B_r(x_\varepsilon)$ when $r > 0$ is small enough. Since v is a Monge supersolution of (1.3) and hence satisfies (4.2), there exists a sequence $\{y_n\} \subset \Omega$ such that

$$\lim_{y_n \to x_\varepsilon} \frac{[v(y_n) - v(x_\varepsilon)]_-}{d(x_\varepsilon, y_n)} \geq f(x_\varepsilon) > 0.$$

Note that here it is crucial to have $f(x_\varepsilon) > 0$ so that for large integers n we have $v(y_n) < v(x_\varepsilon)$. Hence, by (4.6) we have

$$\begin{aligned} f(x_\varepsilon) &\le \lim_{y_n \to x_\varepsilon} \frac{\lambda[u(y_n) - u(x_\varepsilon)]_-}{d(y_n, x_\varepsilon)} + \varepsilon \\ &\le \limsup_{x \to x_\varepsilon} \frac{\lambda[u(x) - u(x_\varepsilon)]_-}{d(x, x_\varepsilon)} + \varepsilon \\ &= \lambda|\nabla^- u|(x_\varepsilon) + \varepsilon \end{aligned}$$

Using the fact that u is a Monge subsolution, we get

$$f(x_\varepsilon) \le \lambda f(x_\varepsilon) + \varepsilon,$$

which contradicts the choice of $\varepsilon > 0$ as in (4.5). Our proof is thus complete. □

In the above theorem we cannot replace $|\nabla^- u|$ with $|\nabla u|$. A simple counterexample with $|\nabla u|$ is as follows: in $[-1, 1] \subset \mathbb{R}$, both $1 - |x|$ and $|x| - 1$ would be two different solutions of (1.3) with $f \equiv 1$ that satisfy the boundary condition $u(\pm 1) = 0$, but only the former is the unique Monge solution.

Remark 4.3. It can be easily seen that the continuity assumption on f is not utilized at all in the proof above. Hence the comparison principle in Theorem 4.2 still holds even if we drop the continuity assumption for f. In [12] we study in detail Monge solutions for discontinuous Hamiltonians in general metrics measure spaces.

Remark 4.4. An analogous comparison principle can be established for locally Lipschitz Monge solutions of (1.1) provided that $p \mapsto H(x, \rho, p)$ is continuous in $[0, \infty)$ uniformly for all $(x, \rho) \in \Omega \times \mathbb{R}$ and $\rho \mapsto H(x, \rho, p)$ is strictly increasing in the sense that there exists $\mu > 0$ such that

$$\rho \mapsto H(x, \rho, p) - \mu\rho$$

is nondecreasing for all $x \in \Omega$ and $p \ge 0$. The proof is similar to that of Theorem 4.2.

Concerning the existence of Monge solutions, Theorem 1.2 shows that any c-solution of (1.3) is a Monge solution. In particular, the formula (2.5) provides a unique c- and Monge solution of (1.3) and (1.4) if (3.11) holds.

4.2. **Local equivalence of solutions of eikonal equations.**

We study the local relation between Monge solutions, c-solutions and s-solutions. Let us first discuss the subsolution properties.

Proposition 4.5 (Relation between c- and Monge subsolutions)**.** *Let* $(\mathbf{X}, d)$ *be a complete length space and* Ω *be an open set in* $\mathbf{X}$*. Assume that* $f \in C(\Omega)$ *with* $f \ge 0$*. Let* $u \in C(\Omega)$*. Then* u *is a c-subsolution of* (1.3) *if and only if it is a Monge subsolution of* (1.3)*.*

Proof. We begin with a proof of the implication "⇒". Since the local Lipschitz continuity of c-subsolutions u is provided in Lemma 3.3, it suffices to verify that $|\nabla^- u|(x_0) \le f(x_0)$ for every $x_0 \in \Omega$. To this end, we fix $x_0 \in \Omega$. Since u is a c-subsolution of (1.3), using (3.2) with $s = 0$ and $\xi(0) = x_0$ we have

$$u(x_0) - u(\xi(t)) \le \int_0^t f(\xi(s))\, ds$$

for any $\xi \in \mathcal{A}_{x_0}(\mathbb{R}, B_r(x_0))$ and $0 \leq t \leq T^+_{B_r(x_0)}[\xi]$. Therefore

$$\frac{u(x_0) - u(x)}{d(x, x_0)} \leq \frac{1}{d(x, x_0)} \int_0^t f(\xi(s))\, ds \leq \frac{1}{d(x, x_0)} \ell(\xi) \sup_{B_r(x_0)} f$$

for any $x \in B_r(x_0)$ and any curve $\xi \subset B_r(x_0)$ joining x_0 and x. By the continuity of f and by the fact that $\mathbf{X}$ is a length space, taking the infimum over all ξ and then sending $r \to 0$ we obtain

$$|\nabla^- u|(x_0) \leq f(x_0)$$

as desired.

We next prove the reverse implication "$\Leftarrow$". We again fix $x_0 \in \Omega$ arbitrarily. We take an arbitrary curve $\xi \in \mathcal{A}_{x_0}(\mathbb{R}, \Omega)$; in particular $\xi(0) = x_0$. Suppose that there is a function $\phi \in C^1(\mathbb{R})$ such that $t \mapsto u(\xi(t)) - \phi(t)$ attains a local maximum at $t = 0$. Then there is some $t_0 > 0$ such that we have

$$\phi(t) - \phi(0) \geq u(\xi(t)) - u(\xi(0))$$

when $-t_0 < t < t_0$. If $\phi'(0) = 0$, then we obtain immediately the desired inequality (2.2). If $\phi'(0) \neq 0$, then without loss we may assume that $\phi'(0) < 0$, in which case $\phi(t) - \phi(0) < 0$ for all $t \in (0, t_1)$ for sufficiently small $t_1 \in (0, t_0)$. (If $\phi'(0) > 0$, then we can consider $t \in (-t_1, 0)$ instead below.) It follows that

$$|\phi'(0)| \leq \limsup_{t \to 0+} \frac{u(\xi(0)) - u(\xi(t))}{t} \leq |\nabla^- u|(x_0).$$

Since u is a Monge subsolution, we have $|\nabla^- u|(x_0) \leq f(x_0)$ and thus deduce (2.2) again. □

Proposition 4.6 (Relation between s- and Monge subsolutions)**.** *Let $(\mathbf{X}, d)$ be a complete length space and Ω be an open set in $\mathbf{X}$. Assume that f is locally uniformly continuous and $f \geq 0$ in Ω. Let $u \in C(\Omega)$. Then the following results hold.*

(i) *If u is a Monge subsolution of* (1.3)*, then it is an s-subsolution of* (1.3)*.*
(ii) *If u is a locally uniformly continuous s-subsolution of* (1.3)*, then it is a Monge subsolution of* (1.3)*.*

Proof. (i) It is an immediate consequence of Proposition 3.4 and Proposition 4.5.

(ii) Take $\delta > 0$ arbitrarily and $f_\delta = f + \delta$ in Ω. Fix $x_0 \in \Omega$ and $r > 0$ small such that $B_{4r}(x_0) \subset \Omega$ and u, f are both bounded on $B_{4r}(x_0)$. For $s \in [0, 4r)$ we set

$$M_s := \sup_{B_s(x_0)} f_\delta,$$

and choose $M > 0$ such that

$$M > \max\left\{M_{2r}, \frac{\sup_{B_{3r}(x_0)} u - u(x_0)}{r}\right\}.$$

Define a continuous function $g_r : [0, 3r) \to [0, \infty)$ by

$$g_r(t) := M_r t + M(t - r)_+$$

for $t \in [0, 3r)$. Due to the choice of M above, the function defined by

$$v_r(x) = u(x_0) + g_r(d(x_0, x))$$

for $x \in B_{3r}(x_0)$, satisfies $v_r \geq u$ on $\partial B_{2r}(x_0)$. Moreover, for any $x \in B_{2r}(x_0)$ with $x \neq x_0$ and any $\varepsilon > 0$, we can find an arc-length parametrized curve ξ such that $\xi(0) = x$, $\xi(t_\varepsilon) = x_0$ and $t_\varepsilon - \varepsilon \leq d(x, x_0) \leq t_\varepsilon$. For any $t \in [0, t_\varepsilon]$, we have

$$d(x_0, \xi(t)) \geq d(x_0, x) - d(x, \xi(t)) \geq (t_\varepsilon - \varepsilon) - (t_\varepsilon - t) = t - \varepsilon.$$

Therefore

$$t - \varepsilon \leq d(x_0, \xi(t)) \leq \ell(\xi[0,t]) \leq d(x, x_0) + \varepsilon - d(x, \xi(t)).$$

Taking $x_\varepsilon = \xi(\sqrt{\varepsilon})$, we have

$$\frac{v_r(x) - v_r(x_\varepsilon)}{d(x, x_\varepsilon)} \geq \frac{g_r(d(x_0,x)) - g_r(d(x_0,x_\varepsilon))}{d(x,x_0) - d(x_0,x_\varepsilon) + \varepsilon} \geq \frac{1}{1 + 2\sqrt{\varepsilon}} \frac{g_r(d(x_0,x)) - g_r(d(x_0,x_\varepsilon))}{d(x,x_0) - d(x_0,x_\varepsilon)}$$

when $\varepsilon > 0$ is sufficiently small. Hence if $0 < d(x, x_0) \leq r$, then for sufficiently small $\varepsilon > 0$ we have $g_r(d(x_\varepsilon, x_0)) \geq M_r d(x_\varepsilon, x_0)$, and then by the choice of M_r we have

$$\limsup_{x_\varepsilon \to x} \frac{g_r(d(x_0,x)) - g_r(d(x_0,x_\varepsilon))}{d(x,x_0) - d(x_0,x_\varepsilon)} = M_r \geq f_\delta(x).$$

If $r < d(x, x_0) < 2r$, then for sufficently small ε we have $d(x_\varepsilon, x_0) > r$ as well, and so we get

$$\frac{g_r(d(x_0,x)) - g_r(d(x_0,x_\varepsilon))}{d(x,x_0) - d(x_0,x_\varepsilon)} = \frac{(M_r + M)d(x_0,x) - (M_r + M)d(x_0,x_\varepsilon)}{d(x,x_0) - d(x_0,x_\varepsilon)}.$$

Therefore as $d(x_0, x) < 2r$, we have

$$\limsup_{x_\varepsilon \to x} \frac{g_r(d(x_0,x)) - g_r(d(x_0,x_\varepsilon))}{d(x,x_0) - d(x_0,x_\varepsilon)} \geq M > M_{2r} \geq f_\delta(x).$$

In either case, we see that $|\nabla^- v_r|(x) \geq f_\delta(x)$; in other words, v_r is a Monge supersolution of

$$|\nabla u| = f_\delta \quad \text{in } B_{2r}(x_0) \setminus \{x_0\}.$$

In view of Proposition 4.7(i), we see that v_r is an s-supersolution of the same equation (keep in mind also that by its construction, v_r is M-Lipschitz); in particular, we have

$$v_r(x) - u(y) \geq v_r(x) - v_r(y) \geq -M d(x, y) \tag{4.7}$$

for all $x \in B_{2r}(x_0)$ and $y \in \partial B_{2r}(x_0) \cup \{x_0\}$.

On the other hand, u is an s-subsolution and is uniformly continuous in $B_{2r}(x_0)$ with some modulus σ_0. We have

$$u(x) - u(y) \leq \sigma_0(d(x, y))$$

for all $x \in B_{2r}(x_0)$ and $y \in \partial B_{2r}(x_0) \cup \{x_0\}$. Combining this with (4.7), we have shown that the condition (1.8) holds with $\Omega = B_{2r}(x_0) \setminus \{x_0\}$, $v = v_r$, $\zeta = u$ and $\sigma(s) = \max\{Ms, \sigma_0(s)\}$ for $s \geq 0$. Since $f_\delta = f + \delta$ in Ω, the function u must also be an s-subsolution for the eikonal equation related to the function f_δ. We thus can use the comparison result [21, Theorem 5.3] to get $u \leq v_r$ in $B_{2r}(x_0) \setminus \{x_0\}$. Letting $\delta \to 0$, we are led to

$$u(x) \leq u(x_0) + d(x, x_0) \sup_{B_r(x_0)} f \quad \text{for all } x \in B_r(x_0).$$

One can use the same argument to show that for all $x, y \in B_{r/4}(x_0)$ (and therefore $d(x, y) \leq r/2$),

$$u(y) \leq u(x) + d(x, y) \sup_{B_{r/2}(x)} f \leq u(x) + d(x, y) \sup_{B_r(x_0)} f,$$

which yields (recalling that we chose $r > 0$ small enough so that f is bounded on $B_{4r}(x_0)$)

$$|u(x) - u(y)| \leq d(x, y) \sup_{B_r(x_0)} f.$$

This immediately implies that

$$|\nabla^- u|(x_0) \le |\nabla u|(x_0) \le f(x_0). \tag{4.8}$$

Hence, we can conclude that u is a Monge subsolution of (1.3), since x_0 is arbitrarily taken. □

In the proof of (ii) above, the local uniform continuity of u and f (especially near x_0) enables us to adopt the comparison principle. The uniform continuity can be removed if the space $(\mathbf{X}, d)$ has some compactness a priori.

We next turn to the relation between supersolutions.

Proposition 4.7 (Relation between supersolutions)**.** *Let $(\mathbf{X}, d)$ be a complete length space and Ω be an open set in $\mathbf{X}$. Assume that f is locally uniformly continuous and $f \ge 0$ in Ω. Let $u \in \mathrm{Lip}_{loc}(\Omega)$. Then*

(i) *u is a Monge supersolution of* (1.3) *if and only if u is an s-supersolution of* (1.3)*.*
(ii) *Assume in addition that $f > 0$ on Ω. If u is a local c-supersolution of* (1.3)*, then u is a Monge supersolution of* (1.3)*.*

Proof. (i) Let us first show the equivalence between a Monge supersolution and an s-supersolution. We begin with the implication "⇒". Let u be a Monge supersolution. Suppose that there exist $\psi_1 \in \overline{\mathcal{C}}(\Omega)$ and $\psi_2 \in \mathrm{Lip}_{loc}(\Omega)$ such that $u - \psi_1 - \psi_2$ attains a local minimum at x_0. If $f(x_0) = 0$, then the desired inequality

$$|\nabla \psi_1|(x_0) \ge -|\nabla \psi_2|^*(x_0)$$

is trivial. It thus suffices to consider the case $f(x_0) > 0$. By the definition of Monge supersolutions, for any $x_0 \in \Omega$,

$$|\nabla^- u|(x_0) \ge f(x_0) > 0.$$

Then for any $\delta > 0$, for each $\varepsilon > 0$ we can find $x_\varepsilon \in \Omega$ with $x_\varepsilon \to x_0$ as $\varepsilon \to 0$ such that

$$u(x_\varepsilon) - u(x_0) \le (-f(x_0) + \delta) d(x_0, x_\varepsilon). \tag{4.9}$$

It follows from (4.9) and the maximality of $u - \psi_1 - \psi_2$ at x_0 that

$$\psi_1(x_\varepsilon) - \psi_1(x_0) + \psi_2(x_\varepsilon) - \psi_2(x_0) \le (-f(x_0) + \delta) d(x_0, x_\varepsilon),$$

which implies that

$$\frac{|\psi_1(x_\varepsilon) - \psi_1(x_0)|}{d(x_0, x_\varepsilon)} \ge \frac{\psi_1(x_0) - \psi_1(x_\varepsilon)}{d(x_0, x_\varepsilon)} \ge f(x_0) - \delta - \frac{|\psi_2(x_\varepsilon) - \psi_2(x_0)|}{d(x_0, x_\varepsilon)}.$$

Letting $\varepsilon \to 0$ and then $\delta \to 0$, we obtain

$$|\nabla \psi_1|(x_0) \ge f(x_0) - |\nabla \psi_2|^*(x_0).$$

It follows that u is an s-supersolution.

The proof for the reverse implication "⇐" is given next. So we assume that u is an s-supersolution. Suppose that u is not a Monge supersolution. Then there exists $x_0 \in \Omega$ such that

$$\limsup_{x \to x_0} \frac{[u(x_0) - u(x)]_+}{d(x, x_0)} < f(x_0).$$

A contradiction is immediately obtained if $f(x_0) = 0$. We thus only consider $f(x_0) > 0$ below. Then there exists $\delta > 0$ such that for all $x \in B_\delta(x_0)$,

$$\frac{u(x_0) - u(x)}{d(x, x_0)} - f(x_0) \le -2\delta. \tag{4.10}$$

By the continuity of f, for any $0 < \varepsilon < \min\{\delta, f(x_0)/2\}$, we can choose $0 < r < \delta$ such that

$$|f(x) - f(x_0)| \leq \varepsilon \quad \text{for all } x \in B_r(x_0). \tag{4.11}$$

Observe that $f(x) \geq f(x_0)/2 > 0$ whenever $x \in B_r(x_0)$. Let

$$v(x) := u(x_0) - (f(x_0) - \varepsilon)d(x, x_0) + \delta r.$$

Then in view of (4.10), we have $v(x) \leq u(x)$ for all $x \in \partial B_r(x_0)$. Moreover, we claim that v is an s-subsolution of

$$|\nabla v| = f \quad \text{in } B_r(x_0).$$

Indeed, for any $x \in B_r(x_0)$, if $v - \psi_1 - \psi_2$ achieves a maximum at x, where $\psi_1 \in \underline{\mathcal{C}}(\Omega)$ and $\psi_2 \in \mathrm{Lip}_{loc}(\Omega)$, then

$$\psi_1(x) - \psi_1(y) \leq v(x) - v(y) + \psi_2(y) - \psi_2(x).$$

It follows that

$$\begin{aligned}|\nabla \psi_1|(x) = \limsup_{y\to x} \frac{\psi_1(x) - \psi_1(y)}{d(x,y)} &\leq \limsup_{y\to x} \frac{v(x) - v(y)}{d(x,y)} + \limsup_{y\to x} \frac{\psi_2(y) - \psi_2(x)}{d(x,y)} \\ &\leq f(x_0) - \varepsilon + |\nabla \psi_2|^*(x) \\ &\leq f(x) + |\nabla \psi_2|^*(x).\end{aligned}$$

The claim has been proved. Applying the comparison principle for s-solutions, we have $v \leq u$ in $B_r(x_0)$. This contradicts the fact that

$$v(x_0) = u(x_0) + \delta r > u(x_0).$$

Our proof for the equivalence of supersolutions is now complete.

(ii) It follows immediately from (i) and Remark 3.8. Note that Remark 3.8 requires $\inf_{B_r(x)} f > 0$ for any $x \in \Omega$ and $r > 0$ small, which is implied by the continuity and positivity of f in Ω. □

We are not able to show that any Monge supersolution of (1.3) is a local c-supersolution. However, if u is a Monge solution, then we can show that u must be a local c-solution.

Proposition 4.8 (Relation between Monge and local c-solutions)**.** *Let* $(\mathbf{X}, d)$ *be a complete length space and* Ω *be an open set in* $\mathbf{X}$*. Assume that* f *is locally uniformly continuous and* $f > 0$ *in* Ω*. If* u *is a Monge solution of* (1.3)*, then* u *is a local c-solution of* (1.3)*.*

Proof. Suppose that u is a Monge solution of (1.3) in Ω. By Proposition 4.5, we know that u must be a c-subsolution. It suffices to show that u is a local c-supersolution of (1.3).

For any $x_0 \in \Omega$, take $r > 0$ small such that u is Lipschitz in $\overline{B_r(x_0)}$, that is, there exists $L > 0$ such that

$$|u(x) - u(y)| \leq L d(x, y) \quad \text{for any } x, y \in \overline{B_r(x_0)}. \tag{4.12}$$

Letting $\zeta(y) = u(y)$ for all $y \in \partial B_r(x_0)$, by (2.5) with $\Omega = B_r(x_0)$ we have the unique c-solution U in $B_r(x_0)$ given by

$$U(x) := \inf\left\{\int_0^{t_r^+} f(\xi(s))\, ds + u\left(\xi(t_r^+)\right) \,:\, \xi \in \mathcal{A}_x(\mathbb{R}, \mathbf{X}) \text{ with } 0 < T^+_{B_r(x_0)}[\xi] < \infty\right\}. \tag{4.13}$$

It follows from Proposition 4.5 and Proposition 4.7(ii) that U is a Monge solution of the eikonal equation in $B_r(x_0)$. Note also that, by Proposition 3.9(1),

$$U(x) - u(y) \leq d(x,y) \max\left\{L, \sup_{B_r(x_0)} f\right\} \quad \text{for any } x \in B_r(x_0) \text{ and } y \in \partial B_r(x_0).$$

We then can adopt the comparison principle, Theorem 4.2, to get $U \leq u$ in $B_r(x_0)$. In view of (4.13), it follows that for any $x \in B_r(x_0)$ and any $\varepsilon > 0$ small, there exists a curve $\xi_\varepsilon \in A_x(\mathbb{R}, \mathbf{X})$ such that

$$u(x) \geq U(x) \geq \int_0^{t_r^+} f(\xi_\varepsilon(s))\, ds + u\left(\xi_\varepsilon(t_r^+)\right) - \varepsilon, \tag{4.14}$$

where $t_r^+ = T^+_{B_r(x_0)}[\xi_\varepsilon]$ denotes the exit time of ξ_ε from $B_r(x_0)$. On the other hand, since u is a c-subsolution, we can use Proposition 3.2 to get, for any $0 \leq t \leq t_r^+$,

$$u(\xi_\varepsilon(t)) \leq u\left(\xi_\varepsilon(t_r^+)\right) + \int_t^{t_r^+} f(\xi_\varepsilon(s))\, ds. \tag{4.15}$$

Combining (4.14) and (4.15), we deduce that for any $0 \leq t \leq t_r^+$,

$$u(x) \geq u(\xi_\varepsilon(t)) + \int_0^t f(\xi_\varepsilon(s))\, ds - \varepsilon.$$

Setting

$$\xi(t) = \begin{cases} \xi_\varepsilon(t) & \text{if } t \geq 0, \\ \xi_\varepsilon(-t) & \text{if } t < 0, \end{cases} \quad \text{and} \quad w(t) = u(x) - \int_0^t f(\xi(s))\, ds$$

for $t \in \mathbb{R}$, we easily see that (ξ, w) satisfies the conditions for local c-supersolutions in Definition 2.2. Indeed, w is of class C^1 in $(-t_r^+, t_r^+) \setminus \{0\}$ with $w' = f \circ \xi$ in $(-t_r^+, t_r^+) \setminus \{0\}$. If there is $\phi \in C^1(-t_r^+, t_r^+)$ such that $w - \phi$ achieves a minimum at some $t_0 \in (-t_r^+, t_r^+)$, then $t_0 \neq 0$ since $f > 0$ in Ω. It then follows that $\phi'(t_0) = w'(t_0)$, which yields

$$|\phi'|(t_0) = |w'|(t_0) = f(\xi(t_0)).$$

Hence, u is a local c-supersolution and therefore a local c-solution. □

We now complete the proof of Theorem 1.2.

Proof of Theorem 1.2. The proof consists of the results in Propositions 4.5, 4.6, 4.7 and 4.8. In addition, combining (4.8) and the definition of Monge supersolutions, we have (1.9) if any of (a), (b) and (c) holds. □

Remark 4.9. It is worth pointing out that, due to Proposition 4.6 and Proposition 4.7(i), the equivalence between Monge solutions and locally uniformly continuous s-solutions still holds even if the assumption $f > 0$ is relaxed to $f \geq 0$ in Ω. See Theorem 4.11 a nd 4.12 below for a more general result focusing on the relation between Monge and s-solutions.

The local uniform continuity of f and u in Theorem 1.2 can be dropped if the space $(\mathbf{X}, d)$ is assumed to proper, that is, any closed bounded subset of $\mathbf{X}$ is compact.

Corollary 4.10 (Local equivalence in a proper space)**.** *Let $(\mathbf{X}, d)$ be a proper complete geodesic space and Ω be a bounded open set in $\mathbf{X}$. Assume that $f \in C(\Omega)$ and $f > 0$ in Ω. Let $u \in C(\Omega)$. Then the following statements are equivalent:*

(a) *u is a c-solution of* (1.3)*;*
(b) *u is a s-solution of* (1.3)*;*

(c) *u is a Monge solution of* (1.3).

In addition, if any of (a)–(c) holds, then $u \in \mathrm{Lip}_{loc}(\Omega)$ *and satisfies* (1.9).

4.3. **General Hamilton-Jacobi equations.** We next turn to a more general class of Hamilton-Jacobi equations. In this case, c-solutions are no longer defined. We can still show the equivalence between Monge solutions and s-solutions.

Theorem 4.11 (Equivalence of Monge and s-solutions of general equations)**.** *Let* $(\mathbf{X}, d)$ *be a complete length space and* $\Omega \subset \mathbf{X}$ *be an open set. Let* $H : \Omega \times \mathbb{R} \times [0, \infty) \to \mathbb{R}$ *be continuous and satisfy the following conditions:*

(1) $(x, \rho) \mapsto H(x, \rho, p)$ *is locally uniformly continuous, that is, for any* $x_0 \in \Omega$ *and* $\rho_0 \in \mathbb{R}$*, there exist* $\delta > 0$ *small and a modulus of continuity* ω *such that*
$$|H(x_1, \rho_1, p) - H(x_2, \rho_2, p)| \leq \omega\left(d(x_1, x_2) + |\rho_1 - \rho_2|\right)$$
for all $x_1, x_2 \in B_\delta(x_0)$*,* $\rho_1, \rho_2 \in [\rho_0 - \delta, \rho_0 + \delta]$ *and* $p \geq 0$*.*

(2) *For any* $x_0 \in \Omega$ *and* $\rho_0 \in \mathbb{R}$*, there exist* $\delta > 0$ *and* $\lambda_0 > 0$ *such that*
$$p \mapsto H(x, \rho, p) - \lambda_0 p$$
is increasing for every $(x, \rho) \in B_\delta(x_0) \times [\rho_0 - \delta, \rho_0 + \delta]$*.*

(3) $p \mapsto H(x, \rho, p)$ *is coercive in the sense that*
$$\inf_{(x,\rho)\in\Omega\times[-R,R]} H(x, \rho, p) \to \infty \quad \text{as } p \to \infty \text{ for any } R > 0. \tag{4.16}$$

Then u is a Monge solution of (1.1) *if and only if u is a locally uniformly continuous s-solution of* (1.1)*. In addition, such u is locally Lipschitz in* Ω*.*

Proof. Let u be either a Monge solution or a locally uniformly continuous s-solution. We first claim that
$$H(x, u(x), 0) \leq 0 \quad \text{for any } x \in \Omega. \tag{4.17}$$
Thanks to the condition (2), this is clearly true when u is a Monge solution. It thus suffices to show (4.17) for a locally uniformly continuous s-solution u. Fix $x_0 \in \Omega$ arbitrarily. Let
$$\psi_1(x) = \frac{1}{\varepsilon} d(x, x_0)^2$$
for $\varepsilon > 0$ small. Then, due to the local boundedness of u, there exist $\delta > 0$ and $y_\varepsilon \in B_\delta(x_0) \subset \Omega$ such that
$$(u - \psi_1)(y_\varepsilon) \geq \sup_{B_\delta(x_0)} (u - \psi_1) - \varepsilon^2$$
and $y_\varepsilon \to x_0$ as $\varepsilon \to 0$. By Ekeland's variational principle (cf. [18, Theorem 1.1], [19, Theorem 1]), there is a point $x_\varepsilon \in B_\varepsilon(y_\varepsilon)$ such that
$$(u - \psi_1)(x_\varepsilon) \geq (u - \psi_1)(y_\varepsilon)$$
and $u - \psi_1 - \psi_2$ attains a local maximum in $B_\delta(x_0)$ at $x_\varepsilon \in B_\varepsilon(y_\varepsilon)$, where
$$\psi_2(x) = \varepsilon d(x_\varepsilon, x).$$
It is clear that $x_\varepsilon \to x_0$ as $\varepsilon \to 0$. Since u is an s-subsolution, we have
$$\inf_{|\rho|\leq\varepsilon} H\left(x_\varepsilon, u(x_\varepsilon), \frac{2}{\varepsilon} d(x_\varepsilon, x_0) + \rho\right) \leq 0,$$
which, by the condition (2), yields
$$H\left(x_\varepsilon, u(x_\varepsilon), 0\right) \leq 0.$$

Letting $\varepsilon \to 0$, by the continuity of H, we deduce (4.17) at $x = x_0$. We have completed the proof of the claim.

By the coercivity condition (3) we can define a function $h : \Omega \to [0, \infty)$ to be

$$h(x) := \inf\{p \geq 0 : H(x, u(x), p) > 0\}, \tag{4.18}$$

and, thanks to the continuity of H and the condition (2), we see that for each $x \in \Omega$, $h(x) \geq 0$ is the unique value satisfying

$$H(x, u(x), h(x)) = 0. \tag{4.19}$$

We next claim that h is locally uniformly continuous in Ω. To see this, fix $x_0 \in \Omega$ and an arbitrarily small $\delta > 0$. We take $x, y \in B_{\delta_1}(x_0)$ with $\delta_1 \leq \delta$ sufficiently small such that $u(x), u(y) \in [u(x_0) - \delta, u(x_0) + \delta]$. Then by the condition (1) we have

$$H(x, u(x), p) - H(y, u(y), p) \leq \omega(d(x, y) + |u(x) - u(y)|) \tag{4.20}$$

for any $p \geq 0$. Denote $W(x, y) := \omega(d(x, y) + |u(x) - u(y)|)$ for simplicity. Since H satisfies the condition (2), we can use (4.20) to get, for any $x, y \in B_{\delta_1}(x_0)$,

$$H\left(x, u(x), h(y) + \frac{1}{\lambda_0} W(x, y)\right) \geq H(x, u(x), h(y)) + W(x, y) \geq H(y, u(y), h(y)) = 0,$$

which, by (4.18) and (4.19), yields

$$h(x) \leq h(y) + \frac{1}{\lambda_0} W(x, y).$$

We can analogously show that

$$h(x) \geq h(y) - \frac{1}{\lambda_0} W(x, y)$$

and therefore h is uniformly continuous in $B_{\delta_1}(x_0)$.

From (4.19) it is clear that u is a Monge solution of (1.1) if and only if $|\nabla^- u|(x) = h(x)$, that is, u is a Monge solution of (1.3) with $f = h$. Note however that the function h depends on u implicitly in general.

We now show that u is an s-solution of (1.1) if and only if u is an s-solution of (1.3) with $f = h$. To see this, suppose that there exist $x_0 \in \Omega$, $\psi_1 \in \underline{\mathcal{C}}(\Omega)$ and $\psi_2 \in \mathrm{Lip}_{loc}(\Omega)$ such that $u - \psi_1 - \psi_2$ attains a maximum in Ω at x_0. Then by the monotonicity of $p \to H(x, \rho, p)$, the viscosity inequality (2.9) holds at x_0 if and only if

$$H\left(x_0, u(x_0), (|\nabla\psi_1|(x_0) - |\nabla\psi_2|^*(x_0)) \vee 0\right) \leq 0,$$

which, due to (4.18), amounts to saying that

$$|\nabla\psi_1|(x_0) - |\nabla\psi_2|^*(x_0) \leq h(x_0),$$

that is, u is an s-subsolution of (1.3) with $f = h$. Analogous results for supersolutions can be similarly proved.

Noticing that Monge solutions and locally uniformly continuous s-solutions of (1.3) have been proved to be (locally) equivalent in Theorem 1.2, we immediately obtain the equivalence of both notions for (1.1) and local Lipschitz continuity. □

As in Corollary 4.10, if $(\mathbf{X}, d)$ is additionally assumed to be proper, then in Theorem 4.11 we can drop the assumption on the local uniform continuity of s-solutions. Moreover, when $(\mathbf{X}, d)$ is proper, in the proof above we only need to show the continuity of h as in (4.18), since continuity implies local uniform continuity. Thus the condition (1) can be removed and (2) can be weakened by merely assuming that for every $(x, \rho) \in \Omega \times \mathbb{R}$, the map $p \mapsto H(x, \rho, p)$ is strictly increasing in $(0, \infty)$. Below we state the result without proofs.

Theorem 4.12 (Equivalence of Monge and s-solutions in a proper space)**.** *Let $(\mathbf{X}, d)$ be a complete proper geodesic space and $\Omega \subset \mathbf{X}$ be an open set. Let $H : \Omega \times \mathbb{R} \times [0, \infty) \to \mathbb{R}$ be continuous and be coercive as in* (4.16)*. Assume that for every $(x, \rho) \in \Omega \times \mathbb{R}$, $p \mapsto H(x, \rho, p)$ is strictly increasing in $(0, \infty)$. Then u is a Monge solution of* (1.1) *if and only if u is an s-solution of* (1.1)*. In addition, such u is locally Lipschitz in Ω.*

The strict monotonicity of $p \to H(x, \rho, p)$ assumed in Theorem 4.11 and Theorem (4.12) enables us to apply an implicit function argument. Although it is not clear to us if one can weaken the requirement, the examples below show that the equivalence result fails to hold in general if H is not monotone in p.

Example 4.13. Let $\Omega = \mathbf{X} = \mathbb{R}$ with the standard Euclidean metric. Let

$$H(p) = 1 - |p - 2| + \max\{p - 3, 0\}^2, \quad p \geq 0.$$

One can easily verify that this Hamiltonian satisfies all assumptions in Theorem 4.11 except for the monotonicity.

It is not difficult to see that the function u given by $u(x) = -3|x|$ for $x \in \mathbb{R}$ is a Monge solution of

$$H(|\nabla u|) = 0 \quad \text{in } \mathbb{R}, \tag{4.21}$$

since $|\nabla^- u| = 3$ in $\mathbb{R}$. It is however not a conventional viscosity solution or an s-solution, though it is an s-supersolution.

Example 4.14. Let $\Omega = \mathbf{X} = \mathbb{R}$ again. Set

$$H(p) = 1 - |p| + \max\{p - 3, 0\}^2, \quad p \geq 0,$$

which again satisfies all assumptions in Theorem 4.11 but the monotonicity. This time we have

$$u(x) = |x|, \quad x \in \mathbb{R}$$

as a viscosity solution or s-solution of (4.21). But it is not a Monge solution, since $|\nabla^- u|(0) = 0$ and $H(0) \neq 0$.

4.4. **Further regularity.** Motivated by the observation (1.9) in Theorem 1.2, we consider a higher regularity than the Lipschitz continuity related to the Monge solutions of (1.1). One can slightly strengthen Definition 4.1 by further requiring that the solution u satisfy (1.10). We say that a solution u is regular in Ω if (1.10) holds. The reason why we regard this condition as regularity will be clarified below.

The property (1.10) is studied for time-dependent Hamilton-Jacobi equations on metric spaces [31, Proof of Theorem 2.5 and Remark 2.27]. The authors of [31] show that at any fixed time, such spatial regularity holds for the Hopf-Lax formula almost everywhere if $\mathbf{X}$ is a metric measure space that satisfies a doubling condition and a local Poincaré inequality, or everywhere if $\mathbf{X}$ is a finite dimensional space with Alexandrov curvature bounded from below. In our stationary setting, we have shown that any Monge solution of the eikonal equation (1.3) is regular in a complete length space without using *any* measure structure or imposing any assumptions on the space dimension and curvature, see Theorem 1.2. Now we consider the more general Hamiltonian H.

Proposition 4.15 (Regularity of Monge solutions)**.** *Suppose that the assumptions in Theorem 4.11 or Theorem 4.12 hold. If u is a Monge solution of* (1.1)*, then u is regular in Ω and $u \in \underline{\mathcal{C}}(\Omega)$.*

Proof. We have shown in the proof of Theorem 4.11 that u is locally a Monge solution of

$$|\nabla u| = h(x),$$

where $h \in C(\Omega)$ is given by (4.18). Also, we have $h \geq c$ in Ω due to (**??**). The regularity (1.10) is thus an immediate consequence of (1.9) in Theorem 1.2. We also see that $u \in \underline{\mathcal{C}}(\Omega)$. One can also apply the same proof to obtain the conclusion under the assumptions of Theorem 4.12. □

We emphasize that for (1.10) the strict monotonicity of $p \to H(x, \rho, p)$ cannot be relaxed, as indicated by the following simple example.

Example 4.16. Let

$$H(p) = \begin{cases} p & \text{when } 0 \leq p < 1, \\ 1 & \text{when } 1 \leq p < 2, \\ p - 1 & \text{when } p \geq 2. \end{cases}$$

In this case, it is easily seen that

$$u(x) = \begin{cases} x & \text{for } x \leq 0, \\ 2x & \text{for } x > 0 \end{cases}$$

is an s-solution (usual viscosity solution) of

$$H(|\nabla u|) = 1 \quad \text{in } \mathbf{X} = \mathbb{R},$$

but $|\nabla u|(0) = 2 \neq 1 = |\nabla^- u|(0)$.

We finally remark that (1.10) can be regarded as a type of regularity that is related to the semi-concavity in the Euclidean spaces. Let us below briefly recall several results on semi-concave functions in $\mathbb{R}^N$; we refer the reader to [11] for detailed introduction to the classical results and to [2, 3] for recent developments in more general sub-Riemannian contexts.

Recall (cf. [11, Definition 2.1.1]) that for any open set $\Omega \subset \mathbb{R}^N$, $u \in C(\Omega)$ is said to be a (generalized) semi-concave function in Ω if there exists a nondecreasing upper semicontinuous function $\omega : [0, \infty) \to [0, \infty)$ such that

$$\lim_{r \to 0+} \omega(r) = 0$$

and

$$u(x + \eta) + u(x - \eta) - 2u(x) \leq \omega(|\eta|)|\eta| \tag{4.22}$$

for any $x \in \Omega$ and any $\eta \in \mathbb{R}^N$ with $|\eta|$ sufficiently small. A more well-known notion of semi-concave functions is to ask $u \in C(\Omega)$ to satisfy (4.22) with $\omega(r) = cr$ for some $c > 0$; in this case, u is semi-concave in Ω if and only if $x \mapsto u(x) - c|x|^2/2$ is concave in Ω.

It is well known [11, Theorem 5.3.7] that viscosity solutions of (1.1) are locally semi-concave (in the generalized sense) provided that $H(x, \rho, p)$ is locally Lipschitz and strictly convex in p; one can apply this result to (1.3) by changing the form into

$$|\nabla u|^2 = f(x)^2 \quad \text{in } \Omega$$

in order to meet the requirement of strict convexity. Moreover, any such semi-concave function has nonempty superdifferentials; in other words, it can be touched everywhere from above by a C^1 function; see [11, Proposition 3.3.4]. The condition (1.10) is slightly weaker than this property in $\mathbb{R}^N$.

Proposition 4.17 (Regularity and upper testability in Euclidean spaces)**.** *Let $x_0 \in \mathbb{R}^N$ and assume that $u : \mathbb{R}^N \to \mathbb{R}$ is Lipschitz near x_0. If there exists a function ψ differentiable at x_0 such that $u - \psi$ attains a local maximum at x_0, then*

$$|\nabla u|(x_0) = |\nabla^- u|(x_0). \tag{4.23}$$

Proof. By assumptions, we have

$$u(x) - \psi(x) \leq u(x_0) - \psi(x_0)$$

for any $x \in \Omega$ near x_0. It follows that

$$\begin{aligned} \min\{u(x) - u(x_0), 0\} &\leq \min\{\psi(x) - \psi(x_0), 0\}, \\ \max\{u(x) - u(x_0), 0\} &\leq \max\{\psi(x) - \psi(x_0), 0\}. \end{aligned}$$

These relations imply that

$$\begin{aligned} |\nabla^- u|(x_0) &\geq |\nabla^- \psi|(x_0), \\ |\nabla^+ u|(x_0) &\leq |\nabla^+ \psi|(x_0). \end{aligned} \tag{4.24}$$

Noticing that ψ is differentiable at x_0, we have

$$|\nabla^+ \psi|(x_0) = |\nabla^- \psi|(x_0) = |\nabla \psi(x_0)|. \tag{4.25}$$

We can use (4.24) to obtain (4.23) immediately. □

The statement converse to that of Proposition 4.17 also holds when $N = 1$ but it fails to hold in higher dimensions, as indicated by the example below.

Example 4.18. Let $u : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$u(x) = \min\{x_1, 0\} + |x_2| \quad \text{for } x = (x_1, x_2) \in \mathbb{R}^2.$$

It is clear that u is Lipschitz in $\mathbb{R}^2$ and $|\nabla u|(0) = |\nabla^- u|(0) = 1$ but u cannot be tested at $x = 0$ from above by any function that is differentiable at $x = 0$.

However, for $x_0 \in \mathbb{R}^N$, if (1.10) holds in a neighborhood Ω of x_0 and $|\nabla u|$ is continuous in Ω, then u can be touched from above everywhere in Ω by a C^1 function. Indeed, under these assumptions, u can be regarded as a Monge solution of (1.3) with f continuous in Ω. This in turn implies that u is an s-solution and a conventional viscosity solution in the Euclidean space by Corollary 4.10. Then its local semi-concavity and testability from above follow immediately.

These suggest that (1.10) can be adopted as a generalized semi-concavity in more general metric spaces.

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