A SHORT PROOF OF THE INFINITESIMAL HILBERTIANITY OF THE WEIGHTED EUCLIDEAN SPACE

SIMONE DI MARINO, DANKA LUČIĆ, AND ENRICO PASQUALETTO

Abstract. We provide a quick proof of the following known result: the Sobolev space associated with the Euclidean space, endowed with the Euclidean distance and an arbitrary Radon measure, is Hilbert. Our new approach relies upon the properties of the Alberti–Marchese decomposability bundle. As a consequence of our arguments, we also prove that if the Sobolev norm is closable on compactly-supported smooth functions, then the reference measure is absolutely continuous with respect to the Lebesgue measure.

Introduction

In recent years, the theory of weakly differentiable functions over an abstract metric measure space $(\mathrm{X},\mathsf{d},\mu)$ has been extensively studied. Starting from the seminal paper [6], several (essentially equivalent) versions of Sobolev space $W^{1,2}(\mathrm{X},\mathsf{d},\mu)$ have been proposed in [13, 4, 12]. The definition we shall adopt in this paper is the one via test plans and weak upper gradients, which has been introduced by L. Ambrosio, N. Gigli and G. Savaré in [4]. In general, $W^{1,2}(\mathrm{X},\mathsf{d},\mu)$ is a Banach space, but it might be non-Hilbert: for instance, consider the Euclidean space endowed with the ℓ^∞-norm and the Lebesgue measure. Those metric measure spaces whose associated Sobolev space is Hilbert – which are said to be *infinitesimally Hilbertian*, cf. [9] – play a very important role. We refer to the introduction of [11] for an account of the main advantages and features of this class of spaces.

The aim of this manuscript is to provide a quick proof of the following result (cf. Theorem 2.3):

$$(\mathbb{R}^d,\mathsf{d}_{\mathrm{Eucl}},\mu)\ \text{ is infinitesimally Hilbertian for any Radon measure } \mu\geq 0 \text{ on } \mathbb{R}^d, \tag{$\star$}$$

where $\mathsf{d}_{\mathrm{Eucl}}(x,y):=|x-y|$ stands for the Euclidean distance on $\mathbb{R}^d$. This fact has been originally proven in [10], but it can also be alternatively considered as a special case of the main result in [8]. The approach we propose here is more direct and is based upon the differentiability theorem [1] for Lipschitz functions in $\mathbb{R}^d$ with respect to a given Radon measure, as we are going to describe.

Let $\mu\geq 0$ be any Radon measure on $\mathbb{R}^d$. G. Alberti and A. Marchese proved in [1] that it is possible to select the maximal measurable sub-bundle $V(\mu,\cdot)$ of $T\mathbb{R}^d$ – called the *decomposability bundle* of μ – along which all Lipschitz functions are μ-a.e. differentiable. This way, any given Lipschitz function $f\colon\mathbb{R}^d\to\mathbb{R}$ is naturally associated with a gradient $\nabla_{\mathrm{AM}}f$, which is an L^∞-section of $V(\mu,\cdot)$. Being ∇_{AM} a linear operator, its induced Dirichlet energy functional E_{AM} on $L^2(\mu)$ is a quadratic form. Hence, the proof of $(\star)$ presented here follows along these lines:

a) The maximality of $V(\mu,\cdot)$ ensures that the curves selected by a test plan $\boldsymbol{\pi}$ on $(\mathbb{R}^d,\mathsf{d}_{\mathrm{Eucl}},\mu)$ are 'tangent' to $V(\mu,\cdot)$, namely, $\dot\gamma_t\in V(\mu,\gamma_t)$ for $(\boldsymbol{\pi}\otimes\mathcal{L}^1)$-a.e. (γ,t). See Lemma 2.1.

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b) Given any Lipschitz function $f\colon \mathbb{R}^d \to \mathbb{R}$, we can deduce from item a) that the modulus of the gradient $\nabla_{\rm AM} f$ is a weak upper gradient of f; cf. Proposition 2.2.
c) Since Lipschitz functions with compact support are dense in energy in $W^{1,2}(\mathbb{R}^d, \mathsf{d}_{\rm Eucl}, \mu)$ – cf. Theorem 1.3 below – we conclude from b) that the Cheeger energy $\mathsf{E}_{\rm Ch}$ is the lower semicontinuous envelope of $\mathsf{E}_{\rm AM}$. This grants that $\mathsf{E}_{\rm Ch}$ is a quadratic form, thus accordingly the space $(\mathbb{R}^d, \mathsf{d}_{\rm Eucl}, \mu)$ is infinitesimally Hilbertian. See Theorem 2.3 for the details.

Finally, by combining our techniques with a structural result for Radon measures in the Euclidean space by De Philippis–Rindler [7], we eventually prove (in Theorem 3.5) the following claim:

$$\text{The Sobolev norm } \|\cdot\|_{W^{1,2}(\mathbb{R}^d,\mathsf{d}_{\rm Eucl},\mu)} \text{ is closable on } C^\infty_c\text{-functions} \quad \Longrightarrow \quad \mu \ll \mathcal{L}^d.$$

Cf. Definition 3.1 for the notion of closability we are referring to. This result solves a conjecture that has been posed by M. Fukushima (according to V.I. Bogachev [5, Section 2.6]).

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1. Preliminaries

1.1. **Sobolev calculus on metric measure spaces.** By *metric measure space* $(\mathrm{X}, \mathsf{d}, \mu)$ we mean a complete, separable metric space (X, d) together with a non-negative Radon measure $\mu \neq 0$.

We denote by $\mathrm{LIP}(\mathrm{X})$ the space of all real-valued Lipschitz functions on X, whereas $\mathrm{LIP}_c(\mathrm{X})$ stands for the family of all elements of $\mathrm{LIP}(\mathrm{X})$ having compact support. Given any $f \in \mathrm{LIP}(\mathrm{X})$, we shall denote by $\mathrm{lip}(f)\colon \mathrm{X} \to [0,+\infty)$ its *local Lipschitz constant*, which is defined as

$$\mathrm{lip}(f)(x) := \begin{cases} \overline{\lim}_{y\to x} \big|f(x)-f(y)\big|/\mathsf{d}(x,y) & \text{if } x \in \mathrm{X} \text{ is an accumulation point,} \\ 0 & \text{otherwise.} \end{cases}$$

The metric space (X, d) is said to be *proper* provided its bounded, closed subsets are compact.

To introduce the notion of Sobolev space $W^{1,2}(\mathrm{X}, \mathsf{d}, \mu)$ that has been proposed in [4], we first need to recall some terminology. The space $C\big([0,1],\mathrm{X}\big)$ of all continuous curves in X is a complete, separable metric space if endowed with the sup-distance $\mathsf{d}_\infty(\gamma,\sigma) := \max\big\{\mathsf{d}(\gamma_t,\sigma_t)\;\big|\; t \in [0,1]\big\}$. We say that $\gamma \in C\big([0,1],\mathrm{X}\big)$ is *absolutely continuous* provided there exists a function $g \in L^1(0,1)$ such that $\mathsf{d}(\gamma_s,\gamma_t) \le \int_s^t g(r)\,\mathrm{d}r$ holds for all $s,t \in [0,1]$ with $s < t$. The *metric speed* $|\dot\gamma|$ of γ, defined as $|\dot\gamma_t| := \lim_{h\to 0} \mathsf{d}(\gamma_{t+h},\gamma_t)/|h|$ for $\mathcal{L}^1$-a.e. $t \in [0,1]$, is the minimal integrable function (in the $\mathcal{L}^1$-a.e. sense) that can be chosen as g in the previous inequality; cf. [2, Theorem 1.1.2]. A *test plan* over $(\mathrm{X}, \mathsf{d}, \mu)$ is a Borel probability measure $\boldsymbol{\pi}$ on $C\big([0,1],\mathrm{X}\big)$, concentrated on absolutely continuous curves, such that the following properties are satisfied:

- Bounded compression. There exists $\mathrm{Comp}(\boldsymbol{\pi}) > 0$ such that $(\mathrm{e}_t)_*\boldsymbol{\pi} \le \mathrm{Comp}(\boldsymbol{\pi})\,\mu$ holds for all $t \in [0,1]$, where $\mathrm{e}_t\colon C\big([0,1],\mathrm{X}\big) \to \mathrm{X}$ stands for the evaluation map $\gamma \mapsto \mathrm{e}_t(\gamma) := \gamma_t$.
- Finite kinetic energy. It holds that $\iint_0^1 |\dot\gamma_t|^2\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}(\gamma) < +\infty$.

Let $f\colon \mathrm{X} \to \mathbb{R}$ be a given Borel function. We say that $G \in L^2(\mu)$ is a *weak upper gradient* of f provided for any test plan $\boldsymbol{\pi}$ on $(\mathrm{X}, \mathsf{d}, \mu)$ it holds that $f \circ \gamma \in W^{1,1}(0,1)$ for $\boldsymbol{\pi}$-a.e. γ and that

$$\big|(f\circ\gamma)'_t\big| \le G(\gamma_t)\,|\dot\gamma_t| \quad \text{for } (\boldsymbol{\pi}\otimes\mathcal{L}^1)\text{-a.e. } (\gamma,t).$$

The minimal such function G (in the μ-a.e. sense) is called the *minimal weak upper gradient* of f and is denoted by $|Df| \in L^2(\mu)$.

Definition 1.1 (Sobolev space [4]). *The* Sobolev space $W^{1,2}(\mathrm{X},\mathsf{d},\mu)$ *is defined as the family of all those functions* $f\in L^2(\mu)$ *that admit a weak upper gradient* $G\in L^2(\mu)$*. We endow the vector space* $W^{1,2}(\mathrm{X},\mathsf{d},\mu)$ *with the Sobolev norm* $\|f\|^2_{W^{1,2}(\mathrm{X},\mathsf{d},\mu)}:=\|f\|^2_{L^2(\mu)}+\big\||Df|\big\|^2_{L^2(\mu)}$.

The Sobolev space $\big(W^{1,2}(\mathrm{X},\mathsf{d},\mu),\|\cdot\|_{W^{1,2}(\mathrm{X},\mathsf{d},\mu)}\big)$ is a Banach space, but in general it is not a Hilbert space. This fact motivates the following definition, which has been proposed by N. Gigli:

Definition 1.2 (Infinitesimal Hilbertianity [9]). *We say that a metric measure space* $(\mathrm{X},\mathsf{d},\mu)$ *is* infinitesimally Hilbertian *provided its associated Sobolev space* $W^{1,2}(\mathrm{X},\mathsf{d},\mu)$ *is a Hilbert space.*

Let us define the *Cheeger energy* functional $\mathsf{E}_{\mathrm{Ch}}\colon L^2(\mu)\to[0,+\infty]$ as

$$\mathsf{E}_{\mathrm{Ch}}(f):=\begin{cases}\frac{1}{2}\int|Df|^2\,\mathrm{d}\mu & \text{if } f\in W^{1,2}(\mathrm{X},\mathsf{d},\mu),\\ +\infty & \text{otherwise.}\end{cases}\tag{1.1}$$

It holds that the metric measure space $(\mathrm{X},\mathsf{d},\mu)$ is infinitesimally Hilbertian if and only if E_{Ch} satisfies the *parallelogram rule* when restricted to $W^{1,2}(\mathrm{X},\mathsf{d},\mu)$, *i.e.*,

$$\mathsf{E}_{\mathrm{Ch}}(f+g)+\mathsf{E}_{\mathrm{Ch}}(f-g)=2\,\mathsf{E}_{\mathrm{Ch}}(f)+2\,\mathsf{E}_{\mathrm{Ch}}(g)\quad\text{for every } f,g\in W^{1,2}(\mathrm{X},\mathsf{d},\mu).\tag{1.2}$$

Furthermore, we define the functional $\mathsf{E}_{\mathrm{lip}}\colon L^2(\mu)\to[0,+\infty]$ as

$$\mathsf{E}_{\mathrm{lip}}(f):=\begin{cases}\frac{1}{2}\int\mathrm{lip}^2(f)\,\mathrm{d}\mu & \text{if } f\in\mathrm{LIP}_c(\mathrm{X}),\\ +\infty & \text{otherwise.}\end{cases}\tag{1.3}$$

Given any $f\in\mathrm{LIP}_c(\mathrm{X})$, it holds that $f\in W^{1,2}(\mathrm{X},\mathsf{d},\mu)$ and $|Df|\le\mathrm{lip}(f)$ in the μ-a.e. sense. This ensures that the inequality $\mathsf{E}_{\mathrm{Ch}}\le\mathsf{E}_{\mathrm{lip}}$ is satisfied. Actually, E_{Ch} is the $L^2(\mu)$-relaxation of $\mathsf{E}_{\mathrm{lip}}$:

Theorem 1.3 (Density in energy [3]). *Let* $(\mathrm{X},\mathsf{d},\mu)$ *be a metric measure space, with* (X,d) *proper. Then* E_{Ch} *is the* $L^2(\mu)$-lower semicontinuous envelope *of* $\mathsf{E}_{\mathrm{lip}}$, i.e., *it holds that*

$$\mathsf{E}_{\mathrm{Ch}}(f)=\inf\varliminf_{n\to\infty}\mathsf{E}_{\mathrm{lip}}(f_n)\quad\text{for every } f\in L^2(\mu),$$

where the infimum is taken among all sequences $(f_n)_n\subseteq L^2(\mu)$ *such that* $f_n\to f$ *in* $L^2(\mu)$.

1.2. **Decomposability bundle.** Let us denote by $\mathrm{Gr}(\mathbb{R}^d)$ the set of all linear subspaces of $\mathbb{R}^d$. Given any $V,W\in\mathrm{Gr}(\mathbb{R}^d)$, we define the distance $\mathsf{d}_{\mathrm{Gr}}(V,W)$ as the Hausdorff distance in $\mathbb{R}^d$ between the closed unit ball of V and that of W. Hence, $\big(\mathrm{Gr}(\mathbb{R}^d),\mathsf{d}_{\mathrm{Gr}}\big)$ is a compact metric space.

Theorem 1.4 (Decomposability bundle [1]). *Let* $\mu\ge0$ *be a given Radon measure on* $\mathbb{R}^d$*. Then there exists a* μ*-a.e. unique Borel mapping* $V(\mu,\cdot)\colon\mathbb{R}^d\to\mathrm{Gr}(\mathbb{R}^d)$*, called the* decomposability bundle *of* μ*, such that the following properties hold:*

i) *Any function* $f\in\mathrm{LIP}(\mathbb{R}^d)$ *is differentiable at* μ*-a.e.* $x\in\mathbb{R}^d$ *with respect to* $V(\mu,x)$, i.e., *there exists a Borel map* $\nabla_{\mathrm{AM}}f\colon\mathbb{R}^d\to\mathbb{R}^d$ *such that* $\nabla_{\mathrm{AM}}f(x)\in V(\mu,x)$ *for all* $x\in\mathbb{R}^d$ *and*

$$\lim_{V(\mu,x)\ni v\to0}\frac{f(x+v)-f(x)-\nabla_{\mathrm{AM}}f(x)\cdot v}{|v|}=0\quad\text{for } \mu\text{-a.e. } x\in\mathbb{R}^d.\tag{1.4}$$

ii) *There exists a function* $f_0\in\mathrm{LIP}(\mathbb{R}^d)$ *such that for* μ*-a.e. point* $x\in\mathbb{R}^d$ *it holds that* f_0 *is not differentiable at* x *with respect to any direction* $v\in\mathbb{R}^d\setminus V(\mu,x)$.

We refer to $\nabla_{\mathrm{AM}}f$ as the *Alberti–Marchese gradient* of f. It readily follows from (1.4) that $\nabla_{\mathrm{AM}}f$ is uniquely determined (up to μ-a.e. equality) and that for every $f,g\in\mathrm{LIP}(\mathbb{R}^d)$ it holds that

$$\nabla_{\mathrm{AM}}(f\pm g)(x)=\nabla_{\mathrm{AM}}f(x)\pm\nabla_{\mathrm{AM}}g(x)\quad\text{for } \mu\text{-a.e. } x\in\mathbb{R}^d.\tag{1.5}$$

Remark 1.5. Theorem 1.4 was actually proven under the additional assumption of μ being a finite measure. However, the statement depends only on the null sets of μ, not on the measure μ itself. Therefore, in order to obtain Theorem 1.4 as a consequence of the original result in [1], it is sufficient to replace μ with the following Borel probability measure on $\mathbb{R}^d$:

$$\tilde{\mu} \coloneqq \sum_{j=1}^{\infty} \frac{\mu|_{B_j(\bar{x})}}{2^j \mu\big(B_j(\bar{x})\big)}, \qquad \text{for some } \bar{x} \in \mathrm{spt}(\mu).$$

Observe, indeed, that the measure $\tilde{\mu}$ satisfies $\mu \ll \tilde{\mu} \ll \mu$. ■

Remark 1.6. Given any function $f \in \mathrm{LIP}(\mathbb{R}^d)$, it holds that

$$\big|\nabla_{\mathrm{AM}} f(x)\big| \leq \mathrm{lip}(f)(x) \quad \text{ for } \mu\text{-a.e. } x \in \mathbb{R}^d. \tag{1.6}$$

Indeed, fix any point $x \in \mathbb{R}^d$ such that f is differentiable at x with respect to $V(\mu, x)$. Then for all $v \in V(\mu, x) \setminus \{0\}$ it holds that $\nabla_{\mathrm{AM}} f(x) \cdot v = |v| \lim_{h \searrow 0} \big(f(x + hv) - f(x)\big)/|hv| \leq |v| \,\mathrm{lip}(f)(x)$ by (1.4), thus accordingly $\big|\nabla_{\mathrm{AM}} f(x)\big| = \sup\big\{\nabla_{\mathrm{AM}} f(x) \cdot v \;\big|\; v \in V(\mu, x),\, |v| \leq 1\big\} \leq \mathrm{lip}(f)(x)$. ■

2. Universal infinitesimal Hilbertianity of the Euclidean space

The objective of this section is to show that the Euclidean space is *universally infinitesimally Hilbertian*, meaning that it is infinitesimally Hilbertian when equipped with any Radon measure; cf. Theorem 2.3 below. The strategy of the proof we are going to present here is based upon the structure of the decomposability bundle described in Subsection 1.2.

First of all, we prove that any given test plan over the weighted Euclidean space is 'tangent', in a suitable sense, to the Alberti–Marchese decomposability bundle:

Lemma 2.1. *Let $\mu \geq 0$ be a given Radon measure on $\mathbb{R}^d$. Let $\boldsymbol{\pi}$ be a test plan on $(\mathbb{R}^d, \mathsf{d}_{\mathrm{Eucl}}, \mu)$. Then for $\boldsymbol{\pi}$-a.e. γ it holds that*

$$\dot{\gamma}_t \in V(\mu, \gamma_t) \quad \textit{for } \mathcal{L}^1\textit{-a.e. } t \in [0,1].$$

Proof. Let f_0 be an L-Lipschitz function as in ii) of Theorem 1.4. Set $B \subseteq C\big([0,1], \mathbb{R}^d\big) \times [0,1]$ as

$$B \coloneqq \Big\{(\gamma, t) \;\Big|\; \gamma \text{ and } f_0 \circ \gamma \text{ are differentiable at } t, \text{ and } \dot{\gamma}_t \notin V(\mu, \gamma_t)\Big\}.$$

It can be easily shown that B is Borel measurable. We can assume that γ is absolutely continuous (since by definition a test plan is concentrated on absolutely continuous curves); in particular, also $f_0 \circ \gamma$ is absolutely continuous, and thus both γ and $f_0 \circ \gamma$ are differentiable $\mathcal{L}^1$-almost everywhere. In particular, we are done if we can prove that $(\boldsymbol{\pi} \otimes \mathcal{L}^1)(B) = 0$.

Call $B_t \coloneqq \big\{\gamma \;\big|\; (\gamma, t) \in B\big\}$ for every $t \in [0,1]$. Moreover, G stands for the set of all $x \in \mathbb{R}^d$ such that f_0 is not differentiable at x with respect to any direction $v \in \mathbb{R}^d \setminus V(\mu, x)$. Thus, $\mu(\mathbb{R}^d \setminus G) = 0$ by Theorem 1.4. We claim that the inclusion $\mathrm{e}_t(B_t) \subseteq \mathbb{R}^d \setminus G$ holds for every $t \in [0,1]$. Indeed, for every $\gamma \in B_t$ one has that

$$\begin{aligned}\left|\frac{f_0(\gamma_t + h\dot{\gamma}_t) - f_0(\gamma_t)}{h} - (f_0 \circ \gamma)'_t\right| &\leq \left|\frac{f_0(\gamma_t + h\dot{\gamma}_t) - f_0(\gamma_{t+h})}{h}\right| + \left|\frac{f_0(\gamma_{t+h}) - f_0(\gamma_t)}{h} - (f_0 \circ \gamma)'_t\right| \\ &\leq L\left|\frac{\gamma_{t+h} - \gamma_t}{h} - \dot{\gamma}_t\right| + \left|\frac{f_0(\gamma_{t+h}) - f_0(\gamma_t)}{h} - (f_0 \circ \gamma)'_t\right|,\end{aligned}$$

so by letting $h \to 0$ we conclude that f_0 is differentiable at γ_t in the direction $\dot{\gamma}_t$, *i.e.*, $\gamma_t \notin G$. Therefore, we conclude that $\boldsymbol{\pi}(B_t) \leq \boldsymbol{\pi}\big(\mathrm{e}_t^{-1}(\mathbb{R}^d \setminus G)\big) \leq \mathrm{Comp}(\boldsymbol{\pi})\, \mu(\mathbb{R}^d \setminus G) = 0$ for all $t \in [0,1]$. This grants that $(\boldsymbol{\pi} \otimes \mathcal{L}^1)(B) = 0$ by Fubini theorem, whence the statement follows. □

As a consequence of Lemma 2.1, we can readily prove that the modulus of the Alberti–Marchese gradient of a given Lipschitz function is a weak upper gradient of the function itself:

Proposition 2.2. *Let $\mu \geq 0$ be a Radon measure on $\mathbb{R}^d$. Let $f \in \mathrm{LIP}_c(\mathbb{R}^d)$ be given. Then the function $|\nabla_{\mathrm{AM}} f| \in L^2(\mu)$ is a weak upper gradient of f.*

Proof. Let $\boldsymbol{\pi}$ be any test plan over $(\mathbb{R}^d, \mathsf{d}_{\mathrm{Eucl}}, \mu)$. We claim that for $\boldsymbol{\pi}$-a.e. γ it holds

$$(f \circ \gamma)'_t = \nabla_{\mathrm{AM}} f(\gamma_t) \cdot \dot{\gamma}_t \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in [0,1]. \tag{2.1}$$

Indeed, for $(\boldsymbol{\pi} \otimes \mathcal{L}^1)$-a.e. (γ, t) we have that f is differentiable at γ_t with respect to $V(\mu, \gamma_t)$ and that $\dot{\gamma}_t \in V(\mu, \gamma_t)$; this stems from item i) of Theorem 1.4 and Lemma 2.1. Hence, (1.4) yields

$$\nabla_{\mathrm{AM}} f(\gamma_t) \cdot \dot{\gamma}_t = \lim_{h \searrow 0} \frac{f(\gamma_t + h\dot{\gamma}_t) - f(\gamma_t)}{h} = \lim_{h \searrow 0} \frac{f(\gamma_{t+h}) - f(\gamma_t)}{h} = (f \circ \gamma)'_t,$$

which proves the claim (2.1). In particular, for $\boldsymbol{\pi}$-a.e. curve γ it holds

$$\left|(f \circ \gamma)'_t\right| \leq \left|\nabla_{\mathrm{AM}} f(\gamma_t)\right| |\dot{\gamma}_t| \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in [0,1].$$

Given that $|\nabla_{\mathrm{AM}} f| \in L^2(\mu)$ by (1.6), we conclude that $|Df| \leq |\nabla_{\mathrm{AM}} f|$ holds in the μ-a.e. sense. □

We are now in a position to prove the universal infinitesimal Hilbertianity of the Euclidean space, as an immediate consequence of Proposition 2.2 and of the linearity of ∇_{AM}:

Theorem 2.3 (Infinitesimal Hilbertianity of weighted $\mathbb{R}^d$)**.** *Let $\mu \geq 0$ be a Radon measure on $\mathbb{R}^d$. Then the metric measure space $(\mathbb{R}^d, \mathsf{d}_{\mathrm{Eucl}}, \mu)$ is infinitesimally Hilbertian.*

Proof. First of all, let us define the *Alberti–Marchese energy* functional $\mathsf{E}_{\mathrm{AM}} \colon L^2(\mu) \to [0, +\infty]$ as

$$\mathsf{E}_{\mathrm{AM}}(f) \coloneqq \begin{cases} \frac{1}{2} \int |\nabla_{\mathrm{AM}} f|^2 \, \mathrm{d}\mu & \text{if } f \in \mathrm{LIP}_c(\mathbb{R}^d), \\ +\infty & \text{otherwise.} \end{cases}$$

Since $|Df| \leq |\nabla_{\mathrm{AM}} f| \leq \mathrm{lip}(f)$ holds μ-a.e. for any $f \in \mathrm{LIP}_c(\mathbb{R}^d)$ by Proposition 2.2 and (1.6), we have that $\mathsf{E}_{\mathrm{Ch}} \leq \mathsf{E}_{\mathrm{AM}} \leq \mathsf{E}_{\mathrm{lip}}$, where E_{Ch} and $\mathsf{E}_{\mathrm{lip}}$ are defined as in (1.1) and (1.3), respectively. In view of Theorem 1.3, we deduce that E_{Ch} is the $L^2(\mu)$-lower semicontinuous envelope of E_{AM}. Thanks to the identities in (1.5), we also know that E_{AM} satisfies the parallelogram rule when restricted to $\mathrm{LIP}_c(\mathbb{R}^d)$, which means that

$$\mathsf{E}_{\mathrm{AM}}(f+g) + \mathsf{E}_{\mathrm{AM}}(f-g) = 2\,\mathsf{E}_{\mathrm{AM}}(f) + 2\,\mathsf{E}_{\mathrm{AM}}(g) \quad \text{for every } f, g \in \mathrm{LIP}_c(\mathbb{R}^d). \tag{2.2}$$

Fix $f, g \in W^{1,2}(\mathbb{R}^d, \mathsf{d}_{\mathrm{Eucl}}, \mu)$. Let us choose any two sequences $(f_n)_n, (g_n)_n \subseteq \mathrm{LIP}_c(\mathbb{R}^d)$ such that

- $f_n \to f$ and $g_n \to g$ in $L^2(\mu)$,
- $\mathsf{E}_{\mathrm{AM}}(f_n) \to \mathsf{E}_{\mathrm{Ch}}(f)$ and $\mathsf{E}_{\mathrm{AM}}(g_n) \to \mathsf{E}_{\mathrm{Ch}}(g)$.

In particular, observe that $f_n + g_n \to f + g$ and $f_n - g_n \to f - g$ in $L^2(\mu)$. Therefore, it holds that

$$\begin{aligned} \mathsf{E}_{\mathrm{Ch}}(f+g) + \mathsf{E}_{\mathrm{Ch}}(f-g) &\leq \varliminf_{n\to\infty} \big(\mathsf{E}_{\mathrm{AM}}(f_n+g_n) + \mathsf{E}_{\mathrm{AM}}(f_n-g_n)\big) \overset{(2.2)}{=} 2 \lim_{n\to\infty} \big(\mathsf{E}_{\mathrm{AM}}(f_n) + \mathsf{E}_{\mathrm{AM}}(g_n)\big) \\ &= 2\,\mathsf{E}_{\mathrm{Ch}}(f) + 2\,\mathsf{E}_{\mathrm{Ch}}(g). \end{aligned}$$

By replacing f and g with $f+g$ and $f-g$, respectively, we conclude that the converse inequality is verified as well. Consequently, the Cheeger energy E_{Ch} satisfies the parallelogram rule (1.2), thus $W^{1,2}(\mathbb{R}^d, \mathsf{d}_{\mathrm{Eucl}}, \mu)$ is a Hilbert space. This completes the proof of the statement. □

Remark 2.4. As a byproduct of the proof of Theorem 2.3, we see that for all $f \in W^{1,2}(\mathbb{R}^d, \mathsf{d}_{\mathrm{Eucl}}, \mu)$ there exists a sequence $(f_n)_n \subseteq \mathrm{LIP}_c(\mathbb{R}^d)$ such that $f_n \to f$ and $|\nabla_{\mathrm{AM}} f_n| \to |Df|$ in $L^2(\mu)$. ■

Example 2.5. Given an arbitrary Radon measure μ on $\mathbb{R}^d$, it might happen that

$$|Df| \neq |\nabla_{\mathrm{AM}} f| \quad \text{for some } f \in \mathrm{LIP}_c(\mathbb{R}^d).$$

For instance, consider the measure $\mu \coloneqq \mathcal{L}^1|_C$ on $\mathbb{R}$, where $C \subseteq \mathbb{R}$ is any Cantor set of positive Lebesgue measure. Since the support of μ is totally disconnected, one has that every $f \in L^2(\mu)$ is a Sobolev function with $|Df| = 0$. However, it holds $V(\mu, x) = \mathbb{R}$ for $\mathcal{L}^1$-a.e. $x \in C$ by Rademacher theorem, whence for any $f \in \mathrm{LIP}(\mathbb{R})$ we have that $\nabla_{\mathrm{AM}} f(x) = f'(x)$ for $\mathcal{L}^1$-a.e. $x \in C$. ■

3. Closability of the Sobolev norm on smooth functions

The aim of this conclusive section is to address a problem that has been raised by M. Fukushima (as reported in [5, Section 2.6]). Namely, we provide a (negative) answer to the following question: *Does there exist a singular Radon measure μ on $\mathbb{R}^2$ for which the Sobolev norm $\|\cdot\|_{W^{1,2}(\mathbb{R}^2,\mathsf{d}_{\mathrm{Eucl}},\mu)}$ is closable on compactly-supported smooth functions (in the sense of Definition 3.1 below)?*

Actually, we are going to prove a stronger result: *Given any Radon measure μ on $\mathbb{R}^d$ that is not absolutely continuous with respect to $\mathscr{L}^d$, it holds that $\|\cdot\|_{W^{1,2}(\mathbb{R}^d,\mathsf{d}_{\mathrm{Eucl}},\mu)}$ is not closable on compactly-supported smooth functions.* Cf. Theorem 3.5 below.

Let $f \in C^\infty_c(\mathbb{R}^d)$ be given. Then we denote by $\nabla f \colon \mathbb{R}^d \to \mathbb{R}^d$ its classical gradient. Note that the identity $|\nabla f| = \mathrm{lip}(f)$ holds. Given a Radon measure μ on $\mathbb{R}^d$, it is immediate to check that

$$\nabla_{\mathrm{AM}} f(x) = \pi_x\big(\nabla f(x)\big) \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d, \tag{3.1}$$

where $\pi_x \colon \mathbb{R}^d \to V(\mu, x)$ stands for the orthogonal projection map. We denote by $L^2_\mu(\mathbb{R}^d, \mathbb{R}^d)$ the space of all (equivalence classes, up to μ-a.e. equality, of) Borel maps $v \colon \mathbb{R}^d \to \mathbb{R}^d$ with $|v| \in L^2(\mu)$. It holds that $L^2_\mu(\mathbb{R}^d, \mathbb{R}^d)$ is a Hilbert space if endowed with the norm $v \mapsto \big(\int |v|^2 \,\mathrm{d}\mu\big)^{1/2}$.

Definition 3.1 (Closability of the Sobolev norm on smooth functions)**.** *Let μ be a Radon measure on $\mathbb{R}^d$. Then the Sobolev norm $\|\cdot\|_{W^{1,2}(\mathbb{R}^d,\mathsf{d}_{\mathrm{Eucl}},\mu)}$ is* closable on compactly-supported smooth functions *provided the following property is verified: if a sequence $(f_n)_n \subseteq C^\infty_c(\mathbb{R}^d)$ satisfies $f_n \to 0$ in $L^2(\mu)$ and $\nabla f_n \to v$ in $L^2_\mu(\mathbb{R}^d, \mathbb{R}^d)$ for some element $v \in L^2_\mu(\mathbb{R}^d, \mathbb{R}^d)$, then it holds that $v = 0$.*

In order to provide some alternative characterisations of the above-defined closability property, we need to recall the following improvement of Theorem 1.3 in the weighted Euclidean space case:

Theorem 3.2 (Density in energy of smooth functions [10])**.** *Let μ be a Radon measure on $\mathbb{R}^d$. Then E_{Ch} is the $L^2(\mu)$-lower semicontinuous envelope of the functional*

$$L^2(\mu) \ni f \longmapsto \begin{cases} \frac{1}{2}\int |\nabla f|^2 \,\mathrm{d}\mu & \text{if } f \in C^\infty_c(\mathbb{R}^d), \\ +\infty & \text{otherwise.} \end{cases}$$

Lemma 3.3. *Let μ be a Radon measure on $\mathbb{R}^d$. Then the following conditions are equivalent:*

- i) *The Sobolev norm $\|\cdot\|_{W^{1,2}(\mathbb{R}^d,\mathsf{d}_{\mathrm{Eucl}},\mu)}$ is closable on compactly-supported smooth functions.*
- ii) *The functional $\mathsf{E}_{\mathrm{lip}}$ – see (1.3) – is $L^2(\mu)$-lower semicontinuous when restricted to $C^\infty_c(\mathbb{R}^d)$.*
- iii) *The identity $|Df| = |\nabla f|$ holds μ-a.e. on $\mathbb{R}^d$, for every function $f \in C^\infty_c(\mathbb{R}^d)$.*

Proof.
i) $\Longrightarrow$ ii) Fix any $f \in C^\infty_c(\mathbb{R}^d)$ and $(f_n)_n \subseteq C^\infty_c(\mathbb{R}^d)$ such that $f_n \to f$ in $L^2(\mu)$. We claim that

$$\int |\nabla f|^2 \,\mathrm{d}\mu \leq \varliminf_{n\to\infty} \int |\nabla f_n|^2 \,\mathrm{d}\mu. \tag{3.2}$$

Without loss of generality, we may assume the right-hand side in (3.2) is finite. Therefore, we can find a subsequence $(f_{n_k})_k$ of $(f_n)_n$ and an element $v \in L^2_\mu(\mathbb{R}^d, \mathbb{R}^d)$ such that $\lim_k \int |\nabla f_{n_k}|^2 \,\mathrm{d}\mu = \varliminf_n \int |\nabla f_n|^2 \,\mathrm{d}\mu$ and $\nabla f_{n_k} \rightharpoonup v$ in the weak topology of $L^2_\mu(\mathbb{R}^d, \mathbb{R}^d)$. By virtue of Banach–Saks theorem, we can additionally require that $\nabla \tilde{f}_k \to v$ in the strong topology of $L^2_\mu(\mathbb{R}^d, \mathbb{R}^d)$, where we set $\tilde{f}_k \coloneqq \frac{1}{k}\sum_{i=1}^k f_{n_i} \in C^\infty_c(\mathbb{R}^d)$ for all $k \in \mathbb{N}$. Since $\tilde{f}_k - f \to 0$ in $L^2(\mu)$ and $\nabla(\tilde{f}_k - f) \to v - \nabla f$

in $L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)$, we deduce from i) that $v=\nabla f$. Consequently, we have that $\nabla f_n \rightharpoonup \nabla f$ in the weak topology of $L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)$, thus proving (3.2) by semicontinuity of the norm. In other words, it holds that $\mathsf{E}_{\rm lip}(f)\leq \underline{\lim}_n\,\mathsf{E}_{\rm lip}(f_n)$, which yields the validity of item ii).
ii) $\Longrightarrow$ iii) Let $f\in C^\infty_c(\mathbb{R}^d)$ be given. Theorem 3.2 yields existence of a sequence $(f_n)_n\subseteq C^\infty_c(\mathbb{R}^d)$ such that $f_n\to f$ and $|\nabla f_n|\to|Df|$ in $L^2(\mu)$. Therefore, item ii) ensures that

$$\frac{1}{2}\int|\nabla f|^2\,\mathrm{d}\mu=\mathsf{E}_{\rm lip}(f)\leq\varliminf_{n\to\infty}\mathsf{E}_{\rm lip}(f_n)=\lim_{n\to\infty}\frac{1}{2}\int|\nabla f_n|^2\,\mathrm{d}\mu=\frac{1}{2}\int|Df|^2\,\mathrm{d}\mu.$$

Since $|Df|\leq|\nabla f|$ holds μ-a.e. on $\mathbb{R}^d$, we conclude that $|Df|=|\nabla f|$, thus proving item iii).
iii) $\Longrightarrow$ i) We argue by contradiction: suppose that there exists a sequence $(f_n)_n\subseteq C^\infty_c(\mathbb{R}^d)$ such that $f_n\to 0$ in $L^2(\mu)$ and $\nabla f_n\to v$ in $L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)$ for some $v\in L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)\setminus\{0\}$. Fix any $k\in\mathbb{N}$ such that $\|\nabla f_k-v\|_{L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)}\leq\frac{1}{3}\|v\|_{L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)}$. In particular, $\|\nabla f_k\|_{L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)}\geq\frac{2}{3}\|v\|_{L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)}$. Let us define $g_n\coloneqq f_k-f_n\in C^\infty_c(\mathbb{R}^d)$ for every $n\in\mathbb{N}$. Since $g_n\to f_k$ in $L^2(\mu)$ and $\nabla g_n\to\nabla f_k-v$ in $L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)$ as $n\to\infty$, we conclude that

$$\|\nabla f_k\|_{L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)}\geq\frac{2}{3}\,\|v\|_{L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)}>\frac{1}{3}\,\|v\|_{L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)}\geq\|\nabla f_k-v\|_{L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)}=\lim_{n\to\infty}\|\nabla g_n\|_{L^2_\mu(\mathbb{R}^d,\mathbb{R}^d)},$$

whence $\mathsf{E}_{\rm lip}(f_k)>\lim_n\mathsf{E}_{\rm lip}(g_n)$. This contradicts the lower semicontinuity of $\mathsf{E}_{\rm lip}$ on $C^\infty_c(\mathbb{R}^d)$. Consequently, item i) is proven. □

The last ingredient we need is the following result proven by G. De Philippis and F. Rindler:

Theorem 3.4 (Weak converse of Rademacher theorem [7])**.** *Let μ be a Radon measure on $\mathbb{R}^d$. Suppose all Lipschitz functions $f\colon\mathbb{R}^d\to\mathbb{R}$ are μ-a.e. differentiable. Then it holds that $\mu\ll\mathcal{L}^d$.*

We are finally in a position to prove the following statement concerning closability:

Theorem 3.5 (Failure of closability for singular measures)**.** *Let $\mu\geq 0$ be a given Radon measure on $\mathbb{R}^d$. Suppose that μ is not absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^d$. Then the Sobolev norm $\|\cdot\|_{W^{1,2}(\mathbb{R}^d,\mathsf{d}_{\rm Eucl},\mu)}$ is not closable on compactly-supported smooth functions.*

Proof. First of all, Theorem 3.4 grants the existence of a Lipschitz function $f\colon\mathbb{R}^d\to\mathbb{R}$ and a Borel set $P\subseteq\mathbb{R}^d$ such that $\mu(P)>0$ and f is not differentiable at any point of P. Recalling Theorem 1.4, we then see that $V(\mu,x)\neq\mathbb{R}^n$ for μ-a.e. $x\in P$. Therefore, we can find a compact set $K\subseteq P$ and a vector $v\in\mathbb{R}^d$ such that $\mu(K)>0$ and $v\notin V(\mu,x)$ for μ-a.e. $x\in K$. Now pick any $g\in C^\infty_c(\mathbb{R}^d)$ such that $\nabla g(x)=v$ holds for all $x\in K$. Then Proposition 2.2 and (3.1) yield

$$|Dg|(x)\leq|\nabla_{\rm AM}\,g|(x)=\big|\pi_x\big(\nabla g(x)\big)\big|=\big|\pi_x(v)\big|<|v|=|\nabla g|(x)\qquad\text{for }\mu\text{-a.e. }x\in K,$$

thus accordingly $\|\cdot\|_{W^{1,2}(\mathbb{R}^d,\mathsf{d}_{\rm Eucl},\mu)}$ is not closable on compactly-supported smooth functions by Lemma 3.3. Hence, the statement is achieved. □

Remark 3.6. The converse of Theorem 3.5 might fail. For instance, the measure μ described in Example 2.5 is absolutely continuous with respect to $\mathcal{L}^1$, but the Sobolev norm $\|\cdot\|_{W^{1,2}(\mathbb{R},\mathsf{d}_{\rm Eucl},\mu)}$ is not closable on compactly-supported smooth functions as a consequence of Lemma 3.3. ■

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Dipartimento di Matematica (DIMA), Via Dodecaneso 35, 16146 Genova, Università di Genova
Email address: simone.dimarino@unige.it

Department of Mathematics and Statistics, P.O. Box 35 (MaD), FI-40014 University of Jyvaskyla
Email address: danka.d.lucic@jyu.fi

Department of Mathematics and Statistics, P.O. Box 35 (MaD), FI-40014 University of Jyvaskyla
Email address: enrico.e.pasqualetto@jyu.fi