Lecture notes on Variational Mean Field Games

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Abstract

These lecture notes aim at giving the details presented in the short course (6h) given in Cetraro, in the CIME School about MFG of June 2019. The topics which are covered concern first-order MFG with local couplings, and the main goal is to prove that minimizers of a suitably expressed global energy are equilibria in the sense that a.e. trajectory solves a control problem with a running cost depending on the density of all the agents. Both the case of a cost penalizing high densities and of an L^∞ constraint on the same densities are considered. The details of a construction to prove that minimizers actually define equilibria are presented under a boundedness assumption of the running cost, which is proven in the relevant cases.

1 Introduction and modeling

The theory of Mean Field Games was introduced around 2006 at the same time by Lasry and Lions, [23, 24, 25], and by Caines, Huang and Malhamé, [21], in order to describe the evolution of a population of rational agents, each one choosing (or controlling) a path in a state space, according to some preferences which are affected by the presence of other agents nearby in a way physicists call mean-field effect. The evolution is described through a Nash equilibrium in a game with a continuum of players. This can be interpreted as a limit as $N \to \infty$ of a game with N indistinguishable players, each one having a negligible effect as $N \to \infty$ on the mean-field. The class of games we consider, called Mean Field Games (MFG for short), are very particular differential games: typically, in a differential game the role of the time variable is crucial since if a player decides to deviate from a given strategy (a notion which is at the basis of the Nash equilibrium definition), the other can react to this change, so that the choice of a strategy is usually not defined as the choice of a path, but of a function selecting a path according to the information the player has at each given time. Yet, when each player is considered as negligible, any deviation he/she performs will have no effect on the other players, so that they will not react. In this way we have a static game where the space of strategies is a space of paths. Because of indistinguishability, the main tool to describe such equlibria will be the use of measures on paths, and in this setting we will use the terminology of *Lagrangian equilibria*. In fluid mechanics, indeed, the Lagrangian formulation consists in "following" each particle and providing for each of them the corresponding trajectories. On the other hand, fluid mechanics also uses another language, the so-called Eulerian formulation, where certain quantities, and in particular the density and the velocity of the particles, are given as a function of time and space. MFG equilibria can also be described

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through a system of PDEs in *Eulerian variables*, where the key ingredients are the density ρ and the value function φ of the control problem solved by each player, the velocity $\mathbf{v}(t,x)$ of the agents at (t,x) being, by optimality, related to the gradient $\nabla\varphi(t,x)$.

The MFG theory is now studied by may scholars in many countries, with a quickly growing set of references. For a general overview of the theory, it is possible to refer to the 6-years course given by P.-L. Lions at Collège de France, for which videorecording is available in French ([28]) or to the lecture notes by P. Cardaliaguet [9], based on the same course. In the present lecture notes we will only be concerned with a sub-class of MFG, those which are deterministic, have a variational structure, and are in some sense *congestion games*, where the cost for an agent passing through a certain point depends, in an increasing way, on the density $\rho(t,x)$ at such a point. We will also see a variant of this class of problems where the penalization on ρ is replaced by a constraint on it (of the form $\rho(t,x)\leq 1$, for instance), which does not fit exactly this framework but shares most of the ideas and the properties. The topic of this course and these lecture notes were already presented in [5], so that there will be some superposition with such a survey paper, but in these notes we will focus on some more particular cases so as to be able to provide more technical details and proofs. Moreover, not all regularity results were available when [5] was written, and some proofs are simplified here.

1.1 A coupled system of PDEs

Let us describe in a more precise way the simplest MFG models and the sub-class that we consider. First, we look at a population of agents moving inside a domain Ω (which can be a bounded domain in $\mathbb{R}^d$ or, for instance, the flat torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d\dots$), and we suppose that every agent chooses his own trajectory solving a minimization problem

$$\min \int_0^T \left(\frac{|x'(t)|^2}{2} + h(t,x(t))\right) \mathrm{d}t + \Psi(x(T)),$$

with given initial point $x(0)$. The mean-field effect will be modeled through the fact that the function $h(t,\cdot)$ depends on the density ρ_t of the agents at time t. The dependence of the cost on the velocity x' could of course be more general than a simple quadratic function, but in all these lecture notes we will focus on the quadratic case (some results that we present could be generalized, while for some parts of the analysis, in particular the regularity obtained via optimal transport methods, the use of the quadratic cost is important).

For the moment, we consider the evolution of the density ρ_t as an input, i.e. we suppose that agents know it. Hence, we can suppose the function h to be given, and we want to study the above optimization problem. The main tool to analyze it, coming from optimal control theory, is the value function. The value function φ is in this case defined via

$$\varphi(t_0,x_0) := \min\left\{\int_{t_0}^T \left(\frac{|x'(t)|^2}{2} + h(t,x)\right) \mathrm{d}t + \Psi(x(T)),\ x:[t_0,T]\to\Omega, x(t_0)=x_0\right\} \tag{1}$$

and it has some important properties. Firs, it solves the *Hamilton-Jacobi equation*

$$\text{(HJ)} \qquad \begin{cases} -\partial_t\varphi + \frac{1}{2}|\nabla\varphi|^2 = h, \\ \varphi(T,x) = \Psi(x) \end{cases}$$

(in the viscosity sense, but we will not pay attention to this technicality, so far); second, the optimal trajectories $x(t)$ can be computed using φ, since they are the solutions of

$$x'(t) = -\nabla\varphi(t,x(t)).$$

Now if we call $\mathbf{v}$ the velocity field which advects the density ρ (which means that ρ is the density of a bunch of particles each following a trajectory $x(t)$ solving $x'(t) = \mathbf{v}_t(x(t))$), fluid mechanics tells us that the pair $(\rho, \mathbf{v})$ solves the *continuity equation*

$$\text{(CE)} \qquad \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0$$

in the weak sense, together with no-flux boundary conditions $\rho \mathbf{v} \cdot n = 0$, modeling the fact that no mass enters or exits Ω.

In MFG we look for an equilibrium in the sense of Nash equilibria: a configuration where no player would spontaneously decide to change his choice if he/she assumes that the choices of the others are fixed. This means that we can consider the densities ρ_t as an input, compute $h[\rho]$, then compute the optimal trajectories through the (HJ) equation, then the solution of (CE) and get new densities as an output: we have an equilibrium if and only if the output densities coincide with the input. This means solving the following coupled (HJ)+(CE) system: the function φ solves (HJ) with a right-hand side depending on ρ, which on turn evolves according to (CE) with a velocity field depending on $\nabla \varphi(t, x)$.

$$\begin{cases} -\partial_t \varphi + \frac{|\nabla \varphi|^2}{2} = h[\rho] \\ \partial_t \rho - \nabla \cdot (\rho \nabla \varphi) = 0, \\ \varphi(T, x) = \Psi(x), \quad \rho(0, x) = \overline{\rho_0}(x). \end{cases} \tag{2}$$

To be more general, it is possible to conside a stochastic case, where agents follow controlled stochastic differential equations of the form $dX_t = \alpha_t \, \mathrm{d}t + \sqrt{2\nu} dW_t$ and minimize

$$\mathbb{E}\left[\int_0^T \left(\frac{1}{2}\alpha_t^2 + h[\rho_t](X_t)\right) \mathrm{d}t + \Psi(X(T))\right].$$

In this case, a Laplacian appears both in the (HJ) and in the (CE) equations:

$$\begin{cases} -\partial_t \varphi - \nu \Delta \varphi + \frac{|\nabla \varphi|^2}{2} - h[\rho] = 0 \\ \partial_t \rho - \nu \Delta \rho - \nabla \cdot (\rho \nabla \varphi) = 0. \end{cases} \tag{3}$$

1.2 Questions and difficulties

Let us be precise now about the kind of questions that one would like to attack, at the interface between mathematical analysis and modeling.

From the analysis and PDE point of view, the most natural questions to ask in MFG is the existence (and possibly the uniqueness and the regularity properties) of the solutions of systems like the above ones. This is an Eulerian question, as the objects which are involved, the density and the velocity, which is related to the value function, are defined for time-space points (t, x). This question can be intuitively attacked via fixed points methods: given ρ, compute $h[\rho]$, φ, then the solution of the evolution equation on ρ, thus getting a new density evolution $\tilde{\rho}$, and look for $\tilde{\rho} = \rho$. Yet, this requires strong continuity properties of this sequence of operators (and, by the way, uniqueness of those solutions if we want the operators to be univalued) which corresponds to uniqueness, regularity, and stability properties of the solutions of the corresponding PDEs. These properties are not always easy to get, but can be usually obtained when

- either we have $\nu > 0$ in System (3), i.e. the equations are parabolic and the regularization effect of the Laplacian provides the desired estimates (this applies quite easily to the present quadratic case, where a change-of-variable $u = e^{-\varphi/2}$ transforms the Hamilton-Jacobi equation of (3) into a linear parabolic equation; the general case is harder and require to use uniqueness and stability properties which are valid for the Fokker-Planck equations under milder regularity assumptions, and which have been recently proven in [36]);

- or the correspondence $\rho \mapsto h[\rho]$ is strongly regularizing (in particular this happens for non-local operators of the form $h[\rho](x) = \int \eta(x-y)\,\mathrm{d}\rho(y)$ for a smooth kernel η). Indeed, if h is guaranteed to be smooth, then φ satisfies semiconcavity properties implying BV estimates on the drift $\nabla\varphi$; this, in turn, provides uniqueness and stability for the continuity equation thanks to the DiPerna Lions theory [16] (this is more or less the point of view presented in [9]).

One of the main interesting cases which is left out is the case where $\nu = 0$ and $h[\rho] = g \circ \rho$ (the local case, where h at a point directly depends only on ρ at the same point). Whenever g is an increasing function this is a very natural model to describe aversion to overcrowding and recalls in a striking way the models about Wardrop equilibria (see [45, 14, 15]).

From the point of view of modeling and game theory, the other natural question is to provide the existence of an equilibrium in the sense of finding which trajectories are followed by each players (or, since players are considered to be indistinguishable, just finding a measure on possible trajectories). This is on the contrary a Lagrangian question, as individual trajectories are involved. The unknown is then a probability measure on a suitable space of paths, which induces measures ρ_t at each instant of time. From these measures we deduce the function $h(t,\cdot)$, which is an ingredient for the optimization problem solved by every agent. The goal is then to choose such a probability on paths so that a.e. path is optimal for the cost built upon the function h which is induced by such probability. Again, there is a difficulty in the local case with $\nu = 0$. Indeed, if $\rho_t(\cdot)$ is just the density of a measure, it is defined only a.e. and so will be $h(t,\cdot)$. Hence, there will be no meaning in integrating it on a path, unless we choose a precise representative, which is a priori arbitrary unless ρ_t is continuous. Of course this difficulty does not exist whenever h is defined via convolution, and in many cases it can also be overcome in the local case for $\nu > 0$ since parabolic equations have a regularization effect and one can expect ρ_t to be smooth.

For both the Eulerian and the Lagrangian question, an answer comes from a variational interpretation: it happens that a solution to the equilibrium system (2) can be found by an overall minimization problem as first outlined in the seminal work by Lasry and Lions [24]. This allows to prove existence of a solution in a suitable sense, and the optimality conditions go in the direction of a Lagrangian equilibrium, as we will see in Section 2.2 and 2.3.

2 Variational formulation

As we said, solutions to the equilibrium system (2) can be found by an overall minimization problem.

The description that we give below will be focused on the case $h[\rho](x) = V(x) + g(\rho(x))$, where we identify the measure ρ with its density w.r.t. the Lebesgue measure on Ω. The function $V : \Omega \to \mathbb{R}_+$ is a potential taking into account different local costs of different points in Ω.

For the variational formulation, we consider all the possible population evolutions, i.e. pairs $(\rho, \mathbf{v})$ satisfying $\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0$ (note that this is the Eulerian way of describing such a movement; in Section 2.2 we will see how to express it in a Lagrangian language) and we minimize the following energy

$$\mathcal{A}(\rho,\mathbf{v}) := \int_0^T \int_\Omega \left(\frac{1}{2}\rho_t|\mathbf{v}_t|^2 + \rho_t V + G(\rho_t)\right)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \Psi\,\mathrm{d}\rho_T,$$

where G is the anti-derivative of g, i.e. $G'(s) = g(s)$ for $s \in \mathbb{R}^+$ with $G(0) = 0$. We fix by convention $G(s) = +\infty$ for $\rho < 0$. Note in particular that G is convex, as its derivative is the increasing function g.

The above minimization problem recalls, in particular when $V = 0$, the Benamou-Brenier dynamic formulation for optimal transport (see [4]). The main difference with the Benamou-Brenier problem is that here we add to the kinetic energy a congestion cost G; also note that usually in optimal transport the target measure ρ_T is fixed, and here it is part of the optimization (but this is not a crucial difference). Finally, note that the minimization of a Benamou-Brenier energy with a congestion cost was already present in [8] where the congestion term was used to model the motion of a crowd with panic.

As is often the case in congestion games, the quantity $\mathcal{A}(\rho, \mathbf{v})$ is not the total cost for all the agents. Indeed, the term $\int\int \frac{1}{2}\rho|\mathbf{v}|^2$ is exactly the total kinetic energy, and the last term $\int \Psi \, \mathrm{d}\rho_T$ is the total final cost, as well as the cost $\int V \, \mathrm{d}\rho_t$ exactly coincides with the total cost enduced by the potential V; yet, the term $\int G(\rho)$ is not the total congestion cost, which should be instead $\int \rho g(\rho)$. This means that the equilibrium minimizes an overall energy (we have what is called a potential game), but not the total cost; this gives rise to the so-called *price of anarchy*.

Another important point is the fact that the above minimization problem is convex, which was by the way the key idea of [4]. Indeed, the problem is not convex in the variables $(\rho, \mathbf{v})$, because of the product term $\rho|\mathbf{v}|^2$ in the functional and of the product $\rho\mathbf{v}$ in the differential constraint. But if one changes variable, defining $\mathbf{w} = \rho\mathbf{v}$ and using the variables $(\rho, \mathbf{w})$, then the constraint becomes linear and the functional convex. We will write $\bar{\mathcal{A}}(\rho, \mathbf{w})$ for the functional $\mathcal{A}(\rho, \mathbf{v})$ written in these variables. The important point for convexity is that the function

$$\mathbb{R} \times \mathbb{R}^d \ni (s, \mathbf{w}) \mapsto \begin{cases} \frac{|\mathbf{w}|^2}{2s} & \text{if } s > 0, \\ 0 & \text{if } (s, \mathbf{w}) = (0, 0), \\ +\infty & \text{otherwise} \end{cases}$$

is convex (and it is actually obtained as $\sup\{as + b \cdot \mathbf{w} \ : \ a + \frac{1}{2}|b|^2 \le 0\}$).

2.1 Convex duality

In order to convince the reader of the connection between the minization of $\mathcal{A}(\rho, \mathbf{v})$ (or of $\bar{\mathcal{A}}(\rho, \mathbf{w})$) and the equilibrium system (2), we will use some formal argument from convex duality. A rigorous equivalence between optimizers and equilibria will be, instead, presented in the Lagrangian framework in Section 2.3.

In order to formally produce a dual problem to $\min \mathcal{A}$, we wil use a min-max exchange procedure. First, we write the constraint $\partial_t\rho + \nabla \cdot (\rho\mathbf{v}) = 0$ in weak form, i.e.

$$\int_0^T \int_\Omega (\rho\partial_t\phi + \nabla\phi \cdot \rho\mathbf{v}) + \int_\Omega \phi_0\overline{\rho_0} - \int_\Omega \phi_T\rho_T = 0 \tag{4}$$

for every function $\phi \in C^1([0, T] \times \Omega)$ (note that we do not impose conditions on the values of ϕ on $\partial\Omega$, hence this is equivalent to completing (CE) with a no-flux boundary condition $\rho\mathbf{v} \cdot n = 0$). Equation (4) requires, in order to make sense, that we give a meaning at ρ_t for every instant of time t (and in particular for $t = T$), which is possible because whenever the kinetic term is finite then the curve ρ_t is a(n absolutely) continuous curve in the space of measures (continuous for the weak convergence, and absolutely continuous for the W_2 Wasserstein distance, see Section 4.1). However, we do not insist on this now, as the presentation stays quite formal.

Using (4) , we can re-write our problem as

$$\min_{\rho,\mathbf{v}} \mathcal{A}(\rho,\mathbf{v}) + \sup_{\phi} \int_0^T \int_\Omega (\rho\partial_t\phi + \nabla\phi\cdot\rho\mathbf{v}) + \int_\Omega \phi_0\overline{\rho_0} - \int_\Omega \phi_T\rho_T,$$

since the sup in ϕ takes value 0 if the constraint is satisfied and $+\infty$ if not. We now switch the inf and the sup and get

$$\sup_{\phi} \int_\Omega \phi_0\overline{\rho_0} + \inf_{\rho,\,\mathbf{v}} \int_\Omega (\Psi-\phi_T)\rho_T + \int_0^T \int_\Omega \left(\frac{1}{2}\rho_t|\mathbf{v}_t|^2 + \rho_t V + G(\rho_t) + \rho\partial_t\phi + \nabla\phi\cdot\rho\mathbf{v}\right) \mathrm{d}x\, \mathrm{d}t.$$

First, we minimize w.r.t. $\mathbf{v}$, thus obtaining $\mathbf{v} = -\nabla\phi$ (on $\{\rho_t > 0\}$) and we replace $\frac{1}{2}\rho|\mathbf{v}|^2 + \nabla\phi\cdot\rho\mathbf{v}$ with $-\frac{1}{2}\rho|\nabla\phi|^2$. Then we get, in the double integral,

$$\inf_\rho\{G(\rho) - \rho(-V - \partial_t\phi + \frac{1}{2}|\nabla\phi|^2)\} = -\sup_\rho\{p\rho - G(\rho)\} = -G^*(p),$$

where we set $p := -V - \partial_t\phi + \frac{1}{2}|\nabla\phi|^2$ and G^* is defined as the Legendre transform of G. Then, we observe that the minimization in the final cost simply gives as a result 0 if $\Psi \geq \phi_T$ (since the minimization is only performed among positive ρ_T) and $-\infty$ otherwise. Hence we obtain a dual problem of the form

$$\sup\left\{-\mathcal{B}(\phi,p) := \int_\Omega \phi_0\overline{\rho_0} - \int_0^T \int_\Omega G^*(p) \;:\; \phi_T \leq \Psi,\ -\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 = V + p\right\}.$$

Note that the condition $G(s) = +\infty$ for $s < 0$ implies $G^*(p) = 0$ for $p \leq 0$. This in particular means that in the above maximization problem one can suppose $p \geq 0$ (indeed, replacing p with p_+ does not change the G^* part, but improves the value of ϕ_0, considered as a function depending on p). The choice of using two variables (ϕ, p) connected by a PDE constraint instead of only ϕ is purely conventional, and it allows for a dual problem which has a sort of symmetry w.r.t. the primal one. Also the choice of the sign is conventional and due to the computation that we will perform later (in particular in Section 4).

Now, standard arguments in convex duality would allow to say that optimal pairs $(\rho,\mathbf{v})$ are obtained by looking at saddle points $((\rho,\mathbf{v}),(\phi,p))$, provided that there is no duality gap between the primal and the dual problems, and that both problems admit a solution. This would mean that, whenever $(\rho,\mathbf{v})$ minimizes $\mathcal{A}$, then there exists a pair (ϕ,p), solution of the dual problem, such that

- $\mathbf{v}$ minimizes $\frac{1}{2}\rho|\mathbf{v}|^2 + \nabla\phi\cdot\rho\mathbf{v}$, i.e. $\mathbf{v} = -\nabla\phi$ ρ-a.e. This gives (CE): $\partial_t\rho - \nabla\cdot(\rho\nabla\phi) = 0$.
- ρ minimizes $G(\rho) - p\rho$, i.e. $g(\rho) = p$ if $\rho > 0$ or $g(\rho) \geq p$ if $\rho = 0$ (in particular, when we have $g(0) = 0$, we can write $g(\rho) = p_+$); this gives (HJ): $-\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 = V + g(\rho)$ on $\{\rho > 0\}$ (as the reader can see, there are some subtleties where the mass ρ vanishes;).
- ρ_T minimizes $(\Psi - \phi_T)\rho_T$ among $\rho_T \geq 0$. But this is not a condition on ρ_T, but rather on ϕ_T: we must have $\phi_T = \Psi$ on $\{\rho_T > 0\}$, otherwise there is no minimizer. This gives the final condition in (HJ).

This provides an informal justification for the equivalence between the equilibrium and the global optimization. It is only informal because we have not discussed whether we have or not duality gaps and whether or not the maximization in (ϕ,p) admits a solution. Moreover, even once these issues are clarified, what we will get will only be a very weak solution to the coupled system (CE)+(HJ).

Nothing guaranteees that this solution actually encodes the individual minimization problem of each agent. This will be clarified in Section 2.3 where a Lagrangian point of view will be presented.

However, let us first give the duality result which can be obtained from a suitable application of Fenchel-Rockafellar's Theorem, and for which details are presented, in much wider generality, in [11].

Theorem 2.1. *Set* $\mathcal{D} = \{(\phi, p) \in C^1([0,T]\times\Omega)\times C^0([0,T]\times\Omega) : -\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 = V + p, \phi_T \leq \Psi\}$ *and* $\mathcal{P} = \{(\rho, \mathbf{v}) : \partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0,\ \rho_0 = \overline{\rho_0}\}$*, where the continuity equation on* $(\rho, \mathbf{v})$ *is satisfied in the sense of* (4) *and* $(\rho_t)_t$ *is a continuous curve of probability measures on* Ω*, with* $\mathbf{v}_t \in L^2(\rho_t)$ *for every t. We then have*

$$\min\{\mathcal{A}(\rho, \mathbf{v}) : (\rho, \mathbf{v}) \in \mathcal{P}\} = \sup\{-\mathcal{B}(\phi, p) : (\phi, p) \in \mathcal{D}\}.$$

Note that in the above theorem we called $\mathcal{D}$ and $\mathcal{P}$ the domains in the dual and primal problems respectively, with the standard confusion between dual and primal (officially it is the problem on measures which should be the dual of that on functions, and not viceversa) which is often done when we prefer to call "primal" the first problem that we meet and which is the main object of our analysis.

It is important to observe that the above theorem does not require the assumption on the growth rate of the Hamiltonian and of the congestion function G, which translate into this quadratic case into "$G(s) \leq C(s^q + 1)$ for an exponent $q < 1 + 2/d$", which is present in the paper [11]. This restriction was required in order to find a suitable relaxed solution to the dual problem, which has in general no solution in $\mathcal{D}$. This result is the object of the following theorem, where we omit this condition on q, since it has been later removed in more recent papers. Indeed, [12] was the first paper where this assumption disappears, for second-order MFG with possibly degenerate diffusion (which include the first-order case; also refer to [19], where duality was used for regularity purposes, which explicitly focuses on the first-order case).

Theorem 2.2. *Set* $\tilde{\mathcal{D}} = \{(\phi, p) \in BV([0,T]\times\Omega)\times\mathcal{M}([0,T]\times\Omega) : -\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 \leq V + p, \phi_T \leq \Psi\}$*. We then have*

$$\min\{\mathcal{A}(\rho, \mathbf{v}) : (\rho, \mathbf{v}) \in \mathcal{P}\} = \max\{-\mathcal{B}(\phi, p) : (\phi, p) \in \tilde{\mathcal{D}}\}$$

and the max on the right hand side is attained.

A disambiguation is needed, when speaking of BV functions, about the final condition $\phi_T \leq \Psi$. Indeed, a BV function could have a jump exactly at $t = T$ and hence its pointwise value at the final time is not well-defined. The important point is that, if $\phi(T^-)$ does not satisfy the required inequality, but $\phi(T)$ is required to satisfy it, then a jump is needed, i.e. a singular part of the measure p concentrated at $\{t = T\}$, and this part will be considered in the dual cost (this is particularly important when the cost G^* in the dual problem has linear growth, and singular parts are allowed, which will be the case for the density-constrained case of Section 5).

However, most of these notes will not make use of this refined duality, both because we want to consider cases where the growth rate of G does not satisfy this inequality and because we will need to use (in Section 3) smooth test functions and apply the duality. For this sake, it will be more convenient to choose almost-maximizers $(\phi, p) \in C^1 \times C^0$ rathen then maximizers with limited regularity.

We finish this section with a last variant, inspired by the crowd motion model of [30]. We would like to consider a variant where, instead of adding a penalization $g(\rho)$, we impose a capacity constraint $\rho \leq 1$. How to give a proper definition of equilibrium? A first, naive, idea, would be the following: when $(\rho_t)_t$ is given, every agent minimizes his own cost paying attention to the constraint $\rho_t(x(t)) \leq 1$. But if ρ already satisfies $\rho \leq 1$, then the choice of only one extra agent will not violate the constraint (since we have a non-atomic game), and the constraint becomes empty. As already pointed out in [39], this cannot be the correct definition.

In [39] an alternative model is formally proposed, based on the effect of the gradient of the pressure on the motion of the agents, but this model is not variational, and no solution has been proven to exist in general in its local and deterministic form.

A different approach to the question of density constraints in MFG was presented in [13]: the idea is to start from the the variational problem

$$\min\left\{\int_0^T\int_\Omega\left(\frac{1}{2}|\mathbf{v}_t|^2+V\right)\,\mathrm{d}\rho_t+\int_\Omega\Psi\,\mathrm{d}\rho_T\ :\ \rho\le 1\right\}.$$

This means that we use $G=I_{[0,1]}$, i.e. $G(s)=0$ for $s\in[0,1]$ and $+\infty$ otherwise. The dual problem can be computed and we obtain

$$\sup\left\{\int_\Omega\phi_0\overline{\rho_0}-\int_0^T\int_\Omega p_+\ :\ \phi_T\le\Psi,\ -\partial_t\phi+\frac{1}{2}|\nabla\phi|^2=V+p\right\}$$

(note that this problem is also obtained as the limit $m\to\infty$ of $g(\rho)=\rho^m$; indeed the functional $\frac{1}{m+1}\int\rho^{m+1}$ Γ-converges to the constraint $\rho\le 1$ as $m\to\infty$).

By looking at the primal-dual optimality conditions, we get again $\mathbf{v}=-\nabla\phi$ and $\phi_T=\Psi$, but the optimality of ρ means

$$0\le\rho<1\Rightarrow p=0,\quad \rho=1\Rightarrow p\ge 0.$$

This gives the following MFG system

$$\begin{cases}-\partial_t\varphi+\frac{|\nabla\varphi|^2}{2}=V+p,\\ \partial_t\rho-\nabla\cdot(\rho\nabla\varphi)=0,\\ \varphi(T,x)=\Psi(x),\quad \rho(0,x)=\overline{\rho_0}(x),\\ p\ge 0,\ \rho\le 1,\ p(1-\rho)=0.\end{cases}\tag{5}$$

It is important to understand that p is a priori just a measure on $[0,T]\times\Omega$, since it has a sign (and distributions with a sign are measures) and it is penalized in terms of its L^1 norm. Indeed, even if Theorem 2.2 is stated exactly by taking p in the space of measures, in general according to the function G^* it is possible to obtain extra summability (if G has growth of order q then one obtains $p\in L^{q'}$). Here, instead, since G^* is linear we do not obtain more than measure bounds. This means that φ is not better than a BV function, and in particular it could have jumps. From the first equation in System (5) and the positivity of p we see that φ could have a jump at time $t=T$ in the sense that $\varphi(T^-)>\varphi(T)=\Psi$. Hence, an alternative way to write the same system is to remove the possible singular part of p concentrated on $t=T$ but consider as a final value for φ the value that it takes at T^-. In this way, we can re-write System (5) as

$$\begin{cases}-\partial_t\varphi+\frac{|\nabla\varphi|^2}{2}=V+p,\\ \partial_t\rho-\nabla\cdot(\rho\nabla\varphi)=0,\\ \varphi(T,x)=\Psi(x)+P(x),\quad \rho(0,x)=\overline{\rho_0}(x),\\ p\ge 0,\ \rho\le 1,\ p(1-\rho)=0,\\ P\ge 0, P(1-\rho_T)=0.\end{cases}\tag{6}$$

Formally, by looking back at the relation between (HJ) and optimal trajectories, we can guess that each agent solves

$$\min \int_0^T \left(\frac{|x'(t)|^2}{2} + h(t, x(t))\right) \mathrm{d}t + \tilde{\Psi}(x(T)), \tag{7}$$

where $h = p + V$ and $\tilde{\Psi} = \Psi + P$. Here p and P are pressures arising from the incompressibility constraint $\rho \le 1$ and only present in the saturated zone $\{\rho = 1\}$, but they finally act as prices paid by the agents to travel through saturated regions. From the economical point of view this is meaningful: due to a capacity constraint, the most attractive regions develop a positive price to be paid to pass through them, and this price is such that, if the agents keep it into account in their choices, then their mass distribution will indeed satisfy the capacity constraints.

This problem of course presents the same difficulties of the case where congestion is penalized and not constrained: what does it mean to integrate p on a path if p is only a measure? We will see later on a technique to get rid of this diffulty, following an idea by Ambrosio and Figalli ([2]) for applications to the incompressible Euler equation, but this techniques requires at least that p is a sufficiently integrable function. In these notes, we will present Ambrosio and Figalli's ideas in the case where h is L^∞, insisting on he simplifaction that it brings, and in Section 5 we will provide indeed an L^∞ regularity result on both p and P. In the original paper on density-constrained MFG, [13], the L^∞ regularity result on p was not available, and suitable regularity results of the form $p \in L^2_t BV_x$ were proven via a technique similar to that used in Section 3 of these notes. Since we have $L^2_t BV_x \subset L^2_t L^{d/(d-1)}_x$, this BV regularity result was enough to apply, at least to a certain extent, the theory developed in [2].

2.2 Lagrangian formulation

We present now an alternative point of view for the overall minimization problem presented in the previous sections. As far as now, we only looked at an Eulerian point of view, where the motion of the population is described by means of its density ρ and of its velocity field $\mathbf{v}$. The Lagrangian point of view would be, instead, to describe the motion by describing the trajectory of each agent. Since the agents are supposed to be indistinguishable, then we just need to determine, for each possible trajectory, the number of agents following it (and not their names...); this means looking at a measure on the set of possible paths.

Set $C = H^1([0,T];\Omega)$; this will be the space of possible paths that we use. In general, absolutely continuous paths would be the good choice, but we can restrict our attention to H^1 paths because of the kinetic energy term that we have in our minimization. We define the evaluation maps $e_t : C \to \Omega$, given for every $t \in [0,T]$ by $e_t(\omega) = \omega(t)$. Also, we define the kinetic energy functional $K : C \to \mathbb{R}$ given by

$$K(\omega) = \frac{1}{2}\int_0^T |\omega'|^2(t)\,\mathrm{d}t.$$

We endow the space C with the uniform convergence (and not the strong H^1 convergence, so that we have compactness of the sublevel sets of K). For notational simplicity, we will also often write K_Ψ for the kinetic energy augmented by a final cost: $K_\Psi(\omega) := K(\omega) + \Psi(\omega(T))$; similarly, we will denote by $K_{\Psi,h}$ the same quanity when also a running cost is included: $K_{\Psi,h}(\omega) := K_\Psi(\omega) + \int_0^T h(t,\omega(t))\,\mathrm{d}t$.

Proposition 2.3. *Suppose $(\rho, \mathbf{v})$ satisfies the continuity equation $\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0$ and $\int_0^T\int_\Omega \rho|\mathbf{v}|^2 < \infty$. Then there exist a representative of ρ such that $t \mapsto \rho_t$ is weakly continuous, and a probability measure*

$Q \in \mathcal{P}(C)$ such that $\rho_t = (e_t)_\# Q$ and

$$\int_C K(\omega)\, \mathrm{d}Q(\omega) \leq \frac{1}{2}\int_0^T \int_\Omega \rho|\mathbf{v}|^2.$$

Conversely, if we have $\rho_t = (e_t)_\# Q$ for a probability measure $Q \in \mathcal{P}(C)$ with $\int_C K(\omega)\, \mathrm{d}Q(\omega) < \infty$, then $t \mapsto \rho_t$ is weakly continuous and there exists a time-dependent family of vector fields $\mathbf{v}_t \in L^2(\rho_t)$ such that $\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0$ and

$$\frac{1}{2}\int_0^T \int_\Omega \rho|\mathbf{v}|^2 \leq \int_C K(\omega)\, \mathrm{d}Q(\omega).$$

The above proposition comes from optimal transport theory and we will discuss a more refined version of it in Section 4. Its proof can be found combining, for instance, Theorems 5.14 and 5.31 in [40]. It allows to re-write the minimization problem

$$\min \left\{\mathcal{A}(\rho, \mathbf{v}) \ : \ \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0\right\},$$

in the following form:

$$\min \left\{ J(Q) := \int_C K \,\mathrm{d}Q + \int_0^T \mathcal{G}((e_t)_\# Q)\,\mathrm{d}t + \int_\Omega \Psi \,\mathrm{d}(e_T)_\# Q,\ Q \in \mathcal{P}(C), (e_0)_\# Q = \overline{\rho_0} \right\}, \tag{8}$$

where $\mathcal{G} : \mathcal{P}(\Omega) \to \overline{\mathbb{R}}$ is defined through

$$\mathcal{G}(\rho) = \begin{cases} \int (V(x)\rho(x) + G(\rho(x)))\,\mathrm{d}x & \text{if } \rho \ll \mathcal{L}^d, \\ +\infty & \text{otherwise.} \end{cases}$$

The functional $\mathcal{G}$ is a typical local functional defined on measures (see [6]). It is lower-semicontinuous w.r.t. weak convergence of probability measures provided $\lim_{s\to\infty} G(s)/s = +\infty$ (which is the same as $\lim_{s\to\infty} g(s) = +\infty$), see, for instance, Proposition 7.7 in [40].

Under these assumptions, it is easy to prove, by standard semicontinuity arguments in the space $\mathcal{P}(C)$, that a minimizer of (8) exists. We summarize this fact, together with the corresponding optimality conditions, in the next proposition. The optimality conditions are obtained by standard convex perturbations: if $\bar{Q}$ is an optimizer and Q a competitor with finite energy, then one sets $Q_\varepsilon := (1-\varepsilon)\bar{Q} + \varepsilon Q$ and differentiates the cost w.r.t. ε at $\varepsilon = 0$. The idea is just that a point optimizes a convex functional on a convex set if and only if it optimizes its linearization around itself.

Proposition 2.4. *Suppose that Ω is compact, that G is a convex and superlinear function, and that V and Ψ are continuous functions on Ω. Then Problem (8) admits a solution $\bar{Q}$.*

Moreover, $\bar{Q}$ is a solution if and only if for any other competitor $Q \in \mathcal{P}(C)$ with $J(Q) < +\infty$ with $(e_0)_\# Q = \overline{\rho_0}$ we have

$$J_h(Q) \geq J_h(\bar{Q}),$$

where $J_{\Psi,h}$ is the linear functional

$$J_{\Psi,h}(Q) = \int K \,\mathrm{d}Q + \int_0^T \int_\Omega h(t,x)\,\mathrm{d}(e_t)_\# Q + \int_\Omega \Psi \,\mathrm{d}(e_T)_\# Q,$$

the function h being defined through $\rho_t = (e_t)_\# \bar{Q}$ and $h(t,x) = V(x) + g(\rho_t(x))$.

Remark 1. The above optimality condition and its interpretation in terms of equilibria (see below), as well as the efinition of the functional via an antiderivative, strongly recall the setting of *continuous Wardrop equilibria*, studied in [14] (see also [15] for a survey of the theory). Indeed, in [14] a traffic intensity i_Q (a positive measure on Ω) is associated with each measure Q on C, and we define a weighted length on curves ω using i_Q as a weighting factor. We then prove that the measure Q which minimizes a suitable functional minimizes its linearization, which in turn implies that the same Q is concentrated on curves which are geodesic for this weighted length, depending on Q itself. Besides some technical details about the precise mathematical form of the functionals, the main difference between Wardrop equilibria (which are traditionally studied in a discrete framework on networks, see [45]) and MFG is the fact that Wardrop's setting is static: in such a traffic notion we consider a continuous traffic flow, where some mass is constantly injected somewhere in the domain, and at the same time constantly absorbed somewhere else (see Chapter 4 of [40] for other models of this form).

We now consider the functional $J_{\Psi,h}$. Note that the function h is obtained from the densities ρ_t, which means that it is well-defined only a.e. However, the integral $\int_0^T\int_\Omega h(t,x)\,\mathrm{d}(e_t)_\# Q$ is well-defined and does not depend on the representative of h, since $J(Q) < +\infty$ implies that the measures $(e_t)_\# Q$ are absolutely continuous for a.e. t. Hence, this functional is well-defined for $h \geq 0$ measurable.

Formally, we can also write

$$\int_0^T \int_\Omega h(t,x)\,\mathrm{d}(e_t)_\# Q = \int_C \mathrm{d}Q \int_0^T h(t,\omega(t))\,\mathrm{d}t$$

and hence we get

$$J_{\Psi,h}(Q) = \int_C \mathrm{d}Q(\omega)\left(K(\omega) + \int_0^T h(t,\omega(t))\,\mathrm{d}t + \Psi(\omega(T))\right) = \int_C K_{\Psi,h}\,\mathrm{d}Q.$$

It is then tempting to interpret the optimality conditions on $\bar{Q}$ stated in Proposition 2.4 by considering that they can only be satisfied if $\bar{Q}$-a.e. curve ω satisfies

$$K(\omega) + \int_0^T h(t,\omega(t))\,\mathrm{d}t + \Psi(\omega(T)) \leq K(\tilde{\omega}) + \int_0^T h(t,\tilde{\omega}(t))\,\mathrm{d}t + \Psi(\tilde{\omega}(T)) \quad \text{for every } \tilde{\omega} \text{ s.t. } \tilde{\omega}(0) = \omega(0). \tag{9}$$

This would be exactly the equilibrium condition in the MFG. Indeed, the MFG equilibrium condition can be expressed in Lagrangian language in the following way: find Q such that, if we define $\rho_t = (e_t)_\#\bar{Q}$ and $h(t,x) = V(x) + g(\rho_t(x))$, then Q is concentrated on minimizers of $K_{\Psi,h}$ for fixed initial point.

Yet, there are two objections to this way of arguing. The first concers the fact that the functional $K_{\Psi,h}$ does indeed depend on the representative of h that we choose and it looks suspicious that such an equilibrium statement could be true independently of the choice of the representative. Moreover, the idea behind the optimality in (9) would be to choose a measure Q concentrated on optimal, or almost optimal, curves starting from each point, and there is no guarantee that such a measure Q satisfies $J(Q) < +\infty$.

The approach that we present in Section 2.3 below, due to Ambrosio and Figalli ([2]) and first applied to MFG in [13] for the case of MFG with density constraints, is a way to rigorously overcome these difficulties. The goal is to find a suitably chosen representative $\hat{h}$ of h so that we can prove that if $\bar{Q}$ minimizes $J_{\Psi,h}$, then it is concentrated on curves minimizing $K_{\Psi,\hat{h}}$.We develop here this theory under the assumption $h \in L^\infty$, while the original proofs were more general, but required some technicalities that we will briefly adress in a comment. We will explain in which points the L^∞ assumption allows to obtain cleaner and more powerful results. Morevoer, we insist that we are allowed to stick to this more restrictive setting because we will see, in Sections 4 and 5, that we do have $h \in L^\infty$ in the cases of interest for us.

2.3 Optimality conditions on the level of single agent trajectories

In this section we consider a measurable function $h : [0,T] \times \Omega \to \mathbb{R}$ and we suppose that h is upper-bounded by a constant H_0, i.e. $h \le H_0$ a.e. As far as lower bounds are concerned, all the section is written supposing $h \in L^1([0,T] \times \Omega)$, but it is not difficult to adapt it to the case where h (or, rather, its negative part) is only a measure. Let us define then

$$h_r(t,x) := \fint_{B(x,r)} h(t,y)\,\mathrm{d}y \quad \text{ if } B(x,r) \subset \Omega$$

and then

$$\hat{h}(t,x) := \begin{cases} \limsup_{r\to 0} h_r(t,x) & \text{if } x \notin \partial\Omega, \\ H_0 & \text{if } x \in \partial\Omega. \end{cases}$$

First, we observe that $\hat{h}$ is a representative of h, in the sense that we have $h = \hat{h}$ a.e. (in the case where h is a measure then h_r is defined as $h_t(B(x,r))/\mathcal{L}^d(B(x,r))$ and $\hat{h}$ is a representative of the absolutely continuous part of h). Indeed, a.e. point in Ω is a Lebesgue point for h, so that the above lim sup is indeed a limit and equal to h, and the boundary where the definition is not given as a limsup is supposed to be negligible. In some sense, we will obtain the desired result by writing estimates involving h_r and passing to the limit as $r \to 0$.

Proposition 2.5. *Suppose that $\bar{Q}$ minimizes $J_{\Psi,h}$ among measures with $J(Q) < +\infty$ and suppose that G is a convex function with polynomial growth, that Ψ is a continuous function and that Ω is a smooth domain. Define $\hat{h}$ as above and suppose that $(e_t)_\#Q$ is absolutely continuous for a.e. t. Then $\bar{Q}$ is concentrated on curves ω such that*

$$K_{\Psi,\hat{h}}(\omega) \le K_{\Psi,\hat{h}}(\tilde{\omega}) \quad \textit{for every } \tilde{\omega} \textit{ s.t. } \tilde{\omega}(0) = \omega(0).$$

Proof. The proof is an adaptation of those proposed in [2, 13].

Consider a countable set $D \subset H^1_\diamond([0,T])$, where $H^1_\diamond([0,T])$ is the Hilbert space of H^1 functions on $[0,T]$, valued in $\mathbb{R}^d$, and vanishing at $t = 0$ (but not necessarily at $t = T$), dense in $H^1_\diamond([0,T])$ for the H^1 norm,. Also consider a curve $\gamma \in D$, a vector $y \in B(0,1) \subset \mathbb{R}^d$, a number $r > 0$, and a cut-off function $\eta \in C^1([0,T])$, with $\eta(0) = 0$, and $\eta > 0$ on $(0,T]$, with $\eta(T) = 1$. Consider a Borel subset $E \subset C$, with $E \subset \{\omega : \omega(t) + \gamma(t) + B(0,r) \subset \Omega \text{ for every } t\}$ and define a map $S : C \to C$ as follows

$$S(\omega) = \begin{cases} \omega + \gamma + r\eta y & \text{if } \omega \in E \\ \omega & \text{if } \omega \notin E. \end{cases}$$

Defining $Q = S_\#\bar{Q}$ we can easily see that we have $J(Q) < +\infty$ (we use here the polynomial growth of G, since the density of $(e_t)_\#Q$ can be decomposed as the sum of two densities with finite value for $\mathcal{G}$, and we need a bound for the sum, which is not available, for instance, for convex functions with exponential growth).

Comparing $J_{\Psi,h}(Q)$ to $J_{\Psi,h}(\bar{Q})$ and erasing, by linearity, the common terms (those coming from the integration on E^c) we get

$$\int_E K_\Psi(S(\omega))\,\mathrm{d}\bar{Q}(\omega)+\iint h(t,\cdot+\gamma(t)+r\eta(t)y)\,\mathrm{d}(e_t)_\#(\bar{Q}\mathbb{1}_E) \geq \int_E K_\Psi(\omega)\,\mathrm{d}\bar{Q}(\omega)+\iint h(t,\cdot)\,\mathrm{d}(e_t)_\#(\bar{Q}\mathbb{1}_E).$$

The first parameter we get rid of is the parameter y, as we take the average among possible $y \in B(0,1)$. It is important to note that we have

$$\fint_{B(0,1)} K(\omega + r\eta y)\,\mathrm{d}y = K(\omega) + O(r^2)$$

(by symmetry, there is no first-order term in r, even if this is not important and we would only need terms tending to 0 as $r \to 0$, independently of their order) and to use the definition of h_r (and, by analogy, of Ψ_r) in order to obtain

$$\int_E \left(K_{\Psi_r}(\omega+\gamma) + O(r^2)\right)\,\mathrm{d}\bar{Q}(\omega) + \iint h_{r\eta(t)}(t,\cdot+\gamma(t))\,\mathrm{d}(e_t)_\#(\bar{Q}\mathbb{1}_E) \geq$$
$$\geq \int_E K_\Psi(\omega)\,\mathrm{d}\bar{Q}(\omega) + \iint h(t,\cdot)\,\mathrm{d}(e_t)_\#(\bar{Q}\mathbb{1}_E).$$

Now, we observe that $h = \hat{h}$ a.e. together with $(e_t)_\#\bar{Q} \ll \mathcal{L}^d$ ($\mathcal{L}^d$ being the Lebesgue measure) allow to replace, in the right hand side, h with $\hat{h}$. Moreover, we rewrite some terms using the following equalities

$$\begin{aligned}\iint h_{r\eta(t)}(t,\cdot+\gamma(t))\,\mathrm{d}(e_t)_\#(\bar{Q}\mathbb{1}_E) &= \int_E \mathrm{d}\bar{Q}(\omega)\int h_{r\eta(t)}(t,\omega(t)+\gamma(t))\,\mathrm{d}t\\ \iint \hat{h}(t,\cdot)\,\mathrm{d}(e_t)_\#(\bar{Q}\mathbb{1}_E) &= \int_E \mathrm{d}\bar{Q}(\omega)\int \hat{h}(t,\omega(t))\,\mathrm{d}t\end{aligned}$$

We then use the arbitrariness of E, thus obtaining the following fact: for $\bar{Q}$ a.e. ω s.t. $\omega(t)+\gamma(t)+B(0,r)\subset\Omega$ for every t we have

$$K_{\Psi_r}(\omega+\gamma) + O(r^2) + \int_0^T h_{r\eta(t)}(t,\omega(t)+\gamma(t))\,\mathrm{d}t \geq K_\Psi(\omega) + \int_0^T \hat{h}(t,\omega(t))\,\mathrm{d}t.$$

This result is true for a.e. curve for fixed γ, while we would like to obtain inequalities which are valid on a full-measure set which is the same for every γ. This explains the use of the dense set D. On the same full-measure set this inequality is true for every $\gamma \in D$, since D is countable. Then, using the density of D and the continuity, for fixed $r > 0$, of all the quantities on the left-hand side, we obtain the following: for $\bar{Q}$ a.e. ω and for every $\gamma \in H^1_\diamond([0,T])$ s.t. $\omega(t)+\gamma(t)+B(0,2r)\subset\Omega$ for every t we have

$$K_{\Psi_r}(\omega+\gamma) + O(r^2) + \int_0^T h_{r\eta(t)}(t,\omega(t)+\gamma(t))\,\mathrm{d}t \geq K_\Psi(\omega) + \int_0^T \hat{h}(t,\omega(t))\,\mathrm{d}t. \tag{10}$$

In order to obtain this result every $\gamma \in H^1_\diamond([0,T])$ is approximated by a sequence $\gamma_k \in D$, with both H^1 and uniform convergence, so that we have $|\gamma_k - \gamma| \leq r$ (which explains the condition $\omega(t)+\gamma(t)+B(0,2r)\subset\Omega$ with a different radius now): the kinetic term $K(\omega+\gamma_k)$ passes to the limit because of strong H^1 convergence, while the integral term passes to the limit via dominated convergence (if h is bounded from above and below, otherwise we use a Fatou's lemma with a limsup, since we have at least $h \leq H_0$), since each function $h_{r\eta(t)}(t, \cdot)$ is continuous, and they are all bounded above by a same constant. This is a first point where we use the L^∞ upper bound $h \leq H_0$. Indeed, if we do not have a suitable bound on h, the L^∞ norm of $h_{r\eta(t)}$ could explose as $t \to 0$ since $\eta(t) \to 0$ (and, unfortunately, it is not possible in general to guarantee integrablity in time of this bound if we want $\eta \in H^1$).

Inequality (10) is true for fixed $r > 0$, but taking a countable sequence tending to 0 we can pass to the limit as $r \to 0$ on a full-measure set, thus obtaining the following: for $\bar{Q}$ a.e. ω and for every $\gamma \in H^1_\diamond([0, T])$ s.t. $\omega(t) + \gamma(t) \in \mathring{\Omega}$ for every t we have

$$K_\Psi(\omega + \gamma) + \int_0^T \hat{h}(t, \omega(t) + \gamma(t))\, \mathrm{d}t \geq K_\Psi(\omega) + \int_0^T \hat{h}(t, \omega(t))\, \mathrm{d}t.$$

Also this limit uses $h \leq H_0$ as an assumption to apply Fatou's lemma with limsup (we need to upper bound the terms with $h_{r\eta(t)}$). Of course some integrability on the curve of the maximal function of h would be enough, but this is a much trickier condition (see below in Remark 3). Note that on the left hand side we used the continuity of Ψ.

This shows optimality of a.e. ω compared to every curve lying in the interior of the domain Ω. In order to handle curves touching $\partial\Omega$, let us take a family of maps $\zeta_\delta : \Omega \to \mathring{\Omega}$ with the following properties: $\mathrm{Lip}(\zeta_\delta) \to 1$ as $\delta \to 0$, $|\zeta_\delta(x) - x| \leq C\delta$ for every $x \in \Omega$, and $\zeta_\delta(x) = x$ if $d(x, \partial\Omega) \geq \delta$. We just observe now that, for a given curve $\omega : [0, T] \to \Omega$, we have

$$\begin{aligned} K_{\Psi,\hat{h}}(\zeta_\delta \circ \omega) \quad \leq \quad & \mathrm{Lip}(\zeta_\delta)^2 K(\omega) + \Psi(\omega(T)) + \int_0^T \hat{h}(t, \omega(t))\, \mathrm{d}t \\ & + |\Psi(\zeta_\delta(\omega(T)) - \Psi(\omega(T)| + H_0 |\{t \,:\, 0 < d(\omega(t), \partial\Omega) < \delta\}| \to K_{\Psi,\hat{h}}(\omega). \end{aligned}$$

Again we used $h \leq H_0 < \infty$. As a result, we obtain that $\bar{Q}$-a.e. curve ω optimizes $K_{\Psi,\hat{h}}$ in the class of H^1 curves staying in Ω and sharing the same starting point. □

Remark 2. The proof can be easily adapted to the case where the function Ψ is not continuous but only bounded, but we need in this case to suppose that $(e_T)_\# \bar{Q}$ is absolutely continuous. It is then possible to treat Ψ exactly as h, replacing it with its representative $\hat{\Psi}$. This will be useful in the density-constrained case where Ψ is replaced by a new function $\Psi + P$.

Remark 3. Both in [2] and [13] h is not required to be bounded, but the statement is slightly different and makes use of the *Maximal function* $Mh := \sup_r h_r$. The result which is obtained is the optimality of $\bar{Q}$-a.e. curve in the class of curves $\tilde{\omega}$ with $\int Mh(t, \tilde{\omega}(t))\, \mathrm{d}t < +\infty$, and moreover the result is local in time (perturbations are only allowed to start from $t_0 > 0$). Besides this small technicality about locality in time, the optimality which is obtained is only useful if there are many curves $\tilde{\omega}$ satisfying this integrability condition on Mh. A typical statement is then "for Q-a.e. curve $\tilde{\omega}$ this is the case", but it is not straightforward for which measure Q should one require this. Again, the typical approach is to prove that this is the case for all measures Q with $J(Q) < +\infty$ (which are in some sense the relevant measures for this problem, and this corresponds to some integrability property of the densities $\rho_t := (e_t)_\# Q$). In this case, we can compute

$$\int \int Mh(t, \omega(t))\, \mathrm{d}t\ \mathrm{d}Q(\omega) = \int\ \mathrm{d}t \int_\Omega Mh(t, x)\, \mathrm{d}(e_t)_\# Q.$$

We would like to guarantee that every Q with $J(Q) < +\infty$ is such that $\int \int Mh(t, \omega(t))\, \mathrm{d}t\ \mathrm{d}Q(\omega) < \infty$. Since we know that $G((e_t)_\# Q)$ is integrable, it is enough to guarantee $G^*(Mh) \in L^1$. In the case where

$G(s) \approx s^q$ (hence $g(s) \approx s^{q-1}$ we need $Mh \in L^{q'}$. Since in this case we know $\rho \in L^q$, then $h \approx g(\rho) \in L^{q'}$ and this implies $Mh \in L^{q'}$ from standard theorems in harmonic analysis, as soon as $q' > 1$.

As we can see, the analysis of these equilibrium conditions motivates a deeper study of regularity issues, for several reasons. Indeed, in order to apply the previous considerations it would be important to obtain upper bounds on $h[\rho]$; when this is not possible, at least obtaining higher integrability (in particular when we only know $h \in L^1$, passing to $L^{1+\varepsilon}$ would be crucial) would be important in order to deal with the integrability of Mh. Higher integrability can sometimes be obtained via higher-order estimates (proving BV or Sobolev estimates). More generally, better regularity on ρ (or on the dual variable φ) could give "better" solutions to the (HJ) equation (instead of just a.e. solutions).

This is why in the next sections we will see some regularity techniques. In Section 3 we will prove Sobolev results on the optimal density ρ which are interesting in themselves, and also imply higher integrability. Then in Section 4 we will see how to directly obtain L^∞ results with a different technique. Finally, Section 5 is devoted to the density-constrained case: for this case, [13] presented a non-trivial variant of the technique used here in section 3 and obtained BV estimates on the pressure, which implied that the pressure is a function belonging to a certain L^q space, $q > 1$: here, instead, we will present the approach of [27] which provides $p \in L^\infty$ (yet, we will choose an easier proof, not available in [27]).

3 Regularity via duality

We present here a technique to prove Sobolev regularity results for the optimal density ρ. This technique, based on duality, is inspired from the work of [7], and has been applied to MFG in [13]. It is actually very general, and [41] shows how it can be used to prove (or re-prove) many regularity results in elliptic equations coming from convex variational problems.

We start from a lemma related to the duality results of Section 2.1.

Lemma 3.1. *For any $(\phi, p) \in \mathcal{D}$ and $(\rho, \mathbf{v}) \in \mathcal{P}$ we have*

$$\mathcal{B}(\phi, p) + \mathcal{A}(\rho, \mathbf{v}) = \int_\Omega (\Psi - \phi_T)\, \mathrm{d}\rho_T + \int_0^T \int_\Omega (G(\rho) + G^*(p) - \rho p)\ \mathrm{d}x\, \mathrm{d}t + \frac{1}{2}\int_0^T \int_\Omega \rho |\mathbf{v} + \nabla\phi|^2\, \mathrm{d}x\, \mathrm{d}t.$$

Proof. We start from

$$\mathcal{B}(\phi, p) + \mathcal{A}(\rho, \mathbf{v}) = \int_0^T \int_\Omega \left(\frac{1}{2}\rho|\mathbf{v}|^2 + G(\rho) + G^*(p) + V\rho\right) \mathrm{d}x\, \mathrm{d}t + \int_\Omega \Psi\ \mathrm{d}\rho_T - \int_\Omega \phi_0\, d\overline{\rho_0}. \tag{11}$$

Then we use

$$\int_\Omega \Psi\ \mathrm{d}\rho_T - \int_\Omega \phi_0\, d\overline{\rho_0} = \int_\Omega (\Psi - \phi_T)\ \mathrm{d}\rho_T + \int_\Omega \phi_T\ \mathrm{d}\rho_T - \int_\Omega \phi_0\, d\overline{\rho_0}$$

and

$$\begin{aligned}\int_\Omega \phi_T\ \mathrm{d}\rho_T - \int_\Omega \phi_0\, d\overline{\rho_0} &= \int_0^T \int_\Omega (-\phi\nabla\cdot(\rho\mathbf{v}) + \rho\partial_t\phi)\ \mathrm{d}x\, \mathrm{d}t \\ &= \int_0^T \int_\Omega \left(\nabla\phi\cdot(\rho\mathbf{v}) + \rho\left(\frac{1}{2}|\nabla\phi|^2 - (p+V)\right)\right) \mathrm{d}x\, \mathrm{d}t.\end{aligned}$$

If we insert this into (11) we get the desired result. □

It is important to stress that we used the fact that ϕ is C^1 since $(\rho, \mathbf{v})$ only satisfies (CE) in a weak sense, i.e. tested against C^1 functions. The same computations above would not be possible for $(\phi, p) \in \tilde{\mathcal{D}}$.

The regularity proof will come from the previous computations applied to suitable translations in space and/or time.

In order to simplify the exposition, we will suppose that $\Omega = \mathbb{T}^d$ is the d-dimensional flat torus, which avoids boundary issues. To handle the case of a domain Ω with boundary, we refer to the computations in [41] which suggest how to adapt the method below. Finally, for simplicity, we will only prove in this paper local results in $(0, T)$, so that also the time boundary does not create difficulties.

Here is the intuition behind the proof in this spatially homogeneous case. First, we use Lemma 3.1 to deduce

$$\mathcal{B}(\phi, p) + \mathcal{A}(\rho, \mathbf{v}) \geq \int_0^T \int_\Omega (G(\rho) + G^*(p) - \rho p) \, \mathrm{d}x \, \mathrm{d}t$$

(since the other terms appearing in Lemma 3.1 are positive). Then, let us suppose that there exist two function $J, J_* : \mathbb{R} \to \mathbb{R}$ and a positive constant $c_0 > 0$ such that for all $a, b \in \mathbb{R}$ we have

$$G(a) + G^*(b) \geq ab + c_0 |J(a) - J_*(b)|^2. \tag{12}$$

Remark 4. Of course, this is always satisfied by taking $J = J_* = 0$, but there are less trivial cases. For instance, if $G(\rho) = \frac{1}{q}\rho^q$ for $q > 1$, then $G^*(p) = \frac{1}{q'} q^{r'}$, with $q' = q/(q-1)$ and

$$\frac{1}{q}|a|^q + \frac{1}{q'}|b|^{q'} \geq ab + \frac{1}{2\max\{q, q'\}} |a^{q/2} - b^{q'/2}|^2,$$

i.e. we can use $J(a) = a^{q/2}$ and $J_*(b) = b^{q'/2}$. Another easy case to consider is the one where $G'' \geq c_0 > 0$. In this case we can choose $J = \mathrm{Id}$ and $J^* = (G^*)'$.

We wish to show that if $(\rho, \mathbf{v})$ is a minimizer of $\mathcal{A}$ then $J(\rho) \in H^1_{\mathrm{loc}}((0, T) \times \Omega)$. Should $\mathcal{B}$ admit a C^1 minimizer ϕ (more precisely, a pair (ϕ, p)), then by the Duality Theorem 2.1, we would have $\mathcal{B}(\phi, p) + \mathcal{A}(\rho, \mathbf{v}) = 0$. Using Lemma 3.1, we get $J(\rho) = J_*(p)$. If we manage to show that $\tilde{\rho}(t, x) := \rho(t + \eta, x + \delta)$ with a corresponding velocity field $\tilde{\mathbf{v}}$ satisfies

$$\mathcal{A}(\tilde{\rho}, \tilde{\mathbf{v}}) \leq \mathcal{A}(\rho, \mathbf{v}) + C(|\eta|^2 + |\delta|^2) \tag{13}$$

for small $\eta \in \mathbb{R}, \delta \in \mathbb{R}^d$, then we would have

$$C(|\eta|^2 + |\delta|^2) \geq \mathcal{A}(\tilde{\rho}, \tilde{\mathbf{v}}) + \mathcal{B}(\phi, p) \geq c \|J(\tilde{\rho}) - J_*(p)\|^2_{L^2}.$$

Using then $J_*(p) = J(\rho)$, we would get

$$C(|\eta|^2 + |\delta|^2) \geq c \|J(\tilde{\rho}) - J(\rho)\|^2_{L^2},$$

which would mean that $J(\rho)$ is H^1 as we have estimated the squared L^2 norm of the difference between $J(\rho)$ and its translation by the squared length of the translation. Of course, there are some technical issues that need to be taken care of, for instance $\tilde{\rho}$ is not even well-defined (as we could need the value of ρ outside $[0, T] \times \Omega$), does not satisfy the initial condition $\tilde{\rho}(0) = \overline{\rho_0}$, we do not know if $\mathcal{B}$ admits a minimizer, and we do not know whether (13) holds.

To perform our analysis, let us fix $t_0 < t_1$ and a cut-off function $\zeta \in C^\infty_c(]0, T[)$ with $\zeta \equiv 1$ on $[t_0, t_1]$. Let us define

$$\begin{cases} \rho^{\eta,\delta}(t, x) := \rho(t + \zeta(t)\eta, x + \zeta(t)\delta), \\ \mathbf{v}^{\eta,\delta}(t, x) := \mathbf{v}(t + \zeta(t)\eta, x + \zeta(t)\delta)(1 + \zeta'(t)\eta) - \zeta'(t)\delta. \end{cases} \tag{14}$$

It is easy to check that the pair $(\rho^{\eta,\delta}, \mathbf{v}^{\eta,\delta})$ satisfies the continuity equation together with the initial condition $\rho^{\eta,\delta}(0) = \overline{\rho_0}$. Therefore it is an admissible competitor in $\mathcal{A}$ for any choice of (η, δ). We may then consider the function

$$M : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}, \quad M(\eta, \delta) := \mathcal{A}(\rho^{\eta,\delta}, \mathbf{v}^{\eta,\delta}).$$

The key point here is to show that M is $C^{1,1}$.

Lemma 3.2. *Suppose $V \in C^{1,1}$. Then, the function $(\eta, \delta) \mapsto M(\eta, \delta)$ defined above is also $C^{1,1}$.*

Proof. We have

$$\mathcal{A}(\rho^{\eta,\delta}, \mathbf{v}^{\eta,\delta}) = \int_0^T \int_{\mathbb{T}^d} \frac{1}{2} \rho^{\eta,\delta} |\mathbf{v}^{\eta,\delta}|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{T}^d} V \,\mathrm{d}\rho^{\eta,\delta} + \int_0^T \int_{\mathbb{T}^d} G(\rho^{\eta,\delta}) \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{T}^d} \Psi(x)\,\mathrm{d}\rho_T^{\eta,\delta}.$$

Since $\rho^{\eta,\delta}(T, x) = \rho(T, x)$, the last term does not depend on (η, δ). For the other terms, we use the change-of-variable

$$(s, y) = (t + \zeta(t)\eta, x + \zeta(t)\delta)$$

which is a C^∞ diffeomorphism for small η. Then we can write

$$\begin{aligned}\int_0^T \int_{\mathbb{T}^d} \Big(G(\rho^{\eta,\delta}(x, t)) + V(x)\rho^{\eta,\delta}(x, t)\Big) \,\mathrm{d}x\,\mathrm{d}t &= \int_0^T \int_{\mathbb{T}^d} (G(\rho(y, s)) + V(y - \zeta(t)\delta)\rho(y, s)) \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \int_{\mathbb{T}^d} (G(\rho(y, s))V(y - \zeta(t)\delta)\rho(y, s))\, K(\eta, \delta, s)\,\mathrm{d}y\,\mathrm{d}s,\end{aligned}$$

where $K(\eta, \delta, s)$ is a smooth Jacobian factor (which does not depend on y since the change of variable is only a translation in space). Hence, this term depends smoothly on (η, δ), with the same regularity as that of V.

We also have

$$\begin{aligned}\int_0^T \int_{\mathbb{T}^d} \rho^{\eta,\delta} |\mathbf{v}^{\eta,\delta}|^2 \,\mathrm{d}x\,\mathrm{d}t &= \int_0^T \int_{\mathbb{T}^d} \rho(s, y)|(1 + \eta\zeta'(t))\mathbf{v}(s, y) - \delta\zeta'(t)|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \int_{\mathbb{T}^d} \rho(s, y)|(1 + \eta\zeta'(t(\eta, s)))\mathbf{v}(s, y) - \delta\zeta'(t(\eta, s))|^2 K(\eta, \delta, s)\,\mathrm{d}y\,\mathrm{d}s,\end{aligned}$$

where $K(\eta, \delta, s)$ is the same Jacobian factor as before, and $t(\eta, s)$ is obtained by inversing, for fixed $\eta > 0$, the relation $s = t + \eta\zeta'(t)$, and is also a smooth map. Hence, this term is also smooth. □

We can now apply the previous lemma to the estimate we need.

Proposition 3.3. *There exists a constant C, independent of (η, δ), such that for $|\eta|, |\delta| \leq 1$, we have*

$$|M(\eta, \delta) - M(0, 0)| = |\mathcal{A}(\rho^{\eta,\delta}, \mathbf{v}^{\eta,\delta}) - \mathcal{A}(\rho, \mathbf{v})| \leq C(|\eta|^2 + |\delta|^2).$$

Proof. We just need to use Lemma 3.2 and the optimality of $(\rho, \mathbf{v})$. This means that M achieves its minimum at $(\eta, \delta) = (0, 0)$, therefore its first derivative must vanish at $(0, 0)$ and we may conclude by a Taylor expansion, using boundedness of the second derivatives (as a consequence of the $C^{1,1}$ regularity). □

With this result in mind, we may easily prove the following

Theorem 3.4. *If* $(\rho, \mathbf{v})$ *is a solution to the primal problem* $\min \mathcal{A}$*, if* $\Omega = \mathbb{T}^d$ *and if* J *satisfies* (12)*, then* $J(\rho)$ *satisfies, for every* $t_0 < t_1$*,*

$$||J(\rho(\cdot + \eta, \cdot + \delta)) - J(\rho)||^2_{L^2([t_0,t_1]\times\mathbb{T}^d)} \le C(|\eta|^2 + |\delta|^2)$$

(where the constant C *depends on* t_0, t_1 *and on the data), and hence is of class* $H^1_{loc}(]0, T[\times\mathbb{T}^d))$*.*

Proof. Let us take a minimizing sequence (ϕ_n, p_n) for the dual problem, i.e. $\phi_n \in C^1$, $p_n + V = -\partial_t\phi_n + \frac{1}{2}|\nabla\phi_n|^2$ and

$$\mathcal{B}(\phi_n, p_n) \le \inf_{(\phi,p)\in\mathcal{F}} \mathcal{B}(\phi, p) + \frac{1}{n}.$$

We use $\tilde{\rho} = \rho^{\eta,\delta}$ and $\tilde{\mathbf{v}} = \mathbf{v}^{\eta,\delta}$ as in the previous discussion. Using first the triangle inequality and then Lemma 3.1 we have (where the L^2 norme denotes the norm in $L^2((0, T) \times \mathbb{T}^d)$)

$$\begin{aligned} c_0||J(\rho^{\eta,\delta}) - J(\rho)||^2_{L^2} &\le 2c_0(||J(\rho^{\eta,\delta}) - J_*(p_n)||^2_{L^2} + ||J(\rho) - J_*(p_n)||^2_{L^2}) \\ &\le 2(\mathcal{B}(\phi_n, p_n) + \mathcal{A}(\rho^{\eta,\delta}, \mathbf{v}^{\eta,\delta}) + \mathcal{B}(\phi_n, p_n) + \mathcal{A}(\rho, \mathbf{v})), \end{aligned}$$

hence

$$||J(\rho^{\eta,\delta}) - J(\rho)||^2_{L^2} \le C(\mathcal{B}(\phi_n, p_n) + \mathcal{A}(\rho, \mathbf{v})) + C(|\eta|^2 + |\delta|^2) \le \frac{C}{n} + C(|\eta|^2 + |\delta|^2).$$

Letting n go to infinity and restricting the L^2 norm to $[t_0, t_1] \times \mathbb{T}^d$ gives the claim. □

Remark 5. If one restricts to the case $\eta = 0$, then it is also possible to use a cut-off function $\zeta \in C^\infty_c(]0, T])$ with $\zeta(T) = 1$, as we only perform space translations. In this case, however, the final cost $\int_{\mathbb{T}^d} \Psi(x)\, d\rho^{\eta,\delta}_T$ depends on δ, and one needs to assume $\Psi \in C^{1,1}$ to prove $M \in C^{1,1}$. This allows to deduce H^1 regularity in space, local in time far from $t = 0$, i.e. $J(\rho) \in L^2_{loc}(]0, T]; H^1(\mathbb{T}^d))$.

A finer analysis of the behavior at $t = T$ also allows to extend the above H^1 regularity result in space time till $t = T$, but needs extra tools (in particular defining a suitable extension of ρ for $t > T$). This is developed in [37]. Moreover, it is also possible to obtain regularity results till $t = 0$, under additional assumptions on $\overline{\rho_0}$ and at the price of some extra technical work, as it is done in [19].

Remark 6. From $J(\rho) = J_*(p)$, the above regularity result on ρ can be translated into a corresponding regularity result on p whenever an optimal pair (ϕ, p) exists (even if the dual problem is stated in $\tilde{\mathcal{D}}$: we could indeed prove that there exists a maximizing sequence composed of smooth functions, satisfying suitable H^1 bounds, which would imply the same regularity for the maximizer of the relaxed dual problem).

Remark 7. When $G(\rho) = \rho^q$, $q > 1$, the above H^1 result can be applied to $\rho^{q/2}$ and combined with the Sobolev injection $H^1 \subset L^{2^*}$. This shows that we have $\rho \in L^{\tilde{q}}_{loc}((0, T) \times \Omega)$ for an exponent $\tilde{q} > q$, given by $q(d+1)/(d-1)$ in dimension $d > 1$ (and any exponent $\tilde{q} < \infty$ if $d = 1$). This is a better integrability than just L^q, which came from the finiteness of the functional. The exponent has been computed using the Sobolev injection in dimension $d + 1$, the dimension of $(0, T) \times \Omega$. If we distinguish the behavior in time and space, just using $J(\rho) \in L^2_t H^1_x$, we get $\rho \in L^2_t L^{qd/(d-2)}_x$ for $d > 2$, $\rho \in L^2_t L^{\tilde{q}}_x$ for any $\tilde{q} < \infty$ in dimension $d = 2$, and $L^2_t L^\infty_x$ in dimension $d = 1$.

Finally, we finish this section by underlining the regularity results in the density-constrained case ([13]): the same kind of strategy, but with many more technical issues, which follow the same scheme as in [7] and [1], and the result is much weaker. Indeed, it is only possible to prove in this case $p \in L^2_{loc}((0, T); BV(\mathbb{T}^d))$ (exactly as in [1]). Even if very weak, this result is very important in what it gives higher integrability on p, which was a priory only supposed to be a measure and this allows to get the necessary summability of the maximal function that we briefly mentioned in Section 2.3.

4 Regularity via OT, time discretization, and flow interchange

In this section we will interpret the Eulerian variational forumulation as the search for an optimal curve in the Wasserstein space, i.e. the space of probability measures endowed with a particular distance coming from optimal transport. This will lead to a very efficient time discretization on which we are able to perform suitable computations providing strong bounds.

4.1 Tools from Optimal Transport and Wasserstein spaces

The space $\mathcal{P}(\Omega)$ of probability measures on Ω can be endowed with the Wasserstein distance: if μ and ν are two elements of $\mathcal{P}(\Omega)$, the 2-Wasserstein distance $W_2(\mu,\nu)$ between μ and ν is defined via

$$W_2(\mu,\nu) := \sqrt{\min\left\{\int_{\Omega\times\Omega} |x-y|^2 \,\mathrm{d}\gamma(x,y) \ : \ \gamma \in \mathcal{P}(\Omega\times\Omega) \text{ and } (\pi_x)_\#\gamma = \mu,\ (\pi_y)_\#\gamma = \nu\right\}}. \tag{15}$$

In the formula above, π_x and $\pi_y : \Omega\times\Omega \to \Omega$ stand for the projections on respectively the first and second component of $\Omega\times\Omega$. If $T : X \to Y$ is a measurable application and μ is a measure on X, then the image measure of μ by T, denoted by $T_\#\mu$, is the measure defined on Y by $(T_\#\mu)(B) = \mu(T^{-1}(B))$ for any measurable set $B \subset Y$. For general results about optimal transport, the reader might refer to [44, 3], or [40].

The Wasserstein distance admits a dual formulation, the dual variables being the so-called Kantorovich potentials. The main properties of these potentials, in the case which is of interest to us, are summarized in the proposition below. We restrict to the cases where the measures have a strictly positive density a.e., as in this particular case the potentials are unique (up to a global additive constant). The proof of these results can be found, for instance, in [40, Chapters 1 and 7].

Proposition 4.1. *Let $\mu,\nu \in \mathcal{P}(\Omega)$ be two absolutely continuous probability measures with strictly positive density. Then there exists a unique (up to adding a constant to φ and subtracting it from ψ) pair (φ,ψ) of Kantorovich potentials satisfying the following properties.*

1. *The squared Wasserstein distance $W_2^2(\mu,\nu)$ can be expressed as*

$$\frac{1}{2}W_2^2(\mu,\nu) = \int_\Omega \varphi\mu + \int_\Omega \psi\nu.$$

2. *The "vertical" derivative of $W_2^2(\cdot,\nu)$ at μ is φ: if $\tilde{\mu} \in \mathcal{P}(\Omega)$ is any probability measure, then*

$$\lim_{\varepsilon\to 0} \frac{\frac{1}{2}W_2^2((1-\varepsilon)\mu+\varepsilon\tilde{\mu},\nu) - \frac{1}{2}W_2^2(\mu,\nu)}{\varepsilon} = \int_\Omega \varphi(\tilde{\mu}-\mu).$$

3. *The potentials φ and ψ are one the c-transform of the other, meaning that we have*

$$\begin{cases} \varphi(x) & = \inf_{y\in\Omega} \frac{|x-y|^2}{2} - \psi(y) \\ \psi(y) & = \inf_{x\in\Omega} \frac{|x-y|^2}{2} - \varphi(x). \end{cases}$$

4. *There holds $(\mathrm{Id}-\nabla\varphi)_\#\mu = \nu$ and the transport plan $\gamma := (\mathrm{Id},\mathrm{Id}-\nabla\varphi)_\#\mu$ is optimal in the problem* (15). *We also say that the map $x \mapsto x - \nabla\varphi(x)$ is the optimal transport map from μ to ν.*

The function φ (resp. ψ) is called the Kantorovich potential from μ to ν (resp. from ν to μ).

We will denote by Γ the space of absolutely continuous curves from $[0,1]$ to $\mathcal{P}(\Omega)$ endowed with the Wasserstein distance W_2.

Definition 4.2. *We say that a curve ρ is absolutely continuous if there exists a function $a \in L^1([0,1])$ such that, for every $0 \leqslant t \leqslant s \leqslant 1$,*

$$W_2(\rho_t, \rho_s) \leqslant \int_t^s a(r)\, \mathrm{d}r.$$

We say that ρ is $2-$absolutely continuous if the function a above can be taken in $L^2([0,1])$

This space will be equipped with the distance d_Γ of the uniform convergence, i.e.

$$d_\Gamma(\rho^1, \rho^2) := \max_{t\in[0,1]} W_2(\rho^1_t, \rho^2_t).$$

The main interest of the notion of absolute continuity for curves in the Wasserstein space lies in the following theorem, which we recall without proof (but we refer to [3] or to Chapter 5 in [40]).

Theorem 4.3. *For $\rho \in \Gamma$ the quantity*

$$|\dot{\rho}_t| := \lim_{h\to 0} \frac{W_2(\rho_{t+h}, \rho_t)}{h}$$

exists and is finite for a.e. t. Moreover, we have the following

- *if $\rho \in \Gamma$ is a 2-absolutely continuous curve, there exists for a.e. t a vector field $\mathbf{v}_t \in L^2(\rho_t)$ such that $\|\mathbf{v}_t\|_{L^2(\rho_t)} \leq |\dot{\rho}_t|$ and such that the continuity equation $\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0$ holds in distributional sense;*
- *if $\rho \in \Gamma$ is such that there exists a family of vector fields $\mathbf{v}_t \in L^2(\rho_t)$ satisfying $\int_0^T \int_\Omega |\mathbf{v}_t|^2\, \mathrm{d}\rho_t\, \mathrm{d}t < +\infty$ and $\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0$, then ρ is a 2-absolutely continuous curve and $\|\mathbf{v}_t\|_{L^2(\rho_t)} \geq |\dot{\rho}_t|$ for a.e. t.*

Finally, we can represent $\int_0^1 |\dot{\rho}_t|^2\, \mathrm{d}t$ in various ways such as

$$\int_0^1 |\dot{\rho}_t|^2\, \mathrm{d}t = \sup_{N\geqslant 2} \sup_{0\leqslant t_1<t_2<\ldots<t_N\leqslant 1} \sum_{k=2}^N \frac{W_2^2(\rho_{t_{k-1}}, \rho_{t_k})}{t_k - t_{k-1}} \tag{16}$$

$$= \min\left\{ \int_0^1 \int_\Omega |\mathbf{v}_t|^2\, \mathrm{d}\rho_t\, \mathrm{d}t \;:\; \partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0 \right\}. \tag{17}$$

Observe that the kinetic energy in (16) is exactly the same quantity appearing in Section 2.3.

4.2 Discretization in time of variational MFG and optimality conditions

We first start from the observation that the above tools from optimal transport theory allow to re-write the variational problem defining MFG equilibria into the following form

$$\min\left\{ \int_0^T \frac{1}{2}|\dot{\rho}_t|^2\, \mathrm{d}t + \int_0^T \mathcal{G}(\rho_t)\, \mathrm{d}t + \int_\Omega \Psi\, \mathrm{d}\rho_T \;:\; \rho : [0,T] \to \mathcal{P}(\Omega),\ \rho_0 = \overline{\rho_0} \right\}.$$

A useful approximation can be obtained via time-discretization: we fix a time step $\tau = T/N$ and we look for a sequence $\rho_0 = \overline{\rho_0}, \rho_1, \dots, \rho_N$ solving

$$\min \left\{ \sum_{k=0}^{N-1} \left(\frac{W_2^2(\rho_k, \rho_{k+1})}{2\tau} + \tau \mathcal{G}(\rho_k) \right) + \int_\Omega \Psi \, \mathrm{d}\rho_N \right\}.$$

If $\overline{\rho_0}, \rho_1, \dots, \rho_N$ solves the above minimization problem then, for each $0 < k < N$, the measure ρ_k solves

$$\min \left\{ \frac{W_2^2(\rho, \rho_{k-1})}{2\tau} + \frac{W_2^2(\rho, \rho_{k+1})}{2\tau} + \tau \mathcal{G}(\rho) \quad : \quad \rho \in \mathcal{P}(\Omega) \right\},$$

i.e. it solves a minimization problem similar to what we see in the JKO scheme for gradient flows (see [22, 3, 42]), which would be of the form

$$\min \left\{ \frac{W_2^2(\rho, \rho_{k-1})}{2\tau} + \mathcal{G}(\rho) \right\}.$$

By the way, for $k = N$, we have a true JKO-style problem with one only Wasserstein distance.

From this similarity with the JKO scheme, we are lead to apply techniques which have been previously applied to this other setting, and in particular the notion of *flow interchange*, developed in [31].

Consider the functional $\mathcal{F}_m(\rho) := \int F_m(\rho(x))\, \mathrm{d}x$, where $F_m(s) := s^m$. The important point about this functional, if we suppose Ω to be convex, is that it is a geodesically convex functional on the W_2 Wasserstein space (see [32]). This means that it is convex along constant-speed geodesic interpolations in $\mathbb{W}_2(\Omega)$. Consider now $(\rho_s)_s$ be the gradient flow of $\mathcal{F}_m(\rho)$, i.e. a solution of $\partial_s \rho - \nabla \cdot (\rho \nabla (F_m'(\rho))) = 0$, with initial datum at $s = 0$ equal to the optimal ρ at step k. From the EVI definition of gradient flows ([3]) and the geodesic convexity of F_m we obtain the following inequality, valid for every ν

$$\frac{d}{ds} \frac{W_2^2(\rho_s, \nu)}{2} \leq \mathcal{F}_m(\nu) - \mathcal{F}_m(\rho_s).$$

We can also compute

$$\frac{d}{ds} \mathcal{G}(\rho_s) = - \int \nabla (g(\rho_s) + V) \cdot \nabla (F_m'(\rho_s)) \; \mathrm{d}\rho_s.$$

On the other hand, the optimality of ρ_k implies that the derivative of the sum of the Wasserstein terms and of the $\mathcal{G}$ term should be non-negative, which provides

$$\int \nabla (g(\rho_k) + V) \cdot \nabla (F_m'(\rho_k)) \; \mathrm{d}\rho_k \leq \frac{\mathcal{F}_m(\rho_{k+1}) - 2\mathcal{F}_m(\rho_k) + \mathcal{F}_m(\rho_{k-1})}{\tau^2}.$$

Let us start from the easier case $V = 0$: in this case we get

$$0 \leq \int g'(\rho_k) F_m''(\rho_k) \rho_k |\nabla \rho_k|^2 \quad \leq \quad \frac{\mathcal{F}_m(\rho_{k+1}) - 2\mathcal{F}_m(\rho_k) + \mathcal{F}_m(\rho_{k-1})}{\tau^2}.$$

This shows that $k \mapsto \mathcal{F}_m(\rho_k)$ is (discretely) convex. If $\overline{\rho_0} \in L^m$, and if for some reason we suppose $\rho_T \in L^m$, then, after passing to the limit $\tau \to 0$, we deduce a uniform bound on $\|\rho_t\|_{L^m}$. This also works for $m = \infty$. This was essentially a result proven by P.-L. Lions in his course ([28], lecture of of November 27, 2009), in a more general setting (still with no x-dependence, but with more general Hamiltonians than the quadratic one).

Note that the case where ρ_T is prescribed is known under the name of *planning problem* (see, for instance, [35, 34, 20]) but is out of the scopes of these notes. When, instead, we have a final penalization, the same flow interchange technique provides

$$\int \nabla\Psi \cdot \nabla(F'_m(\rho_N))\ \mathrm{d}\rho_N \le \frac{\mathcal{F}_m(\rho_{N-1}) - \mathcal{F}_m(\rho_N)}{\tau}.$$

After an integration by parts, using $\nabla(F'_m(\rho_N))\rho_N = m(m-1)\rho_N^{m-1}\nabla\rho_N = (m-1)\nabla(F_m(\rho_N))$, and assuming $\Psi \in C^{1,1}$ and $\partial\Psi/\partial n \ge 0$ on $\partial\Omega$, we obtain

$$\mathcal{F}_m(\rho_N) \le \mathcal{F}_m(\rho_{N-1}) + \tau(m-1)\int F_m(\rho_N)\Delta\Psi,$$

i.e.

$$(1 - C\tau)\mathcal{F}_m(\rho_N) \le \mathcal{F}_m(\rho_{N-1}), \qquad \text{for } C = (m-1)\|(\Delta\Psi)_+\|_{L^\infty}. \tag{18}$$

This shows that not only $k \mapsto \mathcal{F}_m(\rho_k)$ is convex, but that we control its final derivative. From a continuous point of view, it is as if we had a function $u \ge 0$, with $u'' \ge 0$ and $u'(T) \le Cu(T)$. This is not enough to provide a bound on $u(T)$ as, for instance, all functions of the form $u(t) = \lambda(1 - C(T-t))_+$ satisfy these assumptions (note by the way that, in case $CT > 1$, we also have $u(0) = 0$, which shows that adding an assumption on the initial data would not be enough). Yet, we can obtain $u(T) \le 2C\int_0^T u$. This can be, for instance, applied to the case where the two functionals G and F_m have the same order of growth: $G \approx F_m$. From the finiteness of the integral of $\mathcal{F}_m$ we would deduce in this case a uniform bound for $\mathcal{F}_m(\rho_T)$ and, if $\mathcal{F}_m(\overline{\rho_0}) < \infty$, a uniform bound in time.

However, we are able, following the non-trivial computations in [26], to obtain much more.

To give an idea of the method, let us stick to the case $V = 0$ and let us impose a very stringent assumption on the congestion function g. We will suppose $g'(s) \ge cs^{-1}$ an assumption which is satisfied in the entropy case $G(s) = s\log s$. We will see that the important assumption is indeed the inequality $g'(s) \ge cs^\alpha$ for $\alpha \ge -1$. The idea is to exploit the positive term $\int g'(\rho_k)F''_m(\rho_k)\rho_k|\nabla\rho_k|^2$. In this case we have

$$\int g'(\rho_k)F''_m(\rho_k)\rho_k|\nabla\rho_k|^2 \ge c\int \rho_k^{m-2}|\nabla\rho_k|^2 = c\|\nabla(\rho_k^{m/2})\|^2_{L^2}.$$

We then apply the Sobolev injection of H^1 into L^β, for an exponent $2\beta > 2$. This allows, for instance, to write

$$\|(\rho_k^{m/2})\|^2_{L^{2\beta}} \le C\|\nabla(\rho_k^{m/2})\|^2_{L^2} + C\int \rho_k^m$$

for a suitable constant C. As the last term in the right hand side is just $\mathcal{F}_m(\rho_k)$, we obtain a bound on $\mathcal{F}_{m\beta}(\rho_k)$ in terms of $\mathcal{F}_m(\rho_k)$ and of its second variation in k. The idea is then to apply Moser's iteration on exponents $m_j \approx \beta^j$. This is delicate, since in order to take care of the second derivative in time (even if it is discrete) we need to integrate in time, and the integral (sum over k in the discrete setting) in time of the $L^{2\beta}$ norms raised to the power 2 is not the $L^{2\beta}$ norm in time-space. This can be dealt with using the fact that all the functionals $\mathcal{F}_m$ are convex in time, which allows to obtain reversed Jensen inequalities: if a function $u \ge 0$ is convex, indeed, we have

$$\left(\int_{T_1}^{T_2} u(t)\,\mathrm{d}t\right)^{1/\beta} \le \frac{(T_2 - T_1)^{1\beta}}{\varepsilon}\int_{T_1-\varepsilon}^{T_2+\varepsilon} u^{1/\beta}(t)\,\mathrm{d}t.$$

This allows hence to obtain an estimate of the form

$$\left(\int \int_{T_1}^{T_2} F_{m\beta}(\rho(t))\, \mathrm{d}t\right)^{1/\beta} \leq C(m,\varepsilon) \int_{T_1-\varepsilon}^{T_2+\varepsilon} F_m(\rho(t))\, \mathrm{d}t$$

and, choosing suitable values of $\varepsilon = \varepsilon_m$ and exploiting the polynomial behaviour of $C(m, \varepsilon)$ in m and ε^{-1}, it is possible to iterate this estimate in the spirit of the work of Moser [33] for elliptic regularity, thus obtaining an estimate on $\|\rho\|_{L^\infty([T_1,T_2]\times\Omega}$ in terms of $\int_{[T_1-\varepsilon,T_2+\varepsilon]\times\Omega} G(\rho)\, \mathrm{d}x\, \mathrm{d}t$.

Even if the computations are less straightforward it is not difficult to see that the assumption $g'(s) \geq cs^{-1}$ can be replaced by a more general one where we use $g'(s) \geq cs^\alpha$ for an exponent $\alpha \geq -1$, and that it is enough, in order to obtain L^∞ bounds, that this inequality is satisfied for $s \geq s_0$ (see [26]).

The situation is trickier when there is an exterior potential V. In this case we have

$$\int g'(\rho_k) F_m''(\rho_k)\rho_k|\nabla\rho_k|^2 \leq \frac{\mathcal{F}_m(\rho_{k+1}) - 2\mathcal{F}_m(\rho_k) + \mathcal{F}_m(\rho_{k-1})}{\tau^2} - \int (\nabla V \cdot \nabla\rho_k) F_m''(\rho_k)\rho_k.$$

The new term needs to be estimated in terms of V and $\mathcal{F}_m$, which can be done in two possible ways. Either we integrate by parts, as we did for the final cost Ψ, and suppose $V \in C^{1,1}$ and $\partial V/\partial n \geq 0$, in which case we use $\nabla\rho_k F_m''(\rho_k)\rho_k = (m-1)\nabla(F_m(\rho)$ and we get

$$-\int (\nabla V \cdot \nabla\rho_k) F_m''(\rho_k)\rho_k \leq (m-1) \int (\Delta V) F_m(\rho_k),$$

or we use a Young inequality:

$$-\int (\nabla V \cdot \nabla\rho_k) F_m''(\rho_k)\rho_k \leq \frac{1}{2}\int |\nabla V|^2 F_m''(\rho_k)\rho_k^2 + \frac{1}{2}\int |\nabla\rho_k|^2 F_m''(\rho_k).$$

The first term in the right-hand side can be bounded by $Cm^2\mathcal{F}_m(\rho_k)$ as soon as V is Lipschitz continuous, and the second can be bounded in terms of $\int g'(\rho_k)F_m''(\rho_k)\rho_k|\nabla\rho_k|^2$ as soon as $g'(\rho) \geq \rho^\alpha$ with $\alpha \geq -1$. We will see in the statement of Theorem 4.4 that this computation (only assuming V to be Lipschitz) can only be exploited for L^∞ regularity under some very restrictive assumptions.

However, a difficulty arising in this case is that $k \mapsto \mathcal{F}_m(\rho_k)$ is no more convex. From a continuous point of view, we do not have anymore a time-dependent function u with $u'' \geq 0$, but rather a solution of $u'' + \omega^2 u \geq 0$, for a constant ω depending on m. Differently from convexity, in general this inequality cannot provide bounds, if we think that functions of the form $u(t) = \lambda \sin \omega t$ solve the equality case for any λ, on intervals of the form $[0, T]$, $T = k\pi/\omega$. Hence, this inequality can only provide bounds on short intervals of time, smaller than π/ω. In particular, when doing Moser's iterations, we need to divide every interval into smaller ones; since the reverse Jensen inequality requires to enlarge these intervals, there will be many new integrals on overlapping intervals. As a result, this will bring to a larger multiplicative constant depending on m (since ω also depends on m, and the parameter ε_m in the enlargement of the intervals also depends on m) in the estimates. This is not a problem as soon as the dependence is polynomial.

A final remark about the case where $g's) \geq s^\alpha$ but $\alpha < -1$. This case is called in [26] "weak congestion". In this case, we only have a control of $\mathcal{F}_m$ in terms of $\mathcal{F}_{\beta(m+1+\alpha)}$. Thus we must start the iterative procedure with a value m such that $m < \beta(m + 1 + \alpha)$, i.e. we must impose *a priori* some L^m regularity on ρ (with an exponent m which depends on α and β, the latter depending itself only on the dimension of the ambient space). Such a regularity can be obtained, for instance, by assuming that $\overline{\rho_0}$ (which is fixed) is in $L^m(\Omega)$ and that T is small enough. Indeed, if this is the case, the boundary condition (18) combined with the interior estimate $u'' + \omega^2 u \geq 0$ show that if T is small enough (given the potentials and the congestion function f), the L^m norm of ρ on $[0, T] \times \Omega$ must be bounded.

We do not develop all the details,which are very technical, here but we summarize here below the L^∞ results which can be found in [26]. The results are based on the above estimates obtained in the time-discrete setting, together with a suitable use of the limit $\tau \to 0$.

Theorem 4.4. *Consider a running cost of the form $h[\rho] = V(x) + g(\rho)$. Suppose that the inequality $g'(s) \geq s^\alpha$ is satisfied for every $s \geq s_0$. Then, we have:*

- *If V is Lipschitz, $\alpha \geq -1$, and $s_0 = 0$ then $\rho \in L^\infty_{loc}((0, T) \times \overline{\Omega})$.*
- *The same result holds if $s_0 > 0$ but $V \in C^{1,1}$ and $\partial V/\partial n \geq 0$.*
- *These results extend to $(0, T]$ if $\Psi \in C^{1,1}$ and $\partial\Psi/\partial n \geq 0$.*
- *If $\alpha < -1$, then the same results, for $V, \Psi \in C^{1,1}$, $\partial V/\partial n \geq 0$ and $\partial\Psi/\partial n \geq 0$, are true if we already know $\rho \in L^{m_0}((0, T) \times \overline{\Omega})$ for $m_0 > d|\alpha + 1|/2$. This is true in particular if $\overline{\rho_0} \in L^{m_0}$ and T is small enough.*

It is now straightforward to apply the L^∞ bounds on ρ to obtain boundedness from above of $h[\rho]$, and then apply the content of Section 2.3 in order to transform the optimality into a the equilibrium condition characterizing optimal tranjectories in MFG.

5 Density-constrained Mean Field Games

In this section we are concerned with the model presentd in [13] (but, compared to such a paper, we will restrict to the case where the cost is quadratic in the velocity): the variational problem to be considered is

$$\min\left\{\int_0^T \int_\Omega \left(\frac{1}{2}|\mathbf{v}_t|^2 + V\right) \mathrm{d}\rho_t \,\mathrm{d}t + \int_\Omega \Psi \,\mathrm{d}\rho_T \;:\; \rho \leq 1\right\}.$$

This can be translated into

$$\min\left\{\int_0^T \frac{1}{2}|\dot\rho_t|^2 \,\mathrm{d}t + \int_0^T \mathcal{G}(\rho_t) \,\mathrm{d}t + \int_\Omega \Psi \,\mathrm{d}\rho_T \;:\; \rho : [0, T] \to \mathcal{P}(\Omega),\ \rho_0 = \overline{\rho_0}\right\},$$

where $\mathcal{G}$ is a very degenerate functional:

$$\mathcal{G}(\rho) := \begin{cases} \int V \,\mathrm{d}\rho & \text{if } \rho \leq 1, \\ +\infty & \text{if not.} \end{cases}$$

We already discussed that this provides the following MFG system

$$\begin{cases} -\partial_t\varphi + \frac{|\nabla\varphi|^2}{2} = V + p, \\ \partial_t\rho - \nabla \cdot (\rho\nabla\varphi) = 0, \\ \varphi(T, x) = \Psi(x) + P(x), \quad \rho(0, x) = \overline{\rho_0}(x), \\ p \geq 0, \rho \leq 1,\ p(1 - \rho) = 0, \\ P \geq 0, P(1 - \rho_T) = 0. \end{cases} \tag{19}$$

and that the running cost of every agent is in the end $V + p$ (note that this is coherent with the general formula $V + G'(\rho)$, where the derivative $G' = g$ should be replaced here by a generic element of the subdifferntial ∂G). Note that in this case we also have an effect on the final cost, where Ψ is replaced by $\Psi + P$. This can be interpreted in two ways. In general, we did not put any density penalization at final time (i.e. the final cost is not of the form $\Psi + g(\rho_T)$ but only of the form Ψ), but here the constraint $\rho_T \le 1$ is also present on the final density, and lets its subdifferential appear. On the other hand, we can consider that the constraint $\rho_t \le 1$ for all $t < T$ is enough to impose the same (by continuity in the Wasserstein space of the curve $t \mapsto \rho_t$) for $t = T$, so that in the final cost functional we can omit the constraint part. If we interpret this in this way, how can we justify the presence of a final cost P? the answer comes from the fact that the natural regularity for the pressure p, which is supposed to be positive, is being a positive measure (since distributions with a sign are measures, and also because in the dual problem p is penalized in a L^1 sense). Hence, the extra cost P represents the singular part of p concentrated on $t = T$. What we will prove in this section is that we have $p \in L^\infty([0, T] \times \Omega)$ and $P \in L^\infty(\Omega)$, thus decomposing the pressure into a bounded density in time-space and a bounded density at the final time.

This problem can also be discretized in the same way as in Section 4, and this discretization technique will be the one which will rigorously provide the estimates we look for. Yet, before looking at the details, we prefer first to give an heuristic derivation of the main idea in continuous time. The key point will consist in proving $\Delta(V + p) \ge 0$ on $\{p > 0\}$. To do this, we consider System (19), and denote by $D_t := \partial_t - \nabla\varphi \cdot \nabla$ the convective derivative. The idea is to look at the quantity $-D_{tt}(\log \rho)$. Indeed, the continuity equation in (19) can be rewritten $D_t(\log \rho) = \Delta\varphi$. On the other hand, taking the Laplacian of the Hamilton-Jacobi equation, it is easy to get, dropping a positive term, $-D_t(\Delta\varphi) \leqslant \Delta(p + V)$. Hence,

$$-D_{tt}(\log \rho) \leqslant \Delta(p + V). \tag{20}$$

Then, we observe that $\log \rho$ is maximal where $\rho = 1$, hence we have $-D_{tt}(\log \rho) \geqslant 0$. This implies $\Delta p \geqslant -\Delta V$ on $\{p > 0\}$, Let us say that the strategy of looking at the convective derivative of $\log \rho$ was already used by Loeper [29] to study a similar problem (related to the reconstruction of the early universe). Moreover, also in [29] the rigorous proof was done by time-discretization.

As the tools which are required to study the L^∞ regularity are much less technical than for the density-penalized case, we will develop here more details. In particular, we will write here the optimality conditions for the discrete problems and see that quantities acting like a pressure appear. For the convergence of these quantities to the true pressures p and P, we refer to [27], whose results are also recalled in Section 5.3.

Some regularity will be needed in order to be able to correctly perform our analysis. In particular, we will assume that $\bar\rho_0$ is smooth and strictly positive and that V and Ψ are C^2 function. We will also add a small entropy penalization to the term $\mathcal{G}$, thus considering

$$\mathcal{G}_\lambda(\rho) =:= \begin{cases} \int V \,\mathrm{d}\rho + \lambda \int \rho \log \rho & \text{if } \rho \le 1, \\ +\infty & \text{if not} \end{cases}$$

and we will also add the same entropy penalization to the final cost, thus solving

$$\min \left\{ \sum_{k=0}^{N-1} \left(\frac{W_2^2(\rho_k, \rho_{k+1})}{2\tau} + \tau \mathcal{G}_\lambda(\rho_k) \right) + \int_\Omega \Psi \,\mathrm{d}\rho_N + \lambda \int_\Omega \rho_N \log \rho_n \,\mathrm{d}x \right\}.$$

Yet, all the estimates that we establish will not depend on the smoothness of $\bar\rho_0, V$ and Ψ or on the value of λ.

5.1 Optimality conditions and regularity of p

In this subsection, we fix $N \geq 1$ and $k \in \{1, 2, \ldots, N-1\}$ a given instant of time. We will fix an optimal sequence $(\rho_0 = \overline{\rho_0}, \rho_1, \ldots \rho_N$ and set $\bar\rho := \rho_k$; we also denote $\mu := \rho_{k-1}$ and $\nu := \rho_{k+1}$. From the same consideration of the previous section, we know that $\bar\rho$ is a minimizer, among all probability measures with density bounded by 1, of

$$\rho \mapsto \frac{W_2^2(\mu,\rho) + W_2^2(\rho,\nu)}{2\tau} + \tau\mathcal{G}_\lambda(\rho)$$

Lemma 5.1. *The density $\bar\rho$ is strictly positive a.e.*

Proof. The proof is based on the fact that the derivative of the function $s \mapsto s\log s$ at $s = 0$ is $-\infty$, so that minimizers avoid the value $\rho = 0$. It can be obtained following the procedure in [40, Lemma 8.6], or of [26, Lemma 3.1], as the construction done in these proofs preserves the constraint of having a density smaller than 1. □

Proposition 5.2. *Let us denote by φ_μ and φ_ν the Kantorovich potentials for the transport from $\bar\rho$ to μ and ν respectively (this potentials are unique up to additive constants because $\bar\rho > 0$). There exists $p \in L^1(\Omega)$, positive, such that $\{p > 0\} \subset \{\bar\rho = 1\}$ and a constant C such that*

$$\frac{\varphi_\mu + \varphi_\nu}{\tau^2} + V + p + \lambda\log(\bar\rho) = C \;\; a.e. \tag{21}$$

Moreover p and $\log(\bar\rho)$ *are Lipschitz continuous.*

Proof. Let $\tilde\rho \in \mathcal{P}(\Omega)$ such that $\tilde\rho \leq 1$. We define $\rho_\varepsilon := (1-\varepsilon)\bar\rho + \varepsilon\tilde\rho$ and use it as a competitor. Clearly $\rho_\varepsilon \leqslant 1$, i.e. it is an admissible competitor. We will obtain the desired optimality conditions comparing the cost of ρ_ε to that of ρ. Using Proposition 4.1, as $\bar\rho > 0$, the Kantorovich potentials φ_μ and φ_ν are unique (up to a constant) and

$$\lim_{\varepsilon\to 0} \frac{W_2^2(\mu,\rho_\varepsilon) - W_2^2(\mu,\bar\rho) + W_2^2(\rho_\varepsilon,\nu) - W_2^2(\bar\rho,\nu)}{2\tau^2} = \int_\Omega \frac{\varphi_\mu + \varphi_\nu}{\tau}(\tilde\rho - \bar\rho).$$

The term involving V is straightforward to handle as it is linear. The only remaining term is the one involving the entropy. For this term (following, for instance, the reasoning in [26, Proposition 3.2]), we can obtain the inequality

$$\limsup_{\varepsilon\to 0} \frac{\int \rho_\varepsilon \log\rho_\varepsilon - \int \bar\rho\log\bar\rho}{\varepsilon} \leqslant \int_\Omega \log(\bar\rho)(\tilde\rho - \bar\rho).$$

Putting the pieces together, we see that $\int_\Omega u\,(\tilde\rho - \bar\rho) \geqslant 0$ for any $\tilde\rho \in \mathcal{P}(\Omega)$ with $\tilde\rho \leqslant 1$, provided that u is defined by

$$u := \frac{\varphi_\mu + \varphi_\nu}{\tau^2} + V + \lambda\log(\bar\rho)$$

It is known, analogously to [30, Lemma 3.3], that this leads to the existence of a constant C such that

$$\begin{cases} \bar\rho = 1 & \text{on } \{u < C\} \\ \bar\rho \leqslant 1 & \text{on } \{u = C\} \\ \bar\rho = 0 & \text{on } \{u > C\} \end{cases} \tag{22}$$

Specifically, C is defined as the smallest real $\tilde{C}$ such that $\mathcal{L}^d(\{u \leqslant \tilde{C}\}) \geqslant 1$, and it is quite straightforward to check that this choice works. Note that the case $\{u > C\}$ can be excluded by Lemma 5.1. We then define the pressure p as $p = (C - u)_+$, thus (21) holds. It satisfies $p \geqslant 0$, and $\bar{\rho} < 1$ implies $p = 0$.

It remains to answer the question of the Lipschitz regularity of p and $\log(\bar{\rho})$. Notice that p is positive, and non zero only on $\{\bar{\rho} = 1\}$. On the other hand, $\log(\bar{\rho}) \leqslant 0$ and it is non zero only on $\{\bar{\rho} < 1\}$. Hence, one can write

$$p = \left(C - \frac{\varphi_\mu + \varphi_\nu}{\tau^2} + V\right)_+ \text{ and } \log(\bar{\rho}) = -\frac{1}{\lambda}\left(C - \frac{\varphi_\mu + \varphi_\nu}{\tau^2} + V\right)_- . \tag{23}$$

Given that the Kantorovich potentials and V are Lipschitz, it implies the Lipschitz regularity for p and $\log(\bar{\rho})$. □

Let us note that φ_μ and φ_ν have additional regularity properties, even though this regularity heavily depends on τ.

Lemma 5.3. *The Kantorovich potentials φ_μ and φ_ν belong to $C^{2,\alpha}(\mathring{\Omega}) \cap C^{1,\alpha}(\Omega)$ and $p \in C^{2,\alpha}(\{p > 0\})$.*

Proof. If $k \in \{2, \dots, N\}$, thanks to Proposition 5.2 (applied in $k - 1$ and $k + 1$), we know that μ and ν have a Lipschitz density and are bounded from below. Using the regularity theory for the Monge Ampère-equation [44, Theorem 4.14], we can conclude that φ_μ and φ_ν belong to $C^{2,\alpha}(\mathring{\Omega}) \cap C^{1,\alpha}(\Omega)$.

Once we have the regularity of $\varphi_\mu + \varphi_\nu$, as we were supposing $V \in C^2$, we get $C^{2,\alpha}$ regularity for $p + \lambda \log \bar{\rho}$, which in turns implies the same regularity for $p = (p + \lambda \log \bar{\rho})_+$ in the open set $\{p > 0\}$. □

Theorem 5.4. *We have the following L^∞ estimate:*

$$p \leq \max V - \min V.$$

Proof. First we will prove that, on the open set $\{p > 0\}$, we have $\Delta(p + V) \geq 0$.

In order to do this, we consider the (optimal) transport map from $\bar{\rho}$ to μ given by $\mathrm{Id} - \nabla\varphi_\mu$, and similarly for ν. Let us define the following quantity:

$$D(x) := -\frac{\log(\mu(x - \nabla\varphi_\mu(x))) + \log(\nu(x - \nabla\varphi_\nu(x))) - 2\log(\bar{\rho}(x))}{\tau^2}.$$

Notice that if $\bar{\rho}(x) = 1$, then by the constraint $\mu(x - \nabla\varphi_\mu(x)) \leqslant 1$ and $\nu(x - \nabla\varphi_\nu(x)) \leqslant 1$ the quantity $D(x)$ is positive. On the other hand, using $(\mathrm{Id} - \nabla\varphi_\mu)_\#\bar{\rho} = \mu$ and the Monge-Ampère equation, for all $x \in \mathring{\Omega}$ there holds

$$\mu(x - \nabla_\mu\varphi_\mu(x)) = \frac{\bar{\rho}(x)}{\det(I - D^2\varphi_\mu(x))},$$

and a similar identity holds for φ_ν. Hence the quantity $D(x)$ is equal, for all $x \in \mathring{\Omega}$, to

$$D(x) = \frac{\log(\det(I - D^2\varphi_\mu(x))) + \log(\det(I - D^2\varphi_\nu(x)))}{\tau^2}.$$

Diagonalizing the matrices $D^2\varphi_\mu, D^2\varphi_\nu$ and using the convexity inequality $\log(1 - y) \leqslant -y$, we end up with

$$D(x) \leqslant -\frac{\Delta(\varphi_\mu(x) + \varphi_\nu(x))}{\tau^2}.$$

This shows that, on the region $\{p > 0\}$, we have the desired inequality $\Delta(p + V) \geq 0$, thanks to (21).

We want now to determine where does $p + V$ attain its maximum. Because of subharmonicity this should be on the boundary of $\{p > 0\}$. This boundary is composed by points on $\partial\Omega$ and by points where $p = 0$.

To handle the boundary $\partial\Omega$, recall that $\nabla\varphi_\mu$ is continuous up to the boundary and that $x-\nabla\varphi_\mu(x) \in \Omega$ for every $x \in \Omega$ as $(\mathrm{Id} - \nabla\varphi_\mu)_\#\bar{\rho} = \mu$. Given the convexity of Ω, it implies $\nabla\varphi_\mu(x) \cdot \mathbf{n}_\Omega(x) \geqslant 0$ for every point $x \in \partial\Omega$, where $\mathbf{n}_\Omega(x)$ is the corresponding outward normal vector. This translates, applying this first to φ_μ and then to φ_ν, into $\nabla(p + V)(x) \cdot \mathbf{n}_\Omega(x) \leq 0$. We are then in this situation: a certain function u satisfies $\Delta u \geq 0$ in the interior of a domain (which is here $\{p > 0\}$) and $\partial u/\partial n \leq 0$ on a part of the boundary. By applying an easy maximum principle to $u_\varepsilon := u+\varepsilon v$ where v is a fixed harmonic function with $\partial v/\partial n < 0$ on the same part of the boundary shows that the maximum of u is attained on the other part of the boundary (we prefer not to evoke Hopf's lemma as we do not want to discuss the regularity of $\partial\Omega$, and we do not need the strong maximum principle). We then deduce that the maximum of $p+V$ is attained on $\{p = 0\}$.

This easily implies

$$\max_{\{p>0\}} p + \min_{\{p>0\}} V \leq \max_{\{p>0\}}(p + V) \leq \max_{\{p=0\}} V,$$

which gives the claim. □

Remark 8. The same proof actually shows the stronger inequality $p + V \leq \max V$.

5.2 Optimality conditions and regularity of P

We look now at the optimality conditions satisfied by ρ_N. The situation is even simple than the one in Section 5.1. Set $\bar{\rho} := \rho_N$ and $\mu := \rho_{N-1}$. We can see that $\bar{\rho}$ is a minimizer, among all probability measures with density bounded by 1, of

$$\rho \mapsto \frac{W_2^2(\mu,\rho)}{2\tau} + \int_\Omega \Psi \,\mathrm{d}\rho + \lambda \int_\Omega \rho \log(\rho)\,\mathrm{d}x.$$

This time, we will assume that Ψ is smooth, but the estimates on P will not depend on its smoothness. As most of the arguments are the same as in Section 5.1 we resume the results in just two statements.

Proposition 5.5. *The optimal $\bar{\rho}$ is strictly positive a.e.. Denoting by φ_μ the Kantorovich potential for the transport from $\bar{\rho}$ to μ (which is unique up to additive constants), there exists $P \in L^1(\Omega)$, positive, such that $\{p > 0\} \subset \{\bar{\rho} = 1\}$ and a constant C such that*

$$\frac{\varphi_\mu}{\tau} + \Psi + P + \lambda \log(\bar{\rho}) = C \;\; a.e. \tag{24}$$

Moreover $\varphi_\mu \in C^{2,\alpha}(\mathring{\Omega}) \cap C^{1,\alpha}(\Omega)$, P and $\log(\bar{\rho})$ are Lipschitz continuous, and $P \in C^{2,\alpha}(\{P > 0\})$.

Proof. The proof is just an adaptation of those of Lemma 5.1, Proposition 5.2, and Lemma 5.3. □

Theorem 5.6. *We have the following L^∞ estimate:*

$$P \leq \max\Psi - \min\Psi.$$

Proof. The proof is just an adaptation of that of Theorem 5.4, defining now

$$D(x) := -\frac{\log(\mu(x - \nabla\varphi_\mu(x))) - \log(\bar{\rho}(x))}{\tau}. \quad \square$$

Another useful result concerns the H^1 regularity of P. This results could have also be obtained in the case of p, and improves the result of [13] (since it consists in $L^\infty_t H^1_x$ regularity under the only assumption $V \in H^1$ compared to $L^2_t BV_x$ for $V \in C^{1,1}$, but in [13] more general cost functions (with non-quadratic Hamiltonians) were also considered. Anyway, it is only for P that we will use it.

Theorem 5.7. *Suppose $\Psi \in H^1(\Omega)$. We then have $P \in H^1(\Omega)$ and*

$$\int_\Omega |\nabla P|^2 \leq \int_\Omega |\nabla \Psi|^2.$$

Proof. In the proof of Theorem 5.6, which is based on that of Theorem 5.4, we also obtained $\Delta(\Psi+P) \geq 0$ on $\{P > 0\}$. By multiplyng times P and integrating by parts, we obtain

$$\int_\Omega |\nabla P|^2 \leq -\int_\Omega \nabla\Psi \cdot \nabla P, \tag{25}$$

from which the claim follows. ◻

Remark 9. From the inequality (25) wa can also obtain $\int |\nabla P|^2 + \int |\nabla(P+\Psi)|^2 \leq \int |\nabla\Psi|^2$, which is a stronger result.

5.3 Approximation and conclusions

We now want to explain how to deduce results on the continuous-time pressure p from the estimates that we detailed in the discrete case. We fix a, integer number $N > 1$ and take $\tau = T/N$ as a time step. We will build an approximate value function ϕ^N together with an approximate pressure p^N which will converge, as $N \to +\infty$, to a pair which solves the (continuous) dual problem.

Let us start from the solution of the discrete problem $\bar\rho^N := (\bar\rho^N_0, \bar\rho^N_1, \ldots, \bar\rho^N_N)$. For any $k \in \{0, 1, \ldots, N-1\}$, we choose (φ^N_k, ψ^N_k) a pair of Kantorovich potential between $\bar\rho^N_k$ and $\bar\rho^N_{k+1}$, such choice being unique up to an additive constant.We then know that there exist a pressure p^N_k and P^N, positive and Lipschitz, and constants C^N_k and C^N such that

$$\begin{cases} \frac{\psi^N_{k-1}+\varphi^N_k}{\tau^2} + V_N + p^N_k + \lambda_N \log(\bar\rho^N_k) = C^N_k & k \in \{1, 2, \ldots, N-1\}, \\ \frac{\psi^N_{k-1}}{\tau} + \Psi_N + P^N + \lambda_N \log(\bar\rho^N_k) = C^N & k = N. \end{cases} \tag{26}$$

We define the following value function, defined on the whole interval $[0, T]$ which can be thought as a function which looks like a solution of what could be called a discrete dual problem.

Definition 5.8. *Let ϕ^N the function defined as follows. The "final" value is given by*

$$\phi^N(T^-, \cdot) := \Psi + P^N. \tag{27}$$

Provided that the value $\phi^N((k\tau)^-, \cdot)$ is defined for some $k \in \{1, 2, \ldots, N\}$, the value of ϕ^N on $((k-1)\tau, k\tau) \times \Omega$ is defined by

$$\phi^N(t, x) := \inf_{y\in\Omega} \left(\frac{|x-y|^2}{2(k\tau - t)} + \phi^N((k\tau)^-, y) \right). \tag{28}$$

If $k \in \{1, 2, \ldots, N-1\}$, the function ϕ^N has a temporal jump at $t = k\tau$ defined by

$$\phi^N((k\tau)^-, x) := \phi^N((k\tau)^+, x) + \tau \left(V_N + p^N_k \right)(x) \tag{29}$$

We now also define a measure $\pi \in \mathcal{M}([0,T] \times \Omega)$ which will play the role of the continuous pressure.

Definition 5.9. *Let π^N be the positive measures on $[0,T] \times \Omega$ defined in the following way: for any test function $a \in C([0,1] \times \Omega)$, we set*

$$\int_{[0,1]\times\Omega} a \, \mathrm{d}\pi^N := \tau \sum_{k=1}^{N-1} \int_\Omega a(k\tau, \cdot) p_k^N + \int_\Omega a(T, \cdot) P^N.$$

In other words, π^N is a sum of singular measures corresponding to the jumps of the value function ϕ^N.

Provided that we set $\phi^N(0^-, \cdot) = \phi^N(0^+, \cdot)$ and $\phi^N(T^+, \cdot) = \Psi_N$, the following equation holds in the sense of distributions on $[0,1] \times \Omega$:

$$-\partial_t \phi^N + \frac{1}{2}|\nabla \phi^N|^2 \leqslant \pi^N + V \tag{30}$$

It is then possible to prove the following (see Section 4 in [27]).

Theorem 5.10. *The sequence (ϕ^N, π^N) is bounded in $\Big(BV([0,T] \times \Omega) \cap L^2([0,T]; H^1(\Omega))\Big) \times \mathcal{M}([0,T] \times \Omega)$ and converges, up to subsequences, to a pair $(\bar\phi, \bar\pi) \in \tilde{\mathcal{D}}$, the convergence being in the sense of distributions. This limit pair $(\bar\phi, \bar\pi) \in \tilde{\mathcal{D}}$ is optimal in the relaxed dual problem. When the functions p_k^N and P^N are uniformly bounded in L^∞ then the measure $\bar\pi$ is the sum of an L^∞ density (w.r.t. the space-time Lebesgue measure $\mathcal{L}^{d+1}$) p on $[0,T] \times \Omega$ and of a singular part on $t = T$ with an L^∞ density (with respect to the space Lebesgue measure $\mathcal{L}^d$) P, and we can write System* (19)*. Moreover, $\bar\phi$ is the value function of the value function of an optimization problem of the form* (1) *for a running cost given by $\widehat{V+p}$ and a final cost given by $\widehat{\Psi+P}$.*

Remark 10. The reader can obeserve that we obtain here the existence of an optimal pair $(\phi, \pi) \in \tilde{\mathcal{D}}$, as in Theorem 2.2. This was already proven in [13] without passing through the discrete approximation.

It remains to be convinced that the optimal measure Q in the Lagrangian problem, in the present case of density constraints, optimizes a functional of the form $J_{\Psi,h}$. This was obtained in the density-penalized case by differentiating along perturbations Q_ε but here the additional term in h is not obtained as a derivative of $G(\rho)$ but comes from the constraint and is in some sense a Lagrange multiplier (and a similar term appears at $t = T$). This makes the proof more difficult, but we can obtain the desired result by using the duality.

Theorem 5.11. *Suppose that (ϕ, π) is an optimal pair in the relaxed dual problem and that π decomposes into a density $p \in L^1([0,T] \times \Omega)$ and a singular measure on $\{t = T\}$ with a density $P \in L^1(\Omega)$. Then we have*

- *for every measure $Q \in \mathcal{P}(C)$ such that $(e_t)_\# Q$ is uniformly L^∞ and $(e_0)_\# Q = \overline{\rho_0}$, we have $J_{\Psi+P,V+p}(Q) \geq \int \phi(0^+) d\overline{\rho_0}$,*
- *if $\bar{Q}$ is optimal in* (8) *for the density-constrained problem (i.e. when $G = I_{[0,1]}$), then we have $J_{\Psi+P,V+p}(\bar{Q}) = \int \phi(0^+) d\overline{\rho_0}$,*

In particular $\bar{Q}$ optimizes $J_{\Psi+P,V+p}$ among measures on curves such that $(e_t)_\# Q$ is uniformly L^∞ and, when $\Psi + P$ and $V + p$ are L^∞, it is concentrated on curves optimizing $K_{\widehat{\Psi+P},\widehat{V+p}}$.

Proof. In order to prove the first statement, we consider a pairs of functions $\phi \in C^1([0,T]\times\Omega)$ and $h \in C^0([0,T]\times\Omega)$ such that $-\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 \le h$. We then have

$$J_{\phi(T),h}(Q) = \int \mathrm{d}Q(\gamma)\left(\int_0^T \left(\frac{1}{2}|\gamma'(t)|^2 + h(t,\gamma(t))\right)\mathrm{d}t + \phi(T,\gamma(T))\right)$$

and for every curve γ, using $-\partial_t\phi + \frac{1}{2}|\nabla|^2 \le h$ and $\frac{1}{2}|\gamma'(t)|^2 + \frac{1}{2}|\nabla\phi(t,\gamma(t))|^2 \ge -\nabla\phi(t,\gamma(t))\cdot\gamma'(t)$, we have

$$\int_0^T \left(\frac{1}{2}|\gamma'(t)|^2 + h(t,\gamma(t))\right)\mathrm{d}t + \phi(T,\gamma(T)) \ge \int_0^T \frac{d}{dt}\phi(t,\gamma(t))\,\mathrm{d}t + \phi(T,\gamma(T)) = \phi(0,\gamma(0)).$$

This would be sufficient to prove the desired inequality if we had enough regularity. The same inequality in the case of the optimal relaxed function ϕ together with $h = V + p$ can be obtained if we regularize by space-time convolution. Let us consider a convolution kernel η supported in $[0,1]\times B_1$, and use convolutions with rescaled versions of this kernel $\eta_\delta(t,x) = \delta^{-(d+1)}\eta(t/\delta, x/\delta)$, so that we do not need to look at times $t < 0$. On the other hand, this requires first to extend ϕ for $t > T$, and it can be done by taking $\phi(t,x) = \Psi(x) + P(x)$ for every $t > T$. As a consequence, one should also extend $h := V + p$, and in this case we use $h(t,x) := \frac{1}{2}|\nabla(\Psi + P)|^2$, which belongs to L^1 thanks to Theorem 5.7 (this explains why we prefer to do an asymmetric convolution looking at the future and not at the past, since we do not know whether $\phi_0 \in H^1$ or not). It is then necessary to extend ϕ and h outside Ω as well, for space convolution. As we assumed that the boundary of Ω is smooth, there exists a C^1 map R, defined on a neighborhood of Ω and valued into Ω, such that its jacobian $DR(x)$ has a determinant bounded from below and from above close to $\partial\Omega$ and its operator norm $||DR(x)||$ tends to 1 as $d(x,\partial\Omega) \to 0$ (a typical example is the reflection map when $\Omega = \{x_1 > 0\}$, possibly composed with a diffeomorphism which rectifies the boundary). Then, It is enough to define $\tilde{\phi}_\varepsilon(t,x) := \phi((1+\varepsilon)t, R(x))$ and $\tilde{h}_\varepsilon(t,x) := (1+\varepsilon)h((1+\varepsilon)t, R(x))$ and take $\phi_\varepsilon := \eta_\delta * \tilde{\phi}_\varepsilon$ and $h_\varepsilon := \eta_\delta * \tilde{h}_\varepsilon$, for a suitable choice $\delta = \delta_\varepsilon$, provided δ_ε is such that $||DR(x)|| \le \sqrt{1+\varepsilon}$ for x such that $d(x,\partial\Omega) \le \delta_\varepsilon$. In this way we obtain smooth functions $(\phi_\varepsilon, h_\varepsilon)$ such that $-\partial\phi_\varepsilon + \frac{1}{2}|\nabla\phi_\varepsilon|^2 \le h_\varepsilon$. This allows to write

$$J_{\phi_\varepsilon(T),h_\varepsilon}(Q) \ge \int \phi_\varepsilon(0)\,\mathrm{d}\overline{\rho_0}.$$

We then need to pass to the limit as $\varepsilon \to 0$. We have $h_\varepsilon \to h$ in L^1 which, together with the L^∞ bound on $(e_t)_\# Q$, allows to deal with the h-term. The kinetic term does not depend on h, and we are only left to consider the terms with $\phi_\varepsilon(T)$ and $\phi_\varepsilon(0)$: since ϕ is a BV function, these functions converge in $L^1(\Omega)$ to $\phi(0^+)$ and $\phi(T^+) = \Psi + P$, respectively, which provides the desired inequality.

We are now left to prove that we have equality if we choose $Q = \bar{Q}$, the optimal measure on curves. For this, we use the equality between the primal and the dual problem (knowing that the value of the primal can be expressed either in its Eulerian formulation or in its Lagrangian one). We then have

$$\int_C K_{\Psi,V}\,\mathrm{d}\bar{Q} = \int \phi_\varepsilon(0)\,\mathrm{d}\overline{\rho_0} - \int_{[0,T]\times\Omega} \mathrm{d}\pi = \int \phi_\varepsilon(0)d\overline{\rho_0} - \int_{[0,T]\times\Omega} p - \int_\Omega P.$$

We then use the fact that we have, by primal-dual optimality conditions (which can also be seen in System (19)), $p_t(1-\rho_t) = 0$ and $P(1-\rho_T) = 0$, where $\rho_t = (e_t)_\#\bar{Q}$. Then we obtain

$$\int_C K_{\Psi,V}\,\mathrm{d}\bar{Q} = \int \phi_\varepsilon(0)\,\mathrm{d}\overline{\rho_0}\int_{[0,T]\times\Omega} p\,\mathrm{d}(e_t)_\#\bar{Q} - \int_\Omega P\,\mathrm{d}(e_T)_\#\bar{Q},$$

which can be re-written in terms of $J_{\Psi+P,V+p}$ and gives the claim. □

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