Displacement convexity of Entropy and the distance cost Optimal Transportation

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Abstract

During the last decade Optimal Transport had a relevant role in the study of geometry of singular spaces that culminated with the Lott-Sturm-Villani theory. The latter is built on the characterisation of Ricci curvature lower bounds in terms of displacement convexity of certain entropy functionals along W_2-geodesics. Substantial recent advancements in the theory (localization paradigm and local-to-global property) have been obtained considering the different point of view of L^1-Optimal transport problems yielding a different curvature dimension $\mathsf{CD}^1(K,N)$ [8] formulated in terms of one-dimensional curvature properties of integral curves of Lipschitz maps. In this note we show that the two approaches produce the same curvature-dimension condition reconciling the two definitions. In particular we show that the $\mathsf{CD}^1(K,N)$ condition can be formulated in terms of displacement convexity along W_1-geodesics.

1 Introduction

The formulation of an appropriate version of Ricci curvature lower bounds valid for possibly singular spaces has been a central topic of research for several years. During the last decade Optimal Transport had a relevant role in the topic that culminated with the successful theory of Lott-Villani [17] and Sturm [22, 23] of metric measure spaces verifying a lower bound on the Ricci curvature in a synthetic sense.

The theory is formulated in terms of displacement convexity of the Renyi entropy. The latter is defined on the set of probability measures $S_N(\cdot|\mathfrak{m}) : \mathcal{P}_2(X, \mathsf{d}) \to \mathbb{R}$ as follows

$$S_N(\mu|\mathfrak{m}) := -\int_X \rho^{-1/N}\, d\mu,$$

where ρ denotes the density of the absolutely continuous part of μ with respect to $\mathfrak{m}$. In rough terms, a space will satisfy the $\mathsf{CD}(K,N)$ condition if the entropy evaluated along W_2-geodesics is more convex than the entropy evaluated along W_2-geodesics of the model space with constant curvature K and dimension N in an appropriate sense (see Definition 2.4).

The theory had a huge impact and a detailed discussion on its development would be beyond the scope of the note. For our purposes, we mention that substantial recent advancements in the theory (localization paradigm and local-to-global property) have been obtained considering the different point of view of L^1-Optimal transport problems yielding a different curvature dimension $\mathsf{CD}^1(K,N)$ [8] formulated in terms of one-dimensional curvature properties of integral curves of Lipschitz maps.

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Motivated by the proof of the local-to-global property for the curvature-dimension condition, in [8] has been shown that a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ verifies $\mathsf{CD}(K,N)$ if and only if it satisfies $\mathsf{CD}^1(K,N)$, provided X is essentially non-branching (see Definition 2.1) and the total space to have finite mass (i.e. $\mathfrak{m}(X) < \infty$).

Moreover it was recently addressed whether or not the CD condition really depends on the special exponent $p = 2$ used to check displacement convexity of entropy. While for smooth manifold it is clear that it does not (being equivalent to a lower bound on the Ricci tensor) the general case of metric measure spaces has been considered in the recent [1] where complete equivalence will be proved. It remained however unclear if the $\mathsf{CD}^1(K,N)$ condition could be equivalently formulated in terms of displacement convexity of the Entropy functional along W_1-geodesics.

In this note we show that this is the case and the two approaches produce the same curvature-dimension condition reconciling the two definitions. We report here the main result of the paper.

Theorem 1.1. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an essentially non-branching metric measure space and further assume* $\mathfrak{m}(X) = 1$. *Then* $(X, \mathsf{d}, \mathfrak{m})$ *satisfies the* $\mathsf{CD}^1(K,N)$ *condition if and only if it satisfies the* $\mathsf{CD}_1(K,N)$ *condition.*

The $\mathsf{CD}_1(K,N)$ condition is formulated, in analogy with the classical $\mathsf{CD}(K,N)$, as displacement convexity of entropy along W_1-geodesics; its precise formulation is given in Definition 2.5.

2 Background material

In this section we will recall some basic notions used throughout the paper.

A triple $(X, \mathsf{d}, \mathfrak{m})$ is called a *metric measure space* if (X, d) is a Polish space (i.e. a complete and separable metric space) and $\mathfrak{m}$ is a positive Radon measure over X. In what follows we will always deal with m.m.s. in which $\mathfrak{m}$ is a probability measure, i.e. $\mathfrak{m}(X) = 1$; we will denote with $\mathcal{P}(X)$ the space of all Borel probability measures over X.

A curve $\gamma \in C([0,1], X)$ is called a *constant speed geodesic* if

$$\mathsf{d}(\gamma_s, \gamma_t) = |s-t| d(\gamma_0, \gamma_1), \quad \forall s, t \in [0,1].$$

From now on the set of all constant speed geodesics will be denoted with $\mathrm{Geo}(X)$ while $\mathrm{e}_t : \mathrm{Geo}(X) \to X$ will denote the *evaluation map* defined by $\mathrm{e}_t(\gamma) = \gamma_t$. Moreover we will call $(X, \mathsf{d}, \mathfrak{m})$ *geodesic* if, for any choice of $x, y \in X$, there exists $\gamma \in \mathrm{Geo}(X)$ with $\gamma_0 = x$, $\gamma_1 = y$.

As usual, for any $p \geq 1$, $\mathcal{P}_p(X)$ will denote the space of probability measures with finite p-moment, i.e.

$$\mathcal{P}_p(X) = \{\mathfrak{m} \in \mathcal{P}(X) : \int_X \mathsf{d}^p(x, x_0)\, \mathfrak{m}(dx) < +\infty, \text{for some } x_0 \in X\},$$

and with $\mathcal{P}_p(X, \mathsf{d}, \mathfrak{m})$ its subspace of $\mathfrak{m}$-absolutely continuous probability. The space $\mathcal{P}_p(X)$ will be endowed with the L^p-Wasserstein distance W_p defined by

$$W_p(\mu_0, \mu_1) = \left(\inf_{\pi} \int_{X \times X} \mathsf{d}^p(x,y) \pi(dxdy) \right)^{1/p}, \tag{2.1}$$

where the infimum is taken in the class of all probability measures in $\mathcal{P}(X \times X)$ with first and second marginal given by μ_0 and μ_1 respectively.

It is a classical fact that if (X, d) is geodesic then $(\mathcal{P}_p(X), W_p)$ is geodesic too and a curve $[0,1] \ni t \mapsto \mu_t \in \mathcal{P}_p(X)$ is a geodesic if and only if there exists $\nu \in \mathcal{P}(\mathrm{Geo}(X))$ such that $(\mathrm{e}_0, \mathrm{e}_1)_\sharp \nu$ realizes the minimum in (2.1) and $\mu_t = \mathrm{e}_{t\sharp}\nu$. We will summarize these two properties saying that ν is an optimal dynamical plan $\nu \in \mathrm{OptGeo}_p(\mu_0, \mu_1)$. Finally, $A \subset \mathrm{Geo}(X)$ is called *a set of non-branching geodesics* if for any $\gamma^1, \gamma^2 \in A$

$$\exists\, \bar{t} \in (0,1) : \gamma^1(s) = \gamma^2(s),\ \forall s \in [0, \bar{t}] \implies \gamma^1(t) = \gamma^2(t),\ \forall t \in [0,1].$$

Finally we recall the classical definition of essentially non-branching. This notion has been firstly introduced in [20] and considers only the case $p = 2$.

Definition 2.1 (Essentially non-branching)**.** Let $(X, \mathsf{d}, \mathfrak{m})$ be a m.m.s.. We say that $(X, \mathsf{d}, \mathfrak{m})$ is W_2-essentially non-branching if for any $\mu_0, \mu_1 \in \mathcal{P}_2(X, \mathsf{d}, \mathfrak{m})$ any element of $\mathrm{OptGeo}_2(\mu_0, \mu_1)$ is concentrated on a set of non-branching geodesics.

2.1 L^1-Optimal Transport

To any 1-Lipschitz function $u : X \to \mathbb{R}$ can be naturally associated a d-cyclically monotone set Γ_u defined in the following way:

$$\Gamma_u := \{(x, y) \in X \times X : u(x) - u(y) = \mathsf{d}(x, y)\}.$$

We define the *transport relation* R_u and the *transport set* $\mathcal{T}_u$ in the following way:

$$R_u := \Gamma_u \cup \Gamma_u^{-1}\ ,\ \ \mathcal{T}_u := P_1(R_u \setminus \{x = y\}), \tag{2.2}$$

where $\{x = y\}$ denotes the diagonal $\{(x, y) \in X^2 : x = y\}$, P_i the projection onto the i-th component and $\Gamma_u^{-1} = \{(x, y) \in X \times X : (y, x) \in \Gamma_u\}$.
Since u is 1-Lipschitz, Γ_u, Γ_u^{-1} and R_u are closed sets, and so are $\Gamma_u(x)$ and $R_u(x)$ (recall that $\Gamma_u(x) = \{y \in X\ ;\ (x, y) \in \Gamma_u\}$ and similarly for $R_u(x)$). Consequently $\mathcal{T}_u$ is a projection of a Borel set and hence it is analytic; it follows that it is universally measurable, and in particular, $\mathfrak{m}$-measurable [21] .

The transport “flavor” of the previous definitions can be seen in the next property that is immediate to verify: for any $\gamma \in \mathrm{Geo}(X)$ such that $(\gamma_0, \gamma_1) \in \Gamma_u$, then

$$(\gamma_s, \gamma_t) \in \Gamma_u, \quad \forall\ 0 \le s \le t \le 1.$$

Finally, recall the definition of the sets of *forward and backward branching points* introduced in [6]:

$$\begin{aligned} A_{+,u} &:= \{x \in \mathcal{T}_u : \exists\, z, w \in \Gamma_u(x), (z, w) \notin R_u\}, \\ A_{-,u} &:= \{x \in \mathcal{T}_u : \exists\, z, w \in \Gamma_u(x)^{-1}, (z, w) \notin R_u\}. \end{aligned}$$

Once branching points are removed, we obtain the *non-branched transport set* and the *non-branched transport relation*,

$$\mathcal{T}_u^b := \mathcal{T}_u \setminus (A_{+,u} \cup A_{-,u}), \quad R_u^b := R_u \cap (\mathcal{T}_u^b \times \mathcal{T}_u^b); \tag{2.3}$$

the following was obtained in [6] and highlights the motivation to remove branching points.

Proposition 2.2. *The set of transport rays $R_u^b \subset X \times X$ is an equivalence relation on the set $\mathcal{T}_u^b$.*

Noticing that once we fix $x \in \mathcal{T}_u^b$, for any choice of $z, w \in R_u(x)$, there exists $\gamma \in \mathrm{Geo}(X)$ such that

$$\{x, z, w\} \subset \{\gamma_s : s \in [0,1]\},$$

it is not hard to deduce that each equivalence class is a geodesic.

The next step is to use this partition of the transport set made of equivalence classes to obtain a corresponding decomposition of the ambient measure $\mathfrak{m}$ restricted to $\mathcal{T}_u^b$. Disintegration Theorem (for an account on it see [4]) will be the appropriate technical tool to use. The first step is to obtain an $\mathfrak{m}$-measurable quotient map f for the equivalence relation R_u^b over $\mathcal{T}_u^b$ whose construction is by now a classical procedure. It is worth stressing that the quotient set will be identified with with a subset of $\mathcal{T}_u^b$ containing a point for each equivalence class, i.e. for each geodesic forming $\mathcal{T}_u^b$. In particular, there will be an $\mathfrak{m}$-measurable quotient set $Q \subset \mathcal{T}_u^b$, image of f. The Disintegration Theorem (for an account on it see [4]) then implies the following disintegration formula:

$$\mathfrak{m}\llcorner_{\mathcal{T}_u^b} = \int_Q \mathfrak{m}_\alpha \mathfrak{q}(d\alpha), \tag{2.4}$$

where $\mathfrak{q} = f_\sharp \mathfrak{m}\llcorner_{\mathcal{T}_u^b}$, and for $\mathfrak{q}$-a.e. $\alpha \in Q$ we have $\mathfrak{m}_\alpha \in \mathcal{P}(X)$, $\mathfrak{m}_\alpha(X \setminus X_\alpha) = 0$, where we have used the notation X_α to denote the equivalence class of the element $\alpha \in Q$ (indeed $X_\alpha = R(\alpha)$). In [6], it was proved that under $\mathsf{RCD}(K,N)$ condition the measure of the sets of branching points is zero. As already observed several times in the literature, the proof only requires existence and uniqueness of optimal maps for $p = 2$.

Theorem 2.3. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a m.m.s. such that for any $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ with $\mu_0 \ll \mathfrak{m}$ any W_2-optimal transference plan is concentrated on the graph of a function. Then for every 1-Lipschitz function $u : X \to \mathbb{R}$ we have*

$$\mathfrak{m}(A_{+,u}) = \mathfrak{m}(A_{-,u}) = 0.$$

It is worth here recalling that if $(X, \mathsf{d}, \mathfrak{m})$ verifies $\mathsf{MCP}(K,N)$ and is essentially non-branching, then [10] implies that $(X, \mathsf{d}, \mathfrak{m})$ verifies the assumptions of Theorem 2.3 implying $\mathfrak{m}(A_{+,u}) = \mathfrak{m}(A_{-,u}) = 0$, for any $u : X \to \mathbb{R}$ 1-Lipschitz function.

2.2 Curvature-Dimension conditions

We conclude this section by quickly recalling the main definitions of synthetic Ricci curvature lower bounds relevant to this note. We start with the first one that has been given by Lott-Villani [17] and Sturm [22],[23].

Given a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ and $N \in \mathbb{R}, N \geq 1$, we define the *Renyi entropy functional* $S_N(\cdot|\mathfrak{m}) : \mathcal{P}_2(X, \mathsf{d}) \to \mathbb{R}$ as follows

$$S_N(\mu|\mathfrak{m}) := -\int_X \rho^{-1/N} \, d\mu,$$

where ρ denotes the density of the absolutely continuous part of μ with respect to $\mathfrak{m}$. We also recall the definition of distortion coefficients. For every $K, N \in \mathbb{R}$ with $N \geq 1$, we set

$$D_{K,\mathcal{N}} := \begin{cases} \frac{\pi}{\sqrt{K/\mathcal{N}}} & K > 0 \ , \ \mathcal{N} < \infty, \\ +\infty & \text{otherwise.} \end{cases}$$

Given $t \in [0,1]$ and $0 < \theta < D_{K,\mathcal{N}}$, the *distortion coefficients* $\sigma^{(t)}_{K,N}(\theta)$ are defined by

$$\sigma^{(t)}_{K,N}(\theta) := \begin{cases} \infty & \text{if } K\theta^2 \geq N\pi^2, \\ \frac{\sin(t\theta\sqrt{K/N})}{\sin(\theta\sqrt{K/N})} & \text{if } 0 < K\theta^2 < N\pi^2, \\ t & \text{if } K\theta^2 < 0 \text{ and } N = 0, \text{ or if } K\theta^2 = 0, \\ \frac{\sinh(t\theta\sqrt{-K/N})}{\sinh(\theta\sqrt{-K/N})} & \text{if } K\theta^2 \leq 0 \text{ and } N > 0. \end{cases}$$

Finally, given $K \in \mathbb{R}$, $N \in (1,\infty]$ and $(t,\theta) \in [0,1] \times \mathbb{R}_+$, $\tau^{(t)}_{K,N}(\theta) := t^{\frac{1}{N}} \sigma^{(t)}_{K,N-1}(\theta)^{1-\frac{1}{N}}$. When $N = 1$, set $\tau^{(t)}_{K,1}(\theta) = t$ if $K \leq 0$ and $\tau^{(t)}_{K,1}(\theta) = +\infty$ if $K > 0$.

Definition 2.4. [[23]] Given two numbers $K, N \in \mathbb{R}$ with $N \geq 1$ we say that a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ satisfies the *curvature-dimension condition* $\mathsf{CD}(K,N)$ if and only if for each pair of $\mu_0, \mu_1 \in \mathcal{P}_2(X, \mathsf{d}, \mathfrak{m})$ there exist an optimal coupling π of $\mu_0 = \rho_0\mathfrak{m}$ and $\mu_1 = \rho_1\mathfrak{m}$ and a W_2-geodesic $\{\mu_t\}$ interpolating the two such that

$$S_{N'}(\mu_t|\mathfrak{m}) \leq -\int_{X\times X} \big[\tau^{(1-t)}_{K,N'}(\mathsf{d}(x,y))\rho_0^{-1/N'}(x) + \tau^{(t)}_{K,N'}(\mathsf{d}(x,y))\rho_1^{-1/N'}(y)\big]\, d\pi(x,y) \tag{2.5}$$

for all $t \in [0,1]$ and all $N' \geq N$.

One can also prescribe the convexity inequality (2.5) to hold along a W_p-geodesic, getting to the more general definition of $\mathsf{CD}_p(K,N)$. In this note we will deal with the case $p = 1$, that, due to the lack of strict convexity of the exponent, needs a more refined definition.

Definition 2.5. Given two numbers $K, N \in \mathbb{R}$ with $N \geq 1$ we say that a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ satisfies the $\mathsf{CD}_1(K,N)$ if and only if for each pair of $\mu_0, \mu_1 \in \mathcal{P}_1(X, \mathsf{d}, \mathfrak{m})$ there exists a Borel probability measure $\boldsymbol{\pi} \in \mathcal{P}(C([0,1], X))$ concentrated on constant speed geodesics, such that $\int \mathsf{d}(\gamma_0, \gamma_1)\, d\boldsymbol{\pi}(\gamma) = W_1(\mu_0, \mu_1)$ for which the inequality

$$S_{N'}(\mu_t|\mathfrak{m}) \leq -\int_{X\times X} \big[\tau^{(1-t)}_{K,N'}(\mathsf{d}(\gamma_0,\gamma_1))\rho_0^{-1/N'}(\gamma_0) + \tau^{(t)}_{K,N'}(\mathsf{d}(\gamma_0,\gamma_1))\rho_1^{-1/N'}(\gamma_1)\big]\, d\boldsymbol{\pi}(\gamma) \tag{2.6}$$

holds for all $t \in [0,1]$ and all $N' \geq N$, where $\mu_t := (\mathrm{e}_t)_*\boldsymbol{\pi}$ and $\mu_t \ll \mathfrak{m}$ and $(\mathrm{e}_i)_\sharp\boldsymbol{\pi} = \mu_i$ for $i = 0, 1$.

Remark 2.6. Notice that since we are dealing with the 1-transportation distance, there are dynamic transport plans which are not concentrated on constant speed geodesics. Insisting on this property in the definition above seems the natural choice to make in connection with the analogous definitions for $p > 1$, see e.g. Lemma 3.2.

We now recall the definition of the $\mathsf{CD}^1(K,N)$ condition introduced in [8] and based on another principle: the localization of Ricci curvature lower bounds along integral curves associated to 1-Lipschitz function.

Definition 2.7. ($\mathsf{CD}^1(K,N)$ when $\mathrm{supp}(\mathfrak{m}) = X$) Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space such that $\mathrm{supp}(\mathfrak{m}) = X$. Let us consider $K, N \in \mathbb{R}$, $N > 1$ and let $u : (X, \mathsf{d}) \to \mathbb{R}$ be a 1-Lipschitz function. We say that $(X, \mathsf{d}, \mathfrak{m})$ satisfies the $\mathsf{CD}^1_u(K,N)$ condition if there exists a family $\{X_\alpha\}_{\alpha\in Q} \subset X$ such that :

(1) There exists a disintegration of $\mathfrak{m}\llcorner_{\mathcal{T}_u}$ on $\{X_\alpha\}_{\alpha\in Q}$:

$$\mathfrak{m}\llcorner_{\mathcal{T}_u} = \int_Q \mathfrak{m}_\alpha\, \mathfrak{q}(d\alpha), \quad \text{where}\, \mathfrak{m}_\alpha(X_\alpha) = 1, \text{ for}\, \mathfrak{q}\text{-a.e.}\, \alpha \in Q.$$

(2) For $\mathfrak{q}$-a.e. $\alpha \in Q$, X_α is a *transport ray* for Γ_u.

(3) For $\mathfrak{q}$-a.e. $\alpha \in Q$, the metric measure space $(X_\alpha, \mathsf{d}, \mathfrak{m}_\alpha)$ satisfies $\mathsf{CD}(K,N)$.

We say that $(X,\mathsf{d},\mathfrak{m})$ satisfies the $\mathsf{CD}^1(K,N)$ condition it is satisfies the $\mathsf{CD}^1_u(K,N)$ condition for every $u : X \to \mathbb{R}$ 1-Lipschitz.

By transport ray, we mean that X_α is the image of a closed non-null geodesic γ parametrized by arc length on an interval I in such a way the function $u \circ \gamma$ is affine with slope -1 on I, moreover it is maximal with respect to inclusion.

Remark 2.8. It is well known that the last condition of Definition 2.7 is equivalent to ask $\mathfrak{m}_\alpha \sim h_\alpha \mathcal{L}^1\llcorner_{[0,|X_\alpha|]}$ where $|X_\alpha|$ denotes the length of the transport ray X_α ($\sim$ means up to isometry of the space) and the density h_α has to satisfy

$$\left(h_\alpha^{1/(N-1)}\right)'' + \frac{K}{N-1} h_\alpha^{1/(N-1)} \le 0, \tag{2.7}$$

in the distributional sense. In turn this is equivalent to the fact that the continuous representative of h_α - which exists by (2.7) and that we shall continue to denote by h_α - satisfies

$$h_\alpha((1-t)R_0 + tR_1)^{\frac{1}{N-1}} \ge \sigma^{(1-t)}_{K,N-1}(R_1 - R_0) h_\alpha(R_0)^{\frac{1}{N-1}} + \sigma^{(t)}_{K,N-1}(R_1 - R_0) h_\alpha(R_1)^{\frac{1}{N-1}},$$

for any $R_0, R_1 \in [0, |X_\alpha|]$, $R_0 \le R_1$, and $t \in [0,1]$.

3 Equivalent Formulations of Ricci Curvature bounds

In this section we obtain the equivalence between $\mathsf{CD}^1(K,N)$ and $\mathsf{CD}_1(K,N)$. Recall that to avoid pathologies we assume $\mathrm{supp}(\mathfrak{m}) = X$.

Theorem 3.1. *Let* $(X,\mathsf{d},\mathfrak{m})$ *be an essentially non-branching metric measure space and further assume* $\mathfrak{m}(X) = 1$*. Then* $(X,\mathsf{d},\mathfrak{m})$ *satisfies the* $\mathsf{CD}^1(K,N)$ *condition if and only if it satisfies the* $\mathsf{CD}_1(K,N)$ *condition.*

We will present separately the two implications needed for the proof of Theorem 3.1.

3.1 $\mathsf{CD}_1(K,N) \implies \mathsf{CD}^1(K,N)$

So consider fixed $u : X \to \mathbb{R}$ a 1-Lipschitz function and $(X,\mathsf{d},\mathfrak{m})$ be essentially non-branching and verifying $\mathsf{CD}_1(K,N)$ with $\mathfrak{m}(X) = 1$.

Step 1. Disintegration formula.
First notice that $\mathsf{CD}_1(K,N)$ implies, reasoning for instance like [23] in the case $p = 2$, that the space is proper. Moreover $\mathsf{CD}_1(K,N)$ implies the following variant of $\mathsf{MCP}(K,N)$ (for the definition of MCP we refer to [23] and [18]):

Lemma 3.2. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a m.m.s. with $\mathfrak{m}(X) = 1$ and satisfying $\mathsf{CD}_1(K, N)$. Then $(X, \mathsf{d}, \mathfrak{m})$ satisfies the following version of $\mathsf{MCP}(K, N)$. For any $\mu_0 \in \mathcal{P}(X)$ with $\mu_0 \ll \mathfrak{m}$ and $x_0 \in X$ there exists a curve (μ_t) which is a W_p-geodesic for any $p \in [1, \infty)$ such that $\mu_t = \rho_t \mathfrak{m} + \mu_t^s$ for all $t \in [0, 1)$ then*

$$\int_X \rho_t^{-1/N'} \mu_t \geq \int_X \tau_{K,N'}^{(1-t)}(\mathsf{d}(x, x_0)) \rho_0^{-1/N'}(x) \mu_0(dx), \tag{3.1}$$

for all $t \in [0, 1)$ and $N' \geq N$.

Proof. Let $\mu_0 \in \mathcal{P}(X, \mathsf{d}, \mathfrak{m})$ and $x \in X$ be given. Since $\mathrm{supp}(\mathfrak{m}) = X$, we can consider $\mu_{1,\varepsilon} := c_\varepsilon \mathfrak{m}\llcorner_{B_\varepsilon(x_1)}$, with $c_\varepsilon > 0$ normalisation constant. Let $\boldsymbol{\pi}_\varepsilon$ be given by Definition 2.5 and put $\mu_t, \varepsilon := (\mathrm{e}_t)_* \boldsymbol{\pi}_\varepsilon$. It is classical to check that properness of X implies that $\boldsymbol{\pi}_\varepsilon$ is precompact and therefore we can obtain a limit dynamical plan $\boldsymbol{\pi}$ inducing a geodesic from μ_0 to δ_0. Validity of (3.1) simply follows by lower semicontinuity of entropy and the claim follows. □

The version of $\mathsf{MCP}(K, N)$ obtained in Lemma 3.2 is actually equivalent to the classical one, provided the space is essentially non-branching: we refer for its proof to [8, Lemma 6.13] (see also [19, Section 5]). Hence in our framework we can directly use the classical $\mathsf{MCP}(K, N)$.

Immediately we deduce that X is a geodesic space. Hence, as discussed in Section 2.1, the following disintegration formula is valid:

$$\mathfrak{m}\llcorner_{\mathcal{T}_u} = \mathfrak{m}\llcorner_{\mathcal{T}_u^b} = \int_Q \mathfrak{m}_\alpha \, \mathfrak{q}(d\alpha), \tag{3.2}$$

where $\mathfrak{q} = f_\sharp(\mathfrak{m}\llcorner_{\mathcal{T}_u^b})$, and for $\mathfrak{q}$-a.e. $\alpha \in Q$ we have $\mathfrak{m}_\alpha \in \mathcal{P}(X)$, with $\mathfrak{m}_\alpha(X \setminus X_\alpha) = 0$: the notation X_α is used to denote the equivalence class of the element $\alpha \in Q$ that is, in particular, a transport ray. Notice that the first identity follows from the essentially non-branching assumption and the discussion after Theorem 2.3.

Hence it is only left to show that $(X_\alpha, \mathsf{d}, \mathfrak{m}_\alpha)$ satisfy $\mathsf{CD}(K, N)$.

Step 2. Intermediate regularity of conditional measures.
It is already present in the literature how to improve the validity of (3.1) to any $\mu_1 \in \mathcal{P}(X)$, provided the space is essentially non-branching and the geodesic $(\mu_t)_{t \in [0,1]}$ is a W_2-geodesic.

This will be enough to deduce a first result on the regularity of $\mathfrak{m}_\alpha$. Indeed localization for $\mathsf{MCP}(K, N)$ was, in a different form, already known in 2009, see [5, Theorem 9.5], for non-branching m.m.s.. The case of essentially non-branching m.m.s.'s and an effective reformulation (after the work of Klartag [16]) have been recently discussed in [11, Section 3] to which we refer for all the missing details (see in particular [11, Theorem 3.5]). Here we briefly report the following fact:

If $(X, \mathsf{d}, \mathfrak{m})$ is an essentially non-branching m.m.s. with $\mathrm{supp}(\mathfrak{m}) = X$ and satisfying $\mathsf{MCP}(K, N)$, for some $K \in \mathbb{R}, N \in (1, \infty)$, then, for $\mathfrak{q}$-a.e.α, $\mathfrak{m}_\alpha = h_\alpha \mathcal{H}^1 \llcorner_{X_\alpha}$ and the one-dimensional metric measure space $(X_\alpha, \mathsf{d}, \mathfrak{m}_\alpha)$ verifies $\mathsf{MCP}(K, N)$; in particular h_α is strictly positive in the relative interior of X_α and locally Lipschitz.

Step 3. $\mathsf{CD}(K, N)$ estimates for one-dimensional spaces.
In order to conclude, it remains to show that for $\mathfrak{q}$-a.e. $\alpha \in Q$, the one-dimensional metric measure space $(X_\alpha, \mathsf{d}, \mathfrak{m}_\alpha)$ satisfies $\mathsf{CD}(K, N)$. It is useful to introduce the following *ray map* $g : \mathrm{Dom}\,(g) \subset Q \times \mathbb{R} \to \mathcal{T}_u$, defined as follows:

$$\begin{aligned} \mathrm{graph}(g) :=& \big\{(\alpha, t, x) \in Q \times [0, +\infty) \times \mathcal{T}_u^b : (\alpha, x) \in \Gamma, \mathsf{d}(\alpha, x) = t\big\} \\ & \cup \big\{(\alpha, t, x) \in Q \times (-\infty, 0] \times \mathcal{T}_u^b : (x, \alpha) \in \Gamma, \mathsf{d}(x, \alpha) = t\big\}. \end{aligned}$$

The ray map g enjoys several properties already obtained in [6, Proposition 5.4]:

- g is a Borel map;
- $t \mapsto g(\alpha, t)$ is an isometry. If $s, t \in \mathit{Dom}(\alpha, \cdot)$ with $s \leq t$, then $(g(\alpha, s), g(\alpha, t)) \in \Gamma$;
- $\mathrm{Dom}\,(g) \ni (\alpha, t) \mapsto g(\alpha, t)$ is bijective on $f^{-1}(Q) \subset \mathcal{T}_u^b$.

In particular, via g we will identify the set of definition of the densities h_α with real intervals.

We start with the following preliminary result.

Lemma 3.3. *For any $\bar{Q} \subseteq Q$ Borel set with positive $\mathfrak{q}$-measure and for $R_0, R_1, L_0, L_1 \in \mathbb{R}$ such that $R_0 < R_1$, $L_0, L_1 > 0$ and $[R_0, R_1 + L_1]$ belongs to the domain of $\mathfrak{q}$-a.e. h_α, it holds:*

$$\begin{aligned}(L_t)^{\frac{1}{N}} &\sup_{\bar{Q}} h_\alpha^{\frac{1}{N}}(R_t) \\ &\geq (L_0)^{\frac{1}{N}} \tau_{K,N}^{(1-t)}(\mathsf{d}(R_0, R_1)) \inf_{\bar{Q}} h_\alpha^{\frac{1}{N}}(R_0) + (L_1)^{\frac{1}{N}} \tau_{K,N}^{(t)}(\mathsf{d}(R_0, R_1)) \inf_{\bar{Q}} h_\alpha^{\frac{1}{N}}(R_1), \end{aligned} \tag{3.3}$$

for every $t \in [0, 1]$, where $R_t = (1-t)R_0 + tR_1$ (the same holds for L_t).

Proof. **Step 1.**
Fix $\bar{Q} \subseteq Q$ Borel set with positive $\mathfrak{q}$-measure and consider $R_0, R_1, L_0, L_1 \in \mathbb{R}$ such that $R_0 < R_1$ and $L_0, L_1 > 0$. Define for $i = 1, 2$ the probability measures:

$$\mu_i = \frac{1}{\mathfrak{q}(\bar{Q})} \int_{\bar{Q}} g(\alpha, \cdot)_\sharp \left(\frac{1}{\varepsilon L_i} \mathcal{L}^1 \llcorner_{[R_i, R_i + \varepsilon L_i]} \right) \mathfrak{q}(d\alpha).$$

First of all observe that, for such measures, the transport has to be performed along the rays $\{X_\alpha\}_{\alpha \in \bar{Q}}$. For sure an optimal plan with this property exists, since the plan π rearranging the mass monotonically along each ray is optimal; hence $\mathrm{supp}\,\pi \subset \Gamma$, so it is d-cyclically monotone and therefore W_1-optimal. The aim is to prove that all the other optimal plans enjoy the same property.

Indeed, if not, there would exist at least one optimal plan $\bar{\pi}$ such that, for some $\bar{Q}_1 \subset \bar{Q}$ of positive $\mathfrak{q}$-measure and for some $S \subset \mathbb{R}$, it holds

$$\bar{\pi}\{(g(\alpha, s), g(\alpha', s')) : \alpha, \alpha' \in \bar{Q}_1, s, s' \in S \,\text{with}\, \alpha \neq \alpha'\} > 0,$$

with $\bar{Q}_1 \times S \subset \mathrm{Dom}\,(g)$. Let us consider the plan

$$\pi^* = \frac{\pi + \bar{\pi}}{2};$$

trivially, it is still optimal for the couple μ_0, μ_1. By construction this plan splits some points, generating in this way a set of branching points with positive measure. This will lead to a contradiction. Consider indeed the Kantorovich potential v associated to the W_1-optimal transport problem between μ_0 and μ_1, possibly different from the 1-Lipschitz function u we fixed above. Theorem 2.3 applied to v implies that necessarily that $\mathfrak{m}(A_{\pm,v}) = 0$. Since $A_{\pm,v}$ will contain $P_1(\{(g(\alpha, s), g(\alpha', s')) : \alpha, \alpha' \in \bar{Q}_1, s, s' \in S \,\text{with}\, \alpha \neq \alpha'\})$ considered above, and $\mu_0 \ll \mathfrak{m}$, the contradiction with $\bar{\pi}(\{(g(\alpha, s), g(\alpha', s')) : \alpha, \alpha' \in \bar{Q}_1, s, s' \in S \,\text{with}\, \alpha \neq \alpha'\}) > 0$ follows. Hence, every optimal plan will have support contained in the set

$$A^\varepsilon_{\bar{Q}} := \cup_{\alpha \in \bar{Q}} g(\alpha, [R_0, R_0 + \varepsilon L_0]) \times g(\alpha, [R_1, R_1 + \varepsilon L_1]).$$

Step 2.

Since by definition $\mu_0, \mu_1 \ll \mathfrak{m}$, there exists a dynamic transport plan $\boldsymbol{\pi}$ as in Definition 2.5 such that for $\mu_t := (\mathrm{e}_t)_* \boldsymbol{\pi} = \rho_t \mathfrak{m}$ the inequality (2.6) holds true. Step 1 above and the fact that $\boldsymbol{\pi}$ is concentrated on constant speed geodesics completely characterize $\boldsymbol{\pi}$; in particular we have that for q-a.e. $\alpha \in Q$ the function ρ_t is 0 m_α-a.e. outside the 'interval' $g(\alpha, [R_t, R_t + \varepsilon L_t])$. Hence using the Disintegration Theorem and Jensen inequality we can estimate the left-hand side of (2.6) by:

$$
\begin{aligned}
\int_X \rho_t^{1-\frac{1}{N}}\, d\mathfrak{m} &= \int_{\bar{Q}} \int_{X_\alpha} \rho_t(x)^{1-\frac{1}{N}} m_\alpha(dx)\mathfrak{q}(d\alpha) = \int_{\bar{Q}} \int_{R_t}^{R_t+\varepsilon L_t} \rho_t(g(\alpha,s))^{1-\frac{1}{N}} h_\alpha(s) ds \mathfrak{q}(d\alpha) \\
&\le (\varepsilon L_t) \int_{\bar{Q}} \sup_{[R_t, R_t+\varepsilon L_t]} h_\alpha^{\frac{1}{N}} \fint_{R_t}^{R_t+\varepsilon L_t} \big(\rho_t(g(\alpha,s)) h_\alpha(s)\big)^{1-\frac{1}{N}} ds \mathfrak{q}(d\alpha) \\
&\le (\varepsilon L_t)^{\frac{1}{N}} \int_{\bar{Q}} \sup_{[R_t, R_t+\varepsilon L_t]} h_\alpha^{\frac{1}{N}} \left(\int_{R_t}^{R_t+\varepsilon L_t} \rho_t(g(\alpha,s)) h_\alpha(s) ds \right)^{1-\frac{1}{N}} \mathfrak{q}(d\alpha) \\
&\le (\varepsilon L_t \mathfrak{q}(\bar{Q}))^{\frac{1}{N}} \sup_{\bar{Q}} \left(\sup_{[R_t, R_t+\varepsilon L_t]} h_\alpha^{\frac{1}{N}} \right).
\end{aligned}
$$

Arguing similarly, the right-hand side of (2.6) can be estimated in the following way where $\pi = (\mathrm{e}_0, \mathrm{e}_1)_\sharp \boldsymbol{\pi}$:

$$
\begin{aligned}
&\int_{X \times X} \rho_0^{-\frac{1}{N}}(x) \tau_{K,N}^{(1-t)}(\mathsf{d}(x,y)) + \rho_1^{-\frac{1}{N}}(y) \tau_{K,N}^{(t)}(\mathsf{d}(x,y)) \pi(dx, dy) \\
&\ge \inf_{A^{\varepsilon}_{\bar{Q}}} \tau_{K,N}^{(1-t)}(\mathsf{d}(x,y)) \int_X \rho_0^{1-\frac{1}{N}}(x) \mathfrak{m}(dx) + \inf_{A^{\varepsilon}_{\bar{Q}}} \tau_{K,N}^{(t)}(\mathsf{d}(x,y)) \int_X \rho_1^{1-\frac{1}{N}}(y) \mathfrak{m}(dy) \\
&\ge (\varepsilon \mathfrak{q}(\bar{Q}))^{\frac{1}{N}} \left[\inf_{A^{\varepsilon}_{\bar{Q}}} \tau_{K,N}^{(1-t)}(\mathsf{d}(x,y)) \inf_{\bar{Q}} \left(\inf_{[R_0, R_0+\varepsilon L_0]} h_\alpha^{\frac{1}{N}} \right) (L_0)^{\frac{1}{N}} \right. \\
&\qquad \left. + \inf_{A^{\varepsilon}_{\bar{Q}}} \tau_{K,N}^{(t)}(\mathsf{d}(x,y)) \inf_{\bar{Q}} \left(\inf_{[R_1, R_1+\varepsilon L_1]} h_\alpha^{\frac{1}{N}} \right) (L_1)^{\frac{1}{N}} \right].
\end{aligned}
$$

Hence, considering both the estimates obtained so far, we get

$$
\begin{aligned}
(L_t)^{\frac{1}{N}} \sup_{\bar{Q}} \left(\sup_{[R_t, R_t+\varepsilon L_t]} h_\alpha^{\frac{1}{N}} \right) \ge{}& \inf_{A^{\varepsilon}_{\bar{Q}}} \tau_{K,N}^{(1-t)}(\mathsf{d}(x,y)) \inf_{\bar{Q}} \left(\inf_{[R_0, R_0+\varepsilon L_0]} h_\alpha^{\frac{1}{N}} \right) (L_0)^{\frac{1}{N}} \\
&+ \inf_{A^{\varepsilon}_{\bar{Q}}} \tau_{K,N}^{(t)}(\mathsf{d}(x,y)) \inf_{\bar{Q}} \left(\inf_{[R_1, R_1+\varepsilon L_1]} h_\alpha^{\frac{1}{N}} \right) (L_1)^{\frac{1}{N}}.
\end{aligned}
$$

Sending $\varepsilon \to 0$, we obtain

$$
(L_t)^{\frac{1}{N}} \sup_{\bar{Q}} h_\alpha^{\frac{1}{N}}(R_t) \ge (L_0)^{\frac{1}{N}} \inf_{A_{\bar{Q}}} \tau_{K,N}^{(1-t)}(\mathsf{d}(x,y)) \inf_{\bar{Q}} h_\alpha^{\frac{1}{N}}(R_0) + (L_1)^{\frac{1}{N}} \inf_{A_{\bar{Q}}} \tau_{K,N}^{(t)}(\mathsf{d}(x,y)) \inf_{\bar{Q}} h_\alpha^{\frac{1}{N}}(R_1),
$$

where $A_{\bar{Q}} := \cup_{\alpha \in \bar{Q}} \{(g(\alpha, R_0), g(\alpha, R_1))\}$. Since $g(\alpha, \cdot)$ is an isometry, (3.3) is proved. □

We are now ready to prove the following:

Proposition 3.4. *For $\mathfrak{q}$-a.e. $\alpha \in Q$, the metric measure space $(X_\alpha, \mathsf{d}, \mathfrak{m}_\alpha)$ satisfies $\mathsf{CD}(K,N)$.*

Proof. By remark 2.8, to prove the claim is sufficient to show that:

$$h_\alpha((1-t)R_0+tR_1)^{\frac{1}{N-1}} \geq \sigma^{(1-t)}_{K,N-1}(R_1-R_0)h_\alpha(R_0)^{\frac{1}{N-1}} + \sigma^{(t)}_{K,N-1}(R_1-R_0)h_\alpha(R_1)^{\frac{1}{N-1}}, \quad (3.4)$$

for all $t \in [0,1]$ and for $R_0, R_1 \in [0, L_\alpha]$ with $R_0 < R_1$, where we have identified the transport ray X_α with the real interval $[0, L_\alpha]$ having the same length.
As already did in [9], it is sufficient to show that for every $R_0, R_1 \in [0, L_\alpha]$ with $R_0 < R_1$ and $L_0, L_1 > 0$, we have that for $\mathfrak{q}$-a.e. $\alpha \in Q$

$$(L_t)^{\frac{1}{N}} h_\alpha^{\frac{1}{N}}(R_t) \geq (L_0)^{\frac{1}{N}} \tau^{(1-t)}_{K,N}(\mathsf{d}(R_0,R_1)) h_\alpha^{\frac{1}{N}}(R_0) + (L_1)^{\frac{1}{N}} \tau^{(t)}_{K,N}(\mathsf{d}(R_0,R_1)) h_\alpha^{\frac{1}{N}}(R_1), \quad (3.5)$$

for all $t \in [0,1]$, where $L_t = (1-t)L_0 + tL_1$ (the same for R_t). Indeed, if this is the case taking also into account the already established continuity of h_α, one can make the choice

$$L_0 = \frac{\sigma^{(1-t)}_{K,N-1}(\mathsf{d}(R_0,R_1))h(R_0)^{\frac{1}{N-1}}}{1-t}, \quad L_1 = \frac{\sigma^{(t)}_{K,N-1}(\mathsf{d}(R_0,R_1))h(R_1)^{\frac{1}{N-1}}}{t},$$

obtaining exactly (3.4). Thus, our aim will be proving (3.5). Arguing by contraddiction, let us assume that there exist $R_0, R_1 \in [0, L_\alpha]$, $L_0, L_1 > 0$ with $R_0 + L_0, R_1 + L_1 < L_\alpha$ and a Borel set $Q_1 \subseteq Q$ with positive $\mathfrak{q}$-measure such that for every $\alpha \in Q_1$ it holds:

$$(L_t)^{\frac{1}{N}} h_\alpha^{\frac{1}{N}}(R_t) < (L_0)^{\frac{1}{N}} \tau^{(1-t)}_{K,N}(\mathsf{d}(R_0,R_1)) h_\alpha^{\frac{1}{N}}(R_0) + (L_1)^{\frac{1}{N}} \tau^{(t)}_{K,N}(\mathsf{d}(R_0,R_1)) h_\alpha^{\frac{1}{N}}(R_1). \quad (3.6)$$

By Lusin Theorem, there exists a Borel set $Q_2 \subset Q_1$ with positive $\mathfrak{q}$-measure on which the maps $\alpha \mapsto h_\alpha(R_i)$, for $i = 0, t, 1$ are continuous. Hence, fixed $\delta > 0$, there exists $Q_3 \subset Q_2$ with positive $\mathfrak{q}$-measure such that

$$(L_t)^{\frac{1}{N}} h_\alpha^{\frac{1}{N}}(R_t) < (L_0)^{\frac{1}{N}} \tau^{(1-t)}_{K,N}(\mathsf{d}(R_0,R_1)) h_\alpha^{\frac{1}{N}}(R_0) + (L_1)^{\frac{1}{N}} \tau^{(t)}_{K,N}(\mathsf{d}(R_0,R_1)) h_\alpha^{\frac{1}{N}}(R_1) - \delta, \quad \forall \alpha \in Q_3.$$

In particular, for every $\bar{Q} \subset Q_3$ compact set with positive $\mathfrak{q}$-measure:

$$(L_t)^{\frac{1}{N}} \sup_{\bar{Q}} h_\alpha^{\frac{1}{N}}(R_t) < (L_0)^{\frac{1}{N}} \tau^{(1-t)}_{K,N}(\mathsf{d}(R_0,R_1)) \sup_{\bar{Q}} h_\alpha^{\frac{1}{N}}(R_0) + (L_1)^{\frac{1}{N}} \tau^{(t)}_{K,N}(\mathsf{d}(R_0,R_1)) \sup_{\bar{Q}} h_\alpha^{\frac{1}{N}}(R_1) - \delta.$$

Combining the latter inequality with (3.3), we deduce that for any $\bar{Q} \subset Q_3$ Borel set with positive $\mathfrak{q}$-measure

$$\begin{aligned}
&(L_0)^{\frac{1}{N}} \tau^{(1-t)}_{K,N}(\mathsf{d}(R_0,R_1)) \inf_{\bar{Q}} h_\alpha^{\frac{1}{N}}(R_0) + (L_1)^{\frac{1}{N}} \tau^{(t)}_{K,N}(\mathsf{d}(R_0,R_1)) \inf_{\bar{Q}} h_\alpha^{\frac{1}{N}}(R_1) < \\
&(L_0)^{\frac{1}{N}} \tau^{(1-t)}_{K,N}(\mathsf{d}(R_0,R_1)) \sup_{\bar{Q}} h_\alpha^{\frac{1}{N}}(R_0) + (L_1)^{\frac{1}{N}} \tau^{(1-t)}_{K,N}(\mathsf{d}(R_0,R_1)) \sup_{\bar{Q}} h_\alpha^{\frac{1}{N}}(R_1) - \delta.
\end{aligned}$$

Since the parameter δ does not depend on $\bar{Q}$, we obtain a contradiction. □

This concludes the proof of the implication: from $\mathsf{CD}_1(K,N)$ to $\mathsf{CD}^1(K,N)$. We will next move to the opposite implication.

3.2 $\mathsf{CD}^1(K,N) \implies \mathsf{CD}_1(K,N)$

Notice that $\mathsf{CD}^1(K,N)$ implies that $(X,\mathsf{d},\mathfrak{m})$ is a proper geodesic space and verifies $\mathsf{MCP}(K,N)$ (see for all the details [8]).

Let $\mu_0,\mu_1 \in \mathcal{P}_1(X,\mathsf{d},\mathfrak{m})$ be given. We will construct a W_1-geodesic verifying the Entropy inequality. Consider therefore $u : X \to \mathbb{R}$ a Kantorovich potential associated to the transport problem between μ_0,μ_1 with cost d. Consider the associated Γ_u; then any optimal transport plan π has to be concentrated over Γ_u, i.e. $\pi(\Gamma_u)=1$. Moreover, with no loss in generality we can assume that μ_0 is concentrated over the transport set $\mathcal{T}_u^b$: indeed the part of μ_0 outside of $\mathcal{T}_u^b$ is left in place by π; in particular, it will not give any contribution in the Entropy inequality as $\tau_{K,N}^{(1-t)}(0)=0$.

Since u is 1-Lipschitz, by the $\mathsf{CD}^1_u(K,N)$ condition there exist a family of rays $\{X_\alpha\}_{\alpha\in Q} \subset X$ and a disintegration of $\mathfrak{m}\llcorner_{\mathcal{T}_u}$ on $\{X_\alpha\}_{\alpha\in Q}$ such that:

$$\mathfrak{m}\llcorner_{\mathcal{T}_u} = \mathfrak{m}\llcorner_{\mathcal{T}_u^b} = \int_Q \mathfrak{m}_\alpha\, \mathfrak{q}(d\alpha), \quad \text{where}\, \mathfrak{m}_\alpha(X_\alpha)=1, \ \text{for}\, \mathfrak{q}\text{-a.e.}\, \alpha \in Q, \tag{3.7}$$

where the first identity is given by Theorem 2.3 and $(X_\alpha,\mathsf{d},\mathfrak{m}_\alpha) \in \mathsf{CD}(K,N)$. It follows that

$$\mu_0 = \rho_0\mathfrak{m} = \int_Q \rho_0\, \mathfrak{m}_\alpha\, \mathfrak{q}(d\alpha) = \int_Q \mu_{0,\alpha}\mathfrak{q}_0(d\alpha) \tag{3.8}$$

where $\mu_{0,\alpha} = \rho_0\mathfrak{m}_\alpha \cdot (\int \rho_0\mathfrak{m}_\alpha)^{-1}$ and $\mathfrak{q}_0 = f_\sharp(\mu_0)$ with f the quotient map. Then we claim that for any Borel set $C \subseteq Q$ it holds:

$$(f^{-1}(C)\times X)\cap(\Gamma_u\setminus\{x=y\})\cap(\mathcal{T}_u^b\times\mathcal{T}_u^b) = (X\times f^{-1}(C))\cap(\Gamma_u\setminus\{x=y\})\cap(\mathcal{T}_u^b\times\mathcal{T}_u^b).$$

Indeed, since $\mu_0(\mathcal{T}_u^b)=\mu_1(\mathcal{T}_u^b)=1$, then $\pi((\Gamma_u\setminus\{x=y\})\cap\mathcal{T}_u^b\times\mathcal{T}_u^b)=1$; hence if $x,y\in\mathcal{T}_u^b$ with $(x,y)\in\Gamma_u$, then it must be $f(x)=f(y)$ since $\mathcal{T}_u^b$ does not admit forward or backward branching points. This implies that

$$\begin{aligned}\mu_0(f^{-1}(C)) &= \pi((f^{-1}(C)\times X)\cap(\Gamma_u\setminus\{x=y\}))\\ &= \pi(X\times f^{-1}(C))\cap(\Gamma_u\setminus\{x=y\})\\ &= \mu_1(f^{-1}(C));\end{aligned}$$

in particular $\mathfrak{q}_0=\mathfrak{q}_1 := f_\sharp(\mu_1)$. Hence, we can write the following disintegration: $\mu_1=\rho_1\mathfrak{m} = \int_Q \rho_1\, \mathfrak{m}_\alpha\, \mathfrak{q}(d\alpha) = \int_Q \mu_{1,\alpha}\mathfrak{q}_0(d\alpha)$, where $\mu_{1,\alpha}=\rho_1\mathfrak{m}_\alpha\cdot(\int\rho_0\mathfrak{m}_\alpha)^{-1}$ and, $\mathfrak{q}_0$-a.e., $\mu_{0,\alpha},\mu_{1,\alpha}$ are probability measures on X_α. Furthermore, by construction they are absolutely continuous with respect to $\mathfrak{m}_\alpha$. By the $\mathsf{CD}^1_u(K,N)$ condition, the metric measure space $(X_\alpha,\mathsf{d},\mathfrak{m}_\alpha)$ satisfies $\mathsf{CD}(K,N)$ and hence there exists an optimal dynamical plan ν_α such that $\rho_{t,\alpha}\mathfrak{m}_\alpha = \mu_{t,\alpha} = (\mathrm{e}_t)_\sharp\nu_\alpha$ is a W^1-geodesic interpolating $\mu_{0,\alpha}$ ad $\mu_{1,\alpha}$ and

$$\rho_{t,\alpha}^{-\frac{1}{N'}}(\gamma_t) \geq \tau_{K,N'}^{(1-t)}(\mathsf{d}(\gamma_0,\gamma_1))\rho_{0,\alpha}^{-\frac{1}{N'}}(\gamma_0) + \tau_{K,N}^{(t)}(\mathsf{d}(\gamma_0,\gamma_1))\rho_{1,\alpha}^{-\frac{1}{N'}}(\gamma_1), \ \text{for}\ \nu_\alpha\ \text{a.e.}\, \gamma. \tag{3.9}$$

It is then natural to proceed gluing 1-dimensional geodesics: define $\nu = \int_Q \nu_\alpha\mathfrak{q}_0(d\alpha)$ and set $\mu_t = (\mathrm{e}_t)_\sharp\nu$. Observe that, it holds $\mu_t = \int_Q \mu_{t,\alpha}\mathfrak{q}_0(d\alpha)$ and we claim that $\{\mu_t\}$ is a W_1-geodesic interpolating μ_0 and μ_1. Indeed:

$$\begin{aligned}
W_1(\mu_t,\mu_s) &\leq \int_{X\times X} \mathsf{d}(x,y)(\mathrm{e}_t,\mathrm{e}_s)_\sharp\nu(dxdy) \\
&= \int_Q\int_{X_\alpha\times X_\alpha} \mathsf{d}(x,y)(\mathrm{e}_t,\mathrm{e}_s)_\sharp\nu_\alpha(dxdy)\mathfrak{q}_0(d\alpha) \\
&= |t-s|\int_Q\int_{X_\alpha\times X_\alpha} \mathsf{d}(x,y)(\mathrm{e}_0,\mathrm{e}_1)_\sharp\nu_\alpha(dxdy)\mathfrak{q}_0(d\alpha) \\
&= |t-s|\int_{X\times X} \mathsf{d}(x,y)(\mathrm{e}_0,\mathrm{e}_1)_\sharp\nu(dxdy) \\
&= |t-s|W_1(\mu_0,\mu_1).
\end{aligned}$$

The last equality follows from the optimality of the plan: indeed $(\mathrm{e}_0,\mathrm{e}_1)_\sharp\nu$ is concentrated on a d-cyclically monotone with marginals μ_0 and μ_1. To conclude, we show the convexity inequality (2.6) along the geodesic μ_t.

If $\mu_t=\rho_t\mathfrak{m}$, it follows from (3.7) that for each $t\in[0,1]$ it holds $\rho_{t,\alpha}=\frac{\rho_t}{\int\rho_0 m_\alpha}$. Hence the inequality (3.9) can be rewritten in the following way:

$$\rho_t^{-\frac{1}{N'}}(\gamma_t)\geq\tau_{K,N'}^{(1-t)}(\mathsf{d}(\gamma_0,\gamma_1))\rho_0^{-\frac{1}{N'}}(\gamma_0)+\tau_{K,N}^{(t)}(\mathsf{d}(\gamma_0,\gamma_1))\rho_1^{-\frac{1}{N'}}(\gamma_1),\quad\text{for }\nu_\alpha\text{-a.e.}\,\gamma. \tag{3.10}$$

Since for $\mathfrak{q}_0$-a.e. α the inequality (3.10) holds for $\nu_\alpha-$a.e. γ, a fortiori it holds true for $\nu-$a.e. γ; hence, the claim is proved.

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