LINEARIZATION FOR FINITE PLASTICITY UNDER DISLOCATION-DENSITY TENSOR REGULARIZATION

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Abstract. Finite-plasticity theories often feature nonlocal energetic contributions in the plastic variables. By introducing a length-scale for plastic effects in the picture, these nonlocal terms open the way to existence results [25]. We focus here on a reference example in this direction, where a specific energetic contribution in terms of dislocation-density tensor is considered [30]. When external forces are small and dissipative terms are suitably rescaled, the finite-strain elastoplastic problem converges toward its linearized counterpart. We prove a Γ-convergence result making this asymptotics rigorous, both at the incremental level and at the level of quasistatic evolution.

1. Introduction

Elastoplastic materials accumulate permanent deformations during mechanical treatment. This is the macroscopic manifestation of a complex microscopic phenomenology, depending indeed on the actual material system, and possibly including slip and twinning along crystallographic planes, dislocation dynamics, microcracks, grain boundary motion, or microstructure evolution [21, 23]. At large strains, the modelization of the macroscopic elastoplastic response of a solid has originated a number of competing options as of kinematic and dynamic assumptions [38]. A mainstay of all finite-strain elastoplastic models is however that they should recover the classical, *linearized*, infinitesimal theory when strains are small and activation thresholds are suitably rescaled. In the engineering community, this fact is usually ascertained at the *material point* level, namely by checking by Taylor expansion that the rescaled energy and dissipation-potential densities of the finite-strain model reduce to that of the linearized case.

A more rigorous approach to linearization in elastoplasticity has been presented in [34] where the convergence of *solutions* of a specific finite-strain model to *solutions* of linearized elastoplasticity is proved by *evolutionary* Γ*-convergence* techniques [29, 33]. The result in [34] is the quasistatic, elastoplastic counterpart to the pioneering theory by Dal Maso, Negri, & Percivale [8], who obtained the convergence of finite to linearized elasticity. In the stationary elastic setting the reader is also referred to [1, 39, 40, 42] for various refinements, to [16, 37] for applications in homogenization, nematic elastomers [2], atomistic models [5], dislocation singularities [41]. As for the elastoplastic setting, rigorous linearization results have been obtained for elastoplastic plates by Davoli [9, 10], for one-dimensional perfect plasticity by Giacomini & Musesti [15], for plasticity in terms of the symmetrized plastic-strain tensor in [18], and for a model for shape-memory alloys in [17].

Existence theories for finite-plasticity in the multidimensional setting [18, 17, 25, 31] call for including in the energy a nonlocal regularization term in the plastic strain. From the modeling

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viewpoint, this term describes nonlocal plastic effects and is inspired to the now classical *strain-gradient plasticity* theory [11, 12, 36]. Its occurrence turns out to be crucial in order to introduce a length scale in the model and control the formation of plastic microstructures. Ultimately, such regularization ensures the necessary mathematical compactness for the analysis. To our knowledge, the only existence results without plastic-strain regularization currently available are one-dimensional [15, 28] or rely on specific structural restrictions [24, 28, 43].

We move here in the setting addressed by Mielke & Müller [30], providing the existence of *incremental* solutions in multiple dimensions by assuming a specific energetic contribution in terms of the *dislocation-density tensor*. Let $\varphi:\Omega\to\mathbb{R}^3$ denote the *deformation* of the elastoplastic body with reference configuration $\Omega\subset\mathbb{R}^3$. Assume the classical *multiplicative* strain decomposition $\nabla\varphi=F_eF_p$, where $F_e\,F_p\in\mathbb{R}^{3\times3}$ with $\det F_e>0$ and $F_p\in SL(3)=\{F\in\mathbb{R}^{3\times3}\ :\ \det F=1\}$ stand for the *elastic* and *plastic* strains respectively [22]. In [30] the stored energy of the body is assumed to have the form

$$\int_\Omega W_{el}(\nabla\varphi F_p^{-1})dx+\int_\Omega W_h(F_p)dx+\int_\Omega V((\operatorname{curl}F_p)F_p^T)dx-\int_\Omega f(t)\cdot\varphi\,dx. \tag{1.1}$$

Here, W_{el} represents the frame-indifferent *elastic* energy density, W_h is the *hardening* energy density, and $f(t)$ is a time-dependent body force. The tensor $\operatorname{curl}F_p\in\mathbb{R}^{3\times3}$ is computed from the plastic strain F_p by taking the curl row by row and the tensor $(\operatorname{curl}F_p)F_p^T$ represents the density of *geometrically necessary* dislocations in the medium [6, 30]. Correspondingly, the function V represents the *stored defect* energy density and is assumed to have polynomial growth.

The dissipation of the system is described by means of the *dissipation* distance

$$\mathcal{D}(F_p,\widehat{F}_p)=\mathcal{D}(I,\widehat{F}_pF_p^{-1})=\inf\int_\Omega\int_0^1R(\dot{P}P^{-1})dtdx, \tag{1.2}$$

where the infimum is computed on all the possible trajectories $P:[0,1]\to\mathbb{R}^{3\times3}$ such that $P(0)=F_p$ and $P(1)=\widehat{F}_p$ and the positively 1-homogeneous function $R:\mathbb{R}^{3\times3}\to[0,\infty]$ is finite only on trace-free symmetric matrices. The main result in [30] states that, under specific qualifications on the nonlinearities and boundary condition, the incremental problem driven by the above introduced energy and dissipation admits a solution.

This paper extends the linearization analysis of [34] to the dislocation-density tensor regularization setting of [30]. In the limit of infinitesimal strains, we show that *incremental* and *energetic solutions* [32, 35] to the finite-strain model converge to the unique corresponding incremental and energetic solutions of the linearized model, respectively. More precisely, for small $\epsilon>0$ we rescale the deformation and the plastic strain as $\varphi_\epsilon=Id+\epsilon u_\epsilon$ and $F_{p,\epsilon}=I+\epsilon z_\epsilon$, where u_ϵ and z_ϵ are interpreted as the displacement and the linearized plastic strain, respectively. Given the small force $f_0(t)=\epsilon f(t)$, the corresponding energy functional related to small deformations is the following rescaled version of (1.1)

$$\begin{aligned}\mathcal{E}_\epsilon(t,u_\epsilon,z_\epsilon)&=\frac{1}{\epsilon^2}\int_\Omega W_{el}((I+\epsilon\nabla u_\epsilon)(I+\epsilon z_\epsilon)^{-1})dx+\frac{1}{\epsilon^2}\int_\Omega W_h(I+\epsilon z_\epsilon)dx\\&+\frac{1}{\epsilon^2}\int_\Omega V((\operatorname{curl}(I+\epsilon z_\epsilon))(I+\epsilon z_\epsilon)^T)dx-\int_\Omega f_0(t)\cdot u_\epsilon dx,\end{aligned}$$

and the rescaled version of the dissipation in (1.2) is

$$\mathcal{D}_\epsilon(z_\epsilon,\widehat{z}_\epsilon)=\frac{1}{\epsilon}\int_\Omega D(I+\epsilon z_\epsilon,I+\epsilon\widehat{z}_\epsilon)dx$$

where D stands for the dissipation distance between plastic states, to be discussed in Subsection 2.4 below. The specific scaling above corresponds to quadratic expansions around the identity.

The corresponding functionals for linearized elastoplasticity read

$$\begin{aligned}\mathcal{E}_0(t,u,z) &= \frac{1}{2}\int_\Omega (\nabla u - z) : \mathbb{C}(\nabla u - z)dx + \frac{1}{2}\int_\Omega z : \mathbb{H}z\, dx \\ &\quad + \mu \int_\Omega (\operatorname{curl} z) : \mathbb{K}(\operatorname{curl} z)dx - \int_\Omega f_0(t)\cdot u\, dx, \\ \mathcal{D}_0(z,\widehat{z}) &= \int_\Omega R(\widehat{z} - z)dx\end{aligned}$$

where $\mathbb{C}$, $\mathbb{H}$, $\mathbb{K}$, and μ are related to the second derivatives at the identity of W_{el}, W_h, and V, respectively.

Given a prior plastic strain $\bar{z}$ and a fixed time t, for $\epsilon \geq 0$ an *incremental solution* is a pair (u_ϵ, z_ϵ) solving

$$(u_\epsilon, z_\epsilon) \in \operatorname{Arg\,min}\big(\mathcal{E}_\epsilon(t,u,z) + \mathcal{D}_\epsilon(z,\bar{z})\big) \tag{1.3}$$

under prescribed boundary conditions. Our first convergence result asserts that, up to not relabeled subsequences,

- **Theorem 3.1:** $(u_\epsilon, z_\epsilon) \rightharpoonup (u_0, z_0)$ weakly in $\mathcal{Q}$ as $\epsilon \searrow 0$,

where (u_0, z_0) is the unique incremental solution to the corresponding incremental problem for linearized elastoplasticity and $\mathcal{Q}$ denotes the state space, see (2.9). This is obtained by proving the Γ-convergence [7] of the functional in (1.3) to $\mathcal{E}_0(t,u,z)+\mathcal{D}_0(z,\bar{z})$ and checking equicoercivity.

Energetic solutions [32, 35] are instead continuous-time trajectories $t \mapsto (u_\epsilon(t), z_\epsilon(t)) \in \mathcal{Q}$ such that $(u_\epsilon(0), z_\epsilon(0)) = (u^0, z^0)$ (prescribed initial values) and for all t the two conditions hold

$$\text{(Global stability)} \quad \mathcal{E}_\epsilon(t, u_\epsilon(t), z_\epsilon(t)) \leq \mathcal{E}_\epsilon(t, \widehat{u}_\epsilon, \widehat{z}_\epsilon) + \mathcal{D}_\epsilon(z_\epsilon(t), \widehat{z}_\epsilon) \quad \forall (\widehat{u}_\epsilon, \widehat{z}_\epsilon) \in \mathcal{Q},$$

$$\text{(Energy conservation)} \quad \mathcal{E}_\epsilon(t, u_\epsilon(t), z_\epsilon(t)) + \mathrm{Diss}_{\mathcal{D}_\epsilon}(z_\epsilon; 0, t) = \mathcal{E}_\epsilon(0, u^0, z^0) - \int_\Omega f_0(t)\cdot u_\epsilon,$$

where $\mathrm{Diss}_{\mathcal{D}_\epsilon}(z_\epsilon; 0, t)$ is the total dissipation in $[0, t]$, see Definition 3.2 below. By *assuming* that energetic solutions (u_ϵ, z_ϵ) for $\epsilon > 0$ exist, our second main result, states that, up to not relabeled subsequences,

- **Theorem 3.3:** $(u_\epsilon(t), z_\epsilon(t)) \rightharpoonup (u_0(t), z_0(t))$ weakly in $\mathcal{Q}$ as $\epsilon \searrow 0$ for all t ,

where $t \mapsto (u_0(t), z_0(t))$ is the unique energetic solution to linearized elastoplasticity [20]. In order to prove convergence we resort in applying the abstract theory from [33]. This convergence relies on checking two separate Γ-liminf inequalities for energy and dissipation, as well as on constructing a *mutual recovery sequence*. Apart from the dislocation-density tensor term, which was not present in [34] and is now addressed in Lemma 5.2, Γ-liminf inequalities have been proved in [34]. The construction of the mutual recovery sequence, see (7.5)-(7.6), differs from that in [34]. On the one hand, one has to check that one can pass to the limit in the dislocation-density term. Since this term involves derivatives, truncations are not allowed in the definition of the recovery sequence, posing additional technical difficulties. On the other hand, we handle a weaker coercivity setting with respect to [34] by allowing the hardening energy density to be of polynomial growth. This requires some generalization of results in [8] and [34], based on appropriate coercivity conditions. Note however that the existence of energetic solutions for $\epsilon > 0$ is still unknown, so that Theorem 3.3 is presently a mere convergence result.

The paper is organized as follows. We present the mechanical model and comment on its mathematical setting in Section 2. The main results are then stated in Section 3. The coercivity of the energy is discussed in Section 4 and all Γ-lim inf inequalities are presented in Section 5. Eventually, Sections 6 and 7 focus on the proof of Theorems 3.1 and 3.3, respectively.

2. Finite-plasticity model

We devote this section to introducing the model and detailing the corresponding assumptions. Our main results are stated in Section 3.

2.1. **Tensor notation.** Let I denote the identity matrix in $\mathbb{R}^{d\times d}$ ($d=2,3$). For all matrices $A\in\mathbb{R}^{d\times d}$ we use the symbols $A^{\rm sym}$ and $A^{\rm anti}$ to denote the symmetric and antisymmetric parts, namely

$$A^{\rm sym}:=\frac{A+A^T}{2},\qquad A^{\rm anti}:=A-A^{\rm sym},$$

where the superscript T denotes transposition. The symbols $\mathbb{R}^{d\times d}_{\rm sym}$ and $\mathbb{R}^{d\times d}_{\rm anti}$ stand for the space of symmetric and antisymmetric $d\times d$ real matrices, respectively. Moreover $\mathbb{R}^{d\times d}_{\rm dev}$ denotes the space of deviatoric $d\times d$ symmetric real matrices, namely $A\in\mathbb{R}^{d\times d}_{\rm sym}$ with null trace ${\rm tr}(A)=A_{ii}=0$. Here and in the following the convention on summation over repeated indices is assumed. The matrix sets

$$GL_+(d)=\{A\in\mathbb{R}^{d\times d}:\det A>0\},\quad SL(d)=\{A\in\mathbb{R}^{d\times d}:\det A=1\},$$
$$SO(d)=\{A\in SL(d):AA^T=A^TA=I\},$$

will also be used.

Given a 4-tensor $\mathbb{T}\in\mathbb{R}^{d\times d\times d\times d}$ which is major symmetric ($\mathbb{T}_{ijlk}=\mathbb{T}_{lkij}$) and positive semidefinite, we denote by $|\cdot|_{\mathbb{T}}$ the seminorm

$$|A|^2_{\mathbb{T}}:=\frac12(A:\mathbb{T}A),$$

where we also used the standard notation for the contraction product $A:B:={\rm tr}(AB^T)$.

Let now a suitably regular map $x\mapsto A(x)\in\mathbb{R}^{d\times d}$ be given. We introduce the differential operator curl as follows

- For $d=2$, ${\rm curl}\,A\in\mathbb{R}^2$ is defined as
$$({\rm curl}\,A)_1=\partial_2A_{11}-\partial_1A_{12},\qquad({\rm curl}\,A)_2=\partial_2A_{21}-\partial_1A_{22}.$$
- For $d=3$, ${\rm curl}\,A\in\mathbb{R}^{3\times3}$ is defined as
$$({\rm curl}\,A)_{ij}=\varepsilon_{jlk}\partial_lA_{ik}$$
where ε_{jlk} is the classical Levi-Civita symbol.

Owing to these positions, one can compute the ${\rm curl}\,(AB)$ of a product. For $d=2$ this reads

$$({\rm curl}\,(AB))_i=A_{ik,2}B_{k1}-A_{ik,1}B_{k2}+A_{ik}({\rm curl}\,B)_k.\tag{2.1}$$

whereas for $d=3$ one has

$$({\rm curl}\,(AB))_{ij}=\varepsilon_{jlk}A_{iq,l}B_{qk}+A_{iq}({\rm curl}\,B)_{qj}=\varepsilon_{jlk}A_{iq,l}B_{qk}+A_{iq}({\rm curl}\,B)_{qj}.\tag{2.2}$$

One can rewrite these formulas via a differential operator $\mathbb{D}$ by distinguishing between the cases $d=2$ and $d=3$. Namely, if $d=2$ we let $\mathbb{D}:\mathcal{D}(\Omega;\mathbb{R}^{2\times2})\to\mathcal{D}(\Omega;\mathbb{R}^{2\times2\times2})$, be defined as

$$(\mathbb{D}A)_{ij1}=A_{ij,2}\qquad(\mathbb{D}A)_{ij2}=-A_{ij,1}.$$

For $d=3$, we define $\mathbb{D}:\mathcal{D}(\Omega;\mathbb{R}^{3\times3})\to\mathcal{D}(\Omega;\mathbb{R}^{3\times3\times3\times3})$ as

$$(\mathbb{D}A)_{ijkl}=\varepsilon_{jql}A_{ik,q}.$$

Along with this notation, (2.1) and (2.2) can be respectively rewritten as

$$(\mathrm{curl}\,(AB))_i = (\mathbb{D}A)_{ikl}B_{kl} + A_{ik}(\mathrm{curl}\,B)_k \quad \text{if } d = 2, \tag{2.3}$$

$$(\mathrm{curl}\,(AB))_{ij} = (\mathbb{D}A)_{ijqk}B_{qk} + A_{iq}(\mathrm{curl}\,B)_{qj} \quad \text{if } d = 3. \tag{2.4}$$

Observe that in dimension $d = 2$ and $d = 3$, respectively, we have

$$(\mathbb{D}A)_{ijk}I_{jk} = (\mathrm{curl}\,A)_i \qquad (\mathbb{D}A)_{ijkl}I_{kl} = (\mathrm{curl}\,A)_{ij}. \tag{2.5}$$

To shortcut notation we denote by $\mathbb{D}A : B$ the first term in both relations (2.3)-(2.4) above. Note that this is a slight abuse of notation for $d = 2$.

We now introduce the *dislocation-density tensor* $G(P)$ as

$$G(P) := \frac{\mathrm{curl}\,P}{\det P} \quad \text{for } d = 2, \qquad G(P) := \frac{(\mathrm{curl}\,P)P^T}{\det P} \quad \text{for } d = 3. \tag{2.6}$$

The tensor $G(P)$ measures the density of *geometrically necessary dislocations* in the intermediate configuration, namely, dislocations arising solely from the underlying kinematics. Different expressions for such a measure have been proposed in the literature, a critical review can be found in [6]. We follow here the approach from [6, 19] in the specific setting given by [44, Formula (154)]. In the reference configuration, the incompatibility of P is classically measured by $\mathrm{curl}\,P$. The surface element in the intermediate configuration is given by

$$n_i\,da_i = (\det P)P^{-T}n_r\,da_r$$

where da is the infinitesimal area element, n is the corresponding normal, and the subscripts i and r stand for *intermediate* and *referential*, respectively (notation refers to $d = 3$). Definition (2.6) ensures that

$$(\mathrm{curl}\,P)\,n_r\,da_r = G(P)\,n_i\,da_i.$$

Hence, in the intermediate configuration the tensor $G(P)$ corresponds to the referential incompatibility measure $\mathrm{curl}\,P$.

Owing to [30, Lemma 2.1] one can find a constant $c_d > 0$ such that

$$|G(P)| \geq c_d\frac{|\mathrm{curl}\,P|}{|P|^2}.$$

Moreover,

$$|G(P)| \geq \frac{|\mathrm{curl}\,P|}{|\det P|} \quad \text{for } d = 2, \qquad |G(P)| \geq \frac{|\mathrm{curl}\,P|}{|\det P||P^{-1}|} \quad \text{for } d = 3. \tag{2.7}$$

In the following, we denote by c any positive constant, possibly depending from data and changing from line to line. Occasionally, dependencies of such constants will be made explicit.

2.2. **States.** Let the open, bounded, connected, and Lipschitz set $\Omega \subset \mathbb{R}^d$ represent the reference configuration of the elastoplastic body. We assume that the boundary $\partial\Omega$ is decomposed into the union of the Dirichlet boundary $\partial_D\Omega$ and the Neumann one $\partial_N\Omega$, which are relatively open in $\partial\Omega$ with $\mathcal{H}^{d-1}(\partial_D\Omega) > 0$, with $\mathcal{H}^{d-1}$ being the $(d-1)$-dimensional Hausdorff measure. Furthermore we assume that $\partial_D\Omega$ fulfills the following geometric property: by letting $\mathrm{aff}(\partial_D\Omega)$ be the smallest affine space containing $\partial_D\Omega$ we require that

$$\mathrm{aff}(\partial_D\Omega) = \mathbb{R}^d. \tag{2.8}$$

The state of the system is determined by its deformation $u \in H^1(\Omega;\mathbb{R}^d)$, which is required to satisfy the Dirichlet condition $u = 0$ on $\partial_D\Omega$, and its plastic strain $P \in L^2(\Omega;\mathbb{R}^{d\times d})$. By letting

$$H^1_D(\Omega;\mathbb{R}^d) := \{u \in H^1(\Omega;\mathbb{R}^d) : u = 0 \text{ on } \partial_D\Omega\}$$

we denote by $\mathcal{Q}$ the space of states (u,P), namely

$$\mathcal{Q} := H^1_D(\Omega;\mathbb{R}^d) \times L^2(\Omega;\mathbb{R}^{d\times d}) \tag{2.9}$$

endowed with the weak topology of $H^1(\Omega;\mathbb{R}^d) \times L^2(\Omega;\mathbb{R}^{d\times d})$.

2.3. **Energy.** Given the state $(u,P) \in \mathcal{Q}$, we define the stored elastic energy and the energy related to hardening as

$$\mathcal{W}_{\rm el}(u,P) := \int_\Omega W_{\rm el}(\nabla u P^{-1})dx, \qquad \mathcal{W}_h(P) := \int_\Omega W_h(P)dx.$$

The total energy at a fixed time $t \in [0,T]$ reads then

$$\mathcal{E}(t,u,P) = \int_\Omega W_{el}(\nabla u P^{-1})dx + \int_\Omega W_h(P)dx + \int_\Omega V(G(P))dx - \langle \ell(t), u\rangle, \tag{2.10}$$

with $G(P)$ defined in (2.6). The last term in (2.10) above features the time-dependent functional $\ell : [0,T] \to (H^1_D(\Omega;\mathbb{R}^d))'$ and represents external actions, possibly of the form

$$\langle \ell(t), u\rangle = \int_\Omega f(t)\cdot u\, dx + \int_{\partial_N\Omega} g(t)\cdot u\, dS,$$

where $f(t) \in L^2(\Omega;\mathbb{R}^d)$ and $g(t) \in L^2(\partial_N\Omega;\mathbb{R}^d)$ are a given body force and a traction on $\partial_N\Omega$, respectively.

We assume that the elastic energy density $W_{el} : \mathbb{R}^{d\times d} \to [0,\infty]$ satisfies, for given positive constants c_i, $i = 1,\dots,4$, and $q_e > 2$, the following

$$W_{el} \in C^1(GL_+(d)),\ W_{el} = \infty \text{ on } \mathbb{R}^{d\times d}\setminus GL_+(d), \tag{2.11a}$$

$$\forall F \in GL_+(d),\ \forall Q \in SO(d):\ W_{el}(QF) = W_{el}(F), \tag{2.11b}$$

$$\forall F \in GL_+(d):\ W_{el}(F) \geq c_1 \mathrm{dist}\,(F, SO(d))^2 + c_2 \mathrm{dist}\,(F,SO(d))^{q_e}, \tag{2.11c}$$

$$\forall F \in GL_+(d): |F^T\partial_F W_{el}(F)| \leq c_3(W_{el}(F) + c_4), \tag{2.11d}$$

$$\exists \mathbb{C} \in \mathbb{R}^{d\times d\times d\times d},\ \mathbb{C} > 0,\ \forall \delta > 0,\ \exists c_{el}(\delta) > 0,\ \forall A \in \mathbb{R}^{d\times d},\ |A| \leq c_{el}(\delta):$$
$$|W_{el}(I+A) - |A|_{\mathbb{C}}^2| \leq \delta |A|^2_{\mathbb{C}}. \tag{2.11e}$$

Relation (2.11b) corresponds to frame indifference whereas (2.11c) is a nondegenaracy condition expressing that the energy grows more than quadratically far from the identity. Assumption (2.11d) is classical and corresponds to the controllability of the Mandel tensor $F^T\partial_F W_{el}(F)$ in terms of the energy [3, 4]. This control was already used in the context of rate-independent processes in [13, 25, 34], among others. Finally, condition (2.11e) encodes the quadratic behavior of the energy for small deformations and is instrumental in order to quantify the small-strain limit. Indeed, by following the discussion in [34], one can check that (2.11e) implies

$$W_{el}(I) = 0, \quad \partial_F W_{el}(I) = 0, \quad \partial_F^2 W_{el}(I) = \mathbb{C}.$$

Note that $\mathbb{C}$ is major symmetric, namely $\mathbb{C}_{ijlk} = \mathbb{C}_{lkij}$. By taking frame indifference (2.11b) into account, the minor symmetries of $\mathbb{C}$ follow, namely $\mathbb{C}_{ijkl} = \mathbb{C}_{jikl} = \mathbb{C}_{ijlk}$, so that $\mathbb{C}A = \mathbb{C}A^{\rm sym}$ holds for all $A \in \mathbb{R}^{d\times d}$. The nondegeneracy condition (2.11c) implies that the tensor $\mathbb{C}$ is positive definite so that the seminorm $|\cdot|_{\mathbb{C}}$ is actually equivalent to the Frobenius norm $|A| = (A:A)^{1/2}$.

For the hardening density $W_h : \mathbb{R}^{d\times d} \to [0,\infty]$ we assume that, for given positive constants c_i, $i = 5, \dots 8$, and $q_h > d$ large (see the qualification (2.19) below), the following holds

$$W_h(P) = \begin{cases} \tilde{W}_h(P) & \text{if } P \in SL(d), \\ \infty & \text{if } P \notin SL(d), \end{cases} \tag{2.12a}$$

$$\tilde{W}_h : \mathbb{R}^{d\times d} \to \mathbb{R} \text{ is locally Lipschitz continuous,} \tag{2.12b}$$

$$\forall P \in \mathbb{R}^{d\times d} : \ \tilde{W}_h(P) \geq c_5|P|^{q_h} - c_6, \tag{2.12c}$$

$$\exists \mathbb{H} \in \mathbb{R}^{d\times d\times d\times d},\ \mathbb{H} > 0,\ \forall \delta > 0,\ \exists c_h(\delta) > 0,\ \forall A \in \mathbb{R}^{d\times d},\ |A| \leq c_h(\delta) :$$

$$|\tilde{W}_h(I + A) - |A|_{\mathbb{H}}|^2| \leq \delta |A|^2_{\mathbb{H}}, \tag{2.12d}$$

$$\forall A \in \mathbb{R}^{d\times d} : \ \tilde{W}_h(I + A) \geq c_7|A|^2 + c_8|A|^{q_h}. \tag{2.12e}$$

As consequence of assumptions (2.12c) and (2.12e), there exists a constant $c_K > 0$ such that

$$P \in SL(d) \ \Rightarrow \ \|P\|^{q_h}_{L^{q_h}} + \|P^{-1}\|^{q_h/(d-1)}_{L^{q_h/(d-1)}} \leq c_K(W_h(P) + 1), \tag{2.13}$$

$$\tilde{W}_h(I + A) \geq c_K|A|^p \ \text{ for any } p \in [2, q_h]. \tag{2.14}$$

Notice that conditions (2.12a) and (2.12c) are weaker than assumption (2.6a) in [34], where both P and P^{-1} were constrained to a compact set containing the identity. Namely, bound (2.13) is assumed there to hold in L^∞.

Following the setting introduced in [30], we assume that there exists positive constants c_i, $i = 9, \dots, 11$, $\mu \geq 0$, and $q_c > 2$ such that

$$V : \mathbb{R}^2 \to [0,\infty] \ \text{ if } d = 2, \quad V : \mathbb{R}^{3\times 3} \to [0,\infty] \ \text{ if } d = 3, \tag{2.15a}$$

$$V \text{ is a function of class } C^1, \tag{2.15b}$$

$$\forall G : \ V(G) \geq c_9|G|^{q_c} + c_{10}|G|^2, \tag{2.15c}$$

$$\forall G : \ |\nabla V(G)| \leq c_{11}(|G|^{q_c-1} + 1), \tag{2.15d}$$

$$\exists \mathbb{K} \in \mathbb{R}^{d\times d\times d\times d},\ \mathbb{K} > 0,\ \forall \delta > 0,\ \exists c_v(\delta) > 0,\ \forall G \in \mathbb{R}^{d\times d},\ |G| \leq c_v(\delta) :$$

$$|V(G) - \mu|G|^2_{\mathbb{K}}| \leq \delta|G|^2_{\mathbb{K}}. \tag{2.15e}$$

For $\epsilon > 0$ given, it is convenient to introduce the function

$$\tilde{V}_\epsilon(y, z) := \frac{1}{\epsilon^2}V(\epsilon y) \ \text{ for } d = 2, \qquad \tilde{V}_\epsilon(y, z) := \frac{1}{\epsilon^2}V(\epsilon y(I + \epsilon z)^T) \ \text{ for } d = 3 \tag{2.16}$$

where $(y, z) \in \mathbb{R}^2 \times \mathbb{R}^{2\times 2}$ or $(y, z) \in \mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3}$, respectively for $d = 2$ or $d = 3$. By using this notation, assumption (2.15e) implies that

- in the case $d = 2$,

$$\exists \mu > 0,\ \forall \delta > 0,\ \exists c_v(\delta) > 0,\ |\epsilon \mathrm{curl}\, z| \leq c_v(\delta) :$$

$$|\tilde{V}_\epsilon(\mathrm{curl}\, z, z) - \mu|\mathrm{curl}\, z|^2_{\mathbb{K}}| \leq \delta|\mathrm{curl}\, z|^2_{\mathbb{K}}, \tag{2.17}$$

- in the case $d = 3$,

$$\exists \mu > 0,\ \forall \delta > 0,\ \exists c_v(\delta) > 0,\ |(\epsilon \mathrm{curl}\, z)(I + \epsilon z)| \leq c_v(\delta) :$$

$$|\tilde{V}_\epsilon(\mathrm{curl}\, z, z) - \mu|\mathrm{curl}\, z(I + \epsilon z)|^2_{\mathbb{K}}| \leq \delta|\mathrm{curl}\, z(I + \epsilon z)|^2_{\mathbb{K}}. \tag{2.18}$$

Some qualification on the exponents q_e, q_h, and q_c appearing in the growth conditions of the energy functionals is in order. We will ask for

$$q_h > \max\left\{\frac{12q_e}{q_e-2}, 2q_c\right\}, \qquad q_c \geq 3, \qquad q_e > 2. \tag{2.19}$$

This complies for instance with the choice $q_c = 3$, $q_h > 6$, $q_e \in (2, 2q_h/(q_h-4))$.

2.4. **Dissipation.** The distance between plastic states is measured by means of a dissipation-density function $R : \mathbb{R}^{d\times d} \to [0,\infty]$ satisfying

$$R(z) := \begin{cases} R^{\rm dev}(z) & \text{if } z \in \mathbb{R}^{d\times d}_{\rm dev}, \\ \infty & \text{otherwise.} \end{cases} \tag{2.20}$$

Here, $R^{\rm dev} : \mathbb{R}^{d\times d}_{\rm dev} \to \mathbb{R}^+$ is a convex and positively 1-homogeneous function with

$$\rho_0|z| \leq R^{\rm dev}(z) \leq \rho_1|z|, \qquad \text{for all } z \in \mathbb{R}^{d\times d}_{\rm dev}, \tag{2.21}$$

for two positive constants $0 < \rho_0 < \rho_1$. We define the dissipation distance $D : \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times d} \to [0,\infty]$ by imposing the plastic-invariance property $D(P,\widehat{P}) = D(I, \widehat{P}P^{-1})$ for all $P, \widehat{P} \in \mathbb{R}^{d\times d}$, by prescribing

$$D(I,P) := \inf\left\{\int_0^1 R(\dot{S}S^{-1})dt : S \in C^1(0,1;\mathbb{R}^{d\times d}),\ S(0) = I,\ S(1) = P\right\},$$

and by letting $D(P,\widehat{P}) = \infty$ if P is not invertible. Let us stress that $D(I,P) < \infty$ implies $\det P = 1$. In fact, by definition there exists a path S such that $\int_0^1 R(\dot{S}S^{-1})dt < \infty$. This implies that $\mathrm{tr}(\dot{S}S^{-1}) = 0$ almost everywhere by (2.20). The Jacobi formula ensures that $\partial_t(\det S) = \mathrm{tr}(\dot{S}S^{-1}/\det S) = \mathrm{tr}(\dot{S}S^{-1})/\det S = 0$ and $\det P = 1$ follows.

We have that [26, Lemma 2.1]

$$D(P,\widehat{P}) \leq c(1 + |P| + |\widehat{P}|), \tag{2.22}$$

for some constant $c > 0$ depending only on the dimension d and on ρ_1 in (2.21).

The dissipation distance between two plastic states $P, \widehat{P} \in L^1(\Omega;\mathbb{R}^{d\times d})$ is then defined by

$$\mathcal{D}(P,\widehat{P}) := \int_\Omega D(P(x),\widehat{P}(x))dx.$$

Note that properties (2.11)-(2.12), (2.15), (2.19), and (2.21) will be tacitly assumed in the following, without specific mentioning.

3. Main results

We aim at stating our main linearization results, describing the limit for deformations and the plastic strains close to identity. The elastoplastic problem is hence reformulated in terms of deviations from identity and rescaled. More precisely, for $\epsilon > 0$ we define

$$u = \frac{1}{\epsilon}(y - \mathrm{id}), \quad z = \frac{1}{\epsilon}(P - I).$$

Correspondingly, one has that

$$\nabla y = I + \epsilon\nabla u, \quad F = (I + \epsilon\nabla u)(I + \epsilon z)^{-1},$$

as well as

$$G(P) = G(I+\epsilon z) = \begin{cases} \epsilon \operatorname{curl} z & \text{if } d=2, \\ \epsilon \operatorname{curl} z (I+\epsilon z)^T & \text{if } d=3. \end{cases}$$

Then, we define the rescaled energy densities as

$$W_{el}^{\epsilon}(u,z) := \frac{1}{\epsilon^2} W_{el}(I+\epsilon\nabla u, I+\epsilon z), \quad W_h^{\epsilon}(z) := \frac{1}{\epsilon^2} W_h(I+\epsilon z),$$
$$\tilde{W}_h^{\epsilon}(z) := \frac{1}{\epsilon^2} \tilde{W}_h(I+\epsilon z), \quad V^{\epsilon}(G) = \frac{1}{\epsilon^2} V(G),$$

and, accordingly, the stored energies

$$\mathcal{W}_{el}^{\epsilon}(F) = \int_\Omega W_{el}^{\epsilon}(F)dx, \quad \mathcal{W}_h^{\epsilon}(z) = \int_\Omega W_h^{\epsilon}(z)dx, \quad \mathcal{V}^{\epsilon}(G) = \int_\Omega V^{\epsilon}(G)dx,$$
$$\mathcal{W}_{\epsilon}(u,z) = \mathcal{W}_{el}^{\epsilon}((I+\epsilon\nabla u)(I+\epsilon z)^{-1}) + \mathcal{W}_h^{\epsilon}(z) + \mathcal{V}^{\epsilon}(G).$$

The limiting stored energy then reads

$$\mathcal{W}_0(u,z) = \int_\Omega |\nabla u^{\mathrm{sym}} - z^{\mathrm{sym}}|_{\mathbb{C}}^2 dx + \int_\Omega |z|_{\mathbb{H}}^2 dx + \mu \int_\Omega |\operatorname{curl} z|_{\mathbb{K}}^2 dx.$$

As far as the dissipation is concerned, we introduce the dissipation functionals $\mathcal{D}_\epsilon, \mathcal{D}_0 : L^1(\Omega;\mathbb{R}^{d\times d}) \to [0,\infty]$ defined as

$$\mathcal{D}_\epsilon(z,\widehat{z}) := \int_\Omega D_\epsilon(z(x),\widehat{z}(x))dx, \qquad \mathcal{D}_0(z,\widehat{z}) := \int_\Omega D_0(z(x),\widehat{z}(x))dx$$

where

$$D_\epsilon(z,\widehat{z}) := \frac{1}{\epsilon} D(I+\epsilon z, I+\epsilon\widehat{z}), \qquad D_0(z,\widehat{z}) := R(z-\widehat{z}), \qquad \forall z,\widehat{z} \in \mathbb{R}^{d\times d}.$$

Given the previous plastic state $\bar{z} \in L^2(\Omega;\mathbb{R}^{d\times d})$, these definitions allow the specification of the incremental problems

$$\min_{(u,z)\in\mathcal{Q}} \mathcal{F}_\epsilon(u,z;\bar{z}) := \min_{(u,z)\in\mathcal{Q}} \left(\mathcal{W}_\epsilon(u,z) + \mathcal{D}_\epsilon(z,\bar{z})\right),$$

for $\epsilon \geq 0$. Note that, under assumptions (2.11)-(2.12), (2.15), (2.19), and (2.21), for all $\epsilon \geq 0$ the functionals $(u,z) \mapsto W_\epsilon(u,z) + \mathcal{D}_\epsilon(z,\bar{z})$ are coercive and lower semicontinuous with respect to the weak topology of $\mathcal{Q}$. Hence, the Direct Method of the calculus of variations ensures that the incremental problems above admit a solution for all $\epsilon \geq 0$. As $\mathcal{W}_0$ is uniformly convex, the solution for $\epsilon = 0$ is unique.

Our first convergence result concerns the convergence of the incremental problems as $\epsilon \to 0$.

Theorem 3.1 (Convergence of the incremental problems). *Let $\bar{z} \in L^2(\Omega;\mathbb{R}^{d\times d})$ be given. Then $\mathcal{F}_\epsilon(\cdot;\bar{z})$ Γ-converge to $\mathcal{F}_0(\cdot;\bar{z})$ with respect to the weak topology of $\mathcal{Q}$.*

The proof of Theorem 3.1 is given in Section 6.

Our second convergence result concerns quasistatic evolution. Let us start by defining the total dissipation associated to $z : [0,T] \to L^1(\Omega;\mathbb{R}^{d\times d})$ on the subinterval $[0,t]$ as

$$\mathrm{Diss}_{\mathcal{D}_\epsilon}(z;0,t) := \sup\left\{ \sum_{i=1}^n \mathcal{D}_\epsilon(z(t^i),z(t^{i-1}) : 0 = t^0 < t^1 < \cdots < t^n = t \right\},$$

where the supremum is taken on all partitions of $[0,t]$. We assume the external load

$$\ell \in W^{1,1}(0,T;(H^1_D(\Omega;\mathbb{R}^d)') \tag{3.1}$$

to be given. For all $\epsilon \geq 0$, the total energy of the system at time t is specified as

$$\mathcal{E}_\epsilon(t,u,z) := \mathcal{W}_\epsilon(u,z) - \langle \ell(t), u\rangle.$$

In the following, we will refer to the triple $(\mathcal{Q}, \mathcal{E}_\epsilon, \mathcal{D}_\epsilon)$ as a *rate-independent system*. Correspondingly, we introduce the set $\mathcal{S}_\epsilon(t)$ of stable states at time t by letting

$$\mathcal{S}_\epsilon(t) := \{(u,z) \in \mathcal{Q} \ : \ \mathcal{E}_\epsilon(t,u,z) < \infty, \ \ \mathcal{E}_\epsilon(t,u,z) \leq \mathcal{E}_\epsilon(t,\widehat{u},\widehat{z}) + \mathcal{D}_\epsilon(z,\widehat{z}) \ \ \forall (\widehat{u},\widehat{z}) \in \mathcal{Q}\},$$

and analogously for the rate-independent system $(\mathcal{Q}, \mathcal{E}_0, \mathcal{D}_0)$.

Our assumptions on initial data read

$$(u^0_\epsilon, z^0_\epsilon) \rightharpoonup (u^0_0, z^0_0) \quad \text{weakly in } \mathcal{Q}, \quad \mathcal{E}_\epsilon(0, u^0_\epsilon, z^0_\epsilon) \to \mathcal{E}_0(0, u^0_0, z^0_0). \tag{3.2}$$

Along with these provisions, we recall the following classical definition of quasistatic evolution for the rate-independent system $(\mathcal{Q}, \mathcal{E}_\epsilon, \mathcal{D}_\epsilon)$ [32].

Definition 3.2 (Energetic solutions)**.** Let $\epsilon \geq 0$ and $T > 0$. A trajectory $t \in [0,T] \mapsto (u_\epsilon(t), z_\epsilon(t)) \in \mathcal{Q}$ is called an energetic solution for the rate-independent system $(\mathcal{Q}, \mathcal{E}_\epsilon, \mathcal{D}_\epsilon)$ if $(u_\epsilon(0), z_\epsilon(0)) = (u^0_\epsilon, z^0_\epsilon)$, the map $t \mapsto \langle \dot{\ell}(t), u_\epsilon(t)\rangle$ is integrable, and the following two conditions are fulfilled for all $t \in [0,T]$

(S) $(u_\epsilon(t), z_\epsilon(t)) \in \mathcal{S}_\epsilon(t)$,
(E) $\mathcal{E}_\epsilon(t, u_\epsilon(t), z_\epsilon(t)) + \mathrm{Diss}_{\mathcal{D}_\epsilon}(z_\epsilon; 0, t) = \mathcal{E}_\epsilon(0, u^0_\epsilon, z^0_\epsilon) - \int_0^t \langle \dot{\ell}, u_\epsilon\rangle ds.$

Our second main result concerns the *evolutionary* Γ*-convergence* [29] of the rate-independent system $(\mathcal{Q}, \mathcal{E}_\epsilon, \mathcal{D}_\epsilon)$ to $(\mathcal{Q}, \mathcal{E}_0, \mathcal{D}_0)$ in terms of convergence of energetic solutions. More precisely, we have the following.

Theorem 3.3 (Convergence of energetic solutions)**.** *Assume* (3.1)-(3.2) *and let* (u_ϵ, z_ϵ) *be an energetic solution for the rate-independent system* $(\mathcal{Q}, \mathcal{E}_\epsilon, \mathcal{D}_\epsilon)$*. Then, for all* $t \in [0,T]$ *we have that* $(u_\epsilon(t), z_\epsilon(t)) \rightharpoonup (u_0(t), z_0(t))$ *weakly in* $\mathcal{Q}$ *where* (u_0, z_0) *is the energetic solution of the rate-independent system* $(\mathcal{Q}, \mathcal{E}_0, \mathcal{D}_0)$*.*

Theorem 3.3 is proved in Section 7. In the spirit of [34], let us emphasize that Theorem 3.3 is purely a convergence result, for the existence of energetic solutions for the rate-independent system $(\mathcal{Q}, \mathcal{E}_\epsilon, \mathcal{D}_\epsilon)$ is *assumed*. In fact, in the present setting existence of energetic solutions is still unknown, unless one resorts to including in the energy some compactifying terms, for instance a full gradient term in ∇z instead of the control on $\mathrm{curl}\, z$. Existence results under such stronger compactness assumptions are in [18, 30]. Existence result in the absence of gradient regularizations are in [27] in the one-dimensional setting and in [24, 43] in the dislocation-free setting.

4. Coercivity and compactness

In preparation for the proofs of Theorems 3.1 and 3.3, let us collect some preliminary remarks on the coercivity of the energy.

A *caveat* on notation: in the following we will use the same symbol c to indicate a positive universal constant, possibly depending on data and changing even within the same line. Occasionally, dependencies of constants on specific parameters will be indicated.

Lemma 4.1 (Coercivity)**.** *For all* $\epsilon \in (0,1)$ *and* $p \in (2, q_h)$ *one has*

$$\begin{aligned} &\|\nabla u\|^2_{L^2} + \epsilon^{q_h - 2}\|z\|^{q_h}_{L^{q_h}} + \epsilon^{p-2}\|z\|^p_{L^p} + \|z\|^2_{L^2} + \|(I+\epsilon z)^{-1}\|^{q_h/2}_{L^{q_h/2}} \\ &\leq c(1 + \mathcal{W}_\epsilon(u,z)). \end{aligned} \tag{4.1}$$

Additionally, the following hold, $d = 2$,

$$\|\operatorname{curl} z\|_{L^2}^2 + \epsilon^{q_c-2}\|\operatorname{curl} z\|_{L^{q_c}}^{q_c} \le c(1 + \mathcal{W}_\epsilon(u, z)), \tag{4.2}$$

and in case $d = 3$,

$$\|(\operatorname{curl} z)(I + \epsilon z^T)\|_{L^2}^2 + \epsilon^{q_c-2}\|(\operatorname{curl} z)(I + \epsilon z^T)\|_{L^{q_c}}^{q_c} + \|\operatorname{curl} z\|_{L^q}^q \\ \le c(1 + \mathcal{W}_\epsilon(u, z)), \tag{4.3}$$

with $q = 2q_h/(q_h + 4)$.

Proof. Assume that $\mathcal{W}_\epsilon(u, z) < c < \infty$. By assumption (2.12a) we deduce that $I + \epsilon z \in SL(d)$ so that $W_h(I + \epsilon z) = \tilde{W}_h(I + \epsilon z)$. From the coercivity assumption (2.12e) we then obtain

$$|z|^2 + \epsilon^{q_h-2}|z|^{q_h} \le c\tilde{W}_h^\epsilon(z). \tag{4.4}$$

This directly implies the L^2 and L^{q_h} control of z in (4.1). Concerning the L^p control of z with $p \in (2, q_h)$ we argue by interpolation. Let $\alpha := 2(q_h - p)/(q_h - 2)$, $q := 2/\alpha$, $q' := q/(q - 1) = 2/(2 - \alpha)$. One has that

$$\|z_\epsilon\|_{L^p}^p = \int_\Omega |z_\epsilon|^\alpha |z_\epsilon|^{p-\alpha} dx \le \|z_\epsilon\|_{L^2}^\alpha \|z_\epsilon\|_{L^{q_h}}^{\frac{q_h(2-\alpha)}{2}} = \|z_\epsilon\|_{L^2}^\alpha \|z_\epsilon\|_{L^{q_h}}^{\frac{q_h(p-2)}{q_h-2}} \le \epsilon^{2-p} c\mathcal{W}_\epsilon(u, z).$$

The boundedness of the L^{q_h} norm of z from (4.4) also implies that $\|\epsilon z\|_{L^{q_h}}^{q_h} \le c\mathcal{W}_h^\epsilon(z)$ and thus we infer $\|I + \epsilon z\|_{L^{q_h}}^{q_h} < c(1 + \mathcal{W}_\epsilon(u, z))$. By using the bound (2.13) we obtain the control of all terms including z in (4.1).

For $d = 2$ we simply have $G = \epsilon\operatorname{curl} z$ and (4.2) follows from property (2.15c). In case $d = 3$, we similarly infer (4.3), from (2.7), (4.1), and the Hölder inequality.

We are left with estimating ∇u. Our argument here is similar to [34, Lemma 3.1] but requires some modifications due to the different coerciveness properties of the energies. Define $\varphi = \mathrm{id} + \epsilon u$ and let $Q \in SO(d)$. We write

$$|\nabla\varphi - Q|^2 \le |\nabla\varphi - Q(I + \epsilon z) + \epsilon Q z|^2 = |(F_{el} - Q)(I + \epsilon z) + \epsilon Q z|^2 \\ \le c|F_{el} - Q|^2 + c|F_{el} - Q|^2|\epsilon z|^2 + c|\epsilon z|^2,$$

where $F_{el} = (I + \epsilon\nabla u)(I + \epsilon z)^{-1}$. Taking the infimum with respect to $Q \in SO(d)$ in the above left-hand side we get

$$\operatorname{dist}^2(\nabla\varphi, SO(d)) \le c(\operatorname{dist}^2(F_{el}, SO(d)) + \operatorname{dist}^2(F_{el}, SO(d))|\epsilon z|^2 + |\epsilon z|^2).$$

Integrating over Ω and using the Hölder inequality with exponents p and p' such that $p = q_e/2$ (thus $p > 1$, and $2p' = 2q_e/(q_e - 2) < q_h$ by (2.19)), we find

$$\int_\Omega \operatorname{dist}^2(\nabla\varphi, SO(d))dx \\ \le c\left(\int_\Omega \operatorname{dist}^2(F_{el}, SO(d))dx + \left(\int_\Omega \operatorname{dist}^{q_e}(F_{el}, SO(d))dx\right)^{\frac{1}{p}} \|\epsilon z\|_{L^{2p'}}^2 + \|\epsilon z\|_{L^2}^2\right) \\ \le c\big(\mathcal{W}_{el}(u, z) + (\mathcal{W}_{el}(u, z))^{\frac{1}{p}}(\mathcal{W}_h(I + \epsilon z))^{\frac{1}{p'}} + \mathcal{W}_h(I + \epsilon z)\big) =: B_\epsilon,$$

where we have used (2.11c), (2.12e), and (2.14) with exponent $2p' \le q_h$. The Rigidity Lemma [14, Theorem 3.1] implies that there exists $\widehat{Q} \in SO(d)$ such that

$$\|\nabla\varphi - \widehat{Q}\|_{L^2}^2 \le B_\epsilon.$$

Moreover, following the argument in [8] one can prove that $\|I-\widehat{Q}\|^2_{L^2}\le B_\epsilon$, which entails

$$\|\epsilon\nabla u\|^2_{L^2}\le\|I-\nabla\varphi\|^2_{L^2}\le 2\|I-\widehat{Q}\|^2_{L^2}+2\|\widehat{Q}-\nabla\varphi\|^2_{L^2}\le 4B_\epsilon.$$

By dividing by ϵ^2 we finally obtain

$$\|\nabla u\|^2_{L^2}\le c\big(\mathcal{W}^\epsilon_{el}(u,z)+(\mathcal{W}^\epsilon_{el}(u,z))^{\frac{1}{p}}(\mathcal{W}^\epsilon_h(z))^{\frac{1}{p'}}+\mathcal{W}^\epsilon_h(z)\big)$$

and the bound on ∇u in (4.1) follows. □

We can refine the estimate on ∇u by using an L^p version of the Rigidity Lemma and a slight modification of the previous proof. To this aim we first need to check that [8, Lemma 3.3] also works with an exponent $p>1$ instead of 2.

Lemma 4.2 (Boundary control)**.** *Let $d=2,3$ and $p>1$. The quantity*

$$|F|_{\partial_D\Omega}:=\left(\min_{\zeta\in\mathbb{R}^d}\int_{\partial_D\Omega}|Fx-\zeta|^p d\mathcal{H}^{d-1}(x)\right)^{1/p}\qquad\forall F\in\mathbb{R}^{d\times d}\tag{4.5}$$

satisfies

$$|F|\le c_D|F|_{\partial_D\Omega}\tag{4.6}$$

for some constant $c_D>0$ independent of $F\in\mathbb{R}^{d\times d}$.

Proof. We firstly check that, for all $F\in\mathbb{R}^{d\times d}$, there exists ζ_F achieving the minimum in (4.5). Indeed, for F fixed let ζ_n be a minimizing sequence for (4.5). One readily checks that $|\zeta_n|(\mathcal{H}^{d-1}(\partial_D\Omega))^{1/p}\le\|Fx-\zeta_n\|_{L^p(\partial_D\Omega)}+c$ so that such minimizing sequence is bounded in $\mathbb{R}^d$. Then, we can conclude by lower-semicontinuity of the L^p norm.

We now prove (4.6) by contradiction, following the lines of [8, Lemma 3.3]. Assume (4.6) to be false. Then, we can find a sequence F_k with $|F_k|=1$, converging to some F, with $|F|=1$, such that

$$\frac{1}{k}=\frac{1}{k}|F_k|^p>\int_{\partial_D\Omega}|F_kx-\zeta_{F_k}|^p d\mathcal{H}^{d-1}(x)\ge 0,$$

where ζ_{F_k} is the minimum of (4.5) associated to F_k. Arguing as before we find that the sequence ζ_{F_k} is bounded, and we can assume with no loss of generality that it admits a limit ζ. By continuity we thus infer that

$$\int_{\partial_D\Omega}|Fx-\zeta|^p d\mathcal{H}^{d-1}(x)=0,$$

from which it follows that $Fx=\zeta$ for $\mathcal{H}^{d-1}$-a.e. in $\partial_D\Omega$. By continuity and linearity we deduce that $Fx=\zeta$ on $\mathrm{aff}(\partial_D\Omega)=\mathbb{R}^d$ (by hypothesis (2.8)) which entails $\zeta=0$ and $F=0$, contradicting the fact that $|F|=1$. □

By using relation (4.6) we can now establish the following refined estimate.

Lemma 4.3 (Coercivity 2)**.** *For all $\epsilon\in(0,1)$ and $p\in[2,q_eq_h/(q_e+q_h)]$ one has*

$$\epsilon^{p-2}\|\nabla u\|^p_{L^p}\le c\mathcal{W}_\epsilon(u,z).\tag{4.7}$$

Proof. Let again $\varphi=Id+\epsilon u$ and $Q\in SO(d)$. We write

$$\begin{aligned}|\nabla\varphi-Q|^p&\le|\nabla\varphi-Q(I+\epsilon z)+\epsilon Qz|^p=|(F_{el}-Q)(I+\epsilon z)+\epsilon Qz|^p\\&\le c|F_{el}-Q|^p+c|F_{el}-Q|^p|\epsilon z|^p+c|\epsilon z|^p.\end{aligned}$$

Arguing as in the proof of Lemma 4.1 we get that

$$\operatorname{dist}^p(\nabla\varphi, SO(d)) \le c\big(\operatorname{dist}^p(F_{el}, SO(d)) + \operatorname{dist}^p(F_{el}, SO(d))|\epsilon z|^p + |\epsilon z|^p\big).$$

By integrating over Ω and using the Hölder inequality with exponents $q = q_e/p$ and $q' = q_e/(q_e - p)$ (since $p \le q_e q_h/(q_e + q_h)$, it follows by (2.19) that $r := pq' \le q_h$), we find

$$\begin{aligned}
&\int_\Omega \operatorname{dist}^p(\nabla\varphi, SO(d))dx \\
&\le c\left(\int_\Omega \operatorname{dist}^p(F_{el}, SO(d))dx + \left(\int_\Omega \operatorname{dist}^{q_e}(F_{el}, SO(d))dx\right)^{1/q} \|\epsilon z\|^p_{L^r} + \|\epsilon z\|^p_{L^p}\right) \\
&\le c\big(\mathcal{W}_{el}(u,z) + [\mathcal{W}_{el}(u,z)]^{\frac{1}{q}}[\mathcal{W}_h(I+\epsilon z)]^{\frac{1}{q'}} + \mathcal{W}_h(I+\epsilon z)\big) := B_\epsilon,
\end{aligned}$$

where we have used (2.11c), (2.12e), and (2.14) with exponent r. The Rigidity Lemma implies that there exists $\widehat{Q} \in SO(d)$ such that

$$\|\nabla\varphi - \widehat{Q}\|^p_{L^p} \le B_\epsilon. \tag{4.8}$$

Following the argument of [8] we obtain that, by letting $\zeta := \fint_\Omega(\varphi - \widehat{Q})dx$, the continuity of the trace and the Poincaré inequality imply that

$$\|\varphi - \widehat{Q}x - \zeta\|^p_{L^p(\partial\Omega)} \le c\|\varphi - \widehat{Q}x - \zeta\|^p_{W^{1,p}(\Omega)} \le cB_\epsilon.$$

As $\varphi = x$ on $\partial\Omega$ we eventually obtain

$$\|x - \widehat{Q}x - \zeta\|^p_{L^p(\partial\Omega)} \le cB_\epsilon.$$

Now, by virtue of Lemma 4.2 we get that $|I - \widehat{Q}|^p \le c|I - \widehat{Q}|^p_{\partial_D\Omega} \le c|I - \widehat{Q} - \zeta|^p$, so that $\|I - \widehat{Q}\|^p_{L^p(\partial_D\Omega)} \le cB_\epsilon$. Thanks to (4.8), this yields

$$\|\epsilon\nabla u\|^p_{L^p} \le cB_\epsilon$$

whence (4.7) follows. □

A consequence of coercivity is the relative compactness of sequences with equibounded energy.

Lemma 4.4 (Compactness)**.** *Let $(u_\epsilon, z_\epsilon) \in \mathcal{Q}$ be a sequence such that $\sup_\epsilon \mathcal{W}_\epsilon(u_\epsilon, z_\epsilon) < \infty$. Then, there exists $(u, z) \in \mathcal{Q}$ such that, up to not relabeled subsequences,*

$$u_\epsilon \rightharpoonup u \quad \text{weakly in } H^1(\Omega;\mathbb{R}^d), \tag{4.9}$$

$$z_\epsilon \rightharpoonup z \quad \text{weakly in } L^2(\Omega;\mathbb{R}^{d\times d}). \tag{4.10}$$

Moreover, if $d = 2$,

$$\operatorname{curl} z_\epsilon \rightharpoonup \operatorname{curl} z \quad \text{weakly in } L^2(\Omega;\mathbb{R}^2), \tag{4.11}$$

while, if $d = 3$,

$$(\operatorname{curl} z_\epsilon)(I + \epsilon z_\epsilon^T) \rightharpoonup \operatorname{curl} z \quad \text{weakly in } L^2(\Omega;\mathbb{R}^{3\times3}), \tag{4.12}$$

$$\operatorname{curl} z_\epsilon \rightharpoonup \operatorname{curl} z \quad \text{weakly in } L^q(\Omega;\mathbb{R}^{3\times3}), \tag{4.13}$$

with q satisfying $1/q = 1/2 + 2/q_h$.

Proof. The first two convergences are easily obtained from the coercivity of Lemma 4.1. Assume $d = 2$. By the boundedness $\|\operatorname{curl} z_\epsilon\|_{L^2} \le c$ we find $\zeta \in L^2$ such that

$$\operatorname{curl} z_\epsilon \rightharpoonup \zeta \quad \text{weakly in } L^2(\Omega;\mathbb{R}^2),$$

so that $\zeta = \operatorname{curl} z$ from condition (4.10). In the case $d = 3$, using bound (4.3) we infer convergence (4.13). To check for (4.12) we firstly establish convergence in L^s with

$$\frac{1}{s} = \frac{1}{2} + \frac{3}{q_h} = \frac{1}{q} + \frac{1}{q_h},$$

(note that $s > 1$ as $q_h > 6$ from (2.19)). To this aim it suffices to observe that $\epsilon(\operatorname{curl} z_\epsilon)z_\epsilon^T$ converges to 0 strongly in L^s, since by the Hölder inequality

$$\epsilon\|(\operatorname{curl} z_\epsilon)z_\epsilon^T\|_{L^s} \leq \epsilon\|\operatorname{curl} z_\epsilon\|_{L^q}\|z_\epsilon\|_{L^{q_h}} \leq c\epsilon^{2/q_h},$$

where in the last inequality we have used (4.1). Now convergence (4.12) follows from the fact that $(\operatorname{curl} z_\epsilon)(I + \epsilon z_\epsilon^T)$ is bounded in L^2 and from the uniqueness of the limit. □

5. Γ-lim inf inequalities

As last preparatory step, we prove in this section some inequalities, to be used in the proof of Theorems 3.1 and 3.3.

Under assumptions (2.11e) and (2.12e), the convergence of energy densities holds, as proved in [34, Lemma 3.2].

Lemma 5.1 (Convergence of densities)**.** *We have*

$$W_{el}^\epsilon \to |\cdot|_{\mathbb{C}}^2 \quad \text{and} \quad \tilde W_h^\epsilon \to |\cdot|_{\mathbb{H}}^2 \quad \text{locally uniformly on } \mathbb{R}^{d\times d}, \tag{5.1}$$

$$|z|_{\mathbb{H}}^2 \leq \inf\{\liminf_{\epsilon\to 0} \tilde W_h^\epsilon(z_\epsilon) : z_\epsilon \to z\}. \tag{5.2}$$

We now use assumption (2.15e) in order to prove a convergence lemma for the dislocation-tensor density function $\tilde V_\epsilon$.

Lemma 5.2 (Convergence of $\tilde V_\epsilon$)**.** *The function $\tilde V_\epsilon$ in* (2.16) *converges locally uniformly on $\mathbb{R}^2 \times \mathbb{R}^{2\times 2}$ (if $d = 2$) and on $\mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3}$ (if $d = 3$) to the function $f(\xi, z) := \mu|\xi|_{\mathbb{K}}^2$ as $\epsilon \to 0$.*

Proof. We prove the result in the case $d = 3$, the case $d = 2$ being similar. Let the compact $K \subset \mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3}$ be given and assume that $(\xi, z) \in K$. Let $\delta > 0$ be fixed and let $c_v(\delta)$ be the corresponding value in (2.18). We estimate

$$\big|\tilde V_\epsilon(\xi, z) - \mu|\xi|_{\mathbb{K}}^2\big| \leq \big|\tilde V_\epsilon(\xi, z) - \mu|\xi(I + \epsilon z)^T|_{\mathbb{K}}^2\big| + \mu\big||\xi(I + \epsilon z)^T|_{\mathbb{K}}^2 - |\xi|_{\mathbb{K}}^2\big|.$$

From (2.18), for ϵ small enough so that $\epsilon K \subset B_{c_v(\delta)}(0)$, we have

$$\big|\tilde V_\epsilon(\xi, z) - \mu|\xi(I + \epsilon z)^T|_{\mathbb{K}}^2\big| \leq \delta|\xi(I + \epsilon z)^T|_{\mathbb{K}}^2.$$

On the other hand

$$\big||\xi(I + \epsilon z)^T|_{\mathbb{K}}^2 - |\xi|_{\mathbb{K}}^2\big| \leq \epsilon c|\xi|^2(|z| + \epsilon|z|^2).$$

In particular, the local uniform convergence as $\epsilon \to 0$ follows by arbitrariness of δ. □

Assume now $(u_\epsilon, z_\epsilon) \in \mathcal{Q}$ be a sequence satisfying the hypotheses of the compactness Lemma 4.4 and let $(u, z) \in \mathcal{Q}$ be a limit for the convergences (4.9) and (4.10). Let us set

$$w_\epsilon := \frac{1}{\epsilon}((I + \epsilon z_\epsilon)^{-1} - I + \epsilon z_\epsilon) = \epsilon(I + \epsilon z_\epsilon)^{-1} z_\epsilon^2, \tag{5.3}$$

in such a way that $(I + \epsilon z_\epsilon)^{-1} = I - \epsilon z_\epsilon + \epsilon w_\epsilon$. Thanks to the coercivity (4.1) it follows that

$$\|\epsilon w_\epsilon\|_{L^{q_h/2}} \leq c. \tag{5.4}$$

As $q_h/2 \geq 3$ we obtain
$$\|\epsilon w_\epsilon\|_{L^3} \leq c.$$
From the second equality in (5.3) we estimate by the Hölder inequality
$$\|w_\epsilon\|_{L^1} \leq \epsilon \|z_\epsilon\|_{L^2} \|z_\epsilon\|_{L^p} \|(I+\epsilon z_\epsilon)^{-1}\|_{L^{q_h/2}} \leq c\epsilon^{2/p} \leq c\epsilon^{1/3},$$
where $1/p + 2/q_h = 1/2$, i.e. $p = 2q_h/(q_h-4) \leq q_h$ (since $q_h \geq 6$ by (2.19)), hence $w_\epsilon \to 0$ in L^1. We can now interpolate $\|w_\epsilon\|_{L^2} \leq \|w_\epsilon\|^{\alpha}_{L^1} \|w_\epsilon\|^{1-\alpha}_{L^3}$ with $\alpha = 1/4$ in order to obtain
$$\|w_\epsilon\|_{L^2} \leq c\epsilon^{\alpha/3} \epsilon^{-3(1-\alpha)/4} = c\epsilon^{-2/3}. \tag{5.5}$$
Finally, since ϵw_ϵ converges to zero strongly in L^1 and boundedness (5.4) holds, we easily see that
$$\epsilon w_\epsilon \to 0 \quad \text{strongly in } L^q, \ \text{ for all } q < \frac{q_h}{2}. \tag{5.6}$$

We introduce the tensors
$$A_\epsilon := \frac{1}{\epsilon}(F_{el,\epsilon} - I) = \frac{1}{\epsilon}\big((I+\epsilon \nabla u_\epsilon)(I+\epsilon z_\epsilon)^{-1} - I\big),$$
and, writing $(I+\epsilon z_\epsilon)^{-1} = I - \epsilon z_\epsilon + \epsilon w_\epsilon$, we obtain
$$A_\epsilon = \frac{1}{\epsilon}\big((I+\epsilon\nabla u_\epsilon)(I - \epsilon z_\epsilon + \epsilon w_\epsilon) - I\big) = \nabla u_\epsilon - z_\epsilon + w_\epsilon - \epsilon(\nabla u_\epsilon z_\epsilon - \nabla u_\epsilon w_\epsilon). \tag{5.7}$$
Let us consider $v_\epsilon := \epsilon(\nabla u_\epsilon z_\epsilon - \nabla u_\epsilon w_\epsilon)$ in order to check that
$$\|v_\epsilon\|_{L^1} \leq \epsilon \|\nabla u_\epsilon\|_{L^2}(\|z_\epsilon\|_{L^2} + \|w_\epsilon\|_{L^2}) \leq c\epsilon^{1/3},$$
where we have used (4.1) and (5.5). Owing to the fact that $w_\epsilon \to 0$ strongly in L^1 and to the convergences (4.9) and (4.10) we finally infer
$$A_\epsilon \rightharpoonup \nabla u - z \quad \text{weakly in } L^1(\Omega;\mathbb{R}^{d\times d}). \tag{5.8}$$

Thanks to Lemmas 5.1 and 5.2 we are now in position to prove the following Γ-lim inf inequalities for the energy and for the dissipation.

Lemma 5.3 (Γ-lim inf for the energy)**.** *For all $(u,z) \in \mathcal{Q}$ we have*
$$\mathcal{W}_0(u,z) \leq \inf\{\liminf_{\epsilon\to 0} \mathcal{W}_\epsilon(u_\epsilon, z_\epsilon) : (u_\epsilon, z_\epsilon) \rightharpoonup (u,z) \ \text{ weakly in } \mathcal{Q}\}.$$

Proof. To prove the lemma we follow the argument in [34, Lemma 3.3]. This relies in the lower-semicontinuity result [34, Lemma 4.2]. Using (4.10) and (5.8) and the convergences (5.1)-(5.2) the Γ-liminf inequalities
$$\int_\Omega |\nabla u - z|^2_{\mathbb{C}} dx \leq \liminf_{\epsilon\to 0} \int_\Omega W^\epsilon_{el}(A_\epsilon) dx = \liminf_{\epsilon\to 0} \frac{1}{\epsilon^2}\int_\Omega W_{el}((I+\epsilon\nabla u_\epsilon)(I+\epsilon z_\epsilon)^{-1}) dx,$$
$$\int_\Omega |z|^2_{\mathbb{H}} dx \leq \liminf_{\epsilon\to 0} \int_\Omega W^\epsilon_h(z_\epsilon) dx = \liminf_{\epsilon\to 0} \frac{1}{\epsilon^2}\int_\Omega W_h(I+\epsilon z_\epsilon) dx,$$
are readily checked. Arguing similarly, owing to Lemma 5.2 and using (4.11) and (4.12), we also find
$$\int_\Omega |\operatorname{curl} z|^2_{\mathbb{K}} dx \leq \liminf_{\epsilon\to 0} \int_\Omega V^\epsilon(G_\epsilon) dx = \liminf_{\epsilon\to 0} \frac{1}{\epsilon^2}\int_\Omega V(\epsilon(\operatorname{curl} z_\epsilon)(I+\epsilon z_\epsilon)^T) dx,$$
and the result follows. □

The convergence of the dissipation term is proved in [34, Lemma 3.5] which we report here for completeness.

Lemma 5.4 (Γ-lim inf for the dissipation)**.** *We have*

$$\mathcal{D}_0(z,\widehat{z}) \leq \inf\{\liminf_{\epsilon\to 0}\mathcal{D}_\epsilon(z_\epsilon,\widehat{z}_\epsilon) : (z_\epsilon,\widehat{z}_\epsilon) \rightharpoonup (z,\widehat{z}) \ \ \textit{weakly in } L^2\}.$$

6. Proof of Theorem 3.1

This section is devoted to checking that $\mathcal{F}_\epsilon(\cdot;\bar{z})$ Γ-converge to $\mathcal{F}_0(\cdot;\bar{z})$ with respect to the weak topology of $\mathcal{Q}$. This amounts to proving the two classical conditions

(Γ-lim inf inequality)

$$\forall (u_\epsilon,z_\epsilon) \rightharpoonup (u,z) \text{ weakly in } \mathcal{Q}: \ \mathcal{F}_0(u,z;\bar{z}) \leq \liminf_{\epsilon\to 0}\mathcal{F}_\epsilon(u_\epsilon,z_\epsilon;\bar{z}), \tag{6.1}$$

(Recovery sequence)

$$\forall (\widehat{u},\widehat{z}) \in \mathcal{Q},\ \exists (\widehat{u}_\epsilon,\widehat{z}_\epsilon) \rightharpoonup (\widehat{u},\widehat{z}) \text{ weakly in } \mathcal{Q}: \ \mathcal{F}_\epsilon(\widehat{u}_\epsilon,\widehat{z}_\epsilon;\bar{z}) \to \mathcal{F}_0(\widehat{u},\widehat{z};\bar{z}). \tag{6.2}$$

In fact, the Γ-lim inf inequality follows from the lower semicontinuity results obtained in Lemmas 5.3-5.4, so that we just need to check the existence of a recovery sequence, i.e. (6.2). We proceed in steps.

Step 1: Reduction to smooth competitors. Let us first note that, in order to prove condition (6.2) it suffices to consider the case of smooth competitors $(\widehat{u},\widehat{z}) \in C^\infty_c(\Omega;\mathbb{R}^d)\times C^\infty_c(\Omega;\mathbb{R}^{d\times d})$. In fact, the general case can be then covered by the following argument: let $(\widehat{u},\widehat{z})$ be generic in $\mathcal{Q}$ and take a sequence $(\widehat{u}_k,\widehat{z}_k) \in C^\infty_c(\Omega;\mathbb{R}^d)\times C^\infty_c(\Omega;\mathbb{R}^{d\times d})$ approaching $(\widehat{u},\widehat{z})$ in $\mathcal{Q}$ and such that

$$\mathcal{F}_0(\widehat{u},\widehat{z};\bar{z}) = \lim_{k\to\infty}\mathcal{F}_0(\widehat{u}_k,\widehat{z}_k;\bar{z}). \tag{6.3}$$

Consider now, for all k, a sequence $(\widehat{u}^\epsilon_k,\widehat{z}^\epsilon_k)$ which converges to $(\widehat{u}_k,\widehat{z}_k)$ in $\mathcal{Q}$ and such that

$$\mathcal{F}_0(\widehat{u}_k,\widehat{z}_k;\bar{z}) \geq \limsup_{\epsilon\to 0}\mathcal{F}_\epsilon(\widehat{u}^\epsilon_k,\widehat{z}^\epsilon_k;\bar{z}). \tag{6.4}$$

By a diagonal extraction argument one can find a subsequence $(\widehat{u}^\epsilon_{k_\epsilon},\widehat{z}^\epsilon_{k_\epsilon})$ converging to $(\widehat{u},\widehat{z})$ in $\mathcal{Q}$ and condition (6.2) is easily checked thanks to conditions (6.3) and (6.4).

Step 2: Case $\bar{z}$ smooth. Let us first assume $\bar{z} \in C^\infty_c(\Omega;\mathbb{R}^{d\times d})$ and $(\widehat{u},\widehat{z}) \in C^\infty_c(\Omega;\mathbb{R}^d)\times C^\infty_c(\Omega;\mathbb{R}^{d\times d})$ be such that $\mathcal{F}_0(\widehat{u},\widehat{z};\bar{z}) < \infty$. We define the recovery sequence as follows:

$$\begin{aligned}
\widehat{z}_\epsilon &:= \frac{1}{\epsilon}\big(\exp\left(\epsilon(\widehat{z}-\bar{z})\right)(I+\epsilon\bar{z}) - I\big),\\
\widehat{u}_\epsilon &:= \widehat{u},\\
\widehat{G}_\epsilon &:= G(\widehat{z}_\epsilon).
\end{aligned}$$

Let us first check that $(\widehat{u}_\epsilon,\widehat{z}_\epsilon) \rightharpoonup (\widehat{u},\widehat{z})$ in $\mathcal{Q}$. By writing the exponential as aseries we find that

$$\widehat{z}_\epsilon = \widehat{z} + \epsilon(\widehat{z}-\bar{z})\bar{z} + \frac{1}{\epsilon}\left(\sum_{k=2}^\infty \frac{(\epsilon(\widehat{z}-\bar{z}))^k}{k!}\right)(I+\epsilon\bar{z}) =: \widehat{z} + \epsilon(\widehat{z}-\bar{z})\bar{z} + \widehat{R}_\epsilon.$$

We have that $\epsilon\|(\widehat{z}-\bar{z})\bar{z}\|_{L^\infty} \leq \epsilon\|\widehat{z}-\bar{z}\|_{L^\infty}\|\bar{z}\|_{L^\infty} \to 0$, whereas

$$|\widehat{R}_\epsilon| = \left|\frac{1}{\epsilon}\left(\sum_{k=2}^{\infty}\frac{(\epsilon(\widehat{z}-\bar{z}))^k}{k!}\right)(I+\epsilon\bar{z})\right| = \frac{1}{\epsilon}\left|\epsilon^2\sum_{k=2}^{\infty}\frac{\epsilon^{k-2}(\widehat{z}-\bar{z})^k}{k!} + \epsilon^3\left(\sum_{k=2}^{\infty}\frac{\epsilon^{k-2}(\widehat{z}-\bar{z})^k}{k!}\right)\bar{z}\right|$$
$$\leq \epsilon\left(|\widehat{z}-\bar{z}|^2\sum_{k=2}^{\infty}\frac{\epsilon^{k-2}|\widehat{z}-\bar{z}|^{k-2}}{k!} + \epsilon|\bar{z}||\widehat{z}-\bar{z}|^2\sum_{k=2}^{\infty}\frac{\epsilon^{k-2}|\widehat{z}-\bar{z}|^{k-2}}{k!}\right)$$
$$\leq \epsilon\left(|\widehat{z}-\bar{z}|^2\frac{1}{2}\exp(\epsilon|\widehat{z}-\bar{z}|)(1+\epsilon|\bar{z}|)\right).$$

This ensures that $\widehat{z}_\epsilon \to \widehat{z}$ strongly in L^∞.

Now, for all $\zeta \in \mathbb{R}^{d\times d}$ we know that $D(I,\exp(\zeta)) \leq R(\zeta)$, so that, from the relation $(I+\epsilon\widehat{z}_\epsilon)(I+\epsilon\bar{z})^{-1} = \exp(\epsilon(\widehat{z}-\bar{z}))$ we infer

$$\limsup_{\epsilon\to 0}\mathcal{D}_\epsilon(\bar{z},\widehat{z}_\epsilon) = \limsup_{\epsilon\to 0}\frac{1}{\epsilon}\int_\Omega D(I,\exp(\epsilon(\widehat{z}-\bar{z})))dx \leq \int_\Omega R(\widehat{z}-\bar{z})dx = \mathcal{D}_0(\bar{z},\widehat{z}).$$

Let us check that

$$\limsup_{\epsilon\to 0}\mathcal{V}^\epsilon(\widehat{G}_\epsilon) = \limsup_{\epsilon\to 0}\int_\Omega V^\epsilon(\widehat{G}_\epsilon)dx \leq \mu\int_\Omega |\mathrm{curl}\,\widehat{z}|^2_{\mathbb{K}}dx, \tag{6.5}$$

where we recall that

$$\mathcal{V}^\epsilon(\widehat{G}_\epsilon) = \int_\Omega V^\epsilon(\widehat{G}_\epsilon)dx = \frac{1}{\epsilon^2}\int_\Omega V(\mathrm{curl}\,\widehat{z}_\epsilon,\widehat{z}_\epsilon)dx.$$

The case $d=2$ being much simpler, we only analyze the more involved case $d=3$. In such a case we have

$$\widehat{G}_\epsilon := \epsilon\mathrm{curl}\,\widehat{z}_\epsilon(I+\epsilon\widehat{z}_\epsilon) = \mathrm{curl}\,\big(\exp(\epsilon(\widehat{z}-\bar{z})-I)(I+\epsilon\bar{z})\big)(I+\epsilon\bar{z})^T\exp(\epsilon(\widehat{z}-\bar{z})^T).$$

We now aim at obtaining a decomposition of $\widehat{G}_\epsilon$. In the following, we use the notation

$$\tilde{z} = \widehat{z}-\bar{z}$$

for short. Let us start by noting that

$$\mathrm{curl}\,\big(\exp(\epsilon\tilde{z})(I+\epsilon\bar{z})\big) = \mathbb{D}\mathrm{exp}\,(\epsilon\tilde{z}) : (I+\epsilon\bar{z}) + \epsilon\exp(\epsilon\tilde{z})\mathrm{curl}\,\bar{z}, \tag{6.6}$$

and setting

$$l_\epsilon := \mathbb{D}\mathrm{exp}\,(\epsilon\tilde{z}) : \bar{z},$$

we can also write, by using (2.5),

$$\mathbb{D}\mathrm{exp}\,(\epsilon\tilde{z}) : (I+\epsilon\bar{z}) = \mathrm{curl}\,\big(\exp(\epsilon\tilde{z})\big) + \epsilon l_\epsilon. \tag{6.7}$$

Let us now estimate the term l_ϵ. By remembering that

$$(\mathbb{D}\mathrm{exp}\,(\epsilon\tilde{z}) : \bar{z})_{ij} = \varepsilon_{j\ell k}\exp(\epsilon\tilde{z})_{iq,\ell}(\bar{z})_{qk}$$

we can control l_ϵ as

$$|l_\epsilon| \leq \sum_\ell |\partial_\ell\exp(\epsilon\tilde{z})||\bar{z}| \leq \sum_\ell\left|\sum_{k=0}^{\infty}\frac{\epsilon^k\partial_\ell\tilde{z}^k}{k!}\right||\bar{z}|$$
$$\leq \epsilon\left|\sum_{k=1}^{\infty}\frac{\epsilon^{k-1}\tilde{z}^{k-1}}{(k-1)!}\right||\nabla\tilde{z}||\bar{z}| = \epsilon e^{|\epsilon\tilde{z}|}|\nabla\tilde{z}||\bar{z}|.$$

Hence $\|l_\epsilon\|_{L^\infty} \leq c\epsilon$. As for the term $\operatorname{curl}(\exp(\epsilon\tilde z))$ in (6.7) we expand the exponential in order to obtain that

$$\begin{aligned}\operatorname{curl}(\exp(\epsilon\tilde z)) &= \epsilon\operatorname{curl}\tilde z + \sum_{k=2}^{\infty}\frac{\epsilon^k}{k!}\operatorname{curl}(\tilde z^k)\\ &= \epsilon\operatorname{curl}\tilde z + \sum_{k=2}^{\infty}\frac{\epsilon^k}{k!}\big(\mathbb{D}(\tilde z^{k-1}):\tilde z + \tilde z^{k-1}\operatorname{curl}\tilde z\big)\\ &= \epsilon\operatorname{curl}\tilde z + \epsilon h_\epsilon + (\exp(\epsilon\tilde z) - I - \epsilon\tilde z)\tilde z^{-1}\operatorname{curl}\tilde z = \epsilon\operatorname{curl}\tilde z + \epsilon h_\epsilon + \epsilon m_\epsilon, \qquad (6.8)\end{aligned}$$

where we have used (2.2) and set

$$h_\epsilon := \sum_{k=2}^{\infty}\frac{\epsilon^{k-1}}{k!}\mathbb{D}(\tilde z^{k-1}):\tilde z, \qquad m_\epsilon = \frac{1}{\epsilon}(\exp(\epsilon\tilde z) - I - \epsilon\tilde z)\tilde z^{-1}\operatorname{curl}\tilde z.$$

We now estimate h_ϵ as

$$(h_\epsilon)_{ij} = \sum_{k=2}^{\infty}\frac{\epsilon^{k-1}}{k!}\varepsilon_{jlk}(\tilde z^{k-1})_{iq,l}\tilde z_{qh},$$

and we can write

$$|h_\epsilon| \leq \sum_{k=2}^{\infty}\frac{\epsilon^{k-1}}{(k-1)!}\frac{k-1}{k}|\nabla\tilde z||\tilde z|^{k-1} \leq \sum_{k=2}^{\infty}\frac{\epsilon^{k-1}|\tilde z|^{k-1}}{(k-1)!}|\nabla\tilde z| \leq (e^{|\epsilon\tilde z|}-1)|\nabla\tilde z| \leq c\epsilon.$$

In particular, $h_\epsilon \to 0$ strongly in L^∞. On the other hand,

$$|m_\epsilon| \leq \frac{1}{\epsilon}\left|\sum_{k=2}^{\infty}\frac{\epsilon^k\tilde z^{k-1}}{k!}\right||\operatorname{curl}\tilde z| \leq \epsilon|\tilde z|e^{|\epsilon\tilde z|}|\operatorname{curl}\tilde z| \leq c\epsilon,$$

so that $m_\epsilon \to 0$ strongly in L^∞ as well. All in all, we have obtained

$$l_\epsilon,\ h_\epsilon,\ m_\epsilon \to 0 \quad \text{strongly in } L^\infty,$$

and, going back to (6.6) and using (6.7) and (6.8), we can rewrite

$$\begin{aligned}\widehat G_\epsilon &= \epsilon(h_\epsilon + l_\epsilon + m_\epsilon + \operatorname{curl}\tilde z + \exp(\epsilon\tilde z)\operatorname{curl}\bar z)(I+\epsilon\bar z)^T\exp(\epsilon\tilde z)^T\\ &= \epsilon(r_\epsilon + \operatorname{curl}\tilde z + \exp(\epsilon\tilde z)\operatorname{curl}\bar z)(I+\epsilon\bar z)^T\exp(\epsilon\tilde z)^T,\end{aligned}$$

where

$$r_\epsilon = h_\epsilon + l_\epsilon + m_\epsilon.$$

In particular, $\epsilon^{-1}\widehat G_\epsilon \to \operatorname{curl}\widehat z$ holds strongly in L^∞ and

$$\|\widehat G_\epsilon\|_{L^\infty} \leq c\epsilon. \qquad (6.10)$$

We now turn to the proof of the limsup relation (6.5). By (6.10) and (2.15e) we find that, for any $\delta > 0$,

$$\int_\Omega V^\epsilon(\widehat G_\epsilon)dx \leq \frac{\mu+\delta}{\epsilon^2}\int_\Omega |\widehat G_\epsilon|^2_{\mathbb{K}}dx.$$

Passing to the limsup and using the strong convergence of $\epsilon^{-1}\widehat G_\epsilon$ to $\operatorname{curl}\widehat z$ we infer

$$\limsup_{\epsilon\to 0}\mathcal{V}^\epsilon(\widehat G_\epsilon) \leq (\mu+\delta)\int_\Omega |\operatorname{curl}\widehat z|^2_{\mathbb{K}}dx$$

and condition (6.5) follows as $\delta > 0$ is arbitrary. An analogous argument using the strong convergence $\widehat{z}_\epsilon \to \widehat{z}$ in L^∞ entails that

$$\limsup_{\epsilon\to 0} \frac{1}{\epsilon^2}\int_\Omega W_h(I+\epsilon\widehat{z}_\epsilon)dx \le \frac{1}{2}\int_\Omega |\widehat{z}|^2_{\mathbb{H}}dx.$$

Let us now address the convergence of the elastic part of the energy, namely

$$\limsup_{\epsilon\to 0} \frac{1}{\epsilon^2}\int_\Omega W_{el}((I+\epsilon\nabla\widehat{u})(I+\epsilon\widehat{z}_\epsilon)^{-1})dx \le \frac{1}{2}\int_\Omega |\nabla\widehat{u}-\widehat{z}|^2_{\mathbb{C}}dx.$$

To this aim, following the same strategy used for proving (6.5), it suffices to show that

$$\widehat{A}_\epsilon := \frac{1}{\epsilon}\big((I+\epsilon\nabla\widehat{u})(I+\epsilon\widehat{z}_\epsilon)^{-1} - I\big) \to (\nabla\widehat{u}-\widehat{z}) \quad \text{strongly in } L^\infty. \tag{6.11}$$

Using the notation from (5.7), we find

$$\widehat{A}_\epsilon = \nabla\widehat{u} - \widehat{z}_\epsilon + \widehat{w}_\epsilon - \epsilon(\nabla\widehat{u}\widehat{z}_\epsilon - \nabla\widehat{u}\widehat{w}_\epsilon),$$

with $\widehat{w}_\epsilon = \epsilon(I+\epsilon\widehat{z}_\epsilon)^{-1}\widehat{z}_\epsilon^2$. The convergence (6.11) follows as soon as we prove that $\widehat{w}_\epsilon \to 0$ strongly in L^∞, so that it is sufficient to show that $(I+\epsilon\widehat{z}_\epsilon)^{-1}$ is uniformly bounded in L^∞ for ϵ small enough. This however follows from the fact that the map $\mathbb{R}^{d\times d} \ni A \mapsto A^{-1}$ is continuous in a neighborhood of the identity I and $\widehat{z}_\epsilon$ is uniformly bounded in L^∞.

Step 3: Case $\bar z$ nonsmooth. Let us drop the smoothness assumption on $\bar z$ by arguing by density. Assume $\bar z \in L^2(\Omega;\mathbb{R}^{d\times d})$ and let $(\widehat{u},\widehat{z}) \in \mathcal{Q}$ be such that $\mathcal{F}_0(\widehat{u},\widehat{z};\bar z) < \infty$ and assume $(\widehat{u},\widehat{z}) \in C^\infty_c(\Omega;\mathbb{R}^d)\times C^\infty_c(\Omega;\mathbb{R}^{d\times d})$. Choose a sequence $\bar z_k \in C^\infty_c(\Omega;\mathbb{R}^{d\times d})$ such that

$$\mathcal{D}_0(\bar z,\bar z_k) \le \frac{1}{k}.$$

Hence, by Step 1, for all fixed $k>0$ we can find a sequence $(\widehat{u}_{\epsilon,k},\widehat{z}_{\epsilon,k})$ such that

$$\begin{aligned}\mathcal{E}_0(\widehat{u},\widehat{z}) + \mathcal{D}_0(\widehat{z},\bar z_k) &= \mathcal{F}_0(\widehat{u},\widehat{z};\bar z_k) \ge \limsup_{\epsilon\to 0}\mathcal{F}_\epsilon(\widehat{u}_{\epsilon,k},\widehat{z}_{\epsilon,k};\bar z_k)\\ &= \limsup_{\epsilon\to 0}\Big(\mathcal{E}_\epsilon(\widehat{u}_{\epsilon,k},\widehat{z}_{\epsilon,k}) + \mathcal{D}_\epsilon(\widehat{z}_{\epsilon,k},\bar z_k)\Big).\end{aligned}$$

We hence write

$$\begin{aligned}\mathcal{E}_0(\widehat{u},\widehat{z}) + \mathcal{D}_0(\widehat{z},\bar z) &\ge \limsup_{\epsilon\to 0}\Big(\mathcal{E}_\epsilon(\widehat{u}_{\epsilon,k},\widehat{z}_{\epsilon,k}) + \mathcal{D}_\epsilon(\widehat{z}_{\epsilon,k},\bar z)\Big)\\ &+ \limsup_{\epsilon\to 0}\Big(\mathcal{D}_\epsilon(\widehat{z}_{\epsilon,k},\bar z_k) - \mathcal{D}_\epsilon(\widehat{z}_{\epsilon,k},\bar z)\Big) + \mathcal{D}_0(\widehat{z},\bar z) - \mathcal{D}_0(\widehat{z},\bar z_k),\end{aligned} \tag{6.12}$$

and by triangle inequalities for $\mathcal{D}_\epsilon$ and $\mathcal{D}_0$ we infer

$$\begin{aligned}|\mathcal{D}_\epsilon(\widehat{z}_{\epsilon,k},\bar z_k) - \mathcal{D}_\epsilon(\widehat{z}_{\epsilon,k},\bar z)| &\le \mathcal{D}_\epsilon(\bar z_k,\bar z),\\ |\mathcal{D}_0(\widehat{z},\bar z_k) - \mathcal{D}_0(\widehat{z},\bar z)| &\le \mathcal{D}_0(\bar z_\epsilon,\bar z) \le \frac{1}{k}.\end{aligned}$$

If we prove that

$$\limsup_{\epsilon\to 0}\mathcal{D}_\epsilon(\bar z_k,\bar z) \le \frac{2}{k}, \tag{6.13}$$

from (6.12) we conclude

$$\mathcal{E}_0(\widehat{u},\widehat{z}) + \mathcal{D}_0(\widehat{z},\bar z) \ge \limsup_{\epsilon\to 0}\Big(\mathcal{E}_\epsilon(\widehat{u}_{\epsilon,k},\widehat{z}_{\epsilon,k}) + \mathcal{D}_\epsilon(\widehat{z}_{\epsilon,k},\bar z)\Big) - \frac{3}{k}$$

and, as $k>0$ is arbitrary, we would hence infer

$$\mathcal{F}_0(\widehat{u},\widehat{z};\bar{z}) \geq \limsup_{\epsilon\to 0} \mathcal{F}_\epsilon(\widehat{u}_\epsilon,\widehat{z}_\epsilon;\bar{z}),$$

along some diagonally extracted subsequence $(\widehat{u}_\epsilon,\widehat{z}_\epsilon) = (\widehat{u}_{k_\epsilon,\epsilon},\widehat{z}_{k_\epsilon,\epsilon})$.

We are hence left to prove (6.13). Let $k>0$ be fixed and write

$$\mathcal{D}_\epsilon(\bar{z}_k,\bar{z}) = \frac{1}{\epsilon}\int_\Omega D(I+\epsilon\bar{z}_k, I+\epsilon\bar{z})dx = \frac{1}{\epsilon}\int_\Omega D(I,(I+\epsilon\bar{z})(I+\epsilon\bar{z}_k)^{-1})dx,$$

and $(I+\epsilon\bar{z})(I+\epsilon\bar{z}_k)^{-1} = I+\epsilon(\bar{z}-\bar{z}_k)+\epsilon\bar{w}_k-\epsilon^2\bar{z}(\bar{z}_k-\bar{w}_k) =: I+\epsilon\zeta_k^\epsilon$ with $\bar{w}_k = \epsilon(I+\epsilon\bar{z}_k)^{-1}\bar{z}_k^2 \to 0$ strongly in L^∞ as $\epsilon\to 0$, so that

$$\zeta_k^\epsilon \to (\bar{z}-\bar{z}_k) \quad \text{strongly in } L^2 \text{ as } \epsilon\to 0.$$

By letting $\Omega_M := \{x\in\Omega : |\zeta_k^\epsilon| > M > 0\}$, the Markov inequality and the L^2 integrability of ζ_k^ϵ entail $|\Omega_M|^{\frac{1}{2}} \leq c/M$. Hence

$$\begin{aligned}\frac{1}{\epsilon}\int_\Omega D(I,I+\epsilon\zeta_k^\epsilon)dx &= \frac{1}{\epsilon}\int_{\Omega_M} D(I,I+\epsilon\zeta_k^\epsilon)dx + \frac{1}{\epsilon}\int_{\Omega\setminus\Omega_M} D(I,I+\epsilon\zeta_k^\epsilon)dx \\ &\leq \frac{c}{\epsilon M}\left(\int_{\Omega_M}(1+|I+\epsilon\zeta_k^\epsilon)|)^2dx\right)^{1/2} + \frac{1}{\epsilon}\int_{\Omega\setminus\Omega_M} D(I,I+\epsilon\zeta_k^\epsilon)dx. \quad (6.14)\end{aligned}$$

where we have used the Schwartz inequality and estimate (2.22). Setting $M = M_\epsilon := \delta/\epsilon$ for δ small enough we see that, on $\Omega\setminus\Omega_M$, $I+\epsilon\zeta_k^\epsilon$ almost everywhere belongs to a neighborhood of the identity I, since $|\epsilon\zeta_k^\epsilon| \leq \delta$. In particular we can write $(I+\epsilon\zeta_k^\epsilon)\chi_{\Omega\setminus\Omega_M} = \exp(\epsilon T_k^\epsilon)$ for some T_k^ϵ. We now claim that there is a constant $c(\delta)>0$ with $c(\delta)\to 0$ as $\delta\to 0$ and such that

$$T_\epsilon^k = \zeta_k^\epsilon(I+Q_k^\epsilon)\chi_{\Omega\setminus\Omega_M}, \quad \text{for some } Q_k^\epsilon \in L^\infty(\Omega\setminus\Omega_M;\mathbb{R}^{d\times d}), \quad |Q_k^\epsilon| < c(\delta). \qquad (6.15)$$

In order to proceed, let us prove here an auxiliary lemma.

Lemma 6.1 (Representation of the logarithm)**.** *Let $\delta_0 > 0$ be small enough so that for all $A\in\mathbb{R}^{d\times d}$ with $|A|\leq\delta_0$ there is $B\in\mathbb{R}^{d\times d}$ such that*

$$I + A = \exp(B). \qquad (6.16)$$

Then, there exists a function $\omega = \omega(\delta) : (0,\delta_0)\to\mathbb{R}^+$ tending to 0 as $\delta\searrow 0$ with the following property: for all $\delta\in(0,\delta_0)$, for any $A\in\mathbb{R}^{d\times d}$ with $|A|\leq\delta$, then $B\in\mathbb{R}^{d\times d}$ satisfying (6.16) *can be written as $B = A(I+Q)$, for some $Q\in\mathbb{R}^{d\times d}$ satisfying $|Q|\leq\omega(\delta)$.*

Proof. Formula (6.16) corresponds to $B = \log(I+A)$. By the continuity of the logarithm at I one can find a modulus of continuity $C_1(\delta)$ (namely, $C_1(\delta)\searrow 0$ for $\delta\searrow 0$) and such that $|B|\leq C_1(\delta)$ if $|A|\leq\delta$. Writing $\exp(B)$ as a series one obtains

$$A = B + \sum_{h=2}^\infty \frac{B^h}{h!},$$

so that

$$B = A\left(I + \sum_{h=2}^\infty \frac{B^{h-1}}{h!}\right)^{-1}.$$

We have that

$$\left|\sum_{h=2}^\infty \frac{B^{h-1}}{h!}\right| \leq \sum_{h=2}^\infty \frac{|B|^{h-1}}{h!} \leq \sum_{h=1}^\infty \frac{|B|^h}{h!} \leq e^{C_1(\delta)} - 1 =: C_2(\delta),$$

and $C_2(\delta) \searrow$ as $\delta \searrow 0$. We now exploit the continuity at 0 of the function $P \mapsto (I+P)^{-1} - I$ which entails the existence of a modulus of continuity $C_3(s)$ such that if $|P| \le s$ then $Q = (I+P)^{-1} - I$ satisfies $|Q| \le C_3(s)$. Therefore, since $P := \sum_{h=2}^{\infty} \frac{B^{h-1}}{h!}$ is such that $|P| \le C_2(\delta)$ we infer that $Q := (I + \sum_{h=2}^{\infty} \frac{B^{h-1}}{h!})^{-1} - I$ satisfies $|Q| \le C_3(C_2(\delta))$. By defining $\omega(\delta) = C_3(C_2(\delta))$ the assertion follows since $B = A(I+Q)$. $\square$

By applying Lemma 6.1 with choices $A = \epsilon\zeta_k^\epsilon$, $B = \epsilon T_\epsilon^k$, and $c(\delta) = \omega(\delta)$, and observing that $|A| \le \delta$ holds uniformly on $\Omega \setminus \Omega_M$, claim (6.15) follows. From it, we infer

$$\frac{1}{\epsilon}\int_{\Omega\setminus\Omega_{M_\epsilon}} D(I, I+\epsilon\zeta_k^\epsilon)dx \le \int_{\Omega\setminus\Omega_{M_\epsilon}} R(T_k^\epsilon) \le \int_\Omega R(\zeta_k^\epsilon)\chi_{\Omega\setminus\Omega_M}dx + \int_\Omega c(\delta)R(\zeta_k^\epsilon)\chi_{\Omega\setminus\Omega_M}dx,$$

and the right-hand side converges to $(1+c(\delta))\int_\Omega R(\bar z - \bar z_k)dx$ as $\epsilon \to 0$ thanks to the Dominated Convergence theorem since $R(\zeta_k^\epsilon)\chi_{\Omega^c_{M_\epsilon}} \le \rho|\zeta_k^\epsilon| \to \rho|\bar z - \bar z_k|$ in L^1. Putting this into (6.14) and remembering that $M = M_\epsilon = \delta/\epsilon$ we get

$$\frac{1}{\epsilon}\int_\Omega D(I, I+\epsilon\zeta_k^\epsilon)dx \le \frac{c}{\delta}\left(\int_{\Omega_M}(1+|I+\epsilon\zeta_k^\epsilon)|\right)^2 dx\Big)^{1/2} + (1+c(\delta))\int_\Omega R(\bar z - \bar z_k)dx + o(1),$$

with $o(1)$ vanishing as $\epsilon \to 0$, so that, thanks to the fact that $\left(\int_{\Omega_M}(1+|I+\epsilon\zeta_k^\epsilon)|)^2 dx\right)^{\frac12} = o(1)$ since $|\Omega_M| \to 0$ as $\epsilon \to 0$, we conclude

$$\limsup_{\epsilon\to 0}\frac{1}{\epsilon}\int_\Omega D(I, I+\epsilon\zeta_k^\epsilon)dx = (1+c(\delta))\mathcal{D}_0(\bar z, \bar z_k) \le \frac{(1+c(\delta))}{k} \le \frac{2}{k},$$

where the last inequality holds true for δ small enough. This concludes the proof of (6.13), whence the assertion of Theorem 3.1 follows.

7. Proof of Theorem 3.3

In order to prove Theorem 3.3 we follow the general theory of evolutionary Γ-convergence for rate-independent systems from [33]. Given the above proved coercivity of the energy and compactness of infimizing sequences, this actually reduces in checking the two conditions:

- (Γ-liminf inequalities) For all $(u,z) \in \mathcal{Q}$ and all sequences $(u_\epsilon, z_\epsilon) \in \mathcal{Q}$ such that $(u_\epsilon, z_\epsilon) \rightharpoonup (u,z)$ weakly in $\mathcal{Q}$, it holds

$$\mathcal{W}_0(u,z) \le \liminf_{\epsilon\to 0}\mathcal{W}_\epsilon(u_\epsilon, z_\epsilon). \tag{7.1}$$

  Moreover, whenever $(z_\epsilon, \widehat z_\epsilon) \in (L^2(\Omega;\mathbb{R}^{d\times d}))^2$ converge to $(z, \widehat z)$ weakly in $(L^2(\Omega;\mathbb{R}^{d\times d}))^2$ we have

$$\mathcal{D}_0(z, \widehat z) \le \liminf_{\epsilon\to 0}\mathcal{D}_\epsilon(z_\epsilon, \widehat z_\epsilon). \tag{7.2}$$

- (Mutual recovery sequence) Let $(\widehat u_0, \widehat z_0) := (u_0, z_0) + (\tilde u, \tilde z)$ with $(\tilde u, \tilde z) \in \mathcal{C}_c^\infty(\Omega;\mathbb{R}^d) \times \mathcal{C}_c^\infty(\Omega;\mathbb{R}^{d\times d}_{\rm dev})$, $t \in [0,T]$, and $(u_\epsilon, z_\epsilon) \rightharpoonup (u,z)$ weakly in $\mathcal{Q}$ such that

$$\mathcal{E}_\epsilon(u_\epsilon, z_\epsilon) < c < \infty. \tag{7.3}$$

  Then, there exists $(\widehat u_\epsilon, \widehat z_\epsilon) \in \mathcal{Q}$ such that

$$\begin{aligned}&\limsup_{\epsilon\to 0}\big(\mathcal{E}_\epsilon(t, \widehat u_\epsilon, \widehat z_\epsilon) - \mathcal{E}_\epsilon(t, u_\epsilon, z_\epsilon)) + \mathcal{D}_\epsilon(z_\epsilon, \widehat z_\epsilon)\big)\\ &\le \big(\mathcal{E}_0(t, \widehat u_0, \widehat z_0) - \mathcal{E}_0(t, u_0, z_0)) + \mathcal{D}_0(z_0, \widehat z_0)\big).\end{aligned} \tag{7.4}$$

The same approach was followed in [34], where nevertheless no gradient term was considered. This calls here for an extension of those techniques.

Since the Γ-$\liminf$ inequalities (7.1)-(7.2) have been already established in Lemmas 5.3-5.4, respectively, we are left with the proof of the existence of a mutual recovery sequence, as in (7.4). As already remarked in Step 1 of Section 6, it is sufficient to check the existence of such mutual recovery sequence in the smooth case $(\tilde u,\tilde z)\in \mathcal{C}^\infty_c(\Omega;\mathbb{R}^d)\times \mathcal{C}^\infty_c(\Omega;\mathbb{R}^{d\times d}_{\rm dev})$, for the general case can be readily deduced by density.

Assume hence to be given $(u_\epsilon,z_\epsilon)\in\mathcal{Q}$ converging to (u_0,z_0) weakly in $\mathcal{Q}$ and such that the bound (7.3) is satisfied for some constant $c>0$. Let $\tilde u\in C^\infty_c(\Omega;\mathbb{R}^d)$ and $\tilde z\in C^\infty_c(\Omega;\mathbb{R}^{d\times d}_{\rm dev})$. We will provide a recovery sequence $\widehat u_\epsilon,\widehat z_\epsilon$ for the limit state

$$\widehat u_0 = u_0+\tilde u,$$
$$\widehat z_0 = z_0+\tilde z.$$

Let $\varphi_\epsilon := id+\epsilon u_\epsilon$ and $\psi_\epsilon := id+\epsilon\tilde u$. By letting the symbol $\circ$ denote composition, we define the mutual recovery sequence by

$$\widehat u_\epsilon = \frac{1}{\epsilon}(\psi_\epsilon\circ\varphi_\epsilon - id) = u_\epsilon + \tilde u\circ\varphi_\epsilon, \tag{7.5}$$

$$\widehat z_\epsilon = \frac{1}{\epsilon}\big(\exp(\epsilon\tilde z)(I+\epsilon z_\epsilon)-I\big). \tag{7.6}$$

Notice that

$$I+\epsilon\widehat z_\epsilon = \exp(\epsilon\tilde z)(I+\epsilon z_\epsilon).$$

Let us define also

$$\widehat G_\epsilon := G(I+\epsilon\widehat z_\epsilon),\qquad G_\epsilon := G(I+\epsilon z_\epsilon),$$

where G is defined in (2.6). Namely, it turns out

$$\widehat G_\epsilon = \epsilon\,\mathrm{curl}\,\widehat z_\epsilon,\quad G_\epsilon=\epsilon\,\mathrm{curl}\,z_\epsilon,\qquad \text{if } d=2,$$
$$\widehat G_\epsilon = \epsilon\,\mathrm{curl}\,\widehat z_\epsilon(I+\epsilon\widehat z_\epsilon)^T,\quad G_\epsilon=\epsilon\,\mathrm{curl}\,z_\epsilon(I+\epsilon z_\epsilon)^T,\qquad \text{if } d=3.$$

We are going to prove that the pair $(\widehat u_\epsilon,\widehat z_\epsilon)$ satisfies condition (7.4). This will follow by separately proving the following inequalities:

$$\limsup_{\epsilon\to0}\Big(\int_\Omega W^\epsilon_{el}(\widehat A_\epsilon)-W^\epsilon_{el}(A_\epsilon)dx\Big)\le\int_\Omega|\nabla\widehat u_0^{sym}-\widehat z_0^{sym}|^2_{\mathbb{C}}dx-\int_\Omega|\nabla u_0^{sym}-z_0^{sym}|^2_{\mathbb{C}}dx, \tag{7.7a}$$

$$\limsup_{\epsilon\to0}\Big(\int_\Omega W^\epsilon_{h}(\widehat z_\epsilon)-W^\epsilon_{h}(z_\epsilon)dx\Big)\le\int_\Omega|\widehat z_0|^2_{\mathbb{H}}dx-\int_\Omega|z_0|^2_{\mathbb{H}}dx, \tag{7.7b}$$

$$\limsup_{\epsilon\to0}\Big(\int_\Omega V^\epsilon(\widehat G_\epsilon)-V^\epsilon(G_\epsilon)dx\Big)\le\int_\Omega|\mathrm{curl}\,\widehat z_0|^2_{\mathbb{K}}dx-\int_\Omega|\mathrm{curl}\,z_0|^2_{\mathbb{K}}dx, \tag{7.7c}$$

$$\limsup_{\epsilon\to0}\mathcal{D}_\epsilon(z_\epsilon,\widehat z_\epsilon)\le\mathcal{D}_0(z_0,\widehat z_0). \tag{7.7d}$$

Before proceeding to the proof of the limsup inequalities (7.7), let us prepare a lemma.

Lemma 7.1 (Admissibility and convergence)**.** *Given* (7.5)-(7.6) *there exists* $\epsilon_0>0$ *such that, for all* $\epsilon<\epsilon_0$ *it holds*

$$\det(I+\epsilon\nabla\widehat u_\epsilon)>0,\qquad I+\epsilon\widehat z_\epsilon\in SL(d)\ \ \text{a.e. on }\Omega. \tag{7.8}$$

Moreover as $\epsilon \to 0$, we have

$$\widehat{u}_\epsilon \rightharpoonup \widehat{u}_0 = u_0 + \tilde{u} \text{ weakly in } H^1(\Omega;\mathbb{R}^d), \tag{7.9a}$$

$$(\nabla\widehat{u}_\epsilon - \nabla u_\epsilon) \to \nabla\tilde{u} \quad \text{strongly in } L^p(\Omega;\mathbb{R}^{d\times d}) \quad \text{for } p = \frac{q_e q_h}{q_e + q_h}, \tag{7.9b}$$

$$\widehat{z}_\epsilon \rightharpoonup \widehat{z}_0 := \tilde{z} + z_0, \text{ weakly in } L^2(\Omega;\mathbb{R}^{d\times d}). \tag{7.9c}$$

Proof. We have $I + \epsilon\nabla\widehat{u}_\epsilon = \nabla\psi_\epsilon(\varphi_\epsilon)\cdot\nabla\varphi_\epsilon = (I + \epsilon\nabla\tilde{u}(\varphi_\epsilon))(I + \epsilon\nabla u_\epsilon)$, so that, using the fact that $\det(I + \epsilon\nabla u_\epsilon) > 0$, we infer that for ϵ small enough $\det(I + \epsilon\nabla\widehat{u}_\epsilon) > 0$ as well.

As for $I + \epsilon\widehat{z}_\epsilon$, its determinant is equal to $\det(I + \epsilon\widehat{z}_\epsilon) = \det(\exp(\epsilon\tilde{z}))\det(I + \epsilon z_\epsilon)$. Since $\det(\exp(\epsilon\tilde{z})) = \exp(\operatorname{tr}(\epsilon\tilde{z})) = \exp(0) = 1$ and $I + \epsilon z_\epsilon \in SL(d)$ we conclude that $\det(I + \epsilon\widehat{z}_\epsilon) = 1$, namely $I + \epsilon\widehat{z}_\epsilon \in SL(d)$, so that (7.8) holds.

Let us now prove the convergences (7.9). It is easily seen that

$$\nabla\widehat{u}_\epsilon = \nabla u_\epsilon + \nabla\tilde{u}(\varphi_\epsilon) + \epsilon\nabla\tilde{u}(\varphi_\epsilon)\nabla u_\epsilon,$$

from which, taking $1 < p < q_e$,

$$\|(\nabla\widehat{u}_\epsilon - \nabla u_\epsilon) - \nabla\tilde{u}\|_{L^p} \le \|\nabla\tilde{u}(\varphi_\epsilon) - \nabla\tilde{u}\|_{L^p} + \|\epsilon\nabla\tilde{u}(\varphi_\epsilon)\nabla u_\epsilon\|_{L^p}. \tag{7.10}$$

By the Poincaré inequality and the bound (4.7), we infer that ϵu_ϵ converges to zero in L^p. In particular, by the Lipschitz continuity of $\tilde{u}$ we deduce

$$\|\nabla\tilde{u}(\varphi_\epsilon) - \nabla\tilde{u}\|_{L^p} \le c\|\varphi_\epsilon - id\|_{L^p} = c\|\epsilon u_\epsilon\|_{L^p} \to 0.$$

Finally the boundedness in L^∞ of $\nabla\tilde{u}$ and again (4.7) imply $\|\epsilon\nabla\tilde{u}(\varphi_\epsilon)\nabla u_\epsilon\|_{L^p} \to 0$, so that from (7.10) we conclude

$$(\nabla\widehat{u}_\epsilon - \nabla u_\epsilon) \to \nabla\tilde{u} \quad \text{strongly in } L^p(\Omega;\mathbb{R}^{d\times d}), \tag{7.11}$$

for $p = q_e q_h/(q_e + q_h)$. In particular,

$$\widehat{u}_\epsilon - u_\epsilon \to \tilde{u} \quad \text{strongly in } W^{1,p}(\Omega;\mathbb{R}^d).$$

Now, using the fact that ∇u_ϵ is bounded in L^2, from convergence (7.11) we deduce that

$$\widehat{u}_\epsilon \rightharpoonup \widehat{u}_0 = u_0 + \tilde{u} \text{ weakly in } H^1(\Omega;\mathbb{R}^d).$$

It remains to show (7.9c). Let us check that

$$\widehat{z}_\epsilon - z_\epsilon \to \tilde{z} \text{ strongly in } L^{q_h}(\Omega;\mathbb{R}^{d\times d}_{\text{dev}}). \tag{7.12}$$

Indeed, we have

$$\widehat{z}_\epsilon - z_\epsilon = \frac{1}{\epsilon}(\exp(\epsilon\tilde{z}) - I)(I + \epsilon z_\epsilon),$$

and $(1/\epsilon)(\exp(\epsilon\tilde{z}) - I) \to \tilde{z}$ in $L^\infty(\Omega;\mathbb{R}^{d\times d}_{\text{dev}})$ whereas, by (4.1), $\epsilon z_\epsilon \to 0$ in $L^{q_h}(\Omega;\mathbb{R}^{d\times d}_{\text{dev}})$. Since z_ϵ converges to z_0 weakly in $L^2(\Omega;\mathbb{R}^{d\times d}_{\text{dev}})$, convergence (7.12) implies the assertion. □

By inspecting the last proof one realizes that, since $q_h > 2$, we have proved that

$$(\widehat{z}_\epsilon - z_\epsilon) \to \tilde{z} \quad \text{strongly in } L^2(\Omega;\mathbb{R}^{d\times d}_{\text{dev}}), \tag{7.13}$$

while it follows

$$(\widehat{z}_\epsilon + z_\epsilon) \rightharpoonup \widehat{z}_0 + z_0 \quad \text{weakly in } L^2(\Omega;\mathbb{R}^{d\times d}_{\text{dev}}). \tag{7.14}$$

We have already introduced the tensor A_ϵ in (5.7). Analogously, we set

$$\widehat{A}_\epsilon := \frac{1}{\epsilon}(\widehat{F}_{el,\epsilon} - I) = \frac{1}{\epsilon}\big((I + \epsilon\nabla\widehat{u}_\epsilon)(I + \epsilon\widehat{z}_\epsilon)^{-1} - I\big),$$

where we have denoted by $\widehat{F}_{el,\epsilon}$ the tensor $\widehat{F}_{el,\epsilon} = (I+\epsilon\nabla\widehat{u}_\epsilon)(I+\epsilon\widehat{z}_\epsilon)^{-1}$. By using notation (5.3), we can also write

$$\begin{aligned}\widehat{A}_\epsilon := &\frac{1}{\epsilon}\big((I+\epsilon\nabla\widehat{u}_\epsilon)(I-\epsilon z_\epsilon+\epsilon w_\epsilon)\exp(-\epsilon\tilde{z})-I\big)\\ =& (\nabla\widehat{u}_\epsilon - z_\epsilon + w_\epsilon - \epsilon\nabla\widehat{u}_\epsilon z_\epsilon + \epsilon\nabla\widehat{u}_\epsilon w_\epsilon)\exp(-\epsilon\tilde{z}) + \frac{1}{\epsilon}(\exp(-\epsilon\tilde{z})-I). \end{aligned}\tag{7.15}$$

Lemma 7.2 (Convergence of the elastic strains)**.** *We have*

$$\widehat{A}_\epsilon - A_\epsilon \to \nabla\tilde{u}-\tilde{z} \quad \textit{strongly in } L^p \quad \textit{for all } p<\frac{q_e q_h}{3q_e+q_h}. \tag{7.16}$$

Moreover, letting $s := 2q_h/(6+q_h)$, then

$$\widehat{A}_\epsilon + A_\epsilon \rightharpoonup (\nabla\widehat{u}_0-\widehat{z}_0)+(\nabla u_0 - z_0) \qquad \textit{weakly in } L^s. \tag{7.17}$$

As a consequence, thanks to (2.19)*, by the Hölder inequality*

$$(\widehat{A}_\epsilon - A_\epsilon):\mathbb{C}(\widehat{A}_\epsilon + A_\epsilon) \rightharpoonup (\nabla\tilde{u}-\tilde{z}):\mathbb{C}(\nabla(\widehat{u}_0+u_0)+(\widehat{z}_0-z_0)) \quad \textit{weakly } L^1. \tag{7.18}$$

Proof. We have

$$\begin{aligned}\widehat{A}_\epsilon - A_\epsilon =&(\nabla\widehat{u}_\epsilon-\nabla u_\epsilon)(I-\epsilon z_\epsilon+\epsilon w_\epsilon)+\frac{1}{\epsilon}(\exp(-\epsilon\tilde{z})-I)\\ &+(\nabla\widehat{u}_\epsilon - z_\epsilon + w_\epsilon - \epsilon\nabla\widehat{u}_\epsilon z_\epsilon + \epsilon\nabla\widehat{u}_\epsilon w_\epsilon)(\exp(-\epsilon\tilde{z})-I).\end{aligned}\tag{7.19}$$

Let us first estimate the norm of $(\exp(-\epsilon\tilde{z})-I)$. By expanding the exponential we have

$$\exp(-\epsilon\tilde{z})-I = \sum_{k=1}^{\infty}\frac{(-\epsilon)^k}{k!}\tilde{z}^k = -\epsilon\tilde{z}\sum_{k=0}^{\infty}\frac{(-\epsilon)^k}{(k+1)!}\tilde{z}^k,$$

so that, taking the L^∞ norm and using that $\|\tilde{z}\|_{L^\infty}\le c$ we find

$$\|\exp(-\epsilon\tilde{z})-I\|_{L^\infty} \le \epsilon\|\tilde{z}\|_{L^\infty}\sum_{k=0}^{\infty}\frac{\epsilon^k}{(k+1)!}\|\tilde{z}\|_{L^\infty}^k \le \epsilon c\sum_{k=0}^{\infty}\frac{\epsilon^k}{k!}c^k \le c\epsilon.$$

This allows us to prove that the last term in (7.19) converges to 0 strongly in L^q for any $q\in[1,q_h/2)$. Indeed, we have $\|\nabla\widehat{u}_\epsilon\|_{L^\infty}\le c$ and

$$\|z_\epsilon\|_{L^q}\le c\epsilon^{(2-q)/q},$$

by (4.1), so that $\epsilon z_\epsilon \to 0$ in L^q. Moreover, $\epsilon w_\epsilon \to 0$ by (5.6) and

$$\|\epsilon\nabla\widehat{u}_\epsilon z_\epsilon + \epsilon\nabla\widehat{u}_\epsilon w_\epsilon\|_{L^q} \le \|\nabla\widehat{u}_\epsilon\|_{L^\infty}\|\epsilon w_\epsilon + \epsilon z_\epsilon\|_{L^q}\to 0,$$

Notice that

$$\frac{q_h}{2}\ge\frac{q_e q_h}{2q_e+q_h}>\frac{q_e q_h}{3q_e+q_h},$$

so that the convergence of the last term in (7.19) occurs in L^p for any $p<q_e q_h/(2q_e+q_h)$. In particular, since we easily check that $(1/\epsilon)(\exp(-\epsilon\tilde{z})-I)\to -\tilde{z}$ in L^p, in order to prove (7.16) it remains to show that $(\nabla\widehat{u}_\epsilon-\nabla u_\epsilon)(I-\epsilon z_\epsilon+\epsilon w_\epsilon)\to\nabla\tilde{u}$ in L^p. Let us rewrite

$$(\nabla\widehat{u}_\epsilon-\nabla u_\epsilon)(I-\epsilon z_\epsilon+\epsilon w_\epsilon) = (\nabla\widehat{u}_\epsilon-\nabla u_\epsilon)+(\nabla\widehat{u}_\epsilon-\nabla u_\epsilon)(\epsilon w_\epsilon-\epsilon z_\epsilon).$$

By (7.9b), we have that the first addend in the right-hand side converges to $\nabla\tilde{u}$ in L^p (since $q_e q_h/(q_e+q_h) > q_e q_h(3q_e+q_h))$. Let us prove that $(\nabla\widehat{u}_\epsilon-\nabla u_\epsilon)(\epsilon w_\epsilon-\epsilon z_\epsilon)$ converges to 0 in

L^p. In fact, we can estimate $\nabla\widehat{u}_\epsilon - \nabla u_\epsilon$ in $L^{q_e q_h/(q_e+q_h)}$, as

$$\begin{aligned}\|\nabla\widehat{u}_\epsilon - \nabla u_\epsilon\|_{L^{\frac{q_e q_h}{q_e+q_h}}} &= \|\nabla\tilde{u}(\varphi_\epsilon) + \epsilon\nabla\tilde{u}(\varphi_\epsilon)\nabla u_\epsilon\|_{L^{\frac{q_e q_h}{q_e+q_h}}} \\ &\le \|\nabla\tilde{u}(\varphi_\epsilon)\|_{L^\infty} + \epsilon\|\nabla\tilde{u}(\varphi_\epsilon)\|_{L^\infty}\|\nabla u_\epsilon\|_{L^{\frac{q_e q_h}{q_e+q_h}}} \le c,\end{aligned}$$

where we have used (4.7). On the other hand, we have already checked that $\epsilon w_\epsilon - \epsilon z_\epsilon$ converges to 0 strongly in L^r for all $r < q_h/2$. Then, by the Hölder inequality, it follows that $(\nabla\widehat{u}_\epsilon - \nabla u_\epsilon)(\epsilon w_\epsilon - \epsilon z_\epsilon)$ converges to 0 in L^q for all q such that $1/q > (q_e+q_h)/q_e q_h + 2/q_h$, and (7.16) follows.

Let us now prove convergence (7.17). From (5.7) and (7.15) we have

$$\begin{aligned}A_\epsilon + \widehat{A}_\epsilon =& \nabla u_\epsilon - z_\epsilon + w_\epsilon - \epsilon(\nabla u_\epsilon z_\epsilon - \nabla u_\epsilon w_\epsilon) \\ &+ (\nabla\widehat{u}_\epsilon - z_\epsilon + w_\epsilon - \epsilon\nabla\widehat{u}_\epsilon z_\epsilon + \epsilon\nabla\widehat{u}_\epsilon w_\epsilon)\exp(-\epsilon\tilde{z}) + \frac{1}{\epsilon}(\exp(-\epsilon\tilde{z}) - I).\end{aligned} \tag{7.20}$$

Since $\nabla u_\epsilon \rightharpoonup \nabla u_0$ and $z_\epsilon \rightharpoonup z_0$ weakly in L^2, and $2 > s$ we have that the same convergences hold weakly in L^s. Moreover by (7.9a) and (7.9c) we also have $\nabla\widehat{u}_\epsilon\exp(-\epsilon\tilde{z}) \rightharpoonup \nabla\widehat{u}_0$ and $\widehat{z}_\epsilon\exp(-\epsilon\tilde{z}) \rightharpoonup \widehat{z}_0$ weakly in L^s. We have to prove that all the other terms in (7.20) converge to zero weakly in L^s. First we claim that

$$w_\epsilon \rightharpoonup 0 \text{ strongly in } L^s.$$

Indeed, by (5.3) and (4.1) via the Hölder inequality we get

$$\|w_\epsilon\|_{L^s} \le \|(I+\epsilon z_\epsilon)^{-1}\|_{L^{q_h/2}}\|\epsilon z^\epsilon\|_{L^{q_h}}\|z_\epsilon\|_{L^2} \le c\epsilon^{2/q_h}.$$

Furthermore, again by (4.1) and (5.4), $\|\epsilon\nabla u_\epsilon w_\epsilon\|_{L^{2q_h/(4+q_h)}} \le \|\nabla u_\epsilon\|_{L^2}\|\epsilon w_\epsilon\|_{L^{q_h/2}} \le c$, and (5.6) implies that

$$\epsilon\nabla u_\epsilon w_\epsilon \rightharpoonup 0 \qquad \text{weakly in } L^{\frac{2q_h}{4+q_h}}.$$

Since $2q_h/(4+q_h) > s$, the latter convergence holds in L^s as well. Estimating the remaining terms is now easy. They all converge to 0 in L^s, as well. This implies that convergence (7.17) holds.

It remains to prove convergence (7.18). This is a consequence of the Hölder inequality, which holds provided that $1 \ge 1/p + 1/s$ for some $p < q_e q_h/(3q_e + q_h)$. This is equivalent to requiring

$$q_h > \frac{12q_e}{q_e - 2},$$

which is guaranteed by condition (2.19). Note that in the above argument we have used the fact that $q_e q_h/(3q_e + q_h) > 1$. This is equivalent to $q_h > 3q_e/(q_e - 1)$ and follows from $q_e > 2$ and $q_h > 6$, which follow from condition (2.19). □

Let us now go back to the proof of the limsup inequalities (7.7). Following [34], for $\delta > 0$ we introduce the sets

$$\begin{aligned}U_\epsilon^\delta &:= \{x \in \Omega : |\epsilon A_\epsilon(x)| + |\epsilon\widehat{A}_\epsilon(x)| \le c_{el}(\delta)\}, \\ Z_\epsilon^\delta &:= \{x \in \Omega : |\epsilon z_\epsilon(x)| + |\epsilon\widehat{z}_\epsilon(x)| \le c_h(\delta)\}, \\ V_\epsilon^\delta &:= \{x \in \Omega : |G_\epsilon(x)| + |\widehat{G}_\epsilon(x)| \le c_v(\delta)\},\end{aligned}$$

where the constants $c_{el}(\delta)$, $c_h(\delta)$, $c_v(\delta)$, come from (2.11e), (2.12e), and (2.15e), respectively. The measure of $\Omega \setminus U_\epsilon^\delta$ can be controlled by recalling that A_ϵ and $\widehat{A}_\epsilon$ are uniformly bounded in

L^s with $s = 2q_h/(6+q_h)$ in the following way

$$|\Omega \setminus U_\epsilon^\delta| = \int_{\Omega\setminus U_\epsilon^\delta} 1dx \le \frac{\epsilon^s}{c_{el}(\delta)^s}\int_\Omega (|A_\epsilon| + |\widehat{A}_\epsilon|)^s dx \le \frac{c\epsilon^s}{c_{el}(\delta)^s}.$$

Similarly, using the L^2 boundedness of z_ϵ and $\widehat{z}_\epsilon$, and of G_ϵ and $\widehat{G}_\epsilon$ (given by (4.1), (4.3)) one infers

$$|\Omega \setminus Z_\epsilon^\delta| \le \frac{c\epsilon^2}{c_h(\delta)^2}, \tag{7.21}$$

$$|\Omega \setminus V_\epsilon^\delta| \le \frac{c\epsilon^2}{c_v(\delta)^2}. \tag{7.22}$$

We are now in position to address the limsup inequality for the elastic part, the hardening part of the energy, and the total dissipation.

Proposition 7.3 (Convergence)**.** *The sequence* $(\widehat{u}_\epsilon, \widehat{z}_\epsilon)$ *converges to* $(\widehat{u}_0, \widehat{z}_0)$ *and satisfies* (7.7a)*,* (7.7b)*, and* (7.7d)*.*

The proof is very similar to the corresponding argument in [34]. We only sketch it and emphasize the points where (minor) differences arise.

Sketch of the proof. We first fix $\delta > 0$ and write

$$\begin{aligned} W_{el}^\epsilon(\widehat{A}_\epsilon) - W_{el}^\epsilon(A_\epsilon) &\le |\widehat{A}_\epsilon|^2_{\mathbb{C}} - |A_\epsilon|^2_{\mathbb{C}} + \delta(|\widehat{A}_\epsilon|^2_{\mathbb{C}} + |A_\epsilon|^2_{\mathbb{C}}) \\ &= \frac{1}{2}(\widehat{A}_\epsilon - A_\epsilon) : \mathbb{C}(\widehat{A}_\epsilon + A_\epsilon) + \delta(|\widehat{A}_\epsilon|^2_{\mathbb{C}} + |A_\epsilon|^2_{\mathbb{C}}). \end{aligned}$$

After defining

$$H_{1,\epsilon} := (I + \epsilon\nabla\widehat{u}_\epsilon)(I + \epsilon\nabla u_\epsilon)^{-1}, \qquad H_{2,\epsilon} := (I + \epsilon z_\epsilon)(I + \epsilon\widehat{z}_\epsilon)^{-1},$$

we follow the lines of [34, Lemma 3.6] and check that $\|H_{1,\epsilon} - I\|_{L^\infty(\Omega\setminus U_\epsilon^\delta;\mathbb{R}^{d\times d})} + \|H_{2,\epsilon} - I\|_{L^\infty(\Omega\setminus U_\epsilon^\delta;\mathbb{R}^{d\times d})} \le \epsilon c$. Therefore, arguing as in [34] we obtain

$$\begin{aligned} &\int_{\Omega\setminus U_\epsilon^\delta} W_{el}^\epsilon(\widehat{A}_\epsilon) - W_{el}^\epsilon(A_\epsilon)dx \le \frac{c}{\epsilon^2}\int_{\Omega\setminus U_\epsilon^\delta} (W_{el}(F_{el,\epsilon}) + C)(|H_{1,\epsilon} - I| + |H_{1,\epsilon} - I|)dx \\ &\le c\left(\frac{1}{\epsilon^2}\int_\Omega W_{el}(F_{el,\epsilon})dx + \frac{C}{\epsilon^2}|\Omega \setminus U_\epsilon^\delta|\right)\left(\|H_{1,\epsilon} - I\|_{L^\infty} + \|H_{2,\epsilon} - I\|_{L^\infty}\right) \\ &\le c(\epsilon + \epsilon^{s-1}). \end{aligned}$$

This allows us to focus on the limiting behavior of the elastic energy on the set U_ϵ^δ, for the remainder is negligible as $\epsilon \searrow 0$. In particular, by using [34, formula (3.25)] inequality (7.7a) follows from convergences (7.18) and the fact that $s > 1$.

Inequality (7.7b) on the hardening energy can be proved as in [34], thanks to convergences (7.13) and (7.14). Eventually, inequality (7.7d) concerning the dissipation term follows again from the analysis in [34], by adapting the argument to our case using the fact that $\tilde{z}$ has compact support. □

Given Proposition 7.3, we are just left with the proof of the limsup inequality (7.7c) concerning the dislocation-density term. We do this in the remainder of this section by distinguishing the cases $d = 2$ and $d = 3$.

We first deal with the more involved three-dimensional case.

Step 1. We have

$$\widehat{G}_\epsilon = \epsilon \mathrm{curl}\, \widehat{z}_\epsilon (I+\epsilon \widehat{z}_\epsilon)^T = \mathrm{curl}\, \big(\exp(\epsilon \tilde{z})(I+\epsilon z_\epsilon)\big)(I+\epsilon z_\epsilon)^T \exp(\epsilon\tilde{z})^T.$$

Using (2.2) we infer

$$\mathrm{curl}\,\big(\exp(\epsilon\tilde{z})(I+\epsilon z_\epsilon)\big) = \mathbb{D}\exp(\epsilon\tilde{z}) : (I+\epsilon z_\epsilon) + \epsilon \exp(\epsilon\tilde{z}) \mathrm{curl}\, z_\epsilon, \tag{7.23}$$

and, setting

$$l_\epsilon := \mathbb{D}\exp(\epsilon\tilde{z}) : z_\epsilon,$$

we can also write, by (2.5),

$$\mathbb{D}\exp(\epsilon\tilde{z}) : (I+\epsilon z_\epsilon) = \mathrm{curl}\,\big(\exp(\epsilon\tilde{z})\big) + \epsilon l_\epsilon. \tag{7.24}$$

Let us estimate l_ϵ by using $(\mathbb{D}\exp(\epsilon\tilde{z}) : z_\epsilon)_{ij} = \varepsilon_{jlk} \exp(\epsilon\tilde{z})_{iq,l} (z_\epsilon)_{qk}$ as

$$\begin{aligned} |l_\epsilon| &\leq \sum_l |\partial_l \exp(\epsilon\tilde{z})||z_\epsilon| \leq \sum_l \left| \sum_{k=1}^\infty \frac{\epsilon^k \partial_l \tilde{z}^k}{k!} \right| |z_\epsilon| \\ &\leq \epsilon \left| \sum_{k=1}^\infty \frac{\epsilon^{k-1}\tilde{z}^{k-1}}{(k-1)!} \right| |\nabla\tilde{z}||z_\epsilon| = \epsilon e^{|\epsilon\tilde{z}|} |\nabla\tilde{z}||z_\epsilon|. \end{aligned}$$

Integrating in space and using (4.1) we conclude that

$$\|l_\epsilon\|_{L^{q_h}} \leq c\epsilon^{2/q_h}. \tag{7.25}$$

As for the term $\mathrm{curl}\,(\exp(\epsilon\tilde{z}))$ in (7.24) we expand the exponential to obtain

$$\begin{aligned} \mathrm{curl}\,(\exp(\epsilon\tilde{z})) &= \epsilon \mathrm{curl}\,\tilde{z} + \sum_{k=2}^\infty \frac{\epsilon^k}{k!} \mathrm{curl}\,(\tilde{z}^k) = \epsilon\mathrm{curl}\,\tilde{z} + \sum_{k=2}^\infty \frac{\epsilon^k}{k!} \big(\mathbb{D}(\tilde{z}^{k-1}) : \tilde{z} + \tilde{z}^{k-1}\mathrm{curl}\,\tilde{z}\big) \\ &= \epsilon\mathrm{curl}\,\tilde{z} + \epsilon h_\epsilon + (\exp(\epsilon\tilde{z}) - I - \epsilon\tilde{z})\tilde{z}^{-1}\mathrm{curl}\,\tilde{z} = \epsilon\mathrm{curl}\,\tilde{z} + \epsilon h_\epsilon + \epsilon m_\epsilon, \end{aligned} \tag{7.26}$$

where we have used (2.2) and set

$$h_\epsilon := \sum_{k=2}^\infty \frac{\epsilon^{k-1}}{k!} \mathbb{D}(\tilde{z}^{k-1}) : \tilde{z}, \qquad m_\epsilon = \frac{1}{\epsilon}(\exp(\epsilon\tilde{z}) - I - \epsilon\tilde{z})\tilde{z}^{-1}\mathrm{curl}\,\tilde{z}.$$

Let us now estimate h_ϵ. Since

$$(h_\epsilon)_{ij} = \sum_{k=2}^\infty \frac{\epsilon^{k-1}}{k!} \varepsilon_{jlk} (\tilde{z}^{k-1})_{iq,l} \tilde{z}_{qh},$$

we can obtain the bound

$$|h_\epsilon| \leq \sum_{k=2}^\infty \frac{\epsilon^{k-1}}{(k-1)!} \frac{k-1}{k} |\nabla\tilde{z}||\tilde{z}|^{k-1} \leq \sum_{k=2}^\infty \frac{\epsilon^{k-1}|\tilde{z}|^{k-1}}{(k-1)!} |\nabla\tilde{z}| = (e^{|\epsilon\tilde{z}|} - 1)|\nabla\tilde{z}| \leq c\epsilon. \tag{7.27}$$

In particular the L^∞ norm of h_ϵ converges to 0. Eventually, we easily check that $m_\epsilon \to 0$ uniformly on Ω, namely,

$$|m_\epsilon| < c\epsilon. \tag{7.28}$$

Summarizing, by (7.25), (7.27), and (7.28), setting $r_\epsilon = h_\epsilon + l_\epsilon + m_\epsilon$ we obtain

$$\|r_\epsilon\|_{L^{q_h}} \leq c\epsilon^{2/q_h}, \tag{7.29}$$

and

$$r_\epsilon \to 0 \qquad \text{strongly in } L^{q_h}.$$

Going back to (7.23) and using (7.24) and (7.26) we find

$$\begin{aligned}\widehat{G}_\epsilon &= \epsilon(h_\epsilon + l_\epsilon + m_\epsilon + \operatorname{curl}\tilde z + \exp(\epsilon\tilde z)\operatorname{curl} z_\epsilon)(I+\epsilon z_\epsilon)^T\exp(\epsilon\tilde z)^T\\ &= \epsilon(r_\epsilon + \operatorname{curl}\tilde z + \exp(\epsilon\tilde z)\operatorname{curl} z_\epsilon)(I+\epsilon z_\epsilon)^T\exp(\epsilon\tilde z)^T.\end{aligned}$$

Step 2. We define for $t\in[0,1]$

$$R_\epsilon(t) := \epsilon t(r_\epsilon + \operatorname{curl}\tilde z)(I+\epsilon z_\epsilon)^T(\exp(\epsilon\tilde z))^T + \epsilon\exp(\epsilon t\tilde z)\operatorname{curl} z_\epsilon(I+\epsilon z_\epsilon)^T(\exp(\epsilon t\tilde z))^T,$$

in such a way that

$$R_\epsilon(0) = G_\epsilon, \qquad R_\epsilon(1) = \widehat{G}_\epsilon,$$

and compute

$$\begin{aligned}\partial_t R_\epsilon(t) &= \epsilon(r_\epsilon + \operatorname{curl}\tilde z)(I+\epsilon z_\epsilon)^T(\exp(\epsilon\tilde z))^T + \epsilon^2\exp(\epsilon t\tilde z)\tilde z(\operatorname{curl} z_\epsilon)(I+\epsilon z_\epsilon)^T(\exp(\epsilon t\tilde z))^T\\ &\quad + \epsilon^2\exp(\epsilon t\tilde z)(\operatorname{curl} z_\epsilon)(I+\epsilon z_\epsilon)^T(\exp(\epsilon t\tilde z))^T\tilde z^T\\ &= \epsilon(r_\epsilon + \operatorname{curl}\tilde z)(I+\epsilon z_\epsilon)^T(\exp(\epsilon t\tilde z))^T + \epsilon^2 L_\epsilon,\end{aligned}$$

where we have set

$$L_\epsilon = \exp(\epsilon t\tilde z)\tilde z(\operatorname{curl} z_\epsilon)(I+\epsilon z_\epsilon)^T(\exp(\epsilon t\tilde z))^T + \exp(\epsilon t\tilde z)(\operatorname{curl} z_\epsilon)(I+\epsilon z_\epsilon)^T(\exp(\epsilon t\tilde z))^T\tilde z^T$$

Using the L^{q_c}-boundedness of $\epsilon\operatorname{curl} z_\epsilon(I+\epsilon z_\epsilon)^T$ provided by (4.3), we can easily estimate

$$\epsilon^2\|L_\epsilon\|_{L^{q_c}} \le c\epsilon^{1+2/q_c}.$$

To estimate the remaining term in $\partial_t R_\epsilon$ we observe that, by (4.1), (7.22), and the Hölder inequality,

$$\|(I+\epsilon z_\epsilon)^T\|_{L^{q_c}(\Omega\setminus V_\epsilon^\delta)} \le |\Omega\setminus V_\epsilon^\delta|^{1/q_h}\|I+\epsilon z_\epsilon\|_{L^{q_h}} \le \frac{\epsilon^{2/q_h}}{c_v(\delta)^{2/q_h}}.$$

In particular, it turns out

$$\begin{aligned}\|\partial_t R_\epsilon(t)\|_{L^{q_c}(\Omega\setminus V_\epsilon^\delta)} &\le \epsilon c\big(\|I+\epsilon z_\epsilon\|_{L^{q_c}(\Omega\setminus V_\epsilon^\delta)} + \|r_\epsilon\|_{L^{q_h}}\|I+\epsilon z_\epsilon\|_{L^{q_h}} + \epsilon\|L_\epsilon\|_{L^{q_c}}\big)\\ &\le c(\delta)\epsilon^{1+2/q_h},\end{aligned}$$

for all $t\in[0,1]$, where we have again used that $q_h\ge 2q_c$, and where $c(\delta)>0$ is a constant independent of ϵ but which might depend on δ. As for $R_\epsilon(t)$, we similarly find that

$$\|R_\epsilon(t)\|_{L^{q_c}} \le c\epsilon^{2/q_c}, \qquad \text{for all } t\in[0,1].$$

Taking into account the control (2.15d), we finally estimate

$$\begin{aligned}\int_{\Omega\setminus V_\epsilon^\delta}\frac{1}{\epsilon^2}V(\widehat{G}_\epsilon) - \frac{1}{\epsilon^2}V(G_\epsilon)\,dx &= \frac{1}{\epsilon^2}\int_0^1\int_{\Omega\setminus V_\epsilon^\delta}\nabla V(R_\epsilon(t))\partial_t R_\epsilon(t)\,dxdt\\ &\le \frac{c}{\epsilon^2}\int_0^1\int_{\Omega\setminus V_\epsilon^\delta}(|R_\epsilon(t)|^{q_c-1}+c_{11})|\partial_t R_\epsilon(t)|\,dxdt\\ &\le \int_0^1\frac{c}{\epsilon^2}\|R_\epsilon(t)\|_{L^{q_c}}^{q_c-1}\|\partial_t R_\epsilon(t)\|_{L^{q_c}(\Omega\setminus V_\epsilon^\delta)} + \frac{c}{\epsilon^2}|\Omega\setminus V_\epsilon^\delta|^{\frac{q_c-1}{q_c}}\|\partial_t R_\epsilon(t)\|_{L^{q_c}(\Omega\setminus V_\epsilon^\delta)}\,dt\\ &\le \frac{c(\delta)}{\epsilon^2}\epsilon^{(2q_c-2)/q_c}\epsilon^{1+2/q_h} = c(\delta)\epsilon^{(q_c-2)/q_c}\epsilon^{2/q_h} \le c(\delta)\epsilon^{2/q_h},\end{aligned}\tag{7.30}$$

where we have used (7.22) and the fact that $q_c\ge 2$.

Let us now focus on the treatment of the left-hand side of (7.30). By using (2.15e) on the set V_ϵ^δ we have

$$V(\widehat{G}_\epsilon) - V(G_\epsilon) \le \mu|\widehat{G}_\epsilon|^2_{\mathbb{K}} - \mu|\widehat{G}_\epsilon|^2_{\mathbb{K}} + \delta(|\widehat{G}_\epsilon|^2_{\mathbb{K}} + |G_\epsilon|^2_{\mathbb{K}}).$$

We thus infer

$$\begin{aligned}\int_\Omega \frac{1}{\epsilon^2}V(\widehat{G}_\epsilon) - \frac{1}{\epsilon^2}V(G_\epsilon)\,dx &= \int_{V_\epsilon^\delta} \frac{1}{\epsilon^2}V(\widehat{G}_\epsilon) - \frac{1}{\epsilon^2}V(G_\epsilon)\,dx + \int_{\Omega\setminus V_\epsilon^\delta} \frac{1}{\epsilon^2}V(\widehat{G}_\epsilon) - \frac{1}{\epsilon^2}V(G_\epsilon)\,dx \\ &\le \frac{\mu}{\epsilon^2}\int_{V_\epsilon^\delta}(|\widehat{G}_\epsilon|^2_{\mathbb{K}} - |G_\epsilon|^2_{\mathbb{K}})\,dx + \delta c + \int_{\Omega\setminus V_\epsilon^\delta} \frac{1}{\epsilon^2}V(\widehat{G}_\epsilon) - \frac{1}{\epsilon^2}V(G_\epsilon)\,dx \\ &\le \frac{\mu}{\epsilon^2}\int_{V_\epsilon^\delta}(|\widehat{G}_\epsilon|^2_{\mathbb{K}} - |G_\epsilon|^2_{\mathbb{K}})\,dx + c\delta + c\epsilon^{2/q_h}.\end{aligned}$$

The limsup inequality (7.7c) will follow, as soon as we check that

$$\frac{1}{\epsilon^2}(|\widehat{G}_\epsilon|^2_{\mathbb{K}} - |G_\epsilon|^2_{\mathbb{K}}) \rightharpoonup |\mathrm{curl}\,\widehat{z}_0|^2_{\mathbb{K}} - |\mathrm{curl}\,z_0|^2_{\mathbb{K}} \quad \text{weakly in } L^1(V_\epsilon^\delta). \tag{7.31}$$

Indeed, from (7.31) one readily proves that

$$\limsup_{\epsilon\to 0}\int_\Omega \frac{1}{\epsilon^2}V(\widehat{G}_\epsilon) - \frac{1}{\epsilon^2}V(G_\epsilon)\,dx \le \mu\int_\Omega |\mathrm{curl}\,\widehat{z}_0|^2_{\mathbb{K}} - |\mathrm{curl}\,z_0|^2_{\mathbb{K}}\,dx + c\delta,$$

and (7.7c) follows as δ is arbitrary.

It hence remains to prove the claim (7.31). We write

$$\frac{1}{\epsilon^2}|\widehat{G}_\epsilon|^2_{\mathbb{K}} - \frac{1}{\epsilon^2}|G_\epsilon|^2_{\mathbb{K}} = \frac{1}{\epsilon^2}(\widehat{G}_\epsilon - G_\epsilon)\mathbb{K} : (\widehat{G}_\epsilon + G_\epsilon),$$

and recall that

$$\frac{1}{\epsilon}\widehat{G}_\epsilon = ((r_\epsilon + \mathrm{curl}\,\tilde{z}) + \exp(\epsilon\tilde{z})\mathrm{curl}\,z_\epsilon)(I + \epsilon z_\epsilon)^T \exp(\epsilon\tilde{z})^T.$$

Since $\exp(\epsilon\tilde{z})$ and $\exp(\epsilon\tilde{z})^T$ converge uniformly to I we can write $\exp(\epsilon\tilde{z}) = I + r(\epsilon)$ and $\exp(\epsilon\tilde{z})^T = I + s(\epsilon)$ with $r(\epsilon)$ and $s(\epsilon)$ converging to 0 uniformly. Therefore, we have

$$\begin{aligned}&\frac{1}{\epsilon}(\widehat{G}_\epsilon - G_\epsilon) = \\ &= (r_\epsilon + \mathrm{curl}\,\tilde{z})(I + \epsilon z_\epsilon)^T(I + s(\epsilon)) + (I + r(\epsilon))\mathrm{curl}\,z_\epsilon(I + \epsilon z_\epsilon^T)(I + s(\epsilon)) - \mathrm{curl}\,z_\epsilon(I + \epsilon z_\epsilon^T) \\ &= (r_\epsilon + \mathrm{curl}\,\tilde{z})(I + \epsilon z_\epsilon)^T(I + s(\epsilon)) + \mathrm{curl}\,z_\epsilon(I + \epsilon z_\epsilon^T)t(\epsilon),\end{aligned}$$

where $t(\epsilon)$ converges to 0 strongly in L^∞ as $\epsilon \to 0$. Since ϵz_ϵ converges to 0 strongly in L^{q_h} and $\mathrm{curl}\,z_\epsilon(I + \epsilon z_\epsilon^T)$ is bounded in L^2, thanks to the fact that $q_h > 4$ we find out that

$$\frac{1}{\epsilon}(\widehat{G}_\epsilon - G_\epsilon) \to \mathrm{curl}\,\tilde{z} \quad \text{strongly in } L^2(\Omega;\mathbb{R}^{3\times 3}). \tag{7.32}$$

On the other hand

$$\begin{aligned}\frac{1}{\epsilon}(\widehat{G}_\epsilon + G_\epsilon) = (r_\epsilon + \mathrm{curl}\,\tilde{z})(I + \epsilon z_\epsilon)^T(I + s(\epsilon)) + (I + r(\epsilon))\mathrm{curl}\,z_\epsilon(I + \epsilon z_\epsilon^T)(I + s(\epsilon)) \\ + \mathrm{curl}\,z_\epsilon(I + \epsilon z_\epsilon^T)\end{aligned}$$

so that (4.12) implies

$$\frac{1}{\epsilon}(\widehat{G}_\epsilon + G_\epsilon) \rightharpoonup \mathrm{curl}\,(\tilde{z} + z_0) + \mathrm{curl}\,z_0 = \mathrm{curl}\,(\widehat{z}_0 + z_0) \quad \text{weakly in } L^2(\Omega;\mathbb{R}^{3\times 3}). \tag{7.33}$$

This, together with (7.32), concludes the proof of (7.31). In particular, the limsup inequality (7.7c) holds for $d = 3$.

Step 3. Let us now turn to prove (7.7c) for $d = 2$. This is easier than the three-dimensional case, so that most details can be omitted. We have

$$\widehat{G}_\epsilon = \epsilon \mathrm{curl}\, \widehat{z}_\epsilon = \mathrm{curl}\, \big(\exp(\epsilon\tilde{z})(I + \epsilon z_\epsilon)\big).$$

Using (2.1) and (2.5) we infer

$$\begin{aligned}\mathrm{curl}\, \big(\exp(\epsilon\tilde{z})(I + \epsilon z_\epsilon)\big) &= \mathbb{D}\exp(\epsilon\tilde{z}) : (I + \epsilon z_\epsilon) + \epsilon \exp(\epsilon\tilde{z})\mathrm{curl}\, z_\epsilon \\ &= \mathbb{D}\exp(\epsilon\tilde{z}) : I + \epsilon l_\epsilon + \epsilon \exp(\epsilon\tilde{z})\mathrm{curl}\, z_\epsilon \\ &= \mathrm{curl}\,(\exp(\epsilon\tilde{z})) + \epsilon l_\epsilon + \epsilon\exp(\epsilon\tilde{z})\mathrm{curl}\, z_\epsilon,\end{aligned}$$

where as before $l_\epsilon := \mathbb{D}\exp(\epsilon\tilde{z}) : z_\epsilon$. The estimate for l_ϵ is similar to the one for the three-dimensional case, and leads us to (7.25). Using a similar computation as in (7.26), we infer

$$\mathrm{curl}\,(\exp(\epsilon\tilde{z})) = \epsilon h_\epsilon + \epsilon m_\epsilon + \epsilon \mathrm{curl}\, \tilde{z},$$

where h_ϵ and m_ϵ fulfill (7.27) and (7.28). Thus, setting $r_\epsilon := h_\epsilon + m_\epsilon + l_\epsilon$ we obtain

$$\widehat{G}_\epsilon = \epsilon\mathrm{curl}\, \tilde{z} + \epsilon r_\epsilon + \epsilon\exp(\epsilon\tilde{z})\mathrm{curl}\, z_\epsilon,$$

where r_ϵ satisfies (7.29). We define for $t \in [0, 1]$

$$R_\epsilon(t) := \epsilon t(r_\epsilon + \mathrm{curl}\, \tilde{z}) + \epsilon\exp(\epsilon t\tilde{z})\mathrm{curl}\, z_\epsilon,$$

so that

$$\partial_t R_\epsilon(t) = \epsilon(r_\epsilon + \mathrm{curl}\, \tilde{z}) + \epsilon^2 \exp(\epsilon t\tilde{z})\tilde{z}(\mathrm{curl}\, z_\epsilon) = \epsilon(r_\epsilon + \mathrm{curl}\, \tilde{z}) + \epsilon^2 L_\epsilon,$$

where, this time,

$$L_\epsilon = \exp(\epsilon t\tilde{z})\tilde{z}(\mathrm{curl}\, z_\epsilon).$$

Hence, by (4.2) it follows that

$$\epsilon\|L_\epsilon\|_{L^{q_c}} \leq c\epsilon^{2/q_c}.$$

We are then again led to the estimates

$$\|R_\epsilon(t)\|_{L^{q_c}} \leq c\epsilon^{2/q_c}, \quad \|\partial_t R_\epsilon(t)\|_{L^{q_c}(\Omega\setminus V_\epsilon^\delta)} \leq c(\delta)\epsilon^{1+2/q_c},$$

and we obtain the analogon of (7.30). To conclude the proof, it remains to verify that also in this case convergences (7.32) and (7.33) hold. This is now an easy check since

$$\begin{aligned}\frac{1}{\epsilon}(\widehat{G}_\epsilon - G_\epsilon) &= \mathrm{curl}\, \tilde{z} - \mathrm{curl}\, z_\epsilon + r_\epsilon + \exp(\epsilon\tilde{z})\mathrm{curl}\, z_\epsilon, \\ \frac{1}{\epsilon}(\widehat{G}_\epsilon + G_\epsilon) &= \mathrm{curl}\, \tilde{z} + \mathrm{curl}\, z_\epsilon + r_\epsilon + \exp(\epsilon\tilde{z})\mathrm{curl}\, z_\epsilon,\end{aligned}$$

convergence (4.11) holds, and $I - \exp(\epsilon\tilde{z})$ converges to 0 strongly in L^∞.

Conclusions

We have proved that finite-plasticity incremental and quasistatic solutions converge to incremental and quasistatic solution of classical linearized elastoplasticity as forces are infinitesimally small. The limiting procedure is based on evolutionary Γ-convergence arguments for rate-independent systems [29, 33].

With respect to previous contributions on linearization in finite plasticity [9, 15, 34], the novelty is here that of considering the occurrence of an energetic contribution related to the dislocation-density tensor. This corresponds to the setting of [30], which is the most general multidimensional framework under which incremental existence is known.

A future line of research could target the case of *compatible* plastic strains P, by constraining the evolution to $\operatorname{curl} P = 0$ instead. In this case, the total deformation φ can be decomposed as $\varphi = \varphi_{el} \circ \varphi_p$ and an existence theory is already available [24, 43].

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