ATOMIC OPERATORS, RANDOM DYNAMICAL SYSTEMS AND INVARIANT MEASURES

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Abstract. We prove that the existence of invariant measures for families of so-called atomic operators (nonlinear generalized weighted shifts) defined over spaces of measurable functions follows just from the existence of appropriate invariant bounded sets. Typically such operators come from infinite dimensional stochastic differential equations generating not necessarily regular solution flows, for instance, from stochastic differential equations with time delay in the diffusion term (regular solution flows called also Carathéodory flows are those almost surely continuous with respect to the initial datum). Thus it is proven that to ensure the existence of an invariant measure for a stochastic solution flow it suffices to find a bounded invariant subset, and no regularity requirement for the flow is necessary. This result is based on the possibility to extend atomic operators by continuity to a suitable set of Young measures, which is proven in the paper. A motivating example giving a new result on existence of an invariant measure for a possibly non regular solution flow of some model stochastic differential equation is also provided.

Dedicated to the memory of Professor M.E. Drakhlin

1. Introduction

The paper deals with existence of invariant measures for stochastic solution flows, i.e. flows generated by stochastic differential equations. The invariant measures for such flows are usually random (see, e.g. [2]), i.e. are defined on the product of the conventional phase space with the underlying probability space. It is known that the principal difficulty in studying existence of such invariant measures is the possibility of extending the flow by continuity to the space of random measures endowed with the suitable weak topology. Once this is done, the existence of an invariant bounded subset for the flow (which is customarily obtained by some a priori estimates on solutions of the underlying equations) gives the existence of an invariant measure through the standard Krylov-Bogolyubov procedure [2]. Thus it is the existence of a continuous extension of stochastic solution flows the main subject of the present paper. It is important to remark that it is by now known to be valid only in the case of so-called *regular* random dynamical systems, i.e. those generating Carathéodory solution flows, that is, solution flows which almost surely consist of continuous paths with respect to the initial datum. In fact, only such flows are studied in [2]. Regularity in the above sense is a rather strong requirement for a random dynamical system which, first and foremost, might be difficult to verify (and in fact, is known to be fulfilled only in a quite limited number of situations, e.g. for stochastic ODE's with "nice" right-hand side involving standard Brownian motion), and, what is more important, is known to be false in general. In fact, there are some natural examples of stochastic differential equations, especially in infinite

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dimensional spaces, for which regularity fails, i.e. which produce non-Carathéodory flows. The most prominent example of this kind is a stochastic delay equation, where delay is incorporated in the diffusion term (see e.g. [14], or the recent paper [19]). Another example can be found in the present paper (Example 6.2).

The crucial difference between Carathéodory (i.e. regular) and non-Carathéodory (i.e. non-regular) solution flows is their behavior with respect to the natural topology on the set of measures. Any Carathéodory flow can easily be extended to a continuous solution flow defined on the set of measures equipped with the suitable weak (narrow) topology [2]. This is due to the fact that the measures of interest are linear functionals on the linear space of Carathéodory functions, so that $f_\sharp\mu$ (defined by $f_\sharp\mu(A) := \mu(f^{-1}(A)))$ is again a measure for any Carathéodory function f, and the desired extension is just $\mu \mapsto f_\sharp\mu$. This argument breaks completely down if the flow is non-Carathéodory. This is not surprising since non-regular random dynamical systems usually provide an erratic behavior [14].

In the present paper we solve this problem by proving the existence of a continuous extension of a general non-Carathéodory flow to the appropriate set of random measures. In particular this gives an opportunity to define the very notion of an invariant measure for non-Carathéodory flows coming from non-regular random dynamical systems and to prove results on existence of invariant measures for general random dynamical systems, including those which do not generate Carathéodory solution flows and which therefore cannot be covered by the existing theory presented in [2]. Summing up, the main result of the paper can be stated as follows: to ensure the existence of an invariant measure of a stochastic solution flow it is unnecessary to check the regularity of the flow and thus it suffices to find a bounded invariant subset.

Structure of the paper and principal results. The paper is organized as follows. In Section 2 we recall the definition of the so-called "*atomic*" operator introduced and studied in [10] and provide some examples of these operators.

In Section 3, which is central to this work, we explain how atomic operators can be extended by continuity to the set of measures we are interested in (which roughly speaking is the closure in the narrow topology of the set of random Dirac measures, i.e. measures concentrated over graphs of functions). Namely, we show that

- every continuous atomic operator between Lebesgue-Bochner spaces can be extended by continuity to an operator between the spaces of measurable functions, and the extension is still a continuous atomic operator (Proposition 3.1);
- every continuous atomic operator between spaces of measurable functions can be extended by continuity to a continuous operator defined on Young measures, namely, over the closure in the narrow topology of the set of random Dirac measures (in fact, even to a linear continuous operator defined on a much wider space dual to a special space of Carathéodory functions), and such an extension is unique (Theorem 3.3).

It is the latter result that is most important for the applications to random dynamical systems. The idea behind it is to observe that an extension to the linear space dual to that of Carathéodory functions (containing the space of Young measures) of any atomic operator should be linear (as is, for instance, the respective extension of a Nemytskiĭ operator generated by a Carathéodory function, which is a particular case of an atomic operator). However, in the non-Carathéodory case it seems to be problematic to arrange an explicit formula for the extension. That is why we first look at the predual space consisting of Carathéodory functions and construct there the conjugate operator which again should be linear, and then obtain the desired continuous extension of the given atomic operator just by the standard duality argument. This fact is crucial for Section 4, where we prove the results on the existence of invariant measures for (families of) atomic operators.

We also show by means of a series of examples in Section 3, sometimes even elementary, that the results on extensions of atomic operators are rather sharp in the sense that atomicity is quite essential for such results to hold. Namely, there exist continuous operators which cannot be extended neither from Lebesgue-Bochner spaces to spaces of measurable functions, nor from spaces of measurable functions to Young measures, and there exist as well operators extendible to measures but with extensions not coming from linear operators.

In Section 5 we show that normally stochastic differential equation give rise to solution flows (also called *evolution families*) consisting of *local* operators: for this property to hold for a stochastic differential equation one just needs well-posedness of the respective initial value problem. This fact is in sharp contrast with the problem of existence of a Carathéodory flow, which often requires much more sophisticated analysis of solutions.

Of course, any Carathéodory flow consists of local operators, but the converse is in general false. This is shown via examples in Section 6. The main goal of this section is introducing the notion of a generalized cocycle property. The difference between a classical cocycle property (see e.g. [2, p. 5]) and the generalized cocycle property is exactly the difference between Carathéodory and non-Carathéodory flows.

In Section 7 we show that an invariant measures for a stochastic solution flow is a common fixed point for a family of *atomic* operators constructed from the generalized cocycle property and extended by continuity in the narrow topology to Young measures. This section contains also the existence result for a model stochastic differential equation (Theorem 7.2). This result is only intended as an illustration of applicability of the abstract theory developed in the paper. In fact, it refers to the situation where the solution flow may be not regular. Thus it may be seen as the motivating example for the technique developed in this paper.

Since the results we provide often require quite a lot of technicalities that are not always easy to follow, it is our explicit intention to put all the necessary technical statements in the appendices. In particular, Appendix A contains some auxiliary results on local functionals and local operators. In particular, Corollary A.6 gives the representation of local operators under certain assumptions as Nemytskiĭ operators generated by Carathéodory functions. Though being far less general than the representation theorem for local operators from [16] (provided there without detailed proof), it suffices for our purposes and its proof is independent of that outlined in [16] and shorter than the latter. In Appendix B we provide the lengthy and technical proof of Theorem 7.2 which is a motivating example for this paper. At last, Appendix C contains some not difficult auxiliary results on tightness of sets of functions as well as of local operators that we need in the paper.

2. Atomic operators

2.1. **Notation and preliminaries.** The triple $(\Omega, \Sigma, \mathbb{P})$, where $\mathbb{P}$ is a finite positive (resp. probability) measure defined on a σ-algebra Σ of $\mathbb{P}$-measurable subsets of a set Ω, is as usual called measure (resp. probability) space. By default in the sequel we will assume all measure spaces we will be dealing with to be complete (i.e. the respective σ-algebra Σ is complete with respect to $\mathbb{P}$). Also, throughout the paper, unless explicitly stated otherwise, we assume all the finite measures to be probability measures just as a matter of technical assumption simplifying the notation (in fact, all the results of this paper remain true if probability measures are replaced by finite measures). We recall also that $(\Omega, \Sigma, \mathbb{P})$ is called a *standard measure space*, if Ω is a

Polish space (i.e. separable metrizable with complete distance), $\mathbb{P}$ is a Borel measure on Ω, Σ is either the σ-algebra of Borel subsets of the space Ω or its $\mathbb{P}$-completion. All the metric spaces are also silently assumed to be complete, unless otherwise explicitly stated. A norm in a normed space X is denoted by $\|\cdot\|_X$.

By $L^p(\Omega, \Sigma, \mathbb{P}; X)$ we denote the classical Lebesgue-Bochner space of (classes of $\mathbb{P}$-equivalent) functions summable with exponent $p > 0$ over Ω with respect to the measure $\mathbb{P}$ and taking values in some normed space X; the respective norm is denoted by $\|\cdot\|_p$. By $L^0(\Omega, \Sigma, \mathbb{P}; X)$, where X is a metric space with distance d, we denote the space of (classes of) X-valued measurable functions equipped with the distance

$$d^0(u, v) := \int_\Omega (d(u(\omega), v(\omega)) \wedge 1)\, d\mathbb{P}(\omega),$$

inducing the topology of convergence in measure.

The characteristic function of a subset $e \subset \Omega$ will in the sequel be denoted by $\mathbf{1}_e(\omega)$. We also find it convenient to write $x\,|_e = y\,|_e$ for $\{x, y\} \subset L^0(\Omega, \Sigma, \mathbb{P}; X)$, if $x(\omega) = y(\omega)$ for $\mathbb{P}$-a.e. $\omega \in e \subset \Omega$ (for too rigorous readers: this may be interpreted as $(x - y)\mathbf{1}_e = 0$).

If $(\Omega_1, \Sigma_1, \mathbb{P}_1)$ and $(\Omega_2, \Sigma_2, \mathbb{P}_2)$ are two measure spaces, a map $F\colon \Sigma_1 \to \Sigma_2$ is called a *σ-homomorphism*, if $F(\Omega_1) = \Omega_2$, $F(\Omega_1 \backslash e) = \Omega_2 \backslash F(e)$ whenever $e \in \Sigma_1$ and

$$F\left(\bigsqcup_{i=1}^\infty e_i\right) = \bigsqcup_{i=1}^\infty F(e_i),$$

for any pairwise disjoint collection of $\mathbb{P}_1$-measurable sets $\{e_i\}_{i=1}^\infty$, where $\sqcup$ stands for the disjoint union. It is further called *nullset preserving*, if

$$\mathbb{P}_2(F(e_1)) = 0 \text{ when } \mathbb{P}_1(e_1) = 0.$$

2.2. **Local and Nemytskiĭ operators.** Let $\mathfrak{X}_i := L^0(\Omega, \Sigma, \mathbb{P}; X_i)$, $i = 1, 2$.

Definition 2.1. *An operator $T\colon \mathfrak{X}_1 \to \mathfrak{X}_2$ is called* **local**, *from $x\,|_e = y\,|_e$ for $\{x, y\} \subset \mathfrak{X}_1$ follows $T(x)\,|_e = T(y)\,|_e$.*

The above general definition is due to I.V. Shragin [20]. The following example is also classical and in fact motivated the study of local operators.

Example 2.2. Let X_1 and X_2 be separable metric spaces, $f\colon \Omega \times X_1 \to X_2$ be a sup-measurable function (i.e. $f(\cdot, x(\cdot))$ is $\mathbb{P}$-measurable whenever $x(\cdot)$ is $\mathbb{P}$-measurable). Then the Nemytskiĭ operator $N_f\colon\ L^0(\Omega, \Sigma, \mathbb{P}; X_1) \to L^0(\Omega, \Sigma, \mathbb{P}; X_2)$ (commonly known also under the name of the *superposition operator* [1]), defined by

$$(N_f x)(\omega) := f(\omega, x(\omega))$$

is local. If $f\colon \Omega \times X_1 \to X_2$ is a Carathéodory function (i.e. $f(\omega, \cdot)$ is continuous for $\mathbb{P}$-almost every $\omega \in \Omega$ and $f(\cdot, x)$ is $\mathbb{P}$-measurable for all $x \in X_1$), then the Nemytskiĭ operator N_f becomes continuous in measure (i.e. as an operator in L^0).

2.3. **Atomic operators.** Now we introduce another definition generalizing the notion of a local operator. Here $\mathfrak{X}_i := L^{p_i}(\Omega_i, \Sigma_i, \mathbb{P}_i; X_i)$, $i = 1, 2$ and $0 \le p_i \le +\infty$.

Definition 2.3. *An operator $T\colon \mathfrak{X}_1 \to \mathfrak{X}_2$ is called* **atomic**, *if there is a nullset-preserving σ-homomorphism $F\colon \Sigma_1 \to \Sigma_2$, such that from $x\,|_{e_1} = y\,|_{e_1}$ for $\{x, y\} \subset \mathfrak{X}_1$ follows $T(x)\,\big|_{F(e_1)} = T(y)\,\big|_{F(e_1)}$.*

In the rare case when the reference to the particular σ-homomorphism F in the above definition should be made, we will call the operator T atomic with respect to F, so that a local operator is atomic with respect to the identity σ-homomorphism.

It is worth emphasizing that in [10] first the notions of so-called measure-theoretic memory and comemory of an operator were introduced and then the definition of an atomic operator was given based on such notions. Though being more abstract, this opens the way to an intrinsic definition of the concept of an atomic operator. However, we do not follow this way in the present paper in order not to overburden it with too many abstract notions.

Obviously, every local operator is atomic. However, the class of atomic operators is richer, as one can conclude from the following example.

Example 2.4. Consider the *generalized shift* operator

$$T_F : \ L^0(\Omega_1, \Sigma_1, \mathbb{P}_1; X) \to L^0(\Omega_2, \Sigma_2, \mathbb{P}_2; X),$$

associated with a given nullset-preserving σ-homomorphism $F\colon \Sigma_1 \to \Sigma_2$, where X is a separable metric space. We define T_F by setting

$$T_F(\mathbf{1}_{e_1} z) := \mathbf{1}_{F(e_1)} z \tag{2.1}$$

for all $e_1 \in \Sigma_1$ and $z \in X$, extending it by linearity to all simple (i.e. finite valued) functions, and then by continuity to the whole space $L^0(\Omega_1, \Sigma_1, \mathbb{P}_1; X)$. For this operator, one has $\operatorname{Im} T_F = L^0(\Omega_2, F\Sigma_1, \mathbb{P}_2; X)$ (see lemma 3.1 from [10] for details which are provided there for the case when X is a Banach space, but are also valid without any change for general case of a metric space X).

Clearly, if for $\{x, y\} \subset L^0(\Omega_1, \Sigma_1, \mathbb{P}_1; X)$ one has $x\,|_{e_1} = y\,|_{e_1}$, then $T(x)\,\big|_{F(e_1)} = T(y)\,\big|_{F(e_1)}$, that is, the generalized shift operator T_F is atomic. In particular, we obtain that any *shift* (sometimes also called *inner superposition*) operator

$$T_g : \ L^0(\Omega_1, \Sigma_1, \mathbb{P}_1; X) \to L^0(\Omega_2, \Sigma_2, \mathbb{P}_2; X),$$

defined by

$$(T_g x)(\omega_2) := x(g(\omega_2)) \tag{2.2}$$

where $g\colon \Omega_2 \to \Omega_1$ is a (Σ_2, Σ_1)-measurable function, is atomic. For this operator to be well-defined on the classes of measurable functions we require

$$\mathbb{P}_2(g^{-1}(e_1)) = 0 \text{ for } e_1 \in \Sigma_1,\, \mathbb{P}_1(e_1) = 0. \tag{2.3}$$

The class of atomic operators is obviously closed under compositions.

2.4. **Representation.** The following important result is an easy extension of the analogous one proven in [10].

Theorem 2.5. *Let X_i, $i = 1, 2$, be Polish spaces. Then for every operator*

$$T\colon L^0(\Omega_1, \Sigma_1, \mathbb{P}_1; X_1) \to L^0(\Omega_2, \Sigma_2, \mathbb{P}_2; X_2),$$

atomic with respect to a nullset preserving σ-homomorphism $F\colon \Sigma_1 \to \Sigma_2$, there is a local operator $N\colon L^0(\Omega_2, \Sigma_2, \mathbb{P}_2; X_1) \to L^0(\Omega_2, \Sigma_2, \mathbb{P}_2; X_2)$ such that

$$T = N \circ T_F. \tag{2.4}$$

The operator T is continuous, if and only if so is the restriction of the operator N to the subspace $L^0(\Omega_2, F\Sigma_1, \mathbb{P}_2; X_1) \subset L^0(\Omega_2, \Sigma_2, \mathbb{P}_2; X_1)$. Moreover, if $(\Omega_1, \Sigma_1, \mathbb{P}_1)$ is a standard measure space, then there is a (Σ_1, Σ_2)-measurable function $g\colon \Omega_2 \to \Omega_1$ satisfying (2.3) *such that $T = N \circ T_g$.*

Proof. We rely completely on the proof of theorem 3.1 from [10] which is the analogous result for the case when X_i are Banach spaces. Namely, define on $\operatorname{Im} T_F$ an operator $N\colon \operatorname{Im} T_F \to L^0(\Omega_2, \Sigma_2, \mu_2; X_2)$ by setting $N(y) := T(x)$, if $y = T_F(x)$. To show that the above definition of N is correct, assume that $y = T_F(x) = T_F(x')$ with $x \neq x'$. Let

$$\begin{aligned} e_1' &:= \{\omega_1 \in \Omega_1 \,:\, x(\omega_1) = x'(\omega_1)\}\,, \\ e_1 &:= \Omega_1 \setminus e_1'. \end{aligned}$$

We will prove that $\mu_F(e_1) := \mathbb{P}_2(F(e_1)) = 0$, hence $F(e_1') = \Omega_2$ modulo a $\mathbb{P}_2$-nullset, and $T(x) = T(x')$ thus showing the correctness of the definition of N. In fact, if $\mu_F(e_1) > 0$, then $\mu_F(e_1 \cap E_1^\alpha) > 0$ for some $\alpha > 0$, where $E_1^\alpha \subset \Omega_1$ is defined by

$$E_1^\alpha := \left\{ \omega_1 \in \Omega_1 \,:\, \frac{d\mu_F}{d\mathbb{P}_1} \geq \alpha \right\}.$$

Consider two sequences of simple functions $\{x_\nu\}$ and $\{x'_\nu\}$ in $L^0(\Omega_1, \Sigma_1, \mu_1; X_1)$ converging to x and x' in measure $\mathbb{P}_1$ on Ω_1, respectively, and such that with $x_\nu \big|_{e_1'} = x'_\nu \big|_{e_1'}$. We have

$$\begin{aligned} y_\nu &:= x_\nu \big|_{e_1 \cap E_1^\alpha} = \sum_{i=1}^{N_\nu} \mathbf{1}_{e_1^{i,\nu}} z_{i,\nu}, \\ y'_\nu &:= x'_\nu \big|_{e_1 \cap E_1^\alpha} = \sum_{i=1}^{N_\nu} \mathbf{1}_{e_1^{i,\nu}} z'_{i,\nu}, \end{aligned}$$

for some $\{z_{i,\nu}, z_{i,\nu}\} \subset X_1$ and disjoint sets $e_1^{i,\nu} \in \Sigma_1$. Denoting by d_i^0 the distance in $L^0(\Omega_i, \Sigma_i, \mu_i; X_i)$ and by d_i the distance in X_i, $i = 1, 2$, we get

$$\begin{aligned} d_2^0(T_F(y_\nu), T_F(y'_\nu)) &= \sum_{i=1}^{N_\nu} \mu_F(e_1^{i,\nu}) \delta^{i,\nu} \geq \alpha \sum_{i=1}^{N_\nu} \mathbb{P}_1(e_1^{i,\nu}) \delta^{i,\nu}, \\ d_1^0(y_\nu, y'_\nu) &= \sum_{i=1}^{N_\nu} \mathbb{P}_1(e_1^{i,\nu}) \delta^{i,\nu}, \text{ where } \delta^{i,\nu} := d_1(z_{i,\nu}, z'_{i,\nu}) \wedge 1. \end{aligned}$$

Minding that $d_2^0(T_F(y_\nu), T_F(y'_\nu)) \to 0$ as $\nu \to \infty$, we have $d_1^0(y_\nu, y'_\nu) \to 0$, and hence

$$x \big|_{e_1 \cap E_1^\alpha} = x' \big|_{e_1 \cap E_1^\alpha},$$

which is the desired contradiction.

The rest of the proof is just word-to-word restating the proof of theorem 3.1 from [10]. □

We note that even if T: $L^0(\Omega_1, \Sigma_1, \mathbb{P}_1; X_1) \to L^0(\Omega_2, \Sigma_2, \mathbb{P}_2; X_2)$ is continuous, the operator N defined in Theorem 2.5 needs not to be continuous (only its restriction to a certain subspace is). Therefore, the function f: $\Omega_2 \times X_1 \to X_2$, generating together with g: $\Omega_2 \to \Omega_1$ the operator T, needs not to be a Carathéodory function. An example from the theory of stochastic processes, which we are going to discuss now, shows that there indeed exist atomic operators T not representable by a composition of a Nemytskiĭ operator generated by a Carathéodory function with a shift operator.

Example 2.6. Consider a probability space $(\Omega, \Sigma, \mathbb{P})$, the standard Wiener process W_t, the Wiener shift $g := \theta_{-1}$: $\Omega \to \Omega$ inducing the isomorphism of the σ-subalgebrae Σ_0 and let $\Sigma_1 := g^{-1}(\Sigma_0)$. Let $X := L^2((0,1), \hat{\Sigma}, \mathcal{L}^1)$, where $\hat{\Sigma}$ stands for the Lebesgue σ-algebra of $(0,1)$, $\mathcal{L}^1$ stands for the Lebesgue measure. Define the operator T: $L^0(\Omega, \Sigma_1, \mathbb{P}; X) \to L^0(\Omega, \Sigma_1, \mathbb{P}; X)$ as the stochastic integration with respect to the Wiener process

$$(Tx)(\omega) := \int_0^{(\cdot)} x(s, g(\omega)) dW_s(\omega).$$

Note that we shifted the Σ_1-measurable integrand $x(t, \omega)$ with the help of g. In this way the stochastic process $x(s, g(\omega))$ becomes Σ_0-measurable, so that the stochastic integral is well-defined. The operator T is atomic since it is a composition of the stochastic integral (which is local) and the shift T_g. However, the stochastic integral cannot be represented by a Nemytskiĭ operator generated by a Carathéodory function. Otherwise, the stochastic integral could have been, by the Riesz representation theorem, reduced to the ordinary Lebesgue-Stieltjes integral, which is impossible (see e.g. [16] or [17]).

3. Young measures and atomic operators

Throughout this section X will by default stand for a Polish space, and $\mathscr{B}(X)$ for its Borel σ-algebra.

We will use the following notation.

- $C_b(X)$ stands for the space of real valued continuous bounded functions on X equipped with the supremum norm $\|\cdot\|_\infty$;
- $Car_b(\Omega,\Sigma,\mathbb{P};X)$ stands for the set of real valued $\Sigma\otimes\mathscr{B}(X)$-measurable functions $f\colon \Omega\times X\to\mathbb{R}$ such that for $\mathbb{P}$-a.e. $\omega\in\Omega$ one has $f(\omega,\cdot)\in C_b(X)$ and

$$\|f\|_{Car_b} := \int_\Omega \|f(\omega,\cdot)\|_\infty \, d\mathbb{P}(\omega) < +\infty. \tag{3.1}$$

  Note that all the elements of $Car_b(\Omega,\Sigma,\mathbb{P};X)$ are Carathéodory functions. Further, we observe that (3.1) defines a norm over $Car_b(\Omega,\Sigma,\mathbb{P};X)$ making the latter become naturally isomorphic to $L^1(\Omega,\Sigma,\mathbb{P};C_b(X))$.
- $\mathcal{Y}(\Omega,\Sigma,\mathbb{P};X)$ stands for the set of positive measures ν over $\Omega\times X$ whose projections on Ω (i.e. the image measures $\pi_{\Omega\sharp}\nu$ under the projection map $\pi_\Omega\colon \Omega\times X\to\Omega$ defined by $\pi_\Omega(\omega,x):=\omega$) equal $\mathbb{P}$, i.e. $\nu(A\times X)=\mathbb{P}(A)$ for each $A\in\Sigma$. The elements of $\mathcal{Y}(\Omega,\Sigma,\mathbb{P};X)$ are called Young measures with marginal $\mathbb{P}$.

A lot of basic facts about $\mathcal{Y}(\Omega,\Sigma,\mathbb{P};X)$ for the case of a Polish space X can be found in classical works [8, 21] (a more recent monograph [6] treating the more general case of a generic topological space X has also to be mentioned).

We consider the set of Young measures $\mathcal{Y}(\Omega,\Sigma,\mathbb{P};X)$ to be endowed with the narrow topology [21], i.e. the weakest topology which makes all the maps

$$\nu\in\mathcal{Y}(\Omega,\Sigma,\mathbb{P};X)\mapsto\int_{\Omega\times X} f\,d\nu$$

continuous, where $f\in Car_b(\Omega,\Sigma,\mathbb{P};X)$. This topology is known to be Hausdorff [8, 21]. It is also important to mention that it is, generally speaking, not metrizable, unless Σ is countably generated. This is in sharp contrast with the space of finite Borel measures over X equipped with the weak topology generated by the duality with $C_b(X)$ (the latter topology is also frequently referred to as narrow).

Note that the space $L^0(\Omega,\Sigma,\mathbb{P};X)$ can be considered imbedded in $\mathcal{Y}(\Omega,\Sigma,\mathbb{P};X)$ through a natural identification of every $u\in L^0(\Omega,\Sigma,\mathbb{P};X)$ with the measure $\delta_u\in\mathcal{Y}(\Omega,\Sigma,\mathbb{P};X)$ (usually not quite appropriately called *Dirac random measure*) defined by

$$\delta_u(E):=\mathbb{P}(\{\omega\in\Omega\,:\,u(\omega)\in E\})$$

for every $E\in\Sigma\times\mathscr{B}(X)$. Then, of course,

$$\int_{\Omega\times X} f\,d\delta_u=\int_\Omega f(\omega,u(\omega))\,d\mathbb{P}(\omega)$$

for $f\in Car_b(\Omega,\Sigma,\mathbb{P};X)$. Clearly, with the above identification, $L^0(\Omega,\Sigma,\mathbb{P};X)$ is not closed (in the narrow topology) in $\mathcal{Y}(\Omega,\Sigma,\mathbb{P};X)$.

3.1. Extension of atomic operators.

We first prove that, roughly speaking, every continuous atomic operator

$$T\colon L^p(\Omega_1,\Sigma_1,\mathbb{P}_1;X_1)\to L^q(\Omega_2,\Sigma_2,\mathbb{P}_2;X_2),$$

where $\{p,q\} \subset [1,+\infty]$ and X_1 and X_2 are Banach spaces, may be extended by continuity in a unique way to an operator

$$\bar{T}\colon L^0(\Omega_1,\Sigma_1,\mathbb{P}_1;X_1) \to L^0(\Omega_2,\Sigma_2,\mathbb{P}_2;X_2),$$

the extension still being atomic.

Proposition 3.1. *Let X_1 and X_2 be Banach spaces (not necessarily separable) and $\{p,q\} \subset [1,+\infty]$. Then every atomic operator*

$$T\colon L^p(\Omega_1,\Sigma_1,\mathbb{P}_1;X_1) \to L^q(\Omega_2,\Sigma_2,\mathbb{P}_2;X_2)$$

sending norm convergent sequences in measure convergent ones, admits the unique extension to a continuous (in measure) operator

$$\bar{T}\colon L^0(\Omega_1,\Sigma_1,\mathbb{P}_1;X_1) \to L^0(\Omega_2,\Sigma_2,\mathbb{P}_2;X_2).$$

The extended operator $\bar{T}$ is still atomic.

Proof. For every $u \in L^0(\Omega_1,\Sigma_1,\mathbb{P}_1;X_1)$ and for every $c \in \mathbb{R}^+$ we define

$$u^c(\omega_1) := \begin{cases} u(\omega_1), & \omega_1 \notin E^c(u), \\ 0, & \text{otherwise}, \end{cases} \tag{3.2}$$

where $E^c(u) := \{\omega_1 \in \Omega_1 \,:\, \|u\|_{X_1} \geq c\}$. Set now

$$\bar{T}(u) := \lim_{\nu\to\infty} T(u^\nu), \tag{3.3}$$

where the limit is intended in measure $\mathbb{P}_2$. In fact, the latter limit exists since by atomicity of T for $m \geq n$ one has

$$T(u^m)(\omega_2) = T(u^m)(\omega_2)$$

for all $\omega_2 \in (\Omega_2 \setminus F(E^n(u))) \cup F(E^m(u))$, and hence

$$\begin{aligned} &\mathbb{P}_2(\{\omega_2 \in \Omega_2 \,:\, \|T(u^m)(\omega_2) - T(u^n)(\omega_2)\|_{X_2} \neq 0\}) \\ &\qquad = \mathbb{P}_2(\Omega_2) - \mathbb{P}_2(F(E^m(u))) + \mathbb{P}_2(F(E^n(u))), \end{aligned}$$

which means that the sequence $\{T(u^\nu)\}$ is fundamental in $L^0(\Omega_2,\Sigma_2,\mathbb{P}_2;X_2)$. We show now that

(i) $\bar{T}\colon L^0(\Omega_1,\Sigma_1,\mathbb{P}_1;X_1) \to L^0(\Omega_2,\Sigma_2,\mathbb{P}_2;X_2)$ is atomic;
(ii) $\bar{T}$ is continuous (in measure);
(iii) for every continuous (in measure) extension

$$\tilde{T}\colon L^0(\Omega_1,\Sigma_1,\mathbb{P}_1;X_1) \to L^0(\Omega_2,\Sigma_2,\mathbb{P}_2;X_2)$$

of the operator T one has $\tilde{T} = \bar{T}$.

Clearly, (iii) follows from the fact that the space $L^p(\Omega_1,\Sigma_1,\mathbb{P}_1;X_1)$ is dense in $L^0(\Omega_1,\Sigma_1,\mathbb{P}_1;X_1)$. To show (i), we observe that for $\{u,v\} \subset L^0(\Omega_1,\Sigma_1,\mathbb{P}_1;X_1)$ one has that

$$u|_e = v|_e \text{ implies } u^\nu|_{e\setminus E^\nu(u)} = v^\nu|_{e\setminus E^\nu(v)},$$

and hence,

$$T(u^\nu)|_{F(e\setminus E^\nu(u))} = T(v^\nu)|_{F(e\setminus E^\nu(v))}.$$

Thus, passing to a limit in measure $\mathbb{P}_2$ as $\nu \to \infty$ in the above relationship, and minding that

$$\mathbb{P}_1(E^\nu(u)) \to 0, \qquad \mathbb{P}_1(E^\nu(v)) \to 0,$$

which implies

$$\mathbb{P}_2(F(E^\nu(u))) \to 0, \qquad \mathbb{P}_2(F(E^\nu(v))) \to 0,$$

we get

$$T(u)|_{F(e)} = T(v)|_{F(e)}.$$

Finally, to show (ii), assume that $\{u_\nu\} \subset L^0(\Omega_1,\Sigma_1,\mathbb{P}_1;X_1)$, $u_\nu \to u$ in measure $\mathbb{P}_1$. Then $\{u_\nu^c\} \subset L^p(\Omega_1,\Sigma_1,\mathbb{P}_1;X_1)$ and $u_\nu^c \to u^c$ in norm, hence $T(u_\nu^c) \to T(u^c)$ measure. For every $k \in \mathbb{N}$ we choose a $c = c(k)$ such that $d^0(Tu^c, \bar{T}u) \leq 1/k$, where

d^0 stands for the distance in $L^0(\Omega_2, \Sigma_2, \mathbb{P}_2; X_2)$. Further, let $\nu = \nu(k)$ be such that both

$$d^0(T(u_\nu^{c(k)}), T(u_\nu^{c(k)})) \le 1/k \text{ and } \mathbb{P}_2(F(E^{c(k)}(u_\nu))) \le 1/k.$$

Since the latter relationship implies $d^0(\bar{T}(u_\nu^{c(k)}), \bar{T}(u_\nu)) \le 1/k$ in view of atomicity of $\bar{T}$, then by triangle inequality one has $d^0(\bar{T}(u_{\nu(k)}), \bar{T}(u)) \le 3/k \to 0$ as $k \to \infty$, and the thesis follows since the above argument can be applied to an arbitrary subsequence of the original sequence $\{u_\nu\}$. □

The following elementary example shows that the atomicity of the operator T in Proposition 3.1 is essential, that is, even for very simple non atomic operators it may happen that no continuous extension from the space L^p to the space L^0 exists.

Example 3.2. Let $p \ge 1$, $\Omega := [0,1]$, $\mathbb{P} := \mathcal{L}^1 \llcorner \Omega$ is the Lebesgue measure over Ω, and consider the continuous (in norm) operator $T\colon L^p(\Omega, \Sigma, \mathbb{P}; \mathbb{R}) \to L^1(\Omega, \Sigma, \mathbb{P}; \mathbb{R})$ defined by the formula

$$(Tu)(\omega) := \mathbf{1}_\Omega(\omega) \int_\Omega u(z)\, d\mathbb{P}(z).$$

Clearly this operator cannot be extended to a continuous operator over the whole $L^0(\Omega, \Sigma, \mathbb{P}; \mathbb{R})$. In fact, for the sequence of functions $\{u_\nu\} \subset L^p(\Omega, \Sigma, \mathbb{P}; \mathbb{R})$ defined by

$$u_\nu(\omega) := \begin{cases} 2\nu^3\omega, & \omega \in [0, 1/\nu], \\ 0, & \omega \in (1/\nu, 1], \end{cases}$$

one has that $u_\nu \to 0$ in measure though the sequence Tu_ν clearly does not converge in measure since $(Tu_\nu)(\omega) := \nu \mathbf{1}_\Omega(\omega)$.

We are able to prove now the main theorem of this section.

Theorem 3.3. *Let $\mathcal{K}_i := L^0(\Omega_i, \Sigma_i, \mathbb{P}_i; X_i)$, where X_i are Polish spaces, $i = 1, 2$. Then every nonlinear continuous (in measure) atomic operator $T\colon \mathcal{K}_1 \to \mathcal{K}_2$ admits a linear continuous extension*

$$\bar{T}\colon Car_b'(\Omega_1, \Sigma_1, \mathbb{P}_1; X_1) \to Car_b'(\Omega_2, \Sigma_2, \mathbb{P}_2; X_2),$$

where $Car_b'(\Omega_i, \Sigma_i, \mathbb{P}_i; X_i)$ stand for the duals of $Car_b(\Omega_i, \Sigma_i, \mathbb{P}_i; X_i)$, $i = 1, 2$.

If, further, $\bar{\mathcal{K}}_i \subset \mathcal{Y}(\Omega_i, \Sigma_i, \mathbb{P}_i; X_i)$, $i = 1, 2$, are such that

$$\{\delta_u\}_{u \in \mathcal{K}_1} \subset \bar{\mathcal{K}}_1,$$
$$\bar{T}\bar{\mathcal{K}}_1 \subset \bar{\mathcal{K}}_2,$$

then one has that $\bar{T}\colon \bar{\mathcal{K}}_1 \to \bar{\mathcal{K}}_2$ is continuous in the narrow topology of Young measures. In particular, for $\bar{\mathcal{K}}_i$ one can take the sequential closures of the sets $\{\delta_u\}_{u \in \mathcal{K}_i}$ in the $*$-weak topology of $Car_b'(\Omega_i, \Sigma_i, \mathbb{P}_i; X_i)$ (or, equivalently, in the narrow topology of $\mathcal{Y}(\Omega_i, \Sigma_i, \mathbb{P}_i; X_i)$), $i = 1, 2$.

Remark 3.4. The heart of the proof is the construction of the extension of the original atomic operator as dual to some linear continuous operator between the spaces of Carathéodory functions. The extended operator obtained in this way is a priori defined, continuous and even linear between the respective dual spaces, which clearly include the sets of Young measures $\mathcal{Y}_i := \mathcal{Y}(\Omega_i, \Sigma_i, \mathbb{P}_i; X_i)$, $i = 1, 2$, but are not reduced to the latter. In the sequel however we will be interested in extending this operator to smaller sets which consist only of Young measures, which explains the second part of the above statement.

Proof. We first construct the desired extension of T. This will require several technical steps and will be essentially based on the representation Theorem 2.5.

According to Theorem 2.5, T is a composition $T = N \circ T_F$ of a continuous local operator $N\colon L^0(\Omega_2, F\Sigma_1, \mathbb{P}_2; X_1) \to L^0(\Omega_2, \Sigma_2, \mathbb{P}_2; X_2)$ and a generalized shift operator T_F: $L^0(\Omega_1, \Sigma_1, \mathbb{P}_1; X_1) \to L^0(\Omega_2, F\Sigma_1, \mathbb{P}_2; X_1)$. We will treat local and shift operators separately.

Step 1. We first define a special linear continuous operator

$$\tilde{N} : Car_b(\Omega_2, \Sigma_2, \mathbb{P}_2; X_2) \to Car_b(\Omega_2, \Sigma_2, \mathbb{P}_2; X_1).$$

Consider an arbitrary $f \in Car_b(\Omega_2, \Sigma_2, \mathbb{P}_2; X_2)$. Define the operator

$$h_f : L^0(\Omega_2, \Sigma_F, \mathbb{P}_2; X_1) \to L^1(\Omega_2, \Sigma_F, \mathbb{P}_2; \mathbb{R}),$$

where Σ_F stands for the completion of $F\Sigma_1$ with respect to $\mathbb{P}_2$, by setting

$$(h_f(u))(\cdot) := \mathbb{E}(f(\cdot, (N(u))(\cdot); \Sigma_F), \tag{3.4}$$

where $\mathbb{E}(\cdot; \Sigma_F)$ denotes the conditional expectation with respect to Σ_F. We emphasize that according to the definition of the class Car_b (namely, because of (3.1)), the operator h_f acts into $L^1(\Omega_2, \Sigma_2, \mathbb{P}_2; X_1)$, and, moreover, is continuous (i.e. sends measure convergent sequences in L^1 convergent ones). In fact, when $u_\nu \in L^0(\Omega_2, \Sigma_F, \mathbb{P}_2; X_1)$ and $u_\nu \to u$ in measure, then $N(u_\nu) \to N(u)$ in measure $L^0(\Omega_2, \Sigma_2, \mathbb{P}_2; X_1)$ in view of continuity of N. Therefore, $(N_f \circ N)(u_\nu) \to (N_f \circ N)(u)$ in $L^0(\Omega_2, \Sigma_2, \mathbb{P}_2; \mathbb{R})$, where N_f stands for the Nemytskiĭ operator generated by f. In view of (3.1) and the Lebesgue dominated convergence theorem one has that $(N_f \circ N)(u_\nu) \to (N_f \circ N)(u)$ also in $L^1(\Omega_2, \Sigma_2, \mathbb{P}_2; \mathbb{R})$ which proves the desired continuity of h_f in view of the continuity of the conditional expectation.

We show now that h_f is local. For this purpose, we pick an arbitrary pair of functions $\{u, v\} \subset L^0(\Omega_2, \Sigma_F, \mathbb{P}_2; X_1)$ such that $u(\omega_2) = v(\omega_2)$ for $\mathbb{P}_2$-a.e. $\omega_2 \in A$, where $A \in \Sigma_F$. Since both N and N_f are local, we have that $f(\omega_2, (N(u))(\omega_2)) = f(\omega_2, (N(v))(\omega_2))$ for $\mathbb{P}_2$-a.e. $\omega_2 \in A$, or in other words,

$$\mathbf{1}_A (N_f \circ N)(u) = \mathbf{1}_A (N_f \circ N)(v)$$

$\mathbb{P}_2$-a.e. on Ω_2. Hence

$$\begin{aligned} \mathbf{1}_A \mathbb{E}((N_f \circ N)(u); \Sigma_F) &= \mathbb{E}(\mathbf{1}_A (N_f \circ N)(u); \Sigma_F) \\ &= \mathbb{E}(\mathbf{1}_A (N_f \circ N)(v); \Sigma_F) = \mathbf{1}_A \mathbb{E}((N_f \circ N)(v); \Sigma_F) \end{aligned}$$

$\mathbb{P}_2$-a.e. on Ω_2, which proves locality of h_f.

Since h_f: $L^0(\Omega_2, \Sigma_F, \mathbb{P}_2; X_1) \to L^1(\Omega_2, \Sigma_F, \mathbb{P}; \mathbb{R})$ is local and continuous, while

$$|(h_f(u))(\omega_2)| \le \|f(\omega_2, \cdot)\|_\infty \in L^1(\Omega_2, \Sigma_2, \mathbb{P}_2; \mathbb{R}),$$

then according to Corollary A.6 there exists a Carathéodory function γ_f: $\Omega_2 \times X_1 \to \mathbb{R}$ generating the operator h_f, such that $\gamma_f(\cdot, x)$ is Σ_F-measurable in for all $x \in X_1$. Namely,

$$(h_f(u))(\omega_2) = \gamma_f(\omega_2, u(\omega_2))$$

for all $u \in L^0(\Omega_2, \Sigma_F, \mathbb{P}_2; X_1)$. Moreover, such a function is unique among all the Σ_F-measurable functions representing the operator h_f in the above sense.

We claim now that

$$\gamma_f \in Car_b(\Omega_2, \Sigma_2, \mathbb{P}_2; X_1) \tag{3.5}$$

To prove this, let

$$\hat{\gamma}(\omega_2) := \sup_{x_1 \in X_1} |\gamma_f(\omega_2, x_1)|.$$

The latter function is Σ_F-measurable. In fact, since X_1 is separable, while $\gamma_f(\omega_2, \cdot)$ is continuous for $\mathbb{P}_2$-a.e. $\omega_2 \in \Omega_2$, then $\hat{\gamma}(\omega_2) = \sup_{i \in \mathbb{N}} |\gamma_f(\omega_2, x_1^i)|$, where $\{x_1^i\}_{i=1}^\infty \subset X_1$ is a countable dense subset of X_1. The Σ_F-measurability of each $\gamma_f(\cdot, x_1^i)$ shows then the desired Σ_F-measurability of $\hat{\gamma}$.

Let now $v \in L^1(\Omega_2, \Sigma_F, \mathbb{P}_2; \mathbb{R})$ be a positive integrable (with respect to $\mathbb{P}_2$) and Σ_F-measurable function. Consider the set

$$\{(\omega_2, x_1) : |\gamma_f(\omega_2, x_1)| \ge \hat{\gamma}(\omega_2) - v(\omega_2)\}.$$

It is nonempty and $\Sigma_F \otimes \mathscr{B}(X)$-measurable. Hence by the Aumann measurable selection theorem (theorem III.22 from [7]) there exists a Σ_F-measurable function $u\colon \Omega_2 \to X_1$, the graph of which belongs to this set, namely,

$$|(h_f(u))(\omega_2)| = |\gamma_f(\omega_2, u(\omega_2))| \geq \hat{\gamma}(\omega_2) - v(\omega_2)$$

for $\mathbb{P}_2$-a.e. $\omega_2 \in \Omega_2$. Minding that both $h_f(u)$ and and v belong to $L^1(\Omega_2, \Sigma_F, \mathbb{P}_2; \mathbb{R})$, it follows from the above inequality that also $\hat{\gamma} \in L^1(\Omega_2, \Sigma_F, \mathbb{P}_2; \mathbb{R})$ hence proving (3.1) and therefore also the above claim.

We are able now to define the map

$$\tilde{N} \, : \, Car_b(\Omega_2, \Sigma_2, \mathbb{P}_2; X_2) \to Car_b(\Omega_2, \Sigma_2, \mathbb{P}_2; X_1).$$

by setting

$$\tilde{N} f := \gamma_f.$$

The linearity of γ_f with respect to f, and hence that of $\tilde{N}$ is immediate from the construction, while the boundedness of the latter is practically contained in the proof of the fact that $\gamma_f \in Car_b(\Omega_2, \Sigma_2, \mathbb{P}_2; X_1)$. In fact, the careful inspection of this proof gives

$$\|\gamma_f\|_{Car_b} = \|\hat{\gamma}\|_1 \leq \|v\|_1 + \|h_f(u)\|_1 \leq \|v\|_1 + \|f\|_{Car_b}$$

as desired.

Step 2. We define now another auxiliary linear and continuous map

$$\tilde{T}_F \, : \, Car_b(\Omega_2, \Sigma_2, \mathbb{P}_2; X_1) \to Car_b(\Omega_1, \Sigma_1, \mathbb{P}_1; X_1).$$

Consider an arbitrary function $f \in Car_b(\Omega_2, \Sigma_2, \mathbb{P}_2; X_1)$. We will now use the following lemma.

Lemma 3.5. *With the above notation, there is a function $\tilde{f} \in Car_b(\Omega_1, \Sigma_1, \mathbb{P}_1; X_1)$, satisfying*

$$\int_{F(e_1)} f(\omega_2, (T_F u)(\omega_2))\, d\mathbb{P}_2(\omega_2) = \int_{e_1} \tilde{f}(\omega_1, u(\omega_1))\, d\mathbb{P}_1(\omega_2) \tag{3.6}$$

for all $u \in L^0(\Omega_1, \Sigma_1, \mathbb{P}_1; X_1)$ and for all $e_1 \in \Sigma_1$. Moreover, the function $\tilde{f}$ satisfying (3.6) *is unique in the following sense. Assume that an integrand $g\colon \Omega_2 \times X_1 \to \mathbb{R}$ satisfies*

$$\int_{F(e_1)} f(\omega_2, (T_F u)(\omega_2))\, d\mathbb{P}_2(\omega_2) = \int_{e_1} g(\omega_1, u(\omega_1))\, d\mathbb{P}_1(\omega_2) \tag{3.7}$$

for all $u \in L^0(\Omega_1, \Sigma_1, \mathbb{P}_1; X_1)$ and for all $e_1 \in \Sigma_1$. Then there is a set $N_1 \subset \Omega_1$, such that $\mathbb{P}_1(N_1) = 0$ and

$$g(\omega_1, x_1) = \tilde{f}(\omega_1, x_1) \tag{3.8}$$

for all $(\omega_1, x_1) \in (\Omega_1 \setminus N_1) \times X_1$. Finally, one has $\|\tilde{f}\|_{Car_b} \leq \|f\|_{Car_b}$.

We set now

$$\tilde{T}_F f := \tilde{f}$$

and observe that with this notation the relationship (3.6) implies

$$\int_{\Omega_2} f(\omega_2, (T_F u)(\omega_2))\, d\mathbb{P}_2(\omega_2) = \int_{\Omega_1} (\tilde{T}_F f)(\omega_1, u(\omega_1))\, d\mathbb{P}_1(\omega_1).$$

Linearity of $\tilde{T}_F$ is also immediate from (3.6).

Step 3. We are finally able to define the desired extension of the operator T. For every $\nu_1 \in Car_b\prime(\Omega_1, \Sigma_1, \mathbb{P}_1; X_1)$ we set $\bar{T}\nu_1$ to be such that

$$\left\langle f_2, \bar{T}\nu_1 \rangle_2 := ((\tilde{T}_F \circ \tilde{N}) f_2, \nu_1 \right\rangle_1,$$

for every $f_2 \in Car_b(\Omega_2, \Sigma_2, \mathbb{P}_2; X_2)$, where $\langle \cdot, \cdot \rangle_i$ stands for the duality pairings between $Car_b(\Omega_i, \Sigma_i, \mathbb{P}_i; X_i)$ and the respective dual $Car_b'(\Omega_i, \Sigma_i, \mathbb{P}_i; X_i)$, $i = 1, 2$.

Clearly, $\bar{T}\colon Car_b'(\Omega_1,\Sigma_1,\mathbb{P}_1;X_1)\to Car_b'(\Omega_2,\Sigma_2,\mathbb{P}_2;X_2)$ is linear and continuous, while for $u_1\in\mathcal{K}_1$ one has

$$\begin{aligned}
\left\langle f_2,\bar{T}\delta_{u_1}\right\rangle_2 &= \left\langle \tilde{T}_F\circ\tilde{N}f_2,\delta_{u_1}\right\rangle_1 = \int_{\Omega_1\times X_1}(\tilde{T}_F\circ\tilde{N})f_2\,d\delta_{u_1}\\
&= \int_{\Omega_1}((\tilde{T}_F\circ\tilde{N})f_2)(\omega_1,u_1(\omega_1))\,d\mathbb{P}_1(\omega_1)\\
&= \int_{\Omega_2}(\tilde{N}f_2)(\omega_2,T_Fu_1(\omega_2))\,d\mathbb{P}_2(\omega_2)\\
&= \int_{\Omega_2}(h_{f_2}(T_Fu_1))(\omega_2)\,d\mathbb{P}_2(\omega_2)\\
&= \int_{\Omega_2}f_2(\omega_2,(N(T_Fu_1))(\omega_2))\,d\mathbb{P}_2(\omega_2)\\
&= \int_{\Omega_2\times X_2}f_2\,d\delta_{N\circ T_F(u_1)} = \left\langle f_2,\delta_{N\circ T_F(u_1)}\right\rangle_2
\end{aligned}$$

for every $f_2\in Car_b(\Omega_2,\Sigma_2,\mathbb{P}_2;X_2)$, which means that $\bar{T}\delta_{u_1}=\delta_{Tu_1}$, i.e. $\bar{T}$ is an extension of T. This concludes the proof of the first part of the statement.

Step 4. The second part of the statement follows immediately from the fact that the narrow topology of Young measures is generated by the duality with bounded Carathéodory functions. Finally, if for $\bar{\mathcal{K}}_i$ one takes the sequential closures of the sets $\{\delta_u\}_{u\in\mathcal{K}_i}$ in the $*$-weak topology of $Car_b'(\Omega_i,\Sigma_i,\mathbb{P}_i;X_i)$, $i=1,2$, then by Lemma C.1 one has $\bar{\mathcal{K}}_i\subset\mathcal{Y}(\Omega_i,\Sigma_i,\mathbb{P}_i;X_i)$. Moreover, if $\nu_1\in\bar{\mathcal{K}}_1$, then there is a sequence $\{\delta_{u_1^n}\}\subset\bar{\mathcal{K}}_1$ such that $\delta_{u_1^n}\to\nu_1$, hence

$$\bar{T}\delta_{u_1^n}=\delta_{Tu_1^n}\to\bar{T}\nu_1$$

as $n\to\infty$ (the convergence is each of the cases is meant in the $*$-weak topology of $Car_b'(\Omega_i,\Sigma_i,\mathbb{P}_i;X_i)$, $i=1,2$, which, by Lemma C.1 is equivalent to narrow convergence of measures). This implies $\bar{T}\nu_1\in\bar{\mathcal{K}}_2$ concluding the proof of the last claim. □

Proof of Lemma 3.5:

Let the functional $I\colon L^0(\Omega_1,\Sigma_1,\mathbb{P}_1;X_1)\times\Sigma_1\to\mathbb{R}$ be defined by the relationship

$$I(u,e_1):=\int_{F(e_1)}f(\omega_2,(T_Fu)(\omega_2))\,d\mathbb{P}_2(\omega_2).$$

Clearly, I is local and additive. Moreover, since

$$f\in Car_b(\Omega_2,\Sigma_2,\mathbb{P}_2;X_1)$$

then in view of the Lebesgue theorem for every $e_1\in\Sigma_1$ the functional $I(\cdot,e_1)$: $L^0(\Omega_1,\Sigma_1,\mathbb{P}_1;X_1)\to\mathbb{R}$ is continuous and bounded both from above and from below. According to Corollary A.5, there exists then a Carathéodory function $\tilde{f}$: $\Omega_1\times X_1\to\mathbb{R}$ satisfying (3.6), and, moreover, the latter function is unique in the sense indicated in the statement of the lemma being proven. It remains therefore to show that $\tilde{f}\in Car_b(\Omega_1,\Sigma_1,\mathbb{P}_1;X_1)$. Clearly, (3.6) implies that $\tilde{f}(\cdot,x_1)$ is the Radon-Nikodym derivative of the signed measure $I(\cdot,x_1\mathbf{1}_{\Omega_1})$ with respect to $\mathbb{P}_1$, namely,

$$\tilde{f}(\cdot,x_1)=\frac{dI(x_1\mathbf{1}_{\Omega_1},\cdot)}{d\mathbb{P}_1}.$$

Define a new measure J over Σ_1 by the formula

$$J(e_1)=\int_{F(e_1)}\sup_{x_1\in X_1}|f(\omega_2,x_1)|\,d\mathbb{P}_2(\omega_2),$$

and let $j \in L^1(\Omega_1, \Sigma_1, \mathbb{P}_1; \mathbb{R})$ stand for for the Radon-Nikodym derivative of J with respect to $\mathbb{P}_1$. Let $\{x_1^i\}_{i=1}^\infty \subset X_1$ be a countable dense subset of X_1. Since

$$|I(x_1^i \mathbf{1}_{\Omega_1}, e_1)| \leq J(e_1)$$

for all $e_1 \in \Sigma_1$ and for all $i \in \mathbb{N}$, then

$$|\tilde{f}(\omega_1, x_1^i)| \leq j(\omega_1)$$

for all $i \in \mathbb{N}$ and for all $\omega_1 \in \Omega_1 \setminus N_1^i$, where $N_1^i \subset \Omega_1$ satisfies $\mathbb{P}_1(N_1^i) = 0$. Minding that $\tilde{f}$ is a Carathéodory function, one has for $\mathbb{P}_1$-a.e. $\omega_1 \in \Omega_1 \setminus \cup_i N_1^i$ and hence also for $\mathbb{P}_1$-a.e. $\omega_1 \in \Omega_1$ the estimate

$$\sup_{x_1 \in X_1} |\tilde{f}(\omega_1, x_1))| = \sup_{i \in \mathbb{N}} |\tilde{f}(\omega_1, x_1^i))| \leq j(\omega_1),$$

hence also

$$\|\tilde{f}\|_{Car_b} \leq \int_{\Omega_1} j(\omega_1)\, d\mathbb{P}_1(\omega_1) = J(\Omega_1) = \|f\|_{Car_b},$$

which shows the announced claim.

From the above construction it is clear that the continuous extensions of atomic operators to duals of Carathéodory functions are linear. However, in the example below we show that this is not necessarily the case for all the operators admitting the extension by continuity. Namely, there are operators between spaces of measurable functions which possess continuous extensions to the respective spaces of measures that are not coming from any linear operator on a larger space.

Example 3.6. Consider the operator $T : L^0(\Omega, \Sigma, \mathbb{P}; \mathbb{R}) \to L^0(\Omega, \Sigma, \mathbb{P}; \mathbb{R})$, defined by the formula

$$T(u)(\omega) := \mathbf{1}_\Omega(\omega) \int_\Omega (0 \vee u(z) \wedge 1)\, d\mathbb{P}(z),$$

where $\Omega := [0, 1]$, $\mathbb{P} := \mathcal{L}^1$ is the Lebesgue measure. Clearly, T can be represented as a composition $T = T_0 \circ N_f$ of the operator

$$T_0 \colon L^p(\Omega, \Sigma, \mathbb{P}; \mathbb{R}) \to L^0(\Omega, \Sigma, \mathbb{P}; \mathbb{R})$$

defined by the formula

$$(T_0 u)(\omega) := \mathbf{1}_\Omega(\omega) \int_\Omega u(z)\, d\mathbb{P}(z)$$

(this operator was considered in Example 3.2) with the Nemytskiĭ operator

$$N_f \colon L^0(\Omega, \Sigma, \mathbb{P}; \mathbb{R}) \to L^p(\Omega, \Sigma, \mathbb{P}; \mathbb{R})$$

generated by the function $f(\omega, u) := 0 \vee u \wedge 1$. The latter extends to the linear continuous operator $\bar{N}_f \colon \mathcal{Y}(\Omega \times \mathbb{R}, \Sigma, \mathbb{P}) \to \mathcal{Y}(\Omega \times [0, 1], \Sigma, \mathbb{P})$ defined on Young measures according to the formula

$$\bar{N}_f \mu := f_\sharp \mu \text{ for all } \mu \in \mathcal{Y}(\Omega \times \mathbb{R}, \Sigma, \mathbb{P}).$$

The operator T_0 can be extended to the operator $\bar{T}_0$ defined on the set $\mathcal{Y}_1(\Omega \times \mathbb{R}, \Sigma, \mathbb{P}) \subset \mathcal{Y}(\Omega \times \mathbb{R}, \Sigma, \mathbb{P})$ of Young measures, the second marginal of which has finite first order moment, i.e.

$$\mathcal{Y}_1(\Omega \times \mathbb{R}, \Sigma, \mathbb{P}) := \left\{ \mu \in \mathcal{Y}(\Omega \times \mathbb{R}, \Sigma, \mathbb{P}) \,:\, \int_\mathbb{R} |x|\, d\pi_{\mathbb{R}\sharp}\mu(x) < +\infty \right\},$$

where $\pi_{\mathbb{R}\sharp}\mu$ stands for the second marginal of the measure μ. Minding that in the case $\mu = \delta_u$ one has that $\pi_{\mathbb{R}\sharp}\mu$ is the distribution law of u, i.e. $\pi_{\mathbb{R}\sharp}\mu(e) = \mathbb{P}(\{\omega \in \Omega \,:\, u(\omega) \in e\})$ for every Borel set $e \subset \mathbb{R}$, so that

$$\int_\Omega u(\omega)\, d\mathbb{P}(\omega) = \int_\mathbb{R} x\, d\pi_{\mathbb{R}\sharp}\mu(x),$$

we get for the operator $\bar{T}_0$ the formula

$$\bar{T}_0(\mu) := \delta_{\mathbf{1}_\Omega(\cdot)\int_{\mathbb{R}} x\, d\pi_{\mathbb{R}\sharp}\mu(x)} = \delta_{\int_{\mathbb{R}} x\, d\pi_{\mathbb{R}\sharp}\mu(x)} \otimes \mathbb{P}.$$

Observe that clearly

$$\mathcal{Y}(\Omega \times [0,1], \Sigma, \mathbb{P}) \subset \mathcal{Y}_1(\Omega \times \mathbb{R}, \Sigma, \mathbb{P}),$$

and that the restriction of $\bar{T}_0$ to this set is continuous in the narrow topology of Young measures. Hence the operator $\bar{T} := \bar{T}_0 \circ \bar{N}_f$ extends the operator T by continuity to the space of Young measures, though the extension does not come from a linear operator in the space dual to that of Carathéodory functions. In fact, this operator is given by the formula

$$\bar{T}(\mu) = \delta_{\mathbf{1}_\Omega(\cdot)\int_{\mathbb{R}}(0\vee x\wedge 1)\, d\pi_{\mathbb{R}\sharp}\mu(x)} = \delta_{\int_{\mathbb{R}}(0\vee x\wedge 1)\, d\pi_{\mathbb{R}\sharp}\mu(x)} \otimes \mathbb{P},$$

and hence in general $\bar{T}(\mu_1/2 + \mu_2/2) \neq \bar{T}(\mu_1)/2 + \bar{T}(\mu_2)/2$.

Further, observe that the continuous extension is uniquely determined over the narrow closure of random Dirac measures (i.e. on the whole set of Young measures, since $\mathbb{P}$ is nonatomic [7]).

The last example in this section describes operators that are continuous in measure yet not continuous in the narrow topology, and thus cannot be continuously extended to the space of Young measures.

Example 3.7. Let $\Omega := (0, 2\pi)$ be equipped with the ordinary Lebesgue measure $\mathbb{P} := d\omega$ and the usual Lebesgue σ-algebra Σ. Chosen a number $\lambda \in \mathbb{R}$, $\lambda \neq 1$, consider the operator $T\colon L^0(\Omega, \Sigma, \mathbb{P}; \mathbb{R}) \to L^0(\Omega, \Sigma, \mathbb{P}; \mathbb{R})$ defined as follows:

$$T(u) := \tilde{T}(-1 \wedge u \vee 1), \text{ where}$$

the operator $\tilde{T}\colon L^2(\Omega, \Sigma, \mathbb{P}; \mathbb{R}) \to L^2(\Omega, \Sigma, \mathbb{P}; \mathbb{R})$ sends each function

$$u(\omega) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \sin k\omega + b_k \cos k\omega)$$

to the function

$$(\tilde{T}u)(\omega) := \sum_{k=1}^{\infty} \tilde{a}_k \sin k\omega,$$

with $\tilde{a}_k = a_k$ for k even and $\tilde{a}_k = \lambda a_k$ for k odd. Clearly, the operator T is continuous in measure, since the operator $\tilde{T}$ is linear and bounded (in $L^2(\Omega)$). However, T cannot be extended with continuity to Young measures. In fact, if we consider, for instance, the sequence $u_k(\omega) := \sin k\omega$, then we have that $\delta_{u_k} \rightharpoonup \phi \otimes d\omega$ in the narrow sense of Young measures as $k \to \infty$, where ϕ is the measure on $\mathbb{R}$ concentrated on $[-1, 1]$ and defined by

$$\phi = \frac{1}{\pi\sqrt{1-x^2}}\, dx.$$

On the other hand, setting $v_k := T(u_k) = \tilde{T}u_k$, we have that $v_{2k} = u_{2k}$ and $v_{2k+1} = \lambda u_{2k}$, and hence $\delta_{u_{2k}} \rightharpoonup \phi \otimes d\omega$, but $\delta_{u_{2k+1}} \rightharpoonup \psi \otimes d\omega$, in the narrow sense of Young measures as $k \to \infty$, where ψ is the measure on $\mathbb{R}$ concentrated on $[-\lambda, \lambda]$ and defined by

$$\psi = \begin{cases} \frac{1}{\pi\sqrt{\lambda^2 - x^2}}\, dx, & \lambda \neq 0, \\ \delta_0, & \lambda = 0, \end{cases}$$

and hence $\psi \neq \phi$ as $\lambda \neq 1$.

4. Invariant measures for atomic operators

Throughout this section again by default X will stand for a Polish space. Recall the following notion [8, 21].

Definition 4.1. *A set of Young measures $\mathcal{H} \in \mathcal{Y}(\Omega, \Sigma, \mathbb{P}; X)$ is called tight, if for every $\varepsilon > 0$ there is a compact subset $K_\varepsilon \subset X$ such that*

$$\sup_{\nu \in \mathcal{H}} \nu(\Omega \times (X \setminus K_\varepsilon)) \leq \varepsilon.$$

Further, a set of functions $\mathcal{K} \subset L^0(\Omega, \Sigma, \mathbb{P}; X)$ is called tight, if it is tight as a set of Young measures, i.e. the set $\{\delta_u\}_{u \in \mathcal{K}}$ is tight. In other words, $\mathcal{K} \subset L^0(\Omega, \Sigma, \mathbb{P}; X)$ is tight, if for every $\varepsilon > 0$ there is such a compact subset $K_\varepsilon \subset X$ that

$$\sup_{u \in \mathcal{K}} \mathbb{P}(\{\omega \in \Omega \,:\, u(\omega) \notin K_\varepsilon\}) \leq \varepsilon.$$

Let $\mathbb{T}$ be an additive subset of the set $\mathbb{R}$. Typical examples are $\mathbb{R}^+$, $\gamma\mathbb{Z}^+$ $(\gamma > 0)$ etc.

As an immediate corollary of the extension Theorem 3.3 we obtain the following result.

Theorem 4.2. *Let $\mathcal{K} \subset L^0(\Omega, \Sigma, \mathbb{P}; X)$ be a tight set. Let $T_\tau \colon L^0(\Omega, \Sigma, \mathbb{P}; X) \to L^0(\Omega, \Sigma, \mathbb{P}; X)$, where $\tau \in \mathbb{T}$ is a one parameter family of commuting continuous (in measure) atomic operators sending $\mathcal{K}$ into itself. Then this family admits a common invariant measure $\nu \in \bar{\mathcal{K}}$, where $\bar{\mathcal{K}}$ stands for the narrow closure of the set $\mathcal{K}$ in the space of Young measures. In particular, every continuous (in measure) atomic operator $T\colon L^0(\Omega, \Sigma, \mathbb{P}; X) \to L^0(\Omega, \Sigma, \mathbb{P}; X)$ sending $\mathcal{K}$ into itself admits an invariant measure.*

Proof. Denote by $\bar{\mathcal{K}}$ the closure of the set $\mathcal{K}$ in the $*$-weak topology of the space $Car_b'(\Omega, \Sigma, \mathbb{P}; X)$. By Lemma C.2 one has that $\bar{\mathcal{K}} \subset \mathcal{Y}(\Omega, \Sigma, \mathbb{P}; X)$ and hence it coincides with the narrow closure of the set $\mathcal{K}$ in the space of Young measures. One observes then that $\bar{\mathcal{K}}$ is tight by Lemma 9 from [21], and hence narrow compact in $\mathcal{Y}(\Omega; \Sigma; \mathbb{P}; X)$ by theorem 11 from [21] (or theorem 4.4 from [8]).

In the notation of the extension Theorem 3.3, there is a continuous extension $\bar{T}_\tau$: $Car_b'(\Omega, \Sigma, \mathbb{P}; X) \to Car_b'(\Omega, \Sigma, \mathbb{P}; X)$ sending $\bar{\mathcal{K}}$ into itself. Note that the operators $\bar{T}_\tau$ are linear continuous and still form a commuting family. Minding that $\mathcal{K}$ is also convex, the reference to the Markov-Kakutani fixed point theorem concludes the proof. $\square$

We introduce now the notion of tightness of operators between spaces of measurable functions, which as we will see often is a good substitute for compactness property.

Definition 4.3. *The operator $T\colon L^0(\Omega_1, \Sigma_1, \mathbb{P}_1; X_1) \to L^0(\Omega_2, \Sigma_2, \mathbb{P}_2; X_2)$ is called tight, if it sends bounded sets into tight ones.*

Now we may claim the following result.

Corollary 4.4. *Assume that the family of commuting continuous (in measure) atomic operators*

$$T_\tau \,:\, L^0(\Omega, \Sigma, \mathbb{P}; X) \to L^0(\Omega, \Sigma, \mathbb{P}; X), \qquad\qquad \tau \in \mathbb{T}$$

maps a set $B \subset L^0(\Omega, \Sigma, \mathbb{P}; X)$ into its tight subset. Then all T_τ, $\tau \in \mathbb{T}$ admit a common invariant measure in $\bar{B} \subset \mathcal{Y}(\Omega, \Sigma, \mathbb{P}; X)$, where $\bar{B}$ stands for the narrow closure of the set $\{\delta_u\}_{u \in B}$. In particular, every continuous (in measure) atomic tight operator $T\colon L^0(\Omega, \Sigma, \mathbb{P}; X) \to L^0(\Omega, \Sigma, \mathbb{P}; X)$ mapping a bounded set $B \subset L^0(\Omega, \Sigma, \mathbb{P}; X)$ into itself, has an invariant measure in $\bar{B}$.

Proof. Since the set $D := T(B)$ is tight and T maps D into itself, we may apply Theorem 4.2 to obtain the desired result. $\square$

Corollary 4.5. *Assume that the family of commuting continuous (in measure) atomic operators*

$$T_\tau : L^0(\Omega,\Sigma,\mathbb{P};X) \to L^0(\Omega,\Sigma,\mathbb{P};X), \qquad \tau\in\mathbb{T},$$

maps some bounded set $B \subset L^0(\Omega,\Sigma,\mathbb{P};X)$ into itself. Further, let there exist an $s\in\mathbb{T}$ such that T_s is tight. Then all T_τ, $\tau\in\mathbb{T}$ admit a common invariant measure in $\bar{B}$, where $\bar{B}$ stands for the narrow closure of the set $\{\delta_u\}_{u\in B}$.

Proof. Consider the commuting family of operators

$$\mathcal{A} := \{T_{t_1}\circ T_{t_2}\circ\ldots T_{t_k}\circ T_s\}_{\{t_j\}\subset\mathbb{T},k\in\mathbb{N}}.$$

For every $\{t_j\}_{j=1}^k \subset \mathbb{T}$, $k\in\mathbb{N}$ one has that

$$T_{t_1}\circ T_{t_2}\circ\ldots T_{t_k}\circ T_s\circ T_s(B) = T_s\circ T_{t_1}\circ T_{t_2}\circ\ldots T_{t_k}(B) \subset T_s(B),$$

the latter subset being tight, and hence by Corollary 4.4 the family $\mathcal{A}$ has a common invariant measure μ, i.e. $\bar{A}\mu=\mu$ for every $A\in\mathcal{A}$. We show that this measure is in fact invariant for the original family $\{T_t\}_{t\in\mathbb{T}}$. For this purpose fix a $\tau\in\mathbb{T}$ and observe that both $T_t\circ T_\tau\circ T_s\in\mathcal{A}$ and $T_\tau\circ T_s\in\mathcal{A}$, so that

$$\bar{T}_t\circ\bar{T}_\tau\circ\bar{T}_s\mu=\mu,$$
$$\bar{T}_\tau\circ\bar{T}_s\mu=\mu,$$

which implies $\bar{T}_t\mu=\mu$, concluding the proof. $\square$

5. Stochastic evolution equations

Let $V\subset H\subset V'$ be a Gelfand triple, consisting of a separable Hilbert space H, a reflexive Banach space V and its conjugate V', each embedding being continuous and dense in the respective topologies. The pairing $a\cdot b$ between V and V' coincides with the inner product in H if $b\in H$.

The following equation is considered

$$dx = f(t,x)\,dZ(t), \qquad t\in[0,T] \tag{5.1}$$

where Z is an m-dimensional semimartingale ($m\in\mathbb{N}$), and $F:\Omega\times\mathbb{R}\times V\to(V')^m$, $(V')^m := V'\times\ldots\times V'$ (m times) and $T>0$ is fixed. Assume that for any $s\in[0,T)$ the equation (5.1) has a unique (mild) solution $x(\cdot)\colon\Omega\times[s,T)\to H$ for any $x(s)\in L^p(\Omega,\Sigma_s,\mathbb{P};H)$. It is also assumed that this solution belongs to $L^p(\Omega,\Sigma_t,\mathbb{P};H)$ for each $t\in[0,T)$ (in particular, this implies that the solution flow is *adapted*). Finally, for each $t\in[s,T)$ the value $x(t)$ of the solution continuously depends on the initial values $x(s)$ in the sense of the natural topologies on $L^p(\Omega,\Sigma_s,\mathbb{P};H)$ and $L^p(\Omega,\Sigma_t,\mathbb{P};H)$. These assumptions give rise to the following continuous evolution operator determined by the solution flow $U_t^s\colon L^p(\Omega,\Sigma_s,\mathbb{P};H)\to L^p(\Omega,\Sigma_t,\mathbb{P};H)$.

Proposition 5.1. *The operator U_t^s is local in ω and satisfies the evolution property*

$$U_t^\sigma\circ U_\sigma^s = U_t^s \quad \text{for all } \ 0\le s\le\sigma\le t. \tag{5.2}$$

Proof. We observe first that due to the properties of the stochastic integral

$$\int_s^t \vartheta(u)\mathbf{1}_e\,dZ(u) = \int_s^t \vartheta(u)\,dZ(u)\mathbf{1}_e$$

for all $e\in\Sigma_s$, provided that the stochastic integral exists. This immediately implies that the stochastic process $\varkappa(t)=x(t)\mathbf{1}_e+y(t)\mathbf{1}_{\Omega\setminus e}$ ($t\ge s$) is (a unique) solution of (5.1) satisfying $\varkappa(s)=x(s)\mathbf{1}_e+y(s)\mathbf{1}_{\Omega\setminus e}$. Assume now that $x(s)|_e=y(s)|_e$ a.s. for some $e\in\Sigma_s$. Then $\varkappa(s)=y(s)$ a.s. implying, due to the uniqueness property, that $\varkappa(t) = y(t)$ a.s. for any $t \geq s$. In particular, $x(t)|_e = y(t)|_e$ a.s. This yields locality of the evolution operator U_t^s for $0 \leq s \leq t \leq T$.

The evolution property follows directly from the uniqueness. $\square$

Remark 5.2. Assume in addition that all the solutions to (5.1) belong to the space $L^p(\Omega, \Sigma_T, \mathbb{P}; \tilde{H}_s)$ for some space $\tilde{H}_s$ of functions $x\colon [s,T] \to H$ (i.e. of continuous or of cadlag functions). Then the uniqueness of solutions to (5.1) gives by the same argument the locality of the operator $U^\sigma_{(\cdot)}\colon L^p(\Omega, \Sigma_s, \mathbb{P}; H) \to L^p(\Omega, \Sigma_T, \mathbb{P}; \tilde{H}_s)$ defined by

$$(U^\sigma_{(\cdot)} x)(t) := U_t^\sigma x(s).$$

Using Proposition 3.1 we now obtain the following result.

Corollary 5.3. *For any $s \in [0,T)$ the equation* (5.1) *has a unique (mild) solution $x(\cdot)\colon \Omega \times [s,T) \to H$ for any $x(s) \in L^0(\Omega, \Sigma_s, \mathbb{P}; H)$. For any $t \in [s,T)$ the value $x(t)$ of the solution continuously depends on the initial values $x(s)$ in the sense of the natural topologies on $L^0(\Omega, \Sigma_s, \mathbb{P}; H)$ and $L^0(\Omega, \Sigma_t, \mathbb{P}; H)$. The evolution operator $U_t^s\colon L^0(\Omega, \Sigma_s, \mathbb{P}; H) \to L^0(\Omega, \Sigma_t, \mathbb{P}; H)$ is local and continuous.*

The property of locality refers to the pathwise nature of stochastic differential equations, where the evolution of a bunch of trajectories for $\omega \in e$, where $e \subset \Omega$ is an arbitrary measurable subset of a positive measure, does not depend (up to a $\mathbb{P}$-null set) on the evolution of the trajectories for ω outside e.

From Theorem 3.3 we immediately get the following result.

Corollary 5.4. *The solution flow U_t^s for $0 \leq s \leq t \leq T$ of the equation* (5.1) *extends continuously to the solution flow $\bar{U}_t^s\colon \mathcal{Y}(\Omega, \Sigma_s, \mathbb{P}; H) \to \mathcal{Y}(\Omega, \Sigma_t, \mathbb{P}; H)$.*

Now we outline another example of stochastic evolution coming from stochastic hereditary equations. For the solution's trajectories on an interval $[a,b]$ one can use either the space $D([a,b]; \mathbb{R}^n)$ of cadlag functions (if Z is discontinuous), or its subspace $C([a,b]; \mathbb{R}^n)$ containing continuous functions (if Z is so). In both cases one can use more general trajectory spaces like the so-called the Delfour-Mitter space $L^2([a,b], \mathcal{B}, \mathcal{L}^1; \mathbb{R}^n) \times \mathbb{R}^n$ [14], where $\mathcal{L}^1$ is the linear Lebesgue measure, $\mathcal{B}$ is the Lebesgue σ-algebra of $[a,b]$. To simplify the notation in the latter we will always omit the reference to $\mathcal{B}$ and $\mathcal{L}^1$.

For the sake of simplicity, we denote the space of trajectories by $\mathcal{S}([a,b]; \mathbb{R}^n)$. We use the following notation: $\mathcal{S} := \mathcal{S}([-h+s, s], \mathbb{R}^n)$, $x_t(\sigma) := x(t+\sigma)$, while $\sigma \in [-h+s, s]$ and $t \geq s$.

We study the stochastic functional differential equation (see [14] for the detailed definitions)

$$dx(t) = F(t, x_t) dZ(t), \tag{5.3}$$

where $t \in (s,T)$, $T > s$ being fixed, with the initial condition

$$x(\sigma) = \varphi(\sigma), \qquad \sigma \leq s. \tag{5.4}$$

Here $s \in [0, +\infty)$, $Z(t)$, $t \geq s$ is an m-dimensional semimartingale, and $F\colon \Omega \times [s,T) \times S \to \mathbb{R}^{n \times m}$ is a continuous vector-functional. We assume also that the initial function φ is taken from the space $L^p(\Omega, \Sigma_s, \mathbb{P}; \mathcal{S})$ for some $p \geq 0$.

The solutions of (5.3) should be adapted with respect to the filtration $\{\Sigma_t\}_{t \in [0,T)}$ associated with the semimartingale Z. We denote the set of all n-dimensional $\{\Sigma_t\}$-adapted stochastic processes by $\mathcal{A}$.

Assume that for any $\varphi \in L^p(\Omega, \Sigma_s, \mathbb{P}; \mathcal{S})$ there exists a unique solution $x(\cdot)$ to the equation (5.3) satisfying (5.4) and belonging to the space $\mathcal{A} \cap L^p(\Omega, \Sigma_T, \mathbb{P}; \mathcal{S}([s-h, T); \mathbb{R}^n))$ (equipped with the topology of the second space). As in the previous example, this solution should depend continuously on the initial data in the respective topologies.

Now let us introduce the evolution operator associated with the hereditary equation (5.3)

$$U_t^s:\ L^2(\Omega,\Sigma_s;\mathcal{S})\to L^2(\Omega,\Sigma_t;\mathcal{S}),\qquad t\geq s,$$

defined by

$$U_t^s(\varphi):={}^{\varphi}x_t^s,\quad \varphi\in L^2(\Omega,\Sigma_s;\mathcal{S}), \tag{5.5}$$

where ${}^{\varphi}x^s(t)$ satisfies

$${}^{\varphi}x^s(t)=\begin{cases}\varphi(0)+\displaystyle\int_s^t F(u,{}^{\varphi}x_u^s)dZ(u), & t>s\\ \varphi(t-s), & -h+s\leq t\leq s.\end{cases} \tag{5.6}$$

In quite a similar way as for the equation (5.1), we arrive at the following results.

Proposition 5.5. *The above operator U_t^s is local in ω and satisfies the evolution property*

$$U_t^\sigma\circ U_\sigma^s=U_t^s\quad \textit{for all}\quad s\leq\sigma\leq t\leq T. \tag{5.7}$$

Corollary 5.6. *For any $s\in[0,T)$ the equation (5.3) has a unique solution $x(\cdot)\colon\Omega\times[s,T)\to\mathcal{S}$ for any $\varphi\in L^0(\Omega,\Sigma_s,\mathbb{P};\mathcal{S})$. For any $t\in[s,T)$ the value $x(t)$ of the solution continuously depends on the initial values $x(s)$ in the sense of the topologies on $L^0(\Omega,\Sigma_s,\mathbb{P};\mathcal{S})$ and $L^0(\Omega,\Sigma_t,\mathbb{P};\mathcal{S})$. The evolution operator*

$$U_t^s\colon L^0(\Omega,\Sigma_s,\mathbb{P};\mathcal{S})\to L^0(\Omega,\Sigma_t,\mathbb{P};\mathcal{S})$$

is local and continuous. Moreover, the operator

$$U_{(\cdot)}^\sigma\colon L^0(\Omega,\Sigma_s,\mathbb{P};\mathcal{S})\to L^p(\Omega,\Sigma_T,\mathbb{P};\mathcal{S}[-h+s,T])$$

defined by

$$(U_{(\cdot)}^\sigma x)(t):=U_t^\sigma x(s)$$

is local and continuous as well.

Corollary 5.7. *The solution flow U_t^s for $0\leq s\leq t\leq T$ of the equation (5.3) extends continuously to the solution flow $\bar{U}_t^s\colon\mathcal{Y}(\Omega,\Sigma_s,\mathbb{P};\mathcal{S})\to\mathcal{Y}(\Omega,\Sigma_t,\mathbb{P};\mathcal{S})$.*

Remark 5.8. In applications extensions of evolution operators to the space of Young measures are of great interest. It is for instance known that in the stochastic Hopf bifurcation, even in the plane, the zero solution which passes through a critical point may produce a solution measure, so that the effect of bifurcation is only visible if such generalized solutions are taken into consideration [2]. Thus, the notion of a solution measure is important for understanding the dynamics of the solution of stochastic equations. In the case of the Carathéodory flows the problem of extension is trivial (see [2, p. 28]). In the general case the problem is solved by Corollaries 5.4-5.7.

6. Generalized cocycles

We keep fixed a filtered probability space

$$(\Omega,\Sigma,(\Sigma_t)_{t\in\mathbb{R}^+},\mathbb{P}) \tag{6.1}$$

satisfying the usual conditions (see e.g. [2]). In addition, we assume that $\mathbb{T}$ is a subsemigroup of the additive group $\mathbb{R}$ with the Borel s-algebra on it. In what follows we use a measurable and measure-preserving dynamical system $(\Omega,(\theta(\tau)_{\tau\in\mathbb{T}},\mathbb{P})$, which is consistent with the filtration $(\Sigma_t)_{t\in\mathbb{R}^+}$, i.e. a family $\theta(\tau)\colon\Omega\to\Omega$ satisfies [2] satisfying

(i) $(\omega,\tau)\mapsto\theta(\omega,\tau)$ is measurable,
(ii) $\theta(\cdot,0)=id_\Omega$,
(iii) $\theta(\cdot,\tau+\sigma)=\theta(\cdot,\tau)\theta(\cdot,\sigma)$ for all $\tau,\sigma\in\mathbb{T}$,
(iv) $\theta(\tau)_\sharp\mathbb{P}=\mathbb{P}$ for all $\tau\in\mathbb{T}$,

(v) $\theta(\tau)(\Sigma_t) = \Sigma_{\tau+t}$ for all $\tau \in \mathbb{T},\ t \in \mathbb{R}^+$.

Now, for a family of Carathéodory mappings $V_t\colon \Omega \times X \to X$ $(t \in \mathbb{R}^+)$, defined on a Polish space X, the cocycle property [2] with respect to the semigroup $\mathbb{T}$ means that

$$V_{\tau+t}(\omega, x) = V_t(\theta(\tau,\omega), x) \circ V_\tau(\omega, x) \tag{6.2}$$

a.s. for each $t \in \mathbb{R}^+$, $\tau \in \mathbb{T}$ $(t \geq \tau)$, $x \in X$.

Example 6.1. Let $\mathbb{T} = \mathbb{R}^+$ and $\theta(\cdot, \tau)$ be the Wiener shift satisfying

$$W(\omega, t+\tau) - W(\omega, s+\tau) = W(\theta(\tau,\omega), t) - W(\theta(\tau,\omega), s) \text{ a.s.}$$

for every τ, t, s in $\mathbb{R}^+$ (see e.g. [3]). Here and below $W(\omega, t)$ stands for the scalar Wiener process. Consider an Itô equation in $\mathbb{R}^n$

$$dx(t) = a(x(t))\,dt + b(x(t))\,dW(t) \tag{6.3}$$

under the usual assumptions (e.g. a, b are non-random and uniformly Lipschitz) implying that there is a unique (up to a natural equivalence) solution $x_\alpha(t)$ for any (random) initial value $x(0) = \alpha \in L^0(\Omega, \Sigma_0, \mathbb{P}; B)$. Then the solution flow defined by $V_t(\omega, \alpha) := x_\alpha(\omega, t)$ is well-known to have the cocycle property [2]. Assuming instead in (6.3) that a and b depend on time and are γ-periodic yields a cocycle over the sub-semigroup $\gamma\mathbb{N}$.

On the other hand, it was shown by S.-E. A. Mohammed (see [14]) that the evolution operators constructed for stochastic delay equations (see Section 5) can be *non-regular*, in the sense that they do not give rise to a Carathéodory solution flow (which he calls *regular*). The difference between regular and non-regular cases is crucial: no cocycle property for non-regular equations. Thus, the Lyapunov exponents can only be constructed for regular flows. A typical example of a non-regular equation can be as simple as $dx(t) = x(t-h)dW(t)$.

The following example shows that stochastic evolution equations may also give rise to non-regular solution flows.

Example 6.2. Consider an orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$ in a separable Hilbert space $\mathcal{H}$. Let $B_k(t)$ $(t \geq 0,\ k \in \mathbb{N})$ be independent standard Brownian motions.

We define an (unbounded) linear operator $\mathcal{A}$ by $\mathcal{A}(\sum_{k\in\mathbb{N}} x_k e_k) := \sum_{k\in\mathbb{N}} n x_k e_k$ and an infinite dimensional Wiener process $W(t)$ in $\mathcal{H}$ by $W(t) := \sum_{k\in\mathbb{N}} \frac{1}{k} B_k(t) e_k$. Clearly, the covariance operator Q for this process is given by $Q = \operatorname{diag}[\frac{1}{k^2}]_{k\in\mathbb{N}}$ which is a trace-class operator, so that $W(t)$ is a Q-Wiener process (see e.g. [9, pp. 52-53]). Below we assume that the σ-algebra Σ_t is generated by W_s, $0 \leq s \leq t$.

The stochastic differential equation

$$dx(t) = \mathcal{A}x(t)dW(t) \tag{6.4}$$

is diagonal with the evolution operator given by

$$U(t,s)(x) := \operatorname{diag}[g_k(t,s)]_{k\in\mathbb{N}}(x),$$

where $g_k(t,s) := \exp(B_k(t) - B_k(s) - (t-s)/2)$, $t \geq s \geq 0$.

For all $t \geq s \geq 0$ the operator

$$U(t,s)\colon L^2(\Omega, \Sigma_s, \mathbb{P}; \mathcal{H}) \to L^2(\Omega, \Sigma_t, \mathbb{P}; \mathcal{H})$$

is bounded. To see it, we observe that g_k satisfy $\mathbb{E}g_k(t,s) = 1$ for all $t \geq s \geq 0$ and every $k \in \mathbb{N}$.

Take now an arbitrary $x := \sum_{k\in\mathbb{N}} x_k e_k \in L^2(\Omega, \Sigma_s, \mathbb{P}; \mathcal{H})$, the norm in the latter space denoted by $\|\cdot\|_{L^2(\mathcal{H})}$. As $g_k(t,s)$ is independent of x_k, we have

$$\begin{aligned}\mathbb{E}\left\|\sum g_k(t,s)x_k\right\|^2_{\mathcal{H}} &= \mathbb{E}\sum g_k^2(t,s)x_k^2\\ &= \sum \mathbb{E}g_k^2(t,s)\mathbb{E}x_k^2 \qquad \text{(by independence)}\\ &= \mathbb{E}\sum x_k^2 = \|x\|_{L^2(\mathcal{H})},\end{aligned}$$

so that the operator norm $\|U(t,s)\| = 1$ for all $t \geq s \geq 0$. In addition, $U(t,s)$ is local and thus extends to a continuous operator from $L^0(\Omega, \Sigma_s, \mathbb{P}; \mathcal{H})$ to $L^0(\Omega, \Sigma_t, \mathbb{P}; \mathcal{H})$.

On the other hand, the random variables $h_k(t,s) := g_k(t,s) - 1$ are independent and normally distributed random variables with the law $\mathcal{N}(0, e^{(t-s)} - 1)$, where $t \geq s \geq 0$.

For any $R > 0$, $t \geq s > 0$ we then have

$$\begin{aligned}\mathbb{P}\left\{\sup_{k\in\mathbb{N}} |h_k(t,s)| < R\right\} &= \mathbb{P}\left\{\left(\bigcap_{k\in\mathbb{N}} \{\omega\colon |h_k(t,s)| < R\}\right)\right\}\\ &= \prod_{k\in\mathbb{N}} \mathbb{P}\{\omega\colon |h_k(t,s)| < R\} = \exp\left\{-\frac{\sqrt{2}}{\pi}\sum_{k\in\mathbb{N}} m_R\right\} = 0,\end{aligned}$$

where

$$m_R := \int_{\frac{R}{\sqrt{\exp(t-s)-1}}}^{\infty} \exp\left\{-\frac{1}{2}x^2\right\} dx > 0.$$

Thus, $\sup_{k\in\mathbb{N}} \|U(t,s)(e_k)\|_{\mathcal{H}} = \infty$ a.s. This means that the evolution operator $U(t,s)$ is non-Carathéodory for any $t \geq s > 0$.

In Section 5 we proved that under the existence and uniqueness assumptions the evolution operator is always local in ω (even if it is not Carathéodory). But in the non-Carathéodory case we do not have the cocycle property. Thus, we need a generalization of this concept based on the evolution operators, rather than on solution flows. That is why we have to be more specific about the domains and the range of the involved operators.

Let us assume that we are given an evolution family U_t^s of local operators (e.g. a solution flow to some stochastic differential equation) that for some $p \geq 0$ act continuously from $L^p(\Omega, \Sigma_s, \mathbb{P}; X)$ to $L^p(\Omega, \Sigma_t, \mathbb{P}; X)$, and in addition, we have the isometries

$$T_{\theta(\tau,\cdot)}\colon L^p(\Omega, \Sigma_t, \mathbb{P}; X) \to L^p(\Omega, \Sigma_{t+\tau}, \mathbb{P}; X).$$

Definition 6.3. *The generalized cocycle property with respect to the semigroup* $\mathbb{T}$ *is given by*

$$U_{t+\tau} = T_{\theta(\tau,\cdot)} \circ U_t \circ T^{-1}_{\theta(\tau,\cdot)} \circ U_\tau, \quad (t \in \mathbb{R}^+, \tau \in \mathbb{T};\ t \geq \tau \geq 0). \tag{6.5}$$

In the case when the evolution operators U are given by Carathéodory solution flows V_t, i.e. when $U_t = N_{V_t}$, it is easy to check (e.g. using an arbitrary $x \in L^p(\Omega, \Sigma_s, \mathbb{P}; X)$) that (6.5) gives (6.2).

The following theorem, which deals with the equations (5.1) and (5.3), justifies the above definition.

Theorem 6.4. *Assume that the semimartingale* $Z(t)$ *on the filtered probability space* (6.1) *is a helix with respect to the dynamical system* $(\theta(\tau,\cdot))_{\tau\in\mathbb{T}}$, *i.e.*

$$Z(\omega, t+\tau) - Z(\omega, s+\tau) = Z(\theta(\tau,\omega), t) - Z(\theta(\tau,\omega), s) \ \text{a.s.} \tag{6.6}$$

for every $\tau \in \mathbb{T}$, $t, s \in \mathbb{R}^+$. *We have*

(i) *if* $s = 0$ *and* $f(\theta(\tau,\omega), t, x) = f(\omega, t+\tau, x)$ *a.s. for all* $\tau \in \mathbb{T}$, $t \in \mathbb{R}^+$, $x \in H$, *then the evolution operator* U_t^0 *for the equation* (5.1) *satisfies the generalized cocycle property* (6.5) *in the space* $X = H$*;*

(ii) *if* $s = 0$ *and* $F(\theta(\tau,\omega), t, \varphi) = F(\omega, t+\tau, \varphi)$ *a.s. for all* $\tau \in \mathbb{T}$, $t \in \mathbb{R}^+$, $\varphi \in S$, *then the evolution operator* U_t^0 *for the equation* (5.3) *satisfies the generalized cocycle property* (6.5) *in the space* $X = S$.

Proof. We only verify the statement (i), since (ii) can be proven similarly.

Due to the evolution property (5.2) it is suffices to check that

$$U_t = T^{-1}_{\theta(\tau,\cdot)} \circ U^{\tau}_{t+\tau} \circ T_{\theta(\tau,\cdot)}, \tag{6.7}$$

or in other words, that for any $\varphi \in L^p(\Omega, \Sigma_\tau, \mathbb{P}; H)$, the stochastic process $y(t) := T^{-1}_{\theta(\tau,\cdot)}\left(U^{\tau}_{t+\tau}\varphi\right)$ will be a solution to the equation (5.1) for $t > \tau$, satisfying $y(\tau) = T^{-1}_{\theta(\tau,\cdot)}\varphi$. The latter equality is evident, so we will concentrate on $y(t)$ for $t > 0$.

We will use the following property of the helices:

$$T^{-1}_{\theta(\tau,\cdot)}\left(\int_s^t \vartheta(u)dZ(u)\right) = \int_s^t (T^{-1}_{\theta(\tau,\cdot)}\vartheta)(u)dZ(u-\tau) \tag{6.8}$$

for any $t, s \in \mathbb{R}^+$, $\tau \in \mathbb{T}$, which holds for all predictable stochastic processes ϑ that are integrable with respect to the semimartingale Z (see e.g. [18] for the case $\mathbb{T} = \mathbb{R}^+$, the proof for $\mathbb{T} \neq \mathbb{R}^+$ is similar).

As $Y(t+\tau) = Y(\tau) + \int_\tau^{t+\tau} f(u, Y(u))Z(u)$ where $Y(t) = U_t^\tau\varphi$, we obtain due to (6.8)

$$\begin{aligned}
y(t) &= T^{-1}_{\theta(\tau,\cdot)}(Y(t+\tau)) = T^{-1}_{\theta(\tau,\cdot)}\left(Y(\tau) + \int_\tau^{t+\tau} f(u, Y(u))dZ(u)\right)\\
&= T^{-1}_{\theta(\tau,\cdot)}(Y(\tau)) + \int_\tau^{t+\tau} T^{-1}_{\theta(\tau,\cdot)}\left(f(u, Y(u))\right)dZ(u-\tau)\\
&= T^{-1}_{\theta(\tau,\cdot)}(Y(\tau)) + \int_0^{t} T^{-1}_{\theta(\tau,\cdot)}\left(f(v+\tau, Y(v+\tau))\right)dZ(v)\\
&= T^{-1}_{\theta(\tau,\cdot)}(Y(\tau)) + \int_0^{t} \left(T^{-1}_{\theta(\tau,\cdot)} \circ T_{\theta(\tau,\cdot)}\right)\left(f(v, y(v))\right)dZ(v)\\
&= T^{-1}_{\theta(\tau,\cdot)}(Y(\tau)) + \int_0^{t} \left(f(v, y(v))\right)dZ(v),
\end{aligned}$$

because $T_{\theta(\tau,\cdot)}(f(v,x) = f(v+\tau, x))$ for all $x \in V$ and $y(v) = T^{-1}_{\theta(\tau,\cdot)}Y(v+\tau)$ by assumption. This means that $y(t)$ satisfies (5.1), and the result follows. $\square$

7. Invariant measures for stochastic dynamical systems with the generalized cocycle property

In this section we apply the general results of Section 4 to stochastic equations. First of all, we observe that even when the solution flow is regular, a natural invariant measure will be a Young measure on $\Omega \times X$, where X is the phase space (see e.g. [2], [14]). In the regular case we, however, can naturally extend this solution flow to measures on $\Omega \times X$ by setting $\mu \mapsto (V_t)_\sharp\mu$, which is well-defined and continuous in the narrow topology. As we saw in the previous sections, the problem becomes much more involved in the non-regular case, i.e. when the evolution operators do not come from the Carathéodory solution flows.

In what follows we, as before, use the isometries

$$T_{\theta(\tau,\cdot)} \colon L^0(\Omega, \Sigma_t, \mathbb{P}; X) \to L^0(\Omega, \Sigma_{t+\tau}, \mathbb{P}; X).$$

We also recall that $\bar{T}$ is the continuous extension of an operator T to the set of Young measures μ satisfying $\mu(\Omega \times X) = 1$.

Theorem 7.1. *Assume that for some $p \in [1, +\infty)$ an evolution family U_t^s consisting of local operators that act continuously from $L^p(\Omega, \Sigma_s, \mathbb{P}; X)$ to $L^p(\Omega, \Sigma_t, \mathbb{P}; X)$, possesses the generalized cocycle property* (6.5) *with respect to a given additive semigroup $\mathbb{T}$. Assume further that there exist $R > 0$, $h > 0$ such that for any $u \in L^p(\Omega, \Sigma_0, \mathbb{P}; X)$ and for all $\tau \in \mathbb{T}, \tau > h$*

$$\mathbb{E}\|u\|^p < R \quad \textit{implies} \quad \mathbb{E}\|U_\tau^0 u\|^p < R. \tag{7.1}$$

Finally, assume that for some $s \in \mathbb{T}$ the set $U_s^0(B)$ is tight in $L^0(\Omega, \Sigma, \mathbb{P}; X)$, where $B := \{u \in L^p(\Omega, \Sigma_0, \mathbb{P}; X) \,:\, \mathbb{E}\|u\|^p < R\}$. Then there exists at least one measure in $\bar{B}$, where $\bar{B}$ stands for the narrow closure of the set $\{\delta_u \,:\, u \in B\}$, for which

$$\bar{U}_{t+\tau}^0 \mu = \left(\bar{T}_{\theta(\tau,\cdot)} \circ \bar{U}_t^0\right) \mu \tag{7.2}$$

for any $\tau \in \mathbb{T}$, $t \in R^+$.

Proof. Using Proposition 3.1 we can assume that the family U_t^s consists of local and continuous operators from $L^0(\Omega, \Sigma_s, \mathbb{P}; X)$ to $L^0(\Omega, \Sigma_t, \mathbb{P}; X)$.

We wish to apply Corollary 4.5. To do it, we introduce the family of continuous atomic operators

$$T_\tau := T_{\theta(\tau,\cdot)}^{-1} \circ U_\tau^0 \colon L^0(\Omega, \Sigma_0, \mathbb{P}; X) \to L^0(\Omega, \Sigma_0, \mathbb{P}; X).$$

Due to the cocycle property (6.5) we have

$$\begin{aligned} T_\tau \circ T_\sigma &= T_{\theta(\tau,\cdot)}^{-1} \circ U_\tau^0 \circ T_{\theta(\sigma,\cdot)}^{-1} \circ U_\sigma^0 \\ &= T_{\theta(\tau+\sigma,\cdot)}^{-1} \circ T_{\theta(\sigma,\cdot)} \circ U_\tau^0 \circ T_{\theta(\sigma,\cdot)}^{-1} \circ U_\sigma^0 \\ &= T_{\theta(\tau+\sigma,\cdot)}^{-1} \circ U_{\tau+\sigma}^0 \\ &= T_{\tau+\sigma} \end{aligned}$$

for any $\tau, \sigma \in \mathbb{T}$, which means that this family is commutative. Consider the subfamily T_τ ($\tau \in \mathbb{T}, \tau > h$). For a sufficiently large $\tau \in \mathbb{T}, \tau > h$ we will definitely have that $\tau + s \in T$, $\tau + s > h$ and the operator $T_{\tau+s} = T_s \circ T_\tau$ is tight. Corollary 4.5 gives then a common invariant measure $\mu \in \bar{B}$ for the above subfamily. However, if we take an arbitrary $\eta \in \mathbb{T}$ and sufficiently large $\tau \in \mathbb{T}$ such that $\eta + \tau > h$, then

$$T_\eta \mu = (T_\eta \circ T_\tau)\, \mu = T_{\eta+\tau} \mu = \mu.$$

This proves (7.2) for $t = 0$.

Finally, making advantage of the generalized cocycle property once again yields

$$\bar{U}_{t+\tau}^0 \mu = \left(\bar{T}_{\theta(\tau,\cdot)} \circ \bar{U}_t \circ \bar{T}_{\theta(\tau,\cdot)}^{-1} \circ \bar{U}_\tau^0\right) \mu = \left(\bar{T}_{\theta(\tau,\cdot)} \circ \bar{U}_t^0\right) \bar{T}_\tau \mu = \left(\bar{T}_{\theta(\tau,\cdot)} \circ \bar{U}_t^0\right) \mu,$$

and the result follows. □

The equality (7.2) says that if $\mathbb{T} = \mathbb{R}^+$, then the solution measure starting at μ will be stationary (in distribution), while in the case $\mathbb{T} = \gamma\mathbb{N}$ the solution measure will be γ-periodic (again in the sense of distributions).

We consider at last a model example where the assumptions of Theorem 7.1 can easily be verified. The result we provide is not meant to be of the most general character, and is intended just to illustrate the application of the abstract theory developed in the paper. However, we stress that it is new and covers many interesting cases, including those where very little or nothing is known about invariant measures. In the example we use the Delfour-Mitter space $S := L^2([-h, 0)]; \mathbb{R}^n) \times \mathbb{R}^n$, with the norm

$$\|\varphi\|_S^2 := |\varphi(0)|^2 + \int_{-h}^0 |\varphi(\sigma)|^2 d\sigma = \int_{-h}^0 |\varphi(\sigma)|^2 d\lambda(\sigma), \tag{7.3}$$

where $\varphi := (\varphi(\cdot), \varphi(0))$, $\varphi(\cdot) \in S$, $\varphi(0) \in \mathbb{R}^n$ and λ is the sum of the Lebesgue measure on $[-h, 0]$ and the Dirac measure at $\sigma = 0$.

Theorem 7.2. *Assume that the equation* (5.3) *satisfies the following conditions:*

(i) $Z(t) = (t, W^1(t), \ldots, W^k(t))^T$, *where* $t \geq 0$ *and* $W^i(t)$, $i = 1, \ldots, k$ *are independent Wiener processes;*

(ii) $F(\omega, t, \varphi) = A\varphi + F_0(\omega, t, \varphi)$, *where* $t \geq 0$, $\varphi \in S$, $\omega \in \Omega$, A *is a stable (Hurwitz)* $n \times n$ *matrix and* $F_0 \colon \Omega \times \mathbb{R}^+ \times S \to \mathbb{R}^{n \times m}$, $m = k + 1$, *is a (nonlinear) operator continuous in the third variable (i.e. in* φ*) and measurable in the first two variables, being in addition adapted in* ω *and satisfying* $F_0(\theta(\tau, \omega), t, \varphi) = F_0(\omega, t + \tau, \varphi)$ *a.s. for all* $\tau \in \mathbb{T}$, $t \in \mathbb{R}^+$, $\varphi \in C$, *where either* $\mathbb{T} = \gamma\mathbb{N}$ *for some* $\gamma > 0$, *or* $\mathbb{T} = \mathbb{R}^+$, *and* $\theta(\tau, \cdot)$ *is the standard Wiener shift;*

(iii) *for some* $\tau_0 \in \mathbb{T}$, $\tau_0 > 0$, *one has*

$$\mathbb{E}|F_0(t, \varphi)|^2 \leq K \qquad (t \geq 0,\ \varphi \in S)$$

for some $K > 0$*;*

(iv) *for every* $T > 0$, *and* $\varphi \in S$ *there exists a unique solution* $x(\cdot)$ *of the equation* (5.3) *satisfying* (5.4) *and belonging to* $\mathcal{A} \cap L^2(\Omega, \Sigma_T, \mathbb{P}; L^2([0, T]; \mathbb{R}^n))$ *(*$\mathcal{A}$ *is the set of all* n*-dimensional* Σ_t*-adapted stochastic processes) and depending continuously (in the respective topologies) on* φ *(in particular, this is true if* F_0 *is locally Lipschitz in the third variable).*

Then there exists a generalized invariant measure μ *for the solution flow of the equation* (5.3), *i.e. a Young measure satisfying*

$$\bar{U}^0_{t+\tau}\mu = \left(\bar{T}_{\theta(\tau,\cdot)} \circ \bar{U}^0_t\right)\mu \tag{7.4}$$

for any $\tau \in \mathbb{T}$, $t \in \mathbb{R}^+$, *where* $U^0_{t+\tau}$ *stands for the family of evolution operators corresponding to the solution flow of* (5.3).

The proof of this theorem will be given in the appendix B.

Appendix A. Representation of local functionals and operators

Here and below we assume that $(\Omega, \Sigma, \mathbb{P})$ is a probability space with complete σ-algebra Σ, and X is a Polish space. The space $L^1(\Omega, \Sigma, \mathbb{P}; X)$ will be then abbreviated to $L^1(\Omega; X)$. We recall the following definitions from [4].

Definition A.1. *A functional* $I \colon L^1(\Omega; X) \times \Sigma \to \bar{\mathbb{R}} := \mathbb{R} \times \{\pm\infty\}$ *is called*

(i) *local, if for every* $\{u, v\} \subset L^0(\Omega; X)$ *and every* $A \in \Sigma$ *one has*

$$I(u, A) = I(v, A) \text{ whenever } u(\omega) = v(\omega) \text{ for } \mathbb{P}\text{-a.e. } \omega \in A;$$

(ii) *additive, if when* $A \in \Sigma$ *and* $B \in \Sigma$ *are disjoint, i.e.* $A \cap B = \emptyset$, *then*

$$I(u, A \cup B) = I(u, A) + I(u, B)$$

for every $u \in L^1(\Omega; X)$.

Definition A.2. *A function* $f \colon \Omega \times X \to \bar{\mathbb{R}}$ *is called*

(i) *integrand, if it is* $\Sigma \otimes \mathscr{B}(X)$*-measurable;*

(ii) *normal integrand, if it is an integrand, while* $f(\omega, \cdot)$ *is l.s.c. for* $\mathbb{P}$*-a.e.* $\omega \in \Omega$.

We also need the following lemma which is a slightly adapted version of proposition 2.1.3 from [4].

Lemma A.3. *Let* f, $g \colon \Omega \times X \to \bar{\mathbb{R}}$ *be two nonnegative integrands such that*

$$\int_e f(\omega, u(\omega))\, d\mathbb{P}(\omega) \leq \int_e g(\omega, u(\omega))\, d\mathbb{P}(\omega) \tag{A.1}$$

for every $(u, e) \in L^0(\Omega; X) \times \Sigma$. *Then there is a* $N \subset \Omega$ *with* $\mathbb{P}(N) = 0$ *such that*

$$f(\omega, x) \leq g(\omega, x)$$

for all $(\omega, x) \in (\Omega \setminus N) \times X$*. In particular, if*

$$\int_e f(\omega, u(\omega))\, d\mathbb{P}(\omega) = \int_e g(\omega, u(\omega))\, d\mathbb{P}(\omega),$$

then one has

$$f(\omega, x) = g(\omega, x)$$

for all $(\omega, x) \in (\Omega \setminus N) \times X$*.*

Proof. Suppose (A.1) holds. To prove the claim, it is enough to show for every $k \in \mathbb{N}$ that

$$f_k(\omega, x) \leq g_k(\omega, x)$$

for $(\omega, x) \in (\Omega \setminus N_k) \times X$, where $f_k := f \wedge k$, $g_k := g \wedge k$, and $N_k \subset \Omega$ is a $\mathbb{P}$-nullset. For this purpose for every $l \in \mathbb{N}$ define

$$\begin{aligned} S_{l,k} &:= \{(\omega, x) \in \Omega \times X \,:\, f_k(\omega, x) \geq g_k(\omega, x) + 1/l\}, \\ S_{l,k}(\omega) &:= \{x \in X \,:\, (\omega, x) \in S_{l,k}\}. \end{aligned}$$

Since Σ is supposed to be complete, then by the projection theorem (theorem III.23 from [7]) the set

$$\Omega_{l,k} := \{\omega \in \Omega \,:\, S_{l,k}(\omega) \neq \emptyset\} \in \Sigma,$$

while by the Aumann measurable selection theorem (theorem III.22 from [7]) there is a Σ-measurable function $s_{l,k}\colon \Omega_{l,k} \to X$ such that $s_{l,k}(\omega) \in S_{l,k}(\omega)$ for all $\omega \in \Omega_{l,k}$. We extend this map to the whole Ω setting $s_{l,k}(\omega) := x_0$ for all $\omega \notin \Omega_{l,k}$, where $x_0 \in X$ is an arbitrarily chosen element of X.

By definition of $s_{l,k}$ one has then

$$f_k(\omega, s_{l,k}(\omega)) \geq g_k(\omega, s_{l,k}(\omega)) + 1/l$$

for all $\omega \in \Omega_{l,k}$. But since $f_k \leq k$, then $g_k(\omega, s_{l,k}(\omega)) < k$, and hence, according to the definition of g_k, one gets

$$g_k(\omega, s_{l,k}(\omega)) = g(\omega, s_{l,k}(\omega))$$

for $\omega \in \Omega_{l,k}$. The latter equality implies

$$g(\omega, s_{l,k}(\omega)) + 1/l = g_k(\omega, s_{l,k}(\omega)) + 1/l \leq f_k(\omega, s_{l,k}(\omega)) \leq f(\omega, s_{l,k}(\omega)).$$

Integrating the above inequality over $\Omega_{l,k}$ provides

$$\int_{\Omega_{l,k}} g(\omega, s_{l,k}(\omega))\, d\mathbb{P}(\omega) + \mathbb{P}(\Omega_{l,k})/l \leq \int_{\Omega_{l,k}} f(\omega, s_{l,k}(\omega))\, d\mathbb{P}(\omega).$$

Taking into account that in view of (A.1) one has

$$\int_{\Omega_{l,k}} f(\omega, s_{l,k}(\omega))\, d\mathbb{P}(\omega) \leq \int_{\Omega_{l,k}} g(\omega, s_{l,k}(\omega))\, d\mathbb{P}(\omega),$$

we get

$$\int_{\Omega_{l,k}} g(\omega, s_{l,k}(\omega))\, d\mathbb{P}(\omega) + \mathbb{P}(\Omega_{l,k})/l \leq \int_{\Omega_{l,k}} g(\omega, s_{l,k}(\omega))\, d\mathbb{P}(\omega).$$

Minding now that

$$0 \leq \int_{\Omega_{l,k}} g(\omega, s_{l,k}(\omega))\, d\mathbb{P}(\omega) \leq k,$$

since $0 \leq g = g_k \leq k$ over $\Omega_{l,k}$, we arrive finally at the conclusion that $\mathbb{P}(\Omega_{l,k}) = 0$. We may set now

$$N_k := \bigcup_{l \in \mathbb{N}} \Omega_{l,k}$$

yielding $\mathbb{P}(N_k) = 0$ and

$$f_k(\omega, x) \leq g_k(\omega, x)$$

for $(\omega, x) \in (\Omega \setminus N_k) \times X$ which concludes the proof. $\square$

We claim now the following statements.

Proposition A.4. *Let the functional $I\colon L^0(\Omega;X)\times\Sigma\to\mathbb{R}$ be local, additive and bounded from below, i.e. $I(u)\geq c$ for some $c\in\mathbb{R}$ and for all $u\in L^1(\Omega;X)$. Assume, moreover, that*

(i) *there is an $u_0\in L^1(\Omega;X)$ such that $I(u_0,\cdot)$ is a signed measure absolutely continuous with respect to $\mathbb{P}$;*

(ii) *the functional $I(\cdot,\Omega)$ is l.s.c.*

Then there is a normal integrand $f\colon\Omega\times X\to\bar{\mathbb{R}}$ such that

$$I(u,A)=\int_A f(\omega,u(\omega))\,d\mathbb{P}(\omega) \tag{A.2}$$

for all $(u,A)\in L^1(\Omega,X)\times\Sigma$. Moreover, such an integrand is unique in the sense that whenever for some integrand $g\colon\Omega\times X\to\bar{R}$ one has

$$I(u,A)=\int_A g(\omega,u(\omega))\,d\mathbb{P}(\omega)$$

for all $(u,A)\in L^1(\Omega,X)\times\Sigma$, then $g(\omega,x)=f(\omega,x)$ for all $(\omega,x)\in(\Omega\setminus N)\times X$, where $N\subset\Omega$ satisfies $\mathbb{P}(N)=0$.

Proof. The proof follows the lines of that of the analogous theorem 2.4.2 from [4] which is formulated for functionals defined over a Lebesgue space (instead of L^0) of functions with values in a finite-dimensional space $\mathbb{R}^n$ (instead of a generic Polish space X). The generalization for our case is quite straightforward though technical and therefore we provide here the proof just for the readers convenience.

Step 1. Without loss of generality we may assume $c=0$, hence $I(u,e)\geq 0$ for all $(u,e)\in L^0(\Omega;X)\times\Sigma$. For every $k\in\mathbb{N}$ define the Moreau-Yosida transform $I_k(\cdot,e)$ of the functional $I(\cdot,e)$ by the formula

$$I_k(u,e):=\inf\left\{I(v,e)+k\int_e d(u(\omega),v(\omega))\wedge 1\,d\mathbb{P}(\omega)\ :\ v\in L^0(\Omega;X)\right\},$$

where d stands for the distance in X. One has then

$$0\leq I_k(u,e)\leq k\int_e d(u(\omega),u_0(\omega))\wedge 1\,d\mathbb{P}(\omega). \tag{A.3}$$

According to proposition 1.3.7 from [4] one has that each $I_k(\cdot,e)$ is Lipschitz continuous over $L^0(e,\Sigma\cap e,\mathbb{P};X)$ with Lipschitz constant k, namely,

$$|I_k(u,e)-I_k(v,e)|\leq k\int_e d(u(\omega),v(\omega))\wedge 1\,d\mathbb{P}(\omega). \tag{A.4}$$

For every $x\in X$ the set function $I_k(x\mathbf{1}_\Omega,\cdot)$ is additive on disjoint sets and, in view of (A.3) and (i), is bounded from above by a finite measure which is absolutely continuous with respect to $\mathbb{P}$. Therefore, $I_k(x\mathbf{1}_\Omega,\cdot)$ is a finite measure which is also absolutely continuous with respect to $\mathbb{P}$. Denote by $f_k(\omega,x)$ the Radon-Nikodym derivative of $I_k(x\mathbf{1}_\Omega,\cdot)$ with respect to $\mathbb{P}$, i.e.

$$I_k(x\mathbf{1}_\Omega,e)=\int_e f_k(\omega,x)\,d\mathbb{P}(\omega).$$

Let $D:=\{x_i\}_{i=1}^\infty\subset X$ stand for a countable dense subset of X. From (A.3) and (A.4) it follows then that

$$\begin{aligned}&0\leq f_k(\omega,x_i)\leq a(\omega)+k(d(x_i,u_0(\omega))\wedge 1),\\&|f_k(\omega,x_i)-f_k(\omega,x_j)|\leq k(d(x_i,x_j)\wedge 1)\end{aligned} \tag{A.5}$$

for some $a\in L^1(\Omega;\mathbb{R})$ and for all $\{i,j\}\in\mathbb{N}$ and $\omega\in\Omega\setminus N_{ij}$ where $N_k\subset\Omega$ is some set satisfying $\mathbb{P}(N_{ij})=0$. Let

$$N:=\bigcup_{\{i,j\}\subset\mathbb{N}} N_{ij}.$$

Fix an $\omega \in \Omega \setminus N$. Since the function $f_k(\omega, \cdot)\colon D \to \mathbb{R}$ is Lipschitz continuous over D according to (A.5), then it admits a Lipschitz continuous extension to the whole X. Namely, there is a function $g_k(\omega, \cdot)\colon X \to \mathbb{R}$ which is still Lipschitz continuous and $f_k(\omega, x) = g_k(\omega, x)$ for all $x \in D$. We set now

$$\tilde{f}_k(\omega, x) := \begin{cases} g_k(\omega, x), & \omega \in \Omega \setminus N, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to verify that $\tilde{f}_k$ is a Carathéodory function (moreover, $\tilde{f}_k(\omega, \cdot)$ is Lipschitz continuous), while

$$0 \le \tilde{f}_k(\omega, x) \le a(\omega) + k(d(x, u_0(\omega)) \wedge 1) \tag{A.6}$$

for all $\omega \in \Omega$ and $x \in X$.

Finally, we observe that

$$I_k(u, e) = \int_e \tilde{f}_k(\omega, u(\omega))\, d\mathbb{P}(\omega) \tag{A.7}$$

for every $u \in L^0(\Omega; X)$ and $e \in \Sigma$. In fact, (A.7) is clearly valid for any simple function (i.e. the function with the finite number of values) with values in D. But the latter form a dense subset of $L^0(\Omega; X)$. Thus approximating an arbitrary $u \in L^0(\Omega; X)$ by a sequence $\{u_i\}_{i=1}^{\infty} \subset L^0(\Omega; X)$ of simple functions with values in D, i.e. $u_i \to u$ in $L^0(\Omega; X)$ as $i \to \infty$, we get

$$I_k(u_i, e) = \int_e \tilde{f}_k(\omega, u_i(\omega))\, d\mathbb{P}(\omega) \to \int_e \tilde{f}_k(\omega, u(\omega))\, d\mathbb{P}(\omega)$$

as $i \to \infty$ in view of the estimate (A.6) and the Lebesgue theorem. On the other hand, since $I_k(\cdot, e)$ is Lipschitz continuous over $L^0(\Omega; X)$ as follows from (A.4), then $I_k(u_i, e) \to I_k(u, e)$ as $i \to \infty$, which shows (A.7).

Step 2. For $l \ge k$ we have $0 \le I_k(u, e) \le I_l(u, e)$ for all $(u, e) \in L^0(\Omega; X) \times \Sigma$. This implies in view of Lemma A.3 the existence of a set $\tilde{N} \subset \Omega$ with $\mathbb{P}(\tilde{N}) = 0$ such that

$$0 \le \tilde{f}_k(\omega, x) \le \tilde{f}_l(\omega, x)$$

for all $(\omega, x) \in (\Omega \setminus \tilde{N}) \times X$. We set now

$$f(\omega, x) := \begin{cases} \sup_{k \in \mathbb{N}} \tilde{f}_k(\omega, x), & \omega \in \Omega \setminus \tilde{N}, \\ 0, & \text{otherwise.} \end{cases} \tag{A.8}$$

Obviously, f is a normal integrand as a supremum of an increasing sequence of Carathéodory functions. But (ii) in view of additivity and locality of $I(u, \cdot)$ implies that $I(\cdot, e)$ is l.s.c. for every $e \in \Sigma$, and hence, by proposition 1.3.7 from [4] one has

$$I(u, e) = \sup_{k \in \mathbb{N}} I_k(u, e)$$

for every $(u, e) \in L^0(\Omega; X) \times \Sigma$. Minding the representation (A.7) and the relationship (A.8), we get with the help of the Beppo Levi theorem

$$I(u, e) = \int_e f(\omega, u(\omega))\, d\mathbb{P}(\omega),$$

which concludes the proof of existence.

The uniqueness follows immediately from the second claim of Lemma A.3. □

Corollary A.5. *Let the functional $I\colon L^0(\Omega; X) \times \Sigma \to \mathbb{R}$ be local, additive and bounded both from above and from below, i.e. $c \le I(u) \le C$ for some $c, C \in \mathbb{R}$ and for all $u \in L^1(\Omega; X)$. Assume, moreover, that*

(i) *there is an $u_0 \in L^1(\Omega; X)$ such that $I(u, \cdot)$ is a signed measure absolutely continuous with respect to $\mathbb{P}$;*
(ii) *the functional $I(\cdot, \Omega)$ is continuous.*

Then there is a Carathéodory function $f\colon \Omega \times X \to \bar{\mathbb{R}}$ *such that*

$$I(u,e) = \int_e f(\omega, u(\omega))\, d\mathbb{P}(\omega) \tag{A.9}$$

for all $(u,e) \in L^0(\Omega, X) \times \Sigma$. *Moreover, such a function is unique in the sense announced in Proposition A.4.*

Proof. According to Proposition A.4, there is a unique normal integrand $f\colon \Omega\times X \to \bar{\mathbb{R}}$ such that (A.9) is valid. Analogously, since $-I$ also satisfies the conditions of Proposition A.4, then

$$-I(u,e) = \int_e g(\omega, u(\omega))\, d\mathbb{P}(\omega)$$

for a unique normal integrand g: $\Omega \times X \to \bar{\mathbb{R}}$. On the other hand, from (A.9) one gets

$$-I(u,e) = \int_e (-f(\omega, u(\omega)))\, d\mathbb{P}(\omega),$$

and the uniqueness of the integrand g representing the functional $-I$ provides

$$g(\omega, x) = -f(\omega, x)$$

for all $x \in X$ and for $\mathbb{P}$-a.e. $\omega \in \Omega$. Since $g(\omega, \cdot)$ is l.s.c. for such ω, then the latter relationship implies that $f(\omega,\cdot)$ is u.s.c. Combined with lower semicontinuity of $f(\omega,\cdot)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$, this proves that f is a Carathéodory function. ☐

The following corollary is the main representation result of this section; though being less general than theorem 1 from [16], it has a much shorter and easier proof and quite suffices for our purposes.

Corollary A.6. *Let the operator* $N\colon L^0(\Omega; X) \to L^1(\Omega; \mathbb{R})$ *be local and continuous (in measure) such that*

$$\int_\Omega |(N(u))(\omega)|\, d\mathbb{P}(\omega) \le C$$

for some $C \ge 0$ *and for all* $u \in L^0(\Omega; X)$. *Then there exists a Carathéodory function* $f\colon \Omega \times X \to \bar{\mathbb{R}}$ *such that*

$$(N(u))(\omega) = f(\omega, u(\omega)) \tag{A.10}$$

for all $u \in L^0(\Omega, X)$ *and* $\mathbb{P}$*-a.e.* $\omega \in \Omega$. *Moreover, such a function is unique in the sense announced in Proposition A.4.*

Proof. Define the functional I: $L^0(\Omega; X) \times \Sigma \to \mathbb{R}$ by the formula

$$I(u,e) := \int_e (N(u))(\omega)\, d\mathbb{P}(\omega).$$

Clearly, I satisfies conditions of Corollary A.5, and hence there is a Carathéodory function f: $\Omega \times X \to \bar{\mathbb{R}}$ such that

$$I(u,e) = \int_e f(\omega, u(\omega))\, d\mathbb{P}(\omega)$$

for all $(u,e) \in L^0(\Omega, X) \times \Sigma$. Therefore, both $f(\cdot, u(\cdot))$ and $(N(u))(\cdot)$ is a Radon-Nikodym derivative of $I(u,\cdot)$ with respect to $\mathbb{P}$, and hence (A.10) is valid. If there is an integrand g: $\Omega \times X \to \mathbb{R}$ such that

$$(N(u))(\omega) = g(\omega, u(\omega))$$

for all $u \in L^0(\Omega, X)$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, then

$$I(u,e) = \int_e g(\omega, u(\omega))\, d\mathbb{P}(\omega),$$

and hence $g(\omega, x) = f(\omega, x)$ for all $x \in X$ and for all $\omega \in \Omega \setminus N$, where $N \subset \Omega$ is $\mathbb{P}$-negligible, according to Proposition A.4. ☐

Appendix B. Proof of Theorem 7.2

In view of Theorem 6.4(ii) and Theorem 7.1 applied with $p := 2$ we see that the existence of an invariant measure is ensured provided that

(A) for some $C > 0$ and some norm $|||\cdot|||$ in the space S one has

$$\mathbb{E}|||\phi|||^2 \leq C \quad \text{implies} \quad \mathbb{E}|||U_\tau^0\phi|||^2 \leq C$$

for all $\phi \in L^2(\Omega, \Sigma_0, \mathbb{P}; S)$, $\tau \in \mathbb{T}$, $\tau > h$, where U_τ^0 is defined by (5.5);

(B) U_T^0 is tight for some $T > h$.

We divide therefore the proof into two steps starting with part (A).

Step 1. As the matrix A is stable, there exists a symmetric and positive matrix P such that $A^*P + PA = -I$, where I is the $n \times n$-identity matrix (for instance, one can put $P := \int_0^\infty \exp(A^*s)\exp(As)ds$, see [13]). We define the quadratic form $v(x) = x^*Px$ (the dot product of x and Px) on $\mathbb{R}^n$ and the Lyapunov functional V on S by the formula

$$V(\phi) := v(\phi(0)) + \int_{-h}^0 v(\phi(\sigma))d\sigma = \int_{-h}^0 v(\sigma)d\lambda(\sigma), \tag{B.1}$$

where λ is the sum of the Lebesgue measure on $[-h, 0]$ and the Dirac measure at $\sigma = 0$ (i.e. is the same measure used in (7.3)). Clearly, $\varphi \mapsto \sqrt{V(\varphi)}$ is a norm on S.

Below we will always assume that $t \geq h$, so that due to (5.6) we have the following representation:

$$(U_t^0)(\phi)(\sigma) = x_t(\sigma) = x(t+\sigma) = \int_0^{t+\sigma} F(s, x_s(\cdot))dZ_s,$$

where

$$F(s, x_s(\cdot))\,dZ(s) = Ax(s)ds + F_{01}(s, x_s(\cdot))\,ds + F_{02}(s, x_s(\cdot))dW(s). \tag{B.2}$$

Applying the stochastic integration by parts formula [15] (which is a particular case of the Itô formula)

$$d\,(u^*w) = u^*dw + (du)^*w + (du)^*dw$$

to $v(x(t+\sigma))$ (with $u := x$ and $w := Px$) for an arbitrary $\sigma \in [-h, 0]$ and any $t \geq h$, we get

$$\begin{aligned} dv(x(t+\sigma)) = x^*(t+\sigma)\,d(Px(t+\sigma)) + dx^*(t+\sigma)Px(t+\sigma) \\ + dx^*(t+\sigma)\,d(Px(t+\sigma)). \end{aligned}$$

Using (B.2) and minding that, according to the formal calculation rules with the stochastic differential, one has $dt\,dW = 0$ and $(dW)^*dW = 0$ [15], we get

$$\begin{aligned} dv(x(t+\sigma)) = x^*(t+\sigma)\,PAx(t+\sigma)dt + x^*(t+\sigma)\,PF_{01}(t+\sigma, x_{t+\sigma})dt \\ + x^*(t+\sigma)\,PF_{02}(t+\sigma, x_{t+\sigma})dW(t+\sigma) \\ + [Ax(t+\sigma)]^*\;Px(t+\sigma)dt + F_{01}^*(t+\sigma, x_{t+\sigma})\,Px(t+\sigma)\,dt \\ + [F_{02}(t+\sigma, x_{t+\sigma})dW(t+\sigma)]^*\;Px(t+\sigma) \\ + [F_{02}(t+\sigma, x_{t+\sigma})dW(t+\sigma)]^*\;PF_{02}(t+\sigma, x_{t+\sigma})dW(t+\sigma). \end{aligned}$$

Since $a^*b = b^*a$ when a and b are vectors, while $P^* = P$ and $dW^*QdW = \operatorname{tr} Q$, we get

$$\begin{aligned} dv(x(t+\sigma)) = x^*(t+\sigma)(PA + A^*P)x(t+\sigma)dt + 2x^*(t+\sigma)PF_{01}(t+\sigma, x_{t+\sigma})dt \\ + \operatorname{tr}\,(F_{02}^*(t+\sigma, x_{t+\sigma})\,PF_{02}(t+\sigma, x_{t+\sigma}))\,dt \\ + x^*(t+\sigma)\,PF_{02}(t+\sigma, x_{t+\sigma})dW(t+\sigma) \\ + [Px(t+\sigma)]^*\,F_{02}(t+\sigma, x_{t+\sigma})dW(t+\sigma), \end{aligned}$$

where $-h \le \sigma \le 0$ and $t \ge h$. Thus, integrating the above relationship and minding that $\mathbb{E}W_t = 0$, we get

$$\begin{aligned}\mathbb{E}v(x(t+\sigma)) = \mathbb{E}v(x(h+\sigma)) &- \int_h^t \mathbb{E}|x(s+\sigma)|^2 ds \\ &+ 2\int_h^t \mathbb{E}x^*(s+\sigma)PF_{01}(s+\sigma, x_{s+\sigma})ds \\ &+ \int_h^t \mathbb{E}\mathrm{tr}\,\left(F_{02}^*(s+\sigma, x_{s+\sigma})\, PF_{02}(s+\sigma, x_{s+\sigma})\right)\, ds,\end{aligned}$$

where $-h \le \sigma \le 0$ and $t \ge h$. Now we integrate the last equality with respect to the measure λ, which gives

$$\begin{aligned}\mathbb{E}V(x_t) = \mathbb{E}V(x_h) &- \int_h^t \mathbb{E}\|x_s\|_S^2 ds \\ &+ 2\int_h^t \mathbb{E}\int_{-h}^0 x^*(s+\sigma)PF_{01}(s+\sigma, x_{s+\sigma})dsd\lambda(\sigma) \\ &+ \int_h^t \mathbb{E}\int_{-h}^0 \mathrm{tr}\,\left(F_{02}^*(s+\sigma, x_{s+\sigma})\, PF_{02}(s+\sigma, x_{s+\sigma})\right) dsd\lambda(\sigma) \qquad (t \ge h).\end{aligned}$$

In particular, this shows that the function $\gamma(t) := \mathbb{E}V(x_t)$ is differentiable for $t > h$ and

$$\begin{aligned}\gamma'(t) = -\mathbb{E}\|x_t\|_S^2 &+ 2\mathbb{E}\int_{-h}^0 x^*(t+\sigma)PF_{01}(t+\sigma, x_{t+\sigma})d\lambda(\sigma) \\ &+ \mathbb{E}\int_{-h}^0 \mathrm{tr}\,\left(F_{02}^*(t+\sigma, x_{t+\sigma})\, PF_{02}(t+\sigma, x_{t+\sigma})\right) d\lambda(\sigma) \qquad (t \ge h).\end{aligned}$$

The assumption (iii) of the theorem being proven together with the Hölder inequality implies then that

$$\gamma'(t) \le -\mathbb{E}\|x_t\|_S^2 + C_1\mathbb{E}\|x_t\|_S + C_2 \qquad (t > h),$$

where C_1 and C_2 are some positive constants. Therefore $\gamma'(t) < 0$ as soon as

$$\mathbb{E}\|x_t\|_S^2 \ge R > R_0 := \frac{C_1 + \sqrt{C_1^2 + 4C_2}}{2} \quad \text{and} \quad t > h.$$

On the other hand, $v(x) \le \|P\| \cdot |x|^2$, where $\|P\|$ stands for the matrix norm of P, so that $V(\varphi) \le \|P\| \cdot \|\varphi\|_S^2$. Hence, for any $R > R_0$ and any $t > h$, the inequality $\mathbb{E}V(x_t) \ge R\|P\|$ always implies $\frac{d}{dt}\mathbb{E}V(x_t) < 0$. In other words,

$$\mathbb{E}V(\phi) < R\|P\| \quad \text{implies} \quad \mathbb{E}V(x_t) < R\|P\| \quad (t > h).$$

Minding that $x_t = U_t^0\phi$, this completes the proof of part (A) (with $C := R\|P\|$ and $|||\cdot|||^2 := V(\cdot)$).

Step 2. We now prove the tightness condition (B). Below we assume that $T > h$ is kept fixed.

The assumption (iii) of Theorem 7.2 gives immediately the following estimate:

$$\mathbb{E}\|F(t,\phi)\|_S^2 \le C(1 + \mathbb{E}\|\phi\|_S^2) \qquad (t \ge 0,\ \phi \in L^0(\Omega, \Sigma_t, \mathbb{P}; S)). \tag{B.3}$$

We use again the representation (5.6) to obtain

$$(U_t^0\phi)(\sigma) = \phi(t+\sigma)\mathbf{1}_{\{t+\sigma<0\}} + \left(\phi(0) + \int_0^{t+\sigma} F(s, (U_s^0\phi)(\sigma))\, dZ(s)\right)\mathbf{1}_{\{t+\sigma\ge 0\}}.$$

Squaring the latter equality, minding that $(a+b)^2 \leq 2a^2 + 2b^2$ and finally taking the expectation, we get

$$\begin{aligned}
\mathbb{E}|U_t^0\phi|^2 &\leq \mathbb{E}|\phi(t+\sigma)|^2 \mathbf{1}_{\{t+\sigma<0\}} \\
&\qquad + \left(2\mathbb{E}|\phi(0)|^2 + 2\mathbb{E}\left|\int_0^{t+\sigma} F(s,(U_s^0\phi)(\sigma))\,dZ(s)\right|^2\right)\mathbf{1}_{\{t+\sigma\geq 0\}} \\
&\leq \mathbb{E}|\phi(t+\sigma)|^2 \mathbf{1}_{\{t+\sigma<0\}} \\
&\qquad + \left(2\mathbb{E}|\phi(0)|^2 + C_1\int_0^{t+\sigma} \mathbb{E}\left|F(s,(U_s^0\phi)(\sigma))\right|^2\,ds\right)\mathbf{1}_{\{t+\sigma\geq 0\}} \\
&\leq 2\mathbb{E}|\phi(t+\sigma)|^2 \mathbf{1}_{\{t+\sigma<0\}} \\
&\qquad + 2\left(\mathbb{E}|\phi(0)|^2 + C_2\int_0^{t+\sigma} \mathbb{E}(1+\mathbb{E}\|U_s^0\phi\|_S^2)\,ds\right)\mathbf{1}_{\{t+\sigma\geq 0\}},
\end{aligned}$$

where the latter estimate follows from (B.3). Integrating over the interval $[-h,0]$ with respect to the measure λ and minding (7.3) gives

$$\mathbb{E}\|U_t^0\phi\|_S^2 \leq 2\mathbb{E}\|\phi\|_S^2 + C_3\int_0^t (1+\mathbb{E}\|U_s^0\phi\|_S^2)\,ds.$$

We apply now the Gronwall inequality obtaining thus the estimate

$$\mathbb{E}\|U_t^0\phi\|_S^2 \leq C_4\left(1+\mathbb{E}\|\phi\|_S^2\right),$$

so that

$$\mathbb{E}\|F(t,U_t^0\phi)\| \leq C_5\left(1+\mathbb{E}\|\phi\|_S^2\right)$$

for any $t\in[0,T]$ and any $\phi\in L^0(\Omega,\Sigma_0,\mathbb{P};S)$ due to (B.3). Finally,

$$\int_0^T \mathbb{E}|F(t,U_t^0\phi)|^2 dt \leq C_6\left(1+\mathbb{E}\|\phi\|_S^2\right). \tag{B.4}$$

Let now $B\subset L^0(\Omega,\Sigma_0,\mathbb{P};S)$ be an arbitrary bounded set. The estimate (B.4) says that $h\colon \phi\mapsto F(t,U_t^0\phi)$ is bounded as an operator between $L^2(\Omega,\Sigma_0,\mathbb{P};S)$ and $L^2(\Omega,\Sigma_T,\mathbb{P};L^2([0,T];\mathbb{R}^{n\times m}))$ (as before, for the sake of brevity we always omit the reference to the Lebesgue σ-algebra and the Lebesgue measure of $[0,T]$). On the other hand, the operator h is a composition of the Nemytskiĭ operator F and the local operator U_t^0 (the latter is local due to Corollary 5.6). Using the property of locality and boundedness in L^2 we can actually prove that h is bounded as an operator between $L^0(\Omega,\Sigma_0,\mathbb{P};S)$ and $L^0(\Omega,\Sigma_T,\mathbb{P};L^2([0,T];\mathbb{R}^{n\times m}))$. To see it, we pick any $\phi\in B$ and any $\varepsilon>0$. Then there exists a $\phi_\varepsilon\in L^2(\Omega,\Sigma_0,\mathbb{P};S)$ such that $\mathbb{P}(\{\phi\neq\phi_\varepsilon\})<\varepsilon$. Locality of h gives then

$$\{h(\phi)\neq h(\phi_\varepsilon)\}\subset\{\phi\neq\phi_\varepsilon\},$$

hence $\mathbb{P}(\{h(\phi)\neq h(\phi_\varepsilon)\})<\varepsilon$ as well. It remains to refer to Lemma C.3 which provides boundedness of h in L^0.

Continuing the proof of the tightness claim (B) we recall that

$$(U_t^0\phi)(\sigma) = \phi(0) + \int_0^{t+\sigma} H_\phi(s)\,dZ(s),$$

where $H_\phi(s) := F(s,U_s^0\phi)$, $\phi\in B$, $\sigma\in[-h,0]$. We know already that the set

$$\mathcal{H} := \{H_\phi \,:\, \phi\in B\}$$

is bounded in $L^0(\Omega,\Sigma_T,\mathbb{P};L^2([0,T];\mathbb{R}^{n\times m}))$. It remains thus to prove the tightness of the stochastic integral operator

$$\mathcal{T}(H,\phi)(t) := \phi(0) + \int_0^t H(s)\,dZ(s)$$

as a mapping from $L^0(\Omega, \Sigma_T, \mathbb{P}; L^2([0,T]; \mathbb{R}^{n\times m})) \cap \mathcal{A}$ ($\mathcal{A}$ is the set of adapted processes on $[0,T]$). This follows from Lemmata C.4 and C.5.

Remark B.1. It is well-known that the evolution operator U_t^0 for deterministic delay equations is compact for $t > h$ for reasonable right hand-side nonlinearities, if the delay does not exceed h. Part (B) above is the stochastic counterpart of this general statement. The suggested proof of part (B) is only based on the assumptions (i) and (iv) of the Theorem 7.2 and the linear growth assumption. This means that in the case of stochastic delay equations with reasonable nonlinearities the evolution operator should be always expected to be tight for $t > h$, if the delay does not exceed h.

Appendix C. Some properties of local operators and tight sets

Here we collect some auxiliary technical statements on tightness of sets and of local operators, which are used in the paper and are also of some independent interest.

Throughout this section $(\Omega, \Sigma, \mathbb{P})$ is a measure space and X is a Polish space. For every ψ in the dual of $Car_b(\Omega, \Sigma, \mathbb{P}; X)$ denote by $\pi_X\psi$ the functional over $C_b(X)$ defined by

$$\langle \pi_X\psi, f\rangle := \langle \psi, f \circ \pi_X\rangle$$

(we use the same notation $\langle \cdot, \cdot\rangle$ both for pairing between $Car_b(\Omega, \Sigma, \mathbb{P}; X)$ and its dual and between $C_b(X)$ and its dual since the context in each case is quite clear). Clearly, π_X is a continuous operator between the respective duals (equipped with their $*$-weak topologies). Note that if ψ is a measure, i.e. $\psi \in \mathcal{Y}(\Omega, \Sigma, \mathbb{P}; X)$, then π_X is the usual push-forward operator with respect to the projection map $\pi_X : \Omega \times X \to X$ defined by $\pi_X(\omega, x) := x$ (we again slightly abuse the notation by using same symbols for formally different objects), in other words, $\pi_X\psi = \pi_{X\sharp}\psi$ in this case.

The first two statements are rather general. Although being of somewhat folkloric character, they cannot be easily found in the literature (at least in the explicit form as presented below). The first one is a direct generalization of the corollary to theorem 1 from [12][vol. I, ch. VI, § 1].

Lemma C.1. *Let a sequence $\{\mu_n\} \subset \mathcal{Y}(\Omega, \Sigma, \mathbb{P}; X)$, be such that for every $f \in Car_b(\Omega, \Sigma, \mathbb{P}; X)$ there is a limit*

$$L(f) := \lim_{n\to\infty} \int_{\Omega\times X} f(\omega, x)\, d\mu_n(\omega, x).$$

Then the sequence $\{\mu_n\}$ is a tight set, and hence, in particular, $\mu_n \rightharpoonup \mu$ in the narrow sense of measures for some measure $\mu \in \mathcal{Y}(\Omega, \Sigma, \mathbb{P}; X)$ as $n \to \infty$. In other words, the space of Young measures $\mathcal{Y}(\Omega, \Sigma, \mathbb{P}; X)$ is sequentially closed in the $*$-weak topology of the space $Car_b'(\Omega, \Sigma, \mathbb{P}; X)$ dual to $Car_b(\Omega, \Sigma, \mathbb{P}; X)$, while the narrow topology of $\mathcal{Y}(\Omega, \Sigma, \mathbb{P}; X)$ is inherited from the $*$-weak topology of $Car_b'(\Omega, \Sigma, \mathbb{P}; X)$.

Proof. It is clearly enough suffices to prove the tightness of $\{\mu_n\}$ and then refer to Lemma C.2. The tightness of $\{\mu_n\}$ follows from the tightness of the set of measures $\{\pi_{X\sharp}\mu_n\}$ over X. Since for every $f \in C_b(X)$ one has

$$\lim_{n\to\infty} \int_X f(x)\, d\pi_{X\sharp}\mu_n(x) = \lim_{n\to\infty} \int_{\Omega\times X} f(\pi_X(\omega, x))\, d\mu_n(\omega, x),$$

then this limit exists and therefore by the corollary to theorem 1 from [12][vol. I, ch. VI, § 1] the sequence $\{\pi_{X\sharp}\mu_n\}$ is tight as requested. $\square$

Lemma C.2. *Let a set $K \subset \mathcal{Y}(\Omega, \Sigma, \mathbb{P}; X)$ be tight. Then its closure $\bar{K}$ in the $*$-weak topology of the dual space to $Car_b(\Omega, \Sigma, \mathbb{P}; X)$ consists of Young measures, i.e. $\bar{K} \subset \mathcal{Y}(\Omega, \Sigma, \mathbb{P}; X)$.*

Proof. Let a generalized (Moore-Smith) sequence $\{\mu_\alpha\}_{\alpha\in A}\subset K$, where A is some directed set, be such that $\mu_\alpha\to\mu$ in the $*$-weak topology of the dual space Car_b' to $Car_b(\Omega,\Sigma,\mathbb{P};X)$, for some $\mu\in Car_b'$. Since the set of Borel measures $\{\pi_{X\sharp}\mu_\alpha\}_{\alpha\in A}$ over X is tight, one has that for every $\varepsilon>0$ there is a compact set $C_\varepsilon\subset X$ such that for every $u\in C_b(X)$ with $0\le u\le 1$ and $u=0$ over C_ε one has

$$\langle u,\pi_{X\sharp}\mu_\alpha\rangle\le\varepsilon \text{ for all } \alpha\in A.$$

Passing to a limit in α, we get $\langle u,\pi_X\mu\rangle\le\varepsilon$, which means that $\pi_X\mu$ is a Borel measure (say, by Proposition B.7 from [5]). Hence, μ is a Young measure by proposition 4.12 from [8] (the Stone-Daniell characterization of random measures). □

Now we present some results regarding tight sets and local operators.

Lemma C.3. *A set $B\subset L^0(\Omega,\Sigma,\mathbb{P};X)$, is bounded (resp. tight) if and only if for all $\varepsilon>0$ there is a bounded (resp. tight) set $B_\varepsilon\subset L^0(\Omega,\Sigma,\mathbb{P};X)$ such that for every $f\in B$ there is an $f_\varepsilon\in B_\varepsilon$ for which $\mathbb{P}(\{f\neq f_\varepsilon\})<\varepsilon$.*

Proof. The "if" part is trivial, because one can always choose $B_\varepsilon:=B$. To prove the "only if" part, choose an arbitrary $\varepsilon>0$ and an arbitrary $f\in B$. This gives an $f_\varepsilon\in B_\varepsilon$ such that $\mathbb{P}(\{f\neq f_\varepsilon\})<\varepsilon$. Let a ball (resp. compact set) $K_\varepsilon\subset X$ be chosen such that $\mathbb{P}(\{f_\varepsilon\notin K_\varepsilon\})<\varepsilon$ (such a set exists in view of the assumption on B_ε). Since since $f(\omega)\in K_\varepsilon$ when $f(\omega)=f_\varepsilon(\omega)$ and $f_\varepsilon(\omega)\in K_\varepsilon$, we get

$$\mathbb{P}(\{f\notin K_\varepsilon\})\le\mathbb{P}(\{f\neq f_\varepsilon\})+\mathbb{P}(\{f_\varepsilon\notin K_\varepsilon\})<2\varepsilon,$$

which means that B is bounded (resp. tight). □

Lemma C.4. *The Nemytskiĭ operator h defined by the formula*

$$h(x)(\omega):=f(\omega,x(\omega)),$$

where $f\in Car_b(\Omega,\Sigma,\mathbb{P};X)$, is tight in $L^0(\Omega,\Sigma,\mathbb{P};X)$, if $f(\omega,\cdot)$ is compact for almost all $\omega\in\Omega$.

Proof. Clearly, for any bounded $S\in X$ the image of the set $B=L^0(\Omega,\Sigma,\mathbb{P};S)$ is contained in the relatively compact, random set $A(\omega,S)$. From proposition 2.15 in [8] it follows that for any $\varepsilon>0$ there is a compact subset $Q\subset X$ for which $\mathbb{P}\left(\{\omega: A(\omega,S)\not\subset Q\}\right)<\varepsilon$, so that $\mathbb{P}\left(\{h(x)\notin Q\}\right)<\varepsilon$ for all $x\in B$. Applying now Lemma C.3 completes the proof. □

In particular, the above result implies tightness of the operator

$$\mathcal{J}(H)(t):=\int_0^t H(s)\,ds$$

in the space $L^0(\Omega,\Sigma_T,\mathbb{P};L^2([0,T];\mathbb{R}))\cap\mathcal{A}$, where $\mathcal{A}$ is the set of all adapted processes on $[0,T]$.

The tightness of the corresponding Itô integral follows from the following statement.

Lemma C.5. *The stochastic integral operator $\mathcal{I}$:*

$$\mathcal{I}(H)(t):=\int_0^t H(s)\,dW(s),$$

where $W(t)$ stands for the scalar Brownian motion, is tight as mapping from the space $L^0(\Omega,\Sigma_T,\mathbb{P};L^2([0,T];\mathbb{R}))\cap\mathcal{A}$ into itself, where $\mathcal{A}$ is the set of all adapted processes on $[0,T]$.

Proof. Below $L^2:=L^2([0,T];\mathbb{R})$. We set

$$g_t^\nu:=\sum_{k=0}^{\nu-1}\frac{kT}{\nu}\mathbf{1}_{[\frac{kT}{\nu},\frac{(k+1)T}{\nu})}(t).$$

Clearly, $g_t^\nu \le t$ and

$$\mathbb{E}\left|\int_0^T \mathbf{1}_{[g_t^\nu,t]}(s)H(s)dW(s)\right|^2 = \mathbb{E}\int_0^T \mathbf{1}_{[g_t^\nu,t]}(s)H^2(s)ds.$$

Therefore,

$$\begin{aligned}\mathbb{E}\int_0^T\left|\int_0^T \mathbf{1}_{[g_t^\nu,t]}(s)H(s)dW(s)\right|^2 dt &= \int_0^T dt\,\mathbb{E}\left|\int_0^T \mathbf{1}_{[g_t^\nu,t]}(s)H(s)dW(s)\right|^2\\ &= \int_0^T dt\,\mathbb{E}\int_0^T \mathbf{1}_{[g_t^\nu,t]}(s)H^2(s)ds\\ &= \mathbb{E}\int_0^T dt\int_0^T \mathbf{1}_{[g_t^\nu,t]}(s)H^2(s)ds\\ &= \mathbb{E}\int_0^T H^2(s)ds\int_0^T \mathbf{1}_{[g_t^\nu,t]}(s)dt\\ &\le \left(\mathbb{E}\int_0^T H^2(s)ds\right)\sup_{0\le s\le T}\int_0^T \mathbf{1}_{[g_t^\nu,t]}(s)dt\\ &\le \frac{CT}{\nu}\mathbb{E}\|H\|_2^2,\end{aligned}$$

and finally

$$\mathbb{E}\left\|\mathcal{I}(H)-\mathcal{I}_\nu(H)\right\|_2^2 \le \frac{CT}{\nu}\mathbb{E}\|H\|_2^2,$$

where

$$\mathcal{I}_\nu(H)(t) := \int_0^{g_\nu(t)} H(s)dW(s).$$

The operator $\mathcal{I}_\nu$ is a finite-dimensional linear Nemytskiĭ operator. According to Lemma C.4 it is tight.

To prove tightness of the operator $\mathcal{I}$ we first observe that due to its linearity and thanks to Lemma C.3 we may assume below that $\mathbb{E}\|H\|_2 \le 1$. We denote the set of all such H by B.

Let $\epsilon > 0$ be given. For any $\nu \in \mathbb{N}$ we choose $k(\nu) \in \mathbb{N}$ such that

$$\mathbb{E}\left\|\mathcal{I}(H)-\mathcal{I}_{k(\nu)}(H)\right\|_2^2 \le \frac{\varepsilon 2^{-\nu-1}}{\nu} \tag{C.1}$$

for any $H \in B$. Using tightness of $\mathcal{I}_\nu$ we choose a compact set $Q_\nu \in L^2$ for which

$$\mathbb{P}\left(\left\{\mathcal{I}_{k(\nu)}(H) \notin Q_\nu\right\}\right) \le \varepsilon 2^{-\nu-1} \tag{C.2}$$

for any $H \in B$ and $\nu \in \mathbb{N}$.

Letting Q_ν^δ stand for the closed neighborhood of Q_ν we put

$$Q := \bigcap_{\nu=1}^{\infty} Q_\nu^{1/\nu}$$

and observe that Q is a compact in L^2, as any $1/\nu$-net for Q_ν will be a $2/\nu$-net for Q.

Now for any $H \in B$ we use the estimates (C.1), (C.2) and Chebyshev inequality providing

$$\begin{aligned}
\mathbb{P}\left(\{\mathcal{I}(H) \notin Q\}\right) &\leq \sum_{\nu=1}^{\infty} \mathbb{P}\left(\left\{\mathcal{I}(H) \notin Q_\nu^{1/\nu}\right\}\right) \\
&\leq \sum_{\nu=1}^{\infty} \left(\mathbb{P}\left(\left\{\left\|\mathcal{I}(H) - \mathcal{I}_{k(\nu)}(H)\right\|_2^2 \geq \frac{1}{\nu}\right\}\right) + \mathbb{P}\left(\left\{\mathcal{I}_{k(\nu)}(H) \notin Q_\nu\right\}\right)\right) \\
&\leq \sum_{\nu=1}^{\infty} \left(\nu \mathbb{E}\left\|\mathcal{I}(H) - \mathcal{I}_{k(\nu)}(H)\right\|_2^2 + \mathbb{P}\left(\left\{\mathcal{I}_{k(\nu)}(H) \notin Q_\nu\right\}\right)\right) \\
&\leq \sum_{\nu=1}^{\infty} \left(\nu \frac{\varepsilon}{\nu} 2^{-\nu-1} + \varepsilon 2^{-\nu-1}\right) \leq \varepsilon.
\end{aligned}$$

Therefore, the set $\mathcal{I}(B)$ is tight, and the lemma is proven. □

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