

BLOW-UP ANALYSIS OF CONFORMAL METRICS OF THE DISK WITH PRESCRIBED GAUSSIAN AND GEODESIC CURVATURES

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ABSTRACT. This paper is concerned with the compactness of metrics of the disk with prescribed Gaussian and geodesic curvatures. We consider a blowing-up sequence of metrics and give a precise description of its asymptotic behavior. In particular, the metrics blow-up at a unique point on the boundary and we are able to give necessary conditions on its location. It turns out that such conditions depend locally on the Gaussian curvatures but they depend on the geodesic curvatures in a nonlocal way. This is a novelty with respect to the classical Nirenberg problem where the blow-up conditions are local, and this new aspect is driven by the boundary condition.

1. INTRODUCTION

Let (Σ, g) denote a compact surface Σ equipped with a certain metric g . The classical Kazdan-Warner problem (see [5, 32]) consists of determining whether a prescribed function K is the Gaussian curvature of a new metric \tilde{g} conformal to g . If $\tilde{g} = e^u g$ and K, K_g are respectively the Gaussian curvatures relative to these metrics, then the following relation holds:

$$(1.1) \quad -\Delta_g u + 2K_g(x) = 2K(x)e^u \quad \text{in } \Sigma.$$

Hence the Kazdan-Warner problem reduces to solving this equation.

The problem of prescribing the Gaussian curvature on the standard sphere \mathbb{S}^2 is particularly delicate and receives the name of Nirenberg problem. This question has been addressed by a large number of papers (see for instance [1, 8, 10, 11, 13, 14, 16, 30, 27, 28, 26, 42, 45]), some of whom are commented below. For instance, an obstruction to existence was found by Kazdan and Warner ([32]). On the other hand, if K has antipodal symmetry (i.e. $K(-x) = K(x)$) and it is somewhere positive then there exists a solution, see [40]. Other results under symmetry assumptions are given in [13]. In [42] solutions of (1.1) have been found as the asymptotic limit of a suitable flow. This problem is also related to a hyperbolic version of the Christoffel problem, see [22].

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A related and significant issue in this kind of problems is the study of compactness of solutions, starting from [7, 36]. Roughly speaking, given $\{u_n\}$ a sequence of solutions, we desire to find conditions that allow us to pass to the limit. By regularity, it is enough to show L^∞ boundedness. This problem is typically studied by means of a blow-up analysis, which determines whether the sequence u_n is uniformly bounded or may blow-up. After rescaling, a blowing-up sequence of solutions resembles locally a limit entire solution, and such solutions are classified in [15]. In particular this implies a quantization result for general Liouville equations, see [36].

Being more specific, let us consider the problem:

$$(1.2) \quad -\Delta u_n + 2 = 2K_n(x)e^{u_n} \quad \text{in } \mathbb{S}^2,$$

where K_n converges in C^2 sense to a strictly positive function $K(x)$. Observe that if $K_n = 1$, the problem is invariant by the group of conformal maps of the sphere, which is not compact. In the non-constant case, this invariance is lost but concentration of solutions may still occur. This is the so-called ‘‘bubbling phenomena’’; the solutions concentrate all their mass around a certain singular point.

Not only the asymptotic behavior of the sequence is relevant, but also the location of the point of concentration. It has been shown ([12, 8]) that if the sequence u_n is blowing-up, then the point of concentration satisfies:

$$(1.3) \quad \nabla K(p) = 0, \quad \Delta K(p) = 0.$$

As a consequence, if those conditions are never satisfied for any point $p \in \mathbb{S}^2$, one concludes compactness for (1.2). This result is also a key point in the proof of existence of solutions for the Nirenberg problem, which was given for the first time in [10] via a variational argument and revisited under a different approach in [30].

If Σ is a surface with boundary, one usually imposes boundary conditions. A natural geometric problem consists in prescribing also the geodesic curvature of the boundary; in this way we are led to the problem:

$$(1.4) \quad \begin{cases} -\Delta u + 2K_g = 2Ke^u & \text{in } \Sigma, \\ \frac{\partial u}{\partial \nu} + 2h_g = 2he^{u/2} & \text{on } \partial\Sigma, \end{cases}$$

where ν is the outward normal vector to $\partial\Sigma$. Here h_g is the geodesic curvature of $\partial\Sigma$ under the metric g , and h is the geodesic curvature to be prescribed for the new metric $\tilde{g} = e^u g$.

In the literature there are some results on the latter problem. The case of constant K, h has been considered in [6], where the author used a parabolic flow to obtain solutions in the limit. Some classification results for the half-plane are also available in [23, 37, 46]. The case of nonconstant curvatures was addressed for the first time in [17], but the results there are partial and in some of them an unknown Lagrange multiplier appears in the equation.

In [39] the case of surfaces topologically different from the disk is studied when $K < 0$. In that paper a new type of blow-up phenomenon appears, where length and area diverge. In the existence results the authors exploit the variational formulation of the problem and solutions are obtained by minimization and min-max techniques. For more general mean field problems with boundary terms a quantization result has been recently given in [4].

The case of the disk $\Sigma = \mathbb{D}$ can be seen as a natural generalization of the Nirenberg problem. Indeed, the effect of the noncompact group of conformal maps of the disk plays a fundamental role. The problem becomes:

$$(1.5) \quad \begin{cases} -\Delta u = 2Ke^u & \text{in } \mathbb{D}, \\ \frac{\partial u}{\partial \nu} + 2 = 2he^{u/2} & \text{on } \partial\mathbb{D}. \end{cases}$$

Integrating (1.5) and applying the Gauss-Bonnet Theorem, one obtains

$$(1.6) \quad \int_{\mathbb{D}} Ke^u + \int_{\partial\mathbb{D}} he^{u/2} = 2\pi,$$

which implies that K or h must be positive somewhere.

Some works have also addressed problem (1.5). For example, the case $h = 0$ has been treated in [11] (see also [24]), while the case $K = 0$ has been considered in [9, 35, 38]. If $K = 0$ a blow-up analysis has been performed in [25], and in [19] a new approach is given under mild conditions on the function h .

Up to our knowledge, the only result on (1.5) for non-constant curvatures is [18]. In that paper an existence result for (1.5) with positive symmetric curvatures is given, in the spirit of the aforementioned result of Moser ([40]).

The aim of this paper is to provide a complete blow-up analysis of problem (1.5) for non-constant functions K, h . We emphasize that we do not impose any sign restriction on the functions K and h . We are interested in the precise asymptotic behavior of blowing-up solutions, but also on the location of the point of concentration, which turns out to be located on $\partial\mathbb{D}$. Indeed, we shall find an analogue of the conditions in (1.3) for equation (1.5). Such conditions involve both curvatures K and h , and studying their interaction is the original motivation of this work. It turns out that the blow-up point p depends on K in a local way but it depends on h in a nonlocal fashion. Indeed the $1/2$ laplacian of h appears which, as is well known, depends on all values of h on $\partial\mathbb{D}$. This was rather unexpected, at least for us, since the original problem is local in nature.

Being more specific, our main result is the following:

Theorem 1.1. *Let u_n be a sequence of solutions of the problem*

$$(1.7) \quad \begin{cases} -\Delta u_n = 2K_n e^{u_n} & \text{in } \mathbb{D}, \\ \frac{\partial u_n}{\partial \nu} + 2 = 2h_n e^{u_n/2} & \text{on } \partial\mathbb{D}, \end{cases}$$

where $K_n \rightarrow K$ in $C^2(\overline{\mathbb{D}})$ and $h_n \rightarrow h$ in $C^2(\partial\mathbb{D})$ as $n \rightarrow +\infty$. Assume moreover that u_n is blowing-up, namely, $\sup\{u_n\} \rightarrow +\infty$, with bounded global mass,

$$(1.8) \quad \int_{\mathbb{D}} e^{u_n} + \int_{\partial\mathbb{D}} e^{u_n/2} \leq C.$$

Then there exists a unique point $p \in \partial\mathbb{D}$ such that

- i) If $K(p) \leq 0$ then $h(p) > \sqrt{-K(p)}$.
- ii) There exists $a_n \rightarrow p$ so that

$$u_n(z) = u_{a_n}(z) + \psi_n(z),$$

$$u_{a_n}(z) := 2 \log \left\{ \frac{2\hat{\phi}_n(1 - |a_n|^2)}{\hat{\phi}_n^2 |1 - \bar{a}_n z|^2 + \hat{k}_n |z - a_n|^2} \right\},$$

with

$$\hat{\phi}_n := \phi_n \left(\frac{a_n}{|a_n|} \right), \quad \hat{k}_n := K_n \left(\frac{a_n}{|a_n|} \right),$$

where

$$(1.9) \quad \phi_n(z) := h_n(z) + \sqrt{h_n(z)^2 + K_n(z)}.$$

Moreover, the error term ψ_n satisfies:

$$\|\psi_n\|_{C^{0,\alpha}(\mathbb{D})} \rightarrow 0 \text{ for any } \alpha \in (0, 1/2).$$

iii) Let H denote the harmonic extension of h , that is,

$$\begin{cases} \Delta H = 0 & \text{in } \mathbb{D}, \\ H = h & \text{on } \partial\mathbb{D}. \end{cases}$$

Define:

$$(1.10) \quad \Phi(z) := H(z) + \sqrt{H(z)^2 + K(z)},$$

which, by *i*), is positive and well defined at least in a neighborhood of p . Then:

$$(1.11) \quad \nabla\Phi(p) = 0.$$

Let us observe that condition (1.11) above is equivalent to:

$$h_\tau(p) = -\frac{K_\tau(p)}{2\Phi(p)}, \quad (-\Delta)^{1/2}h(p) = -\frac{K_\nu(p)}{2\Phi(p)}.$$

Here τ denotes the tangent vector to $\partial\mathbb{D}$ at p . As a consequence we obtain a compactness criterion: if such a point p does not exist, then the sequence u_n is bounded from above and it is precompact. As mentioned before, this criterion involves local terms on K , h , but also the half-laplacian of h at the blow-up point.

Observe that if $K = 0$, then $h(p) > 0$ and condition *iii*) reads as $h_\tau(p) = 0$, $(-\Delta)^{1/2}h(p) = 0$ (in other words, $\nabla H(p) = 0$). Instead, if $h = 0$ then $K(p) > 0$ and we obtain that $\nabla K(p) = 0$. Even these particular results were not known in the literature.

The proof of Theorem 1.1 involves a quite detailed blow-up analysis. Indeed one needs good global estimates on the blow-up sequence, and not only local estimates around the blow-up point. By a suitable rescaling, we can pass to a limit problem posed in a half-plane, whose solutions have been classified. But for a global estimate one needs to make use of the conformal group of the disk and make this approach match with the previous rescaling argument.

In our proofs we have to bypass several technical difficulties since no assumption on the sign of K , h is made; the possible compensation of terms gives some troubles when passing to the limit. In general this can be very problematic, see [20, 21]. We are able to overcome these difficulties by exploiting the finite mass assumption and using some ideas from [39].

Remarks 1.2.

- (1) Condition (1.8) has a clear geometric interpretation: both the area and the length are assumed to be bounded. This hypothesis is necessary for Theorem 1.1 to hold. Otherwise, a different phenomenon blow-up could appear, as has been shown in [39] for $K < 0$.

Observe that if $K > 0$ and $h > 0$, assumption (1.8) is automatically satisfied by (1.6). We are able to give other conditions ensuring (1.8), see Proposition 6.2. This proposition is a consequence of the isoperimetric inequality and a classical length bound due to Toponogov ([44]). Indeed we will show that (1.8) holds if one of the curvatures is identically equal to 0 and the other one is strictly positive.

- (2) As commented above, condition (1.11) is an analogue of (1.3) for problem (1.7). It is to be expected that one can use it to give existence results for equation (1.5). This will be pursued in a future work.
- (3) Condition (1.11) admits a variational interpretation, see Remark 5.4.

The rest of the paper is organized as follows. In Section 2 we collect and prove some preliminary results. In particular the limit problems in the disk, in the plane and in the half-plane are considered, together with a Pohozaev-type identity. In Section 3 we start the blow-up analysis and we prove *i*) of the Theorem 1.1. This analysis is refined in Section 4 by giving a precise description of the asymptotic behavior of u_n : namely, assertion *ii*) of Theorem 1.1. In Section 5 we give the proof of condition *iii*) in Theorem 1.1. A final Appendix contains some asymptotic computations and also Proposition 6.2, where conditions for the validity of (1.8) are given. 6.

Notations.

Let us fix some notations. The metric distance between two points $z_1, z_2 \in \overline{\mathbb{D}}$ will be written as $\text{dist}(z_1, z_2)$. We will denote an open ball centered at a point $p \in \overline{\mathbb{D}}$ of radius $r > 0$ as

$$B_r(p) := \{z \in \overline{\mathbb{D}} : \text{dist}(z, p) < r\}.$$

At any point $z = (x, y) \in \partial\mathbb{D}$ we fix a tangent vector $\tau(x, y) = (-y, x)$ or, in complex notation, $\tau(z) = iz$. We will use the following notation for some subsets of $B_r(p) \subset \mathbb{R}^2$:

$$\begin{aligned} B_r^+(p) &:= \{z = (x, y) \in \mathbb{R}^2 : \text{dist}(z, p) < r, y \geq 0\}; \\ \Gamma_r(p) &:= \{(x, y) \in \partial B_r^+(p) : y = 0\}; \\ \partial^+ B_r(p) &:= \partial B_r^+(p) \setminus \Gamma_r(p). \end{aligned}$$

The same notions will be used in \mathbb{R}^2 with its underlying metric.

In the estimates we will denote C as a positive constant, independent of the parameters, that may vary from line to line. If some dependence respect to certain parameters must be pointed out, we will indicate it in the subscript, such as C_ε or $C_{\varepsilon, \delta}$.

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2. PRELIMINARIES

In this section we collect and derive some useful results which will be used later on.

2.1. The limit problem in the disk. We are devoted to the properties of the limit problem:

$$(2.1) \quad \begin{cases} -\Delta v = 2K_0 e^v & \text{in } \mathbb{D}, \\ \frac{\partial v}{\partial \nu} + 2 = 2h_0 e^{v/2} & \text{on } \partial\mathbb{D}, \end{cases}$$

where K_0, h_0 are real constants. The content of this section is rather known, but we have not been able to find a specific reference.

Let us introduce the group of conformal maps of the disk, namely:

$$(2.2) \quad \mathcal{G} := \{e^{i\theta} f_a : \mathbb{D} \rightarrow \mathbb{D}; \theta \in [0, 2\pi), f_a(z) = \frac{a+z}{1+\bar{a}z}, a \in \mathbb{D}\}.$$

It is well-known that problem (2.1) is conformally invariant, i.e., given a solution v of (2.1) and $f \in \mathcal{G}$, then:

$$v_f(z) := v(f(z)) + 2 \log |f'(z)|,$$

is also a solution of the same problem. In the next lemma we show that the unique solutions of (2.1) are those coming from conformal maps from the disk to the standard surfaces with constant curvatures.

Lemma 2.1. *Problem (2.1) admits a solution if and only if:*

$$(2.3) \quad \text{either } K_0 > 0 \text{ or } K_0 \leq 0 \text{ and } h_0 > \sqrt{-K_0}.$$

In such case, all solutions are determined by the formula

$$(2.4) \quad v_a(z) := 2 \log \left\{ \frac{2\phi_0(1-|a|^2)}{\phi_0^2|1-\bar{a}z|^2 + K_0|z-a|^2} \right\},$$

where $\phi_0 := h_0 + \sqrt{h_0^2 + K_0}$, and $a \in \mathbb{D}$.

Proof. If $K_0 \neq 0$, by considering the function $v + \log |K_0|$, we pass to a problem with $K_0 = \pm 1$. Then, we can restrict ourselves to the cases $K_0 = \pm 1$ or $K_0 = 0$. Observe that under the metric $g = e^v dz$, \mathbb{D} has constant gaussian curvature equal to K_0 . Hence it is locally isometric to compact subdomains of \mathbb{S}^2 , \mathbb{H}^2 or \mathbb{R}^2 , depending on the case $K_0 = 1$, $K_0 = -1$, or $K_0 = 0$, respectively.

Observe also that $\partial\mathbb{D}$ has constant geodesic curvature equal to h_0 , and this property translates via the local isometry. Since \mathbb{D} is simply connected, we conclude that the local isometry is a global one. In other words, we have a global isometry $\Upsilon : \mathbb{D} \rightarrow U$, where U is a subdomain of either \mathbb{S}^2 , \mathbb{H}^2 or \mathbb{R}^2 , with geodesic curvature equal to h_0 .

Observe that if the ambient domain is \mathbb{R}^2 , then h_0 needs to be strictly positive. Moreover, in \mathbb{H}^2 such domains are bounded only if $h_0 > 1$. As a consequence we obtain (2.3).

In other words, U is a disk in either \mathbb{S}^2 , \mathbb{H}^2 or \mathbb{R}^2 and, by composing with a symmetry, we can assume that:

$$U = \begin{cases} \left\{ (x_1, x_2, x_3) \in \mathbb{S}^2 : x_3 \geq \frac{h_0}{\sqrt{1+h_0^2}} \right\} & \text{if } K_0 = 1, \\ \left\{ (x_1, x_2) \in \mathbb{H}^2 : \sqrt{x_1^2 + x_2^2} \leq h_0 - \sqrt{h_0^2 - 1} \right\} & \text{if } K_0 = -1, \\ \left\{ (x_1, x_2) \in \mathbb{R}^2 : \sqrt{x_1^2 + x_2^2} \leq 1/h_0 \right\} & \text{if } K_0 = 0. \end{cases}$$

Above we have expressed \mathbb{H}^2 via coordinates in the Poincaré disk.

On the other hand, the identity map $I : (\mathbb{D}, dz) \rightarrow (\mathbb{D}, e^v dz)$ is clearly a conformal map. As a consequence, the composition $\Upsilon \circ I : (\mathbb{D}, dz) \rightarrow U$ is a conformal map. Those maps are classified, and hence the conformal factor e^v is given by one of those maps. This gives the expression (2.4). \square

Remark 2.2. *In particular, there exists a unique solution to (2.1) satisfying the extra condition:*

$$\int_{\mathbb{D}} x e^{v(z)} = 0, \quad \int_{\mathbb{D}} y e^{v(z)} = 0,$$

which is just the solution given in (2.4) for $a = 0$, that is:

$$(2.5) \quad v_0(z) = 2 \log \left\{ \frac{2\phi_0}{\phi_0^2 + K_0|z|^2} \right\}.$$

Next lemma addresses the question of nondegeneracy of such solution:

Lemma 2.3. *Let us consider the linearized problem:*

$$(2.6) \quad \begin{cases} -\Delta\psi = 2K_0 e^{v_0}\psi & \text{in } \mathbb{D}, \\ \frac{\partial\psi}{\partial\nu} = h_0 e^{v_0/2}\psi & \text{on } \partial\mathbb{D}, \end{cases}$$

where v_0 is given in (2.5). Then the vector space of solutions of (2.6) is spanned by the functions:

$$\psi_1(z) := \frac{x}{\phi_0^2 + K_0|z|^2}, \quad \psi_2(z) := \frac{y}{\phi_0^2 + K_0|z|^2},$$

where $\phi_0 := h_0 + \sqrt{h_0^2 + K_0}$. In particular, a solution of (2.6) satisfying the extra assumptions:

$$\int_{\mathbb{D}} x e^{v_0(z)} \psi(z) = 0, \quad \int_{\mathbb{D}} y e^{v_0(z)} \psi(z) = 0,$$

must be necessarily equal to 0.

Proof. We show the proof only in the case $K_0 = 1$, the other cases being analogous. As shown in the previous lemma, $(\mathbb{D}, e^{v_0} dz)$ is isomorphic to a spherical cap in \mathbb{S}^2 , given by $U = \{(x_1, x_2, x_3) \in \mathbb{S}^2 : x_3 \geq \frac{h_0}{\sqrt{1+h_0^2}}\}$. Via that conformal map, problem (2.6) becomes:

$$(2.7) \quad \begin{cases} -\Delta\psi = 2\psi & \text{in } U, \\ \frac{\partial\psi}{\partial\nu} = h_0\psi & \text{on } \partial U. \end{cases}$$

As it is well known, the coordinate functions x, y are generators of the vector space of solutions of that problem. Via the conformal map, this translates into ψ_1, ψ_2 . \square

Next lemma is a standard regularity result:

Lemma 2.4. *Let ψ be a solution of*

$$(2.8) \quad \begin{cases} -\Delta\psi = c(z) & \text{in } \mathbb{D}, \\ \frac{\partial\psi}{\partial\nu} = d(z) & \text{on } \partial\mathbb{D}. \end{cases}$$

Then for any $q > 1$ there exists $C_q > 0$ such that

$$\|\psi\|_{W^{1+1/q, q}(\mathbb{D})} \leq C_q (\|c\|_{L^q(\mathbb{D})} + \|d\|_{L^q(\partial\mathbb{D})}).$$

The next result is a quantitative version of the nondegeneracy result stated in Lemma 2.3.

Lemma 2.5. *For any $q > 1$ there exists $C_q > 0$ such that the following holds: for any solution ψ of the nonhomogeneous linearized problem*

$$(2.9) \quad \begin{cases} -\Delta\psi = 2K_0e^{v_0}\psi + c(z) & \text{in } \mathbb{D}, \\ \frac{\partial\psi}{\partial\nu} = h_0e^{v_0/2}\psi + d(z) & \text{on } \partial\mathbb{D}, \end{cases}$$

satisfying

$$(2.10) \quad \int_{\mathbb{D}} xe^{v_0(z)}\psi(z) = 0, \quad \int_{\mathbb{D}} ye^{v_0(z)}\psi(z) = 0,$$

we have that

$$\|\psi\|_{W^{1+1/q,q}(\mathbb{D})} \leq C_q(\|c\|_{L^q(\mathbb{D})} + \|d\|_{L^q(\partial\mathbb{D})}).$$

Proof. Consider the solution ψ_n to the problem

$$\begin{cases} -\Delta\psi_n = 2K_0e^{v_0}\psi_n + c_n(z) & \text{in } \mathbb{D}, \\ \frac{\partial\psi_n}{\partial\nu} = h_0e^{v_0/2}\psi_n + d_n(z) & \text{on } \partial\mathbb{D}. \end{cases}$$

We assume that the solutions ψ_n are reasonably smooth, otherwise one can argue by density. Reasoning by contradiction, suppose that there exists c_n, d_n and solutions ψ_n with

$$\|c_n\|_{L^q(\mathbb{D})} + \|d_n\|_{L^q(\partial\mathbb{D})} = 1, \quad \|\psi_n\|_{W^{1+1/q,q}(\mathbb{D})} \rightarrow +\infty.$$

Define $\tilde{\psi}_n := \|\psi_n\|_{W^{1+1/q,q}(\mathbb{D})}^{-1}\psi_n$. Up to a subsequence, we can assume that $\tilde{\psi}_n \rightharpoonup \tilde{\psi}_0$ in $W^{1+1/q,q}(\mathbb{D})$. Clearly, by compactness,

$$\begin{aligned} 2K_0e^{v_0}\tilde{\psi}_n + \frac{c_n}{\|\psi_n\|_{W^{1+1/q,q}(\mathbb{D})}} &\rightarrow 2K_0e^{v_0}\tilde{\psi}_0 \text{ in } L^q(\mathbb{D}), \\ 2h_0e^{v_0/2}\tilde{\psi}_n + \frac{d_n}{\|\psi_n\|_{W^{1+1/q,q}(\mathbb{D})}} &\rightarrow 2h_0e^{v_0/2}\tilde{\psi}_0 \text{ in } L^q(\partial\mathbb{D}). \end{aligned}$$

By Lemma 2.4 we conclude $\tilde{\psi}_n \rightarrow \tilde{\psi}_0$ in $W^{1+1/q,q}(\mathbb{D})$. In particular, $\|\tilde{\psi}_0\|_{W^{1+1/q,q}(\mathbb{D})} = 1$. Moreover, if ψ_n satisfy (2.10), so it does $\tilde{\psi}_0$. By Lemma 2.3 we conclude that $\tilde{\psi}_0 = 0$, a contradiction. \square

2.2. The limit problem in the plane and the half-plane. As it is well known, the study of blowing-up solutions of a nonlinear PDE can be put in relation with limit problems in the entire space or in half-spaces. The classification of the finite mass solutions of Liouville problems was first made in [15] in the whole space \mathbb{R}^2 , and is rather known. Indeed, the unique solutions of the problem:

$$(2.11) \quad -\Delta v = 2K_0e^v \text{ in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} e^v < +\infty,$$

are given by the expression:

$$(2.12) \quad v(x) = 2 \log \left\{ \frac{2\lambda}{K_0\lambda^2 + |x - x_0|^2} \right\}, \quad x_0 \in \mathbb{R}^2, \lambda > 0.$$

All those solutions are known to satisfy the quantization property:

$$(2.13) \quad K_0 \int_{\mathbb{R}^2} e^v = 4\pi.$$

Since our problem considers a boundary condition, we will also be interested in this paper in the solutions of the problem posed in the halfplane: that is, given real constants K_0, h_0 , the problem

$$(2.14) \quad \begin{cases} -\Delta v = 2K_0 e^v & \text{in } \mathbb{R}_+^2, \\ \frac{\partial v}{\partial \nu} = 2h_0 e^{\frac{v}{2}} & \text{on } \partial\mathbb{R}_+^2, \end{cases} \quad \int_{\mathbb{R}_+^2} e^v + \int_{\partial\mathbb{R}_+^2} e^{v/2} < +\infty.$$

It is known (see [37, 46]) that (2.14) is solvable for any $K_0 > 0$, and also if $K_0 \leq 0$ and $h_0 > \sqrt{-K_0}$. In any case, any solution is given by the expression:

$$(2.15) \quad v(w_1, w_2) = 2 \log \left\{ \frac{2\lambda}{K_0 \lambda^2 + (w_1 - w_0)^2 + (w_2 + \lambda h_0)^2} \right\}, \quad w_0 \in \mathbb{R}, \lambda > 0,$$

that satisfies

$$(2.16) \quad h_0 \int_{\partial\mathbb{R}_+^2} e^{v/2} = \beta, \quad K_0 \int_{\mathbb{R}_+^2} e^v = 2\pi - \beta,$$

where β is given by

$$(2.17) \quad \beta := 2\pi \frac{h_0}{\sqrt{h_0^2 + K_0}}.$$

2.3. Kazdan-Warner type conditions. In this subsection we state and prove the Kazdan-Warner conditions for problem (1.5). There is a version already available for boundary problems in this way, posed in the half-sphere, see [26]. To keep the paper self-contained we give our own version of it, which suits for our later purposes.

Let us recall the following Pohozaev-type identity, depending on an arbitrary field F .

Lemma 2.6. *Let u be a solution of (1.5). Then, given any vector field $F : \overline{\mathbb{D}} \rightarrow \mathbb{R}^2$, there holds:*

$$\begin{aligned} & \int_{\partial\mathbb{D}} [2Ke^u(F \cdot \nu) + (2he^{u/2} - 2)(\nabla u \cdot F) - \frac{|\nabla u|^2}{2} F \cdot \nu] \\ &= \int_{\mathbb{D}} [2e^u(\nabla K \cdot F + K \nabla \cdot F) + DF(\nabla u, \nabla u) - \nabla \cdot F \frac{|\nabla u|^2}{2}]. \end{aligned}$$

Proof. The proof follows by multiplying the equation (1.5) by $\nabla u \cdot F$ and integrating by parts (see for instance [39, Lemma 5.5]). \square

From this we can obtain the following Kazdan-Warner identity.

Proposition 2.7. *If u is a solution of (1.5) then*

$$\int_{\mathbb{D}} e^u \nabla K \cdot F = 4 \int_{\partial\mathbb{D}} h_\tau e^{u/2} y,$$

where $F(x, y) := (1 - x^2 + y^2, -2xy)$ (with complex notation, $F(z) = 1 - z^2$).

Proof. The idea is to consider the variation along the conformal transformations (see (2.2)) that keep fixed the point $p = (1, 0)$. This corresponds to

$$f_\lambda(z) := \frac{\lambda + z}{1 + \lambda z},$$

with $\lambda \in (-1, 1)$. By taking into account the invariance of (2.1), we define

$$v_\lambda(z) := u(f_\lambda(z)) + 2 \log(|f'_\lambda(z)|).$$

In order to study its variation with respect to λ we compute first

$$F(z) := \left. \frac{d}{d\lambda} f_\lambda(z) \right|_{\lambda=0} = \left. \frac{1-z^2}{(1+\lambda z)^2} \right|_{\lambda=0} = 1-z^2 = (1-x^2+y^2, -2xy),$$

where we have used the standard notation $z = x + iy$. From these identities we can conclude that

$$\left. \frac{d}{d\lambda} v_\lambda(z) \right|_{\lambda=0} = \nabla u \cdot F - 4x.$$

Hence, we test in the problem with $\nabla u \cdot F - 4x$. We first multiply it by $\nabla u \cdot F$, which is just to insert F in Lemma 2.6. Observe that on $\partial\mathbb{D}$, F is a tangential vector field, and $F \cdot \tau = -2y$, where $\tau(x, y) = (-y, x)$. Moreover, since F is holomorphic, the Cauchy-Riemann conditions yield

$$DF(\nabla u, \nabla u) - \nabla \cdot F \frac{|\nabla u|^2}{2} = 0.$$

Then, we obtain

$$(2.18) \quad \int_{\partial\mathbb{D}} (2he^{u/2} - 2)(-2y)u_\tau = \int_{\mathbb{D}} 2e^u(\nabla K \cdot F - 4xK).$$

We now multiply (1.5) by $4x$ and integrate to obtain

$$(2.19) \quad \begin{aligned} 8 \int_{\mathbb{D}} Ke^u x &= 4 \int_{\mathbb{D}} u_x - 4 \int_{\partial\mathbb{D}} x \nabla u \cdot \nu = 4 \int_{\mathbb{D}} u_x - 4 \int_{\partial\mathbb{D}} x(2he^{u/2} - 2) \\ &= 4 \int_{\mathbb{D}} u_x + 4 \int_{\partial\mathbb{D}} y(2h_\tau e^{u/2} + he^{u/2} u_\tau). \end{aligned}$$

Observe also that, integrating by parts,

$$(2.20) \quad \int_{\partial\mathbb{D}} yu_\tau = - \int_{\partial\mathbb{D}} xu = - \int_{\mathbb{D}} u_x.$$

Putting together (2.18), (2.19) and (2.20) we conclude. \square

Remark 2.8. *The same equality holds by interchanging the roles of x and y , if we also change orientation $\tau \rightarrow -\tau$.*

Remark 2.9. *Let us observe that the Kazdan-Warner identity given in Proposition 2.7 can be written as*

$$\int_{\mathbb{D}} e^u(1+x^2+y^2)^2 \nabla K \cdot \nabla T = -8 \int_{\partial\mathbb{D}} h_\tau T_\tau e^{u/2},$$

where $T(x, y) = x/(1+x^2+y^2)$. As a consequence, if $\nabla K \cdot \nabla T$, $h_\tau T_\tau$ are both positive (or negative), then (1.5) does not admit any solution. This is a typical Kazdan-Warner type obstruction; in the case of the semisphere, this was observed in [26, Theorem 1].

3. A BLOW-UP ANALYSIS

In this section we begin the blow-up analysis given in Theorem 1.1. The main goal here is to give a proof of the Proposition 3.1 below.

Under the assumptions of Theorem 1.1, define:

$$(3.1) \quad u_n(\tilde{z}_n) := \max_{\mathbb{D}} u_n(z) \rightarrow +\infty.$$

We also define the singular set of the blowing-up sequence as in [7]:

$$(3.2) \quad \mathcal{S} := \{p \in \overline{\mathbb{D}} : \exists z_n \in \overline{\mathbb{D}}, z_n \rightarrow p \text{ such that } u_n(z_n) \rightarrow +\infty\},$$

By (3.1), up to a subsequence, we can assume that $\tilde{z}_n \rightarrow p \in \mathcal{S}$ as $n \rightarrow +\infty$.

The main result of this section is the following:

Proposition 3.1. *Under the assumptions of Theorem 1.1, $\mathcal{S} = \{p\} \subset \partial\mathbb{D}$ and, up to a subsequence,*

$$(3.3) \quad \begin{aligned} u_n &\rightarrow -\infty && \text{locally uniform in } \overline{\mathbb{D}} \setminus \{p\}, \\ h_n e^{u_n/2} &\rightarrow \beta \delta_p && \text{and } K_n e^{u_n} \rightarrow (2\pi - \beta) \delta_p, \end{aligned}$$

in the sense of measures, where

$$(3.4) \quad \beta := 2\pi \frac{h(p)}{\sqrt{h^2(p) + K(p)}}.$$

Moreover,

$$(3.5) \quad u_n(z) = 2 \log \left\{ \frac{2\phi_n(p)(1 - |\zeta_n|^2)}{\phi_n^2(p)|1 - \overline{\zeta_n}z|^2 + K_n(p)|z - \zeta_n|^2} \right\} + O(1),$$

where $z \in \overline{\mathbb{D}}$, the function ϕ_n is defined in (1.9) and $\zeta_n \in \mathbb{D}$, $\zeta_n \rightarrow p$.

This proposition is a first step in order to conclude the asymptotics of Theorem 1.1. In next section we will give a more accurate description (passing from $O(1)$ of (3.5) to $o(1)$). This will be needed in order to conclude the necessary conditions on the point p in Section 5.

Proposition 3.1 will follow from the next result, which addresses more general Liouville-type equations:

Proposition 3.2. *Let u_n be a sequence of solutions of the problems*

$$(3.6) \quad \begin{cases} -\Delta u_n + 2\hat{K}_n = 2K_n e^{u_n} & \text{in } \mathbb{D}, \\ \frac{\partial u_n}{\partial \nu} + 2\hat{h}_n = 2h_n e^{u_n/2} & \text{on } \partial\mathbb{D}, \end{cases}$$

where $K_n \rightarrow K$, $\hat{K}_n \rightarrow \hat{K}$ in $C^2(\overline{\mathbb{D}})$ and $h_n \rightarrow h$, $\hat{h}_n \rightarrow \hat{h}$ in $C^2(\partial\mathbb{D})$ as $n \rightarrow +\infty$. Assume moreover $\sup\{u_n\} \rightarrow +\infty$, but it has bounded global mass:

$$(3.7) \quad \int_{\mathbb{D}} e^{u_n} + \int_{\partial\mathbb{D}} e^{u_n/2} \leq C.$$

Then the singular set \mathcal{S} is finite. Moreover,

- i) If $p \in \mathcal{S} \setminus \partial\mathbb{D}$, then $K(p) > 0$;
- ii) If $p \in \mathcal{S} \cap (\partial\mathbb{D})$ and $K(p) \leq 0$, then $h(p) > \sqrt{-K(p)}$;
- iii) $u_n \rightarrow -\infty$ locally uniformly in $\overline{\mathbb{D}} \setminus \mathcal{S}$;

iv)

$$(3.8) \quad K_n e^{u_n} \rightharpoonup \sum_{p \in \mathcal{S} \setminus (\partial \mathbb{D})} 4\pi \delta_p + \sum_{p \in \mathcal{S} \cap \partial \mathbb{D}} \gamma_p \delta_p,$$

and

$$(3.9) \quad h_n e^{u_n/2} \rightharpoonup \sum_{p \in \mathcal{S} \cap \partial \mathbb{D}} \gamma'_p \delta_p,$$

where $\gamma_p + \gamma'_p = 2\pi$ and

$$(3.10) \quad \gamma_p := 2\pi \frac{h(p)}{\sqrt{h^2(p) + K(p)}}.$$

Remark 3.3. Proposition 3.2 holds true also for domains with boundary different from the disk, as well as for 2-manifolds with boundaries, with the obvious modifications in the proofs. However, since the goal of the paper is the study of the case of the disk, we have preferred a statement in this setting.

First of all, let us introduce a minimal mass lemma, which is just a version for the case of boundaries of a well-known lemma by Brezis and Merle ([7], see Theorem 1 and Corollary 3).

Lemma 3.4. Under the assumptions of Proposition 3.2, let $p \in \mathbb{D}$ and $r > 0$ such that $B_r(p) \subset \mathbb{D}$. If

$$(3.11) \quad \int_{B_r(p)} K_n^+ e^{u_n} \leq \varepsilon < 2\pi,$$

then u_n^+ is uniformly bounded in $L^\infty(B_{\frac{r}{2}}(p))$.

If, instead, $p \in \partial \mathbb{D}$ and

$$(3.12) \quad \int_{B_r(p) \cap \overline{\mathbb{D}}} K_n^+ e^{u_n} \leq \varepsilon < \frac{\pi}{2}, \quad \int_{B_r(p) \cap \partial \mathbb{D}} h_n^+ e^{\frac{u_n}{2}} \leq \varepsilon < \frac{\pi}{2},$$

then u_n^+ is uniformly bounded in $L^\infty(B_{\frac{r}{2}}(p) \cap \overline{\mathbb{D}})$.

A proof of the previous result can be found in [4, Lemma 2.4].

As a consequence of this lemma, the set \mathcal{S} (defined in (3.2)) must be finite. Moreover:

$$(3.13) \quad K_n e^{u_n} \rightharpoonup \sum_{p \in \mathcal{S} \cap \mathbb{D}} \alpha_p \delta_p + \sum_{p \in \mathcal{S} \cap \partial \mathbb{D}} \gamma_p \delta_p + \Psi,$$

and

$$(3.14) \quad h_n e^{u_n/2} \rightharpoonup \sum_{p \in \mathcal{S} \cap \partial \mathbb{D}} \gamma'_p \delta_p + \Psi',$$

where $\alpha_p \geq 2\pi$, $\gamma_p \geq \frac{\pi}{2}$ or $\gamma'_p \geq \frac{\pi}{2}$, $\Psi \in L^1(\mathbb{D}) \cap L_{loc}^\infty(\overline{\mathbb{D}} \setminus \mathcal{S})$, $\Psi' \in L^1(\partial \mathbb{D}) \cap L_{loc}^\infty(\partial \mathbb{D} \setminus \mathcal{S})$.

Taking into account (3.7), for any point $p \in \mathcal{S} \cap \mathbb{D}$ we have:

$$\begin{aligned} 2\pi &\leq \int_{B_r(p)} K_n^+ e^{u_n} \leq \int_{B_r(p)} (K_n^+(p) + r \|\nabla K_n^+\|_{L^\infty(B_r(p))}) e^{u_n} \\ &\leq C(K_n^+(p) + r \|\nabla K_n^+\|_{L^\infty}). \end{aligned}$$

By choosing $r > 0$ sufficiently small we conclude that $K(p) > 0$, so that *i*) of Proposition 3.2 holds.

Let us now point out that, outside the set \mathcal{S} we can assume that e^{u_n} is uniformly bounded. Via a Green representation formula (or via local regularity estimates for the Neumann problem) one can deduce that u_n has bounded oscillation far from \mathcal{S} : being more specific,

$$(3.15) \quad |u_n(z_1) - u_n(z_2)| \leq C \quad \forall z_1, z_2 \in \overline{\mathbb{D}} \setminus \bigcup_{p \in \mathcal{S}} B_r(p),$$

for some $r > 0$ fixed.

3.1. Proof of Proposition 3.2. The proof of Proposition 3.2 is the result of a more detailed study of the behavior of u_n around the singular points. We shall focus on the boundary points, since the theory for interior blow-up points is much more developed (see for instance [34, 36]).

Consider hence a point $p \in \mathcal{S} \cap \partial\mathbb{D}$ and a neighborhood of it which does not intersect any other singular point. Via a conformal map, we can pass to a problem in a half-ball. Let us be more specific, and assume without loss of generality that $p = (1, 0)$. By the Möbius transformation

$$f_1 : \mathbb{R}_+^2 \longrightarrow \mathbb{D} \\ w \longmapsto z = \frac{i-w}{w+i}.$$

we can map a semiball $B_1^+(0) \subset \mathbb{R}_+^2$ into the right half disk, where the point p corresponds to the origin. Consider the transformation

$$v_n(w) := u_n(f_1(w)) + 2 \log |f_1'(w)| = u_n(f_1(w)) + 2 \log \frac{2}{|w+i|^2}.$$

Then v_n satisfies the problem

$$(3.16) \quad \begin{cases} -\Delta v_n + 2\hat{K}_n = 2K_n e^{v_n} & \text{in } B_{r_0}^+(0), \\ \frac{\partial v_n}{\partial \nu} + 2\hat{h}_n = 2h_n e^{\frac{v_n}{2}} & \text{on } \Gamma_{r_0}(0) := (-r_0, r_0) \times \{0\}, \end{cases}$$

for some $r_0 > 0$ small, where $\mathcal{S} = \{0\}$. Here the functions $\hat{K}_n, \hat{h}_n, K_n$ and h_n come from the original data composed with the transformation f_1 : for the sake of clarity we keep the same notation. Besides, by (3.1), there exists a sequence $\{w_n\} \subset B_{r_0}^+(0)$ such that

$$(3.17) \quad v_n(w_n) = \max_{B_{r_0}^+(0)} v_n(w) \rightarrow +\infty, \quad w_n \rightarrow 0.$$

Notice that w_n may not match $f_1(\tilde{z}_n)$.

The following selection process has been applied many times in the literature starting from [36] (see in particular Lemma 4). The case with boundary have been treated in [39, Lemma 7.4], to which we refer for details (see also [3]).

Lemma 3.5. *There exists a finite number of sequences $\Sigma_n := \{w_1^n, \dots, w_\ell^n, \hat{w}_1^n, \dots, \hat{w}_k^n\} \rightarrow 0$ and positive sequences $\varepsilon_1^n, \dots, \varepsilon_\ell^n \rightarrow 0, \hat{\varepsilon}_1^n, \dots, \hat{\varepsilon}_k^n \rightarrow 0$ such that*

- $v_n(w_i^n) = \max\{v_n(w), w \in B_{\varepsilon_i^n}(w_i^n) \cap \overline{\mathbb{R}_+^2}\}$ for $i = 1, \dots, \ell$,
- $\hat{\varepsilon}_i^n \leq \text{dist}(\hat{w}_i^n, \Gamma_{r_0}(0))$ and $v_n(\hat{w}_i^n) = \max\{v_n(w), w \in B_{\hat{\varepsilon}_i^n}(\hat{w}_i^n)\}$ for $i = 1, \dots, k$,

- c) If $K(0) \leq 0$ then $k = 0$ and $h(0) > \sqrt{-K(0)}$.
d) $\int_{B_r^+(0)} K_n e^{v_n} + \int_{\Gamma_r(0)} h_n e^{v_n/2} \rightarrow 2\pi\ell + 4\pi k$.
e) $v_n \rightarrow -\infty$ uniformly in compact sets of $B_r^+(0) \setminus \{0\}$.

The points w_i^n represent points of local maxima for which the rescaled problem converge to a limit problem posed in a half-plane (2.14), whereas for the points \hat{w}_i^n the limit problem is posed in the entire plane (2.11). The restrictions in c) come from the obstructions to the existence of solutions for those problems. Moreover, the mass quantization in d) comes from (2.13) and (2.16).

Remark 3.6. In the analysis performed in [39, Lemma 7.4] only the points w_i^n appear. This is because in that paper K is strictly negative and hence no solutions in the entire plane exist. However, the main point in the arguments of [39] is to show that the terms $\int_{B_r^+(0)} K_n e^{v_n}$, $\int_{\Gamma_r(0)} h_n e^{v_n/2}$ do not give any contribution apart from that coming from their corresponding limit problems. That proof depends on the decay of the limit solutions (2.15) and (2.12), and works equally well in our framework.

Let us point out that the coefficients γ, γ' of the expansions (3.13) and (3.14) (we omit the dependence on p) satisfy:

$$(3.18) \quad \gamma := \lim_{r \rightarrow 0} \lim_{n \rightarrow +\infty} \int_{B_r^+(0)} K_n e^{v_n},$$

$$(3.19) \quad \gamma' := \lim_{r \rightarrow 0} \lim_{n \rightarrow +\infty} \int_{\Gamma_r(0)} h_n e^{v_n/2}.$$

Next we prove the quantization of this values using a proper Pohozaev type identity and the previous asymptotic estimates.

Lemma 3.7. Let γ and γ' given in (3.18) and (3.19). Then

$$\gamma + \gamma' = 2\pi.$$

Proof. Applying a Pohozaev type identity in $B_r^+(0)$ for $0 < r < r_0$ with r_0 fixed in (3.16), as Lemma 2.6 with $F = w$, we have that

$$\begin{aligned} & \int_{\partial B_r^+(0)} [2K_n e^{v_n} (w \cdot \nu) + (2h_n e^{v_n/2} - 2\hat{h}_n)(\nabla v_n \cdot w) - \frac{|\nabla v_n|^2}{2} w \cdot \nu] dw \\ &= \int_{B_r^+(0)} [2e^{v_n} (\nabla K_n \cdot w + 2K_n) + 2\hat{K}_n \nabla v_n \cdot w] dw. \end{aligned}$$

If we divide $\partial B_r^+ = \partial^+ B_r \cup \Gamma_r$, we immediately obtain

$$\begin{aligned} & \int_{\partial B_r^+(0)} [2K_n e^{v_n} (w \cdot \nu) + (2h_n e^{v_n/2} - 2\hat{h}_n)(\nabla v_n \cdot w) - \frac{|\nabla v_n|^2}{2} w \cdot \nu] dw \\ &= \int_{\partial^+ B_r(0)} [2r K_n e^{v_n} + (2h_n e^{v_n/2} - 2\hat{h}_n)(\nabla v_n \cdot w) - \frac{|\nabla v_n|^2}{2} r] dw \\ &+ \int_{\Gamma_r(0)} (2h_n e^{v_n/2} - 2\hat{h}_n) \partial_1 v_n w_1 dw, \end{aligned}$$

where we have used the notation $w = w_1 + iw_2$. Taking into account that $|\nabla K_n| \leq C$ uniformly in $B_r^+(0)$, then

$$(3.20) \quad \int_{B_r^+(0)} (\nabla K_n \cdot w) e^{v_n} dw \leq r \int_{B_r^+(0)} |\nabla K_n| e^{v_n} dw = O(r).$$

On the other hand, integrating by parts

$$\begin{aligned} \int_{\Gamma_r(0)} 2h_n e^{v_n/2} \nabla v_n \cdot w \, dw &= 4 \left[h_n e^{v_n/2} w_1 \right]_{w=(-r,0)}^{(r,0)} - 4 \int_{\Gamma_r(0)} \nabla h_n \cdot w e^{v_n/2} \, dw \\ &\quad - 4 \int_{\Gamma_r(0)} h_n e^{v_n/2} \, dw. \end{aligned}$$

Again, since $|\nabla h_n| \leq C$, we have

$$(3.21) \quad \int_{\Gamma_r(0)} \nabla h_n \cdot w e^{v_n/2} \, dw = O(r).$$

Moreover, by Lemma 3.5, e), we deduce that

$$(3.22) \quad \left[h_n e^{v_n/2} w_1 \right]_{w=(-r,0)}^{(r,0)} \rightarrow 0, \quad \int_{\partial^+ B_r(0)} K_n e^{v_n} r \, dw \rightarrow 0, \quad \text{as } n \rightarrow +\infty.$$

By Green representation formula,

$$\begin{aligned} \nabla v_n(w) &= \frac{1}{\pi} \int_{B_r^+(0)} \frac{w-z}{|w-z|^2} 2K_n(z) e^{v_n(z)} \, dz \\ &\quad + \frac{1}{\pi} \int_{\Gamma_r(0)} \frac{w-z}{|w-z|^2} 2h_n(z) e^{v_n(z)/2} \, dz + O(r), \end{aligned}$$

and, using (3.13) and (3.14),

$$\nabla v_n(w) \rightarrow \nabla \mathcal{G} = \frac{2}{\pi} (\gamma + \gamma') \frac{w}{|w|^2} + O(r) \text{ outside the origin,}$$

$$\int_{\partial^+ B_r(0)} \nabla v_n \cdot w \, dw \rightarrow \int_{\partial^+ B_r(0)} \nabla \mathcal{G} \cdot w \, dw = O(r),$$

$$\int_{B_r(0)} \nabla v_n \cdot w \, dw \rightarrow \int_{B_r(0)} \nabla \mathcal{G} \cdot w \, dw = O(r),$$

$$\int_{\partial^+ B_r(0)} \frac{|\nabla v_n|^2}{2} r \, dw \rightarrow \int_{\partial^+ B_r(0)} \frac{|\nabla \mathcal{G}|^2}{2} r \, dw = 2(\gamma + \gamma')^2 \frac{1}{\pi} + O(r).$$

Plugging the expressions above we arrive at

$$2(\gamma + \gamma')^2 \frac{1}{\pi} = 4(\gamma + \gamma') + O(r),$$

giving the desired conclusion. \square

Observe that the previous lemma, together with Lemma 3.5, d), implies that $\ell = 1$ and $k = 0$. Moreover, by (2.17),

$$\gamma = 2\pi \frac{h(0)}{h^2(0) + K(0)}.$$

This finishes the proof of Proposition 3.2.

3.2. Proof of Proposition 3.1. We finally turn our attention back to problem (1.7). In this specific case we first show that the singular set is formed by a unique point located at $\partial\mathbb{D}$.

Lemma 3.8. *The blow-up set $S = \{p\} \subset \partial\mathbb{D}$.*

Proof. By (1.6) we know

$$\int_{\mathbb{D}} K_n e^{u_n} + \int_{\partial\mathbb{D}} h_n e^{u_n/2} = 2\pi.$$

Furthermore, Proposition 3.2 implies that

$$\int_{\mathbb{D}} K_n e^{u_n} + \int_{\partial\mathbb{D}} h_n e^{u_n/2} \rightarrow 4\pi N + 2\pi M,$$

with $N = |\mathcal{S} \cap \mathbb{D}|$ and $M = |\mathcal{S} \cap \partial\mathbb{D}|$, meaning the cardinals of those sets. Thus, the only possibility is $N = 0, M = 1$. \square

Next, we establish global pointwise estimates of the bubbling sequence around 0. This type of estimate was first derived by Y.Y. Li in [34] by the method of moving planes. Another argument was given by Bartolucci-Chen-Lin-Tarantello in [2] for singular problems, which is more suited to our framework (see also Section 4.2. in [43] for details). Since this argument is rather standard for Liouville's type problems, we will be sketchy.

Lemma 3.9. *If $K(p) \leq 0$, then $h(p) > \sqrt{-K(p)}$. Moreover, the solutions v_n of (3.16) satisfy:*

$$(3.23) \quad v_n(w_1, w_2) = 2 \log \left\{ \frac{2\lambda_n}{K_n(0)\lambda_n^2 + (w_1 - w_{1,n})^2 + (w_2 - w_{2,n} + h_n(0)\lambda_n)^2} \right\} + O(1),$$

in $B_{r_0}^+(0)$ where $w_n = (w_{1,n}, w_{2,n}) \rightarrow 0$ defined in (3.17) and:

$$(3.24) \quad \lambda_n := \frac{2\delta_n}{K_n(0) + h_n^2(0)} \quad \text{with} \quad \delta_n := e^{-\frac{v_n(w_n)}{2}}.$$

Proof. Define the rescaled function

$$\tilde{v}_n(w) := v_n(w_n + \delta_n w) + 2 \log \delta_n,$$

that satisfies the problem

$$(3.25) \quad \begin{cases} -\Delta \tilde{v}_n = 2K_n(w_n + \delta_n w) e^{\tilde{v}_n} & \text{in } B_{r_0/\delta_n}^+ \left(-\frac{w_n}{\delta_n} \right), \\ \frac{\partial \tilde{v}_n}{\partial \nu} = 2h_n(w_n + \delta_n w) e^{\frac{\tilde{v}_n}{2}} & \text{on } \Gamma_{r_0/\delta_n} \left(-\frac{w_n}{\delta_n} \right), \end{cases}$$

where

$$B_{r_0/\delta_n}^+ \left(-\frac{w_n}{\delta_n} \right) := \left\{ (w_1, w_2) \in B_{r_0/\delta_n} \left(-\frac{w_n}{\delta_n} \right) : w_2 \geq -\frac{w_n}{\delta_n} \right\},$$

$$\Gamma_{r_0/\delta_n} \left(-\frac{w_n}{\delta_n} \right) := (-r_0/\delta_n, r_0/\delta_n) \times \left\{ -\frac{w_n}{\delta_n} \right\}.$$

By simplicity we assume $-\frac{w_n}{\delta_n} \rightarrow q = 0$. Thus, by the Harnack inequality (see for instance [31]):

$$(3.26) \quad \tilde{v}_n \text{ is uniformly bounded in } L_{loc}^\infty(\mathbb{R}_+^2),$$

and

$$(3.27) \quad \tilde{v}_n \rightarrow v_0 \quad \text{in } C_{loc}^2(\mathbb{R}_+^2),$$

where v_0 is the entire solution of the problem

$$(3.28) \quad \begin{cases} -\Delta v_0 = 2K(0)e^{v_0} & \text{in } \mathbb{R}_+^2, \\ \frac{\partial v_0}{\partial \nu} = 2h(0)e^{\frac{v_0}{2}} & \text{on } \partial\mathbb{R}_+^2, \\ v_0(0) = 0, \end{cases}$$

with

$$\int_{\mathbb{R}_+^2} e^{v_0} + \int_{\partial\mathbb{R}_+^2} e^{v_0/2} < C.$$

According to Subsection 2.2, if $K(0) \leq 0$, then $h(0) > \sqrt{-K(0)}$. Moreover, v_0 takes the form:

$$(3.29) \quad v_0(w_1, w_2) = 2 \log \left\{ \frac{2\lambda_0}{K(0)\lambda_0^2 + w_1^2 + (w_2 + \lambda_0 h(0))^2} \right\},$$

where $\lambda_0 := \frac{2}{K(0)+h(0)^2} > 0$. In addition, we have that

$$(3.30) \quad |w_n| \leq C\delta_n.$$

We are now concerned with the global $O(1)$ estimate. By the analysis in [39, Lemma 7.4] (see in particular Step 4), we conclude that for every ε there exists $R_\varepsilon > 1$ such that

$$(3.31) \quad \tilde{v}_n(w) \leq (4 - \varepsilon) \log |w| + C_\varepsilon, \quad \text{for } |w| \geq 2R_\varepsilon.$$

Taking into account (3.15) and (3.27), by Green representation formula one has that

$$\begin{aligned} \tilde{v}_n(w) + \frac{M_n}{\pi} \log |w| &= \frac{1}{\pi} \int_{B_{r_0/\delta_n}^+(-\frac{w_n}{\delta_n})} \log \left(\frac{|w||z|}{|w-z|} \right) 2K_n(w_n + \delta_n z) e^{\tilde{v}_n(z)} dz \\ &\quad + \frac{1}{\pi} \int_{\Gamma_{r_0/\delta_n}(-\frac{w_n}{\delta_n})} \log \left(\frac{|w||z|}{|w-z|} \right) 2h_n(w_n + \delta_n z) e^{\frac{\tilde{v}_n(z)}{2}} dz + \Psi_n, \end{aligned}$$

where Ψ_n is uniformly bounded in $\overline{B_{r_0/\delta_n}^+(-\frac{w_n}{\delta_n})}$ and M_n is defined as

$$\begin{aligned} M_n &:= \int_{B_{r_0}^+(0)} 2K_n e^{v_n} + \int_{\Gamma_{r_0}(0)} 2h_n e^{\frac{v_n}{2}} \\ &= \int_{B_{r_0/\delta_n}^+(-\frac{w_n}{\delta_n})} 2K_n(w_n + \delta_n w) e^{\tilde{v}_n} + \int_{\Gamma_{r_0/\delta_n}(-\frac{w_n}{\delta_n})} 2h_n(w_n + \delta_n w) e^{\frac{\tilde{v}_n}{2}}. \end{aligned}$$

We can estimate the expression above using the asymptotic behavior given by (3.31) in order to conclude that

$$(3.32) \quad \left| \tilde{v}_n(w) + \frac{M_n}{\pi} \log |w| \right| = O(1) \quad \text{for } 3 \leq |w| \leq \frac{r_0}{\delta_n}.$$

See Lemma 4.2.4. in [43] for further details on the computations of the logarithmic terms.

Finally, using a Pohozaev type identity in $B_{-\delta_n \log \delta_n}^+(0)$ one can obtain that

$$(3.33) \quad |M_n - 4\pi| = O(|\log \delta_n|^{-1}).$$

We refer the reader to Lemma 4.2.5. in [43] for more details.

Combining (3.26), (3.27), (3.32) and (3.33), we conclude

$$\left| \tilde{v}_n(w_1, w_2) - 2 \log \left\{ \frac{2\lambda_0}{K(0)\lambda_0^2 + w_1^2 + (w_2 + h(0)\lambda_0)^2} \right\} \right| \leq C, \quad \text{in } \overline{B_{r_0/\delta_n}^+}(0).$$

Using the definition of \tilde{v}_n and (3.17), scaling back to v_n we obtain (3.23). \square

We now conclude the proof of Proposition 3.1 by showing the estimate (3.5). We transform properly the solution of (3.16) into a solution in the set

$$\mathcal{D} := \left\{ z = x + iy \in \overline{\mathbb{D}} : x > \frac{1 - r_0}{1 + r_0} \right\}.$$

In order to do that, note that

$$u_n(z) = v_n(f_1^{-1}(z)) + 2 \log \frac{2}{|z + 1|^2}$$

where f_1^{-1} is a Möbius transformation, inverse of f_1 , namely

$$\begin{aligned} f_1^{-1} : \overline{\mathbb{D}} &\longrightarrow \mathbb{R}_+^2 \\ z &\longmapsto w = i \frac{1-z}{z+1}, \end{aligned}$$

that is,

$$u_n(z) = 2 \log \left\{ \frac{4\lambda_n}{\left[K_n(0)\lambda_n^2 + \left(\frac{2y}{|z+1|^2} - w_{1,n} \right)^2 + \left(\frac{1-|z|^2}{|z+1|^2} - w_{2,n} + h_n(0)\lambda_n \right)^2 \right] |z + 1|^2} \right\}.$$

To complete the proof, define a sequence ζ_n as

$$\zeta_n := \left(1 - \lambda_n \frac{\phi_n^2(p) + K_n(p)}{\phi_n(p)} \right) p.$$

Taking into account (3.24), observe that $\zeta_n \rightarrow p$ as $n \rightarrow +\infty$ and $|\zeta_n| < 1$. Moreover, (3.30) holds by the choice of ζ_n . Considering the function

$$(3.34) \quad \tilde{u}_n(z) := 2 \log \left\{ \frac{2\phi_n(p)(1 - |\zeta_n|^2)}{\phi_n^2(p)|1 - \overline{\zeta_n}z|^2 + K_n(p)|z - \zeta_n|^2} \right\},$$

where $z \in \mathcal{D}$, it can be proved that

$$|\tilde{u}_n(z) - u_n(z)| = O(1),$$

that is,

$$u_n(z) = \tilde{u}_n(z) + O(1), \quad z \in \mathcal{D}.$$

We finally extend the previous expression to the whole disk. Letting $z \in \mathbb{D}$ and $z_1 \in \mathcal{D}$, by (3.15) and (3.34) we obtain that

$$\begin{aligned} &\left| u_n(z) - 2 \log \left\{ \frac{2\phi_n(p)(1 - |\zeta_n|^2)}{\phi_n^2(p)|1 - \overline{\zeta_n}z|^2 + K_n(p)|z - \zeta_n|^2} \right\} \right| \\ &\leq \left| u_n(z_1) - 2 \log \left\{ \frac{2\phi_n(p)(1 - |\zeta_n|^2)}{\phi_n^2(p)|1 - \overline{\zeta_n}z|^2 + K_n(p)|z - \zeta_n|^2} \right\} \right| + |u_n(z) - u_n(z_1)| \\ &\leq C, \end{aligned}$$

that gives the desired conclusion.

4. ESTIMATE OF THE ERROR TERM

Proposition 4.1. *Under the assumptions of Theorem 1.1, there exist $a_n \in \mathbb{D}$ such that, defining*

$$v_n(z) := u_n(f_{a_n}(z)) + 2 \log \frac{1 - |a_n|^2}{|1 + \overline{a_n}z|^2},$$

with $f_a(z) := \frac{a+z}{1+\overline{a}z}$, there holds:

a) v_n solves the problem

$$(4.1) \quad \begin{cases} -\Delta v_n = 2K_n(f_{a_n}(z))e^{v_n(z)} & \text{in } \mathbb{D}, \\ \frac{\partial v_n}{\partial \nu} + 2 = 2h_n(f_{a_n}(z))e^{v_n(z)/2} & \text{on } \partial\mathbb{D}, \end{cases}$$

b) $\int_{\mathbb{D}} x e^{v_n(z)} dz = \int_{\mathbb{D}} y e^{v_n(z)} dz = 0$,

c) $a_n \rightarrow p$ where $p \in \partial\mathbb{D}$ is the blow-up point,

d) v_n is uniformly bounded,

e) $v_n(z) = 2 \log \left(\frac{2\hat{\phi}_n}{\hat{\phi}_n^2 + \hat{k}_n |z|^2} \right) + \xi_n(z)$, where

$$\hat{\phi}_n := \phi_n \left(\frac{a_n}{|a_n|} \right), \hat{k}_n := K_n \left(\frac{a_n}{|a_n|} \right),$$

with ϕ_n defined in (1.9) and $\|\xi_n\|_{C^{0,\alpha}(\overline{\mathbb{D}})} \leq C(1 - |a_n|)^{1-\alpha}$ for any $\alpha \in (0, 1/2)$.

Proof. Assertion a) is immediate. In order to prove b), given any continuous function u we define $\Gamma : \mathbb{D} \rightarrow \mathbb{R}^2$ as

$$\Gamma(a) := \int_{\mathbb{D}} z e^{v_a(z)} dz,$$

where

$$v_a(z) := u(f_a(z)) + 2 \log |f'_a(z)|.$$

We claim that there exists $a \in \mathbb{D}$ such that $\Gamma(a) = 0$. Observe that if $a_n \rightarrow a \in \partial\mathbb{D}$,

$$\begin{aligned} \int_{\mathbb{D}} z e^{v_{a_n}(z)} dz &= \int_{\mathbb{D}} z e^{u(f_{a_n}(z))} |f'_{a_n}(z)|^2 dz = \int_{\mathbb{D}} f_{-a_n}(z') e^{u(z')} dz' \\ &\rightarrow -a \int_{\mathbb{D}} e^{u(z')} dz', \end{aligned}$$

by Lebesgue Theorem. Then, the map Γ can be extended in a continuous way to the boundary by $\Gamma(a) = -a \int_{\mathbb{D}} e^{u(z')} dz'$. As a consequence its Brouwer degree is 1 and we conclude the claim.

We now prove c). Assume by contradiction that, up to a subsequence, $a_n \rightarrow a_0 \in \overline{\mathbb{D}}$, $a_0 \neq p$. Observe that f_{a_n} converges to f_{a_0} uniformly in $B_p(r)$, for small $r > 0$. Recall that by Proposition 3.1, $K_n e^{u_n} \rightarrow \beta \delta_p$, with $p \in \partial\mathbb{D}$. Then

$$0 = \int_{\mathbb{D}} z e^{v_n(z)} dz = \int_{\mathbb{D}} z e^{u_n(f_{a_n}(z))} |f'_{a_n}(z)|^2 dz = \int_{\mathbb{D}} f_{-a_n}(z) e^{u_n(z)} dz \rightarrow c_0 \frac{-a_0 + p}{1 - \overline{a_0}p},$$

with $c_0 \neq 0$ a constant independent of a , a contradiction.

Hence, let $a_n \rightarrow p$. Taking u_n as in Proposition 3.1, define

$$v_n(z) := u_n(f_{a_n}(z)) + 2 \log \frac{1 - |a_n|^2}{|1 + \overline{a_n}z|^2},$$

which implies

$$v_n(z) = 2 \log \left\{ \frac{2\phi_n(p)(1 - |\zeta_n|^2)(1 - |a_n|^2)}{\phi_n^2(p)|1 - \bar{\zeta}_n a_n + z(\bar{a}_n - \bar{\zeta}_n)|^2 + K_n(p)|a_n - \zeta_n + z(1 - \bar{a}_n \zeta_n)|^2} \right\} + O(1).$$

Letting $\tilde{a}_n = \frac{a_n - \zeta_n}{1 - \bar{a}_n \zeta_n} \in \mathbb{D}$ we can rewrite the previous expression as

$$v_n(z) = 2 \log \left\{ \frac{2\phi_n(p)(1 - |\tilde{a}_n|^2)}{\phi_n^2(p)|1 - \bar{\tilde{a}}_n z|^2 + K_n(p)|z - \tilde{a}_n|^2} \right\} + O(1).$$

Suppose that $\tilde{a}_n \rightarrow \tilde{a} \in \partial\mathbb{D}$. Then $v_n(\tilde{a}_n) \rightarrow +\infty$ and $e^{v_n} \rightarrow 0$ locally in $\mathbb{D} \setminus \{\tilde{a}\}$. However, v_n has barycenter 0, which contradicts $\tilde{a} \in \partial\mathbb{D}$. Consequently, $|\tilde{a}_n| < 1 - \varepsilon$ uniformly for $\varepsilon > 0$ and v_n remains uniformly bounded, what proves *d*).

We finally prove *e*). By *d*) v_n is bounded from above and hence the terms

$$2K_n(f_{a_n}(z))e^{v_n(z)}, \quad 2h_n(f_{a_n}(z))e^{v_n(z)/2},$$

are bounded in L^∞ . By Lemma 2.4, v_n is bounded in $W^{1+1/q,q}(\mathbb{D})$ for all $q > 1$. By the Sobolev embeddings, v_n is bounded in $C^{0,\alpha}(\bar{\mathbb{D}})$ for all $\alpha \in (0, 1)$ and thus, up to a subsequence, we can assume that $v_n \rightarrow v_0$ in $C^{0,\alpha}(\bar{\mathbb{D}})$, where v_0 is a solution of

$$\begin{cases} -\Delta v_0 = 2K(p)e^{v_0(z)} & \text{in } \mathbb{D}, \\ \frac{\partial v_0}{\partial \nu} + 2 = 2h(p)e^{v_0(z)/2} & \text{on } \partial\mathbb{D}. \end{cases}$$

Furthermore, $\int_{\mathbb{D}} x e^{v_0(z)} dz = 0$, $\int_{\mathbb{D}} y e^{v_0(z)} dz = 0$. Applying Remark 2.2 we conclude that v_0 is given by (2.5).

Denote

$$\tilde{v}_n(z) := 2 \log \left(\frac{2\hat{\phi}_n}{\hat{\phi}_n^2 + \hat{k}_n |z|^2} \right).$$

Thus, $\xi_n(z) = v_n(z) - \tilde{v}_n(z)$, and it converges to 0 in $C^{0,\alpha}(\bar{\mathbb{D}})$ sense, for any $\alpha \in (0, 1)$, by the arguments above. Our aim is to give a quantitative estimate on this convergence. Observe that ξ_n satisfies:

$$\begin{cases} -\Delta \xi_n = 2K_n \left(f_{a_n} \left(\frac{a_n}{|a_n|} \right) \right) e^{\tilde{v}_n} \xi_n + c_n(z) & \text{in } \mathbb{D}, \\ \frac{\partial \xi_n}{\partial \nu} = 2h_n \left(f_{a_n} \left(\frac{a_n}{|a_n|} \right) \right) e^{\tilde{v}_n/2} \xi_n + d_n(z), & \text{on } \partial\mathbb{D}. \end{cases}$$

with

$$\begin{aligned} c_n(z) &:= 2 \left[K_n(f_{a_n}(z)) - K_n \left(f_{a_n} \left(\frac{a_n}{|a_n|} \right) \right) \right] e^{v_n} \\ &\quad + 2K_n \left(f_{a_n} \left(\frac{a_n}{|a_n|} \right) \right) e^{\tilde{v}_n} (e^{\xi_n} - 1 - \xi_n), \\ d_n(z) &:= 2 \left[h_n(f_{a_n}(z)) - h_n \left(f_{a_n} \left(\frac{a_n}{|a_n|} \right) \right) \right] e^{v_n/2} \\ &\quad + 2h_n \left(f_{a_n} \left(\frac{a_n}{|a_n|} \right) \right) e^{\tilde{v}_n/2} (e^{\xi_n/2} - 1 - \xi_n/2). \end{aligned}$$

By the mean value theorem,

$$\left| K_n(f_{a_n}(z)) - K_n \left(f_{a_n} \left(\frac{a_n}{|a_n|} \right) \right) \right| \leq C \left| f_{a_n}(z) - f_{a_n} \left(\frac{a_n}{|a_n|} \right) \right|,$$

$$\left| h_n(f_{a_n}(z)) - h_n\left(f_{a_n}\left(\frac{a_n}{|a_n|}\right)\right) \right| \leq C \left| f_{a_n}(z) - f_{a_n}\left(\frac{a_n}{|a_n|}\right) \right|,$$

and applying Proposition 6.1 we obtain

$$\int_{\mathbb{D}} \left| f_{a_n}(z) - f_{a_n}\left(\frac{a_n}{|a_n|}\right) \right|^q \leq C(1 - |a_n|)^q, \quad \int_{\partial\mathbb{D}} \left| f_{a_n}(z) - f_{a_n}\left(\frac{a_n}{|a_n|}\right) \right|^q \leq C(1 - |a_n|),$$

for any $1 < q < 2$.

Since $\xi_n \rightarrow 0$ uniformly, we have that $\|e^{\xi_n} - 1 - \xi_n\|_{L^q} \leq C\|\xi_n\|_{L^\infty}\|\xi_n\|_{L^q}$. We now apply Lemma 2.5, to conclude that

$$(4.2) \quad \|\xi_n\|_{W^{1+1/q,q}(\mathbb{D})} \leq C_q(1 - |a_n|)^{1/q} + o_n(1)(\|\xi_n\|_{L^q(\mathbb{D})} + \|\xi_n\|_{L^q(\partial\mathbb{D})}).$$

On the other hand, we have that $\|\xi_n\|_{L^q(\mathbb{D})} + \|\xi_n\|_{L^q(\partial\mathbb{D})} \leq C\|\xi_n\|_{W^{1+1/q,q}(\mathbb{D})}$, from which we obtain, making n large enough, that

$$\|\xi_n\|_{W^{1+1/q,q}(\mathbb{D})} \leq (C_q + 2)(1 - |a_n|)^{1/q}.$$

Now it suffices to recall the continuous Sobolev embedding $W^{1+1/q,q}(\mathbb{D}) \subset C^{0,\alpha}(\overline{\mathbb{D}})$ for $\alpha = 1 - 1/q$, to conclude the proof of Proposition 4.1. \square

4.1. Proof of Theorem 1.1, *i*) and *ii*). From the above analysis, *i*) readily follows. Moreover, with the notation of Proposition 4.1 it is enough to define

$$u_n(z) := v_n(f_{-a_n}(z)) - 2 \log \frac{1 - |a_n|^2}{|1 + \overline{a_n}z|^2}.$$

to obtain *ii*). Observe also that

$$\psi_n(z) = \xi_n(f_{-a_n}(z)),$$

and $|f'_{-a_n}(z)| \leq C(1 - |a_n|)$. Hence, the mean value theorem allows us to estimate the $C^{0,\alpha}$ norm of ψ_n as

$$\begin{aligned} |\psi_n(z_1) - \psi_n(z_2)| &= |\xi_n(f_{-a_n}(z_1)) - \xi_n(f_{-a_n}(z_2))| \\ &\leq C(1 - |a_n|)^{1-\alpha} |f_{-a_n}(z_1) - f_{-a_n}(z_2)|^\alpha \\ &\leq C(1 - |a_n|)^{1-2\alpha} |z_1 - z_2|^\alpha. \end{aligned}$$

5. CONCLUSION OF THE PROOF OF THEOREM 1.1

In this section we conclude the proof of the main Theorem 1.1 by showing *iii*). Multiplying (1.7) by a proper vector field and integrating drives to a first blow-up condition.

Proposition 5.1. *Let $p \in S$ and recall the definition of Φ given in 1.10. Then*

$$(5.1) \quad 2h_\tau(p) + \frac{K_\tau(p)}{\Phi(p)} = 0,$$

or, equivalently, $\Phi_\tau(p) = 0$.

Proof. Consider the vector field $F : \overline{\mathbb{D}} \rightarrow \mathbb{R}^2$ defined by $F(x, y) = (-y, x)$. Observe that F is the tangential vector in $\partial\mathbb{D}$. Applying Lemma 2.6 and integrating by parts we obtain

$$(5.2) \quad -2 \int_{\partial\mathbb{D}} (u_n)_\tau = 4 \int_{\partial\mathbb{D}} (h_n)_\tau e^{u_n/2} + 2 \int_{\mathbb{D}} \nabla K_n \cdot F e^{u_n},$$

where u_n is the solution of (1.7) satisfying Theorem 1.1, i , ii). Here the lower index τ means the tangential derivative, i.e. in the direction of F . Observe that we have used the facts that $\nabla \cdot F = 0$ and $DF(\nabla u_n, \nabla u_n) = 0$.

Reasoning similarly as we did to obtain the distributional convergences (3.3) and using the quantization of mass given in Proposition 3.1, we immediately get that

$$(h_n)_\tau e^{u_n/2} \rightharpoonup 2\pi \frac{h_\tau(p)}{\sqrt{h^2(p) + K(p)}} \delta_p,$$

$$\Phi_n e^{u_n} \rightharpoonup 2\pi \frac{1}{\sqrt{h^2(p) + K(p)}} \delta_p,$$

where the function Φ is defined in (1.10).

Multiplying and dividing properly by Φ_n , by the above convergences and the pointwise estimates on u_n the right-hand side in the identity (5.2) converges to

$$(5.3) \quad 8\pi \frac{h_\tau(p)}{\sqrt{h^2(p) + K(p)}} + 4\pi \frac{K_\tau(p)}{\Phi(p) \sqrt{h^2(p) + K(p)}}.$$

Noticing that

$$\int_{\partial\mathbb{D}} (u_n)_\tau = 0,$$

the expression (5.3) vanishes and this implies (5.1). \square

Without loss of generality from now on we will suppose $p = (1, 0)$. We can also assume, by composing with suitable rotations of the functions K_n, h_n , that $\lambda_n \in \mathbb{R}$, $\lambda_n \rightarrow 1$. Consider

$$(5.4) \quad u_{\lambda_n}(x, y) := 2 \log \left\{ \frac{2(1 - \lambda_n^2) \hat{\phi}_n}{\hat{\phi}_n^2(1 - \lambda_n x)^2 + \hat{\phi}_n^2(\lambda_n y)^2 + \hat{k}_n(x - \lambda_n)^2 + \hat{k}_n y^2} \right\},$$

that corresponds to the profile given in Theorem 1.1. Notice that the denominator of the logarithm is positive as a consequence of the fact

$$\Phi(p)^2 + K(p) > 0.$$

Lemma 5.2. *If $u_n = u_{\lambda_n} + \psi_n$ with u_{λ_n} given by (5.4) and $\|\psi_n\|_{C^{0,\alpha}} \leq C(1 - \lambda_n)^{1-2\alpha}$, $\alpha \in (0, 1/2)$, then*

$$(i) \quad \int_{\partial\mathbb{D}} (h_n)_\tau e^{\frac{u_n}{2}} y = e^{\frac{\psi_n(p)}{2}} \int_{\partial\mathbb{D}} (h_n)_\tau e^{\frac{u_{\lambda_n}}{2}} y + o_n(1)(1 - \lambda_n).$$

$$(ii) \quad \int_{\mathbb{D}} ((K_n)_y xy - (K_n)_x(1 - x^2 + y^2)) e^{u_n}$$

$$= e^{\psi_n(p)} \int_{D^2} ((K_n)_y xy - (K_n)_x(1 - x^2 + y^2)) e^{u_{\lambda_n}} + o_n(1)(1 - \lambda_n),$$

where $o_n(1)$ is a quantity that goes to zero as $n \rightarrow \infty$.

Proof. By definition

$$e^{\frac{u_n}{2}} = e^{\frac{u_{\lambda_n} + \psi_n}{2}} = e^{\frac{\psi_n(p)}{2}} e^{\frac{u_{\lambda_n}}{2}} e^{\frac{\psi_n - \psi_n(p)}{2}} = e^{\frac{\psi_n(p)}{2}} e^{\frac{u_{\lambda_n}}{2}} (1 + O(\psi_n - \psi_n(p))),$$

and hence

$$\int_{\partial\mathbb{D}} (h_n)_\tau e^{\frac{u_n}{2}} y = e^{\frac{\psi_n(p)}{2}} \int_{\partial\mathbb{D}} (h_n)_\tau e^{\frac{u_{\lambda_n}}{2}} y + \underbrace{e^{\frac{\psi_n(p)}{2}} \int_{\partial\mathbb{D}} (h_n)_\tau e^{\frac{u_{\lambda_n}}{2}} y O(\psi_n - \psi_n(p))}_{I}.$$

Using that $\|\psi_n\|_{C^{0,\alpha}} \leq C(1 - \lambda_n)^{1-2\alpha}$, $\alpha \in (0, \frac{1}{2})$ we get

$$\begin{aligned} |I| &\leq C(1 - \lambda_n)^{1-2\alpha} \int_{\partial\mathbb{D}} e^{\frac{u_{\lambda_n}}{2}} |y||z - 1|^\alpha dz \\ &\leq C(1 - \lambda_n)^{2-2\alpha} \int_{\partial\mathbb{D}} \frac{|y||z - 1|^\alpha}{\hat{\phi}_n^2(1 - \lambda_n x)^2 + \hat{\phi}_n^2(\lambda_n y)^2 + \hat{k}_n(x - \lambda_n)^2 + \hat{k}_n y^2} dz \\ &\leq C(1 - \lambda_n)^{2-2\alpha}, \end{aligned}$$

and (i) follows. Likewise,

$$\begin{aligned} &\int_{\mathbb{D}} ((K_n)_y xy - (K_n)_x(1 - x^2 + y^2)) e^{u_n} \\ &= e^{\psi_n(p)} \int_{\mathbb{D}} ((K_n)_y xy - (K_n)_x(1 - x^2 + y^2)) e^{u_{\lambda_n}} \\ &\quad + \underbrace{e^{\psi_n(p)} \int_{\mathbb{D}} ((K_n)_y xy - (K_n)_x(1 - x^2 + y^2)) e^{u_{\lambda_n}} O(\psi_n - \psi_n(p))}_{II}, \end{aligned}$$

and

$$\begin{aligned} |II| &\leq C(1 - \lambda_n)^{2-2\alpha} \int_{\mathbb{D}} \frac{(|x||y| + |1 - x| + |y|^2)|z - 1|^\alpha(1 + \lambda_n)}{(\hat{\phi}_n^2(1 - \lambda_n x)^2 + \hat{\phi}_n^2(\lambda_n y)^2 + \hat{k}_n(x - \lambda_n)^2 + \hat{k}_n y^2)^2} dz \\ &\leq C(1 - \lambda_n)^{2-2\alpha}, \end{aligned}$$

what proves (ii). □

The next result finishes the proof of Theorem 1.1.

Theorem 5.3. *Recall the definition of Φ given in (1.10). Then*

$$(5.5) \quad 2(-\Delta)^{1/2}h(p) + \frac{K_x(p)}{\Phi(p)} = 0,$$

or, in other words, $\Phi_\nu(p) = 0$.

Proof. Let $u_n(z) = u_{\lambda_n}(z) + \psi_n(z)$ be the solution of (1.7) that satisfies Theorem 1.1, (ii), with u_{λ_n} defined in (5.4) for $\lambda_n \in (-1, 1)$. By Proposition 2.7 there holds

$$(5.6) \quad 2 \int_{\mathbb{D}} (K_n)_y e^{u_n} xy - \int_{\mathbb{D}} (K_n)_x e^{u_n} (1 - x^2 + y^2) = -4 \int_{\partial\mathbb{D}} (h_n)_\tau e^{u_n/2} y,$$

and, by Lemma 5.2,

$$(5.7) \quad \begin{aligned} &2 \int_{\mathbb{D}} (K_n)_y e^{u_{\lambda_n}} xy - \int_{\mathbb{D}} (K_n)_x e^{u_{\lambda_n}} (1 - x^2 + y^2) \\ &= -4 \int_{\partial\mathbb{D}} (h_n)_\tau e^{u_{\lambda_n}/2} y + o_n(1)(1 - \lambda_n). \end{aligned}$$

Let us study first the term in the right hand side. Integrating by parts and substituting by the precise value of u_{λ_n} we get

$$\begin{aligned} -4 \int_{\partial\mathbb{D}} (h_n)_\tau e^{u_{\lambda_n}/2} y &= -4 \int_{\partial\mathbb{D}} (h_n - h_n(p))_\tau e^{u_{\lambda_n}/2} y \\ &= 4 \int_{\partial\mathbb{D}} (h_n - h_n(p)) \left(e^{u_{\lambda_n}/2} \frac{(u_{\lambda_n})_\tau}{2} y + e^{u_{\lambda_n}/2} x \right) \\ &= 8(1 - \lambda_n^2) \frac{\hat{\phi}_n}{\hat{\phi}_n^2 + \hat{k}_n} \int_{\partial\mathbb{D}} (h_n - h_n(p)) \frac{x(1 - \lambda_n)^2 + 2\lambda_n(x - 1)}{((1 - \lambda_n)^2 - 2\lambda_n(x - 1))^2}. \end{aligned}$$

Consider now the left hand side in (5.7). Again integrating by parts and using the definition of u_{λ_n} it can be seen that

$$\begin{aligned} 2 \int_{\mathbb{D}} (K_n)_y e^{u_{\lambda_n}} xy - \int_{\mathbb{D}} (K_n)_x e^{u_{\lambda_n}} (1 - x^2 + y^2) \\ = \int_{\mathbb{D}} (K_n - K_n(p)) ((e^{u_{\lambda_n}})_x (1 - x^2 + y^2) - 2(e^{u_{\lambda_n}})_y xy - 4xe^{u_{\lambda_n}}) \\ = -16(1 - \lambda_n^2)^2 \hat{\phi}_n^2 (\hat{\phi}_n^2 + \hat{k}_n) \int_{\mathbb{D}} \frac{(K_n - K_n(p))(-\lambda_n((1 - x)^2 + y^2) + (1 - \lambda_n)^2 x)}{(\hat{\phi}_n^2(1 - \lambda_n x)^2 + \hat{\phi}_n^2(\lambda_n y)^2 + \hat{k}_n(x - \lambda_n)^2 + \hat{k}_n y^2)^3}. \end{aligned}$$

Calling

$$\begin{aligned} I_n &:= \int_{\partial\mathbb{D}} (h_n - h_n(p)) \frac{x(1 - \lambda_n)^2 + 2\lambda_n(x - 1)}{((1 - \lambda_n)^2 - 2\lambda_n(x - 1))^2}, \\ II_n &:= (1 - \lambda_n) \int_{\mathbb{D}} \frac{(K_n - K_n(p))(-\lambda_n((1 - x)^2 + y^2) + (1 - \lambda_n)^2 x)}{(\hat{\phi}_n^2(1 - \lambda_n x)^2 + \hat{\phi}_n^2(\lambda_n y)^2 + \hat{k}_n(x - \lambda_n)^2 + \hat{k}_n y^2)^3}, \end{aligned}$$

the identity (5.7) is equivalent to

$$(5.8) \quad I_n = -2(1 + \lambda_n) \hat{\phi}_n (\hat{\phi}_n^2 + \hat{k}_n)^2 II_n + o_n(1).$$

We aim to pass to the limit when $n \rightarrow \infty$ in this expression, that corresponds to the effect of concentration of the *bubble* at the point p .

Let $\varepsilon > 0$ fixed. We assume $1 - \lambda_n < \varepsilon$, and we compute the limit of I_n dividing the integral in three regions: $\partial\mathbb{D} \setminus B_\varepsilon(p)$, $\partial\mathbb{D} \cap (B_\varepsilon(p) \setminus B_{1-\lambda_n}(p))$ and $\partial\mathbb{D} \cap B_{1-\lambda_n}(p)$. Indeed, if we denote

$$f_n(x, y) := \frac{x(1 - \lambda_n)^2 + 2\lambda_n(x - 1)}{((1 - \lambda_n)^2 - 2\lambda_n(x - 1))^2},$$

we write

$$\begin{aligned} I_n &= \int_{\partial\mathbb{D} \setminus B_\varepsilon(p)} (h_n - h_n(p)) f_n + \int_{\partial\mathbb{D} \cap (B_\varepsilon(p) \setminus B_{1-\lambda_n}(p))} (h_n - h_n(p)) f_n \\ &\quad + \int_{\partial\mathbb{D} \cap B_{1-\lambda_n}(p)} (h_n - h_n(p)) f_n \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

Notice that in the region $\partial\mathbb{D} \setminus B_\varepsilon(p)$ the integral is no longer singular. Thus, for λ_n sufficiently close to 1,

$$|(h_n - h_n(p)) f_n| \leq C \|h\|_{L^\infty(\partial\mathbb{D})} \in L^1(\partial\mathbb{D} \setminus B_\varepsilon(p)),$$

where C is a positive constant independent of n . By the Dominated Convergence Theorem,

$$\lim_{n \rightarrow \infty} I_1 = \int_{\partial \mathbb{D} \setminus B_\varepsilon(p)} \frac{h(p) - h}{2(1-x)}.$$

On the other hand, due to the evenness of f_n with respect to the variable y , I_2 can be written as

$$I_2 = \int_{\partial \mathbb{D}} (h_n - h_n(p) - (h_n)_\tau(p)y) f_n \chi_{\partial \mathbb{D} \cap (B_\varepsilon(p) \setminus B_{1-\lambda_n}(p))},$$

and using the facts that h_n is uniformly bounded in $C^2(\partial \mathbb{D})$ and $|x-1| = \frac{|z-1|^2}{2} > \frac{(1-\lambda_n)^2}{2}$ we get

$$|I_2| \leq C \int_{\partial \mathbb{D}} \frac{|y|^2|x-1|}{|x-1|^2} \chi_{B_\varepsilon(p)} \leq C \int_{\partial \mathbb{D} \cap B_\varepsilon(p)} \frac{|z-1|^2|z-1|^2}{|z-1|^4} \leq C\varepsilon.$$

Therefore,

$$\lim_{n \rightarrow \infty} I_2 = O(\varepsilon).$$

Likewise,

$$|I_3| \leq C \int_{\partial \mathbb{D} \cap B_{1-\lambda_n}(p)} \frac{(1-\lambda_n)^2(1-\lambda_n)^2}{(1-\lambda_n)^4} \leq C(1-\lambda_n),$$

where we have used again the evenness of f_n with respect to y and that in this region $|1-x| \leq C(1-\lambda_n)^2$ and $|y|^2 \leq C(1-\lambda_n)^2$. Hence,

$$\lim_{n \rightarrow \infty} I_3 = 0.$$

We compute now the limit of II_n . Naming

$$g_n(x, y) := \frac{(-\lambda_n((1-x)^2 + y^2) + (1-\lambda_n)^2x)}{\left(\hat{\phi}_n^2(1-\lambda_nx)^2 + \hat{\phi}_n^2(\lambda_ny)^2 + \hat{k}_n(x-\lambda_n)^2 + \hat{k}_ny^2\right)^3},$$

we split the integral as

$$\begin{aligned} II_n &= (1-\lambda_n) \int_{\mathbb{D} \setminus B_\rho(p)} (K_n - K_n(p))g_n + (1-\lambda_n) \int_{\mathbb{D} \cap B_\rho(p)} (K_n - K_n(p))g_n \\ &= II_1 + II_2. \end{aligned}$$

We first consider the case of $\mathbb{D} \setminus B_\rho(p)$. Observe that for $0 < \rho < 1$ fixed and $z = (x, y)$ we have $|x-1| \geq c(\rho) > 0$ for some constant dependent on ρ , and hence

$$|x-\lambda_n| = |x-1-\lambda_n+1| \geq |x-1| - |1-\lambda_n| \geq c(\rho) - |1-\lambda_n| \geq \frac{c(\rho)}{2}$$

for λ_n sufficiently close to 1. On the other hand,

$$(1-\lambda_nx)^2 \geq (x-\lambda_n)^2,$$

and thus, provided that $\Phi(p)^2 + K(p) > 0$, for λ_n close enough to 1 we have that

$$|II_1| \leq C(1-\lambda_n)\|K\|_{L^\infty(\mathbb{D})},$$

with C a positive constant independent of n . Thus

$$\lim_{n \rightarrow \infty} II_1 = 0.$$

We finally estimate the term II_2 . Using that

$$K_n(x, y) - K_n(p) - (K_n)_y(p)y = (K_n)_x(p)(x - 1) + O(|z - 1|^2),$$

and the evenness of g_n with respect to y we write the integral as

$$\begin{aligned} II_2 &= (1 - \lambda_n)(K_n)_x(p) \int_{\mathbb{D} \cap B_\rho(p)} (x - 1)g_n + (1 - \lambda_n) \int_{\mathbb{D} \cap B_\rho(p)} O(|z - 1|^2)g_n \\ &=: II_{21} + II_{22}. \end{aligned}$$

Making the change of variables

$$1 - x = \tilde{x}(1 - \lambda_n), \quad y = \tilde{y}(1 - \lambda_n),$$

we obtain

$$II_{21} = (K_n)_x(p) \int_{(\mathbb{R}_+ \times \mathbb{R}) \cap B_{\frac{\rho}{1-\lambda_n}}(0)} \frac{\tilde{x}(\tilde{x}(1 - \lambda_n) - 1 + \lambda_n(\tilde{x}^2 + \tilde{y}^2)) d\tilde{y} d\tilde{x}}{(\hat{\phi}_n^2(1 + \lambda_n\tilde{x})^2 + \hat{\phi}_n^2(\lambda_n\tilde{y})^2 + \hat{k}_n(1 - \tilde{x})^2 + \hat{k}_n\tilde{y}^2)^3}.$$

Assume first $K(p) > 0$. Notice that, since $\tilde{x} \geq 0$, $(1 + \lambda_n\tilde{x})^2 \geq 1 + \lambda_n^2\tilde{x}^2$, and thus, for λ_n close enough to 1,

$$\begin{aligned} &\left| \frac{\tilde{x}(\tilde{x}(1 - \lambda_n) - 1 + \lambda_n(\tilde{x}^2 + \tilde{y}^2))}{(\hat{\phi}_n^2(1 + \lambda_n\tilde{x})^2 + \hat{\phi}_n^2(\lambda_n\tilde{y})^2 + \hat{k}_n(1 - \tilde{x})^2 + \hat{k}_n\tilde{y}^2)^3} \chi_{(\mathbb{R}_+ \times \mathbb{R}) \cap B_{\frac{\rho}{1-\lambda_n}}(0)} \right| \\ &\leq C \frac{\tilde{x}(2(\tilde{x}^2 + \tilde{y}^2) + 1)}{(\Phi(p)^2 + \Phi(p)^2\frac{\tilde{x}^2}{2} + K(p)\tilde{y}^2)^3}, \end{aligned}$$

that belongs to $L^1(\mathbb{R}_+ \times \mathbb{R})$. Otherwise, if $K(p) \leq 0$ there holds (for λ_n close enough to 1)

$$\begin{aligned} \hat{\phi}_n^2(1 + \lambda_n\tilde{x})^2 + \hat{k}_n(1 - \tilde{x})^2 &= \hat{\phi}_n^2 + \hat{k}_n + \tilde{x}^2(\hat{\phi}_n^2\lambda_n^2 + \hat{k}_n) - 2\tilde{x}(-\lambda_n\hat{\phi}_n^2 + \hat{k}_n) \\ &\geq \hat{\phi}_n^2 + \hat{k}_n + \tilde{x}^2(\hat{\phi}_n^2\lambda_n^2 + \hat{k}_n), \end{aligned}$$

since $\tilde{x} \geq 0$ and $\Phi(p)^2 + K(p) > 0$. Thus, there exists $c_0 > 0$, independent of λ_n , such that

$$\begin{aligned} &\left| \frac{\tilde{x}(\tilde{x}(1 - \lambda_n) - 1 + \lambda_n(\tilde{x}^2 + \tilde{y}^2))}{(\hat{\phi}_n^2(1 + \lambda_n\tilde{x})^2 + \hat{\phi}_n^2(\lambda_n\tilde{y})^2 + \hat{k}_n(1 - \tilde{x})^2 + \hat{k}_n\tilde{y}^2)^3} \chi_{(\mathbb{R}_+ \times \mathbb{R}) \cap B_{\frac{\rho}{1-\lambda_n}}(0)} \right| \\ &\leq c_0 \frac{\tilde{x}(2(\tilde{x}^2 + \tilde{y}^2) + 1)}{(1 + \tilde{x}^2 + \tilde{y}^2)^3} \in L^1(\mathbb{R}_+ \times \mathbb{R}). \end{aligned}$$

Passing to the limit in the integral we conclude

$$\begin{aligned} \lim_{n \rightarrow \infty} II_{21} &= K_x(p) \left[\int_{\mathbb{R}_+ \times \mathbb{R}} \frac{\tilde{x}((\tilde{x}^2 + \tilde{y}^2) - 1) d\tilde{y} d\tilde{x}}{(\Phi(p)^2(1 + \tilde{x})^2 + (\Phi(p)^2 + K(p))\tilde{y}^2 + K(p)(1 - \tilde{x})^2)^3} \right] \\ &= \frac{\pi}{8\Phi(p)^2(K(p) + \Phi(p)^2)} K_x(p). \end{aligned}$$

The above integral in the half-plane has been computed with the help of Mathematica.

Moreover,

$$\begin{aligned} |II_{22}| &\leq C(1 - \lambda_n) \int_{(\mathbb{R}_+ \times \mathbb{R}) \cap B_{\frac{\rho}{1-\lambda_n}}(0)} \frac{(\tilde{x}^2 + \tilde{y}^2)|\tilde{x}(1 - \lambda_n) - 1 + \lambda_n(\tilde{x}^2 + \tilde{y}^2)| d\tilde{y} d\tilde{x}}{|\hat{\phi}_n^2(1 + \lambda_n\tilde{x})^2 + \hat{\phi}_n^2(\lambda_n\tilde{y})^2 + \hat{k}_n(1 - \tilde{x})^2 + \hat{k}_n\tilde{y}^2|^3} \\ &\leq C(1 - \lambda_n) \int_{(\mathbb{R}_+ \times \mathbb{R}) \cap B_{\frac{\rho}{1-\lambda_n}}(0)} \frac{(\tilde{x}^2 + \tilde{y}^2)(2(\tilde{x}^2 + \tilde{y}^2) + 1) d\tilde{y} d\tilde{x}}{(1 + \tilde{x}^2 + \tilde{y}^2)^3} \\ &= O((1 - \lambda_n) \log(1 - \lambda_n)), \end{aligned}$$

and therefore

$$\lim_{n \rightarrow \infty} II_{22} = 0.$$

Passing to the limit in (5.8) we get

$$\int_{\partial\mathbb{D} \setminus B_\varepsilon(p)} \frac{h(p) - h}{2(1 - x)} + O(\varepsilon) = -\frac{\pi}{2\Phi(p)} K_x(p).$$

Finally, making $\varepsilon \rightarrow 0$,

$$\text{p.v.} \int_{\partial\mathbb{D}} \frac{h(p) - h}{2(1 - x)} = -\frac{\pi}{2\Phi(p)} K_x(p).$$

Taking into account the definition of $(-\Delta)^{1/2}$ in \mathbb{S}^1 (see, for instance, [19, Appendix]), we obtain:

$$2(-\Delta)^{1/2}h(p) + \frac{K_x(p)}{\Phi(p)} = 0.$$

□

Remark 5.4. As we have seen, the point of concentration of a blow-up sequence is a critical point of the function Φ . This phenomenon has a nice interpretation. Consider the energy functional associated to (1.5), that is, $I : H^1(\mathbb{D}) \rightarrow \mathbb{R}$,

$$I(u) := \int_{\mathbb{D}} \frac{1}{2} |\nabla u|^2 - 2Ke^u + \int_{\partial\mathbb{D}} 2u - 4he^{u/2}.$$

Let us evaluate the energy of the functions u_{λ_n} defined in (5.4), as $\lambda_n \rightarrow 1$. Because of the concentration phenomena, we have

$$I(u_{\lambda_n}) = \tilde{I}(u_{\lambda_n}) + o_n(1),$$

where \tilde{I} is the limit functional related to constant curvatures

$$\tilde{I}(u) := \int_{\mathbb{D}} \frac{1}{2} |\nabla u|^2 - 2K(p)e^u + \int_{\partial\mathbb{D}} 2u - 4h(p)e^{u/2}.$$

Observe now that $\tilde{I}(u_{\lambda_n})$ is constant in λ_n ; so, for convenience, we may take $\lambda_n = 0$, that is,

$$u_0(z) = 2 \log \left(\frac{2\Phi(p)}{\Phi(p)^2 + K(p)|z|^2} \right).$$

Let us compute the terms

$$\begin{aligned} \int_{\mathbb{D}} \frac{1}{2} |\nabla u_0|^2 &= 8 \frac{K(p)^2}{\Phi(p)^4} 2\pi \int_0^1 \frac{r^3 dr}{(1 + \frac{K(p)}{\Phi(p)^2} r^2)^2} = 8\pi \int_0^{\frac{K(p)}{\Phi(p)^2}} \frac{s ds}{(1 + s)^2} \\ &= 8\pi \left(\log \left(1 + \frac{K(p)}{\Phi(p)^2} \right) + \frac{\Phi(p)^2}{\Phi(p)^2 + K(p)} - 1 \right) \end{aligned}$$

$$\int_{\partial\mathbb{D}} 2u_0 = 8\pi \log \frac{2\Phi(p)}{\Phi(p)^2 + K(p)}.$$

Moreover, we already know the value of the exponential terms, recall (3.3). Then,

$$\begin{aligned} \tilde{I}(u_0) &= 8\pi \left(\log(2/\Phi(p)) + \frac{\Phi(p)^2}{\Phi(p)^2 + K(p)} - 1 \right) - 4\pi \left(1 + \frac{h(p)}{h^2(p) + K(p)} \right) \\ &= -8\pi + 4\pi \left(2\log(2/\Phi(p)) + \frac{2\Phi(p)^2}{\Phi(p)^2 + K(p)} - 1 - \frac{h(p)}{h^2(p) + K(p)} \right). \end{aligned}$$

By definition $\frac{2\Phi(p)^2}{\Phi(p)^2 + K(p)} - 1 - \frac{h(p)}{h^2(p) + K(p)} = 0$ and hence

$$\tilde{I}(u_0) = -8\pi(1 + \log(\Phi(p)/2)).$$

6. APPENDIX

Proposition 6.1. *Let $1 < q < 2$ and $a_n \in \mathbb{D}$ with $|a_n| \rightarrow 1$ as $n \rightarrow +\infty$. There exists $C > 0$, independent of n , such that*

$$\begin{aligned} i) \quad & \int_{\mathbb{D}} \left| f_{a_n}(z) - f_{a_n} \left(\frac{a_n}{|a_n|} \right) \right|^q \leq C(1 - |a_n|)^q, \\ ii) \quad & \int_{\partial\mathbb{D}} \left| f_{a_n}(z) - f_{a_n} \left(\frac{a_n}{|a_n|} \right) \right|^q \leq C(1 - |a_n|), \end{aligned}$$

where $f_a(z) := \frac{a+z}{1+\bar{a}z}$.

Proof. By definition of $f_a(z)$,

$$\begin{aligned} \left| f_{a_n}(z) - f_{a_n} \left(\frac{a_n}{|a_n|} \right) \right| &= \left| \frac{a_n + z}{1 + \bar{a}_n z} - \frac{a_n}{|a_n|} \right| = \frac{(1 - |a_n|)}{|a_n|} \left| \frac{a_n - |a_n|z}{1 + \bar{a}_n z} \right| \\ &\leq C(1 - |a_n|) \frac{1}{|1 + \bar{a}_n z|}, \end{aligned}$$

for $|a_n|$ sufficiently close to 1. Thus,

$$(6.1) \quad \int_{\mathbb{D}} \left| f_{a_n}(z) - f_{a_n} \left(\frac{a_n}{|a_n|} \right) \right|^q \leq C(1 - |a_n|)^q \int_{\mathbb{D}} \frac{1}{|1 + \bar{a}_n z|^q},$$

that is singular at $z_0 = -\frac{1}{\bar{a}_n} = -\frac{a_n}{|a_n|^2}$. Notice that $|z_0| > 1$ and therefore $z_0 \notin \mathbb{D}$. Since $1 < q < 2$ we immediately obtain *i*).

Let us estimate the integral on $\partial\mathbb{D}$. Notice first that, if $z \in \partial\mathbb{D}$, then

$$|1 + \bar{a}_n z| = |\bar{z}| |1 + \bar{a}_n z| = |\bar{z} + \bar{a}_n| = |z + a_n| = \text{dist}(z, -a_n).$$

Thus, arguing as before we obtain

$$\begin{aligned} (6.2) \quad \int_{\partial\mathbb{D}} |f_{a_n}(z) - f_{a_n} \left(\frac{a_n}{|a_n|} \right)|^q &\leq C(1 - |a_n|)^q \int_{\partial\mathbb{D}} \frac{1}{|1 + \bar{a}_n z|^q} \\ &= C(1 - |a_n|)^q \int_{\partial\mathbb{D}} \frac{1}{\text{dist}(z, -a_n)^q}. \end{aligned}$$

Notice that

$$\text{dist}(z, -a_n) \geq \text{dist} \left(-\frac{a_n}{|a_n|}, -a_n \right) = 1 - |a_n|.$$

We split the integral in two regions,

$$\Sigma_1 := \partial\mathbb{D} \setminus B_1\left(-\frac{a_n}{|a_n|}\right), \quad \Sigma_2 := \partial\mathbb{D} \cap B_1\left(-\frac{a_n}{|a_n|}\right).$$

If $z \in \Sigma_1$ then there exists $c > 0$, independent of n , such that $\text{dist}(z, -a_n) \geq c$. Therefore

$$(6.3) \quad \int_{\Sigma_1} \frac{1}{\text{dist}(z, -a_n)^q} \leq c^{-q} |\Sigma_1| \leq c^{-q} |\partial\mathbb{D}|.$$

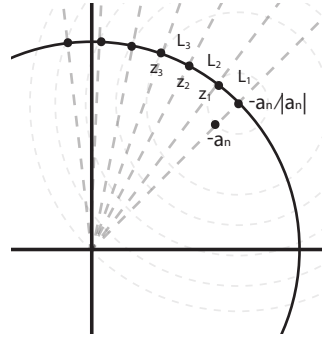
To analyze the region Σ_2 we divide the integral in subintervals in the following form,

$$L_1 := B_{(1-|a_n|)}\left(-\frac{a_n}{|a_n|}\right) \cap \Sigma_2,$$

$$L_j := \left(B_{j(1-|a_n|)}\left(-\frac{a_n}{|a_n|}\right) \setminus B_{(j-1)(1-|a_n|)}\left(-\frac{a_n}{|a_n|}\right) \right) \cap \Sigma_2, \quad j = 2, \dots, J_n,$$

where $J_n := \lceil \frac{1}{1-|a_n|} \rceil$, and

$$\{z_j, \tilde{z}_j\} := \partial B_{j(1-|a_n|)}\left(-\frac{a_n}{|a_n|}\right) \cap \Sigma_2, \quad j = 1, \dots, J_n - 1.$$



Domain Σ_2 .

Notice that

$$|L_j| = 2 \arcsin(1 - |a_n|) \approx 2(1 - |a_n|),$$

and

$$\text{dist}(z, -a_n) \geq 1 - |a_n|, \quad z \in L_1,$$

$$\text{dist}(z, -a_n) \geq \text{dist}(z_{j-1}, -a_n) \geq (j-1)(1 - |a_n|), \quad z \in L_j,$$

for all $j = 2, \dots, J_n$. Thus,

$$\int_{L_1} \frac{1}{\text{dist}(z, -a_n)^q} \leq \frac{2|L_1|}{(1 - |a_n|)^q} \leq C(1 - |a_n|)^{1-q},$$

$$\int_{L_j} \frac{1}{\text{dist}(z, -a_n)^q} \leq \frac{C}{(j-1)^q} (1 - |a_n|)^{1-q}, \quad j = 2, \dots, J_n.$$

Summing up we get

$$(6.4) \quad \begin{aligned} \int_{\Sigma_2} \frac{1}{\text{dist}(z, -a_n)^q} &\leq C(1 - |a_n|)^{1-q} \left(1 + \sum_{j=2}^{J_n} \frac{1}{(j-1)^q} \right) \\ &\leq C(1 - |a_n|)^{1-q} \left(1 + \sum_{j=1}^{+\infty} \frac{1}{j^q} \right) \leq C(1 - |a_n|)^{1-q}. \end{aligned}$$

Replacing (6.3) and (6.4) in (6.2) we conclude the validity of *ii*. \square

6.1. On the validity of assumption (1.8). We finally discuss some conditions under which condition (1.8) of Theorem 1.1 holds. First of all, by the Gauss-Bonnet formula (1.6), if $K > 0$, $h > 0$ (1.8) is trivially satisfied. In order to find other conditions ensuring (1.6) let us recall two classical results:

- (1) **The isoperimetric inequality**, in the following version (see for instance [41, Theorem 4.3, page 1206]):

Let Ω be a simply connected domain in a surface with Gaussian curvature K . Denote by L the length of its boundary and by A its area. Then,

$$L^2 \geq 2A \left(2\pi - \int_{\Omega} K^+ \right).$$

- (2) **Toponogov's length bound:** Let Ω be a simply connected domain in a surface with $h \geq 0$, where h is the geodesic curvature of its boundary. Assume that $K \geq k_0$ for some constant $k_0 > 0$. Then the length of $\partial\Omega$ satisfies the inequality:

$$L \leq \frac{2\pi}{\sqrt{k_0}}.$$

A comment is in order here. The result of Toponogov ([44]) applies to domains on complete manifolds with Gaussian curvature bounded from below. Hence an extension of Ω to a complete surface is needed, but this is indeed possible (see [33]). However, a direct proof can be given, see Theorem 4 of [29]. There the case $k_0 = 1$ is treated, but a simple scaling argument gives the result for any $k_0 > 0$.

We can now prove the following result:

Proposition 6.2. *Let u be a solution of the problem:*

$$(6.5) \quad \begin{cases} -\Delta u = 2Ke^u & \text{in } \mathbb{D}, \\ \frac{\partial u}{\partial \nu} + 2 = 2he^{u/2} & \text{on } \partial\mathbb{D}, \end{cases}$$

for some continuous functions $K(x)$, $h(x)$. Assume that for some $\delta > 0$:

- a) either $K(x) \geq 0$ and $h \geq \delta$,
b) or $K \geq \delta^2$ and $h \geq 0$.

Then,

$$(6.6) \quad \int_{\mathbb{D}} e^{u_n} + \int_{\partial\mathbb{D}} e^{u_n/2} \leq C,$$

where the constant C depends only on δ .

Proof. Assume first a). By (1.6), we already know that

$$\int_{\partial\mathbb{D}} h e^{u/2} \leq 2\pi \Rightarrow \int_{\partial\mathbb{D}} e^{u/2} \leq \frac{2\pi}{\delta}.$$

In order to bound the area term, we apply the isoperimetric inequality to the metric $e^u g_0$:

$$(6.7) \quad \left(\int_{\partial\mathbb{D}} e^{u/2} \right)^2 \geq 2 \int_{\mathbb{D}} e^u \left(2\pi - \int_{\mathbb{D}} K^+ e^u \right).$$

Since $K \geq 0$ and making use of (1.6), $2\pi - \int_{\mathbb{D}} K^+ e^u = 2\pi - \int_{\mathbb{D}} K e^u = \int_{\partial\mathbb{D}} h e^{u/2}$. Hence,

$$\left(\int_{\partial\mathbb{D}} e^{u/2} \right)^2 \geq 2\delta \int_{\mathbb{D}} e^u \int_{\partial\mathbb{D}} e^{u/2} \Rightarrow \int_{\mathbb{D}} e^u \leq \frac{1}{2\delta} \int_{\partial\mathbb{D}} e^{u/2} \leq \frac{\pi}{\delta^2}.$$

Assume now b). In this case (1.6) implies that

$$\int_{\mathbb{D}} e^u \leq \frac{2\pi}{\delta^2}.$$

In order to bound the boundary term, we apply Toponogov result to the metric $e^u g_0$:

$$\int_{\partial\mathbb{D}} e^{u/2} \leq \frac{2\pi}{\delta}.$$

□

REFERENCES

- [1] T. Aubin, Meilleures constantes dans le thorme d'inclusion de Sobolev et un thorme de Fredholm non linéaire pour la transformation conforme de la courbure scalaire, *J. Functional Analysis* 32 (1979), no. 2, 148-174.
- [2] D. Bartolucci, C-C. Chen, C-S. Lin, G. Tarantello, Profile of blow-up solutions to mean field equations with singular data, *Comm. Partial Differential Equations* 29 (2004), no. 7-8, 1241-1265.
- [3] D. Bartolucci, A. Jevnikar, Y. Lee, W. Yang, Uniqueness of bubbling solutions of mean field equations, *J. Math. Pures Appl.*, 123 (2019), 78-126.
- [4] L. Battaglia, R. López-Soriano, A double mean field related to a curvature prescription problem, *J. Differential Equations*, (2020) <https://doi.org/10.1016/j.jde.2020.02.012>.
- [5] M. Berger, On Riemannian structures of prescribed Gaussian curvature for compact 2-manifolds, *J. Diff. Geom.* 5 (1971), 325-332.
- [6] S. Brendle, A family of curvature flows on surfaces with boundary, *Math. Z.* 241 (2002), no. 4, 829-869.
- [7] H. Brezis, F. Merle, Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions, *Commun. Partial Differential Equations* 16 (1991), 1223-1253.
- [8] S.Y.A. Chang, M. Gursky, P.C. Yang, The scalar curvature equation on 2- and 3-spheres, *Calc. Var. Partial Differential Equations* 1 (1993), no. 2, 205-229.
- [9] K.C. Chang, J.Q. Liu, A prescribing geodesic curvature problem, *Math. Z.* 223 (1996), 343-365.
- [10] S.Y.A. Chang, P.C. Yang, Prescribing Gaussian curvature on \mathbb{S}^2 , *Acta Math.* 159 (1987), no. 3-4, 215-259.
- [11] S.Y.A. Chang, P.C. Yang, Conformal deformation of metrics on \mathbb{S}^2 , *J. Diff. Geom.* 27 (1988), 259-296.
- [12] S. Y. A. Chang, P.C. Yang, A perturbation result in prescribing scalar curvature on \mathbb{S}^2 , *Duke Math. J.* 64, 27-69 (1991)
- [13] W. Chen, W. Ding, Scalar curvatures on \mathbb{S}^2 , *TAMS Vol. 303, Number 1* (1987), 365-382.

- [14] W. Chen, W. Ding, A problem concerning the scalar curvature on \mathbb{S}^2 , *Kexue Tongbao* (English Ed.) 33 (1988), no. 7, 533-537.
- [15] W. X. Chen, C. Li, C. (1991), Classification of solutions of some nonlinear elliptic equations, *Duke Math. J.* 63(3):615-622.
- [16] W. Chen, C. Li, A necessary and sufficient condition for the Nirenberg problem. (English summary), *Comm. Pure Appl. Math.* 48 (1995), no. 6, 657-667.
- [17] P. Cherrier, Problemes de Neumann non lineaires sur les variets riemanniennes, *J. Funct. Anal.* 57 (1984), 154-206.
- [18] S. Cruz, D. Ruiz, Prescribing Gaussian and geodesic curvatures on disks, *Adv. Nonlinear Stud.* 18 (2018), no. 3, 453-468.
- [19] F. Da Lio, L. Martinazzi, T. Riviere, Blow-Up Analysis of a Nonlocal Liouville-Type Equation, *Analysis and PDE* 8 (2015), No. 7, 1757-1805.
- [20] F. De Marchis, R. López-Soriano, Existence and non existence results for the singular Nirenberg problem, *Calc. Var. Partial Differential Equations*, 55(2):Art. 36, 35, 2016.
- [21] F. De Marchis, R. López-Soriano, D. Ruiz, Compactness, existence and multiplicity for the singular mean field problem with sign-changing potentials, *J. Math. Pures Appl.* 115 (2018), 237-267.
- [22] J. M. Espinar, J. A. Galvez and P. Mira, Hypersurfaces in $\mathbb{H}^n + 1$ and conformally invariant equations: the generalized Christoffel and Nirenberg problems, *J. Eur. Math. Soc.* 11 (2009), 903-939.
- [23] J.A. Gálvez, P. Mira, The Liouville equation in a half-plane, *J. Differential Equations* 246 (2009), no. 11, 4173-4187.
- [24] K. Guo and S. Hu, Conformal deformation of metrics on subdomains of surfaces, *J. Geom. Anal.* 5 (1995), no. 3, 395-410.
- [25] Y.X. Guo, J. Liu, Blow-up analysis for solutions of the Laplacian equation with exponential Neumann boundary condition in dimension two, *Commun. Contemp. Math.* 8 (2006), 737-761.
- [26] H. Hamza, Sur les transformations conformes des variétés riemanniennes à bord, *J. Funct. Anal.* 92 (1990), no. 2, 403-447.
- [27] Z. C. Han, Prescribing Gaussian curvature on \mathbb{S}^2 , *Duke Math. J.* 61 (1990), no. 3, 679-703.
- [28] Z. C. Han, Y. Li, A note on the Kazdan-Warner type condition, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 13 (1996), no. 3, 283-292.
- [29] F. Hang, X. Wang, Rigidity Theorems for Compact Manifolds with Boundary and Positive Ricci Curvature, *J. Geom. Anal.* (2009), 628642.
- [30] Min Ji, On positive scalar curvature on \mathbb{S}^2 , *Calc. Var.* 19, 165-182 (2004).
- [31] J. Jost, G. Wang, C. Zhou, Metrics of constant curvature on a Riemann surface with two corners on the boundary, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(2):437-456, 2009.
- [32] J.L. Kazdan, F.W. Warner, Curvature functions for compact 2-manifolds, *Ann. of Math.* 99 (1974), 14-47.
- [33] S. Kronwith, Convex manifolds of nonnegative curvature, *J. Diff. Geometry* 14 (1979) 621-628-
- [34] Y. Y. Li, Harnack type inequality: The method of moving planes, *Comm. Math. Phys.* 200 (1999), 421-444.
- [35] Y. Li, P. Liu, A Moser-Trudinger inequality on the boundary of a compact Riemann surface, *Math. Z.* 250 (2005), no. 2, 363-386.
- [36] Y.Y. Li, I. Shafrir, Blow-up analysis for solutions of $-\Delta u = Ve^u$ in dimension two, *Indiana Univ. Math. J.* 43 (1994), no. 4, 1255-1270.
- [37] Y.Y. Li, M. Zhu, Uniqueness theorems through the method of moving spheres, *Duke Math. J.* 80 (1995), no. 2, 383-417.
- [38] P. Liu, W. Huang, On prescribing geodesic curvature on D^2 , *Nonlinear Analysis* 60 (2005) 465-473.
- [39] R. López-Soriano, A. Malchiodi, D. Ruiz, Conformal metrics with prescribed gaussian and geodesic curvatures, submitted, [arXiv:1806.11533](https://arxiv.org/abs/1806.11533).
- [40] J. Moser, On a non-linear Problem in Differential Geometry and Dynamical Systems, Academic Press, N.Y. (ed M. Peixoto) 1973.
- [41] R. Osserman, The isoperimetric inequality, *Bulletin of the AMS* 84 (1978), 1182-1238.
- [42] M. Struwe, A flow approach to Nirenberg's problem, *Duke Math. J.* 128 (2005), no. 1, 19-64.
- [43] G. Tarantello, *Analytical aspects of Liouville-type equations with singular sources*, Stationary partial differential equations. Vol I, 491-592, *Handb. Differ. Equ.*, North-Holland, Amsterdam, (2004)
- [44] V. A. Toponogov, A bound for the length of a convex curve on a two-dimensional surface. (Russian) *Sibirsk. Mat. .* 4 1963 1189-1193.

- [45] X. Xu, P.C. Yang, Remarks on prescribing Gauss curvature, TAMS 336 (1993), 831-840.
- [46] L. Zhang, Classification of conformal metrics on \mathbb{R}_+^2 with constant Gauss curvature and geodesic curvature on the boundary under various integral finiteness assumptions, Calc. Var. Partial Differential Equations 16 (2003), no. 4, 405–430.

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