Sliding mode control for a diffuse interface tumor growth model coupling a Cahn-Hilliard equation with a reaction-diffusion equation

Michele Colturato
Università degli Studi di Brescia
V.le Europa 11, 25121 Brescia, Italy
E-mail: michele.colturato@unibs.it

Abstract

We consider the sliding mode control (SMC) problem for a diffuse interface tumor growth model coupling a Cahn-Hilliard equation with a reaction-diffusion equation perturbed by a maximal monotone nonlinearity. We prove existence and regularity of strong solutions and, under further assumptions, a uniqueness result. Then, we show that the chosen SMC law forces the system to reach within finite time a sliding manifold that we chose in order that the tumor phase remains constant in time.

Key words: Sliding mode control, Cahn-Hilliard system, Reaction-diffusion equation, Tumor growth, Nonlinear boundary value problem, State-feedback control law.

AMS (MOS) subject classification: 35K61, 35K25, 35D30, 34H05, 80A22.

1 Introduction

The study of tumor growth dynamics has recently become a major issue in applied mathematics and numerical simulations of diffuse interface models have been carried out in several papers, e.g., [6, 10, 12, 13, 19, 21, 25–29]; nonetheless, a rigorous mathematical analysis of the resulting systems of PDEs is still in its infancy. Recently, the authors of [47] consider a continuum diffuse interface model of multispecies tumor growth using the Cahn-Hilliard framework. Moreover, let us quote [20, 30, 31, 33, 34, 38, 42], where chemotaxis and transport effects was introduced and rigorous sharp interface limits was obtained.

In the present contribution we consider the sliding mode control problem for a tumor growth model consisting of a Cahn-Hilliard equation coupled with a reaction-diffusion equation:

$$\partial_t\varphi - \Delta\mu = (\gamma_1\sigma - \gamma_2)p(\varphi) \quad \text{a.e. in } Q = \Omega \times (0,T), \tag{1.1}$$

$$\mu = \ell\partial_t\varphi - \Delta\varphi + \xi + \pi(\varphi) + \mu_{\mathcal{S}} \quad \text{a.e. in } Q, \tag{1.2}$$

$$\partial_t\sigma - \Delta\sigma + \zeta = -\gamma_3\sigma p(\varphi) + \gamma_4(\sigma_s - \sigma) + g \quad \text{a.e. in } Q, \tag{1.3}$$

$$\zeta(t) \in \rho\,\mathrm{Sign}(a\sigma(t) + b\varphi(t) + \eta^*) \text{ for a.e. } t \in (0,T), \tag{1.4}$$

$$\xi \in \beta(\varphi) \text{ a.e. in } Q, \tag{1.5}$$

where Ω is the domain where the evolution takes place and T is some final time. The function φ is an health status indicator: $\varphi = 1$ and $\varphi = 0$ represent the tumor phase and the healthy phase, respectively. The function μ is the chemical potential, while σ is the concentration of the nutrient for the tumor cells (e.g., oxygen or glucose). Beside, $\mu_{\mathcal{S}}$ is a given smooth function, ℓ is a viscosity coefficient, while γ_i for $i = 1, \cdots, 4$ denotes the positive constant proliferation rate, apoptosis rate, nutrient consumption rate, and nutrient supply rate, respectively. The terms $\gamma_1 p(\varphi)\sigma$, $\gamma_2 p(\varphi)$ and $\gamma_3 p(\varphi)\sigma$ model the proliferation of tumor cells, the death of tumor cells and the depletion of nutrient, respectively. We assume that γ_i, $i = 1, 2, 3$, are positive parameters and π is a positive, bounded and Lipschitz continuous function. Finally, σ_s denotes the nutrient concentration in the vasculature while $\gamma_4(\sigma_s - \sigma)$ represents the supply of nutrient from the blood vessels. The term $\xi + \pi(\varphi)$, appearing in (1.2), represents the derivative of a double-well potential $\mathcal{W}$ defined as the sum

$$\mathcal{W} = \widehat{\beta} + \widehat{\pi}, \tag{1.6}$$

where

$$\widehat{\beta} : \mathbb{R} \longrightarrow [0, +\infty] \text{ is proper, l.s.c. and convex with } \widehat{\beta}(0) = 0, \tag{1.7}$$

$$\widehat{\pi} : \mathbb{R} \to \mathbb{R},\ \widehat{\pi} \in C^1(\mathbb{R}) \text{ with } \pi := \widehat{\pi}' \text{ Lipschitz continuous.} \tag{1.8}$$

Since $\widehat{\beta}$ is proper, l.s.c. and convex, the subdifferential $\partial\widehat{\beta} =: \beta$ is well defined and is a maximal monotone graph. For a comprehensive discussion of the theory of maximal monotone operators, we refer, e.g., to [1,5].

In our system we consider the maximal monotone operator $\rho\,\mathrm{Sign}$, where $\rho \in (0, +\infty)$ is a parameter and $\mathrm{Sign} : L^2(\Omega) =: H \mapsto 2^H$ is defined by $\mathrm{Sign}(v) = v/\|v\|$, if $v \neq 0$, while $\mathrm{Sign}(v) = B_1(0)$, if $v = 0$, where $B_1(0)$ is the closed unit ball of H. The aim of introducing such a feedback law in (1.3) is to obtain a sliding mode control (SMC) on the system: this technique is one of the most important approaches to the design of robust controllers for nonlinear complex dynamics (see, e.g., [2, 22–24, 35, 40–42, 44, 45, 48]). The design procedure of a SMC system is a two-stage process. The first phase is to choose a set of sliding manifolds such that the original system restricted to the intersection of them has a desired behavior. The second step is to design a SMC law that forces a linear combination of the system trajectories to reach the sliding surface $-\eta^*$ in a finite time. Sliding mode controls are useful in many applications: we cite [7, 37, 46] concerning the control of semilinear PDE systems and the recent contribution [3], where a sliding mode approach is applied for the first time to phase field systems of Caginalp type. We also mention the analysis developed in [8, 16–18]: in particular, the second contribution is devoted to a conserved phase field system with a SMC feedback law for the internal energy in the temperature equation.

Other approaches to the problem of control for tumor growth models are possible, even if a few mathematical results are presently available on this subject in the literature. In the recent papers [11, 14, 32] the authors consider the problem of finding first order necessary optimality conditions for the minimization of a cost functional, forcing the phase to approach the desired target in the best possible way by means of a control variable representing the concentration of cytotoxic drugs or the supply of a nutrient.

The main advantage of sliding mode control is that it strengthens the trajectories of the system to reach the sliding surface and keep it on it in a pointwise way, while, in general, within the classical optimal control theory (cf., e.g., [14, 32]), one can get just necessary optimality conditions and the control is nonlocal in space and/or in time.

The above system is complemented by homogeneous Neumann boundary conditions for σ and φ, for a Dirichlet boundary condition for μ,

$$\partial_\nu \sigma = \partial_\nu \varphi = 0, \qquad \mu = 0 \qquad \text{on } \Sigma := \partial\Omega \times (0,T). \tag{1.9}$$

This choice looks reasonable from the modeling point of view: the Dirichlet boundary condition for μ allows for the free flow of cells across the outer boundary. In particular, let us refer to [4] and [47] where similar conditions are imposed on a chemical potential in a different framework. Finally, initial conditions on σ and φ are prescribed:

$$\sigma(0) = \sigma_0, \qquad \varphi(0) = \varphi_0 \qquad \text{in } \Omega. \tag{1.10}$$

The paper is organized as follows. In the next two sections, we list our assumptions, state the problem in a precise form and present our results. The last seven sections are devoted to the corresponding proofs. Section 4, 5, 6 and 7 deal with well-posedness, while regularity and uniqueness of the solution are proved in in Sect. 8 and Sect. 9, respectively. Finally, the existence of the sliding mode is proved in Sect. 10.

2 Preliminary assumptions

2.1 Initial statements

We assume $\Omega \subseteq \mathbb{R}^3$ to be open, bounded, connected, of class C^1 and we write $|\Omega|$ for its Lebesgue measure. Moreover, Γ and ∂_ν still stand for the boundary of Ω and the outward normal derivative, respectively. Given a finite final time $T > 0$, for every $t \in (0,T]$ we set

$$Q_t = (0,t) \times \Omega, \quad Q = Q_T, \tag{2.1}$$

$$\Sigma_t = (0,t) \times \Gamma, \quad \Sigma = \Sigma_T. \tag{2.2}$$

In the following, we set for brevity:

$$H = L^2(\Omega), \quad V = H^1(\Omega), \quad V_0 = H^1_0(\Omega), \quad W = \{u \in H^2(\Omega) : \ \partial_\nu u = 0 \text{ on } \partial\Omega\}, \tag{2.3}$$

with usual norms $\|\cdot\|_H$, $\|\cdot\|_V$ and inner products $(\cdot,\cdot)_H$, $(\cdot,\cdot)_V$, respectively. The symbols V^* and V_0^* denotes the dual space of V and V_0^*, respectively, while the pair $\langle\cdot,\cdot\rangle_{V^*,V}$ and $\langle\cdot,\cdot\rangle_{V_0^*,V_0}$ represents the duality pairing between V^* and V, and between V^* and V_0^*, respectively. Moreover, we identify H with its dual space.

2.2 The potential $\mathcal{W}$

We introduce the potential $\mathcal{W}$ as the sum

$$\mathcal{W} = \tilde{\beta} + \tilde{\pi}, \tag{2.4}$$

where

$$\tilde{\beta} : \mathbb{R} \longrightarrow [0, +\infty] \text{ is proper, l.s.c. and convex with } \tilde{\beta}(0) = 0, \tag{2.5}$$

$$\tilde{\pi} : \mathbb{R} \to \mathbb{R},\ \tilde{\pi} \in C^1(\mathbb{R}) \text{ with } \pi := \tilde{\pi}' \text{ Lipschitz continuous.} \tag{2.6}$$

Since $\tilde{\beta}$ is proper, l.s.c. and convex, the subdifferential $\beta := \partial\tilde{\beta}$ is well defined. We denote by $D(\beta)$ and $D(\tilde{\beta})$ the effective domains of β and $\tilde{\beta}$, respectively, and also assume that $\text{int}(D(\beta)) \neq \emptyset$. Thanks to these assumptions, β is a maximal monotone graph. Moreover, as $\tilde{\beta}$ takes its minimum in 0, we have that $0 \in \beta(0)$. Now, we introduce the operator $\mathcal{B}$, i.e., the realization of β in $L^2(Q)$ in the following way:

$$\mathcal{B} : L^2(Q) \longrightarrow L^2(Q) \tag{2.7}$$

$$\xi \in \mathcal{B}(\varphi) \Longleftrightarrow \xi(x,t) \in \beta(\varphi(x,t)) \quad \text{for a.e. } (x,t) \in Q. \tag{2.8}$$

We notice that

$$\beta = \partial\tilde{\beta}, \qquad\qquad \mathcal{B} = \partial\Phi, \tag{2.9}$$

where

$$\Phi : \ L^2(Q) \longrightarrow (-\infty, +\infty] \tag{2.10}$$

$$\Phi(u) = \begin{cases} \int_Q \tilde{\beta}(u) & \text{if } u \in L^2(Q) \text{ and } \tilde{\beta}(u) \in L^1(Q), \\ +\infty & \text{elsewhere, with } u \in L^2(Q). \end{cases} \tag{2.11}$$

2.3 The operator Sign

We consider the maximal monotone operator

$$\text{Sign} : \qquad H \longrightarrow 2^H \tag{2.12}$$

$$\text{Sign}(v) = \begin{cases} \frac{v}{\|v\|} & \text{if } v \neq 0, \\ B_1(0) & \text{if } v = 0, \end{cases} \tag{2.13}$$

where $B_1(0)$ is the closed unit ball of H. The operator Sign is the subdifferential of the map $\|\cdot\| : H \to \mathbb{R}$. Moreover,

$$0 \in \text{Sign}(0). \tag{2.14}$$

and it is trivial to prove that, denoting by $\rho \in (0, +\infty)$ a positive parameter, there exists a positive constant $C_A > 0$ such that

$$\|v\|_H \leq C_A(1 + \|\eta\|_H) \quad \text{for every } \eta \in H,\, v \in \rho\,\text{Sign}(\eta). \tag{2.15}$$

3 Setting of the problem and results

We consider

$$\gamma_i \in [0, +\infty), \text{ for } i = 1, 2, 3, \qquad \rho,\ \ell,\ \sigma_s,\ \gamma_4,\ a \in (0, +\infty), \qquad b \in \mathbb{R}, \tag{3.1}$$

$$\eta^* \in W, \qquad \sigma_{\mathcal{S}} \in L^2(0,T;H), \qquad g \in L^2(0,T;H), \qquad \mu_\Gamma \in L^2(0,T;H^{1/2}(\Gamma)), \tag{3.2}$$

$$\varphi_0,\ \ \sigma_0 \in V, \qquad \tilde{\beta}(\varphi_0) \in L^1(\Omega), \tag{3.3}$$

$$\pi : \mathbb{R} \longrightarrow [0, +\infty) \qquad \text{is a bounded and Lipschitz continuous function} \tag{3.4}$$

We also assume that

$$\exists\, C_B > 0 \text{ such that } |\beta'(r)| \le C_B(1 + |r|^2) \text{ for every } r \in D(\beta'). \tag{3.5}$$

Then, we introduce the harmonic extension $\mu_{\mathcal{S}}$ of μ_Γ defined as the unique solution to the problem

$$\mu_{\mathcal{S}}(t) \in H^1(\Omega), \quad -\Delta\mu_{\mathcal{S}}(t) = 0 \text{ in } \mathcal{D}'(\Omega), \quad \mu_{\mathcal{S}}(t)_{|\Gamma} = \mu_\Gamma(t) \quad \text{for a.a. } t \in (0, T). \tag{3.6}$$

We look for a triplet (σ, φ, μ) satisfying at least the regularity requirements

$$\sigma \in H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W), \tag{3.7}$$

$$\varphi \in H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W), \tag{3.8}$$

$$\mu \in L^\infty(0, T; V_0) \cap L^2(0, T; H^2(\Omega)), \tag{3.9}$$

and solving the Problem (P) defined as

$$\partial_t\varphi - \Delta\mu = (\gamma_1\sigma - \gamma_2)p(\varphi) \ \text{ a.e. in } Q, \tag{3.10}$$

$$\mu = \ell\partial_t\varphi - \Delta\varphi + \xi + \pi(\varphi) + \mu_{\mathcal{S}} \ \text{ a.e. in } Q, \tag{3.11}$$

$$\partial_t\sigma - \Delta\sigma + \zeta = -\gamma_3\sigma p(\varphi) + \gamma_4(\sigma_s - \sigma) + g \ \text{ a.e. in } Q, \tag{3.12}$$

$$\zeta(t) \in \rho\,\mathrm{Sign}(a\sigma(t) + b\varphi(t) + \eta^*) \text{ for a.e. } t \in (0, T), \tag{3.13}$$

$$\xi \in \beta(\varphi) \text{ a.e. in } Q, \tag{3.14}$$

$$\partial_\nu\sigma = \partial_\nu\varphi = 0, \qquad \mu = 0 \qquad \text{on } \Sigma, \tag{3.15}$$

$$\sigma(0) = \sigma_0, \qquad \varphi(0) = \varphi_0 \qquad \text{in } \Omega. \tag{3.16}$$

Theorem 3.1 (Existence) *Assume* (2.5)–(2.6), (2.14)–(2.15) *and* (3.1)–(3.6). *Then Problem* (P) *defined in* (3.10)–(3.16) *has at least a solution* (σ, φ, μ) *satisfying* (3.7)–(3.9).

Theorem 3.2 (Regularity) *Let* (2.5)–(2.6), (2.14)–(2.15) *and* (3.1)–(3.6) *hold. Denoting by* $\beta^0(\varphi_0)$ *the element of the range of* β *having minimum modulus and assuming, in addition, that*

$$\sigma_0, \ \varphi_0 \in W, \qquad \mu_{\mathcal{S}} \in H^1(0, T; H), \tag{3.17}$$

then Problem (P) *defined in* (3.10)–(3.16) *has at least a solution* (σ, φ, μ) *such that*

$$\sigma \in H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W), \tag{3.18}$$

$$\varphi \in W^{1,\infty}(0, T; H) \cap H^1(0, T; V) \cap L^\infty(0, T; W), \tag{3.19}$$

$$\mu \in L^\infty(0, T; V_0) \cap L^2(0, T; H^2(\Omega)). \tag{3.20}$$

Theorem 3.3 (Uniqueness) *Assume* (2.5)–(2.6), (2.14)–(2.15) *and* (3.1)–(3.6). *If* $b = 0$, *then the solution of Problem* (P) *stated by* (3.10)–(3.16) *is unique.*

Let us remark that we guarantee the uniqueness of the solution only if $b = 0$. On the other hand, if $b \neq 0$, it turns out to be difficult to obtain a uniqueness result because a monotonicity argument can not be applied in the proof.

Theorem 3.4 (Sliding mode control) *Let* (2.5)–(2.6), (2.14)–(2.15), (3.1)–(3.6) *and* (3.17)–(5.17) *hold and*

$$g \in L^\infty(0,T,H), \quad \kappa(t) \in \mathrm{Sign}(a\sigma(t) + b\varphi(t) + \eta^*) \text{ for a.e. } t \in (0,T). \tag{3.21}$$

We assume, in addition, that at least one among (3.5) *and the following condition*

$$\text{the regularity properties (3.17) hold and } D(\beta) = D(\beta') = \mathbb{R} \tag{3.22}$$

is satisfied. Then, for some $\rho^* > 0$ and for every $\rho > \rho^*$, there exists a solution (σ, φ, μ) to Problem (P) (see (3.10)–(3.16)*) and a time T^* such that, for every $t \in [T^*, T]$*

$$a\sigma(t) + b\varphi(t) = -\eta^* \qquad \text{a.e. in } \Omega. \tag{3.23}$$

We observe that, if $a = 0$, the feedback control term κ defined in (3.21) depends only on the evolution of φ and the SMC low can be obtained as in [15].

4 Moreau-Yosida regularization of the operators

Yosida regularization of Sign. Let us introduce the operator $\mathrm{Sign}_\varepsilon : H \to H$ as the Yosida regularization at level $\varepsilon > 0$ of the operator Sign. For $\varepsilon > 0$ we define

$$\mathrm{Sign}_\varepsilon : H \longrightarrow H, \qquad \mathrm{Sign}_\varepsilon = \frac{I - (I + \varepsilon\,\mathrm{Sign})^{-1}}{\varepsilon}, \tag{4.1}$$

where I denotes the identity operator. Note that $\mathrm{Sign}_\varepsilon$ is Lipschitz continuous and maximal monotone, with Lipschitz constant $1/\varepsilon$, and satisfies the following properties. Denoting by $J_\varepsilon = (I + \varepsilon\,\mathrm{Sign})^{-1}$ the resolvent operator, for all $\delta > 0$ we have that

$$\mathrm{Sign}_\varepsilon\,\eta \in \mathrm{Sign}(J_\varepsilon \eta), \tag{4.2}$$

$$(\mathrm{Sign}_\varepsilon)_\delta = \mathrm{Sign}_{\varepsilon+\delta}, \tag{4.3}$$

$$\|\mathrm{Sign}_\varepsilon\,\eta\|_H \le \|\mathrm{Sign}^0\,\eta\|_H, \tag{4.4}$$

$$\lim_{\varepsilon \to 0} \|\mathrm{Sign}_\varepsilon\,\eta\|_H = \|\mathrm{Sign}^0\,\eta\|_H, \tag{4.5}$$

where $\mathrm{Sign}^0\,\eta$ is the element of the range of $\mathrm{Sign}\,\eta$ having minimum norm. We also point out a key property of $\mathrm{Sign}_\varepsilon$, which is a consequence of (2.15):

$$\|v\|_H \le C_A(1 + \|\eta\|_H) \qquad \text{for all } \eta \in H,\ v \in \mathrm{Sign}_\varepsilon\,\eta. \tag{4.6}$$

Indeed notice that $0 \in \mathrm{Sign}(0)$ and $0 \in I(0)$: consequently, for every $\varepsilon > 0$, $0 \in (I + \varepsilon\,\mathrm{Sign})(0)$. This fact implies that $J_\varepsilon(0) = 0$. Since Sign is a maximal monotone operator, J_ε is a contraction. Then, from (2.15) and (4.2), it follows that

$$\begin{aligned} \|\mathrm{Sign}_\varepsilon\,\eta\|_H &\le C_A(\|J_\varepsilon \eta\|_H + 1) \\ &\le C_A(\|J_\varepsilon \eta - J_\varepsilon 0\|_H + \|J_\varepsilon 0\|_H + 1) \\ &\le C_A(\|\eta\|_H + 1). \end{aligned}$$

Finally, we observe that $\mathrm{Sign}_\varepsilon(v)$ is the gradient at v of the C^1 functional $\|\cdot\|_{H,\varepsilon}$ defined as

$$\|v\|_{H,\varepsilon} := \min_{w\in H} \{\frac{1}{2\varepsilon}\|w-v\|_H^2 + \|w\|_H\} = \int_0^{\|v\|_H} \min\{s/\varepsilon, 1\}\, ds \quad \text{for every } v \in H. \tag{4.7}$$

We also recall that

$$\mathrm{Sign}_\varepsilon(v) = \frac{v}{\max\{\varepsilon, \|v\|_H\}} \quad \text{for every } v \in H. \tag{4.8}$$

Moreau-Yosida regularization of β and $\tilde{\beta}$. We introduce the Yosida regularization of β. For every $\varepsilon > 0$ we define

$$\beta_\varepsilon : \mathbb{R} \longrightarrow \mathbb{R}, \qquad \beta_\varepsilon = \frac{I - (I + \varepsilon\beta)^{-1}}{\varepsilon}. \tag{4.9}$$

We remark that β_ε is Lipschitz continuous with Lipschitz constant $1/\varepsilon$ and satisfies the following properties. Denoting by $R_\varepsilon = (I + \varepsilon\beta)^{-1}$ the resolvent operator, for all $\delta > 0$ and for every $\varphi \in D(\beta)$ we have that

$$\beta_\varepsilon(\varphi) \in \beta(R_\varepsilon\varphi), \tag{4.10}$$

$$(\beta_\varepsilon)_\delta = \beta_{\varepsilon+\delta}, \tag{4.11}$$

$$|\beta_\varepsilon(\varphi)| \leq |\beta^0(\varphi)|, \tag{4.12}$$

$$\lim_{\varepsilon\to 0} \beta_\varepsilon(\varphi) = \beta^0(\varphi), \tag{4.13}$$

where $\beta^0(\varphi)$ is the element of the range of β having minimum modulus. For $\varepsilon > 0$, we also introduce $\tilde{\beta}_\varepsilon : \mathbb{R} \to [0, +\infty]$ as the standard Moreau-Yosida regularization of $\tilde{\beta}$

$$\tilde{\beta}_\varepsilon := \min_{y\in\mathbb{R}} \left\{ \tilde{\beta}(x) + \frac{1}{2\varepsilon}|x - y| \right\} \tag{4.14}$$

and we recall that, for every $\varphi \in D(\tilde{\beta})$,

$$\tilde{\beta}_\varepsilon(\varphi) \leq \tilde{\beta}(\varphi). \tag{4.15}$$

Moreover, β_ε is the Frechet derivative of $\tilde{\beta}_\varepsilon$. Then, for every $\varphi_1, \varphi_2 \in D(\tilde{\beta})$, we have that

$$\tilde{\beta}_\varepsilon(\varphi_2) = \tilde{\beta}_\varepsilon(\varphi_1) + \int_{\varphi_1}^{\varphi_2} \beta_\varepsilon(s)\, ds. \tag{4.16}$$

5 The approximating problem (P_τ)

In order to prove Theorem 3.1, we first introduce a backward finite differences scheme. Assume that N is a positive integer and let Z be any normed space. By fixing the time step

$$\tau = T/N, \quad N \in \mathbb{N},$$

we introduce the interpolation maps from Z^{N+1} into either $L^\infty(0,T;Z)$ or $W^{1,\infty}(0,T;Z)$. For $(z^0,z^1,\dots,z^N)\in Z^{N+1}$, we define the piecewise constant functions $\overline{z}_\tau$ and the piecewise linear functions $\widehat{z}_\tau$, respectively:

$$\begin{aligned}&\overline{z}_\tau\in L^\infty(0,T;Z),\quad \overline{z}((i+s)\tau)=z^{i+1},\\ &\widehat{z}_\tau\in W^{1,\infty}(0,T;Z),\quad \widehat{z}((i+s)\tau)=z^i+s(z^{i+1}-z^i),\end{aligned}\tag{5.1}$$

if $0<s<1$ and $i=0,\dots,N-1$. By a direct computation, we have that

$$\|\overline{z}_\tau-\widehat{z}_\tau\|_{L^\infty(0,T;Z)} = \max_{i=0,\dots,N-1}\|z_{i+1}-z_i\|_Z=\tau\|\partial_t\widehat{z}_\tau\|_{L^\infty(0,T;Z)},\tag{5.2}$$

$$\|\overline{z}_\tau-\widehat{z}_\tau\|^2_{L^2(0,T;Z)} = \frac{\tau}{3}\sum_{i=0}^{N-1}\|z_{i+1}-z_i\|_Z^2=\frac{\tau^2}{3}\|\partial_t\widehat{z}_\tau\|^2_{L^2(0,T;Z)},\tag{5.3}$$

$$\|\overline{z}_\tau-\widehat{z}_\tau\|^2_{L^\infty(0,T;Z)} \le \sum_{i=0}^{N-1}\tau^2\left\|\frac{z_{i+1}-z_i}{\tau}\right\|_Z^2\le\tau\|\partial_t\widehat{z}_\tau\|^2_{L^2(0,T;Z)}.\tag{5.4}$$

Now, we introduce the approximating problem (P_τ). We set

$$g^i:=\frac{1}{\tau}\int_{(i-1)\tau}^{i\tau}g(s)\,ds,\qquad\text{for } i=1,\dots,N,\tag{5.5}$$

$$\mu_{\mathcal{S}}^i:=\frac{1}{\tau}\int_{(i-1)\tau}^{i\tau}\mu_{\mathcal{S}}(s)\,ds,\qquad\text{for } i=1,\dots,N,\tag{5.6}$$

and we look for a triplet $(\sigma_\tau,\varphi_\tau,\mu_\tau)$ satisfying at least the regularity requirements

$$\sigma_\tau\in H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W),\tag{5.7}$$

$$\varphi_\tau\in H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W),\tag{5.8}$$

$$\mu_\tau\in L^\infty(0,T;V_0)\cap L^2(0,T;H^2(\Omega)),\tag{5.9}$$

and solving the approximating problem (P_τ):

$$\frac{\varphi^i-\varphi^{i-1}}{\tau}-\Delta\mu^i=(\gamma_1\sigma^{i-1}-\gamma_2)p(\varphi^{i-1}),\tag{5.10}$$

$$\mu^i=\ell\frac{\varphi^i-\varphi^{i-1}}{\tau}-\Delta\varphi^i+\xi^i+\pi(\varphi^i)-\mu_{\mathcal{S}}^i,\tag{5.11}$$

$$\frac{\sigma^i-\sigma^{i-1}}{\tau}-\Delta\sigma^i+\zeta^i=-\gamma_3\sigma^{i-1}p(\varphi^{i-1})+\gamma_4(\sigma_s-\sigma^{i-1})+g^i,\tag{5.12}$$

$$\zeta^i(t)\in\rho\,\mathrm{Sign}_\tau(a\sigma^i(t)+b\varphi^i(t)+\eta^*),\tag{5.13}$$

$$\xi^i\in\beta_\tau(\varphi^i),\tag{5.14}$$

$$\partial_\nu\sigma^i=\partial_\nu\varphi^i=0,\qquad\mu^i=0\qquad\text{on }\Sigma,\tag{5.15}$$

$$\sigma^0=\sigma_0,\qquad\varphi^0=\varphi_0\qquad\text{in }\Omega,\tag{5.16}$$

where β_τ is the Yosida regularization of β at level τ. If at least one of the conditions (3.5)–(3.22) hold and

$$\exists\,\xi_0\in\beta(\varphi_0)\ \text{ a.e. in }\Omega:\quad \xi_0\in H\Longrightarrow\quad\lim_{\tau\searrow0}\beta_\tau(\varphi)=\beta^0(\varphi_0)\quad\text{in }H,\tag{5.17}$$

then the solution of (P_τ) is unique.

6 A priori estimates on (P_τ)

In the remainder of the paper we often owe to the Hölder inequality and to the elementary Young inequalities in performing our a priori estimates. For every $x, y > 0$, $\alpha \in (0,1)$ and $\delta > 0$ there hold

$$xy \leq \alpha x^{\frac{1}{\alpha}} + (1-\alpha) y^{\frac{1}{1-\alpha}}, \tag{6.1}$$

$$xy \leq \delta x^2 + \frac{1}{4\delta} y^2. \tag{6.2}$$

Moreover, we also use the inequality deduced from the compactness of the embedding $V \subset H \subset V^*$ (see [43, Lemma 8, p. 84]): for all $\delta > 0$ there exists a constant $K > 0$ such that

$$\|z\|_H \leq \delta \|z\|_V + K \|z\|_{V^*} \qquad \text{for all } z \in H. \tag{6.3}$$

In the following, the symbol c stands for different positive constants which depend only on $|\Omega|$, on the final time T, on the shape of the nonlinearities and on the constants and the norms of the functions involved in the assumptions of our statements.

First a priori estimate. We sum (5.10), (5.11) and (5.12) tested by $\tau\mu^i$, $(\varphi^i - \varphi^{i-1})$ and $\tau\sigma^i$, respectively. Then, we sum up for $i = 1, \ldots, n$ and obtain that

$$\ell\tau \sum_{i=1}^n \left\| \frac{\varphi^i - \varphi^{i-1}}{\tau} \right\|_H^2 + \tau \sum_{i=1}^n \|\nabla \mu^i\|_H^2 + \sum_{i=1}^n \Big(\nabla\varphi^i, \nabla(\varphi^i - \varphi^{i-1}) \Big)_H$$

$$+ \sum_{i=1}^n \Big(\beta(\varphi^i), \varphi^i - \varphi^{i-1} \Big)_H + \sum_{i=1}^n \Big(\sigma^i, \sigma^i - \sigma^{i-1} \Big)_H + \tau \sum_{i=1}^n \|\sigma^i\|_H^2 = \sum_{i=1}^n \Big(-\pi(\varphi^i), \varphi^i - \varphi^{i-1} \Big)_H$$

$$+ \tau \sum_{i=1}^n \Big((\gamma_1 \sigma^{i-1} - \gamma_2) p(\varphi^{i-1}), \mu^i \Big)_H - \tau \sum_{i=1}^n \Big(\zeta^i, \sigma^i \Big)_H - \sum_{i=1}^n \Big(\mu_{\mathcal{S}}^i, \varphi^i - \varphi^{i-1} \Big)_H$$

$$- \tau \sum_{i=1}^n \gamma_3 \Big(\sigma^{i-1} p(\varphi^{i-1}), \sigma^i \Big)_H + \tau \sum_{i=1}^n \gamma_4 \Big(\sigma_s - \sigma^{i-1}, \sigma^i \Big)_H + \tau \sum_{i=1}^n \Big(g^i, \sigma^i \Big)_H. \tag{6.4}$$

Now, the third and the fifth term on the left hand side of (6.4) can be rewritten as

$$\sum_{i=1}^n \Big(\nabla\varphi^i, \nabla(\varphi^i - \varphi^{i-1}) \Big)_H = \frac{1}{2} \|\nabla\varphi^n\|_H^2 - \frac{1}{2} \|\nabla\varphi_0\|_H^2 + \sum_{i=1}^n \frac{1}{2} \|\nabla(\varphi^i - \varphi^{i-1})\|_H^2, \tag{6.5}$$

$$\sum_{i=1}^n \Big(\sigma^i, (\sigma^i - \sigma^{i-1}) \Big)_H = \frac{1}{2} \|\sigma^n\|_H^2 - \frac{1}{2} \|\sigma_0\|_H^2 + \sum_{i=1}^n \frac{1}{2} \|\sigma^i - \sigma^{i-1}\|_H^2. \tag{6.6}$$

Recalling that β is the subdifferential of $\tilde{\beta}$, for the fourth term on the left hand side of (6.4) we have that

$$\sum_{i=1}^n \Big(\beta(\varphi^i), \varphi^i - \varphi^{i-1} \Big)_H \geq \int_\Omega \tilde{\beta}(\varphi^n) - \int_\Omega \tilde{\beta}(\varphi_0). \tag{6.7}$$

Moreover, due to the Lipschitz continuity of π, the first term on the right hand side of (6.4) can be estimated as follows

$$\begin{aligned}\sum_{i=1}^{n}\Big(-\pi(\varphi^i),\varphi^i-\varphi^{i-1}\Big)_H &\le \tau\sum_{i=1}^{n}C_\pi(1+\|\varphi^i\|_H)\left\|\frac{\varphi^i-\varphi^{i-1}}{\tau}\right\|_H\\ &\le \frac{\ell\tau}{2}\sum_{i=1}^{n}\left\|\frac{\varphi^i-\varphi^{i-1}}{\tau}\right\|_H^2+\tau c\sum_{i=1}^{n}(1+\|\varphi^i\|_H^2). \qquad (6.8)\end{aligned}$$

By applying the Young inequality and the Poincaré inequality to the second term on the right hand side of (6.4), we obtain that

$$\begin{aligned}\tau\sum_{i=1}^{n}\Big((\gamma_1\sigma^{i-1}-\gamma_2)p(\varphi^{i-1}),\mu^i\Big)_H &\le \frac{\tau}{4c_p}\sum_{i=1}^{n}\|\mu^i\|_H^2+\tau\sum_{i=1}^{n}\|\sigma^{i-1}\|_H^2+c\\ &\le \frac{\tau}{4}\sum_{i=1}^{n}\|\nabla\mu^i\|_H^2+\tau c\Bigg(1+\sum_{i=1}^{n}\|\sigma^{i-1}\|_H^2\Bigg), \qquad (6.9)\end{aligned}$$

while, using (4.6) and the Young inequality to estimate the third term on the right hand side of (6.4), we infer that

$$\begin{aligned}-\tau\sum_{i=1}^{n}\Big(\zeta^i,\sigma^i\Big)_H &\le \tau\sum_{i=1}^{n}C_A(1+\|a\sigma^i+b\varphi^i+\eta^*\|_H)\|\sigma^i\|_H\\ &\le \tau c\sum_{i=1}^{n}\Big(\|\sigma^i\|_H^2+\|\varphi^i\|_H^2+1\Big). \qquad (6.10)\end{aligned}$$

Finally, due to (3.1)–(3.4), by applying the Young inequality to the last four terms on the right hand side of (6.4), we have that

$$-\sum_{i=1}^{n}\Big(\mu_{\mathcal{S}}^i,\varphi^i-\varphi^{i-1}\Big)_H\le\frac{1}{4c_p}\sum_{i=1}^{n}\|(\varphi^i-\varphi^{i-1})\|_H^2+c\le\frac{1}{4}\sum_{i=1}^{n}\|\nabla(\varphi^i-\varphi^{i-1})\|_H^2+c, \qquad (6.11)$$

$$-\tau\sum_{i=1}^{n}\gamma_3\Big(\sigma^{i-1}p(\varphi^{i-1}),\sigma^i\Big)_H\le\tau c\Bigg(1+\sum_{i=1}^{n}\|\sigma^i\|_H^2+\sum_{i=1}^{n}\|\sigma^{i-1}\|_H^2\Bigg), \qquad (6.12)$$

$$\tau\sum_{i=1}^{n}\gamma_4\Big(\sigma_s-\sigma^{i-1},\sigma^i\Big)_H\le\tau c\Big(1+\|\sigma^i\|_H^2+\|\sigma^{i-1}\|_H^2\Big), \qquad (6.13)$$

$$\tau\sum_{i=1}^{n}\Big(g^i,\sigma^i\Big)_H\le\tau\sum_{i=1}^{n}\|g^i\|_H^2+\tau\sum_{i=1}^{n}\|\sigma^i\|_H^2\le\tau c\Bigg(1+\sum_{i=1}^{n}\|\sigma^i\|_H^2\Bigg). \qquad (6.14)$$

Consequently, using (6.5)–(6.14), from (6.4) we infer that,

$$\begin{aligned}&\frac{\ell}{2}\tau\sum_{i=1}^{n}\left\|\frac{\varphi^i-\varphi^{i-1}}{\tau}\right\|_H^2+\frac{\tau}{4}\sum_{i=1}^{n}\|\nabla\mu^i\|_H^2+\frac{1}{2}\|\nabla\varphi^n\|_H^2+\frac{1}{4}\sum_{i=1}^{n}\|\nabla(\varphi^i-\varphi^{i-1})\|_H^2\\ &+\|\tilde\beta(\varphi^n)\|_{L^1(\Omega)}+\frac{1}{2}\|\sigma^n\|_H^2+\sum_{i=1}^{n}\frac{1}{2}\|\sigma^i-\sigma^{i-1}\|_H^2+\tau\sum_{i=1}^{n}\|\nabla\sigma^i\|_H^2\le\frac{1}{2}\|\nabla\varphi_0\|_H^2\end{aligned}$$

$$+\|\tilde{\beta}(\varphi_0)\|_{L^1(\Omega)} + \frac{1}{2}\|\sigma_0\|_H^2 + \tau c\left(1 + \sum_{i=1}^n \left(\|\sigma^i\|_H^2 + \|\sigma^{i-1}\|_H^2 + \|\varphi^i\|_H^2 + \|\varphi^{i-1}\|_H^2\right)\right). \quad (6.15)$$

whence, using (3.3) and applying the Gronwall lemma, we conclude that

$$\|\partial_t \widehat{\varphi}_\tau\|_{L^2(0,T;H)} + \|\overline{\varphi}_\tau\|_{L^\infty(0,T;V)} \leq c, \quad (6.16)$$

$$\|\overline{\mu}_\tau\|_{L^2(0,T;V_0)} \leq c, \quad (6.17)$$

$$\|\tilde{\beta}(\overline{\varphi}_\tau)\|_{L^\infty(0,T;L^1(\Omega))} \leq c, \quad (6.18)$$

$$\|\overline{\sigma}_\tau\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \leq c, \quad (6.19)$$

and, due to the sublinear growth of $\operatorname{Sign}\tau$ stated by (4.6),

$$\|\overline{\zeta}_\tau\|_{L^\infty(0,T;H)} \leq c. \quad (6.20)$$

Finally, by comparison in (5.10), we have that

$$\|\overline{\mu}_\tau\|_{L^2(0,T;H^2(\Omega)\cap V_0)} \leq c. \quad (6.21)$$

Second a priori estimate. We test (5.12) by $(\sigma^i - \sigma^{i-1})$ and sum up for $i = 1, \ldots, n$. We obtain that

$$\tau \sum_{i=1}^n \left\|\frac{\sigma^i - \sigma^{i-1}}{\tau}\right\|_H^2 + \sum_{i=1}^n \left(\nabla\sigma^i, \nabla(\sigma^i - \sigma^{i-1})\right)_H = \sum_{i=1}^n \left(\zeta^i, \sigma^i - \sigma^{i-1}\right)_H$$

$$-\gamma_3 \sum_{i=1}^n \left(\sigma^{i-1} p(\varphi^{i-1}), \sigma^i - \sigma^{i-1}\right)_H + \gamma_4 \sum_{i=1}^n \left(\sigma_s - \sigma^{i-1} + g^i, \sigma^i - \sigma^{i-1}\right)_H. \quad (6.22)$$

We observe that the second term on the left hand side of (6.22) can be rewritten as

$$\sum_{i=1}^n \left(\nabla\sigma^i, \nabla(\sigma^i - \sigma^{i-1})\right)_H = \frac{1}{2}\|\nabla\sigma^n\|_H^2 - \frac{1}{2}\|\nabla\sigma_0\|_H^2 + \sum_{i=1}^n \frac{1}{2}\|\nabla(\sigma^i - \sigma^{i-1})\|_H^2. \quad (6.23)$$

Moreover, using (3.4), (6.16)–(6.20) and the Young inequality, the three terms on the right hand side of (6.22) can be estimated as follows:

$$\sum_{i=1}^n \left(\zeta^i, \sigma^i - \sigma^{i-1}\right)_H \leq \sum_{i=1}^n \|\zeta^i\|_H^2 + \sum_{i=1}^n \|\sigma^i - \sigma^{i-1}\|_H^2 \leq c, \quad (6.24)$$

$$-\gamma_3 \sum_{i=1}^n \left(\sigma^{i-1} p(\varphi^{i-1}), \sigma^i - \sigma^{i-1}\right)_H \leq \sum_{i=1}^n \|\sigma^{i-1} p(\varphi^{i-1})\|_H^2 + \sum_{i=1}^n \|\sigma^i - \sigma^{i-1}\|_H^2 \leq c, \quad (6.25)$$

$$\gamma_4 \sum_{i=1}^n \left(\sigma_s - \sigma^{i-1} + g^i, \sigma^i - \sigma^{i-1}\right)_H, \leq \sum_{i=1}^n \|\sigma_s - \sigma^{i-1} + g^i\|_H^2 + \sum_{i=1}^n \|\sigma^i - \sigma^{i-1}\|_H^2 \leq c. \quad (6.26)$$

Due to (3.3), (6.23)–(6.26), from (6.22) we infer that

$$\tau \sum_{i=1}^n \left\|\frac{\sigma^i - \sigma^{i-1}}{\tau}\right\|_H^2 \frac{1}{2}\|\nabla\sigma^n\|_H^2 + \sum_{i=1}^n \frac{1}{2}\|\nabla(\sigma^i - \sigma^{i-1})\|_H^2 \leq c.$$

Then, we conclude that

$$\|\partial_t \widehat{\sigma}_\tau\|_{L^2(0,T;H)} + \|\overline{\sigma}_\tau\|_{L^\infty(0,T;V)} \leq c, \tag{6.27}$$

whence, by comparison in (5.10) and (5.12), respectively, we infer that

$$\|\overline{\varphi}_\tau\|_{L^2(0,T;W)} \leq c, \tag{6.28}$$

$$\|\overline{\sigma}_\tau\|_{L^2(0,T;W)} \leq c, \tag{6.29}$$

$$\|\overline{\mu}_\tau\|_{L^2(0,T;H^2(\Omega)\cap V_0)} \leq c. \tag{6.30}$$

Third a priori estimate. We test (5.11) by $\tau\beta_\tau(\varphi^i)$ and sum up for $i = 1, \ldots, n$. We obtain that

$$\tau\sum_{i=1}^n \|\beta_\tau(\varphi^i)\|_H^2 + \tau\sum_{i=1}^n \beta_\tau'(\varphi^i)|\nabla\varphi^i|^2 = -\ell\sum_{i=1}^n \Big(\beta_\tau(\varphi^i), \varphi^i - \varphi^{i-1}\Big)_H$$

$$+\tau\sum_{i=1}^n \Big(\beta_\tau(\varphi^i), \mu^i\Big)_H - \tau\sum_{i=1}^n \Big(\beta_\tau(\varphi^i), \pi(\varphi^i)\Big)_H - \tau\sum_{i=1}^n \Big(\beta_\tau(\varphi^i), \mu_{\mathcal{S}}^i\Big)_H. \tag{6.31}$$

Due to the monotonicity of β_τ, the last term on the left hand side of (6.31) is nonnegative. Moreover, by applying the Young inequality to the four term on the right hand side of (6.31) and using (3.3) and (6.16), we have that

$$-\tau\ell\sum_{i=1}^n \Big(\beta_\tau(\varphi^i), \varphi^i - \varphi^{i-1}\Big)_H \leq \frac{\tau}{5}\sum_{i=1}^n \|\beta_\tau(\varphi^i)\|_H^2 + c, \tag{6.32}$$

$$\tau\sum_{i=1}^n \Big(\beta_\tau(\varphi^i), \mu^i\Big)_H \leq \frac{\tau}{5}\sum_{i=1}^n \|\beta_\tau(\varphi^i)\|_H^2 + c, \tag{6.33}$$

$$-\tau\sum_{i=1}^n \Big(\beta_\tau(\varphi^i), \pi(\varphi^i)\Big)_H \leq \frac{\tau}{5}\sum_{i=1}^n \|\beta_\tau(\varphi^i)\|_H^2 + c, \tag{6.34}$$

$$-\tau\sum_{i=1}^n \Big(\beta_\tau(\varphi^i), \mu_{\mathcal{S}}^i\Big)_H \leq \frac{\tau}{5}\sum_{i=1}^n \|\beta_\tau(\varphi^i)\|_H^2 + c. \tag{6.35}$$

Due to (6.32)–(6.35), from (6.31) we infer that

$$\frac{\tau}{5}\sum_{i=1}^n \|\beta_\tau(\varphi^i)\|_H^2 \leq c, \tag{6.36}$$

whence we conclude that

$$\|\beta_\tau(\overline{\varphi})\|_{L^2(0,T;H)} \leq c, \tag{6.37}$$

and, by comparison in (5.11),

$$\|\overline{\varphi}\|_{L^2(0,T;W)} \leq c. \tag{6.38}$$

Summary of the a priori estimates. Let us collect the previous estimates. From (6.16)–(6.20)(6.21), (6.27)–(6.28) and (6.37)–(6.38) we conclude that there exists a constant $c > 0$, independent of τ, such that

$$\|\overline{\varphi}_\tau\|_{L^\infty(0,T;V)\cap L^2(0,T;W)} \leq c, \tag{6.39}$$

$$\|\partial_t \widehat{\varphi}_\tau\|_{L^2(0,T;H)} \leq c, \tag{6.40}$$

$$\|\overline{\sigma}_\tau\|_{L^\infty(0,T;V)\cap L^2(0,T;W)} \leq c, \tag{6.41}$$

$$\|\partial_t \widehat{\sigma}_\tau\|_{L^2(0,T;H)} \leq c, \tag{6.42}$$

$$\|\overline{\mu}_\tau\|_{L^\infty(0,T;V_0)\cap L^2(0,T;H^2(\Omega))} \leq c, \tag{6.43}$$

$$\|\overline{\xi}_\tau)\|_{L^2(0,T;H)} \leq c, \tag{6.44}$$

$$\|\overline{\zeta}_\tau)\|_{L^\infty(0,T;H)} \leq c. \tag{6.45}$$

7 Passage to the limit as $\tau \searrow 0$

Based on available results (cf., e.g., [9]), it turns out that there exists a solution $(\sigma_\tau, \varphi_\tau, \mu_\tau)$ of (P_τ) satisfying the regularity requirements (5.7)–(5.9). In this section we pass to the limit as $\tau \searrow 0$ and prove that the limit of subsequences of solutions $(\sigma_\tau, \varphi_\tau, \mu_\tau)$ for (P_τ) yields a solution (σ, φ, μ) of (P).

Thanks to (6.39)–(6.45) and to the well-known weak or weak* compactness results, we deduce that, at least for a subsequence of $\tau \searrow 0$, there exist seven limit functions σ, $\widehat{\sigma}$, φ, $\widehat{\varphi}$, μ, ζ, and ξ such that

$$\overline{\sigma}_\tau \rightharpoonup^* \sigma \quad \text{in} \quad L^\infty(0,T;V) \cap L^2(0,T;W), \tag{7.1}$$

$$\widehat{\sigma}_\tau \rightharpoonup^* \widehat{\sigma} \quad \text{in} \quad H^1(0,T;H), \tag{7.2}$$

$$\tau\widehat{\sigma}_\tau \rightharpoonup^* 0 \quad \text{in} \quad H^1(0,T;H), \tag{7.3}$$

$$\overline{\varphi}_\tau \rightharpoonup^* \varphi \quad \text{in} \quad L^\infty(0,T;V) \cap L^2(0,T;W), \tag{7.4}$$

$$\widehat{\varphi}_\tau \rightharpoonup^* \widehat{\varphi} \quad \text{in} \quad H^1(0,T;H), \tag{7.5}$$

$$\tau\widehat{\varphi}_\tau \rightharpoonup^* 0 \quad \text{in} \quad H^1(0,T;H), \tag{7.6}$$

$$\overline{\mu}_\tau \rightharpoonup \mu \quad \text{in} \quad L^\infty(0,T;V_0) \cap L^2(0,T;H^2(\Omega)), \tag{7.7}$$

$$\overline{\zeta}_\tau \rightharpoonup^* \zeta \quad \text{in} \quad L^\infty(0,T;H), \tag{7.8}$$

$$\overline{\xi}_\tau \rightharpoonup \xi \quad \text{in} \quad L^2(0,T;H). \tag{7.9}$$

First, we observe that $\sigma = \widehat{\sigma}$. Indeed, thanks to (5.3) and (7.1)–(7.3), we have that

$$\|\overline{\sigma}_\tau - \widehat{\sigma}_\tau\|_{L^2(0,T;H)} \leq \frac{\tau}{\sqrt{3}} \|\partial_t \widehat{\sigma}_\tau\|_{L^2(0,T;H)} \leq c\tau,$$

and consequently $\overline{\sigma}_\tau - \widehat{\sigma}_\tau \to 0$ strongly in $L^2(0,T;H)$. Similarly, thanks to (5.3) and (7.4)–(7.6), we check that $\varphi = \widehat{\varphi}$. Next, in view of the convergences in (7.1)–(7.2), (7.4)–(7.5) and owing to the strong compactness lemma stated in [43, Lemma 8, p. 84], we have that

$$\widehat{\sigma}_\tau \to \sigma \quad \text{in} \quad C^0([0,T];H), \tag{7.10}$$

$$\widehat{\varphi}_\tau \to \varphi \quad \text{in} \quad C^0([0,T];H), \tag{7.11}$$

whence, due to the Lipschitz continuity of π, we have that

$$\pi(\overline{\varphi}_\tau) \to \pi(\varphi) \quad \text{in } L^\infty(0,T;H).$$

Now, we check that $\xi = \beta(\varphi)$: due to the weak convergence of ξ_τ and to the strong convergence of $\widehat{\varphi}_\tau$ ensured by (7.9) and (7.11), respectively, we have that

$$\limsup_{\tau\searrow 0} \int_0^T\int_\Omega \beta(\overline{\varphi}_\tau)\overline{\varphi}_\tau = \lim_{\tau\searrow 0}\int_0^T \langle\beta(\overline{\varphi}_\tau),\overline{\varphi}_\tau\rangle = \int_0^T \langle \xi,\varphi\rangle = \int_0^T\int_\Omega \xi\varphi,$$

so that a standard tool for maximal monotone operators (cf., e.g., [1, Lemma 1.3, p. 42]) ensure that $\xi = \beta(\varphi)$. With an analogous strategy, we check that $\zeta = \rho\,\mathrm{Sign}(a\sigma + b\varphi + \eta^*)$. We set

$$\eta_\tau = a\overline{\sigma}_\tau + b\overline{\varphi}_\tau + \eta^*.$$

Due to the convergences (7.10)–(7.11), we have that

$$\overline{\eta}_\tau \to \eta := a\sigma + b\varphi + \eta^* \qquad \text{in } L^2(0,T;H), \tag{7.12}$$

whence, thanks to the weak convergence of ζ_τ ensured by (7.8), we infer that

$$\limsup_{\tau\searrow 0}\int_0^T\int_\Omega \overline{\zeta}_\tau\overline{\eta}_\tau = \lim_{\tau\searrow 0}\int_0^T\langle\overline{\zeta}_\tau,\overline{\eta}_\tau\rangle = \int_0^T \langle\zeta,\overline{\eta}\rangle = \int_0^T\int_\Omega \zeta\eta,$$

so that, applying [1, Lemma 1.3, p. 42], we conclude that $\zeta = \rho\,\mathrm{Sign}(a\sigma + b\varphi + \eta^*)$. At this point, using (7.1)–(7.9) and passing to the limit in (5.10), (5.11) and (5.12), we arrive at (3.10), (3.11) and (3.12), respectively. Therefore, the existence of solution to problem (P) stated by Theorem 3.1 is proved.

8 Regularity

This section is devoted to the proof of Theorem 3.2.

First regularity estimate. We consider the approximating problem (P_τ) stated by (5.10)–(5.16) and we assume, in addition, (3.17)–(5.17). Taking into account (6.39)–(6.45), we take the difference between the equation (5.11) written in the steps i and $i-1$, respectively, and we add the resultant expression tested by $(\varphi^i - \varphi^{i-1})$ with (5.10) multiplied by $(\mu^i - \mu^{i-1})$. Taking the sum for $i = 2, \cdots, n$ we infer that

$$\frac{1}{2}\|\nabla\mu^n\|_H^2 + \sum_{i=2}^n \frac{\tau^2}{2}\left\|\nabla\frac{\mu^i-\mu^{i-1}}{\tau}\right\|_H^2 + \frac{\ell}{2}\left\|\frac{\varphi^n-\varphi^{n-1}}{\tau}\right\|_H^2 + \frac{\ell}{2}\sum_{i=2}^n\left\|\frac{\varphi^i-\varphi^{i-1}}{\tau} - \frac{\varphi^{i-1}-\varphi^{i-2}}{\tau}\right\|_H^2$$
$$+ \sum_{i=2}^n \tau\left\|\nabla\frac{\varphi^i-\varphi^{i-1}}{\tau}\right\|_H^2 + \sum_{i=2}^n\left(\frac{\beta_\tau(\varphi^i)-\beta_\tau(\varphi^{i-1})}{\tau}, \frac{\varphi^i-\varphi^{i-1}}{\tau}\right)_H$$
$$\le \frac{1}{2}\|\nabla\mu^1\|_H^2 + \frac{\ell}{2}\left\|\frac{\varphi^1-\varphi^0}{\tau}\right\|_H^2 - \sum_{i=2}^n\left(\frac{\pi(\varphi^i)-\pi(\varphi^{i-1})}{\tau}, \frac{\varphi^i-\varphi^{i-1}}{\tau}\right)_H$$

$$
+\sum_{i=2}^{n}\tau\left\|\frac{\mu_{\mathcal{S}}^{i}-\mu_{\mathcal{S}}^{i-1}}{\tau}\right\|_{H}\left\|\frac{\varphi^{i}-\varphi^{i-1}}{\tau}\right\|_{H}+\int_{\Omega}(\gamma_{1}\sigma^{n-1}-\gamma_{2})p(\varphi^{n-1})\mu^{n}-\int_{\Omega}(\gamma_{1}\sigma^{1}-\gamma_{2})p(\varphi^{1})\mu^{1}
$$

$$
-\sum_{i=1}^{n-1}\Big[(\gamma_{1}\sigma^{i}-\gamma_{2})p(\varphi^{i})-(\gamma_{1}\sigma^{i-1}-\gamma_{2})p(\varphi^{i-1})\Big]\mu^{i}. \tag{8.1}
$$

Due to (6.40) and the Lipschitz continuity of π, the third term on the right hand side of (8.1) can be estimated as follows

$$
-\sum_{i=2}^{n}\left(\frac{\pi(\varphi^{i})-\pi(\varphi^{i-1})}{\tau},\frac{\varphi^{i}-\varphi^{i-1}}{\tau}\right)_{H}\leq c\sum_{i=2}^{n}\tau\left\|\frac{\varphi^{i}-\varphi^{i-1}}{\tau}\right\|_{H}^{2}\leq c. \tag{8.2}
$$

We also observe that the last term on the left hand side of (8.1) is nonnegative due to the monotonicity of β_τ. Moreover, using the Young inequality and recalling (3.6) and (6.40), for the fourth term on the right hand side of (8.1) we have that

$$
\sum_{i=2}^{n}\tau\left\|\frac{\mu_{\mathcal{S}}^{i}-\mu_{\mathcal{S}}^{i-1}}{\tau}\right\|_{H}\left\|\frac{\varphi^{i}-\varphi^{i-1}}{\tau}\right\|_{H}\leq\sum_{i=2}^{n}\tau^{2}\left\|\frac{\mu_{\mathcal{S}}^{i}-\mu_{\mathcal{S}}^{i-1}}{\tau}\right\|_{H}^{2}+\sum_{i=2}^{n}\left\|\frac{\varphi^{i}-\varphi^{i-1}}{\tau}\right\|_{H}^{2}\leq c, \tag{8.3}
$$

while, due to to the boundedness of p, using the Young inequality and (6.39)–(6.43), the fifth, the sixth and the seventh term, can be estimated as follows:

$$
\int_{\Omega}(\gamma_{1}\sigma^{n-1}-\gamma_{2})p(\varphi^{n-1})\mu^{n}\leq c\left(1+\int_{\Omega}|\sigma^{n-1}|^{2}+\int_{\Omega}|\mu^{n}|^{2}\right)\leq c,
$$

$$
-\int_{\Omega}(\gamma_{1}\sigma^{1}-\gamma_{2})p(\varphi^{1})\mu^{1}\leq c\left(1+\int_{\Omega}|\sigma^{1}|^{2}+\int_{\Omega}|\mu^{1}|^{2}\right)\leq c,
$$

$$
-\sum_{i=1}^{n-1}\Big[(\gamma_{1}\sigma^{i}-\gamma_{2})p(\varphi^{i})-(\gamma_{1}\sigma^{i-1}-\gamma_{2})p(\varphi^{i-1})\Big]\mu^{i}\leq c\left(1+\sum_{i=1}^{n-1}\|\sigma^{i}\|_{H}^{2}+\sum_{i=1}^{n-1}\|\mu^{i}\|_{H}^{2}\right)\leq c.
$$

In order to estimate te first two terms on the right hand side of (8.1), we write the equations (5.10) and (5.11) for $i=1$ and we obtain that

$$
\frac{\varphi^{1}-\varphi^{0}}{\tau}-\Delta\mu^{1} = (\gamma_{1}\sigma^{0}-\gamma_{2})p(\varphi^{0})+g^{1}, \tag{8.4}
$$

$$
\mu^{1} = \ell\frac{\varphi^{1}-\varphi^{0}}{\tau}-\Delta\varphi^{1}+\xi^{1}+\pi(\varphi^{1})+\mu_{\mathcal{S}}^{1}. \tag{8.5}
$$

Then, adding (8.4) and (8.5) tested by μ^1 and $(\varphi^1-\varphi^0)/\tau$, respectively, and integrating over Ω, we obtain that

$$
\|\nabla\mu^{1}\|_{H}^{2}+\ell\left\|\frac{\varphi^{1}-\varphi^{0}}{\tau}\right\|_{H}^{2}+\tau\left\|\nabla\frac{\varphi^{1}-\varphi^{0}}{\tau}\right\|_{H}^{2}+\int_{\Omega}(\xi^{1}-\xi^{0})\left(\frac{\varphi^{1}-\varphi^{0}}{\tau}\right)
$$

$$
=\int_{\Omega}\Big[(\gamma_{1}\sigma_{0}-\gamma_{2})p(\varphi_{0})+g^{1}\Big]\mu^{1}+\int_{\Omega}(\Delta\varphi_{0}-\xi_{0}-\pi(\varphi_{0})+\mu_{\mathcal{S}}^{0})\left(\frac{\varphi^{1}-\varphi^{0}}{\tau}\right). \tag{8.6}
$$

Now, we observe that the third term on the left hand side of (8.6) is nonnegative, while the two terms on the right hand side of (8.6) can be estimated using (3.1)–(3.6), (3.17)–(5.17), (6.39)–(6.44) and the Young inequality:

$$\int_\Omega \Big[(\gamma_1\sigma_0-\gamma_2)p(\varphi_0)+g^1\Big]\mu^1 \le c(1+\|\sigma_0\|_H^2+\|\mu^1\|_H^2)\le c, \tag{8.7}$$

$$\begin{aligned}\int_\Omega (\Delta\varphi_0-\xi_0-\pi(\varphi_0)+\mu_{\mathcal S}^0)\Big(\frac{\varphi^1-\varphi^0}{\tau}\Big) &\le \frac{\ell}{2}\Big\|\frac{\varphi^1-\varphi^0}{\tau}\Big\|_H^2+c\Big(1+\|\varphi_0\|_W^2+\|\xi_0\|_H^2\Big)\\ &\le \frac{\ell}{2}\Big\|\frac{\varphi^1-\varphi^0}{\tau}\Big\|_H^2+c. \end{aligned}\tag{8.8}$$

Then we obtain that

$$\|\nabla\mu^1\|_H^2+\frac{\ell}{2}\Big\|\frac{\varphi^1-\varphi^0}{\tau}\Big\|_H^2+\tau\Big\|\nabla\frac{\varphi^1-\varphi^0}{\tau}\Big\|_H^2\le c,$$

whence the first and the second term of the right hand side of (8.1) are bounded by a positive constant. Combining the above estimates with (8.1), we infer that

$$\begin{aligned}&\frac12\|\nabla\mu^n\|_H^2+\sum_{i=2}^n\frac{\tau^2}{2}\Big\|\nabla\frac{\mu^i-\mu^{i-1}}{\tau}\Big\|_H^2+\frac{\ell}{2}\Big\|\frac{\varphi^n-\varphi^{n-1}}{\tau}\Big\|_H^2\\ &+\frac{\ell}{2}\sum_{i=2}^n\Big\|\frac{\varphi^i-\varphi^{i-1}}{\tau}-\frac{\varphi^{i-1}-\varphi^{i-2}}{\tau}\Big\|_H^2+\sum_{i=2}^n\tau\Big\|\nabla\frac{\varphi^i-\varphi^{i-1}}{\tau}\Big\|_H^2\le c,\end{aligned}\tag{8.9}$$

whence, recalling (6.39)–(6.45), we conclude that

$$\|\widehat{\varphi}_\tau\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)\cap L^2(0,T;W)}\le c. \tag{8.10}$$

Second regularity estimate. We test (5.11) by $-\Delta\varphi^i$. Integrating over Ω, we obtain that

$$\begin{aligned}\|\Delta\varphi^i\|_H^2+\int_\Omega\beta_\tau'(\varphi^i)|\nabla\varphi^i|^2 &\le \int_\Omega\nabla\varphi^i\cdot\nabla\mu^i-\ell\int_\Omega\nabla\Big(\frac{\varphi^i-\varphi^{i-1}}{\tau}\Big)\cdot\nabla\varphi^i\\ &-\int_\Omega\pi'(\varphi^i)|\nabla\varphi^i|^2+\int_\Omega\nabla\mu_{\mathcal S}^i\cdot\nabla\varphi^i.\end{aligned}\tag{8.11}$$

Using (6.39)–(6.45), (8.10) and the Young inequality, we have that

$$\int_\Omega\nabla\varphi^i\cdot\nabla\mu^i \le \|\varphi^i\|_V^2+\|\mu^i\|_V^2\le c, \tag{8.12}$$

$$-\ell\int_\Omega\nabla\Big(\frac{\varphi^i-\varphi^{i-1}}{\tau}\Big)\cdot\nabla\varphi^i \le \Big\|\nabla\frac{\varphi^i-\varphi^{i-1}}{\tau}\Big\|_H^2+\ell^2\|\varphi^i\|_V^2\le c, \tag{8.13}$$

$$-\int_\Omega\pi'(\varphi^i)|\nabla\varphi^i|^2 \le C_\pi\|\varphi^i\|_V^2\le c, \tag{8.14}$$

$$\int_\Omega\nabla\mu_{\mathcal S}^i\cdot\nabla\varphi^i \le \|\mu_{\mathcal S}^i\|_V^2+\|\mu^i\|_V^2\le c. \tag{8.15}$$

Then, combining (8.11) with the above estimates (8.12)–(8.15), we infer that $\|\Delta\varphi^i\|_H\le c$, whence we conclude that

$$\|\overline{\varphi}_\tau\|_{L^\infty(0,T;W)}\le c. \tag{8.16}$$

Conclusion. Due to (8.10) and (8.16), passing to the limit as $\tau \searrow 0$ in (P_τ) (see (5.10)–(5.16)), we infer that

$$\sigma \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \tag{8.17}$$

$$\varphi \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W), \tag{8.18}$$

$$\mu \in L^\infty(0,T;V_0) \cap L^2(0,T;H^2(\Omega)), \tag{8.19}$$

whence Theorem 3.2 is completely proved.

9 Uniqueness

9.1 Uniqueness - Problem (P)

Assuming $b = 0$ and integrating (3.10) over $(0,t)$, we obtain that

$$\varphi - \Delta(1 * \mu) = \varphi_0 + \{1 * [(\gamma_1\sigma - \gamma_2)p(\varphi)]\}. \tag{9.1}$$

Let $(\sigma_i, \varphi_i, \mu_i)$, $i = 1,2$, be two solutions of Problem (P) (see (3.10)–(3.16)). We take the difference between (9.1) written for $(\sigma_i, \varphi_i, \mu_i)$, $i = 1,2$, respectively, and test the resultant equation by $(\mu_1 - \mu_2)$. Then, we take the difference between (3.11) written for $(\sigma_i, \varphi_i, \mu_i)$, $i = 1,2$, respectively, and test the resultant equation by $(\varphi_1 - \varphi_2)$. Finally, we take the difference between (3.12) written for $(\sigma_i, \varphi_i, \mu_i)$, $i = 1,2$, respectively, and test the resultant equation by $(\sigma_1 - \sigma_2)$. Summing up, integrating over Q_t and setting

$$\sigma := \sigma_1 - \sigma_2, \qquad \varphi := \varphi_1 - \varphi_2, \qquad \mu := \mu_1 - \mu_2,$$

we obtain that

$$\frac{1}{2}\int_\Omega |\nabla(1*\mu)(t)|^2 + \frac{\ell}{2}\int_\Omega |\varphi(t)|^2 + \int_{Q_t} |\nabla\varphi|^2 + \int_{Q_t} (\xi_1 - \xi_2)(\varphi_1 - \varphi_2) + \frac{1}{2}\int_\Omega |\sigma(t)|^2$$

$$+\frac{1}{2}\int_\Omega |\nabla\sigma(t)|^2 + \frac{1}{a}\int_{Q_t} \big(\rho\,\mathrm{Sign}(a\sigma_1 + \eta^*) - \rho\,\mathrm{Sign}(a\sigma_2 + \eta^*)\big)\big((a\sigma_1 + \eta^*) - (a\sigma_2 + \eta^*)\big)$$

$$+\gamma_4 \int_{Q_t} |\sigma|^2 = -\int_{Q_t} \big(\pi(\varphi_1) - \pi(\varphi_2)\big)(\varphi_1 - \varphi_2) - \gamma_3 \int_{Q_t} \big(\sigma_1 p(\varphi_1) - \sigma_2 p(\varphi_2)\big)\big(\sigma_1 - \sigma_2\big)$$

$$-\int_{Q_t} \{1 * [(\gamma_1\sigma_1 - \gamma_2)p(\varphi_1) - (\gamma_1\sigma_2 - \gamma_2)p(\varphi_2)]\}\mu. \tag{9.2}$$

We observe that the fourth and the seventh term on the left hand side of (9.2) are nonnegative due to the monotonicity of β and Sign, respectively. Due to the Lipschitz continuity of π, the first term on the right hand side of (9.2) can be estimated as follows:

$$-\int_{Q_t} \big(\pi(\varphi_1) - \pi(\varphi_2)\big)(\varphi_1 - \varphi_2) \le c \int_0^t \|\varphi(s)\|_H^2 \, ds. \tag{9.3}$$

Since $V \hookrightarrow L^4(\Omega)$, using (3.7), the Hölder inequality, the Poincaré inequality and the Young inequality, the second term on the right hand side of (9.2) can be estimated as follows:

$$-\gamma_3 \int_{Q_t} \big(\sigma_1 p(\varphi_1) - \sigma_2 p(\varphi_2)\big)\big(\sigma_1 - \sigma_2\big)$$

$$\begin{aligned}
&\le c\int_{Q_t} |\sigma|^2 p(\varphi_1) + c\int_{Q_t} |\sigma_2||p(\varphi_1) - p(\varphi_2)||\sigma| \\
&\le c\|p\|_\infty \int_0^t \|\sigma(s)\|_H^2 \, ds + c\int_0^t \|\sigma_2(s)\|_{L^4(\Omega)}\|\varphi(s)\|_{L^4(\Omega)}\|\sigma(s)\|_{L^2(\Omega)} \, ds \\
&\le c\int_0^t \|\sigma(s)\|_H^2 \, ds + \frac{1}{2}\int_0^t \|\varphi(s)\|_V \, ds + c\|\sigma_2\|_{L^\infty(0,T;V)} \int_0^t \|\sigma(s)\|_H^2 \, ds \\
&\le c\int_0^t \|\sigma(s)\|_H^2 \, ds + \frac{1}{2}\int_0^t \|\varphi(s)\|_V \, ds.
\end{aligned} \tag{9.4}$$

Integrate by parts the last term on the right hand side of (9.2), we obtain that

$$-\int_{Q_t} \{1 * [(\gamma_1\sigma_1 - \gamma_2)p(\varphi_1) - (\gamma_1\sigma_2 - \gamma_2)p(\varphi_2)]\}\mu$$

$$= -\int_\Omega \{1 * [(\gamma_1\sigma_1(t) - \gamma_2)p(\varphi_1(t)) - (\gamma_1\sigma_2 - \gamma_2)p(\varphi_2(t))]\}(1 * \mu)(t)$$

$$+ \int_{Q_t} [(\gamma_1\sigma_1 - \gamma_2)p(\varphi_1) - (\gamma_1\sigma_2 - \gamma_2)p(\varphi_2)](1 * \mu)$$

$$\le -\int_\Omega [1 * (\gamma_1\sigma p(\varphi_1))](1 * \mu)(t) - \int_\Omega \{1 * [(\gamma_1\sigma_2 - \gamma_2)(p(\varphi_1) - p(\varphi_2))]\}(1 * \mu)(t)$$

$$+ \int_{Q_t} \gamma_1\sigma p(\varphi_1)(1 * \mu) + \int_{Q_t} (\gamma_1\sigma_2 - \gamma_2)(p(\varphi_1) - p(\varphi_2))(1 * \mu) \tag{9.5}$$

Now, we estimate every term on the right hand side of (9.5), separately, using (3.7), the Hölder inequality, the Poincaré inequality, the Young inequality and the continuous immersion $V \hookrightarrow L^4(\Omega)$. For the first term on the right hand side of (9.5) we have that

$$\begin{aligned}
-\int_\Omega [1 * (\gamma_1\sigma p(\varphi_1))](t)(1 * \mu)(t) &\le |\gamma_1|\|p\|_\infty \int_\Omega (1 * |\sigma|)(t)(1 * \mu)(t) \\
&\le \frac{1}{8}\|(1 * \mu)(t)\|_H^2 + c\left\|\int_0^t |\sigma(s)| \, ds\right\|_H^2 \\
&\le \frac{1}{8}\|(1 * \mu)(t)\|_H^2 + c\int_0^t \|\sigma(s)\|_H^2 \, ds,
\end{aligned} \tag{9.6}$$

while, for the second term on the right hand side of (9.5) we obtain that

$$-\int_\Omega \{1 * [(\gamma_1\sigma_2 - \gamma_2)(p(\varphi_1) - p(\varphi_2))]\}(1 * \mu)(t) \tag{9.7}$$

$$
\begin{aligned}
&\leq \|1 * [(\gamma_1\sigma_2 - \gamma_2)(p(\varphi_1) - p(\varphi_2))](t)\|_{L^{4/3}(\Omega)} \|(1*\mu)(t)\|_{L^4(\Omega)} \\
&\leq \left(\|p'\|_\infty \int_0^t \|(\gamma_1\sigma_2(s) - \gamma_2)|\varphi(s)|\|_{L^{4/3}(\Omega)} \right) \|(1*\mu)(t)\|_{L^4(\Omega)} \\
&\leq \frac{1}{8}\|\nabla(1*\mu)(t)\|_H^2 + c\left[\int_0^t \left(\int_\Omega (\sigma_2+1)^{4/3}|\varphi|^{4/3} \right)^{3/4} \right]^2 \\
&\leq \frac{1}{8}\|\nabla(1*\mu)(t)\|_H^2 + c\left[\int_0^t \||\sigma_2(s)|+1\|_{L^4(\Omega)} \|\varphi(s)\|_{L^2(\Omega)}\, ds \right]^2 \\
&\leq \frac{1}{8}\|\nabla(1*\mu)(t)\|_H^2 + c\int_0^t \||\sigma_2(s)|+1\|_{L^4(\Omega)}^2 \|\varphi(s)\|_{L^2(\Omega)}^2\, ds \\
&\leq \frac{1}{8}\|\nabla(1*\mu)(t)\|_H^2 + c\Big(1+\|\sigma_2\|_{L^\infty(0,T;V)}^2\Big)\left(\int_0^t \|\varphi(s)\|_H^2\, ds \right) \\
&\leq \frac{1}{8}\|\nabla(1*\mu)(t)\|_H^2 + c\left(\int_0^t \|\varphi(s)\|_H^2\, ds \right). \qquad (9.8)
\end{aligned}
$$

Beside, the third term on the right hand side of (9.5) can be estimated as follows:

$$
\int_{Q_t} \gamma_1\sigma p(\varphi_1)(1*\mu) \leq |\gamma_1|\|p\|_\infty \int_{Q_t} |\sigma||(1*\mu)| \leq c\int_0^t \Big(\|\sigma(s)\|_H^2 + \|(1*\mu)(s)\|_H^2 \Big)\, ds, \quad (9.9)
$$

while, for the last term on the right hand side of (9.5) we have that

$$
\begin{aligned}
&\int_{Q_t} (\gamma_1\sigma_2 - \gamma_2)(p(\varphi_1) - p(\varphi_2))(1*\mu) \\
&\leq c\|p'\|_\infty \int_{Q_t} (|\sigma_2|+1)|\varphi||1*\mu| \\
&\leq c\int_{Q_t} (|\sigma_2|+1)|\varphi||1*\mu| \\
&\leq c\int_0^t (\|\sigma_2(s)\|_{L^4(\Omega)}+1)\|\varphi(s)\|_{L^2(\Omega)}\|(1*\mu)(s)\|_{L^4(\Omega)}\, ds \\
&\leq c\int_0^t (\|\sigma_2(s)\|_V+1)\|\varphi(s)\|_H\|(1*\mu)(s)\|_V\, ds \\
&\leq c\left(\int_0^t \Big(\|\varphi(s)\|_H^2 + \|(1*\mu)(s)\|_V^2 \Big)\, ds \right). \qquad (9.10)
\end{aligned}
$$

Combining (9.6)–(9.10) with (9.5), (9.3) and (9.3), from (9.2) we infer that

$$
\begin{aligned}
&\frac{1}{4}\int_\Omega |\nabla(1*\mu)(t)|^2 + \frac{\ell}{2}\int_\Omega |\varphi(t)|^2 + \int_{Q_t} |\nabla\varphi|^2 + \frac{1}{2}\int_\Omega |\sigma(t)|^2 + \frac{1}{2}\int_\Omega |\nabla\sigma(t)|^2 + \gamma_4\int_{Q_t} |\sigma|^2 \\
&\qquad \leq c\int_0^t \Big(\|\varphi(s)\|_V^2 + \|\sigma(s)\|_H^2 + \|(1*\mu)(s)\|_H^2 \Big)\, ds, \qquad (9.11)
\end{aligned}
$$

Then, by applying the Gronwall lemma, we conclude that the left hand side of (9.11) is null, whence $\sigma = \varphi = \mu = 0$ a.e. in Q.

9.2 Uniqueness - Problem (P_ε)

Assuming (3.5), we consider the Problem (P_ε) obtained from (3.10)–(3.16) by replacing the operators Sign and β by their Yosida regularizations (see (4.1) and (4.9)), respectively, and we denote by $(\sigma_\varepsilon, \varphi_\varepsilon, \mu_\varepsilon)$ its solution. Then, we make a change of variable and set

$$\eta = a\sigma + b\varphi + \eta^*, \qquad \eta_0 = a\sigma_0 + b\varphi_0 + \eta^*, \qquad \eta_\varepsilon = a\sigma_\varepsilon + b\varphi_\varepsilon + \eta^*, \tag{9.12}$$

where (σ, φ, μ) is a solution of Problem (P) stated by (3.10)–(3.16). With this change of variable, we obtain the following system:

$$\partial_t\varphi_\varepsilon - \Delta\mu_\varepsilon = \Big(\frac{\gamma_1}{a}(\eta_\varepsilon - b\varphi_\varepsilon - \eta^*) - \gamma_2\Big)p(\varphi_\varepsilon) \ \text{ a.e. in } Q, \tag{9.13}$$

$$\mu_\varepsilon = \ell\partial_t\varphi_\varepsilon - \Delta\varphi_\varepsilon + \xi_\varepsilon + \pi(\varphi_\varepsilon) + \mu_{\mathcal{S}} \ \text{ a.e. in } Q, \tag{9.14}$$

$$\begin{aligned}\partial_t\eta_\varepsilon - \Delta\eta_\varepsilon + \zeta_\varepsilon &= b\partial_t\varphi_\varepsilon - b\Delta\varphi_\varepsilon - \Delta\eta^* - \gamma_3(\eta_\varepsilon - b\varphi_\varepsilon - \eta^*)p(\varphi_\varepsilon)\\ &\quad + \gamma_4(a\sigma_{\mathcal{S}} - \eta_\varepsilon + b\varphi_\varepsilon + \eta^*) + ag, \quad \text{a.e. in } Q,\end{aligned} \tag{9.15}$$

$$\zeta_\varepsilon(t) \in \rho\,\mathrm{Sign}_\varepsilon(\eta_\varepsilon(t)) \text{ for a.e. } t \in (0,T), \tag{9.16}$$

$$\xi_\varepsilon \in \beta_\varepsilon(\varphi_\varepsilon) \text{ a.e. in } Q, \tag{9.17}$$

$$\partial_\nu\eta_\varepsilon = \partial_\nu\varphi_\varepsilon = 0, \qquad \mu_\varepsilon = 0 \qquad \text{on } \Sigma, \tag{9.18}$$

$$\eta_\varepsilon(0) = \eta_0, \qquad \varphi_\varepsilon(0) = \varphi_0 \qquad \text{in } \Omega. \tag{9.19}$$

Let $(\eta_{\varepsilon,i}, \varphi_{\varepsilon,i}, \mu_{\varepsilon,i})$, $i = 1,2$, be two solutions of the system (9.13)–(9.19). We take the difference between (9.13) written for $(\eta_{\varepsilon,i}, \varphi_{\varepsilon,i}, \mu_{\varepsilon,i})$, $i = 1,2$, respectively, and test the resultant equation by $(\mu_{\varepsilon,1} - \mu_{\varepsilon,2})$. Then, we take the difference between (9.14) written for $(\eta_{\varepsilon,i}, \varphi_{\varepsilon,i}, \mu_{\varepsilon,i})$, $i = 1,2$, respectively, and test the resultant equation by $\partial_t(\varphi_{\varepsilon,1} - \varphi_{\varepsilon,2})$. Finally, we take the difference between (9.15) written for $(\eta_{\varepsilon,i}, \varphi_{\varepsilon,i}, \mu_{\varepsilon,i})$, $i = 1,2$, respectively, and test the resultant equation by $(\eta_{\varepsilon,1} - \eta_{\varepsilon,2})$. Summing up, integrating over Q_t, exploiting the cancellation of the suitable corresponding terms, and setting

$$\eta_\varepsilon := \eta_{\varepsilon,1} - \eta_{\varepsilon,2}, \qquad \varphi_\varepsilon := \varphi_{\varepsilon,1} - \varphi_{\varepsilon,2}, \qquad \mu_\varepsilon := \mu_{\varepsilon,1} - \mu_{\varepsilon,2},$$

we obtain that

$$\begin{aligned}&\int_{Q_t} |\nabla\mu_\varepsilon| + \ell\int_{Q_t} |\partial_t\varphi_\varepsilon|^2 + \frac{1}{2}\|\varphi_\varepsilon(t)\|_V^2 + \frac{1}{2}\|\eta_\varepsilon(t)\|_H^2 + \int_{Q_t} |\nabla\eta_\varepsilon|^2 + \int_{Q_t} \zeta_\varepsilon\eta_\varepsilon\\ &\le \int_{Q_t} \frac{\gamma_1}{a}(\eta_\varepsilon - b\varphi_\varepsilon)p(\varphi_{\varepsilon,1})\mu_\varepsilon + \int_{Q_t} \Big(\frac{\gamma_1}{a}(\eta_{\varepsilon,2} - b\varphi_{\varepsilon,2} - \eta^*) - \gamma_2\Big)\Big(p(\varphi_{\varepsilon,1}) - p(\varphi_{\varepsilon,2})\Big)\mu_\varepsilon\\ &\quad - \int_{Q_t} \Big(\beta_\varepsilon(\varphi_{\varepsilon,1}) + \pi(\varphi_{\varepsilon,1}) - \varphi_{\varepsilon,1} - \beta_\varepsilon(\varphi_{\varepsilon,2}) - \pi(\varphi_{\varepsilon,2}) + \varphi_{\varepsilon,2}\Big)\partial_t\varphi_\varepsilon\\ &\quad -\gamma_3\int_{Q_t} (\eta_{\varepsilon,2} - b\varphi_{\varepsilon,2} - \eta^*)(p(\varphi_{\varepsilon,1}) - p(\varphi_{\varepsilon,2}))\eta_\varepsilon - \gamma_3\int_{Q_t} (\eta_\varepsilon - b\varphi_\varepsilon)p(\varphi_{\varepsilon,1})\eta_\varepsilon\\ &\quad + b\int_{Q_t} \partial_t\varphi_\varepsilon\eta_\varepsilon + b\int_{Q_t} \nabla\varphi_\varepsilon \cdot \nabla\eta_\varepsilon + \gamma_4\int_{Q_t} (b\varphi_\varepsilon - \eta_\varepsilon)\eta_\varepsilon.\end{aligned} \tag{9.20}$$

We notice that the last term on the left hand side of (9.20) is nonnegative, due to the monotonicity of $\mathrm{Sign}_\varepsilon$. Beside, using (3.18)–(3.20), the Poincaré inequality and the Young inequality, the first term and the last four terms on on the right hand side of (9.20) can be estimated as follows:

$$\int_{Q_t} \frac{\gamma_1}{a}(\eta_\varepsilon - b\varphi_\varepsilon)p(\varphi_{\varepsilon,1})\mu_\varepsilon \leq c\|p\|_\infty \int_0^t \Big(\|\eta_\varepsilon(s)\|_H + \|\varphi_\varepsilon(s)\|_H\Big)\|\mu_\varepsilon(s)\|_H \, ds$$

$$\leq c\int_0^t \Big(\|\eta_\varepsilon(s)\|_H + \|\varphi_\varepsilon(s)\|_H\Big)\|\nabla\mu_\varepsilon(s)\|_H \, ds$$

$$\leq c\int_0^t \Big(\|\eta_\varepsilon(s)\|_H^2 + \|\varphi_\varepsilon(s)\|_H^2\Big) \, ds + \frac{1}{4}\int_{Q_t} |\nabla\mu_\varepsilon|^2, \tag{9.21}$$

and, with an analogous technique,

$$\begin{aligned} -\gamma_3\int_{Q_t}(\eta_\varepsilon - b\varphi_\varepsilon)p(\varphi_{\varepsilon,1})\eta_\varepsilon &\leq c\|p\|_\infty \int_{Q_t}(|\eta_\varepsilon| + |\varphi_\varepsilon|)|\eta_\varepsilon| \\ &\leq c\int_0^t \Big(\|\eta_\varepsilon(s)\|_H^2 + \|\varphi_\varepsilon(s)\|_H^2\Big) \, ds, && (9.22) \\ b\int_{Q_t}\partial_t\varphi_\varepsilon\eta_\varepsilon &\leq \frac{\ell}{4}\int_{Q_t}|\partial_t\varphi_\varepsilon|^2 + c\int_0^t \|\eta_\varepsilon(s)\|_H^2 \, ds, && (9.23) \\ b\int_{Q_t}\nabla\varphi_\varepsilon\cdot\nabla\eta_\varepsilon &\leq \frac{1}{4}\int_{Q_t}|\nabla\eta_\varepsilon| + c\int_0^t \|\varphi_\varepsilon(s)\|_V^2 \, ds, && (9.24) \\ \gamma_4\int_{Q_t}(b\varphi_\varepsilon - \eta_\varepsilon)\eta_\varepsilon &\leq c\int_0^t \Big(\|\eta_\varepsilon(s)\|_H^2 + \|\varphi_\varepsilon(s)\|_H^2\Big) \, ds. && (9.25) \end{aligned}$$

Moreover, due to (3.18)–(3.20), the Hölder inequality, the Poincaré inequality, the Young inequality and the continuous immersion $V \hookrightarrow L^4(\Omega)$, for the second term on the right hand side of (9.20) we have that

$$\int_{Q_t}\left(\frac{\gamma_1}{a}(\eta_{\varepsilon,2} - b\varphi_{\varepsilon,2} - \eta^*) - \gamma_2\right)\Big(p(\varphi_{\varepsilon,1}) - p(\varphi_{\varepsilon,2})\Big)\mu_\varepsilon$$

$$\leq c\|p'\|_\infty\int_{Q_t}(|\eta_{\varepsilon,2}| + |\varphi_{\varepsilon,2}| + |\eta^*|)|\varphi_\varepsilon||\mu_\varepsilon| + c\|p'\|_\infty\int_{Q_t}|\varphi_\varepsilon||\mu_\varepsilon|$$

$$\leq c\int_0^t (\|\eta_{\varepsilon,2}(s)\|_{L^4(\Omega)} + \|\varphi_{\varepsilon,2}(s)\|_{L^2(\Omega)} + 1)\|\varphi_\varepsilon(s)\|_{L^2(\Omega)}\|\mu_\varepsilon(s)\|_{L^4(\Omega)} \, ds$$

$$+c\int_0^t \|\varphi_\varepsilon(s)\|_H^2 \, ds + \frac{1}{8}\int_{Q_t}|\nabla\mu_\varepsilon(s)|^2$$

$$\leq c\int_0^t (\|\eta_{\varepsilon,2}(s)\|_V + \|\varphi_{\varepsilon,2}(s)\|_H + 1)\|\varphi_\varepsilon(s)\|_H\|\nabla\mu_\varepsilon(s)\|_H \, ds$$

$$+c\int_0^t \|\varphi_\varepsilon(s)\|_H^2 \, ds + \frac{1}{8}\int_{Q_t}|\nabla\mu_\varepsilon(s)|^2 \leq \frac{1}{4}\int_{Q_t}|\nabla\mu_\varepsilon|^2 + c\int_0^t \|\varphi_\varepsilon(s)\|_H^2 \, ds, \tag{9.26}$$

and, with an analogous strategy, for the fourth term on the right hand side of (9.20) we obtain that

$$-\gamma_3 \int_{Q_t} (\eta_{\varepsilon,2} - b\varphi_{\varepsilon,2} - \eta^*)(p(\varphi_{\varepsilon,1} - p(\varphi_{\varepsilon,2})\eta_\varepsilon \le c\|p'\|_\infty \int_{Q_t} (|\eta_{\varepsilon,2}| + |\varphi_{\varepsilon,2}| + |\eta^*|)|\varphi_\varepsilon||\eta_\varepsilon|$$

$$\le c \int_0^t (\|\eta_{\varepsilon,2}(s)\|_{L^4(\Omega)} + \|\varphi_{\varepsilon,2}(s)\|_{L^2(\Omega)} + 1)\|\varphi_\varepsilon(s)\|_{L^2(\Omega)} \|\eta_\varepsilon(s)\|_{L^4(\Omega)}\, ds$$

$$\le c \int_0^t (\|\eta_{\varepsilon,2}(s)\|_V + \|\varphi_{\varepsilon,2}(s)\|_H + 1)\|\varphi_\varepsilon(s)\|_H \|\eta_\varepsilon(s)\|_H\, ds$$

$$\le c \int_0^t \|\varphi_\varepsilon(s)\|_H \|\nabla\eta_\varepsilon(s)\|_H\, ds \le \frac{1}{4}\int_{Q_t} |\nabla\eta_\varepsilon| + c\int_0^t \|\varphi_\varepsilon(s)\|_H^2\, ds. \tag{9.27}$$

Finally, recalling (3.5) and the continuous immersion $V \hookrightarrow L^6(\Omega)$, the third term on the right hand side of (9.20) can be estimated as follow:

$$-\int_{Q_t} \Big(\beta_\varepsilon(\varphi_{\varepsilon,1}) + \pi(\varphi_{\varepsilon,1}) - \varphi_{\varepsilon,1} - \beta_\varepsilon(\varphi_{\varepsilon,2}) - \pi(\varphi_{\varepsilon,2}) + \varphi_{\varepsilon,2}\Big)\partial_t\varphi_\varepsilon$$

$$\le c\int_{Q_t} \Big(|\beta_\varepsilon'(\varphi_{\varepsilon,1})| + |\beta_\varepsilon'(\varphi_{\varepsilon,2})| + 1\Big)|\varphi_\varepsilon||\partial_t\varphi_\varepsilon| \le c\int_{Q_t} \Big(|\varphi_{\varepsilon,1}|^2 + |\varphi_{\varepsilon,2}|^2 + 1\Big)|\varphi_\varepsilon||\partial_t\varphi_\varepsilon|$$

$$\le c\int_0^t \||\varphi_{\varepsilon,1}(s)|^2 + |\varphi_{\varepsilon,2}(s)|^2 + 1\|_{L^3(\Omega)} \|\varphi_\varepsilon(s)\|_{L^6(\Omega)} \|\partial_t\varphi_\varepsilon\|_{L^2(\Omega)}\, ds$$

$$\le \frac{\ell}{4}\int_{Q_t} |\partial_t\varphi_\varepsilon|^2 + c\int_0^t \||\varphi_{\varepsilon,1}(s)|^2 + |\varphi_{\varepsilon,2}(s)|^2 + 1\|_{L^3(\Omega)}^2 \|\varphi_\varepsilon(s)\|_V^2\, ds$$

$$\le \frac{\ell}{4}\int_{Q_t} |\partial_t\varphi_\varepsilon|^2 + c\int_0^t \left(\int_\Omega \Big(|\varphi_{\varepsilon,1}(s)|^6 + |\varphi_{\varepsilon,2}(s)|^6 + 1\Big)\right)^{2/3} \|\varphi_\varepsilon(s)\|_V^2\, ds$$

$$\le \frac{\ell}{4}\int_{Q_t} |\partial_t\varphi_\varepsilon|^2 + c\int_0^t \Big(\|\varphi_{\varepsilon,1}(s)\|_{L^6(\Omega)} + \|\varphi_{\varepsilon,2}(s)\|_{L^6(\Omega)} + 1\Big)^4 \|\varphi_\varepsilon(s)\|_V^2\, ds$$

$$\le \frac{\ell}{4}\int_{Q_t} |\partial_t\varphi_\varepsilon|^2 + c\int_0^t \Big(\|\varphi_{\varepsilon,1}\|_{L^\infty(0,T;V)} + \|\varphi_{\varepsilon,2}\|_{L^\infty(0,T;V)} + 1\Big)^4 \|\varphi_\varepsilon(s)\|_V^2\, ds$$

$$\le \frac{\ell}{4}\int_{Q_t} |\partial_t\varphi_\varepsilon|^2 + c\int_0^t \|\varphi_\varepsilon(s)\|_V^2\, ds. \tag{9.28}$$

Thanks to (9.21)–(9.28), from (9.20), we infer that

$$\frac{1}{2}\int_{Q_t} |\nabla\mu_\varepsilon| + \frac{\ell}{2}\int_{Q_t} |\partial_t\varphi_\varepsilon|^2 + \frac{1}{2}\|\varphi_\varepsilon(t)\|_V^2 + \frac{1}{2}\|\eta_\varepsilon(t)\|_H^2 + \frac{1}{2}\int_{Q_t} |\nabla\eta_\varepsilon|^2$$

$$\le c\int_0^t \Big(\|\eta_\varepsilon(s)\|_H^2 + \|\varphi_\varepsilon(s)\|_V^2\Big)\, ds, \tag{9.29}$$

whence, by applying the Gronwall lemma, we conclude that the left hand side of (9.29) is null. Then, $\eta_\varepsilon = \varphi_\varepsilon = \mu_\varepsilon = 0$ a.e. in Q.

10 Sliding mode control

The argument we use in the proof of Theorem 3.4 relies in the following Lemma (see [2, Lemma 4.1, p. 20]).

Lemma 10.1 *Let a_0, b_0, ψ_0, $\rho \in \mathbb{R}$ be such that*

$$a_0,\ b_0,\ \psi_0 \geq 0 \quad \textit{and} \quad \rho > a_0^2 + 2b_0 + 2\frac{\psi_0}{T}. \tag{10.1}$$

Let $\psi : [0,T] \to [0,+\infty)$ be an absolutely continuous function satisfying $\psi(0) = \psi_0$ and

$$\psi' + \rho \leq a_0\rho^{1/2} + b_0 \quad \textit{a.e. in the set } P := \{t \in (0,T) : \psi(t) > 0\}. \tag{10.2}$$

Then, the following conditions hold true:

1. *If $\psi_0 = 0$, then ψ vanishes identically.*

2. *If $\psi_0 > 0$, then there exists $T^* \in (0,T)$ satisfying $T^* \leq 2\psi_0/(\rho - a_0^2 - 2b_0)$ such that ψ is strictly decreasing in $(0,T^*)$ and ψ vanishes in $[T^*,T]$.*

10.1 Sliding mode estimates

With the same change of variable used in (9.12), under the assumptions of Theorem 3.4, Problem (P_ε) (see (9.13)–(9.19)) can be rewritten as

$$\partial_t\varphi_\varepsilon - \Delta\mu_\varepsilon = \Big(\frac{\gamma_1}{a}(\eta_\varepsilon - b\varphi_\varepsilon - \eta^*) - \gamma_2\Big)p(\varphi_\varepsilon) \ \text{ a.e. in } Q, \tag{10.3}$$

$$\mu_\varepsilon = \ell\partial_t\varphi_\varepsilon - \Delta\varphi_\varepsilon + \xi_\varepsilon + \pi(\varphi_\varepsilon) + \mu_{\mathcal{S}} \ \text{ a.e. in } Q, \tag{10.4}$$

$$\partial_t\eta_\varepsilon - \Delta\eta_\varepsilon + \rho\kappa_\varepsilon = w_\varepsilon \ \text{ a.e. in } Q, \tag{10.5}$$

$$\kappa_\varepsilon(t) \in \mathrm{Sign}_\varepsilon(\eta_\varepsilon(t)) \text{ for a.e. } t \in (0,T), \tag{10.6}$$

$$\xi_\varepsilon \in \beta_\varepsilon(\varphi_\varepsilon) \text{ a.e. in } Q, \tag{10.7}$$

$$\partial_\nu\eta_\varepsilon = \partial_\nu\varphi_\varepsilon = 0, \qquad \mu_\varepsilon = 0 \qquad \text{on } \Sigma, \tag{10.8}$$

$$\eta_\varepsilon(0) = \eta_0, \qquad \varphi_\varepsilon(0) = \varphi_0 \qquad \text{in } \Omega, \tag{10.9}$$

where

$$w_\varepsilon := b\partial_t\varphi_\varepsilon - b\Delta\varphi_\varepsilon - \Delta\eta^* - \gamma_3(\eta_\varepsilon - b\varphi_\varepsilon - \eta^*)p(\varphi_\varepsilon) + \gamma_4(a\sigma_{\mathcal{S}} - \eta_\varepsilon + b\varphi_\varepsilon + \eta^*) + ag. \tag{10.10}$$

First estimate. We test (10.3), (10.4) and (10.5) by μ_ε, $\partial_t\varphi_\varepsilon$ and η_ε, respectively. Adding the corresponding equations and integrating over Q_t, we obtain that

$$\int_{Q_t}|\nabla\mu_\varepsilon|^2+\ell\int_{Q_t}|\partial_t\varphi_\varepsilon|^2+\frac{1}{2}\int_\Omega|\nabla\varphi_\varepsilon(t)|^2+\int_\Omega\tilde\beta_\varepsilon(\varphi_\varepsilon(t))+\int_\Omega\tilde\pi(\varphi_\varepsilon(t))$$
$$+\frac{1}{2}\int_\Omega|\eta_\varepsilon(t)|^2+\int_{Q_t}|\nabla\eta_\varepsilon|^2+a\int_{Q_t}\rho\kappa_\varepsilon\eta_\varepsilon=\frac{1}{2}\int_\Omega|\nabla\varphi_0|^2+\int_\Omega\tilde\beta_\varepsilon(\varphi_0)|^2+\int_\Omega\tilde\pi(\varphi_0)$$
$$+\frac{1}{2}\int_\Omega|\eta_0|^2+\int_{Q_t}\mu_{\mathcal S}\partial_t\varphi_\varepsilon+b\int_{Q_t}\partial_t\varphi_\varepsilon\eta_\varepsilon+b\int_{Q_t}\nabla\varphi_\varepsilon\cdot\nabla\mu_\varepsilon-\int_{Q_t}\Delta\eta^*\eta_\varepsilon$$
$$-\gamma_3\int_{Q_t}|\eta_\varepsilon|^2p(\varphi_\varepsilon)+b\gamma_3\int_{Q_t}\varphi_\varepsilon\eta_\varepsilon p(\varphi_\varepsilon)+\gamma_3\int_{Q_t}\eta^*\eta_\varepsilon p(\varphi_\varepsilon)+\gamma_4a\int_{Q_t}\sigma_{\mathcal S}\eta_\varepsilon$$
$$-\gamma_4\int_{Q_t}|\eta_\varepsilon|^2+\gamma_4b\int_{Q_t}\varphi_\varepsilon\eta_\varepsilon+\gamma_4\int_{Q_t}\eta^*\eta_\varepsilon+a\int_{Q_t}g\eta_\varepsilon. \tag{10.11}$$

We observe that the first four terms on the right hand side of (10.11) are bounded, due to (3.3). Moreover, thanks to the monotonicity of the operator $\mathrm{Sign}_\varepsilon$, the last term on the left hand side of (3.3) is nonnegative. Finally, the other terms on the right hand side of (3.3) can be estimated using (3.1)–(3.4) and the Young inequality:

$$\int_{Q_t}\mu_{\mathcal S}\partial_t\varphi_\varepsilon\le\frac{\ell}{4}\int_{Q_t}|\partial_t\varphi_\varepsilon|^2+c, \tag{10.12}$$

$$b\int_{Q_t}\partial_t\varphi_\varepsilon\eta_\varepsilon\le\frac{\ell}{4}\int_{Q_t}|\partial_t\varphi_\varepsilon|^2+c\int_{Q_t}|\eta_\varepsilon|^2, \tag{10.13}$$

$$b\int_{Q_t}\nabla\varphi_\varepsilon\cdot\nabla\mu_\varepsilon\le\frac{1}{2}\int_{Q_t}|\nabla\mu_\varepsilon|^2+c\int_{Q_t}|\nabla\varphi_\varepsilon|^2, \tag{10.14}$$

$$\int_{Q_t}\Big(b\gamma_3p(\varphi_\varepsilon)+\gamma_4b\Big)\varphi_\varepsilon\eta_\varepsilon\le c\bigg(\int_{Q_t}|\varphi_\varepsilon|^2+\int_{Q_t}|\eta_\varepsilon|^2\bigg), \tag{10.15}$$

$$\int_{Q_t}\Big(-\Delta\eta^*+ag+\gamma_3p(\varphi_\varepsilon)(-\eta_\varepsilon+\eta^*)+\gamma_4(a\sigma_{\mathcal S}-\eta_\varepsilon+\eta^*)\Big)\eta_\varepsilon\le c\bigg(\int_{Q_t}|\eta_\varepsilon|^2+1\bigg), \tag{10.16}$$

whence, from (10.11) we infer that

$$\frac{1}{2}\int_{Q_t}|\nabla\mu_\varepsilon|^2+\frac{\ell}{4}\int_{Q_t}|\partial_t\varphi_\varepsilon|^2+\frac{1}{2}\int_\Omega|\nabla\varphi_\varepsilon(t)|^2+\int_\Omega\tilde\beta_\varepsilon(\varphi_\varepsilon(t))+\int_\Omega\tilde\pi(\varphi_\varepsilon(t))$$
$$+\frac{1}{2}\int_\Omega|\eta_\varepsilon(t)|^2+\int_{Q_t}|\nabla\eta_\varepsilon|^2\le c\bigg(\int_{Q_t}|\varphi_\varepsilon|^2+\int_{Q_t}|\nabla\varphi_\varepsilon|^2+\int_{Q_t}|\eta_\varepsilon|^2+1\bigg). \tag{10.17}$$

Now, we apply the Gronwall lemma and, by comparison in (10.3), we conclude that

$$\|\mu_\varepsilon\|_{L^2(0,T;W)}\le c, \tag{10.18}$$
$$\|\varphi_\varepsilon\|_{H^1(0,T;H)\cap L^\infty(0,T;V)}\le c, \tag{10.19}$$
$$\|\eta_\varepsilon\|_{L^\infty(0,T;H)\cap L^2(0,T;V)}\le c. \tag{10.20}$$

Second estimate. We derive (10.4) with respect to time. Then, we add the resultant equation tested by $\partial_t\varphi_\varepsilon$ with (10.3) tested by $\partial_t\mu_\varepsilon$. Integrating over Q_t, we obtain that

$$\begin{aligned}&\frac{1}{2}\int_\Omega|\nabla\mu_\varepsilon(t)|^2+\frac{\ell}{2}\int_\Omega|\partial_t\varphi_\varepsilon(t)|^2+\int_{Q_t}|\nabla\partial_t\varphi_\varepsilon|^2+\int_{Q_t}\beta'(\varphi_\varepsilon)|\partial_t\varphi_\varepsilon|^2\\&=\frac{1}{2}\int_\Omega|\nabla\mu_0|^2+\frac{\ell}{2}\int_\Omega|\partial_t\varphi_\varepsilon(0)|^2-\int_{Q_t}\pi'(\varphi_\varepsilon)|\partial_t\varphi_\varepsilon|^2-\int_{Q_t}\partial_t\mu_{\mathcal{S}}\partial_t\varphi_\varepsilon\\&\quad+\int_{Q_t}\Big(\frac{\gamma_1}{a}(\eta_\varepsilon-b\varphi_\varepsilon-\eta^*)-\gamma_2\Big)p(\varphi_\varepsilon)\partial_t\mu_\varepsilon.\end{aligned}\tag{10.21}$$

We observe that the first term on the right hand side of (10.21) is bounded due to (3.17). Then, we estimate each term of the right hand side of (10.21), separately: due to (10.19) and the Lipschitz continuity of π, we have that

$$-\int_{Q_t}\pi'(\varphi_\varepsilon)|\partial_t\varphi_\varepsilon|^2\le c\int_{Q_t}|\partial_t\varphi_\varepsilon|^2\le c.\tag{10.22}$$

Moreover, thanks to (3.17) and (10.19), using the Young inequality we obtain that

$$-\int_{Q_t}\partial_t\mu_{\mathcal{S}}\partial_t\varphi_\varepsilon\le\int_{Q_t}|\partial_t\mu_{\mathcal{S}}|^2+\int_{Q_t}|\partial_t\varphi_\varepsilon|^2\le c.\tag{10.23}$$

In order to estimate the second term on the right hand side of (10.21), we add (10.3) written for $t=0$ and tested by $\partial_t\varphi_\varepsilon(0)$ with (10.4) written for $t=0$ and tested by μ_0. Integrating the resultant equation over Ω, we obtain that

$$\begin{aligned}\frac{1}{2}\int_\Omega|\nabla\mu_0|^2+\ell\int_\Omega|\partial_t\varphi_\varepsilon(0)|^2&=\int_\Omega\Big(\Delta\mu_0-\xi_0-\pi(\varphi_0)-\mu_{\mathcal{S}}\Big)\partial_t\varphi_\varepsilon(0)\\&\quad+\int_\Omega\Big(\frac{\gamma_1}{a}(\eta_0-b\varphi_0-\eta^*)-\gamma_2\Big)p(\varphi_0)\mu_0.\end{aligned}\tag{10.24}$$

Due to (3.17), applying the Young inequality to every term on the right hand side of (10.24), we have that

$$\begin{aligned}&\int_\Omega\Big(\Delta\mu_0-\xi_0-\pi(\varphi_0)-\mu_{\mathcal{S}}\Big)\partial_t\varphi_\varepsilon(0)\\&\le\frac{\ell}{2}\int_\Omega|\partial_t\varphi_\varepsilon(0)|^2+c(1+\|\mu_0\|_W^2+\|\xi_0\|_H^2+\|\varphi_0\|_V^2)\le\frac{\ell}{2}\int_\Omega|\partial_t\varphi_\varepsilon(0)|^2+c,\end{aligned}\tag{10.25}$$

while

$$\begin{aligned}\int_\Omega\Big(\frac{\gamma_1}{a}(\eta_0-b\varphi_0-\eta^*)-\gamma_2\Big)p(\varphi_0)\mu_0&\le\frac{1}{4}\int_\Omega|\nabla\mu_0|^2+c(1+\|\eta_0\|_H^2+\|\varphi_0\|_H^2)\\&\le\frac{1}{4}\int_\Omega|\nabla\mu_0|^2+c.\end{aligned}\tag{10.26}$$

Combining (10.24) with (10.25)–(10.26) we infer that

$$\frac{\ell}{2}\int_\Omega|\partial_t\varphi_\varepsilon(0)|^2\le c,\tag{10.27}$$

whence, the second term on the right hand side of (10.21) is bounded. Finally, integrating by parts the last term on the the right hand side of (10.21), we infer that

$$\int_{Q_t} \Big(\frac{\gamma_1}{a}(\eta_\varepsilon - b\varphi_\varepsilon - \eta^*) - \gamma_2\Big) p(\varphi_\varepsilon)\partial_t\mu_\varepsilon = \int_\Omega \Big(\frac{\gamma_1}{a}(\eta_\varepsilon(t) - b\varphi_\varepsilon(t) - \eta^*) - \gamma_2\Big) p(\varphi_\varepsilon(t))\mu_\varepsilon(t)$$

$$- \int_\Omega \Big(\frac{\gamma_1}{a}(\eta_0 - b\varphi_0 - \eta^*) - \gamma_2\Big) p(\varphi_0)\mu_0 - \int_{Q_t} \frac{\gamma_1}{a}(\partial_t\eta_\varepsilon - b\partial_t\varphi_\varepsilon) p(\varphi_\varepsilon)\mu_\varepsilon$$

$$- \int_{Q_t} \Big(\frac{\gamma_1}{a}(\eta_\varepsilon - b\varphi_\varepsilon - \eta^*) - \gamma_2\Big) p'(\varphi_\varepsilon)\partial_t\varphi_\varepsilon\mu_\varepsilon. \tag{10.28}$$

Now, using (3.17), (10.18)–(10.20) and the Young inequality, the right hand side of (10.28) can be estimated as follows:

$$\int_\Omega \Big(\frac{\gamma_1}{a}(\eta_\varepsilon(t) - b\varphi_\varepsilon(t) - \eta^*) - \gamma_2\Big) p(\varphi_\varepsilon(t))\mu_\varepsilon(t) \le \frac{1}{4}\int_\Omega |\mu_\varepsilon(t)|^2 + c \le \frac{1}{4}\int_\Omega |\nabla\mu_\varepsilon(t)|^2, \tag{10.29}$$

$$- \int_\Omega \Big(\frac{\gamma_1}{a}(\eta_0 - b\varphi_0 - \eta^*) - \gamma_2\Big) p(\varphi_0)\mu_0 \le \|\mu_\varepsilon\|_H^2 + c(\|\eta_0\|_H^2 + \|\varphi_0\|_H^2) \le c, \tag{10.30}$$

$$- \int_{Q_t} \frac{\gamma_1}{a}(\partial_t\eta_\varepsilon - b\partial_t\varphi_\varepsilon) \le c\left(\int_{Q_t} |\partial_t\eta_\varepsilon|^2 + \int_{Q_t} |\partial_t\varphi_\varepsilon|^2\right) \le c, \tag{10.31}$$

$$- \int_{Q_t} \Big(\frac{\gamma_1}{a}(\eta_\varepsilon - b\varphi_\varepsilon - \eta^*) - \gamma_2\Big) p'(\varphi_\varepsilon)\partial_t\varphi_\varepsilon\mu_\varepsilon$$

$$\le c\int_{Q_t} \Big(\frac{\gamma_1}{a}(\eta_\varepsilon - b\varphi_\varepsilon - \eta^*) - \gamma_2\Big)^2 |\partial_t\varphi_\varepsilon|^2 + \int_{Q_t} |\mu_\varepsilon|^2 \le c + \int_{Q_t} |\mu_\varepsilon|^2. \tag{10.32}$$

Combining (10.29)–(10.32) with (10.28) and (10.21), we infer that

$$\frac{1}{4}\int_\Omega |\nabla\mu_\varepsilon(t)|^2 + \frac{\ell}{2}\int_\Omega |\partial_t\varphi_\varepsilon(t)|^2 + \int_{Q_t} |\nabla\partial_t\varphi_\varepsilon|^2 + \int_{Q_t} \beta'(\varphi_\varepsilon)|\partial_t\varphi_\varepsilon|^2$$

$$\le c\left(1 + \int_0^t \Big(\|\mu_\varepsilon(s)\|_V^2 + \|\partial_t\varphi_\varepsilon(s)\|_H^2\Big) ds\right), \tag{10.33}$$

whence, applying the Gronwall lemma, we conclude that

$$\|\varphi_\varepsilon\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)\cap L^\infty(0,T;V)} \le c, \tag{10.34}$$

and, by comparison in (10.3), we also infer that

$$\|\mu_\varepsilon\|_{L^\infty(0,T;W)} \le c. \tag{10.35}$$

Third estimate. We test (10.4) by ξ_ε. Integrating over Q_t, we obtain that

$$\int_{Q_t} |\xi_\varepsilon|^2 + \ell \int_\Omega \tilde{\beta}_\varepsilon(\varphi_\varepsilon(t)) + \int_{Q_t} \beta'_\varepsilon(\varphi_\varepsilon)|\nabla\varphi_\varepsilon|^2$$

$$= \ell \int_\Omega \tilde{\beta}_\varepsilon(\varphi_0) - \int_{Q_t} \pi(\varphi_\varepsilon)\xi_\varepsilon + \int_{Q_t} \mu_{\mathcal{S}}\xi_\varepsilon + \int_{Q_t} \mu_\varepsilon\xi_\varepsilon. \tag{10.36}$$

We notice that the second term on the left hand side of (10.36) is nonnegative, due to the monotonicity of β_ε. Moreover, the first term on the right hand side of (10.36) is bounded, due to (3.3), while the last three terms on the right hand side of (10.36) can be estimated using (10.18)–(10.20) and the Young inequality:

$$\begin{aligned} -\int_{Q_t} \pi(\varphi_\varepsilon)\xi_\varepsilon &= -\int_{Q_t} (\pi(\varphi_\varepsilon) - \pi(\varphi_0))\xi_\varepsilon - \int_{Q_t} \pi(\varphi_0)\xi_\varepsilon \\ &\leq C_\pi \int_{Q_t} |\varphi_\varepsilon - \varphi_0||\xi_\varepsilon| + \int_{Q_t} |\pi(\varphi_0)||\xi_\varepsilon| \\ &\leq \frac{1}{4}\int_{Q_t} |\xi_\varepsilon|^2 + c\int_{Q_t} |\varphi_\varepsilon|^2 + c \\ &\leq \frac{1}{4}\int_{Q_t} |\xi_\varepsilon|^2 + c. \end{aligned} \tag{10.37}$$

Besides, with an analogous technique, we obtain that

$$\int_{Q_t} \mu_{\mathcal{S}}\xi_\varepsilon \leq \frac{1}{4}\int_{Q_t} |\xi_\varepsilon|^2 + c\int_{Q_t} |\mu_{\mathcal{S}}|^2 \leq \frac{1}{4}\int_{Q_t} |\xi_\varepsilon|^2 + c, \tag{10.38}$$

$$\int_{Q_t} \mu_\varepsilon\xi_\varepsilon \leq \frac{1}{4}\int_{Q_t} |\xi_\varepsilon|^2 + c\int_{Q_t} |\mu_\varepsilon|^2 \leq \frac{1}{4}\int_{Q_t} |\xi_\varepsilon|^2 + c. \tag{10.39}$$

Due to (10.37)–(10.39), from (10.36) we obtain that

$$\frac{1}{4}\int_{Q_t} |\xi_\varepsilon|^2 + \ell\int_\Omega \tilde{\beta}_\varepsilon(\varphi_\varepsilon(t)) \leq c, \tag{10.40}$$

whence we conclude that

$$\|\xi_\varepsilon\|_{L^2(0,T;H)} \leq c. \tag{10.41}$$

Finally, by comparison in (10.4), we obtain that

$$\|\varphi_\varepsilon\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \leq c. \tag{10.42}$$

Fourth estimate. We fix $t \in (0,T)$ and test (10.4) by $-\Delta\varphi_\varepsilon(t)$. Integrating the resultant equation over Ω, we obtain that

$$\int_\Omega |\Delta\varphi_\varepsilon(t)|^2 + \int_\Omega \beta'_\varepsilon(\varphi_\varepsilon(t))|\nabla\varphi_\varepsilon(t)|^2 = \int_\Omega \nabla\mu_\varepsilon(t)\cdot\nabla\varphi_\varepsilon(t) - \int_\Omega \pi'(\varphi_\varepsilon(t))|\nabla\varphi_\varepsilon(t)|^2$$

$$+ \int_\Omega \mu_{\mathcal{S}}(t)\Delta\varphi_\varepsilon(t) + \ell\int_\Omega \partial_t\varphi_\varepsilon(t)\Delta\varphi_\varepsilon(t). \tag{10.43}$$

We observe that the second integral of the left hand side of (10.43) is nonnegative, due to the monotonicity of β'_ε. Moreover, due to (10.35) and (10.42), applying the Young inequality to every term on the right hand side of (10.43), we have that

$$\int_\Omega \nabla\mu_\varepsilon(t)\cdot\nabla\varphi_\varepsilon(t) \leq \|\mu_\varepsilon\|^2_{L^\infty(0,T;V)} + \|\varphi_\varepsilon\|^2_{L^\infty(0,T;V)} \leq c, \tag{10.44}$$

$$-\int_\Omega \pi'(\varphi_\varepsilon(t))|\nabla\varphi_\varepsilon(t)|^2 \leq c\|\varphi_\varepsilon\|^2_{L^\infty(0,T;V)} \leq c. \tag{10.45}$$

Besides, we have that

$$\int_\Omega \mu_{\mathcal{S}}(t)\Delta\varphi_\varepsilon(t) \leq \frac{1}{4}\int_\Omega |\Delta\varphi_\varepsilon(t)|^2 + c\|\mu_{\mathcal{S}}\|^2_{L^\infty(0,T;V)} \leq \frac{1}{4}\int_\Omega |\Delta\varphi_\varepsilon(t)|^2 + c, \tag{10.46}$$

$$\ell\int_\Omega \partial_t\varphi_\varepsilon(t)\Delta\varphi_\varepsilon(t) \leq \frac{1}{4}\int_\Omega |\Delta\varphi_\varepsilon(t)|^2 + c\|\varphi_\varepsilon\|^2_{W^{1,\infty}(0,T;H)} \leq \frac{1}{4}\int_\Omega |\Delta\varphi_\varepsilon(t)|^2 + c. \tag{10.47}$$

Combining (10.43) with (10.44)–(10.47), we obtain that

$$\frac{1}{2}\int_\Omega |\Delta\varphi_\varepsilon(t)| \leq c, \tag{10.48}$$

whence we infer that

$$\|\varphi_\varepsilon\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)\cap L^\infty(0,T;W)} \leq c. \tag{10.49}$$

Finally, due to (10.18)–(10.20), (10.35) and (10.49), by comparison in (10.10), we conclude that

$$\|w_\varepsilon\|_{L^\infty(0,T;H)} \leq c. \tag{10.50}$$

Fifth estimate. We test (10.5) by $\partial_t\eta_\varepsilon$. Integrating over Q_t, we obtain that

$$\int_{Q_t} |\partial_t\eta_\varepsilon|^2 + \frac{1}{2}\int_\Omega |\nabla\eta_\varepsilon(t)|^2 + a\rho\int_{Q_t} \kappa_\varepsilon\partial_t\eta_\varepsilon = \frac{1}{2}\int_\Omega |\nabla\eta_0|^2 + \int_{Q_t} w_\varepsilon\partial_t\eta_\varepsilon. \tag{10.51}$$

The first term on the right hand side of (10.51) is bounded thanks to (3.17), while the second term can be estimated using (10.49)–(10.50) and the Young inequality:

$$\int_{Q_t} w_\varepsilon\partial_t\eta_\varepsilon \leq \frac{1}{2}\int_{Q_t} |\partial_t\eta_\varepsilon|^2 + c\int_{Q_t} |w_\varepsilon|^2 \leq \frac{1}{2}\int_{Q_t} |\partial_t\eta_\varepsilon|^2 + c. \tag{10.52}$$

Finally, due to the properties of the operator $\mathrm{Sign}_\varepsilon$ stated by (2.12) and (4.7), the second term on the left hand side of (10.51) can be rewritten as

$$a\rho\int_{Q_t} \kappa_\varepsilon\partial_t\eta_\varepsilon = a(\rho\|\eta_\varepsilon(t)\|_H - \rho\|\eta_0\|_H). \tag{10.53}$$

Combining (10.51) with (10.52)–(10.53), we have that

$$\frac{1}{2}\int_{Q_t} |\partial_t\eta_\varepsilon|^2 + \frac{1}{2}\int_\Omega |\nabla\eta_\varepsilon(t)|^2 + a\rho\|\eta_\varepsilon(t)\|_H \leq c(1+\rho), \tag{10.54}$$

whence

$$\|\eta_\varepsilon\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \le c(1+\rho^{1/2}). \tag{10.55}$$

Then, combining (10.34)–(10.35) with (10.55), we conclude that

$$\partial_t\eta_\varepsilon - \Delta\eta_\varepsilon + \rho\kappa_\varepsilon = w_\varepsilon, \tag{10.56}$$

with

$$\|w_\varepsilon\|_{L^\infty(0,T;H)} \le c(1+\rho^{1/2}). \tag{10.57}$$

10.2 Existence of sliding mode

In order to prove the existence of sliding mode, we fix the constant c appearing in (10.57) and set

$$\rho^* := c^2 + 2c + \frac{2}{T}\|a\sigma_0 + b\varphi_0 + \eta^*\|_H \tag{10.58}$$

and assume $\rho > \rho^*$. We also set

$$\psi_\varepsilon(t) := \|\eta_\varepsilon(t)\|_H \quad \text{for } t \in [0,T]. \tag{10.59}$$

By assuming $h \in (0,T)$ and $t \in (0, T-h)$, we multiply (10.56) by $\kappa_\varepsilon = \mathrm{Sign}_\varepsilon(\eta_\varepsilon)$ and integrate over $(t, t+h)\times\Omega$. We have that

$$\int_t^{t+h} (\partial_t\eta_\varepsilon(s), \kappa_\varepsilon(s))_H \, ds + \int_t^{t+h}\int_\Omega \nabla\eta_\varepsilon\cdot\nabla\kappa_\varepsilon + \rho\int_t^{t+h} \|\kappa_\varepsilon(s)\|_H^2 \, ds$$
$$= \int_t^{t+h} (w_\varepsilon(s), \kappa_\varepsilon(s))_H \, ds. \tag{10.60}$$

Recalling that $\mathrm{Sign}_\varepsilon(v)$ is the gradient at v of the C^1 functional $\|\cdot\|_{H,\varepsilon}$, from (4.7)–(4.8) we deduce that

$$(\partial_t\eta_\varepsilon(s), \kappa_\varepsilon(s))_H = \frac{d}{dt}\int_0^{\psi_\varepsilon(t)} \min\{s/\varepsilon,\ 1\}\, ds \quad \text{for a.a. } t\in(0,T).$$

Then, for the first term on the left hand side of (10.60) we have that

$$\int_t^{t+h} (\partial_t\eta_\varepsilon(s), \kappa_\varepsilon(s))_H \, ds = \int_{\psi_\varepsilon(t)}^{\psi_\varepsilon(t+h)} \min\{s/\varepsilon, 1\}\, ds.$$

We also notice that (4.8) implies that

$$\nabla\eta_\varepsilon(t)\cdot\nabla\kappa_\varepsilon(t) = \frac{|\nabla\eta_\varepsilon(t)|^2}{\max\{\varepsilon, \|\eta_\varepsilon(t)\|_H\}} \ge 0 \quad \text{a.e. in } \Omega, \text{ for a.e. } t\in(0,T),$$

whence the second integral on the left hand side of (10.60) is nonnegative. Moreover, as $\|\kappa_\varepsilon(s)\|_H \le 1$ for every s (see (2.13)) and (10.57) holds, we infer from (10.60) that

$$\int_{\psi_\varepsilon(t)}^{\psi_\varepsilon(t+h)} \min\{s/\varepsilon, 1\}\, ds + \rho\int_t^{t+h} \|\kappa_\varepsilon(s)\|_H^2 \, ds \le hc(\rho^{1/2}+1). \tag{10.61}$$

At this point, we let $\varepsilon \searrow 0$. Due to the strong convergences of σ_ε and φ_ε ensured by (10.18)–(10.20) and by [43, Lemma 8, p. 84], at least for a subsequence, we have that

$$\eta_\varepsilon \to \eta \qquad \text{in } C^0([0,T];H). \tag{10.62}$$

Besides, using standard weak, weakstar and compactness results, from (10.61) we infer that

$$\kappa_\varepsilon \rightharpoonup^* \kappa \qquad \text{in } L^\infty(0,T;H). \tag{10.63}$$

Then, taking the limit as $\varepsilon \searrow 0$ in (10.61) and denoting by

$$\psi(t) := \|\eta(t)\|_H \qquad \text{for } t \in [0,T], \tag{10.64}$$

we obtain that

$$\psi(t+h) - \psi(t) + \rho \int_t^{t+h} \|\kappa(s)\|_H^2 \, ds$$

$$\leq \lim_{\varepsilon \searrow 0} \int_{\psi_\varepsilon(t)}^{\psi_\varepsilon(t+h)} \min\{s/\varepsilon, 1\} \, ds + \rho \liminf_{\varepsilon \searrow 0} \int_t^{t+h} \|\kappa_\varepsilon(s)\|_H^2 \, ds \leq hc(\rho^{1/2}+1) \tag{10.65}$$

for every $h \in (0,T)$ and $t \in (0, T-h)$. Finally, we multiply (10.65) by $1/h$ and let h tend to zero. We conclude that

$$\psi'(t) + \rho\|\kappa(t)\|_H^2 \leq c(\rho^{1/2}+1) \quad \text{for a.a. } t \in (0,T). \tag{10.66}$$

As $\|\kappa(t)\|_H = 1$ if $\|\eta(t)\|_H > 0$ (see (2.13)), we can apply Lemma 10.1 with $a_0 = b_0 = c$ and we observe that our condition $\rho > \rho^*$ completely fits the assumptions by (10.58). Thus, we find $T^* \in [0,T)$ such that $\eta(t) = 0$ for every $t \in [T^*, T]$.

Acknowledgments

This work does not have any conflicts of interest. Partial support from the GNAMPA (Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni) of INdAM is acknowledged.

References

[1] Barbu, V: *Nonlinear differential equations of monotone types in Banach spaces*, Springer, New York, 2010.

[2] Barbu, V, Colli, P, Gilardi, G., Marinoschi, G, Rocca, E: Sliding mode control for a nonlinear phase-field system. SIAM J. Control Optim., **55**, 2108–2133 (2017).

[3] Bartolini, G., Fridman, L., Pisano, A., Usai, E. (eds.): *Modern Sliding Mode Control Theory. New Perspectives and Applications. Lecture Notes in Control and Information Sciences*, vol. 375. Springer, New York (2008).

[4] Bonetti, E., Colli, P., Tomassetti, G.: A non-smooth regularization of a forward-backward parabolic equation. Math. Models Methods Appl. Sci. **27**, 641–661 (2017).

[5] H. Brezis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland Math. Stud., vol. 5, North-Holland, Amsterdam, 1973.

[6] Byrne, H.M., Chaplain, M.A.J.: Growth of nonnecrotic tumors in the presence and absence of inhibitors. Math. Biosci. **130**, 151–181 (1995).

[7] Cheng, M. B., Radisavljevic, V., Su,W. C.: Sliding mode boundary control of a parabolic PDE system with parameter variations and boundary uncertainties. Autom. J. IFAC **47**, 381–387 (2011).

[8] Colli, P., Colturato, M. : Global existence for a singular phase field system related to a sliding mode control problem, Nonlinear Anal. Real World Appl. **41**, 128–151 (2018).

[9] Colli, P., Fukao, T.: Cahn–Hilliard equation with dynamic boundary conditions and mass constraint on the boundary, J. Math. Anal. Appl. **429**, 1190–1213 (2015).

[10] Colli, P., Gilardi, G., Hilhorst, D.: On a Cahn-Hilliard type phase field model related to tumor growth. Discret. Contin. Dyn. Syst. **35**, 2423–2442 (2015).

[11] Colli, P., Gilardi, G., Marinoschi, G., Rocca, E., Sliding mode control for a phase-field system related to tumor growth, Appl. Math. Optim. **79**, 647–670 (2019).

[12] Colli, P., Gilardi, G., Rocca, E., Sprekels, J.:Vanishing viscosities and error estimate for a Cahn-Hilliard type phase-field system related to tumor growth, Nonlinear Anal. RealWorld Appl. **26**, 93–108 (2015).

[13] Colli, P., Gilardi, G., Rocca, E., Sprekels, J.: Asymptotic analyses and error estimates for a Cahn- Hilliard type phase field system modelling tumor growth. Discret. Contin. Dyn. Syst. Ser. S. **10**, 37–54 (2017).

[14] Colli, P., Gilardi, G., Rocca, E., Sprekels, J.: Optimal distributed control of a diffuse interface model of tumor growth. Nonlinearity **30**, 2518–2546 (2017).

[15] Colli, P., Manini, D.: Sliding mode control for a generalization of the Caginalp phase-field system, preprint arXiv:1911.01481 [math.AP], 1–26 (2019).

[16] Colturato, M.: Solvability of a class of phase field systems related to a sliding mode control problem. Appl. Math. **6**, 623–650 (2016).

[17] Colturato, M.: On a class of conserved phase field systems with a maximal monotone perturbation. Appl. Math. Optim. **4**, 1–35 (2017).

[18] Colturato, M.: Well-posedness and longtime behavior for a singular phase field system with perturbed phase dynamics, EECT **7** , 217–245 (2018).

[19] Cristini, V., Lowengrub, J.: *Multiscale Modeling of Cancer. An Integrated Experimental and Mathematical Modeling Approach*, Cambridge University Press, Cambridge (2010).

[20] Cristini, V., Li, X., Lowengrub, J.S., Wise, S.M.: Nonlinear simulations of solid tumor growth using a mixture model: invasion and branching. J. Math. Biol. **58**, 723–763 (2009).

[21] Dai, M., Feireisl, E., Rocca, E., Schimperna, G., Schonbek, M.: Analysis of a diffuse interface model for multispecies tumor growth. Nonlinearity **30**, 1639–1658 (2017).

[22] Edwards, C., Spurgeon, S.: *Sliding Mode Control: Theory and Applications*, Taylor and Francis, London (1999).

[23] Edwards, C., Colet, E., Fridman, L. (eds.): *Advances in Variable Structure and Sliding Mode Control. Lecture Notes in Control and Information Sciences*, vol. 334. Springer, New York (2006).

[24] Fridman, L., Moreno, J., Iriarte, R. (eds.): *Sliding Modes After the First Decade of the 21st Century: State of the Art. Lecture Notes in Control and Information Sciences*, vol. 412. Springer, New York (2011).

[25] Frigeri, S., Grasselli, M., Rocca, E.: On a diffuse interface model of tumor growth. Eur. J. Appl.Math. **26**, 215–243 (2015).

[26] Frigeri, S., Lam, K.F., Rocca, E.: On a diffuse interface model for tumour growth with non-local interactions and degenerate mobilities, to appear. In: Colli, P., Favini, A., Rocca, E., Schimperna, G., Sprekels, J. (eds.) Solvability, Regularity, Optimal Control of Boundary Value Problems for PDEs. Springer INdAM Series, Springer, Milan, 1–28 (2017).

[27] Garcke, H., Lam, K.F.: Global weak solutions and asymptotic limits of a Cahn-Hilliard-Darcy system modelling tumour growth. AIMS Math. **1**, 318–360 (2016).

[28] Garcke, H., Lam, K.F.: Analysis of a Cahn-Hilliard system with non zero Dirichlet conditions modelling tumour growth with chemotaxis. Discret. Contin. Dyn. Syst. **37**, 4277–4308 (2017).

[29] Garcke, H., Lam, K.F.: Well-posedness of a Cahn-Hilliard system modelling tumour growth with chemotaxis and active transport. Eur. J. Appl. Math. **28**, 284–316 (2017).

[30] Garcke, H., Lam, K.F., Sitka, E., Styles, V.: A Cahn-Hilliard-Darcy model for tumour growth with chemotaxis and active transport. Math. Models Methods Appl. Sci. **26**, 1095–1148 (2016).

[31] Garcke, H., Lam, K.F., Nrnberg, R., Sitka, E.: A multiphase Cahn-Hilliard-Darcy model for tumour growth with necrosis, 1–43 (2017).

[32] Garcke, H., Lam, K.F., Rocca, E.: Optimal control of treatment time in a diffuse interface model for tumor growth. Appl. Math. Optim. (2017).

[33] Hawkins-Daarud, A., Van Der Zee, K.G., Oden, J.T.: Numerical simulation of a thermodynamically consistent four-species tumor growth model. Int. J. Numer. Methods Biomed. Eng. **28**, 3–24 (2012).

[34] Hilhorst, D., Kampmann, J., Nguyen, T.N., Van Der Zee, K.G.: Formal asymptotic limit of a diffuseinterface tumor-growth model. Math. Models Methods Appl. Sci. **25**, 1011–1043 (2015).

[35] Itkis, U.: *Control Systems of Variable Structure*, Wiley, New York (1976).

[36] Levaggi, L.: Infinite dimensional systems' sliding motions. Eur. J. Control **8**, 508–516 (2002).

[37] Levaggi, L.: Existence of sliding motions for nonlinear evolution equations in Banach spaces, Discrete Contin. Dyn. Syst. In: 9th AIMS Conference on Dynamical Systems, Differential Equations and Applications, 477–487 (2013).

[38] Melchionna, S., Rocca, E.: Varifold solutions of a sharp interface limit of a diffuse interface model for tumor growth, Interfaces and Free Bound, 1–29 (2017).

[39] Orlov, Y.V.: Application of Lyapunov method in distributed systems. Autom. Remote Control **44**, 426–430 (1983).

[40] Orlov, Y.V., Utkin, V.I.: Sliding mode control in indefinite-dimensional systems. Autom. J. IFAC **23**, 753–757 (1987).

[41] Orlov, Y.V., Utkin, V.I.: Unit sliding mode control in infinite-dimensional systems. Adaptive learning and control using sliding modes. Appl. Math. Comput. Sci. **8**, 7–20 (1998).

[42] Rocca, E., Scala, R.: A rigorous sharp interface limit of a diffuse interface model related to tumor growth. J. Nonlinear Sci. **27**, 847–872 (2017).

[43] J. Simon, Compact sets in the spaces $L^p(0,T;B)$, Ann. Mat. Pura. Appl. (4) **146**, 65–96 (1987).

[44] Utkin, V.: *Sliding Modes in Control and Optimization*, Springer, Berlin (1992).

[45] Utkin, V., Guldner, J., Shi, J.: *Sliding Mode Control in Electro-Mechanical Systems. Automation and Control Engineering*, CRC Press, Boca Raton (2009).

[46] Xing, H., Li, D., Gao, C., Kao, Y.: Delay-independent sliding mode control for a class of quasi-linear parabolic distributed parameter systems with time-varying delay. J. Franklin Inst. **350**, 397–418 (2013).

[47] Wise, S.M., Lowengrub, J.S., Frieboes, H.B., Cristini, V.: Three-dimensional multispecies nonlinear tumor growth-I: model and numerical method. J. Theor. Biol. **253**, 524–543 (2008).

[48] Young, K.D., zgner, . (eds.): *Variable Structure Systems, Sliding Mode and Nonlinear Control*, Springer, New York (1999).