SECOND ORDER LOCAL MINIMAL-TIME MEAN FIELD GAMES

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ABSTRACT. The paper considers a forward-backward system of parabolic PDEs arising in a Mean Field Game (MFG) model where every agent controls the drift of a trajectory subject to Brownian diffusion, trying to escape a given bounded domain Ω in minimal expected time. Agents are constrained by a bound on the drift depending on the density of other agents at their location. Existence for a finite time horizon T is proven via a fixed point argument, but the natural setting for this problem is in infinite time horizon. Estimates are needed to treat the limit $T \to \infty$, and the asymptotic behavior of the solution obtained in this way is also studied. This passes through classical parabolic arguments and specific computations for MFGs. Both the Fokker–Planck equation on the density of agents and the Hamilton–Jacobi–Bellman equation on the value function display Dirichlet boundary conditions as a consequence of the fact that agents stop as soon as they reach $\partial\Omega$. The initial datum for the density is given, and the long-time limit of the value function is characterized as the solution of a stationary problem.

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1. INTRODUCTION

Introduced around 2006 by Jean-Michel Lasry and Pierre-Louis Lions [18–20] and at the same time by Peter Caines, Minyi Huang, and Roland Malhamé [12–14], the theory of Mean Field Games (MFGs, for short) describes the interaction of a continuum of players, assumed to be rational, indistinguishable, and negligible, when each one tries to solve a dynamical control problem influenced only by the average behavior of the other players (through a mean-field type interaction, using the physicists' terminology). The Nash equilibrium in these continuous games is described by a system of PDEs: a Hamilton–Jacobi–Bellman equation for the value function of the control problem of each player, where the distribution (density) of the players appears, coupled with a continuity equation describing the evolution of such a density, where the velocity field is the optimal one in order to solve the control problem, and is therefore related to the gradient of the value

2020 *Mathematics Subject Classification.* 35Q89, 35K40, 35B40, 35A01, 35D30.

Key words and phrases. Mean Field Games, congestion games, parabolic PDEs, MFG system, existence of solutions, asymptotic behavior.

function. This system is typically forward-backward in nature: the density evolves forward in time starting from a given initial datum, and the value function backward in time, according to Bellman's dynamical programming principle, and its final value at a given time horizon T is usually known.

The literature about MFG theory is quickly growing and many references are available. The 6-year course given by P.-L. Lions at Collège de France, for which video-recording is available in French [22], explains well the birth of the theory, but the reader can also refer to the lecture notes by P. Cardaliaguet [6], based on the same course.

In most of the MFG models studied so far the agents consider a fixed time interval $[0, T]$ and optimize a trajectory $x : [0, T] \to \Omega$ (where $\Omega \subset \mathbb{R}^d$ is the state space) trying to minimize a cost of the form $\int_0^T L(t, x(t), x'(t), \rho_t)\,\mathrm{d}t + \Psi(x(T), \rho_T)$, where ρ_t denotes the distribution of players at time t. The function L is typically increasing in $|x'|$ and, in some sense, in ρ. This means that high velocities are costly, and passing through areas where the population is strongly concentrated is also costly. Some MFGs, called *MFGs of congestion* (see, for instance, [1]), consider costs which include a product of the form $\rho_t(x(t))^\alpha |x'(t)|^\beta$ (for some exponents $\alpha, \beta > 0$), which means that high velocities are costly, and that they are even more costly in the presence of high concentrations. These models present harder mathematical difficulties compared to those where the cost is decomposed into $L(t, x(t), x'(t)) + g(t, x(t), \rho_t)$. Indeed, in many cases the latter MFG admits a variational formulation: equilibria can be found by minimizing a global energy among all possible evolutions $(\rho_t)_t$ (hence, they are *potential games*). This allows to prove the existence of the equilibrium via semicontinuity methods, and we refer to [4] and [26] for a detailed discussion of this branch of MFG theory.

When the MFG has no variational interpretation, then the existence of a solution is usually obtained via fixed-point theorems, but these theorems require much more regularity. Roughly speaking, given an evolution ρ one computes the corresponding value function φ as a solution to a Hamilton–Jacobi–Bellman equation and, given φ, one computes a new density evolution $\tilde{\rho}$ by following an evolution equation. We need existence, uniqueness, and stability results for these equations in order to find a fixed point $\tilde{\rho} = \rho$. This usually requires regularity of the velocity field $-\nabla\varphi$, which is difficult to prove, and can be essentially only obtained in two different frameworks: either the dependence of the cost functions on the distribution ρ is highly regularizing (which usually means that it is nonlocal, and passes through averaged quantities such as convolutions $\int \eta(x - y)\,\mathrm{d}\rho(y)$), or diffusion of the agents is taken into account, transforming the optimal control problem into a stochastic one. In this latter case, agents minimize $\mathbb{E}[\int_0^T L(t, X_t, \alpha(t), \rho_t)\,\mathrm{d}t + \Psi(X_T, \rho_T)]$ where the process X follows $\mathrm{d}X_t = \alpha_t\,\mathrm{d}t + \mathrm{d}B_t$ and $(B_t)_{t \geq 0}$ represents a standard Brownian motion.

In [23], the second and third authors of the present paper introduced a different class of models, called *minimal-time MFGs*. The main difference is that instead of considering a cost for the players penalizing both the velocity and the density, and minimizing the integral of such a cost on a fixed time interval $[0, T]$, the dynamics is subject to a constraint where the maximal velocity of the agents cannot exceed a quantity depending on the density ρ_t, and the goal of each agent is to arrive to a given target as soon as possible. In the typical situation, the target of the agents is the boundary $\partial\Omega$ of the domain where the evolution occurs. This can model, for instance, an evacuation phenomenon in crowd motion. The system that one obtains is the following

$$\begin{cases} \partial_t \rho - \nabla \cdot \left(\rho k[\rho] \dfrac{\nabla \varphi}{|\nabla \varphi|} \right) = 0, & \text{in } (0,T) \times \Omega, \\ -\partial_t \varphi + k[\rho] |\nabla \varphi| - 1 = 0, & \text{in } (0,T) \times \Omega, \\ \rho(0,x) = \rho_0(x), & \text{in } \Omega, \\ \varphi(t,x) = 0, & \text{on } (0,T) \times \partial\Omega, \end{cases} \tag{1.1}$$

where the function $k[\rho_t](x)$ denotes the maximal speed that agents can have at point x at time t, i.e., the dynamics is constrained to satisfy $|x'(t)| \leq k[\rho_t](x(t))$. Ideally, one would like to choose k to be a non-increasing function of the density itself, such as $k[\rho](x) = (1 - \rho(x))_+$. This choice is what is done in the well-known Hughes' model for crowd motion [15, 16]. Indeed, this model is very similar to Hughes', which also considers agents who aim at leaving in minimal time a bounded domain under a congestion-dependent constraint on their speeds.

The main difference between the model in [23] (from which the present paper stems) and Hughes' is that, in the latter, at each time, an agent moves in the optimal direction to the boundary assuming that the distribution of agents remains constant, whereas in [23] and here agents take into account the future evolution of the distribution of agents in the computation of their optimal trajectories. This accounts for the time derivative in the Hamilton–Jacobi–Bellman equation from (1.1), which is the main difference between (1.1) and the equations describing the motion of agents in Hughes' model and stands for the anticipation of future behavior of other agents.

Another crucial (and disappointing) similarity between the above MFG system and Hughes' model is the fact that general mathematical results do not exist in the case $k[\rho] = (1-\rho)_+$ and more generally in the local case (except few results in the Hughes case in 1D). Indeed, the lack of regularity makes the model too hard to study, and the MFG case is not variational. In some sense the closest MFG model to this one is the one with multiplicative costs in [1]. Indeed, an L^∞ constraint $|x'| \leq k[\rho]$ can be seen as a limit as $m \to \infty$ of an integral penalization

$$\int \left| \frac{|x'(t)|}{k[\rho_t](x(t))} \right|^m \mathrm{d}t$$

(note that the boundaries of the time interval have been omitted on purpose from the above integral, since the model in [1] is set on a fixed time horizon but this is not part of our setting).

Because of these difficulties, [23] studied the case of a non-local dependence of k w.r.t. ρ (say, $k[\rho](x) = \kappa(\int \eta(x-y)\,\mathrm{d}\rho(y))$, for a non-increasing function κ and a positive convolution kernel η), and proved existence of an equilibrium, characterized it as a solution of a non-local MFG system, and analyzed some examples, including numerical simulations. Instead, in the present paper we want to study the local case with diffusion.

This means that we will consider a local dependence $k[\rho](x) := \kappa(\rho(x))$, and each agent solves a stochastic control problem

$$\inf \left\{ \mathbb{E}[\tau] \,:\, X(\tau) \in \partial\Omega, X(0) = x_0,\, \mathrm{d}X_t = \alpha_t \,\mathrm{d}t + \sqrt{2\nu}\,\mathrm{d}B_t,\, |\alpha_t| \leq \kappa(\rho(t, X_t)) \right\},$$

where $(B_t)_{t \geq 0}$ denotes a standard Brownian motion and the Brownian motions for all players are assumed to be mutually independent. Defining the corresponding value function φ, from classical results on stochastic optimal control (see [10, Chapter IV]), under suitable assumptions, the optimal control is given in feedback form by

$$\alpha_t = -\kappa(\rho(t, X_t)) \frac{\nabla \varphi(t, X_t)}{|\nabla \varphi(t, X_t)|},$$

(a definition which has to be carefully adapted to the case $\nabla\varphi = 0$); moreover, the value function solves the Hamilton–Jacobi–Bellman equation

$$-\partial_t\varphi(t,x) - \nu\Delta\varphi(t,x) + K(t,x)|\nabla\varphi(t,x)| - 1 = 0, \quad (t,x) \in [0,T) \times \Omega,$$

for $K = \kappa(\rho)$. Hence, we know the drift of the optimal stochastic processes followed by each agent, and this allows to write the Fokker–Planck equation solved by the law of this process. Putting together all this information, we obtain the following MFG system

$$\begin{cases} \partial_t\rho - \nu\Delta\rho - \nabla\cdot\left(\rho\kappa(\rho)\frac{\nabla\varphi}{|\nabla\varphi|}\right) = 0, & \text{in } \mathbb{R}_+ \times \Omega, \\ -\partial_t\varphi - \nu\Delta\varphi + \kappa(\rho)|\nabla\varphi| - 1 = 0, & \text{in } \mathbb{R}_+ \times \Omega, \\ \rho(0,x) = \rho_0(x), & \text{in } \Omega, \\ \rho(t,x) = 0, \quad \varphi(t,x) = 0, & \text{on } \mathbb{R}_+ \times \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^d$ is an open and bounded set, whose boundary will be supposed to be of class C^2 in this paper, $\nu > 0$ is a fixed constant, $\kappa : \mathbb{R} \to (0,+\infty)$, and $\rho_0 \geq 0$ is the initial density. The Dirichlet condition on φ comes as usual from the fact that, for agents who are already on the boundary, the remaining time to reach it is zero, and the Dirichlet condition on ρ comes from the fact that we stop the evolution of a particle as soon as it touches the boundary (absorbing boundary conditions).

A crucial difference with the previous paper [23] concerns the time horizon. If we suppose that κ is bounded from below in the model without diffusion, it is not difficult to see that all agents will have left the domain after a common finite time, so that the final value of φ is not really relevant, and the problem can be studied on a finite interval $[0,T]$. This is not the case when there is diffusion, as a density following a Fokker–Planck equation with a bounded drift cannot fully vanish in finite time. As a consequence, the model should be studied on the unbounded interval $[0,\infty)$. For every time $t < \infty$ there is still mass everywhere, but this mass decreases to 0 as $t \to +\infty$, which suggests that the value function φ should converge to a function, that we call Ψ, which is the value function for the corresponding control problem with no mass, i.e. when $\kappa = \kappa(0)$. Since in this control problem κ is independent of time, Ψ is a function of x only and solves a stationary Hamilton–Jacobi–Bellman equation which takes the form of an elliptic PDE

$$-\nu\Delta\Psi + \kappa(0)|\nabla\Psi| - 1 = 0$$

with Dirichlet boundary conditions on $\partial\Omega$. It is then reasonable to investigate whether solutions of the above system satisfy further $\rho_t \to 0$ and $\varphi_t \to \Psi$ as $t \to +\infty$.

In order to study the above system, we will first study an artificial finite-horizon setting, where we stop the game at time T, choose a penalization $\psi : \Omega \to \mathbb{R}_+$ with $\psi = 0$ on $\partial\Omega$, and look at the stochastic optimal control problem

$$\inf\Big\{\mathbb{E}[\min\{\tau,T\} + \psi(X_{\min\{\tau,T\}})] : \\ X(\tau) \in \partial\Omega, X(0) = x_0, \, \mathrm{d}X_t = \alpha_t\,\mathrm{d}t + \sqrt{2\nu}\,\mathrm{d}B_t, \, |\alpha_t| \leq \kappa(\rho(t,X_t))\Big\}.$$

This gives rise to the MFG system

$$\begin{cases} \partial_t\rho - \nu\Delta\rho - \nabla\cdot\left(\rho\kappa(\rho)\frac{\nabla\varphi}{|\nabla\varphi|}\right) = 0, & \text{in } (0,T) \times \Omega, \\ -\partial_t\varphi - \nu\Delta\varphi + \kappa(\rho)|\nabla\varphi| - 1 = 0, & \text{in } (0,T) \times \Omega, \\ \rho(0,x) = \rho_0(x), \qquad \varphi(T,x) = \psi(x), & \text{in } \Omega, \\ \rho(t,x) = 0, \qquad \varphi(t,x) = 0, & \text{on } (0,T) \times \partial\Omega, \end{cases} \tag{1.3}$$

which corresponds to (1.2) with the unbounded time interval $\mathbb{R}_+$ replaced by $(0,T)$ and the additional final condition $\varphi(T,x) = \psi(x)$. We will prove the existence of a solution of the system for finite T, and then consider the limit as $T \to \infty$. In order to guarantee suitable bounds, we just need to choose a sequence of final data ψ_T, possibly depending on T, which is uniformly bounded. We will then get at the limit a solution of the limit system which automatically satisfies $\rho_t \to 0$ (in the sense of uniform convergence) and $\varphi_t \to \Psi$ (this convergence being both uniform and strong in H_0^1).

The paper is organized as follows. After this introduction, Section 2 presents the tools that we need to study the two separate equations appearing in System (1.3) on a finite horizon, which come from the classical theory of parabolic equations. Section 3 is devoted to the existence of solutions of (1.3). After providing a precise definition of solution of (1.3) taking care of the case $\nabla\varphi = 0$, we use the estimates of Section 2 to prove existence via a fixed-point argument based on Kakutani's theorem. Section 4 concerns the limit $T \to \infty$. In this section, some estimates of Section 2 need to be made more precise, in order to see how constants depend on the time horizon T. In this way we are able to prove existence of a limit of the solutions of (1.3) as the time horizon T tends to $+\infty$ and that this limit solves the limit system (1.2). Then we consider the asymptotic behavior of a solution (ρ, φ) of (1.2) as $t \to +\infty$, proving first $\rho_t \to 0$ in L^1 and, thanks to a parabolic regularization argument, also in L^∞. To prove convergence in L^1, which is true for general Fokker–Planck systems under very mild assumptions, we exploit the MFG nature of the system, i.e. the coupling between the two equations, which also provides exponential decrease. We then consider the limit in time of φ, and prove that any bounded solution of this equation, once we know $\kappa(t, x) \to \kappa(0)$, can only converge as $t \to +\infty$ to the stationary function Ψ. This convergence is a priori very weak, but we are able to improve it into $L^\infty \cap H_0^1$, and to prove that the uniform convergences of both ρ and φ occur exponentially fast. The paper is then completed by an appendix, which details some global L^∞ estimates for a large class of parabolic equations, including the estimates that we use to prove uniform convergence in time of ρ_t and φ_t to 0 and Ψ, respectively. These estimates are not surprising and not difficult to prove, using standard Moser iterations, but are not easy to find in the literature under the sole assumption of boundedness of the drift term in the divergence. The computations and the results are essentially the same as in the appendix of [7], but the boundary conditions are different.

2. Preliminary results

This section presents some preliminary results on Fokker–Planck and Hamilton–Jacobi–Bellman equations which are useful for the analysis of the Mean Field Game systems (1.2) and (1.3). We recall that, in the whole paper, Ω denotes an open and bounded set whose boundary $\partial\Omega$ is assumed to be C^2. Even though some of the results presented in this preliminary section also hold without the smoothness assumption on $\partial\Omega$ (such as existence and uniqueness results for both Fokker–Planck and Hamilton–Jacobi–Bellman equations in Propositions 2.2 and 2.5), this assumption is first used to obtain higher regularity of solutions of Hamilton–Jacobi–Bellman equations in Proposition 2.5 and is required for almost all of the subsequent results, including in particular our main results in Sections 3 and 4, as a consequence of the need of higher regularity of φ.

2.1. **Fokker–Planck equation.** We recall some results on the Fokker–Planck equation on a bounded domain $\Omega \subset \mathbb{R}^d$ in finite time horizon $T \in (0, +\infty)$,

$$\begin{cases} \partial_t\rho - \nu\Delta\rho + \nabla\cdot(\rho V) = 0 & \text{in } (0,T)\times\Omega, \\ \rho(0,x) = \rho_0(x) & \text{in } \Omega, \\ \rho(t,x) = 0 & \text{in } [0,T]\times\partial\Omega, \end{cases} \tag{2.1}$$

where $V : (0,T)\times\Omega \to \mathbb{R}^d$ is a given velocity field. We will only focus on the case where V is bounded, an assumption which is satisfied in the cases of interest for this paper and which strongly simplifies the analysis. The results presented in this short section are a mixture of classical results (for which we mainly refer to [8, Section 7.1] or [17, Chapter III]), recent results obtained by Porretta in [24], and extra computations which are not original but are difficult to find in the literature, which we present in the Appendix.

Definition 2.1. Let $\nu > 0$, $V \in L^\infty((0,T)\times\Omega;\mathbb{R}^d)$, and $\rho_0 \in L^1(\Omega)$. We say that $\rho \in L^1((0,T)\times\Omega)$ is a *weak solution* of (2.1) if, for every $\eta \in C^2([0,T]\times\Omega)$ such that $\eta\big|_{[0,T]\times\partial\Omega} = 0$ and $\eta\big|_{\{T\}\times\Omega} = 0$, one has

$$-\int_0^T\int_\Omega \rho\partial_t\eta \,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega (\nu\rho\Delta\eta + \rho V\cdot\nabla\eta)\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega \rho_0(x)\eta(0,x)\,\mathrm{d}x. \tag{2.2}$$

We observe that, whenever equality (2.2) holds for C^2 functions, and if we have further that $\rho \in L^2((t_1,t_2);H^1_0(\Omega)) \cap C^0([t_1,t_2];L^2(\Omega))$ for some $t_1,t_2 \in (0,T)$ with $t_1 < t_2$, then we also have

$$\int_{t_1}^{t_2}\left(\int_\Omega -\rho\partial_t\eta + \nu\nabla\rho\cdot\nabla\eta - \rho V\cdot\nabla\eta\right)\mathrm{d}x\,\mathrm{d}t = \int_\Omega \rho(t_1,x)\eta(t_1,x)\,\mathrm{d}x - \int_\Omega \rho(t_2,x)\eta(t_2,x)\,\mathrm{d}x. \tag{2.3}$$

for every $\eta \in C^1_c([0,T)\times\Omega))$ and, by density, for every $\eta \in L^2((t_1,t_2);H^1_0(\Omega))\cap C^0([t_1,t_2];L^2(\Omega))$ such that $\partial_t\eta \in L^2((t_1,t_2);H^{-1}(\Omega))$. Of course it is well-known that, in case ρ is more regular, other test functions can also be accepted, and that if $\rho \in C^2$ then the equation is satisfied in a classical sense.

We now state a proposition summarizing all the main results that we will use.

Proposition 2.2. *Let $\nu > 0$, $V \in L^\infty((0,T)\times\Omega;\mathbb{R}^d)$, and $\rho_0 \in L^1(\Omega)$ be a given non-negative initial datum. Then* (2.1) *admits a unique weak solution ρ. In addition, we have $\rho \geq 0$ and $\rho \in C^0([0,T];L^1(\Omega))$ with $\|\rho_t\|_{L^1} \leq \|\rho_0\|_{L^1}$, as well as $\nabla\rho \in L^q((0,T)\times\Omega)$ and $\partial_t\rho \in L^q((0,T);W^{-1,q}(\Omega))$ for all $q < \frac{d+2}{d+1}$ and $\rho \in L^r((0,T)\times\Omega)$ for all $r < \frac{d+2}{d}$, and the norms of $\rho, \nabla\rho$ and $\partial_t\rho$ in the above spaces are bounded by quantities only depending on $\|\rho_0\|_{L^1}$. Moreover, for every $t_0 > 0$, we also have $\rho \in L^\infty((t_0,T)\times\Omega)\cap L^2((t_0,T);H^1_0(\Omega))\cap C^0([t_0,T];L^2(\Omega))$ and $\partial_t\rho \in L^2((t_0,T);H^{-1}(\Omega))$.*

Of course we do not provide a full proof of the above results, but we explain below how to deduce the different parts of the statement from the most well-known literature and the relevant references.

Proof. The definition of the solution is exactly the one used in [24], where the key assumption is $\rho|V|^2 \in L^1((0,T)\times\Omega)$. In our case, where V is bounded, this assumption is satisfied as soon as $\rho \in L^1((0,T)\times\Omega)$. One of the main results of [24] is exactly the uniqueness of the solution in this class, and this can be applied to the present setting. The same paper also guarantees the estimates $\nabla\rho \in L^q((0,T)\times\Omega)$, $\partial_t\rho \in L^q((0,T);W^{-1,q}(\Omega))$, $\rho \in L^r((0,T)\times\Omega)$, and the L^1 bound.

Existence is not included in [24] but in the particular case $V \in L^\infty$ it is easy to obtain by regularization and compactness. Indeed, one can apply the classical L^2 theory of [17, Chapter III] to an approximated initial datum, and obtain a sequence of solutions: the L^r bounds of [24], which only depend on the initial L^1 norm in this setting, allow to obtain the compactness we need to pass the PDE to the limit. Note that this argument is specific to the case $V \in L^\infty$ since, otherwise, we would need to control the L^1 norm of $\rho|V|^2$, which is non-trivial.

Using an approximation with smooth ρ_0 and smooth V one can also obtain smooth solutions for which the classical maximum principle guarantees $\rho \geq 0$, and then this property passes to the limit and also applies to the unique weak solution for general ρ_0 and V.

The local L^∞ bound can be obtained thanks to the Appendix of the present paper (even if we stress that similar computations are nowadays standard). For simplicity, the bound is presented under the assumption $\rho_0 \in L^r$, $r > 1$, and not $\rho_0 \in L^1$. Yet, the time-space L^r summability already stated in the claim allows to deduce $\rho_t \in L^r$ for a.e. $t > 0$, and if we choose $t < t_0$ we obtain the desired L^∞ bound. Once we know that ρ is locally (in time) L^∞ (in space), it is also locally (in time) L^2 (in space), and hence the classical L^2 theory of [17, Chapter III] provides the last estimates of the statement. □

2.2. **Hamilton–Jacobi–Bellman equation.** We consider the non-linear Hamilton–Jacobi–Bellman equation in a finite time horizon

$$\begin{cases} -\partial_t\varphi - \nu\Delta\varphi + K|\nabla\varphi| - 1 = 0 & \text{in } (0,T)\times\Omega, \\ \varphi(T,x) = \psi(x) & \text{in } \Omega, \\ \varphi(t,x) = 0 & \text{in } [0,T]\times\partial\Omega, \end{cases} \tag{2.4}$$

where $K : (0,T)\times\Omega \to \mathbb{R}$ is a given function.

Definition 2.3. Let $\nu > 0$, $K \in L^\infty((0,T)\times\Omega;\mathbb{R})$, and $\psi \in L^2(\Omega)$. We say that $\varphi \in L^\infty((0,T);L^2(\Omega)) \cap L^2((0,T);H_0^1(\Omega))$ is a *weak solution* of (2.4) if, for every $\eta \in C^1([0,T]\times\Omega)$ such that $\eta|_{[0,T]\times\partial\Omega} = 0$ and $\eta|_{\{0\}\times\Omega} = 0$, one has

$$\int_0^T\int_\Omega \varphi\partial_t\eta + \nu\int_0^T\int_\Omega \nabla\varphi\cdot\nabla\eta + \int_0^T\int_\Omega (K|\nabla\varphi| - 1)\eta = \int_\Omega \psi(x)\eta(T,x)\,\mathrm{d}x. \tag{2.5}$$

As we did after Definition 2.1, we observe that, if (2.5) holds for every η as before, and if we assume further that $\varphi \in C^0([t_1,t_2];L^2(\Omega))$ for some $t_1, t_2 \in (0,T)$ with $t_1 < t_2$, then we also have

$$\int_{t_0}^{t_1}\int_\Omega (\varphi\partial_t\eta + \nu\nabla\varphi\cdot\nabla\eta + (K|\nabla\varphi| - 1)\eta) \\ = \int_\Omega \varphi(t_1,x)\eta(t_1,x)\,\mathrm{d}x - \int_\Omega \varphi(t_0,x)\eta(t_0,x)\,\mathrm{d}x, \tag{2.6}$$

for every $\eta \in C_c^1((0,T]\times\Omega)$ and, by density, for every $\eta \in L^2((t_1,t_2);H_0^1(\Omega)) \cap C^0([t_1,t_2];L^2(\Omega))$ such that $\partial_t\eta \in L^2((t_1,t_2);H^{-1}(\Omega))$.

Remark 2.4. Note that (2.4), as a Hamilton–Jacobi–Bellman equation of an optimal control problem, is backward in time: the final condition $\varphi(T,x) = \psi(x)$ is given and one solves the equation in the time interval $[0,T]$. One can apply classical results on forward PDEs to (2.4) by using the standard time reversal $t \mapsto T - t$.

The next proposition gathers the main results on solutions of (2.4) that will be needed in the paper.

Proposition 2.5. *Let $\nu > 0$, $K \in L^\infty((0,T)\times\Omega)$, and $\psi \in L^2(\Omega)$. Then* (2.4) *admits a unique weak solution φ. In addition, we have $\varphi \in C^0([0,T];L^2(\Omega))$, and the norms of φ in $L^\infty((0,T);L^2(\Omega))$ and $L^2((0,T);H_0^1(\Omega))$ are bounded by quantities depending only on d, ν, T, Ω, an upper bound on $\|K\|_{L^\infty((0,T)\times\Omega)}$, and $\|\psi\|_{L^2(\Omega)}$.*

Moreover, if $\psi \geq 0$ a.e. in Ω, then the unique solution also satisfies $\varphi \geq 0$ a.e. in $(0,T)\times\Omega$. If $K \geq 0$, $\psi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, and $\psi \geq 0$ a.e. in Ω, then there exists a constant $C > 0$ depending on ν, Ω, and $\|\psi\|_{L^\infty}$ such that $\varphi \leq C$ a.e. on $(0,T)\times\Omega$.

Finally, if $\psi \in H_0^1(\Omega)$, then $\varphi \in C^0([0,T];H_0^1(\Omega)) \cap L^2((0,T);H^2(\Omega))$, $\partial_t\varphi \in L^2((0,T)\times\Omega)$, and the norms of φ in these spaces are bounded by quantities depending only on d, ν, T, Ω, an upper bound on $\|K\|_{L^\infty((0,T)\times\Omega)}$, and $\|\psi\|_{H_0^1(\Omega)}$.

The results stated in Proposition 2.5 are classical and follow from more general results for nonlinear pseudo-monotone operators. Similarly to Proposition 2.2, we explain below how they can be retrieved from the relevant literature.

Proof. Existence of a weak solution φ for $\psi \in L^2(\Omega)$ follows from [25, Theorem 2.1] and the corresponding bounds on the norms of φ are a consequence of [25, Lemma 4.1], whereas uniqueness follows from [9, Theorem 2.4].

The positivity of φ when $\psi \geq 0$ is classical for smooth solutions and can be obtained by an easy application of the maximum principle for parabolic equations. For solutions of HJB obtained as value functions of a stochastic control problem, the result is also straightforward, as the quantity which is minimized is positive. In our context of weak solutions, it can be deduced by applying, for instance, [2, Theorem 1] to $-\varphi$, after changing time orientation and paying attention to the observation at the end of the proof (page 98) that the inequality is enough (indeed, the source term 1 in the HJB equation has the good sign to preserve positivity).

The upper bound on φ under the positivity assumption on ψ and K and the fact that $\psi \in H^1_0(\Omega) \cap L^\infty(\Omega)$ can be obtained by applying a parabolic comparison principle (see [21, Theorem 9.1] for the smooth case) to φ and $\Phi + \|\psi\|_{L^\infty}$, where Φ is the solution of the torsion equation $-\nu\Delta\Phi = 1$ in Ω with Dirichlet boundary conditions.

Finally, higher regularity of φ when $\psi \in H^1_0(\Omega)$ can be obtained in a straightforward manner by noticing that $-\partial_t\varphi - \nu\Delta\varphi = 1 - K|\nabla\varphi|$, i.e., φ satisfies a linear backwards heat equation in Ω with source term $1 - K|\nabla\varphi| \in L^2((0,T)\times\Omega)$. The conclusions then follow from classical improved regularity results for heat equations (such as [8, Section 7.1, Theorem 5] and [17, Chapter III, § 6, Equation (6.10) and Theorem 6.1]). □

We next state, for future reference, a standard parabolic comparison principle for (2.4) (see, e.g., [9, Corollary 2.2]).

Proposition 2.6. *Let φ_1, φ_2 be two solutions of* (2.4) *with $T < +\infty$, with final data such that $\varphi_1(T,\cdot) \geq \varphi_2(T,\cdot)$. Then*

$$\varphi_1 \geq \varphi_2 \quad \text{on} \ (0,T)\times\Omega.$$

3. The MFG system with a finite time horizon

We now consider the MFG system with a finite time horizon (1.3). One of the difficulties in the analysis of (1.3) is that the velocity field in the continuity equation depends on $\frac{\nabla\varphi}{|\nabla\varphi|}$, which is defined only when $\nabla\varphi \neq 0$. In order to handle this difficulty, we make use of the following definition of weak solution.

Definition 3.1. Let $\nu > 0$, $T \in (0,+\infty)$, $\kappa : \mathbb{R} \to (0,+\infty)$ be continuous and bounded, $\rho_0 \in L^1(\Omega)$, and $\psi \in L^2(\Omega)$. We say that $(\rho,\varphi) \in L^1((0,T)\times\Omega) \times L^2((0,T); H^1_0(\Omega))$ is a *weak solution* of (1.3) with initial condition ρ_0 and final condition ψ if there exists $V \in L^\infty((0,T)\times\Omega;\mathbb{R}^d)$ such that $|V(t,x)| \leq \kappa(\rho(t,x))$ and $V(t,x)\cdot\nabla\varphi(t,x) = -\kappa(\rho(t,x))|\nabla\varphi(t,x)|$ a.e. on $(0,T)\times\Omega$ and such that ρ is a solution of the Fokker–Planck equation (2.1) with initial datum ρ_0 and vector field V on $[0,T]\times\Omega$ in the sense of Definition 2.1, and φ is a solution of the Hamilton–Jacobi–Bellman equation (2.4) with final datum ψ and $K = \kappa(\rho)$ in the sense of Definition 2.3 on the same domain.

Remark 3.2. If (ρ,φ) is a weak solution of (1.3) and V is any function satisfying the properties stated in Definition 3.1, then we have $V(t,x) = -\kappa(\rho(t,x))\frac{\nabla\varphi(t,x)}{|\nabla\varphi(t,x)|}$ wherever $\nabla\varphi(t,x) \neq 0$. The introduction of the function V in Definition 3.1 has the advantages of providing a meaning to the first equation of (1.3) and handling its velocity field even when $\nabla\varphi(t,x) = 0$, which might a priori happen in a set of positive measure.

The main result of this section is the following.

Theorem 3.3. *Let $\nu > 0$, $T \in (0,+\infty)$, $\kappa : \mathbb{R} \to (0,+\infty)$ be continuous and bounded, $\rho_0 \in L^1(\Omega)$, and $\psi \in H^1_0(\Omega)$. Then there exists a weak solution (ρ,φ) of* (1.3) *with initial condition ρ_0 and final condition ψ.*

The proof of Theorem 3.3 relies on a fixed-point argument on the velocity field V of the Fokker–Planck equation in (1.3). Before turning to the proof, we need some continuity results on solutions of (2.1) with respect to the velocity field V and on solutions of (2.4) with respect to the function K, which we state and prove now.

Proposition 3.4. *Let $\nu > 0$ and $\rho_0 \in L^1(\Omega)$. Given $V \in L^\infty((0,T)\times\Omega;\mathbb{R}^d)$, let $(V_n)_{n\in\mathbb{N}}$ be a sequence in $L^\infty((0,T)\times\Omega;\mathbb{R}^d)$ such that $V_n \overset{*}{\rightharpoonup} V$ as $n\to\infty$. For $n\in\mathbb{N}$, let ρ_n (resp. ρ) be the unique weak solution of (2.1) in $L^1((0,T)\times\Omega)$ with velocity field V_n (resp. V). Then $\rho_n \to \rho$ in $L^1((0,T)\times\Omega)$ as $n\to\infty$.*

Proof. Since $(V_n)_{n\in\mathbb{N}}$ converges weakly-$*$ to V in L^∞, there exists a constant $M > 0$ such that $\|V_n\|_{L^\infty((0,T)\times\Omega)} \leq M$ for every $n\in\mathbb{N}$ and thus, by Proposition 2.2, there exists $C > 0$ depending only on d, ν, M, and $\|\rho_0\|_{L^1(\Omega)}$ such that, for every $n\in\mathbb{N}$,

$$\|\rho_n\|_{L^\infty((0,T);L^1(\Omega))} + \|\rho_n\|_{L^q((0,T);W^{1,q}(\Omega))} + \|\partial_t\rho_n\|_{L^q((0,T);W^{-1,q}(\Omega))} \leq C. \tag{3.1}$$

It follows from (3.1) and Aubin–Lions Lemma (see, e.g., [28, Corollary 4]) that $(\rho_n)_{n\in\mathbb{N}}$ is relatively compact in $L^1((0,T)\times\Omega)$. Let $\rho^* \in L^1((0,T)\times\Omega)$ be a limit point of $(\rho_n)_{n\in\mathbb{N}}$ and $(\rho_{n_k})_{k\in\mathbb{N}}$ a subsequence of $(\rho_n)_{n\in\mathbb{N}}$ converging to ρ^* in $L^1((0,T)\times\Omega)$.

The weak convergence of V_n in L^∞ together with the strong convergence of ρ_n in L^1 allow to pass to the limit the drift term $\nabla\cdot(\rho_n V_n)$ in the equation and we then easily obtain that ρ^* is a weak solution of (2.1). By the uniqueness of such solution from Proposition 2.2, one concludes $\rho^* = \rho$. In particular, ρ is the unique limit point of the relatively compact sequence $(\rho_n)_{n\in\mathbb{N}}$ in $L^1((0,T)\times\Omega)$, which yields the result. □

Proposition 3.5. *Let $\nu > 0$ and $\psi \in H^1_0(\Omega)$. Given $K \in L^\infty((0,T)\times\Omega)$, let $(K_n)_{n\in\mathbb{N}}$ be a sequence in $L^\infty((0,T)\times\Omega)$ such that $K_n \overset{*}{\rightharpoonup} K$ as $n\to\infty$. For $n\in\mathbb{N}$, let φ_n (resp. φ) be the unique weak solution of (2.4) in $L^\infty((0,T);L^2(\Omega))\cap L^2((0,T);H^1_0(\Omega))$ with K_n (resp. K). Then $\varphi_n \to \varphi$ in $L^2((0,T);H^1_0(\Omega))$ as $n\to\infty$.*

Proof. Again, there exists a constant $M > 0$ such that $\|K_n\|_{L^\infty((0,T)\times\Omega)} \leq M$ for every $n\in\mathbb{N}$ and thus, by Proposition 2.5, there exists $C > 0$ depending only on d, ν, T, Ω, M, and $\|\psi\|_{H^1_0(\Omega)}$ such that

$$\|\varphi_n\|_{L^\infty((0,T);H^1_0(\Omega))} + \|\varphi_n\|_{L^2((0,T);H^2(\Omega))} + \|\partial_t\varphi_n\|_{L^2((0,T)\times\Omega)} \leq C. \tag{3.2}$$

Hence, by Aubin–Lions Lemma (see, e.g., [28, Corollary 4]), $(\varphi_n)_{n\in\mathbb{N}}$ is relatively compact in $L^2((0,T);H^1_0(\Omega))$. Let φ^* be a limit point of $(\varphi_n)_{n\in\mathbb{N}}$ and $(\varphi_{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(\varphi_n)_{n\in\mathbb{N}}$ converging to φ^* in $L^2((0,T);H^1_0(\Omega))$. By (3.2), we also have $\varphi^* \in L^\infty((0,T);H^1_0(\Omega)))$.

Now, because of the non-linearity in the equation, we prefer to provide details on how to pass it to the limit. For every k and every $\eta \in H^1((0,T)\times\Omega)$ such that $\eta\big|_{[0,T]\times\partial\Omega} = 0$ and $\eta\big|_{\{0\}\times\Omega} = 0$, one has

$$\int_0^T\int_\Omega \varphi_{n_k}\partial_t\eta + \nu\int_0^T\int_\Omega \nabla\varphi_{n_k}\cdot\nabla\eta + \int_0^T\int_\Omega (K_{n_k}|\nabla\varphi_{n_k}| - 1)\eta = \int_\Omega \psi(x)\eta(T,x)\,\mathrm{d}x.$$

Since $K_{n_k} \overset{*}{\rightharpoonup} K$ in $L^\infty((0,T)\times\Omega)$ and $\varphi_{n_k} \to \varphi^*$ in $L^2((0,T);H^1_0(\Omega))$, one obtains, letting $k\to\infty$, that

$$\int_0^T\int_\Omega \varphi^*\partial_t\eta + \nu\int_0^T\int_\Omega \nabla\varphi^*\cdot\nabla\eta + \int_0^T\int_\Omega (K|\nabla\varphi^*| - 1)\eta = \int_\Omega \psi(x)\eta(T,x)\,\mathrm{d}x.$$

Hence $\varphi^* \in L^\infty((0,T);L^2(\Omega))\cap L^2((0,T);H^1_0(\Omega))$ is a weak solution of (2.4) and, by the uniqueness of solutions of (2.4) from Proposition 2.5, one deduces that $\varphi^* = \varphi$. Thus φ is the unique limit point in $L^2((0,T);H^1_0(\Omega))$ of the relatively compact sequence $(\varphi_n)_{n\in\mathbb{N}}$, yielding the conclusion. □

We can now turn to the proof of Theorem 3.3.

Proof of Theorem 3.3. Let κ_0 be an upper bound on κ. We endow the space $L^\infty((0,T) \times \Omega; \mathbb{R}^d)$ with its weak-$*$ topology and consider the ball of radius κ_0 given by

$$\mathcal{B} = \left\{V \in L^\infty((0,T) \times \Omega; \mathbb{R}^d) \;\middle|\; \|V\|_{L^\infty((0,T)\times\Omega;\mathbb{R}^d)} \leq \kappa_0\right\}.$$

Note that $\mathcal{B}$ is clearly convex and, by the Banach–Alaoglu theorem, $\mathcal{B}$ is a compact subset of $L^\infty((0,T) \times \Omega; \mathbb{R}^d)$.

Let $\mathcal{S}_{\mathrm{FP}} : L^\infty((0,T)\times\Omega; \mathbb{R}^d) \to L^1((0,T)\times\Omega)$ be the function that associates, with each $V \in L^\infty((0,T) \times \Omega; \mathbb{R}^d)$, the unique weak solution $\rho = \mathcal{S}_{\mathrm{FP}}(V) \in L^1((0,T) \times \Omega)$ of (2.1) with initial condition ρ_0. Note that, by Proposition 3.4, $\mathcal{S}_{\mathrm{FP}}$ is continuous with respect to the weak-$*$ topology of $L^\infty((0,T) \times \Omega; \mathbb{R}^d)$ and the strong topology of $L^1((0,T) \times \Omega)$. Similarly, we define $\mathcal{S}_{\mathrm{HJB}} : L^\infty((0,T) \times \Omega) \to L^2((0,T); H^1_0(\Omega))$ as the function that associates, with each $K \in L^\infty((0,T) \times \Omega)$, the unique weak solution $\varphi = \mathcal{S}_{\mathrm{HJB}}(K) \in L^2((0,T); H^1_0(\Omega))$ of (2.4) with terminal condition ψ. Proposition 3.5 ensures that $\mathcal{S}_{\mathrm{HJB}}$ is continuous with respect to the weak-$*$ topology of $L^\infty((0,T)\times\Omega)$ and the strong topology of $L^2((0,T); H^1_0(\Omega))$.

We define the set-valued map $\mathcal{V}$ that, with each $V \in \mathcal{B}$, associates the set $\mathcal{V}(V) \subset \mathcal{B}$ given by

$$\begin{aligned}\mathcal{V}(V) = \Big\{\widetilde{V} \in \mathcal{B} \;\Big|\; & |\widetilde{V}(t,x)| \leq \kappa(\rho(t,x)) \text{ for a.e. } (t,x) \in (0,T) \times \Omega, \\ & \widetilde{V}(t,x) \cdot \nabla\varphi(t,x) = -\kappa(\rho(t,x))|\nabla\varphi(t,x)| \text{ for a.e. } (t,x) \in (0,T) \times \Omega, \\ & \text{where } \rho = \mathcal{S}_{\mathrm{FP}}(V) \text{ and } \varphi = \mathcal{S}_{\mathrm{HJB}}(\kappa \circ \rho)\Big\}.\end{aligned}$$

In order to prove the existence of a weak solution (ρ, φ) of (1.3), we first prove the existence of a fixed point of the set-valued map $\mathcal{V}$, i.e., of a $V \in L^\infty((0,T) \times \Omega; \mathbb{R}^d)$ such that $V \in \mathcal{V}(V)$. This is done by applying Kakutani's fixed point theorem (see, e.g., [11, §7, Theorem 8.6]) to the set-valued map $\mathcal{V}$. To do so, we first need to verify some properties of $\mathcal{V}$ and its graph $\mathcal{G}$ defined by

$$\mathcal{G} = \left\{(V, \widetilde{V}) \in \mathcal{B} \times \mathcal{B} \;\middle|\; \widetilde{V} \in \mathcal{V}(V)\right\}.$$

Claim 1. *For every $V \in L^\infty((0,T) \times \Omega; \mathbb{R}^d)$, the set $\mathcal{V}(V)$ is non-empty and convex.*

Proof. It is immediate to verify that $\mathcal{V}(V)$ is convex. To prove that it is non-empty, let $V \in L^\infty((0,T) \times \Omega; \mathbb{R}^d)$, $\rho = \mathcal{S}_{\mathrm{FP}}(V)$, and $\varphi = \mathcal{S}_{\mathrm{HJB}}(\kappa \circ \rho)$. Then, the function $\widetilde{V} \in L^\infty((0,T) \times \Omega; \mathbb{R}^d)$ defined for a.e. $(t,x) \in (0,T) \times \Omega$ by

$$\widetilde{V}(t,x) = \begin{cases} -\kappa(\rho(t,x)) \dfrac{\nabla\varphi(t,x)}{|\nabla\varphi(t,x)|} & \text{if } \nabla\varphi(t,x) \neq 0, \\ 0 & \text{otherwise}, \end{cases}$$

clearly satisfies $\widetilde{V} \in \mathcal{V}(V)$. □

Claim 2. *The graph $\mathcal{G}$ is a closed subset of $\mathcal{B} \times \mathcal{B}$.*

Proof. Let $(V_n, \widetilde{V}_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{G}$ converging weakly-$*$ in $\mathcal{B} \times \mathcal{B}$ to a point $(V, \widetilde{V})$. We want to prove $(V, \widetilde{V}) \in \mathcal{G}$, i.e., $\widetilde{V} \in \mathcal{V}(V)$.

Define, for $n \in \mathbb{N}$, the functions $\rho_n \in L^1((0,T) \times \Omega)$ and $\varphi_n \in L^2((0,T); H^1_0(\Omega))$ by $\rho_n = \mathcal{S}_{\mathrm{FP}}(V_n)$ and $\varphi_n = \mathcal{S}_{\mathrm{HJB}}(\kappa \circ \rho_n)$ and, similarly, let $\rho = \mathcal{S}_{\mathrm{FP}}(V)$ and $\varphi = \mathcal{S}_{\mathrm{HJB}}(\kappa \circ \rho)$. Since $\mathcal{S}_{\mathrm{FP}} : L^\infty((0,T) \times \Omega; \mathbb{R}^d) \to L^1((0,T) \times \Omega)$ is continuous with respect to the weak-$*$ topology of $L^\infty((0,T) \times \Omega; \mathbb{R}^d)$ and the strong topology of $L^1((0,T) \times \Omega)$, one deduces $\rho_n \to \rho$ in $L^1((0,T) \times \Omega)$ as $n \to \infty$. Hence, up to extracting subsequences (which we still denote using the same notation for simplicity), one has $\rho_n \to \rho$ a.e. in $(0,T) \times \Omega$. Since κ

is continuous, we deduce $\kappa \circ \rho_n \to \kappa \circ \rho$ a.e. in $(0,T) \times \Omega$, and it follows $\kappa \circ \rho_n \overset{*}{\rightharpoonup} \kappa \circ \rho$ in $L^\infty((0,T) \times \Omega)$. The continuity of $\mathcal{S}_{\mathrm{HJB}} : L^\infty((0,T) \times \Omega; \mathbb{R}^d) \to L^2((0,T); H^1_0(\Omega))$ with respect to the weak-$*$ topology of $L^\infty((0,T) \times \Omega; \mathbb{R}^d)$ and the strong topology of $L^2((0,T); H^1_0(\Omega))$ implies $\varphi_n \to \varphi$ in $L^2((0,T); H^1_0(\Omega))$ as $n \to \infty$.

From the weak convergence of $\widetilde{V}_n$ to $\widetilde{V}$, the convexity of the function $|\cdot|$, and the (strong) convergence of $\kappa(\rho_n)$ to $\kappa(\rho)$, the inequality $|\widetilde{V}_n| \leq \kappa(\rho_n)$ gives at the limit

$$|\widetilde{V}(t,x)| \leq \kappa(\rho(t,x)) \qquad \text{for a.e. } (t,x) \in (0,T) \times \Omega. \tag{3.3}$$

Since $\widetilde{V}_n \in \mathcal{V}(V_n)$ for every $n \in \mathbb{N}$, we have $\widetilde{V}_n(t,x) \cdot \nabla\varphi_n(t,x) = -\kappa(\rho_n(t,x))|\nabla\varphi_n(t,x)|$ for a.e. $(t,x) \in (0,T) \times \Omega$. Then, for every $v \in L^2((0,T) \times \Omega)$, one has

$$\int_0^T \int_\Omega \widetilde{V}_n(t,x) \cdot \nabla\varphi_n(t,x) v(t,x)\, \mathrm{d}x\, \mathrm{d}t = -\int_0^T \int_\Omega \kappa(\rho_n(t,x))|\nabla\varphi_n(t,x)| v(t,x)\, \mathrm{d}x\, \mathrm{d}t.$$

Recalling that, as $n \to \infty$, one has $\widetilde{V}_n \overset{*}{\rightharpoonup} \widetilde{V}$ in $L^\infty((0,T)\times\Omega)$, $\nabla\varphi_n \to \nabla\varphi$ in $L^2((0,T)\times\Omega)$, and $\kappa \circ \rho_n \overset{*}{\rightharpoonup} \kappa \circ \rho$ in $L^\infty((0,T) \times \Omega)$, we obtain, letting $n \to \infty$, that

$$\int_0^T \int_\Omega \widetilde{V}(t,x) \cdot \nabla\varphi(t,x) v(t,x)\, \mathrm{d}x\, \mathrm{d}t = -\int_0^T \int_\Omega \kappa(\rho(t,x))|\nabla\varphi(t,x)| v(t,x)\, \mathrm{d}x\, \mathrm{d}t$$

for every $v \in L^2((0,T) \times \Omega)$, which implies that

$$\widetilde{V}(t,x) \cdot \nabla\varphi(t,x) = -\kappa(\rho(t,x))|\nabla\varphi(t,x)| \qquad \text{for a.e. } (t,x) \in (0,T) \times \Omega. \tag{3.4}$$

Combining (3.3) and (3.4), we conclude that $\widetilde{V} \in \mathcal{V}(V)$, as required. ☐

Claim 3. *For every $V \in L^\infty((0,T) \times \Omega; \mathbb{R}^d)$, the set $\mathcal{V}(V)$ is compact.*

Proof. This is a consequence of the fact that $\mathcal{G}$ is a closed subset of the compact set $\mathcal{B} \times \mathcal{B}$. ☐

Thanks to Claims 2 and 3, it follows from [3, Proposition 1.4.8] that the set-valued map $\mathcal{V}$ is upper semi-continuous. Using this fact and Claims 1 and 3, it follows from Kakutani's fixed point theorem that $\mathcal{V}$ admits a fixed point $V \in \mathcal{B}$. Let $\rho = \mathcal{S}_{\mathrm{FP}}(V)$ and $\varphi = \mathcal{S}_{\mathrm{HJB}}(\kappa \circ \rho)$. Using the facts that ρ and φ are solutions of (2.1) and (2.4), respectively, and that $V \in \mathcal{V}(V)$, it is immediate to verify, using Definitions 2.1, 2.3, and 3.1, that (ρ, φ) is a weak solution of (1.3) with initial condition ρ_0 and final condition ψ, as required. ☐

4. The MFG system with an infinite time horizon

Now that we have established in Section 3 the existence of solutions to the Mean Field Game system (1.3) in a finite time horizon T, we consider in this section the Mean Field Game system (1.2) with an infinite time horizon. Let us first provide the definition of a weak solution in this setting.

Definition 4.1. Let $\nu > 0$, $\kappa : \mathbb{R} \to (0, +\infty)$ be continuous and bounded, and $\rho_0 \in L^1(\Omega)$. We say that $(\rho, \varphi) \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^1(\Omega)) \times L^2_{\mathrm{loc}}(\mathbb{R}_+; H^1_0(\Omega))$ is a *weak solution* of (1.2) with initial condition ρ_0 if $\varphi \in L^\infty(\mathbb{R}_+ \times \Omega)$ and if there exists $V \in L^\infty(\mathbb{R}_+ \times \Omega; \mathbb{R}^d)$ such that $|V(t,x)| \leq \kappa(\rho(t,x))$ and $V(t,x) \cdot \nabla\varphi(t,x) = -\kappa(\rho(t,x))|\nabla\varphi(t,x)|$ a.e. on $\mathbb{R}_+ \times \Omega$ and such that, for every $T > 0$, ρ is a solution of the Fokker–Planck equation (2.1) with initial datum ρ_0 and vector field V on $[0,T] \times \Omega$ in the sense of Definition 2.1 and φ is a solution of the Hamilton–Jacobi–Bellman equation (2.4) with $K = \kappa(\rho)$ in the sense of Definition 2.3 on the same domain[1].

[1] Note that Definition 2.3 requires to fix a final value, and we did not define the notion of solution independently of the final value ψ. This could be formalized as "there exists $\psi \in L^2(\Omega)$ such that φ is a solution of (2.4)". Yet, since the function φ will be finally continuous as a function valued into $L^2(\Omega)$, the final datum on $[0,T]$ will be necessarily given by its own value $\varphi(T,\cdot)$.

Notice that, with respect to Definition 3.1, we make the additional requirement that $\varphi \in L^\infty(\mathbb{R}_+ \times \Omega)$. This is done mainly for three reasons. Firstly, boundedness of the solution of a Hamilton–Jacobi–Bellman equation is a condition usually required in order to ensure that this solution is the value function of an optimal control problem (see, e.g., [5, Theorem 8.1.10] and [10, Chapter II, Corollary 9.1]). Secondly, the strategy we use in this section to prove existence of a solution of (1.2), based on a limit argument from solutions of (1.3) in finite time horizon T as $T \to +\infty$, allows us to ensure that the function $\varphi : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ we construct is indeed bounded. Finally, boundedness of φ is an important property in order to establish the results on the the asymptotic behavior of solutions to (1.2) provided in Theorem 4.2 and Propositions 4.5 and 4.6.

4.1. **Existence of solutions and their asymptotic behavior.** From now on, we let Ψ denote the solution of the (stationary) Hamilton–Jacobi–Bellman equation

$$-\nu\Delta\Psi + \kappa(0)|\nabla\Psi| = 1, \quad x \in \Omega,$$

with Dirichlet boundary conditions $\Psi = 0$ on $\partial\Omega$. Owing to standard results on elliptic equations, Ψ is unique, and it is C^2 and positive in Ω.

The main result of this section is the following.

Theorem 4.2. *Let $\rho_0 \in L^1(\Omega)$. Then, there exists at least one solution (ρ, φ) to the Mean Field Game system with infinite time horizon* (1.2)*.*

In addition, any such solution satisfies

$$\rho_t \underset{t\to+\infty}{\longrightarrow} 0, \quad \varphi_t \underset{t\to+\infty}{\longrightarrow} \Psi,$$

and the above convergences hold uniformly.

The sequel of this section is devoted to the proof of Theorem 4.2. Let us start by giving an idea of the proof. First, we will construct solutions to the problem with infinite time horizon as limits of solutions of the problem with finite time horizon T by letting T go to $+\infty$. Then, to prove the long-time uniform convergence of the solutions, we shall make a crucial use of some regularity results for parabolic equations. More precisely, we will use local maximum principles for Fokker–Planck and for (forward) Hamilton–Jacobi–Bellman equations; roughly speaking, these results state that the $L^\infty(\Omega)$ norm of solutions of such equations at some time t_2 is controlled by some L^p norms of the same solution at some previous time $t_1 < t_2$. The results we use are proved in Appendix A, see Proposition A.1 and Corollaries A.2 and A.3.

We start with a lemma that gathers some useful estimates. These estimates have already been discussed in Section 2, but we need now to track possible dependencies of the constant on the time horizon T.

Lemma 4.3. *Let (ρ, φ) be solution of the finite horizon MFG system* (1.3) *on $[0, T] \times \Omega$ in the sense of Definition 3.1, with final datum $\psi \in H^1_0(\Omega) \cap L^\infty(\Omega)$ with $\psi \geq 0$. Then, there are $C_1, C_2 > 0$, depending on $\|\psi\|_{L^\infty} + \|\psi\|_{H^1_0}$, $\sup \kappa$, ν, Ψ and Ω such that*

$$\|\nabla\varphi(t, \cdot)\|_{L^2} \leq C_1, \quad \text{for all } \; t \in [0, T], \tag{4.1}$$

and

$$\|\varphi\|_{L^2((T_1,T_2);H^2)} \leq C_2(1 + |T_2 - T_1|). \tag{4.2}$$

Proof. Step 1. A preliminary estimate.

Let us start with giving an estimate on the gradient of φ. First, multiplying by φ the equation satisfied by φ and integrating on Ω for a fixed $t \in (0, T)$, we find

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_\Omega \varphi^2\right) = -\nu\int_\Omega |\nabla\varphi|^2 - \int_\Omega \kappa(\rho)|\nabla\varphi|\varphi + \int_\Omega \varphi.$$

Therefore, since φ is bounded, we have

$$\int_{T_1}^{T_2}\int_\Omega |\nabla\varphi|^2 \le C(1+|T_2-T_1|), \tag{4.3}$$

for some $C>0$ depending on $\sup\varphi$, $|\Omega|$, $\sup\kappa$, ν and for every $T_1, T_2 \in [0,T]$ with $0\le T_1\le T_2\le T$. Note that, from Proposition 2.5, $\sup\varphi$ is bounded in terms of ν, Ω, and $\|\psi\|_{L^\infty}$.

Step 2. Bound on $\|\nabla\varphi(t,\cdot)\|_{L^2}$.

We define, for $t\in[0,T]$,

$$u(t) := \frac{1}{2}\int_\Omega |\nabla\varphi(t,x)|^2\,\mathrm{d}x.$$

We differentiate u to obtain

$$u'(t) = \nu\int_\Omega(\Delta\varphi)^2 - \int_\Omega \kappa(\rho)|\nabla\varphi|\Delta\varphi + \int_\Omega \Delta\varphi. \tag{4.4}$$

Using Young's inequality, we find that there are $K_1, K_2>0$ depending only on $|\Omega|$, $\sup\kappa$, ν such that

$$u'(t) + K_1 u(t) + K_2 \ge 0.$$

This implies, for any $0\le t<s\le T$,

$$u(s) + \frac{K_2}{K_1}\left(1-e^{-K_1(s-t)}\right) \ge u(t)e^{-K_1(s-t)}. \tag{4.5}$$

We integrate (4.5) for $s\in(t,t+1)$ to get

$$\frac{1}{2}\int_t^{t+1}\int_\Omega |\nabla\varphi|^2(s,x)\,\mathrm{d}x\,\mathrm{d}s + \frac{K_2}{K_1}\int_0^1(1-e^{-K_1 r})\,\mathrm{d}r \ge \frac{1}{2}\left(\int_0^1 e^{-K_1 r}\,\mathrm{d}r\right)\int_\Omega|\nabla\varphi|^2(t,x)\,\mathrm{d}x.$$

Using (4.3) yields the $L^\infty(H^1)$ bound (4.1) for $t\in[0,T-1)$. To get the $L^\infty(H^1)$ bound (4.1) for $t\in[T-1,T]$, we use (4.5) with $s=T$. The result follows, with a constant also depending on $u(T)=\frac{1}{2}\int_\Omega|\nabla\psi|^2<+\infty$.

Step 3. Bound in $L^2((T_1,T_2);H^2)$.

Let us integrate (4.4) on (T_1,T_2). We find

$$\nu\int_{T_1}^{T_2}\int_\Omega(\Delta\varphi)^2 = u(T_2)-u(T_1) + \int_{T_1}^{T_2}\int_\Omega\kappa(\rho)|\nabla\varphi|\Delta\varphi - \int_{T_1}^{T_2}\int_\Omega\Delta\varphi.$$

Using Young's inequality on $\int_{T_1}^{T_2}\int_\Omega\kappa(\rho)|\nabla\varphi|\Delta\varphi$ and $\int_{T_1}^{T_2}\int_\Omega\Delta\varphi$ and the estimate (4.3), we get the desired bound (4.2) on $L^2((T_1,T_2);H^2)$. □

The next lemma shows that the time derivative of $\int_\Omega\rho\varphi$ is equal to $-\int_\Omega\rho$. Differentiating the average value of the value function is a classical computation in Mean Field Game theory. Since here the value function is an exit time, it is expected that it should decrease with rate 1, and one can guess the result from the fact that the total mass of the agents in this model is not fixed but decreases in time and is equal to $\int_\Omega\rho$.

Lemma 4.4. *Let (ρ,φ) be a solution of the finite-horizon MFG* (1.3) *on $[0,T]\times\Omega$ in the sense of Definition 3.1. Then, for a.e. t, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega\rho(t,x)\varphi(t,x)\,\mathrm{d}x\right) = -\int_\Omega\rho(t,x)\,\mathrm{d}x.$$

Proof. Let us fix two instants of times $t_1<t_2$, with $t_1>0$. On the interval (t_1,t_2) we can use φ as a test function in (2.3) and ρ in (2.6) since both φ and ρ are continuous as curves valued in L^2, belong to $L^2((t_1,t_2);H_0^1(\Omega))$, and their time-derivatives belong to $L^2((t_1,t_2);H^{-1}(\Omega))$. We subtract the two equalities that we obtain, which provides

$$\begin{aligned}\int_{t_1}^{t_2}\int_\Omega \rho\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t - \int_{t_1}^{t_2}\int_\Omega (\nu\nabla\rho-\rho V)\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t \\ + \int_{t_1}^{t_2}\int_\Omega (\varphi\partial_t\rho + \nu\nabla\varphi\cdot\nabla\rho + (\kappa(\rho)|\nabla\varphi|-1)\rho) \\ = 2\int_\Omega \varphi(t_2,x)\rho(t_2,x)\,\mathrm{d}x - 2\int_\Omega \varphi(t_1,x)\rho(t_1,x)\,\mathrm{d}x.\end{aligned}$$

After canceling the terms with $\nabla\rho\cdot\nabla\varphi$ and using $V\cdot\nabla\varphi+\kappa(\rho)|\nabla\varphi|=0$ we are left with

$$\begin{aligned}\int_{t_1}^{t_2}\int_\Omega(\rho\partial_t\varphi+\varphi\partial_t\rho)\,\mathrm{d}x\,\mathrm{d}t - \int_{t_1}^{t_2}\int_\Omega \rho\,\mathrm{d}x\,\mathrm{d}t \\ = 2\int_\Omega \varphi(t_2,x)\rho(t_2,x)\,\mathrm{d}x - 2\int_\Omega \varphi(t_1,x)\rho(t_1,x)\,\mathrm{d}x.\end{aligned}$$

It is then easy to see, by approximation via smooth functions, that for every pair (ρ,φ) such that $\rho,\varphi\in L^2((t_1,t_2);H^1(\Omega))$ and $\partial_t\rho,\partial_t\varphi\in L^2((t_1,t_2);H^{-1}(\Omega))$, we have

$$\int_{t_1}^{t_2}\int_\Omega(\rho\partial_t\varphi+\varphi\partial_t\rho)\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega \varphi(t_2,x)\rho(t_2,x)\,\mathrm{d}x - \int_\Omega \varphi(t_1,x)\rho(t_1,x)\,\mathrm{d}x.$$

We are then left with

$$\int_\Omega \varphi(t_2,x)\rho(t_2,x)\,\mathrm{d}x - \int_\Omega \varphi(t_1,x)\rho(t_1,x)\,\mathrm{d}x = -\int_{t_1}^{t_2}\int_\Omega \rho\,\mathrm{d}x\,\mathrm{d}t,$$

which is equivalent to the claim. □

We are now in position to prove Theorem 4.2.

Proof of Theorem 4.2. Let $\rho_0\in L^1(\Omega)$ be fixed.

Step 1. Existence.

For $T>0$, we let (ρ^T,φ^T) denote a solution of (1.3) with $T>0$, with initial datum ρ_0 for ρ and with final datum ψ^T for φ, where $(\psi^T)_{T>0}$ is any family of non-negative functions, bounded in $L^\infty(\Omega)\cap H_0^1(\Omega)$.

Recall that, by Proposition 2.5, $\|\varphi^T\|_{L^\infty((0,T)\times\Omega)}$ is bounded independently of T. Let $0<T_1<T_2$ be fixed. Lemma 4.3 implies that, as soon as $T>T_2$, φ^T is bounded in $L^2((T_1,T_2);H^2(\Omega))$ independently of $T>0$. Moreover, because $\partial_t\varphi^T\in L^2((T_1,T_2)\times\Omega)$ owing to Proposition 2.5, we can apply Aubin–Lions Lemma to the sequence $(\varphi^T)_{T>0}$ to get that, up to extraction, it converges strongly in $L^2_{loc}((0,+\infty);H^1(\Omega))$ to some limit φ_∞. Up to another extraction, we ensure that the convergence of $\varphi^T,\nabla\varphi^T$ also holds pointwise.

Using Aubin–Lions Lemma for the sequence $(\rho^T)_{T>0}$ as in the proof of Proposition 3.4, we find that, up to another extraction, it converges strongly to a limit ρ_∞ in $L^2((T_1,T_2)\times\Omega)$ and weakly in $L^2((T_1,T_2);H_0^1(\Omega))$. The solutions (ρ^T,φ^T) are associated with a bounded vector field V_T, which will converge weakly-$*$ in L^∞ to a vector field V_∞. Using the same arguments as in the proof of Theorem 3.3, Claim 2, we can pass to the limit $T\to+\infty$ in the equation to find that the pair $(\rho_\infty,\varphi_\infty)$ solves (1.2).

Step 2. Long-time behavior of ρ.

Let (ρ,φ) be a solution of (1.2), as built in the previous step. The integral version of Lemma 4.4, which is valid for (ρ^T,φ^T), also applies to (ρ,φ) at the limit, and we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho(t,x)\varphi(t,x)\,\mathrm{d}x \le \frac{-1}{\sup\varphi}\left(\int_\Omega \rho(t,x)\varphi(t,x)\,\mathrm{d}x\right),$$

hence, for all $t\ge 0$, we have

$$\int_\Omega \rho(t,x)\varphi(t,x)\,\mathrm{d}x \le \left(\int_\Omega \rho_0(x)\varphi(0,x)\,\mathrm{d}x\right)e^{-\frac{1}{\sup\varphi}t}.$$

Moreover, using the fact that $t \mapsto \int_\Omega \rho(t,x)\,\mathrm{d}x$ is non-increasing, we get, integrating the relation from Lemma 4.4,

$$\int_\Omega \rho(t,x)\,\mathrm{d}x \le \int_{t-1}^{t}\int_\Omega \rho(\tau,x)\,\mathrm{d}x d\tau \le \int_\Omega \rho(t-1,x)\varphi(t-1,x)\,\mathrm{d}x,$$

from which we get that there are $\alpha, \beta > 0$ such that

$$\int_\Omega \rho(t,x)\,\mathrm{d}x \le \beta e^{-\alpha t}.$$

Now, let us denote $u(t) := \int_\Omega \rho^2(t,x)\,\mathrm{d}x$. This is well defined for all $t > 0$. We have

$$u'(t) = -2\nu \int_\Omega |\nabla \rho|^2 - 2\int_\Omega \rho V \cdot \nabla \rho,$$

and, using Young's inequality, we get that there is $\delta > 0$ (depending on $\sup \kappa$ and ν) such that

$$u' - 2\delta u \le 0.$$

Hence

$$\int_\Omega \rho^2(t,x)\,\mathrm{d}x \le \left(\int_\Omega \rho^2(1,x)\,\mathrm{d}x\right) e^{2\delta(t-1)}.$$

Now, let $\theta \in (0,1)$ be close enough to 1 so that $\alpha\theta > \delta(1-\theta)$. Let $p_\theta := \theta + 2(1-\theta) > 1$. By classical interpolation arguments on L^p spaces, one has

$$\|\rho(t,\cdot)\|_{L^{p_\theta}} \le \|\rho(t,\cdot)\|_{L^1}^{\theta}\|\rho(t,\cdot)\|_{L^2}^{1-\theta} \le Ae^{-(\alpha\theta-\delta(1-\theta))t},$$

where $A = \beta^\theta e^{-(1-\theta)\delta}\|\rho(1,\cdot)\|_{L^2}^{1-\theta}$. Now that we have that the L^{p_θ} norm of $\rho(t,\cdot)$ goes to zero as t goes to $+\infty$, Corollary A.2 gives us that the L^∞ norm of $\rho(t,\cdot)$ also goes to zero when t goes to $+\infty$.

Step 3. Long-time behavior of φ.

We now turn to the convergence of φ as $t \to +\infty$. Let $(t_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers diverging to $+\infty$. Define

$$\varphi_n(t,x) := \varphi(t+t_n,x).$$

Then, φ_n solves

$$-\partial_t\varphi_n - \nu\Delta\varphi_n + \kappa(\rho(t+t_n,x))|\nabla\varphi_n| - 1 = 0, \quad t > -t_n,\ x \in \Omega.$$

Using the same estimates as in the first step, we find that, up to a subsequence, φ_n converges to some $\overline{\varphi}(t,x)$ in the $L^2_{loc}(H^1)$ sense, that satisfies

$$-\partial_t\overline{\varphi} - \nu\Delta\overline{\varphi} + \kappa(0)|\nabla\overline{\varphi}| - 1 = 0, \quad t \in \mathbb{R},\ x \in \Omega,$$

where we have used the uniform convergence $\rho(t,\cdot) \to 0$ as $t \to +\infty$ from the previous step in order to get the convergence of $\kappa(\rho(t+t_n,x))$ to $\kappa(0)$ as $n \to +\infty$. We now want to prove $\overline{\varphi} = \Psi$. From the boundedness of φ, the function $\overline{\varphi}$ is also bounded.

Let $T > 0$ be fixed. Let u_T, v_T be the solutions of

$$-\partial_t u - \nu\Delta u + \kappa(0)|\nabla u| - 1 = 0, \quad t \in (0,T),\ x \in \Omega, \tag{4.6}$$

with Dirichlet boundary conditions and final data $u_T(T,\cdot) = 0$ and $v_T(T,\cdot) = \Phi|_\Omega + M$, where $M \ge \overline{\varphi}$ and $\Phi|_\Omega \ge 0$ is the restriction to Ω of the solution of the torsion equation $-\nu\Delta\Phi = 1$ in Ω^+ (with Ω^+ a domain that contains Ω, say $\Omega^+ := \Omega + B_1$) with Dirichlet boundary conditions. We recall that the existence of u_T, v_T is guaranteed by Proposition 2.5.

The parabolic comparison principle, Proposition 2.6, implies that, for every $T > 0$,

$$u_T(t,\cdot) \le \overline{\varphi}(t,\cdot) \le v_T(t,\cdot), \qquad \text{for } \ t \in (0,T).$$

Let us prove that u_T, v_T converge to Ψ, the stationary solution of (4.6), as T goes to $+\infty$. To get this, let us show that the sequences of functions $(u_T)_{T>0}$ and $(v_T)_{T>0}$ are non-decreasing and non-increasing respectively, in the sense that $u_T \leq u_{T+h}$ and $v_T \geq v_{T+h}$ on $(0,T) \times \Omega$ for every $h \in (0,T)$.

Let $T > 0$ be fixed and let $h \in (0,T)$. Because (4.6) is autonomous, u_{T+h} and u_T are both solutions of (4.6) on $(0,T) \times \Omega$, with final data $u_{T+h}(T,\cdot)$ and $u_T(T,\cdot) = 0$ respectively. However, because $u_{T+h}(t,\cdot) \geq 0$ for $t \in (0, T+h)$ (as recalled in Proposition 2.5), we have $u_{T+h}(T,\cdot) \geq u_T(T,\cdot)$. To phrase it differently, u_{T+h} and u_T are solutions of the same equation with ordered final data, hence, we can apply the comparison principle Proposition 2.6 to find that $u_{T+h} \geq u_T$ on $(0,T) \times \Omega$.

Similarly , we have that v_{T+h} and v_T solve (4.6) on $(0,T)\times\Omega$, with final data $v_{T+h}(T,\cdot)$ and $v_T(T,\cdot) = \Phi|_\Omega + M$. By a standard comparison principle, we have that $v_{T+h} \leq \Phi|_\Omega + M$. Therefore, we can apply the parabolic comparison principle Proposition 2.6 to get that $v_{T+h} \leq v_T$ on $(0,T) \times \Omega$.

Therefore, owing to theses monotonicities, the sequences $(u_T)_{T>0}$ and $(v_T)_{T>0}$ converge a.e. as T goes to $+\infty$ to functions that do not depend on the t variable (this last fact comes from the equality $u_T(\cdot,\cdot) = u_{T+h}(\cdot + h, \cdot)$, which is true because (4.6) is autonomous and because the solutions are unique). Moreover, arguing as in the first step, we have that these limiting functions are solutions of (4.6). The only stationary solution of (4.6) being Ψ, we get that $\overline{\varphi}(t,\cdot) = \Psi$ for every t. We have thus proven that

$$w_n(t,x) := \varphi(t+t_n, x) - \Psi(x) \underset{n\to+\infty}{\longrightarrow} 0,$$

in the $L^2_{loc}(H^1)$ sense.

Let us prove that this convergence is actually uniform. To this aim, observe that w_n is a weak solution of

$$-\partial_t w_n = \nu \Delta w_n - \kappa(\rho(\cdot + t_n, \cdot)) z_n \cdot \nabla w_n + (\kappa(0) - \kappa(\rho(\cdot + t_n, \cdot)))|\nabla \Psi|,$$

where $z_n := \frac{\nabla \varphi_n + \nabla \Psi}{|\nabla \varphi_n| + |\nabla \Psi|}$ is bounded. Then, for every t_1, t_2 such that $t_2 + 1 < t_1 < t_2 + 2$, using Corollary A.3, we find that

$$\|w_n(t_2,\cdot)\|_{L^\infty} \leq C \left(\|w_n(t_1,\cdot)\|_{L^2} + \|(\kappa(0) - \kappa(\rho(\cdot + t_n, \cdot)))|\nabla\Psi|\|_{L^\infty((t_1,t_2)\times\Omega))} \right).$$

Integrating this for $t_1 \in (t_2+1, t_2+2)$, we find

$$\|w_n(t_2,\cdot)\|_{L^\infty} \leq C \left(\|w_n\|_{L^2((t_2+1,t_2+2)\times\Omega)} + \|(\kappa(0) - \kappa(\rho(\cdot + t_n, \cdot)))|\nabla\Psi|\|_{L^\infty((t_1,t_2)\times\Omega))} \right).$$

Because w_n goes to zero in the $L^2_{loc}(H^1)$ sense and $|\nabla\Psi|$ is bounded, observing that $|\kappa(\rho(\cdot + t_n, \cdot)) - \kappa(0)|$ converges uniformly to zero (this comes from the uniform convergence to zero of ρ from Step 2), we obtain that w_n goes to zero uniformly, whence

$$\varphi(t,x) \underset{t\to+\infty}{\longrightarrow} \Psi(x)$$

in the L^∞ sense. □

4.2. **Improved convergence results.** In the previous section, Theorem 4.2 proved the existence of solutions (ρ, φ) to the MFG system with infinite time horizon (1.2) and characterized the asymptotic behavior of any such solution by providing uniform convergence $\rho_t \to 0$ and $\varphi_t \to \Psi$. We want here to improve this result in two ways: first, we will prove that this convergence is actually exponential (in what concerns φ this requires a very small extra assumption on the function κ); second, we will prove that the convergence of $\varphi(t,\cdot)$ to Ψ as $t \to +\infty$, in addition to being uniform, is also a strong convergence in $H^1_0(\Omega)$. This last result is natural to evoke, because of the role played by $\nabla\varphi$ in the dynamics.

Proposition 4.5. *Suppose that the function $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ is Hölder continuous. Then, there exist constants $C, \alpha > 0$ (depending on κ, ν, and Ω), such that we have, for any $(t, x) \in [0, +\infty) \times \Omega$,*

$$|\rho(t,x)| + |\varphi(t,x) - \Psi(x)| \leq Ce^{-\alpha t}.$$

Proof. The exponential convergence of ρ to 0 is indeed part of the proof of Theorem 4.2, since we proved that, for p close to 1, the L^p norm of ρ_t tends exponentially to 0, and we then used the parabolic regularization estimate $\|\rho_t\|_{L^\infty} \leq C\|\rho_{t-1}\|_{L^p}$.

Thanks to the assumption that κ is Hölder continuous, up to modifying the coefficient in the exponent, we obtain $|K(t,x) - \kappa(0)| \leq Ce^{-\alpha t}$, where $K(t,x) = \kappa(\rho(t,x))$.

We need now to discuss the exponential convergence of φ. Let us fix a time t_1 and define

$$a_\pm := 1 \pm 3C\|\nabla\psi\|_{L^\infty} e^{-\alpha t_1}, \quad \Psi_\pm := a_\pm \Psi \pm e^{-\alpha t_1}.$$

We will use a comparison principle between φ and $\Psi_\pm$. The functions $\Psi_\pm$ solve

$$-\partial_t \Psi_\pm - \nu\Delta\Psi_\pm + \kappa(0)|\nabla\Psi_\pm| - a_\pm = 0,$$

where the time-derivative term is actually 0 since they are functions of the x variable only. If we set $v_\pm := \varphi - \Psi_\pm$, the functions $v_\pm$ solve a linear PDE of the form

$$-\partial_t v_\pm - \nu\Delta v_\pm + w_\pm \cdot \nabla v_\pm \pm 3C\|\nabla\psi\|_{L^\infty} e^{-\alpha t_1} + (K(t,x) - \kappa(0))a_\pm|\nabla\Psi| = 0,$$

where the vector fields $w_\pm$ are such that $|w_\pm(t,x)| \leq K(t,x)$. In particular, if we note that, for t_1 large enough, we have $0 \leq a_\pm \leq 2$, we have $(K(t,x) - \kappa(0))a_\pm|\nabla\Psi| \leq 2C\|\nabla\psi\|_{L^\infty} e^{-\alpha t_1}$. Hence, for v_+ we have

$$-\partial_t v_+ - \nu\Delta v_+ + w_+ \cdot \nabla v_+ < 0$$

and for v_-

$$-\partial_t v_- - \nu\Delta v_- + w_- \cdot \nabla v_- > 0.$$

Let us look now at the boundary conditions of $v_\pm$ relative to the parabolic domain $[t_1, t_2] \times \Omega$. If t_2 is large enough, using the uniform convergence $\varphi_t \to \Psi$, we can infer $v_+(t_2, x) < 0$ for every $x \in \Omega$. Moreover, we also have $v_+(t,x) < 0$ for every t and every $x \in \partial\Omega$. The inequalities are opposite for v_-, i.e. we have $v_-(t,x) > 0$ for $t = t_2$ or $x \in \partial\Omega$. This implies, by the maximum principle in [2] (see [2, Theorem 1], adapted to this backward equation, and using again the version with the inequality presented at the end of the proof, page 98), the inequalities $v_+(t_1, x) \leq 0 \leq v_-(t_1, x)$, i.e.

$$(1 - 3C\|\nabla\psi\|_{L^\infty} e^{-\alpha t_1})\Psi - e^{-\alpha t_1} \leq \varphi \leq (1 + 3C\|\nabla\psi\|_{L^\infty} e^{-\alpha t_1})\Psi + e^{-\alpha t_1}.$$

This shows $\|\varphi_t - \Psi\|_{L^\infty} \leq Ce^{-\alpha t_1}$, for a new constant C. □

We can now pass to the following statement, which proves the convergence of the gradient of φ.

Proposition 4.6. *Let (ρ, φ) be a solution to the Mean Field Game system with infinite time horizon* (1.2). *Then*

$$\varphi(t, \cdot) \underset{t \to +\infty}{\longrightarrow} \Psi$$

in $H^1_0(\Omega)$.

Proof. We first observe that, by Lemma 4.3, the family of functions $(\varphi(t,\cdot))_{t\geq 0}$ is bounded in $H^1_0(\Omega)$. This, together with the uniform convergence to Ψ, implies that one has the weak convergence $\varphi(t,\cdot) \rightharpoonup \Psi$ in $H^1_0(\Omega)$ as $t \to +\infty$.

In order to conclude the proof, it suffices to show $\|\varphi(t,\cdot)\|_{H^1_0(\Omega)} \to \|\Psi\|_{H^1_0(\Omega)}$ as $t \to +\infty$. Since $\varphi(t,\cdot) \rightharpoonup \Psi$ in $H^1_0(\Omega)$ as $t \to \infty$, one has

$$\|\Psi\|^2_{H^1_0(\Omega)} \leq \liminf_{t \to +\infty} \|\varphi(t,\cdot)\|^2_{H^1_0(\Omega)}.$$

Let us argue by contradiction and assume that there exists $\varepsilon > 0$ and an increasing sequence $(s_n)_{n\in\mathbb{N}}$ with $s_n \to +\infty$ as $n \to +\infty$ such that

$$\|\varphi(s_n)\|_{H_0^1(\Omega)}^2 \geq \|\Psi\|_{H_0^1(\Omega)}^2 + 2\varepsilon \tag{4.7}$$

for every $n \in \mathbb{N}$. Recall that, from (4.5) in the proof of Lemma 4.3, there exists $C > 0$ depending only on $\sup\varphi$, $\sup\kappa$, ν, and $|\Omega|$ such that

$$\|\varphi(t,\cdot)\|_{H_0^1(\Omega)}^2 \geq \|\varphi(s_n)\|_{H_0^1(\Omega)}^2 e^{-C(t-s_n)} - C(1 - e^{-C(t-s_n)})$$

for every $n \in \mathbb{N}$ and $t \in (s_n, s_n+1)$. Combining this with (4.7), one obtains that there exists $\delta \in (0,1)$ depending only on $\|\Psi\|_{H_0^1(\Omega)}$, ε, and C such that

$$\|\varphi(t,\cdot)\|_{H_0^1(\Omega)}^2 \geq \|\Psi\|_{H_0^1(\Omega)}^2 + \varepsilon \tag{4.8}$$

for every $n \in \mathbb{N}$ and $t \in (s_n, s_n + \delta)$. By Lemma 4.3, there exists a constant $C' > 0$ depending only on $\sup\varphi$, $\sup\kappa$, ν, $|\Omega|$, and Ψ such that

$$\|\varphi\|_{L^2((s_n,s_n+\delta);H^2(\Omega))}^2 \leq C' \qquad \text{for every } n \in \mathbb{N}.$$

In particular, for every $n \in \mathbb{N}$, there exists $t_n \subset (s_n, s_n+\delta)$ such that $\|\varphi(t_n)\|_{H^2(\Omega)}^2 \leq C'/\delta$. Hence $(\varphi(t_n))_{n\in\mathbb{N}}$ is bounded in $H^2(\Omega)$ and thus, up to extracting subsequences (which we still denote by $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ for simplicity), $(\varphi(t_n))_{n\in\mathbb{N}}$ converges strongly in $H_0^1(\Omega)$ as $n \to \infty$. Since $\varphi(t,\cdot) \rightharpoonup \Psi$ as $t \to +\infty$, the strong limit of $(\varphi(t_n))_{n\in\mathbb{N}}$ in $H_0^1(\Omega)$ is necessarily Ψ, and thus, in particular, $\|\varphi(t_n)\|_{H_0^1(\Omega)}^2 \to \|\Psi\|_{H_0^1(\Omega)}^2$ as $n \to +\infty$. This, however, contradicts (4.8), and establishes the desired result. □

Appendix A. Regularizing effects of parabolic equations

This appendix is concerned with the regularizing properties of a class of parabolic equations including both the Fokker–Planck and the Hamilton–Jacobi–Bellman equations we consider in this paper. More precisely, we consider the increase of the exponent p of the L^p integrability in space of the solution of the system. As recalled in the introduction, the computations and results presented here are very similar to those from the appendix of [7], the main difference lying in the boundary condition. The main result of this appendix is the following.

Proposition A.1. *Let $T \in (0,+\infty]$. Let $V, F, f, g, u \in C^\infty((0,T)\times\Omega)$ with $V, g \in L^\infty((0,T)\times\Omega)$, $u \geq 0$, $u = 0$ on $\partial\Omega$, such that*

$$\partial_t u - \nu\Delta u + \nabla\cdot(uV) + \nabla\cdot F + f + g\cdot\nabla u \leq 0, \quad \text{on} \quad (0,T)\times\Omega. \tag{A.1}$$

Then, for every $p > 1$, every number $a \in (0,1)$ and t_1, t_2 such that $0 < t_1 < t_2 < T$ and $a < |t_1 - t_2| < a^{-1}$, there is $C > 0$, depending only on $p, a, \|V\|_{L^\infty}, \|g\|_{L^\infty}$ such that

$$\|u(t_2,\cdot)\|_{L^\infty} \leq C\left(\|u(t_1,\cdot)\|_{L^p} + \|F\|_{L^\infty((t_1,t_2)\times\Omega)} + \|f\|_{L^\infty((t_1,t_2)\times\Omega)}\right).$$

The same result is true omitting the assumption $u \geq 0$ if the PDE (A.1) *is satisfied as an equality instead of an inequality.*

The proof follows a standard method based on Moser's iterations that will be detailed here. This appendix is included for completeness: the experienced reader will recognize well-known computations, which are simplified in this setting thanks in particular to the Dirichlet boundary conditions we use.

Proof. Let u be as in the proposition. For $k > 1$, we define

$$m_k(t) := \int_\Omega u^k(t,x)\,\mathrm{d}x.$$

We also define $\alpha := \frac{2^\star}{2} = \frac{n}{n-2}$ if $n > 2$ (here $2^\star$ is the Sobolev exponent in dimension n). When $n = 1, 2$ we set $\alpha := 2$ (but any number larger than 1 and smaller than $+\infty$ could be used here). Moreover, we set

$$M := \|F\|_{L^\infty((t_1,t_2)\times\Omega)} + \|f\|_{L^\infty((t_1,t_2)\times\Omega)}$$

Step 1. L^p estimates.

Let us start with proving that, for $k_0 > 1$, there is $C > 0$ depending on k_0 and on the L^∞ norms of V, g, such that, for every $k > k_0 > 1$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(m_k e^{-Ck^2 t}) + \frac{1}{C} m_{\alpha k}^{\frac{1}{\alpha}} e^{-Ck^2 t} \leq C e^{-Ck^2 t} k^2 M^k. \tag{A.2}$$

In order to do so, we differentiate m_k with respect to t, to get

$$\begin{aligned} m_k'(t) &\leq k \int_\Omega (\nu \Delta u - \nabla \cdot (uV) - g \cdot \nabla u - \nabla \cdot F - f) u^{k-1} \\ &\leq -k(k-1)\nu \int_\Omega |\nabla u|^2 u^{k-2} + k(k-1) \int_\Omega (V \cdot \nabla u) u^{k-1} - k \int_\Omega (g \cdot \nabla u) u^{k-1} \\ &\qquad + k(k-1) \int_\Omega (F \cdot \nabla u) u^{k-2} - k \int_\Omega f u^{k-1}. \end{aligned}$$

Now, owing to Young's inequality, we can find $C_1, C_2, C_3 > 0$ depending only on $\|V\|_{L^\infty}$, $\|g\|_{L^\infty}$, k_0, ν such that

$$m_k'(t) \leq -C_1 k^2 \int_\Omega |\nabla u|^2 u^{k-2} + C_2 k^2 \int_\Omega u^k + C_3 k^2 \int_\Omega |F|^2 u^{k-2} + k \int_\Omega |f| u^{k-1}$$

(note that we replaced the coefficient $k(k-1)$ with k^2, as these two numbers are equivalent up to multiplicative constants as far as $k > k_0 > 1$). Moreover, thanks again to a Young inequality, we have

$$|F|^2 u^{k-2} \leq \frac{2}{k} |F|^k + \frac{k-2}{k} u^k \quad \text{and} \quad |f| u^{k-1} \leq \frac{1}{k} |f|^k + \frac{k-1}{k} u^k.$$

Therefore, up to increasing C_2, C_3, we get

$$\begin{aligned} m_k'(t) &\leq -C_1 k^2 \int_\Omega |\nabla u|^2 u^{k-2} + C_2 k^2 \int_\Omega u^k + C_3 k^2 \int_\Omega |F|^k + k \int_\Omega |f|^k \\ &\leq -C_1 k^2 \int_\Omega |\nabla u|^2 u^{k-2} + C_2 k^2 \int_\Omega u^k + C_3 k^2 M^k. \end{aligned}$$

Now, owing to the Gagliardo–Nirenberg–Sobolev inequality, we have, for some $C_4 > 0$,

$$k^2 \int_\Omega |\nabla u|^2 u^{k-2} = 4 \int_\Omega |\nabla (u^{\frac{k}{2}})|^2 \geq C_4 \left(\int_\Omega u^{k\alpha} \right)^{\frac{1}{\alpha}} = C_4 m_{k\alpha}^{\frac{1}{\alpha}}.$$

Hence

$$m_k'(t) + C_1 C_4 m_{\alpha k}^{\frac{1}{\alpha}} \leq C_2 k^2 m_k + C_3 k^2 M^k.$$

Let us denote $C := \max\{\frac{1}{C_1 C_4}, C_2, C_3\}$. Then, the above equation gives us

$$m_k'(t) - Ck^2 m_k + \frac{1}{C} m_{\alpha k}^{\frac{1}{\alpha}} \leq Ck^2 M^k,$$

which we can rewrite in order to get (A.2).

Step 2. Estimates on $m_{\alpha k}$.

We show in this step that, for $k > k_0 > 1$ and for $0 < t_1 < t_2 < T$, we have

$$m_{\alpha k}^{\frac{1}{\alpha}}(t_2) \leq e^{C\alpha k^2 (t_2 - t_1)} \frac{1}{t_2 - t_1} e^{Ck^2 t_2} \int_{t_1}^{t_2} m_{\alpha k}^{\frac{1}{\alpha}}(s) e^{-Ck^2 s} \,\mathrm{d}s + e^{C\alpha k^2 (t_2 - t_1)} M^k, \tag{A.3}$$

for some C depending on k_0 and on the L^∞ norms of V, g.

The relation (A.2) provides

$$\frac{\mathrm{d}}{\mathrm{d}t}(m_k e^{-Ck^2 t}) \leq C e^{-Ck^2 t} k^2 M^k.$$

Let us take $s \in (t_1, t_2)$. We integrate the above inequality for $t \in (s, t_2)$ to get:

$$m_k(t_2)e^{-Ck^2t_2} \leq m_k(s)e^{-Ck^2s} + CM^k \int_s^{t_2} e^{-Ck^2t}\,\mathrm{d}t \leq m_k(s)e^{-Ck^2s} + M^k e^{-Ck^2s}.$$

Taking the power $\frac{1}{\alpha} < 1$ and using its subadditivity yields

$$m_k^{\frac{1}{\alpha}}(t_2)e^{-\frac{C}{\alpha}k^2t_2} \leq m_k^{\frac{1}{\alpha}}(s)e^{-\frac{C}{\alpha}k^2s} + M^{\frac{k}{\alpha}}e^{-\frac{C}{\alpha}k^2s}.$$

We replace k by αk so as to re-write the above inequality as

$$m_{\alpha k}^{\frac{1}{\alpha}}(t_2)e^{-C\alpha k^2t_2} \leq m_{\alpha k}^{\frac{1}{\alpha}}(s)e^{-C\alpha k^2s} + M^k e^{-C\alpha k^2s}.$$

We multiply by $e^{Ck^2s(\alpha-1)}$ and integrate this for $s \in (t_1, t_2)$ in order to obtain

$$m_{\alpha k}^{\frac{1}{\alpha}}(t_2)\int_{t_1}^{t_2} e^{C\alpha k^2(s-t_2)}e^{-Ck^2s}\,\mathrm{d}s \leq \int_{t_1}^{t_2} m_{\alpha k}^{\frac{1}{\alpha}}(s)e^{-Ck^2s}\,\mathrm{d}s + M^k\int_{t_1}^{t_2} e^{-Ck^2s}\,\mathrm{d}s.$$

We then use $e^{C\alpha k^2(s-t_2)} \geq e^{C\alpha k^2(t_1-t_2)}$ in order to obtain

$$m_{\alpha k}^{\frac{1}{\alpha}}(t_2) \leq e^{C\alpha k^2(t_2-t_1)}\left(\int_{t_1}^{t_2} e^{-Ck^2s}\,\mathrm{d}s\right)^{-1}\int_{t_1}^{t_2} m_{\alpha k}^{\frac{1}{\alpha}}(s)e^{-Ck^2s}\,\mathrm{d}s + e^{C\alpha k^2(t_2-t_1)}M^k$$

and finally we use $\int_{t_1}^{t_2} e^{-Ck^2s}\,\mathrm{d}s \geq (t_2-t_1)e^{-Ck^2t_2}$, which provides the desired inequality.

Step 3. Higher integrability estimates.

Let us now show that, for $k > k_0$, there is $C > 0$ (depending on the same quantities as in the previous steps), such that

$$m_{\alpha k}^{\frac{1}{\alpha k}}(t_2) \leq \frac{e^{Ck(t_2-t_1)}}{|t_2-t_1|^{1/k}}\left(m_k(t_1) + M^k\right)^{\frac{1}{k}}, \tag{A.4}$$

First of all, integrating (A.2) for $t \in (t_1, t_2)$, and discharging the final value $m_k(t_2)e^{-Ct_2}$, we obtain

$$\frac{1}{C}\int_{t_1}^{t_2} m_{\alpha k}^{\frac{1}{\alpha}}(t)e^{-Ck^2t}\,\mathrm{d}t \leq m_k(t_1)e^{-Ck^2t_1} + M^k\int_{t_1}^{t_2} Ck^2e^{-Ck^2t}\,\mathrm{d}t \leq e^{-Ck^2t_1}\left(M^k + m_k(t_1)\right).$$

Combining this with (A.3), we get

$$m_{\alpha k}^{\frac{1}{\alpha}}(t_2) \leq e^{C\alpha k^2(t_2-t_1)}\left(\frac{e^{Ck^2(t_2-t_1)}}{t_2-t_1}(m_k(t_1) + M^k) + M^k\right).$$

Up to enlarging the constant C and using $0 < t_2 - t_1 < a^{-1}$, we can write the above inequality in a simpler form, i.e.

$$m_{\alpha k}^{\frac{1}{\alpha}}(t_2) \leq \frac{e^{C(\alpha+1)k^2(t_2-t_1)}}{t_2-t_1}\left(m_k(t_1) + M^k\right),$$

hence, (A.4) holds true, after raising to the power $1/k$ and including $\alpha+1$ in the constant C.

Step 4. Iterations.

We conclude the proof in this step by proving that, for p, t_1, t_2 as in the statement of the proposition, there is $C > 0$ such that

$$\|u(t_2,\cdot)\|_{L^\infty} \leq C(\|u(t_1,\cdot)\|_{L^p} + \|F\|_{L^\infty} + \|f\|_{L^\infty}). \tag{A.5}$$

We denote

$$s_n := t_2 - \frac{t_2-t_1}{(2\alpha)^n},\ k_n := \alpha^n p,\ \beta_n := \frac{e^{Ck_n(s_{n+1}-s_n)}}{(s_{n+1}-s_n)^{\frac{1}{k_n}}},$$

and

$$a_n := m_{k_n}^{\frac{1}{k_n}}(s_n), \quad \tilde{a}_n := \max\{a_n, M\}.$$

Then, (A.4) gives us that

$$a_{n+1} \le \beta_n (a_n^{k_n} + M^{k_n})^{\frac{1}{k_n}} \le \beta_n 2^{\frac{1}{k_n}} \tilde{a}_n.$$

Hence, up to replacing the constant C with a larger one so as to suppose $\beta_n 2^{\frac{1}{k_n}} \ge 1$, we find

$$\tilde{a}_{n+1} \le \beta_n 2^{\frac{1}{k_n}} \tilde{a}_n.$$

We observe that we have $\prod_{n=0}^{+\infty} \beta_n 2^{\frac{1}{k_n}} < +\infty$ as a consequence of the logarithmic estimate

$$\sum_{n=0}^{+\infty} \log(\beta_n 2^{\frac{1}{k_n}}) \le \sum_{n=0}^{+\infty} C k_n \frac{t_2 - t_1}{(2\alpha)^n} + \frac{1}{k_n}(\log 2 + n \log(2\alpha) - \log(t_2 - t_1)) < +\infty.$$

Therefore, we obtain

$$\max\{\lim_{n\to+\infty} a_n, M\} \le \left(\prod_{n=0}^{+\infty} \beta_n 2^{\frac{1}{k_n}}\right) \max\{a_0, M\} \le C(a_0 + M).$$

Therefore, thanks to $\lim_{n\to+\infty} a_n = \|u(t_2,\cdot)\|_{L^\infty}$, we obtain (A.5). This concludes the proof. □

Corollary A.2. *Let $T \in (0, +\infty]$. Let $V \in L^\infty((0,T)\times\Omega)$. Let $u \in L^1((0,T)\times\Omega)$ be a positive distributional solution of*

$$\partial_t u - \nu\Delta u - \nabla\cdot(uV) \le 0, \qquad \text{on} \;\; (0,T)\times\Omega,$$

satisfying the following mild regularity assumption: u is obtained as a measurable curve $(u_t)_t$ of functions of the x variable, which is such that $t \mapsto \int_\Omega \eta(x) u_t(x)\,\mathrm{d}x$ is continuous in time for every $\eta \in C^\infty(\Omega)$ (note that we do not restrict to $\eta \in C_c^\infty(\Omega)$). Then, for every $p > 1$ and $a \in (0,1)$, there is $C > 0$, depending only on p, a, $\|V\|_{L^\infty}$, such that we have

$$\|u(t_2,\cdot)\|_{L^\infty} \le C\|u(t_1,\cdot)\|_{L^p}$$

for every $0 < t_1 < t_2 < T$ with $a < |t_2 - t_1| < a^{-1}$.

Proof. To prove this estimate the only important point is to regularize the equation so as to apply Proposition A.1. In order for the proof to be self-contained, we detail a two-step approximation argument.

We convolve the equation by an approximation of the identity and to apply Proposition A.1. However, convolving will not preserve the Dirichlet boundary conditions, so we first have to extend u by zero on a bigger set.

We define Ω^+ to be a open bounded regular set such that $\Omega + B_1 \subset \Omega^+$, where B_1 is unit ball in $\mathbb{R}^N$.

We define $u^+(t,x) := u(t,x)$ if $x \in \Omega$, and $u^+(t,x) = 0$ elsewhere. Let $\eta_\varepsilon(x)$ be an approximation of the identity whose support is included in B_1. We define $u_\varepsilon := u^+ \star \eta_\varepsilon$ (here, $\star$ is the convolution in space only). It is a function which is smooth in x and continuous in t. We then convolve in time as well, taking $\chi_\delta(t)$ an approximation of the identity whose support is included in $\mathbb{R}_+$. Defining $u_{\varepsilon,\delta} := \chi_\delta \star u_\varepsilon$ we have now a function which is smooth in time and space. It satisfies, in the classical sense,

$$\partial_t u_{\varepsilon,\delta} - \nu\Delta u_{\varepsilon,\delta} - \nabla\cdot(u_{\varepsilon,\delta} V_{\varepsilon,\delta}) \le 0, \qquad \text{for} \;\; t \in (0,T), \; x \in \Omega^+,$$

with $V_{\varepsilon,\delta} := \frac{\chi_\delta \star \eta_\varepsilon \star (uV)}{u_{\varepsilon,\delta}} \in C^\infty$. Moreover, the L^∞ norm of $V_{\varepsilon,\delta}$ is bounded independently of ε and δ. Then, $u_{\varepsilon,\delta}$ is positive, regular and is a (classical) subsolution of a Fokker–Planck equation with regular coefficients, hence we can apply Proposition A.1. We then take the limit $\delta \to 0$, and we observe that we have

$$\|u_\varepsilon(t,\cdot)\|_{L^p} = \lim_{\delta\to 0} \|u_{\varepsilon,\delta}(t,\cdot)\|_{L^p}$$

for every t, since u_ε is continuous. Then, we have

$$\|u(t,\cdot)\|_{L^p} = \lim_{\varepsilon\to 0}\|u_\varepsilon(t,\cdot)\|_{L^p}$$

from standard properties of convolutions (with the possibility, of course, that this limit and this norm take the value $+\infty$). □

Corollary A.3. *Let $T \in (0,+\infty]$. Let $f, g \in L^\infty$ and let $u \in L^\infty((0,T);L^2(\Omega)) \cap L^2((0,T);H^1_0(\Omega))$ be solution (in the weak sense) of*

$$\partial_t u - \nu\Delta u + f + g\cdot\nabla u = 0, \quad \text{on} \quad (0,T)\times\Omega,$$

with Dirichlet boundary conditions and initial datum $u(0,\cdot) = u_0 \in L^2$.

Then, for every $p > 1$, and $a \in (0,1)$ there is $C > 0$, depending only on $p, a, \|g\|_{L^\infty}$ such that

$$\|u(t_2,\cdot)\|_{L^\infty} \le C\left(\|u(t_1,\cdot)\|_{L^p} + \|f\|_{L^\infty}\right)$$

for every $t_1 < t_2$ with $a < |t_2 - t_1| < a^{-1}$.

Proof. Let f_n, g_n be C^∞ and such that $f_n \to f$ and $g_n \to g$ in the L^2 norm. Assume moreover that we have $\|f_n\|_{L^\infty} \to \|f\|_{L^\infty}$ and $\|g_n\|_{L^\infty} \to \|g\|_{L^\infty}$. Let u_n be the solution of

$$\partial_t u_n - \nu\Delta u_n + f_n + g_n\cdot\nabla u_n = 0, \quad \text{on} \quad (0,+\infty)\times\Omega,$$

with Dirichlet boundary condition and with initial datum u_0^n, where u_0^n is a smooth L^2 approximation of u_0.

Then, u_n is smooth enough to apply Proposition A.1 to u_n, to get, for p, t_1, t_2 as in the statement of the corollary,

$$\|u_n(t_2,\cdot)\|_{L^\infty} \le C\left(\|u_n(t_1,\cdot)\|_{L^p} + \|f_n\|_{L^\infty}\right). \tag{A.6}$$

Then, as n goes to $+\infty$, u_n converges (the arguments to prove this are standard and based on the weak L^2 convergence of ∇u_n) to a solution (in the weak sense) of

$$\partial_t u - \nu\Delta u + f + g\cdot\nabla u = 0, \quad \text{on} \quad (0,+\infty)\times\Omega,$$

with Dirichlet boundary conditions and with initial datum u_0. The convergence is also strong in the L^2 sense. Because this solution is unique, it necessarily coincides with the original solution u of the statement. In order to obtain the result, we need to pass to the limit the inequality (A.6). The left-hand side can easily be dealt with by semicontinuity, while for the right-hand side, we suppose $p \le 2$ and we use strong L^2 convergence. Since this convergence is L^2 in space-time, we have convergence of the right-hand side only for a.e. t_1. Yet, using the fact that the solution u is continuous in time as a function valued into $L^2(\Omega)$, the result extends to all t_1. The inequality for $p = 2$ implies that with $p > 2$, up to modifying the constant in a way depending on $|\Omega|$. □

The reader may observe that we used different regularization strategies to prove the two above corollaries. Indeed, the linear behavior of the Fokker–Planck equation allowed to directly regularize the solution (up to modifying the drift vector field: we convolve the solution and define a new drift vector field which preserves the same L^∞ bound, a trick which is completely standard for curves in the Wasserstein space, see for instance [27, Chapter 5]). This is not possible for the Hamilton–Jacobi–Bellman equation. However, when uniqueness of the solution is known, regularizing the coefficients and the data of the equation is another option, and it is what we did in our last corollary.

Acknowledgments. The authors wish to thank many colleagues for useful discussions and suggestions, and in particular Alessio Porretta. Without the comments he made after a talk the second author gave on the topic of the present paper, the strategy to achieve convergence to a solution in the limit $T \to +\infty$ would have been completely different, the result less general, and the time needed to achieve it much longer.

The authors acknowledge the financial support of French ANR project "MFG", reference ANR-16-CE40-0015-01, and of a public grant as part of the "Investissement d'avenir" project, reference ANR-11-LABX-0056-LMH, LabEx LMH, PGMO project VarPDEMFG. The first author was also partially supported by the by the French IDEXLYON project Impulsion "Optimal Transport and Congestion Games" PFI 19IA106udl and the second author was also partially supported by the Hadamard Mathematics LabEx (LMH) through the grant number ANR-11-LABX-0056-LMH in the "Investissement d'avenir" project.

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