KORN AND POINCARÉ-KORN INEQUALITIES FOR FUNCTIONS WITH SMALL JUMP SET

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Abstract. In this paper we prove a regularity and rigidity result for displacements in $GSBD^p$, for every $p > 1$ and any dimension $n \geq 2$. We show that a displacement in $GSBD^p$ with a small jump set coincides with a $W^{1,p}$ function, up to a small set whose perimeter and volume are controlled by the size of the jump. This generalises to higher dimension a result of Conti, Focardi and Iurlano. A consequence of this is that such displacements satisfy, up to a small set, Poincaré-Korn and Korn inequalities. As an application, we deduce an approximation result which implies the existence of the approximate gradient for displacements in $GSBD^p$.

1. Introduction

The modelling and analysis of fracture in the linearised elasticity framework relies on a good understanding of the space BD of functions of bounded deformation. These are vector-valued functions u in L^1, whose symmetric (distributional) gradient Eu is a bounded Radon measure. Over the years, the fine properties of functions in BD, and in the subspace SBD of special functions of bounded deformation (corresponding to the case where Eu has no Cantor part) have been better understood, and the relation between BD, SBD and the space BV of functions of bounded variation has been studied in detail (see e.g., [3, 5, 15], and [17] for the space of generalised functions of bounded deformation). In particular, it has been proved that, for a function $u \in SBD(\Omega)$, Eu admits the decomposition

$$Eu = e(u)\mathcal{L}^n + [u] \odot \nu_u \mathcal{H}^{n-1} \llcorner J_u, \tag{1.1}$$

where $e(u)$ denotes the absolutely continuous part of Eu with respect to the Lebesgue measure $\mathcal{L}^n$, J_u the jump set of u, $[u]$ the jump of u and ν_u the normal to J_u. The decomposition (1.1) has a clear physical meaning: $e(u)$ represents the elastic part of the strain, and J_u the crack set. It is therefore natural that a model of (brittle) fracture, in the linearised setting, would involve an energy of the type

$$\int_\Omega |e(u)|^2 dx + \mathcal{H}^{n-1}(J_u), \tag{1.2}$$

called the Griffith's energy, of which the Mumford-Shah energy in SBV is the scalar counterpart. The energy (1.2) is in fact well defined in the larger space $GSBD(\Omega)$ of generalised special functions of bounded deformation, which has been introduced by Dal Maso in [17], and is essentially designed to contain all the displacements for which the energy is finite (see Section 2 for the definition). Moreover, $GSBD$ is the natural space for (1.2), where one can prove compactness and existence of minimisers under physical assumptions (see, e.g., [11, 12, 13]).

A key difficulty posed by the energy (1.2), compared to scalar models based on functions of bounded variation, is the lack of control on the skew-symmetric part $(Du - Du^T)/2$ of the distributional gradient of u. The classical tool providing a relation between the full gradient and its symmetric part is the Korn inequality.

In this paper we prove Korn and Poincaré-Korn inequalities in $GSBD^p(\Omega)$, the space of functions $u \in GSBD(\Omega)$ for which $e(u) \in L^p(\Omega)$ and $\mathcal{H}^{n-1}(J_u) < +\infty$, for every dimension $n \geq 2$ and any $p > 1$. More precisely, we have the following (see Theorem 4.4).

Theorem 1.1. *Let $n \in \mathbb{N}$ with $n \geq 2$, $p \in (1,\infty)$, and let $\Omega \subset \mathbb{R}^n$ be a bounded, open and connected Lipschitz set. Then, there exists $c = c(n,p,\Omega) > 0$ with the following property. For any $u \in GSBD^p(\Omega)$, there exists a set of finite perimeter $\omega \subset \Omega$ with $\mathcal{L}^n(\omega) + \mathcal{H}^{n-1}(\partial^*\omega) \leq c\mathcal{H}^{n-1}(J_u)$, and an infinitesimal rigid motions a (namely an affine function a, with $e(a) = 0$), such that*

$$\int_{\Omega\setminus\omega} |\nabla u - \nabla a|^p dx \leq c(n,p,\Omega) \int_\Omega |e(u)|^p dx. \tag{1.3}$$

Moreover, there exists $c = c(n,p,q,\Omega) > 0$ such that

$$\|u - a\|_{L^q(\Omega\setminus\omega)} \leq c(n,p,q,\Omega)\|e(u)\|_{L^p(\Omega)}, \tag{1.4}$$

with $q \leq p^*$ if $p < n$, $q < \infty$ if $p = n$, and $q \leq \infty$ for $p > n$.

This result is the generalisation, in dimension $n \geq 2$, of the two-dimensional result in [14] (see also [22]). Theorem 1.1 ensures that $e(u)$ controls $u - a$ and its approximate gradient *outside an exceptional set*[1], and not in the whole set Ω. This is in contrast with the classical Korn and Poincaré-Korn inequalities for functions $u \in W^{1,p}(\Omega;\mathbb{R}^n)$, with $p > 1$, which state that there exists an infinitesimal rigid motion a such that

$$\|Du - Da\|_{L^p(\Omega)} \leq c(n,p,\Omega)\|Eu\|_{L^p(\Omega)}, \tag{1.5}$$

and that, thanks to the Poincaré inequality and Sobolev embeddings,

$$\|u - a\|_{L^q(\Omega)} \leq c(n,p,q,\Omega)\|Eu\|_{L^p(\Omega)}, \tag{1.6}$$

where q depends on n and p (and $q = p^*$ for $p < n$).

Results like (1.5) and (1.6) are clearly out of reach in $(G)SBD$, even for functions u with a small jump set. This is due to the possible presence of small regions of Ω that can be completely (or almost completely) disconnected from the domain, and where u would not necessarily be close to the infinitesimal rigid motion that achieves the smallest distance from u in the majority of the domain. Hence, in general, for a function $u \in (G)SBD(\Omega)$, $e(u)$ cannot control $u - a$ or its approximate gradient in the whole domain Ω, and a result like Theorem 1.1 is the best possible.

The Korn and Poincaré-Korn inequalities in Theorem 1.1 are an immediate corollary of the result below (see Theorem 4.1 and Remark 4.2), which is the main result of this paper.

Theorem 1.2. *Let $n \in \mathbb{N}$ with $n \geq 2$, $p \in (1,\infty)$, and let $\Omega \subset \mathbb{R}^n$ be a bounded and open Lipschitz set. Then, there exists $c = c(n,p,\Omega) > 0$ with the following property. For any $u \in GSBD^p(\Omega)$, there exists a set of finite perimeter $\omega \subset \Omega$ with $\mathcal{L}^n(\omega) + \mathcal{H}^{n-1}(\partial^*\omega) \leq c\mathcal{H}^{n-1}(J_u)$ and $v \in W^{1,p}(\Omega;\mathbb{R}^n)$, such that $u = v$ in $\Omega \setminus \omega$ and $\int_\Omega |e(v)|^p dx \leq c\int_\Omega |e(u)|^p dx$. If in addition u is bounded, then $\|v\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)}$.*

In Theorem 1.2, we prove 'almost' Sobolev regularity for functions in $GSBD^p$. More precisely we show that, given a function $u \in GSBD^p(\Omega)$, we can *replace* it with a function $v \in W^{1,p}(\Omega;\mathbb{R}^n)$ outside an exceptional set $\omega \subset \Omega$, whose volume and perimeter are both controlled by $\mathcal{H}^{n-1}(J_u)$, up to a small cost in Griffith's

[1]Since this exceptional set could in fact be the whole domain in case $\mathcal{H}^{n-1}(J_u)$ is large, we call it a Korn inequality "for functions with small jump set".

energy. The proof of Theorem 1.2 is done by regularising u at several scales, by means of the auxiliary results Lemma 3.1 and Theorem 3.2.

We now illustrate the idea of the proof. As a first step, we cover the domain Ω with a family of disjoint cubes q whose size reduces towards the boundary. The cubes in the partition are then classified into 'good' and 'bad', depending on whether the amount of J_u they contain is smaller or larger than a given threshold. The construction is done so that all the cubes in the covering of Ω are 'good', up to a small neighbourhood of $\partial\Omega$. In this neighbourhood, the 'bad' cubes are cut away from the domain by connecting them to $\partial\Omega$ by means of truncated cones. In this way what remains is still a Lipschitz set (with the same Lipschitz constant as Ω). Moreover, in each 'bad' cube, by definition, the perimeter of the cone is comparable to the perimeter of the cube, and hence is bounded by the measure of the jump set of u in it.

Hence it is sufficient to deal with good cubes. For each of the good cubes q we apply the auxiliary regularity result Theorem 3.2. This ensures that, given a function $\tilde{u} \in GSBD^p(q)$, we can *wipe out* its jump set $J_{\tilde{u}}$ away from the boundary of q, up to a small expense in terms of the Griffith's energy, provided $\mathcal{H}^{n-1}(J_{\tilde{u}})$ is sufficiently smaller than the perimeter of q. This 'smallness' condition is exactly what enters in the definition of 'good' cubes. Applying Theorem 3.2 to $\tilde{u} := u_{|q}$ in every 'good' cube q, we obtain a Sobolev regularisation $\tilde{v}_q$ of $u_{|q}$ and an exceptional set $\tilde{\omega}_q$ with controlled perimeter, such that $\tilde{v}_q = u$ outside $\tilde{\omega}_q$. The function v in the statement of Theorem 1.2 is then obtained by patching together the functions $\tilde{v}_q$ on all the good cubes. The set ω where v needs not coincide with u, is then defined as the union of the exceptional sets $\tilde{\omega}_q$ of the good cubes, together with the truncated cones relative to the bad cubes.

To conclude, we sketch the proof of Theorem 3.2, which is strongly inspired by its two-dimensional version [14], by Conti, Focardi and Iurlano, and involves an iterative regularisation procedure. Starting with $w_0 = \tilde{u}$, we construct a sequence (w_k), where w_{k+1} is obtained by covering a large part of J_{w_k} with a family of disjoint balls, and by replacing w_k in each ball of the covering with a smoother function, provided by Lemma 3.1. The pointwise limit $\tilde{v}$ of the sequence (w_k) has Sobolev regularity in a smaller cube, and satisfies $\tilde{u} = \tilde{v}$ outside an exceptional set $\tilde{\omega}$, which is defined as the union of the coverings of each step.

1.1. **Comparison with previous results.** The foundations of the function spaces SBD and $GSBD$ were laid down in the papers [3, 5], and [17]. Several research avenues have stemmed from them: the derivation of regularity properties for functions in $(G)SBD^p$, and in particular of minimisers of the Griffith's energy, in the spirit of the celebrated result [18] by De Giorgi, Carriero and Leaci for the Mumford-Shah energy (see [15, 9, 11, 12, 4, 13]); of Korn and Poincaré-Korn inequalities with various degrees of generality ([8, 21, 22, 23]); of approximation and density results ([6, 10, 16]); of integral representation for functionals in $(G)SBD^p$ [14].

Our results are in between two of these avenues: we prove Sobolev regularity for functions in $GSBD^p$, for every $p > 1$ and in every dimension $n \geq 2$, outside an exceptional set (see Theorem 1.2) and, as a direct corollary, we obtain a Korn inequality, and a Poincaré-Korn inequality with sharp exponent (see Theorem 1.1), again outside an exceptional set.

Our work has a number of points of contact with previous results, but also a number of differences. In [8] the authors prove a Poincaré-Korn inequality like (1.4) for every $n \geq 2$ and every $p \geq 1$ by means of a slicing argument. Unlike our case, however, they obtain (1.4) with $q = p(1^*)$, rather than $q = p^*$ (which is optimal only for $p = 1$), and no estimate for the gradient of u is provided. Moreover, the exceptional set ω is controlled by the jump set of J_u only in volume, while we also control its perimeter. A Poincaré-Korn inequality like (1.4) is proved also in [21], for $n = 2$ and $p = 2$, with an exceptional set ω whose structure is very simple, and can be related to the measure of J_u. This objective is further pursued in [23], where the author proves a Poincaré-Korn inequality in $GSBD^2$, up to an exceptional set with both perimeter and area bounded by (powers of) the measure of J_u, for $n \geq 2$. The L^2-norm of $e(u)$, however, only controls the distance of u from a rigid motion in the weaker norm $q = 2(1^*)$; additionally, one can obtain an L^∞ bound for such a distance, but the L^2-norm of $e(u)$ has to be weighted with a negative power of the measure of J_u.

The first proofs of a Korn inequality like (1.3), in the $(G)SBD$ context, are due to [22] and [14]. In [22] the proof is done in dimension $n = 2$ and for $p = 2$. Moreover, the distance of ∇u from a skew-symmetric matrix is estimated in a lower L^q-norm, with $q \in [1, 2)$. On the other hand, the exceptional set is estimated, both in perimeter and in area, with the measure of J_u, and the integrability of u is improved to the sharp exponent, with consequent improvement of the Poincaré-Korn inequality. The two-dimensionality of the result is due to an approximation step, done in [21], that is only proved in the planar setting. Also the result in [14] is only proved for $n = 2$, and again this is due to a 'regularisation' step being done by means of a two-dimensional construction. Their approach, like ours, is based on first proving Sobolev regularity outside an exceptional set, and then deducing Korn and Poincaré-Korn inequalities as direct corollaries. Also in [14], like in our result, the exceptional set is bounded in perimeter in terms of J_u, and the Poincaré-Korn inequality is proved with the sharp exponent for every $p > 1$.

In conclusion, our contribution is two-fold. On the one hand our result lifts the restriction to dimension $n = 2$ of the regularisation step from $GSBD^p$ to $W^{1,p}$, up to an exceptional set, which is now valid for every $n \geq 2$ and every $p > 1$. In addition, the exceptional set we provide is bounded both in perimeter and in area with the measure of the jump set of the function. As a consequence, we can deduce the Korn and Poincaré-Korn inequalities up to the sharp exponent for every $n \geq 2$ and $p > 1$, since the regularisation step is not reliant on a planar construction.

1.2. **Conclusion and perspectives.** The main result in this work, asserting the 'almost' Sobolev regularity of $GSBD^p$-functions with small jump set, has some nontrivial consequences which are of independent interest, and which we present here. First of all, we obtain a Korn and a Poincaré-Korn inequality with sharp exponents outside an exceptional set, which is controlled in perimeter and volume by the jump set of the function (Theorem 4.4). We also prove an approximation result (Theorem 5.1) in the spirit of [10, Theorem 3.1]. Theorem 5.1 implies, in particular, the existence of the approximate gradient ∇u for functions in $GSBD^p$ (Corollary 5.2). Note that the existence of ∇u for functions in $GSBD^2$ had already been obtained in [23], as a consequence of the embedding $GSBD^2(\Omega) \subset (GBV(\Omega))^n$ (see [23, Theorem 2.9]), for $n \geq 2$.

It would be interesting to see whether, in analogy with [14], our results can also lead to the integral representation for functionals in SBD^p in higher dimension.

2. Notation

We introduce now some notation that will be used throughout the paper.

(a) $\mathcal{L}^n$ denotes the Lebesgue measure on $\mathbb{R}^n$ and $\mathcal{H}^{n-1}$ the $(n-1)$-dimensional Hausdorff measure on $\mathbb{R}^n$.
(b) $e_1, \ldots, e_n$ is the canonical basis of $\mathbb{R}^n$; $|\cdot|$ denotes the absolute value in $\mathbb{R}$ or the Euclidean norm in $\mathbb{R}^n$, depending on the context, and $\cdot$ denotes the Euclidean scalar product.

(b) $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$.
(c) For $x \in \mathbb{R}^n$ and $\rho > 0$ we define the ball

$$B_\rho(x) := \{y \in \mathbb{R}^n : |y - x| < \rho\}.$$

(d) For $x \in \mathbb{R}^n$, $\mathrm{e} \in \mathbb{S}^{n-1}$, and $\rho > 0$, we define the cylinder

$$C(x, \mathrm{e}, h, \rho) := \{y \in \mathbb{R}^n : |(y - x) \cdot \mathrm{e}| < h, |(y - x) - ((y - x) \cdot \mathrm{e})\mathrm{e}| < \rho\}.$$

(e) For $y \in \mathbb{R}^n$ and $\xi \in \mathbb{S}^{n-1}$, we set

$$\Pi^\xi_y := \{x \in \mathbb{R}^n : (x - y) \cdot \xi = 0\},$$

and use the shortcut $\Pi^\xi = \Pi^\xi_0$.
(f) For $a, b \in \mathbb{R}^n$, we denote with $a \otimes b \in \mathbb{R}^{n\times n}$ the tensor product of a and b, namely the matrix with $(a \otimes b)_{ij} = a_i b_j$ for every $i, j = 1, \dots, n$. Moreover, we denote the symmetrised tensor product as $a \odot b := (a \otimes b + b \otimes a)/2 \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$.
(g) $\mathcal{R}$ denoted infinitesimal rigid motions, namely $a \in \mathcal{R}$ if and only if $a = Ax + b$, with $A \in \mathbb{R}^{n\times n}$ skew-symmetric, and $b \in \mathbb{R}^n$.
(h) An $\mathcal{L}^n$-measurable and bounded set $E \subset \mathbb{R}^n$ is a set of finite perimeter if its characteristic function χ_E is a function of bounded variation. The reduced boundary of E, denoted with $\partial^* E$ is the set of points $x \in \operatorname{supp} |D\chi_E|$ where a generalised normal ν_E is defined.
(i) For $\Omega \subset \mathbb{R}^n$ measurable set $\mathcal{M}_b(\Omega; \mathbb{R}^m)$ denotes the space of bounded Radon measures with values in $\mathbb{R}^m$, for $m \geq 1$. Moreover, for $m = 1$, we denote with $\mathcal{M}^+_b(\Omega)$ the sub-class of positive measures.
(l) For $k \in \mathbb{N}$, $\gamma_k \in \mathbb{R}$ denotes the k-dimensional Lebesgue measure of the unit ball in $\mathbb{R}^k$. In particular, $\mathcal{H}^{n-1}(\mathbb{S}^{n-1}) = n\gamma_n$.

Let $\Omega \subset \mathbb{R}^n$ be an open set. We now introduce the functional spaces we will work with in this paper. We first recall the definition of the space GBD of generalised functions with bounded deformation, which is due to Dal Maso [17] and relies on slicing. Given an $\mathcal{L}^n$-measurable function $u : \Omega \to \mathbb{R}^n$, we say that $u \in GBD(\Omega)$ if there exists $\lambda_u \in \mathcal{M}^+_b(\Omega)$ such that the following is true for every $\xi \in \mathbb{S}^{n-1}$:

- For every $\tau \in C^1(\mathbb{R})$ with $-\frac{1}{2} \leq \tau \leq \frac{1}{2}$ and $0 \leq \tau' \leq 1$

$$D_\xi\big(\tau(u \cdot \xi)\big) = D\big(\tau(u \cdot \xi)\big) \cdot \xi \in \mathcal{M}_b(\Omega);$$

- For every Borel set $B \subset \Omega$

$$\big|D_\xi\big(\tau(u \cdot \xi)\big)\big|(B) \leq \lambda_u(B).$$

We say that $u \in GSBD(\Omega)$ if in addition $\hat{u}^\xi_y(t) \in SBV_{\mathrm{loc}}(\Omega^\xi_y)$ for every $\xi \in \mathbb{S}^{n-1}$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$, where $\Omega^\xi_y := \{t \in \mathbb{R} : y + t\xi \in \Omega\}$ and, for $t \in \Omega^\xi_y$, $\hat{u}^\xi_y(t) := u(y + t\xi) \cdot \xi$ denotes the slice of u in direction ξ. In [17] it is shown that, given a function $u \in GSBD(\Omega)$, one can define an ‘approximate symmetrised gradient’ $e(u) \in L^1(\Omega; \mathbb{R}^{n\times n}_{\mathrm{sym}})$ as well as an $(\mathcal{H}^{n-1}, n-1)$-countably rectifiable jump set J_u, which both coincide with the standard definitions [3] if $u \in BD(\Omega)$.

Finally, we recall the definition of the space $GSBD^p$, namely

$$GSBD^p(\Omega) := \{u \in GSBD(\Omega) : e(u) \in L^p(\Omega; \mathbb{R}^{n\times n}_{\mathrm{sym}}), \mathcal{H}^{n-1}(J_u) < +\infty\}.$$

3. How to wipe out small jump sets

The following Lemma is a variant of [9, Theorem 3], which can be proved by adapting the arguments to the case of a ball.

Lemma 3.1. *Let $n \in \mathbb{N}$ with $n \geq 2$, and let $p \in (1,\infty)$. There exist $\bar{\delta}, c, s$ positive constants, depending only on n and p, with the following property. For every $u \in GSBD^p(B_1)$ with $\delta := \mathcal{H}^{n-1}(J_u)^{1/n} \leq \bar{\delta}$, there exists $\tilde{u} \in GSBD^p(B_1)$ and $R \in (1-\sqrt{\delta}, 1)$ such that*

(1) $\tilde{u} \in C^\infty(B_{1-\sqrt{\delta}})$, $\tilde{u} = u$ in $B_1 \setminus B_R$, and $\mathcal{H}^{n-1}(J_u \cap \partial B_R) = \mathcal{H}^{n-1}(J_{\tilde{u}} \cap \partial B_R) = 0$;

(2) $\mathcal{H}^{n-1}(J_{\tilde{u}} \setminus J_u) \leq c\sqrt{\delta}\mathcal{H}^{n-1}(J_u \cap (B_1 \setminus B_{1-\sqrt{\delta}}))$;

(3) it holds

$$\int_{B_1} |e(\tilde{u})|^p dx \leq (1 + c\delta^s) \int_{B_1} |e(u)|^p dx;$$

(4) if in addition u is bounded, then one can ensure $\|\tilde{u}\|_{L^\infty(B_1)} \leq \|u\|_{L^\infty(B_1)}$.

The last point follows from Remark 6 and Lemma A.1 in [7], which can be used when building the function $\tilde{u}$ in the construction of [9, Theorem 3].

The following theorem is an extension in dimension $n \geq 2$ of a planar result of Conti, Focardi and Iurlano [14]. Our proof is strongly inspired by theirs.

Theorem 3.2. *Let $n \in \mathbb{N}$ with $n \geq 2$, and let $p \in (1,\infty)$. Given $\varepsilon > 0$ and $\sigma \in (0,1)$, there exist $C = C(n,p,\varepsilon) > 0$ and $\tau = \tau(n,p,\varepsilon,\sigma) > 0$ with the following property. For every $\rho > 0$ and $u \in GSBD^p(B_\rho)$ with $\mathcal{H}^{n-1}(J_u) \leq \tau\rho^{n-1}$, there exists $w \in GSBD^p(B_\rho)$ and a set of finite perimeter $\omega \subset B_\rho$, such that $w = u$ in $B_\rho \setminus \omega$, $\mathcal{H}^{n-1}(\partial^*\omega) \leq C\,\mathcal{H}^{n-1}(J_u)$, $w \in W^{1,p}(B_{(1-\sigma)\rho}; \mathbb{R}^n)$, and*

$$\int_{B_\rho} |e(w)|^p dx \leq (1+\varepsilon) \int_{B_\rho} |e(u)|^p dx, \quad \mathcal{H}^{n-1}(J_w) \leq \mathcal{H}^{n-1}(J_u). \tag{3.1}$$

Moreover if u is bounded, one can ensure $\|w\|_{L^\infty(B_\rho)} \leq \|u\|_{L^\infty(B_\rho)}$.

Remark 3.3. A careful inspection of the proof shows that

$$\lim_{\sigma\to 0^+} \tau(n,p,\varepsilon,\sigma) = \lim_{\varepsilon\to 0^+} \tau(n,p,\varepsilon,\sigma) = 0,$$

and

$$\lim_{\varepsilon\to 0^+} C(n,p,\varepsilon) = +\infty.$$

Remark 3.4. The isoperimetric inequality ensures that $\mathcal{L}^n(\omega) \leq C\,\rho\mathcal{H}^{n-1}(J_u)$ (using that ω is small in B_ρ), as well as $\mathcal{L}^n(\omega) \leq C\,\mathcal{H}^{n-1}(J_u)^{n/(n-1)}$ (possibly changing the constant C).

Proof. We start by assuming that

$$\mathcal{H}^{n-1}(J_u) \leq \tau\rho^{n-1},$$

for a $\tau > 0$ to be determined later.

The function w is obtained as the pointwise limit of a sequence $(w_k)_{k\geq 0}$, constructed iteratively starting from $w_0 = u$, which at every step 'regularises' u at the expense of a controlled increase of the L^p norm of the approximate symmetric gradient. We split the proof into several steps.

Step 1: Iterative construction of $(w_k)_{k\geq 0}$. We will now build a sequence of functions $(w_k)_{k\geq 0} \subset GSBD^p(B_\rho)$ by induction.

Step 1.1: Base case. Let $\bar{\delta} = \bar{\delta}(n,p)$ be the constant given by Lemma 3.1, and let $\alpha = \alpha(n,p,\varepsilon) \in (0,1)$ be a constant to be determined later (see (3.27)). We set $w_0 := u$, $\eta_0 := (\alpha\bar{\delta})^n$, $\rho_0 := \rho$ and

$$s_0 := \frac{1}{\rho}\left(\frac{\mathcal{H}^{n-1}(J_u)}{\eta_0}\right)^{\frac{1}{n-1}}. \tag{3.2}$$

Note that by assumption $s_0 \leq (\tau/\eta_0)^{1/(n-1)}$. In order for the iteration to converge, we will need s_0 to be sufficiently small, hence the τ in the statement. We also observe that, by the definition of s_0, we have

$$\mathcal{H}^{n-1}(J_{w_0} \cap B_{\rho_0}) = \mathcal{H}^{n-1}(J_u \cap B_\rho) = \eta_0(\rho_0 s_0)^{n-1}.$$

Step 1.2: Induction step. Let $k \geq 0$, and suppose we are given $w_k \in GSBD^p(B_\rho)$, $s_k \in (0,1)$, $\rho_k \leq \rho$ and $\eta_k \leq \bar{\delta}^n$ which satisfy

$$\mathcal{H}^{n-1}(J_{w_k} \cap B_{\rho_k}) \leq \eta_k(s_k\rho_k)^{n-1}, \tag{3.3}$$

as it is the case for $k = 0$. We will build w_{k+1}, η_{k+1}, s_{k+1} and ρ_{k+1} (explicitly given at the end of the step) such that (3.3) is satisfied for $k+1$. We will divide the proof of the induction step into further substeps.

Our strategy is the following. We construct a function w_{k+1} whose jump set is (in measure) not larger than the one of the function w_k. To do so, we cover a large part of J_{w_k} in the smaller ball $B_{(1-s_k)\rho_k}$ (subsequently defined as $B_{\rho_{k+1}}$) with a family of disjoint balls, and we wipe out a significant part of the jump set of w_k in each ball of the covering.

Step 1.2a: Construction of the covering. We claim that for $\mathcal{H}^{n-1}$-a.e. $x \in J_{w_k} \cap B_{(1-s_k)\rho_k}$ there exists $r_x \in [0, s_k\rho_k]$ such that

$$\begin{cases} \mathcal{H}^{n-1}(J_{w_k} \cap B_{r_x}(x)) = \eta_k r_x^{n-1} \\ \mathcal{H}^{n-1}(J_{w_k} \cap B_r(x)) \geq \eta_k r^{n-1} \quad \text{for } r \leq r_x. \end{cases} \tag{3.4}$$

Indeed, if x is a point of rectifiability of $J_{w_k} \cap B_{(1-s_k)\rho_k}$ and we define

$$\phi(r) := \frac{\mathcal{H}^{n-1}(J_{w_k} \cap B_r(x))}{r^{n-1}}, \quad r \in (0, s_k\rho_k],$$

we have $\lim_{r\to 0^+} \phi(r) = \gamma_{n-1} > \eta_0 \geq \eta_k$, where γ_{n-1} is the $(n-1)$-dimensional Lebesgue measure of the unit ball of $\mathbb{R}^{n-1}$ (it is of course not restrictive to assume $\eta_0 < \gamma_{n-1}$, up to possibly reducing $\bar{\delta}$). Moreover, since $B_{s_k\rho_k}(x) \subset B_{\rho_k}$, from (3.3) it follows that $\phi(s_k\rho_k) \leq \eta_k$. Therefore, we have that $r_x := \inf\{r \in (0, s_k\rho_k) : \phi(r) \leq \eta_k\} > 0$. As ϕ is lower semicontinuous one has $\phi(r_x) \leq \eta_k$, and as it is left-continuous, one has $\phi(r_x) \geq \eta_k$. This shows (3.4). By construction, observe also that

$$\mathcal{H}^{n-1}(J_{w_k} \cap \partial B_{r_x}(x)) = 0. \tag{3.5}$$

By the Besicovitch Covering Theorem (see, for instance, [2, Theorem 2.17]) there exists a positive integer $\xi(n)$, depending only on n, with the following property. There exist $\xi(n)$ countable families of closed balls $(\overline{B}_{r_{x_i^\ell}}(x_i^\ell))$, $i \geq 1$ and $\ell = 1, \dots, \xi(n)$, whose centres satisfy (3.4), with $\overline{B}_{r_{x_i^\ell}}(x_i^\ell) \cap \overline{B}_{r_{x_j^\ell}}(x_j^\ell) = \emptyset$ for $i \neq j$, and such that

$$\mathcal{H}^{n-1}\left((J_{w_k} \cap B_{(1-s_k)\rho_k}) \setminus \left(\bigcup_{\ell=1}^{\xi(n)} \bigcup_{i\geq 1} \overline{B}_{r_{x_i^\ell}}(x_i^\ell) \right) \right) = 0.$$

Let us choose $\bar{\ell} \in \{1, \dots, \xi(n)\}$ such that

$$\mathcal{H}^{n-1}\left((J_{w_k} \cap B_{(1-s_k)\rho_k}) \cap \left(\bigcup_{i\geq 1} \overline{B}_{r_{x_i^{\bar{\ell}}}}(x_i^{\bar{\ell}}) \right) \right)$$

is maximal. Then one has

$$\sum_{i\geq 1}\mathcal{H}^{n-1}\left(J_{w_k}\cap \overline{B}_{r_{x_i^{\bar\ell}}}(x_i^{\bar\ell})\right)\geq \mathcal{H}^{n-1}\left((J_{w_k}\cap B_{(1-s_k)\rho_k})\cap\left(\bigcup_{i\geq 1}\overline{B}_{r_{x_i^{\bar\ell}}}(x_i^{\bar\ell})\right)\right)$$
$$\geq \frac{1}{\xi(n)}\mathcal{H}^{n-1}(J_{w_k}\cap B_{(1-s_k)\rho_k}). \tag{3.6}$$

In what follows we denote, to simplify, $B_i := B_{r_{x_i^{\bar\ell}}}(x_i^{\bar\ell})$, $x_i := x_i^{\bar\ell}$, $r_i := r_{x_i^{\bar\ell}}$.

Step 1.2b: Definition of w_{k+1}. We define w_{k+1} in two different ways, depending on whether the amount of the jump set of w_k in the annulus $B_{\rho_k}\setminus B_{(1-s_k)\rho_k}$ is large or not. Let $\theta\in(0,1)$ (which will be fixed later on). If

$$\mathcal{H}^{n-1}(J_{w_k}\cap B_{(1-s_k)\rho_k})\leq \theta\mathcal{H}^{n-1}(J_{w_k}\cap B_{\rho_k}), \tag{3.7}$$

we let $w_{k+1}:=w_k$. If instead we have the reverse inequality in (3.7), and consequently

$$\mathcal{H}^{n-1}(J_{w_k}\cap(B_{\rho_k}\setminus B_{(1-s_k)\rho_k}))\leq(1-\theta)\mathcal{H}^{n-1}(J_{w_k}\cap B_{\rho_k}), \tag{3.8}$$

we then define w_{k+1} as

$$w_{k+1}(x):=\begin{cases} w_k(x) & \text{if } x\in B_\rho\setminus\left(\bigcup_{i\geq 1}\overline{B}_i\right),\\ \widetilde{w}_{k,i}(x) & \text{if } x\in B_i \text{ for some } i\in\mathbb{N},\end{cases}$$

where $\widetilde{w}_{k,i}\in GSBD^p(B_i)$ denotes the function obtained by applying Lemma 3.1, after suitable translation and rescaling, to the restriction of w_k in each ball B_i for every $i\geq 1$. Note that in this case the value of δ, by definition of the balls B_i (namely by (3.4)), is given by $\eta_k^{1/n}$, and $\eta_k^{1/n}\leq\bar\delta$ by the assumption of the induction step.

Step 1.2c: Proof of the induction step. In case (3.7) is satisfied, we have $w_{k+1}=w_k\in GSBD^p(B_\rho)$, so that

$$\mathcal{H}^{n-1}(J_{w_{k+1}})=\mathcal{H}^{n-1}(J_{w_k}), \tag{3.9}$$

and, by using (3.7) and (3.3),

$$\mathcal{H}^{n-1}(J_{w_{k+1}}\cap B_{(1-s_k)\rho_k})\leq\theta\eta_k(s_k\rho_k)^{n-1}. \tag{3.10}$$

We now assume that (3.8) holds. By Property (1) of Lemma 3.1 we have $w_{k+1}\in GSBD^p(B_\rho)$. Moreover, let $R_{k,i}\in(1-\eta_k^{1/(2n)},1)$ be the radius given by Lemma 3.1 and corresponding to $\widetilde{w}_{k,i}$. Setting $B_i':=B_{(1-\eta_k^{1/(2n)})r_i}(x_i)$ and $B_i'':=B_{R_{k,i}r_i}(x_i)$, we have in particular that $w_{k+1}\in C^\infty(B_i')$, $w_{k+1}=w_k$ in $B_i\setminus B_i''$, and $\mathcal{H}^{n-1}(J_{w_k}\cap\partial B_i'')=\mathcal{H}^{n-1}(J_{w_{k+1}}\cap\partial B_i'')=0$.

Property (2) of Lemma 3.1 provides a control on the (possible) additional jump of w_{k+1} in each B_i (note that this additional jump can only be in $B_i''\setminus B_i'$ by Property (1)):

$$\mathcal{H}^{n-1}((J_{w_{k+1}}\setminus J_{w_k})\cap B_i)=\mathcal{H}^{n-1}((J_{w_{k+1}}\setminus J_{w_k})\cap(B_i''\setminus B_i'))$$
$$\leq c\eta_k^{\frac{1}{2n}}\mathcal{H}^{n-1}(J_{w_k}\cap(B_i\setminus B_i')). \tag{3.11}$$

We now estimate the jump of w_{k+1} in each B_i. By property (1) of Lemma 3.1 and by (3.11)

$$\mathcal{H}^{n-1}(J_{w_{k+1}}\cap B_i)=\mathcal{H}^{n-1}(J_{w_{k+1}}\cap(B_i\setminus B_i'))$$
$$\leq\mathcal{H}^{n-1}((J_{w_{k+1}}\setminus J_{w_k})\cap(B_i''\setminus B_i'))+\mathcal{H}^{n-1}(J_{w_k}\cap(B_i\setminus B_i'))$$
$$\leq\left(1+c\eta_k^{\frac{1}{2n}}\right)\mathcal{H}^{n-1}(J_{w_k}\cap(B_i\setminus B_i')). \tag{3.12}$$

For the last term in (3.12) we have the bound

$$\begin{aligned}\mathcal{H}^{n-1}(J_{w_k}\cap(B_i\setminus B_i')) &= \mathcal{H}^{n-1}(J_{w_k}\cap B_i)-\mathcal{H}^{n-1}(J_{w_k}\cap B_i')\\ &\le \eta_k r_i^{n-1}-\eta_k\big((1-\eta_k^{\frac{1}{2n}})r_i\big)^{n-1}\\ &= \big(1-\big(1-\eta_k^{\frac{1}{2n}}\big)^{n-1}\big)\mathcal{H}^{n-1}(J_{w_k}\cap B_i),\end{aligned}$$

where we have used properties (3.4) and (3.5) for the radii of the balls of the covering. Hence from (3.12) we have

$$\mathcal{H}^{n-1}(J_{w_{k+1}}\cap B_i)\le\big(1+c\eta_k^{\frac{1}{2n}}\big)\big(1-\big(1-\eta_k^{\frac{1}{2n}}\big)^{n-1}\big)\mathcal{H}^{n-1}(J_{w_k}\cap B_i).$$

Possibly reducing $\bar\delta$, we may assume that

$$\big(1+c\eta_k^{\frac{1}{2n}}\big)\left(1-\big(1-\eta_k^{\frac{1}{2n}}\big)^{n-1}\right)\le\left(1+c\sqrt{\bar\delta}\right)\left(1-\left(1-\sqrt{\bar\delta}\right)^{n-1}\right)\le\frac12,$$

so that

$$\mathcal{H}^{n-1}(J_{w_{k+1}}\cap B_i)\le\frac12\mathcal{H}^{n-1}(J_{w_k}\cap B_i). \tag{3.13}$$

Note that (3.13) and (3.5) imply immediately that

$$\mathcal{H}^{n-1}(J_{w_{k+1}})\le\mathcal{H}^{n-1}(J_{w_k}). \tag{3.14}$$

In addition, by (3.13) and (3.5) one has

$$\begin{aligned}&\mathcal{H}^{n-1}(J_{w_{k+1}}\cap B_{\rho_k})-\mathcal{H}^{n-1}(J_{w_k}\cap B_{\rho_k})\\ &\quad\le\sum_{i\ge1}\big(\mathcal{H}^{n-1}(J_{w_{k+1}}\cap B_i)-\mathcal{H}^{n-1}(J_{w_k}\cap B_i)\big)\\ &\quad\le-\frac12\sum_{i\ge1}\mathcal{H}^{n-1}(J_{w_k}\cap B_i)\le-\frac{1}{2\xi(n)}\mathcal{H}^{n-1}(J_{w_k}\cap B_{(1-s_k)\rho_k}),\end{aligned} \tag{3.15}$$

where the last inequality follows from (3.6). We deduce from (3.15) and (3.8) that

$$\begin{aligned}\mathcal{H}^{n-1}(J_{w_{k+1}}\cap B_{\rho_k}) &\le\left(1-\frac{1}{2\xi(n)}\right)\mathcal{H}^{n-1}(J_{w_k}\cap B_{\rho_k})\\ &\qquad+\frac{1}{2\xi(n)}\mathcal{H}^{n-1}(J_{w_k}\cap(B_{\rho_k}\setminus B_{(1-s_k)\rho_k}))\\ &\le\left(1-\frac{\theta}{2\xi(n)}\right)\mathcal{H}^{n-1}(J_{w_k}\cap B_{\rho_k}).\end{aligned}$$

Therefore, if we choose $\theta=2\xi(n)/(1+2\xi(n))<1$, we find that

$$\mathcal{H}^{n-1}(J_{w_{k+1}}\cap B_{(1-s_k)\rho_k})\le\theta\mathcal{H}^{n-1}(J_{w_k}\cap B_{\rho_k})\le\theta\eta_k(s_k\rho_k)^{n-1}. \tag{3.16}$$

In conclusion, whether (3.7) be satisfied or not, one has that, by (3.9) and (3.14),

$$\mathcal{H}^{n-1}(J_{w_{k+1}})\le\mathcal{H}^{n-1}(J_{w_k}), \tag{3.17}$$

and that, by (3.10) and (3.16),

$$\mathcal{H}^{n-1}(J_{w_{k+1}}\cap B_{(1-s_k)\rho_k})\le\theta\mathcal{H}^{n-1}(J_{w_k}\cap B_{\rho_k})\le\theta\eta_k(s_k\rho_k)^{n-1}. \tag{3.18}$$

We now define $\lambda:=(\theta/(1-s_0)^{n-1})^{1/n}$; choosing τ small enough (depending on n,p,σ and α) and recalling the choice (3.2) of s_0, one can ensure that $\lambda\le\sqrt[2n]{\theta}<1$. Then, letting $\rho_{k+1}:=(1-s_k)\rho_k$, $\eta_{k+1}:=\lambda\eta_k$, $s_{k+1}:=\lambda s_k$, we deduce from (3.18) that

$$\mathcal{H}^{n-1}(J_{w_{k+1}}\cap B_{\rho_{k+1}})\le\eta_{k+1}(s_{k+1}\rho_{k+1})^{n-1},$$

which is (3.3) at step $k+1$.

Step 2: Convergence of $(w_k)_{k\ge0}$. We now start the construction of the exceptional set ω given in the statement. To this aim, for every $k\ge0$ we introduce the set ω_k in the following way. If (3.7) is satisfied we let $\omega_k:=\emptyset$, and if not, we let

$\omega_k := \bigcup_{i\geq 1} \overline{B}_i$ (note that $\omega_k \subseteq B_{\rho_k}$). In both cases $\{w_k \neq w_{k+1}\} \subset \omega_k$ and we can estimate the perimeter of ω_k as

$$\mathcal{H}^{n-1}(\partial^* \omega_k) \leq n\gamma_n \sum_{i\geq 1} r_i^{n-1} = \frac{n\gamma_n}{\eta_k} \sum_{i\geq 1} \mathcal{H}^{n-1}(J_{w_k} \cap B_i)$$
$$\leq \frac{n\gamma_n}{\eta_k} \mathcal{H}^{n-1}(J_{w_k} \cap B_{\rho_k}) \leq n\gamma_n (s_k \rho_k)^{n-1} \tag{3.19}$$

thanks to (3.3), where $n\gamma_n = \mathcal{H}^{n-1}(\mathbb{S}^{n-1})$ (see (l)).

We now estimate the L^p-norm of $e(w_{k+1})$ in terms of the norm of $e(w_k)$. Again, this bound is trivial if (3.7) is satisfied. If not, thanks to point (3) in Lemma 3.1, we have that in each B_i of the construction

$$\int_{B_i} |e(w_{k+1})|^p dx \leq \left(1 + c\eta_k^{\frac{s}{n}}\right) \int_{B_i} |e(w_k)|^p dx \tag{3.20}$$

for each $i \geq 1$. As a consequence, by the definition of w_{k+1}, also

$$\int_{B_\rho} |e(w_{k+1})|^p dx \leq \left(1 + c\eta_k^{\frac{s}{n}}\right) \int_{B_\rho} |e(w_k)|^p dx. \tag{3.21}$$

Repeating the construction for all $k \geq 1$ we obtain sequences $(w_k)_{k\geq 0}$, $(s_k)_{k\geq 0}$, $(\eta_k)_{k\geq 0}$, $(\rho_k)_{k\geq 0}$ and $(\omega_k)_{k\geq 0}$ with:

$$\eta_k = \lambda^k (\alpha\bar{\delta})^n, \quad s_k = \lambda^k s_0, \quad \rho_k = \rho \prod_{\ell=0}^{k-1} (1 - \lambda^\ell s_0). \tag{3.22}$$

Since $(\rho_k)_k$ is decreasing, there exists $\rho' := \lim_{k\to\infty} \rho_k$. We now show that ρ' is bounded away from zero. To see this, note that by possibly reducing τ so that $s_0 \leq \frac{1}{2}$,

$$\log \prod_{\ell=0}^{k-1} (1 - \lambda^\ell s_0) = \sum_{\ell=0}^{k-1} \log(1 - \lambda^\ell s_0) \geq -\sum_{\ell=0}^{k-1} \frac{\lambda^\ell s_0}{1 - \lambda^\ell s_0}. \tag{3.23}$$

Moreover, since $s_0 \leq \frac{1}{2}$, we also have that

$$-\frac{1}{1 - \lambda^\ell s_0} \geq -2 \quad \text{for all } \ell \geq 0,$$

so that (3.23) gives

$$\log \prod_{\ell=0}^{k-1} (1 - \lambda^\ell s_0) \geq -2s_0 \sum_{\ell=0}^{k-1} \lambda^\ell \geq -\frac{2s_0}{1-\lambda}.$$
$$\rho' \geq \rho e^{-\frac{2s_0}{1-\lambda}} \geq \rho \exp(-1/(1 - \sqrt[2n]{\theta})) > 0, \tag{3.24}$$

which shows that $\rho' > 0$.

Now we set, for any $\ell \geq 0$, $\tilde{\omega}_\ell := \bigcup_{k\geq \ell} \omega_k$. Then, thanks to (3.19) and to (3.22),

$$\mathcal{H}^{n-1}(\partial^* \tilde{\omega}_\ell) \leq \sum_{k\geq \ell} \mathcal{H}^{n-1}(\partial^* \omega_k) \leq n\gamma_n \sum_{k\geq \ell} (s_k \rho_k)^{n-1}$$
$$\leq n\gamma_n (s_\ell \rho_\ell)^{n-1} \sum_{k\geq \ell} (\lambda^{k-\ell})^{n-1} \leq n\gamma_n (s_\ell \rho_\ell)^{n-1} \frac{1}{1 - \lambda^{n-1}}, \tag{3.25}$$

where we have used the fact that $\rho_k \leq \rho_\ell$ for $k \geq \ell$. Then, since $\rho_\ell \to \rho'$ and $s_\ell \to 0$ as $\ell \to \infty$, it follows that $\mathcal{H}^{n-1}(\partial^* \tilde{\omega}_\ell) \to 0$ as $\ell \to \infty$. Hence by the isoperimetric inequality we also have that $\mathcal{L}^n(\tilde{\omega}_\ell) \to 0$ as $\ell \to \infty$. Since, for $k \geq \ell$, $w_k = w_\ell$ outside $\tilde{\omega}_\ell$, we conclude that, as $k \to \infty$, w_k converges $\mathcal{L}^n$-a.e. in B_ρ to some function w.

Moreover, for every $k \geq 0$, $\mathcal{H}^{n-1}(J_{w_k}) \leq \mathcal{H}^{n-1}(J_u)$ (thanks to (3.17)) and, by (3.21) and (3.22),

$$\int_{B_\rho} |e(w_k)|^p dx \leq \prod_{i=0}^{k-1} \left(1 + c\eta_i^{\frac{s}{n}}\right) \int_{B_\rho} |e(u)|^p dx \leq \prod_{i=0}^{\infty} \left(1 + c(\alpha\bar{\delta})^s \lambda^{\frac{is}{n}}\right) \int_{B_\rho} |e(u)|^p dx. \tag{3.26}$$

Since

$$\log \prod_{i=0}^{\infty} \left(1 + c(\alpha\bar{\delta})^s \lambda^{\frac{is}{n}}\right) = \sum_{i\geq 0} \log \left(1 + c(\alpha\bar{\delta})^s \left(\lambda^{\frac{s}{n}}\right)^i\right)$$
$$\leq c(\alpha\bar{\delta})^s \sum_{i\geq 0} \left(\lambda^{\frac{s}{n}}\right)^i = c(\alpha\bar{\delta})^s \frac{1}{1-\lambda^{\frac{s}{n}}},$$

from (3.26) we deduce that

$$\int_{B_\rho} |e(w_k)|^p dx \leq \exp\left(c(\alpha\bar{\delta})^s \frac{1}{1-\lambda^{\frac{s}{n}}}\right) \int_{B_\rho} |e(u)|^p dx.$$

Then, thanks to [17, Theorem 11.3] (see also [11]) it follows that $w \in GSBD^p(B_\rho)$,

$$\mathcal{H}^{n-1}(J_w) \leq \liminf_{k\to\infty} \mathcal{H}^{n-1}(J_{w_k}) \leq \mathcal{H}^{n-1}(J_u),$$

and

$$\int_{B_\rho} |e(w)|^p dx \leq \liminf_{k\to\infty} \int_{B_\rho} |e(w_k)|^p dx \leq \exp\left(c(\alpha\bar{\delta})^s \frac{1}{1-\lambda^{\frac{s}{n}}}\right) \int_{B_\rho} |e(u)|^p dx.$$

Passing to the limit in (3.3) we have that $\mathcal{H}^{n-1}(J_w \cap B_{\rho'}) = 0$, from which it follows that $w \in W^{1,p}(B_{\rho'};\mathbb{R}^n)$, thanks to Korn's inequality. Note that by (3.22) $\rho' \leq \rho$, and hence $\rho' \to \rho$ as $\mathcal{H}^{n-1}(J_u) \to 0$, thanks to (3.24) (and by the definition of s_0). In particular, provided τ is sufficiently small (depending on n, p, σ and α), we have

$$\rho' \geq \rho \exp\left(-\frac{2s_0}{1-\sqrt[2n]{\theta}}\right) \geq \rho(1-\sigma),$$

hence $w \in W^{1,p}(B_{\rho(1-\sigma)};\mathbb{R}^n)$.

Setting $\omega := \tilde{\omega}_0$, by construction we have that $w = u$ in $B_\rho \backslash \omega$. In addition, from (3.25) we have that $\mathcal{H}^{n-1}(\partial^*\omega) \leq n\gamma_n(s_0\rho)^{n-1}/(1-\lambda^{n-1})$. By our choice (3.2) of s_0, this implies

$$\mathcal{H}^{n-1}(\partial^*\omega) \leq \frac{n\gamma_n}{(\alpha\bar{\delta})^n(1-\lambda^{n-1})} \mathcal{H}^{n-1}(J_u).$$

Using the fact that $\lambda \leq \sqrt[2n]{\theta}$ we finally obtain the estimates

$$\int_{B_\rho} |e(w)|^p dx \leq \exp\left(\frac{c(\alpha\bar{\delta})^s}{1-\theta^{\frac{s}{2n^2}}}\right) \int_{B_\rho} |e(u)|^p dx,$$
$$\mathcal{H}^{n-1}(\partial^*\omega) \leq \frac{n\gamma_n}{(\alpha\bar{\delta})^n(1-\theta^{\frac{n-1}{2n}})} \mathcal{H}^{n-1}(J_u).$$

Now we choose $\alpha = \alpha(n, p, \varepsilon) \in (0, 1)$ such that

$$\exp\left(\frac{c(\alpha\bar{\delta})^s}{1-\theta^{\frac{s}{2n^2}}}\right) \leq (1+\varepsilon). \tag{3.27}$$

Note that now $\tau = \tau(n, p, \varepsilon, \sigma)$. Correspondingly, we define

$$C = C(n, p, \varepsilon) := \frac{n\gamma_n}{(\alpha\bar{\delta})^n(1-\theta^{\frac{n-1}{2n}})}$$

as the constant in the statement of the theorem, and this concludes the proof. □

Remark 3.5. From (3.20), it is easy to show that in fact one can refine (3.1) to

$$\int_\omega |e(w)|^p dx \le (1+\varepsilon) \int_\omega |e(u)|^p dx.$$

In addition, one sees that $C \sim \varepsilon^{-n/s}$, where s is the exponent in Property (3) of Lemma 3.1.

Remark 3.6. It is easy to show (by modifying the proof or, in fact, using the theorem itself) a variant of Theorem 3.2 where B_ρ is replaced with a cube $(-\rho,\rho)^n$.

We can easily deduce that [14, Corollary 3.3] also holds in higher dimension. We repeat the statement here for the reader's convenience.

Corollary 3.7. *Under the same assumptions and notation of Theorem 3.2, there exists an infinitesimal rigid motion $a \in \mathcal{R}$ such that*

$$\int_{B_{(1-\sigma)\rho}\setminus\omega} |\nabla u - \nabla a|^p dx \le c(n,p) \int_{B_\rho} |e(u)|^p dx,$$

and

$$\int_{B_{(1-\sigma)\rho}\setminus\omega} |u - a|^p dx \le c(n,p)\rho^p \int_{B_\rho} |e(u)|^p dx.$$

4. Regularity and rigidity in a general domain

The main result of this section is the following regularity result.

Theorem 4.1. *Let $n \in \mathbb{N}$ with $n \ge 2$, $p \in (1,\infty)$, and let $\Omega \subset \mathbb{R}^n$ be a bounded and open Lipschitz set. There exists $c = c(n,p,\Omega) > 0$ such that, for any $u \in GSBD^p(\Omega)$, there exists a set of finite perimeter $\omega \subset \Omega$ with $\mathcal{L}^n(\omega)+\mathcal{H}^{n-1}(\partial^*\omega) \le c\mathcal{H}^{n-1}(J_u)$ and $v \in W^{1,p}(\Omega;\mathbb{R}^n)$ such that $u = v$ in $\Omega \setminus \omega$ and $\int_\Omega |e(v)|^p dx \le c\int_\Omega |e(u)|^p dx$. If in addition u is bounded, then $\|v\|_{L^\infty(\Omega)} \le \|u\|_{L^\infty(\Omega)}$.*

Remark 4.2. A careful inspection of the proof shows that $c = c(n,p,L)$, where L denotes a Lipschitz constant of $\partial\Omega$.

For the proof of Theorem 4.1 we will use the following lemma, which can be proven by contradiction.

Lemma 4.3. *Let $n \in \mathbb{N}$ with $n \ge 2$, $p \in (1,\infty)$, and let $D \subset \mathbb{R}^n$ be a bounded, open and connected Lipschitz set. Let $\alpha \in (0,1)$. There exist $c > 0$ depending only on D, α and p, such that for any $w \in W^{1,p}(D;\mathbb{R}^n)$ and any Lebesgue measurable set $E \subset D$ with $\mathcal{L}^n(E) \ge \alpha\mathcal{L}^n(D)$, one has*

$$\int_D |w - a_E|^p dx \le c \int_D |e(w)|^p dx,$$

where

$$a_E := \operatorname*{arg\,min}_{a\in\mathcal{R}} \int_E |w-a|^p dx.$$

Proof of Theorem 4.1. It is enough to prove the result in the case $\mathcal{H}^{n-1}(J_u) \le \bar{c}$, for some constant $\bar{c} > 0$ to be determined later on. Indeed, if $\mathcal{H}^{n-1}(J_u) > \bar{c}$, we can simply set $\omega := \Omega$ and $v := 0$, which clearly satisfy the thesis since $\mathcal{L}^n(\omega) \le (\mathcal{L}^n(\Omega)/\bar{c})\mathcal{H}^{n-1}(J_u)$ and, being Ω a set of finite perimeter, $\mathcal{H}^{n-1}(\partial^*\omega) \le (\mathcal{H}^{n-1}(\partial^*\Omega)/\bar{c})\mathcal{H}^{n-1}(J_u)$.

Let L be a Lipschitz constant of $\partial\Omega$. Since Ω is Lipschitz and bounded (see [1, Section 4.9]), there exist $r > 0$ such that for every $x \in \partial\Omega$, there exists $\mathrm{e}(x) \in \mathbb{S}^{n-1}$ such that $\partial\Omega \cap C(x,\mathrm{e}(x),2Lr,r)$ (see (d) in the Notation Section) is the graph of an L-Lipschitz function defined on the $(n-1)$-dimensional ball $\{y \in \Pi_x^{\mathrm{e}(x)} :$

$|(y-x)-((y-x)\cdot \mathrm{e}(x))\mathrm{e}(x)|<r\}$. Setting $\tilde L := \max\{1, L\}$, by possibly reducing r we still have that $\partial\Omega \cap C(x, \mathrm{e}(x), 2\tilde L r, r)$ is the graph of an $\tilde L$-Lipschitz function for every $x \in \partial\Omega$. Consider now the family of open balls $\{B_{r/5}(x)\}_{x\in\partial\Omega}$. By Vitali's Covering Theorem [19, Section 1.5.1], there exist $N \in \mathbb{N}$ and $\{x_1, \dots, x_N\} \subset \partial\Omega$ such that the family $\{B_{r/5}(x_i)\}_{i=1,\dots,N}$ is composed of mutually disjoint balls and

$$\partial\Omega \subset \bigcup_{x\in\partial\Omega} B_{r/5}(x) \subset \bigcup_{i=1}^{N} B_r(x_i) \subset \bigcup_{i=1}^{N} C(x_i, \mathrm{e}(x_i), 2\tilde L r, r).$$

Let us now show that the number of overlapping cylinders at any given point of Ω only depends on L and n. Indeed, let $z \in \Omega$, and let

$$A(z) := \Big\{ x \in \{x_1, \dots, x_N\} : z \in C(x, \mathrm{e}(x), 2\tilde L r, r) \Big\}.$$

Since the diameter of each cylinder is given by $2r\sqrt{1+4\tilde L^2}$, we have

$$|z - x| < 2r\sqrt{1+4\tilde L^2} \quad \text{for every } x \in A(z).$$

Therefore,

$$A(z) \subset B_{2r\sqrt{1+4\tilde L^2}}(z).$$

Recalling that for every $i \in \{1, \dots, N\}$ with $i \neq j$ it holds $|x_i - x_j| > 2r/5$, the cardinality of $A(z)$ is bounded by the maximum number of disjoint balls of radius $r/5$ which can be contained in a ball of radius $2r\sqrt{1+4\tilde L^2}$. By scaling, one can check that this number does not depend on r, but only on $\tilde L$ (i.e. on on L) and on the dimension n.

Let now $\delta_0 = \delta_0(n, r, L) > 0$ be such that $\bigcup_{i=1}^N C_i \supset \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \le \delta_0\}$; we set $C_0 := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \delta_0\}$, so that $\Omega \subset \bigcup_{i=0}^N C_i$. We then introduce a partition of unity of Ω subordinate to the covering $\{C_i\}$, namely a family of maps $\phi_i \in C_c^\infty(C_i, [0,1])$ for $i = 0, \dots, N$, with $\sum_{i=0}^N \phi_i = 1$ in Ω.

Assuming that for every i we can find a function v_i and a set ω_i satisfying the thesis of the theorem in $C_i \cap \Omega$, then $v = \sum_{i=0}^N \big(\phi_{i|C_i\cap\Omega}\big) v_i$ and $\omega = \bigcup_{i=0}^N \omega_i$ satisfy the claim in Ω.

Without loss of generality, we assume that for $i = 1, \dots N$, $C_i = C(0, e_n, r)$, with e_n being the n-th coordinate unit vector, and that $\Omega \cap C_i = \{x = (x', x_n) \in C_i : x_n < g(x')\}$ for a given $\tilde L$-Lipschitz function g defined on the $((n-1)$-dimensional) ball centred at 0 and of radius r in $\Pi_0^{e_n}$, with $g(0) = 0$.

We now build v_i and ω_i for the set $C_i \cap \Omega$, for $i = 0, \dots, N$. Let $\delta > 0$, and let $\mathbf{C}_i$ denote the union of all the n-dimensional cubes $q \in \{z + (0, \delta]^n : z \in \delta\mathbb{Z}^n\}$ with $q \subset C_i$. We assume that δ is small enough so that $\operatorname{supp} \varphi_i \subset \mathbf{C}_i$ and hence, in particular, $\{\mathbf{C}_i\}_{i=0,\dots,N}$ is still a covering of Ω.

Then we build recursively the set $\mathcal{Q}$ of dyadic cubes of edge size $\delta 2^{-k}$, $k \ge 0$, which refine towards the boundary $\partial\Omega$, as follows. As a first step, we denote with $\mathcal{Q}_0$ the set of cubes $q \in \{z+(0,\delta]^n : z \in \delta\mathbb{Z}^n\}$, $q \subset \mathbf{C}_i \cap \Omega$, such that $\mathrm{dist}(q, \partial\Omega) > \delta$. Then, for $k \ge 1$, having built $\mathcal{Q}_\ell$ for $\ell < k$, we define $\mathcal{Q}_k$ as the set of all the smaller cubes $q \in \{z + (0, \delta 2^{-k}]^n : z \in \delta 2^{-k}\mathbb{Z}^n\}$, $q \subset \mathbf{C}_i \cap \Omega$, such that $\mathrm{dist}(q, \partial\Omega) > \delta 2^{-k}$, and q does not intersect cubes of $\bigcup_{\ell<k}\bigcup_{\hat q\in\mathcal{Q}_\ell} \hat q$. Note that we can assume that C_0 is covered by cubes in the family $\mathcal{Q}_0$ only (by e.g. choosing δ sufficiently smaller than δ_0).

Finally, we let $\mathcal{Q} := \bigcup_{k=0}^\infty \mathcal{Q}_k$; note that $\bigcup_{q\in\mathcal{Q}} q = \mathbf{C}_i \cap \Omega \subset C_i \cap \Omega$ covers entirely $\operatorname{supp} \varphi_i \cap \Omega$. Now, for each $q \in \mathcal{Q}$, let q' and q'' denote cubes concentric with q, and with edge size 10% and 20% longer, respectively. Then the cubes q'' (as well as q'), for $q \in \mathcal{Q}$, form a sort of Whitney covering of $C_i \cap \Omega$, at least covering the support of $\varphi_i \cap \Omega$. Moreover, since for every $k \ge 0$ any $q \in \mathcal{Q}_k$ satisfies $\mathrm{dist}(q, \partial\Omega) > \delta 2^{-k}$, clearly also $q', q'' \subset \Omega$. Note that, for fixed $k \geq 0$, an enlarged cube q'' of some cube $q \in \mathcal{Q}_k$ can only intersect cubes belonging to $\mathcal{Q}_k$, $\mathcal{Q}_{k+1}$ and, if $k \geq 1$, $\mathcal{Q}_{k-1}$.

Next, we choose the constant $\bar{c}$ introduced at the start of the proof to be $\bar{c} := \tau(\delta/2)^{n-1}$, where τ is given by Theorem 3.2 (or, more precisely, by the version of Theorem 3.2 for a cube, following Remark 3.6), corresponding to $\sigma = 1/12$. Hence, by the initial assumption $\mathcal{H}^{n-1}(J_u) \leq \bar{c}$ we have

$$\mathcal{H}^{n-1}(J_u) \leq \tau(\delta/2)^{n-1}.$$

Then, by applying Theorem 3.2 (for instance with $\varepsilon = 1$) to $u \in GSBD^p(q'')$, for each $q \in \mathcal{Q}_0$, we find a function $w_q \in GSBD^p(q'')$ and a set of finite perimeter $\omega_q \subset q''$ such that $w_q = u$ in $q'' \backslash \omega_q$, $\int_{q''} |e(w_q)|^p dx \leq C \int_{q''} |e(u)|^p dx$, $\mathcal{H}^{n-1}(\partial^* \omega_q) \leq C\mathcal{H}^{n-1}(J_u \cap q'')$ and $w_q \in W^{1,p}(q'; \mathbb{R}^n)$.

For smaller cubes $q \in \mathcal{Q}_k$, $k \geq 1$, we proceed as follows: if $\mathcal{H}^{n-1}(J_u \cap q'') \leq \tau(\delta/2^{k+1})^{n-1}$, we say that q is "good", we apply Theorem 3.2 to the restriction of u to q'', and find w_q and ω_q as in the case $k = 0$ (note that all the cubes in $\mathcal{Q}_0$ are "good" and that, in particular, C_0 is covered by "good" cubes). In conclusion, for q "good", we find a function $w_q \in GSBD^p(q'')$ and a set of finite perimeter $\omega_q \subset q''$ such that $w_q = u$ in $q'' \setminus \omega_q$ and

$$w_q \in W^{1,p}(q'; \mathbb{R}^n), \tag{4.1}$$

$$\int_{q''} |e(w_q)|^p dx \leq C \int_{q''} |e(u)|^p dx, \tag{4.2}$$

$$\mathcal{H}^{n-1}(\partial^* \omega_q) \leq C\mathcal{H}^{n-1}(J_u \cap q''). \tag{4.3}$$

If instead $\mathcal{H}^{n-1}(J_u \cap q'') > \tau(\delta/2^{k+1})^{n-1}$, we say that q is "bad" and we define

$$\tilde{\omega}_q := \Omega \cap (q + \{x = (x', x_n) : x_n > 2\tilde{L}|x'|\}),$$

namely we connect q with $\partial\Omega$ via a sort of truncated cone with an opening controlled by the Lipschitz constant L of Ω. Scaling arguments (and the fact that $\mathrm{dist}(q, \partial\Omega) \leq \delta 2^{-k+1}$) show that $\mathcal{H}^{n-1}(\partial^* \tilde{\omega}_q) \leq c(\delta 2^{-k})^{n-1}$ where the constant $c = c(n, L)$ depends only on L and the dimension. It follows that in this case, namely for q "bad",

$$\mathcal{H}^{n-1}(\partial^* \tilde{\omega}_q) \leq c(n, L) \frac{2^{n-1}}{\tau} \mathcal{H}^{n-1}(J_u \cap q''). \tag{4.4}$$

We let $\tilde{\omega} := \bigcup_{q \in \mathcal{Q}_b} \tilde{\omega}_q$, $G := (\bigcup_{q \in \mathcal{Q}} q) \setminus \tilde{\omega}$, and $\hat{\omega} := \bigcup_{q \in \mathcal{Q}_g} (\omega_q \cap q')$, where we denoted with $\mathcal{Q}_b, \mathcal{Q}_g \subset \mathcal{Q}$ the "bad" and "good" cubes in $\mathcal{Q}$, respectively. By construction, there exists a $(2\tilde{L})$-Lipschitz function f such that G is the subgraph of f, with $g - 2\delta \leq f \leq g$. Moreover, for some constant c (depending on L, n and τ), and using that the cubes q'' have finite overlap, one has, by (4.3) and (4.4),

$$\mathcal{H}^{n-1}(\partial^* \tilde{\omega}) + \mathcal{H}^{n-1}(\partial^* \hat{\omega}) \leq c\mathcal{H}^{n-1}(J_u \cap (C_i \cap \Omega)).$$

Eventually, we define $\omega_i := \tilde{\omega} \cup \hat{\omega}$; the estimates above show that the perimeter of ω_i is bounded by the measure of the jump of u in $C_i \cap \Omega$.

We now construct a regularised function v_i as a convex combination of the functions w_q relative to "good" cubes $q \in \mathcal{Q}_g$ only. More precisely, let $\psi \in C_c^\infty((0, 1.1)^n; [0, 1])$ be a smooth cut-off function with $\psi = 1$ on $[0, 1]^n$. For any $k \geq 0$ and any $q \in \mathcal{Q}_g \cap \mathcal{Q}_k$ with centre c_q, we define the translated and rescaled version of ψ_q, $\psi_q(x) := \psi((x - c_q)/(\delta 2^{-k})) \in C_c^\infty(q'; [0, 1])$, so that $\psi_q = 1$ on q. Finally, we define the 'normalised' cut-off function $\varphi_q(x) := \psi_q(x)/(\sum_{\hat{q} \in \mathcal{Q}_g} \psi_{\hat{q}}(x))$ for $x \in \cup_{q \in \mathcal{Q}_g} q$.

We then let, for $x \in \cup_{q \in \mathcal{Q}_g} q$, $\tilde{v}_i(x) := \sum_{q \in \mathcal{Q}_g} w_q(x) \varphi_q(x)$. First of all, we extend $\tilde{v}_{i|G}$ from G to $\mathbf{C}_i \cap \Omega$. This can be done, for instance, by following the procedure in [24, Lemma 4], since G is a special Lipschitz set (according to [24, property (49)]) and $\tilde{v}_{i|G} \in W^{1,p}(G;\mathbb{R}^n)$, as each w_q belongs to $W^{1,p}(q';\mathbb{R}^n)$ for $q \in \mathcal{Q}_g$, by (4.1). We denote this extension by v_i. Then $v_i \in W^{1,p}(\mathbf{C}_i \cap \Omega;\mathbb{R}^n)$, $v_i = \tilde{v}_i$ in G, and

$$\int_{\mathbf{C}_i\cap\Omega} |e(v_i)|^p dx \le c \int_G |e(\tilde{v}_i)|^p dx \le c \int_{\cup_{q\in\mathcal{Q}_g} q} |e(\tilde{v}_i)|^p dx, \tag{4.5}$$

with the constant c depends only on the dimension n, on p, and on the Lipschitz constant of G (namely of f), which is $2L$, hence $c = c(n,p,L)$.

Moreover, $v_i = u$ in $(\mathbf{C}_i \cap \Omega) \setminus \omega_i$. Indeed, $\tilde{v}_{i|G} = u$ in $G \setminus \hat{\omega}$ by construction, $v_i = \tilde{v}_i$ in G, and $G = (\mathbf{C}_i \cap \Omega) \setminus \tilde{\omega}$.

It remains to show that $\int_{\mathbf{C}_i\cap\Omega} |e(v_i)|^p dx \le c \int_{C_i\cap\Omega} |e(u)|^p dx$. By (4.5), it is sufficient to show that $\int_{\cup_{q\in\mathcal{Q}_g} q} |e(\tilde{v}_i)|^p dx \le c \int_{C_i\cap\Omega} |e(u)|^p dx$. By the definition of $\tilde{v}_i$, one has

$$e(\tilde{v}_i) = \sum_{q\in\mathcal{Q}_g} \big(e(w_q)\varphi_q + w_q \odot \nabla\varphi_q\big). \tag{4.6}$$

We need therefore to estimate the L^p norm of $\sum_{q\in\mathcal{Q}_g} w_q \odot \nabla\varphi_q$ in terms of the L^p norm of $e(u)$, since the other term in the sum satisfies the bound by (4.2). Notice that as $\sum_q \varphi_q \equiv 1$ in $\cup_{q\in\mathcal{Q}_g} q$, we have that $\sum_q \nabla\varphi_q = 0$ in $\cup_{q\in\mathcal{Q}_g} q$ (where here and in what follows the sums run on cubes in $\mathcal{Q}_g$). Then, if we fix $q \in \mathcal{Q}_g$ and $x \in q' \cap (\cup_{\hat{q}\in\mathcal{Q}_g} \hat{q})$, we have

$$\begin{aligned}\sum_{\hat{q}} w_{\hat{q}}(x) \odot \nabla\varphi_{\hat{q}}(x) &= \sum_{\hat{q}} w_{\hat{q}}(x) \odot \nabla\varphi_{\hat{q}}(x) - w_q(x) \odot \sum_{\hat{q}} \nabla\varphi_{\hat{q}}(x)\\ &= \sum_{\hat{q}} (w_{\hat{q}}(x) - w_q(x)) \odot \nabla\varphi_{\hat{q}}(x)\\ &= \sum_{\hat{q}: q'\cap\hat{q}'\neq\emptyset} (w_{\hat{q}}(x) - w_q(x)) \odot \nabla\varphi_{\hat{q}}(x). \end{aligned}\tag{4.7}$$

Note that the last equality in (4.7) follows since the only terms in the sum that have a non-zero contribution are the ones corresponding to cubes $\hat{q}$ such that $\hat{q}'$ intersects q', whose number is bounded by 2^n.

Now we observe that, if $q' \cap \hat{q}' \neq \emptyset$, then there are two cases: either q and $\hat{q}$ are of the same size, or, alternatively, the edge length of one is twice the edge length of the other one. In either case

$$\mathcal{L}^n(q' \cap \hat{q}') \ge \beta_1 \max\{\mathcal{L}^n(q'), \mathcal{L}^n(\hat{q}')\},$$

where $\beta_1 > 0$ is an explicit constant depending on n and on the choice of σ only. Now, to fix the ideas, assume that $q \in \mathcal{Q}_k$ and $\hat{q} \in \mathcal{Q}_{k+1}$; then, by Remark 3.4 and (4.3), and since $q, \hat{q} \in \mathcal{Q}_g$,

$$\begin{aligned}\mathcal{L}^n(\omega_q \cup \omega_{\hat{q}}) &\le c(n)\delta 2^{-k} \left(\mathcal{H}^{n-1}(\partial^*\omega_q) + \mathcal{H}^{n-1}(\partial^*\omega_{\hat{q}})\right)\\ &\le c(n)C\delta 2^{-k} \left(\mathcal{H}^{n-1}(J_u \cap q'') + \mathcal{H}^{n-1}(J_u \cap \hat{q}'')\right)\\ &\le c(n,\sigma,p)\,\tau \left(\frac{\delta}{2^{k+1}}\right)^n \le c(n,\sigma,p)\,\tau\mathcal{L}^n(q' \cap \hat{q}'),\end{aligned}$$

where $c(n,\sigma,p)$ denotes possibly different constants. In particular, for every $q, \hat{q} \in \mathcal{Q}_g$ with $q' \cap \hat{q}' \neq \emptyset$,

$$\mathcal{L}^n(\omega_q \cup \omega_{\hat{q}}) \le c(n,\sigma,p)\tau\mathcal{L}^n(q' \cap \hat{q}').$$

Hence, up to reducing our choice of σ and consequently τ, we have that

$$\mathcal{L}^n((q' \cap \hat{q}') \setminus (\omega_q \cup \omega_{\hat{q}})) \ge \beta_2\mathcal{L}^n(q' \cap \hat{q}') \ge \beta_1\beta_2 \max\{\mathcal{L}^n(q'), \mathcal{L}^n(\hat{q}')\},$$

for some $\beta_2 > 0$ depending on n, p, and σ. Note that reducing σ has no effect on the size of q', but only on the size of $q'' \setminus q'$ (that gets smaller). Indeed, if we denote with ℓ, ℓ' and ℓ'' the lengths of the edges of q, q' and q'', respectively, and we keep $\ell' = \frac{11}{10}\ell$, then if $\ell'' = a(\sigma)\ell$, with $a(\sigma) \geq \frac{11}{10(1-\sigma)}$, we can still guarantee that $(1-\sigma)\ell'' \geq \ell'$, and hence that $w_q \in W^{1,p}(q';\mathbb{R}^2)$.

We now apply Lemma 4.3 to w_q in q' and to $w_{\hat{q}}$ in $\hat{q}'$, with $E = (q' \cap \hat{q}') \setminus (\omega_q \cup \omega_{\hat{q}})$ and $\alpha = \beta_1\beta_2$. Note that the constant c in the lemma scales with the size of the domain; more precisely, for a dyadic cube q' with side length ℓ', $c = c(n,\alpha,p)(\ell')^p$, with $c(n,\alpha,p)$ being the constant for the unit cube in $\mathbb{R}^n$. Then

$$\int_{q'} |w_q - a_q|^p dx \leq c(\ell')^p \int_{q'} |e(w_q)|^p dx, \tag{4.8}$$

$$\int_{\hat{q}'} |w_{\hat{q}} - a_{\hat{q}}|^p dx \leq c(\hat{\ell}')^p \int_{\hat{q}'} |e(w_{\hat{q}})|^p dx, \tag{4.9}$$

where $c = c(n,\alpha,p)$ is the constant for the unit cube in $\mathbb{R}^n$, and

$$a_q := \operatorname*{arg\,min}_{a\in\mathcal{R}} \int_E |w_q - a|^p dx, \quad a_{\hat{q}} := \operatorname*{arg\,min}_{a\in\mathcal{R}} \int_E |w_{\hat{q}} - a|^p dx.$$

On the other hand, since $w_q = w_{\hat{q}} = u$ in E, we have that $a_q = a_{\hat{q}} = a$, and hence, thanks to (4.8)-(4.9) and (4.2),

$$\begin{aligned}\int_{q'\cap\hat{q}'} |w_q - w_{\hat{q}}|^p dx &\leq c(p)\int_{q'\cap\hat{q}'} |w_q - a|^p dx + c(p)\int_{q'\cap\hat{q}'} |w_{\hat{q}} - a|^p dx \\ &\leq c(p)c\left((\ell')^p \int_{q'} |e(w_q)|^p dx + (\hat{\ell}')^p \int_{\hat{q}'} |e(w_{\hat{q}})|^p dx\right) \\ &\leq c(p)c(\ell)^p \left(\int_{q''} |e(u)|^p dx + \int_{\hat{q}''} |e(u)|^p dx\right).\end{aligned}$$

In conclusion, for a given $q \in \mathcal{Q}_g$, by (4.6), (4.7), (4.2) and the previous estimate,

$$\begin{aligned}\int_q |e(\tilde{v}_i)|^p dx &\leq c \sum_{\hat{q}:q'\cap\hat{q}'\neq\emptyset} \int_{q\cap\hat{q}'} |e(w_{\hat{q}})|^p dx \\ &\quad + c \sum_{\hat{q}:q'\cap\hat{q}'\neq\emptyset} \|\nabla\varphi_{\hat{q}}\|^p_{L^\infty(\hat{q}')} \int_{q\cap\hat{q}'} |w_q - w_{\hat{q}}|^p dx \\ &\leq c \sum_{\hat{q}:q'\cap\hat{q}'\neq\emptyset} \int_{\hat{q}''} |e(u)|^p dx,\end{aligned}$$

since $\|\nabla\varphi_{\hat{q}}\|_{L^\infty(\hat{q}')} \leq c(1/\hat{\ell})$. Using that the cubes q'' have finite overlap, we have

$$\int_{\cup_{q\in\mathcal{Q}_g} q} |e(\tilde{v}_i)|^p dx \leq c \int_{\mathbf{C}_i\cap\Omega} |e(u)|^p dx,$$

and, by (4.5),

$$\int_{\mathbf{C}_i\cap\Omega} |e(v_i)|^p dx \leq c \int_{\mathbf{C}_i\cap\Omega} |e(u)|^p dx.$$

This concludes the proof.

□

An immediate consequence of Theorem 4.1 is the Korn's inequality below, whose proof is a direct adaptation of [14, Corollary 3.3].

Theorem 4.4. *Under the same assumptions and notation of Theorem 4.1, and under the additional requirement that Ω is connected, there exists an affine function $a \in \mathcal{R}$ such that*

$$\int_{\Omega\setminus\omega} |\nabla u - \nabla a|^p dx \leq c(n,p,\Omega) \int_\Omega |e(u)|^p dx. \tag{4.10}$$

Moreover,

$$\left(\int_{\Omega\setminus\omega}|u-a|^q dx\right)^{\frac{1}{q}} \le c(n,p,q,\Omega)\left(\int_\Omega |e(u)|^p dx\right)^{\frac{1}{p}}, \tag{4.11}$$

where $q\in(1,p^*]$ *for* $p<n$ *and* $q\in(1,+\infty)$ *for* $p\ge n$.

Proof. By Korn's inequality applied to $v\in W^{1,p}(\Omega;\mathbb{R}^n)$, there exists $A\in\mathbb{R}^{n\times n}_{\rm skw}$ such that

$$\int_\Omega |\nabla v - A|^p dx \le c(n,p,\Omega)\int_\Omega |e(v)|^p dx;$$

moreover, by applying Poincaré's inequality to the function $x\mapsto v(x)-Ax$, there exists $b\in\mathbb{R}^n$ such that

$$\int_\Omega |v(x)-Ax-b|^p dx \le c(n,p,\Omega)\int_\Omega |\nabla v - A|^p dx. \tag{4.12}$$

We now define $a(x):=Ax+b$; then $a\in\mathcal{R}$. Since $\nabla v=\nabla u$ $\mathcal{L}^n$-a.e. on $\{v=u\}$, we have that

$$\begin{aligned}\int_{\Omega\setminus\omega}|\nabla u-\nabla a|^p dx &= \int_{\Omega\setminus\omega}|\nabla v-\nabla a|^p dx\\ &\le c(n,p,\Omega)\int_\Omega |e(v)|^p dx \le \tilde c(n,p,\Omega)\int_\Omega |e(u)|^p dx,\end{aligned}$$

where the last inequality follows by Theorem 4.1. This proves (4.10). Moreover, we can improve the norm on the left-hand side of (4.12) to the exponent q of the Sobolev embedding of $W^{1,p}$ into L^q. Then, since $v=u$ in $\Omega\setminus\omega$, we have that

$$\begin{aligned}\int_{\Omega\setminus\omega}|u-a|^q dx &= \int_{\Omega\setminus\omega}|v-a|^q dx\\ &\le c(n,p,q,\Omega)\int_\Omega |e(v)|^p dx \le \tilde c(n,p,q,\Omega)\int_\Omega |e(u)|^p dx,\end{aligned}$$

which proves the estimate (4.11). Note that, if $p<n$, we can take $q=p^*$, and that if $p>n$ we can estimate $v-a$ in the Hölder seminorm $C^{0,\alpha}$, with $\alpha=1-\frac{n}{p}$. □

5. An approximation result

In this last section, as an application, we show an approximation result in the spirit of [10, Theorem 3.1].

Theorem 5.1. *Let* $n\in\mathbb{N}$ *with* $n\ge 2$, $p\in(1,\infty)$, *and let* $\Omega\subset\mathbb{R}^n$ *be a bounded open set of finite perimeter. Let* $\varepsilon>0$. *Then, for any* $u\in GSBD^p(\Omega)$, *there exist*

- *a closed set* Γ, *finite union of disjoint* $(n-1)$*-dimensional* C^1 *manifolds with* C^1 *boundary;*
- *a set* $\tilde\omega$, *finite union of cubes;*
- *a set of finite perimeter* $\hat\omega$;

such that

$$\mathcal{H}^{n-1}(J_u\triangle\Gamma)+\mathcal{H}^{n-1}(\partial^*\tilde\omega)+\mathcal{H}^{n-1}(\partial^*\hat\omega)+\mathcal{L}^n(\tilde\omega\cup\hat\omega)<\varepsilon.$$

Moreover, there exists a function $w\in GSBD^p(\Omega)\cap W^{1,p}(\Omega\setminus(\Gamma\cup\overline{\tilde\omega});\mathbb{R}^n)$ *such that* $\{w\ne u\}\subset\tilde\omega\cup\hat\omega$,

$$\int_{\Omega\setminus\tilde\omega}|e(w)|^p\,dx\le(1+\varepsilon)\int_\Omega |e(u)|^p dx,$$

and

$$\mathcal{H}^{n-1}(\Gamma\cap\{w^\pm\ne u^\pm\})<\varepsilon.$$

Corollary 5.2. *Under the same assumptions and notation of Theorem 5.1, for $u \in GSBD^p(\Omega)$ the approximate gradient ∇u exists $\mathcal{L}^n$-a.e. in Ω.*

Note that in the case $p = 2$ the result in Corollary 5.2 has been obtained in [23], as a consequence of the embedding $GSBD^2(\Omega) \subset (GBV(\Omega))^n$ (see [23, Theorem 2.9]), for $n \geq 2$.

Theorem 5.1 will follow as a special case of the following technical proposition.

Proposition 5.3. *Let $n \in \mathbb{N}$ with $n \geq 2$, and $p \in (1, \infty)$. Let $u \in GSBD^p(\mathbb{R}^n)$ and let J be a countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable set with $J_u \subset J$ and $\mathcal{H}^{n-1}(J) < +\infty$. Let $\varepsilon > 0$. Then there exist*

- *a closed set Γ, finite union of disjoint $(n-1)$-dimensional C^1 manifolds with C^1 boundary;*
- *a set $\tilde{\omega}$, finite union of cubes;*
- *a set of finite perimeter $\hat{\omega}$;*
- *a function $w \in GSBD^p(\mathbb{R}^n)$, with $w \in W^{1,p}(B_R(0) \setminus (\Gamma \cup \overline{\tilde{\omega}}); \mathbb{R}^n)$ for any $R > 0$;*

such that $w = u$ $\mathcal{L}^n$-a.e. in $\mathbb{R}^n \setminus (\tilde{\omega} \cup \hat{\omega})$, and

$$\mathcal{H}^{n-1}(J \triangle \Gamma) \leq \varepsilon\,,$$
$$\int_{\mathbb{R}^n \setminus \tilde{\omega}} |e(w)|^p dx \leq (1+\varepsilon) \int_{\mathbb{R}^n} |e(u)|^p dx\,,$$
$$\mathcal{H}^{n-1}(\partial^* \tilde{\omega}) + \mathcal{H}^{n-1}(\partial^* \hat{\omega}) + \mathcal{L}^n(\tilde{\omega} \cup \hat{\omega}) \leq \varepsilon\,.$$

Moreover, $u^\pm(x) = w^\pm(x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma \setminus (\overline{\tilde{\omega}} \cup \hat{\omega}^{(1)} \cup \partial^* \hat{\omega})$, and $\mathcal{H}^{n-1}(\Gamma \cap \{w^\pm \neq u^\pm\}) < \varepsilon$.

We recall that, for $u \in GSBD(\mathbb{R}^n)$, the set J_u is countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable [17, Section 6] (see [20, Section 3.2.14] for the definition), so that the assumption $J_u \subset J$ is not restrictive.

Theorem 5.1 is deduced from Proposition 5.3 in the following way. Let $\Omega \subset \mathbb{R}^n$ and $u \in GSBD^p(\Omega)$ as in the assumptions of Theorem 5.1, and let $\tilde{u}$ denote the extension of u to $\mathbb{R}^n$ obtained by setting $\tilde{u} := 0$ outside Ω. Then $\tilde{u} \in GSBD^p(\mathbb{R}^n)$, and by applying Proposition 5.3 to $\tilde{u}$ and $J = J_{\tilde{u}}$ we obtain the claim.

Proof of Proposition 5.3. Let u, J and ε be as in the statement, and let $\rho > 0$ and $\alpha > 0$ be constants to be determined later. We split the proof into several steps.
Step 1: Covering the jump set. Since J is countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable and $\mathcal{H}^{n-1}(J) < +\infty$, by [20, Theorem 3.2.29] there exists a countable family $(M_k)_{k\in\mathbb{N}}$ of C^1 hypersurfaces such that

$$\mathcal{H}^{n-1}\left(J \setminus \bigcup_{k=1}^{\infty} M_k\right) = 0.$$

With no loss of generality we can assume that for each $k \in \mathbb{N}$ the manifold M_k is a Lipschitz graph with Lipschitz constant less than 1/2 [2, Theorem 2.76]. Then, for every $k \in \mathbb{N}$, $\mathcal{H}^{n-1}$-a.e. point in $J \cap M_k$ is a point of $\mathcal{H}^{n-1}$-density 1 both for J and $J \cap M_k$, namely

$$\lim_{r\to 0^+} \frac{\mathcal{H}^{n-1}(J \cap \overline{B}_r(x))}{\gamma_{n-1} r^{n-1}} = \lim_{r\to 0^+} \frac{\mathcal{H}^{n-1}((J \cap M_k) \cap \overline{B}_r(x))}{\gamma_{n-1} r^{n-1}} = 1,$$

for every $k \in \mathbb{N}$ and $\mathcal{H}^{n-1}$-a.e. $x \in J \cap M_k$. From this, it follows that for every $k \in \mathbb{N}$ and for $\mathcal{H}^{n-1}$-a.e. $x \in J \cap M_k$ there exists $\eta(\alpha, x) \in (0, \rho)$ such that

$$|\mathcal{H}^{n-1}(\overline{B}_r(x) \cap J) - \gamma_{n-1} r^{n-1}| \leq \alpha \gamma_{n-1} r^{n-1},$$
$$|\mathcal{H}^{n-1}(\overline{B}_r(x) \cap (J \cap M_k)) - \gamma_{n-1} r^{n-1}| \leq \alpha \gamma_{n-1} r^{n-1},$$

and

$$\mathcal{H}^{n-1}(\overline{B}_r(x)\cap(J\Delta M_k))\leq\alpha\mathcal{H}^{n-1}(\overline{B}_r(x)\cap J),$$

for every $r\leq\eta(\alpha,x)$. In other words, up to sufficiently restricting the radius of the ball, we can assume that the main content of J in a ball centred at a point $x\in J\cap M_k$ comes from M_k, and not from the other components M_j, for $j\neq k$.

Let $M:=J\cap\cup_k M_k$. Note that the family $\left\{\overline{B}_r(x):x\in M, r\leq\eta(\alpha,x)\right\}$ is a fine cover of M (see [2, Section 2.4]). Then, applying the Vitali-Besicovitch's Covering Theorem [2, Theorem 2.19] to $A=M$ and $\mu=\mathcal{H}^{n-1}\llcorner M$, there exists a *disjoint* subfamily $\left\{\overline{B}_{r(\alpha,x)}(x):x\in M'\right\}$, for some $M'\subset M$ and $r(\alpha,x)\leq\eta(\alpha,x)$, such that

$$\mathcal{H}^{n-1}\left(J\setminus\bigcup_{x\in M'}\overline{B}_{r(\alpha,x)}(x)\right)=0.$$

Moreover, the subfamily above is countable, since it is composed of disjoint sets with nonempty interior. Hence, there exists a sequence $\{x_i\}_{i\in\mathbb{N}}\subset\cup_k M_k$ such that

$$\mathcal{H}^{n-1}\left(J\setminus\bigcup_{i\in\mathbb{N}}\overline{B}_i\right)=0,$$

where $B_i:=B_{r_i}(x_i)$ for every $i\in\mathbb{N}$, and where we set $r_i:=r(\alpha,x_i)$. Finally, note that from the identity above it follows that there exists $N=N(\alpha)\in\mathbb{N}$ such that

$$\mathcal{H}^{n-1}\left(J\setminus\bigcup_{i=1}^{N}\overline{B}_i\right)<\alpha. \tag{5.1}$$

Given $i\in\{1,\dots,N(\alpha)\}$, let $k(i)\in\mathbb{N}$ be such that $x_i\in M_{k(i)}$ and define $\Gamma_i:=M_{k(i)}\cap\overline{B}_i$. Then $B_i\setminus\Gamma_i$ has two (Lipschitz) connected components, and the following properties are satisfied:

a) Γ_i is a Lipschitz graph with constant less than $1/2$;

b) $|\mathcal{H}^{n-1}(\overline{B}_r(x_i)\cap J)-\gamma_{n-1}r^{n-1}|\leq\alpha\gamma_{n-1}r^{n-1}$ for all $r\leq r_i$, where γ_{n-1} is as in (l);

c) $\mathcal{H}^{n-1}(\overline{B}_i\cap(J\triangle\Gamma_i))\leq\alpha\mathcal{H}^{n-1}(\overline{B}_i\cap J)$;

d) $\mathcal{H}^{n-1}\left(J\setminus\bigcup_{i=1}^{N}\Gamma_i\right)\leq\alpha(1+\mathcal{H}^{n-1}(J))$;

e) $\mathcal{L}^n\left(\bigcup_{i=1}^{N}\overline{B}_i\right)\leq\frac{\gamma_n}{\gamma_{n-1}}\frac{\rho}{1-\alpha}\mathcal{H}^{n-1}(J)$.

Properties a), b), c) follow immediately. We now prove property d). First, note that

$$J\setminus\bigcup_{i=1}^{N}\Gamma_i=\left(J\setminus\bigcup_{i=1}^{N}\overline{B}_i\right)\cup\left(\bigcup_{i=1}^{N}J\cap(\overline{B}_i\setminus\Gamma_i)\right).$$

Hence, by (5.1) and by property c)

$$\begin{aligned}\mathcal{H}^{n-1}\left(J\setminus\bigcup_{i=1}^{N}\Gamma_i\right)&<\alpha+\mathcal{H}^{n-1}\left(\bigcup_{i=1}^{N}J\cap(\overline{B}_i\setminus\Gamma_i)\right)\\&\leq\alpha+\sum_{i=1}^{n}\mathcal{H}^{n-1}\left(J\cap(\overline{B}_i\setminus\Gamma_i)\right)\leq\alpha(1+\mathcal{H}^{n-1}(J)),\end{aligned}$$

which shows d). To see e) note that, since the closed balls are disjoint,

$$\begin{aligned}\mathcal{L}^n\Big(\bigcup_{i=1}^N B_i\Big) &= \sum_{i=1}^N \gamma_n r_i^n \le \frac{\gamma_n \rho}{\gamma_{n-1}}\sum_{i=1}^N \gamma_{n-1} r_i^{n-1}\\ &\le \frac{\gamma_n}{\gamma_{n-1}}\frac{\rho}{1-\alpha}\sum_{i=1}^N \mathcal{H}^{n-1}(\overline{B}_i\cap J) \le \frac{\gamma_n}{\gamma_{n-1}}\frac{\rho}{1-\alpha}\mathcal{H}^{n-1}(J),\end{aligned}$$

where we have also used b).

Finally, letting $\Gamma := \bigcup_{i=1}^N \Gamma_i$, one has that Γ is a finite union of disjoint C^1 manifolds with C^1 boundary. Moreover, thanks to c) and d),

$$\mathcal{H}^{n-1}(J\triangle\Gamma) \le \alpha(1+2\mathcal{H}^{n-1}(J)). \tag{5.2}$$

Step 2: Cleaning the jump set in the balls B_i. Let us denote B_i^+, B_i^- the connected components of $B_i\setminus\Gamma_i$. Thanks to Theorem 4.1, in each $B_i^\pm$, $i=1,\dots,N$, there exists a set of finite perimeter $\omega_i^\pm$ and a function $v_i^\pm \in W^{1,p}(B_i^\pm;\mathbb{R}^n)$ such that $v_i^\pm = u$ in $B_i^\pm\setminus\omega_i^\pm$, $\int_{B_i^\pm}|e(v_i^\pm)|^p dx \le c\int_{B_i^\pm}|e(u)|^p dx$, and $\mathcal{L}^n(\omega_i^\pm)+\mathcal{H}^{n-1}(\partial^*\omega_i^\pm) \le c\mathcal{H}^{n-1}(J\cap B_i^\pm)$. Note that, by Remark 4.2, the constant $c=c(n,p)>0$ is independent of i, α and N, since the Lipschitz constants of $B_i^\pm$ are uniformly bounded. Thanks to b) and c) in Step 1, we have that

$$\mathcal{H}^{n-1}(J\cap B_i^\pm) \le \mathcal{H}^{n-1}(\overline{B}_i\cap(J\triangle\Gamma_i)) \le \alpha\mathcal{H}^{n-1}(\overline{B}_i\cap J) \le \alpha(1+\alpha)\gamma_{n-1}r_i^{n-1},$$

and hence $\mathcal{L}^n(\omega_i^\pm)+\mathcal{H}^{n-1}(\partial^*\omega_i^\pm) \le c\,\alpha(1+\alpha)\gamma_{n-1}r_i^{n-1}$. (Note that, in the case where $\omega_i^\pm = B_i^\pm$, we can simply let $v_i^\pm = 0$; however by choosing $\alpha>0$ small enough we can assume with no loss of generality that this does not happen.)

It follows that on ∂B_i the trace of each $v_i^\pm$ coincides with the trace of u, except on a set of total measure at most $2c\,\alpha(1+\alpha)\gamma_{n-1}r_i^{n-1}$. Let now

$$v(x) := \begin{cases} v_i^\pm(x) & \text{if } x\in B_i^\pm\,, i=1,\dots,N\,,\\ u(x) & \text{if } x\in\mathbb{R}^n\setminus\bigcup_{i=1}^N B_i.\end{cases}$$

Then $v\in GSBD^p(\mathbb{R}^n)$, and we have the following properties:

1) $J_v\cap B_i \subset \Gamma_i$ for each $i=1,\dots,N$;
2) $\sum_{i=1}^N \mathcal{H}^{n-1}(J_v\cap\partial B_i) \le \alpha(1+2c)\mathcal{H}^{n-1}(J)$;
3) $\mathcal{H}^{n-1}(J_v\setminus\Gamma) \le \alpha(1+(1+2c)\mathcal{H}^{n-1}(J))$.

Property 1) follows from the definition of v. For property 2), note that by c)

$$\begin{aligned}\sum_{i=1}^N \mathcal{H}^{n-1}\left(J_v\cap\partial B_i\right) &\le \sum_{i=1}^N \mathcal{H}^{n-1}\left(J_u\cap\partial B_i\right)+\sum_{i=1}^N \mathcal{H}^{n-1}\left(\partial B_i\cap(\partial^*\omega_i^+\cup\partial^*\omega_i^-)\right)\\ &\le \sum_{i=1}^N \mathcal{H}^{n-1}\left(J\cap(\overline{B}_i\setminus\Gamma_i)\right)+\sum_{i=1}^N\left(\mathcal{H}^{n-1}(\partial^*\omega_i^+)+\mathcal{H}^{n-1}(\partial^*\omega_i^-)\right)\\ &\le \alpha\mathcal{H}^{n-1}(J)+\sum_{i=1}^N\left(\mathcal{H}^{n-1}(\partial^*\omega_i^+)+\mathcal{H}^{n-1}(\partial^*\omega_i^-)\right)\\ &\le \alpha\mathcal{H}^{n-1}(J)+2c\sum_{i=1}^N\left(\mathcal{H}^{n-1}(J\cap B_i^+)+\mathcal{H}^{n-1}(J\cap B_i^-)\right)\\ &\le \alpha\mathcal{H}^{n-1}(J)+2c\sum_{i=1}^N\left(\mathcal{H}^{n-1}((J\cap B_i)\setminus\Gamma_i)\right)\\ &\le \alpha(1+2c)\mathcal{H}^{n-1}(J).\end{aligned}$$

Let us show property 3). By (5.1), 1) and 2) we have that

$$\begin{aligned}\mathcal{H}^{n-1}\left(J_v\setminus\Gamma\right) &\le \mathcal{H}^{n-1}\left(J\setminus\bigcup_{i=1}^N\overline{B}_i\right)+\sum_{i=1}^N\mathcal{H}^{n-1}\left(J_v\cap\partial B_i\right)\\ &\le \alpha(1+(1+2c)\mathcal{H}^{n-1}(J)).\end{aligned}$$

Moreover, letting $\omega_B:=\bigcup_{i=1}^N(\omega_i^+\cup\omega_i^-)$, one has that $v=u$ $\mathcal{L}^n$-a.e. in $\mathbb{R}^n\setminus\omega_B$ and $\mathcal{L}^n(\omega_B)+\mathcal{H}^{n-1}(\partial^*\omega_B)\le 2c\,\alpha\mathcal{H}^{n-1}(J)$.

Step 3: Cleaning the jump set in the rest of the domain. We now pick $\delta>0$ with

$$0<\delta\le 0.8\alpha\Big(\min_{i=1,\dots,N} r_i\Big)/(2\sqrt{n}),$$

and consider the covering of $\mathbb{R}^n\setminus\bigcup_{i=1}^N B_i$ made of:

- the family $\mathcal{Q}_1$ of cubes $\delta z+[0,\delta]^n$, $z\in\mathbb{Z}^n$, which intersect $\mathbb{R}^n\setminus\bigcup_{i=1}^N B_i$;
- the family $\mathcal{Q}_2$ of cubes $\delta z+[0,\delta]^n$, $z\in\mathbb{Z}^n$, which are not in $\mathcal{Q}_1$, but intersect some cubes in $\mathcal{Q}_1$.

We set $\mathcal{Q}=\mathcal{Q}_1\cup\mathcal{Q}_2$. For each $q\in\mathcal{Q}$, we denote with $q\subset q'\subset q''$ the concentric cubes q' and q'' with edges $(9/8)\delta$ and $(10/8)\delta$, respectively; we also denote with $\ell<\ell'<\ell''$ the lengths of the edges of q, q' and q'', respectively, so that in particular, $\ell'=(1-0.1)\ell''$. For each i, letting $B_i':=B_{(1-\alpha)r_i}(x_i)$, we observe that, since Γ_i are equi-Lipschitz with constant less than $\frac{1}{2}$, one has

$$\mathcal{H}^{n-1}\big(\Gamma_i\cap(B_i\setminus B_i')\big)\le c\,\alpha r_i^{n-1}\le c\,\alpha\mathcal{H}^{n-1}(J\cap\overline{B}_i) \tag{5.3}$$

for some constant $c=c(n)$, where in the last inequality we used property b). Hence, since by the definition of δ we have that $q''\cap B_i'=\emptyset$ for each $q\in\mathcal{Q}$ and for every i, we have that

$$J_v\cap\bigcup_{q\in\mathcal{Q}}q''=\left(J_v\setminus\left(\bigcup_{i=1}^N\overline{B}_i\right)\right)\cup\left(\bigcup_{i=1}^N J_v\cap(B_i\setminus B_i')\right)\cup\left(\bigcup_{i=1}^N(J_v\cap\partial B_i)\right).$$

Then, recalling 1), and using (5.1), (5.3) and 2), we have

$$\mathcal{H}^{n-1}\left(J_v\cap\bigcup_{q\in\mathcal{Q}}q''\right)\le\alpha(1+(1+3c)\mathcal{H}^{n-1}(J)), \tag{5.4}$$

for a constant c depending only on the dimension.

We now invoke Theorem 3.2 (in its version for cubes, as noted in Remark 3.6) for parameters $\varepsilon = 1$ (which thus needs not be the ε of the statement), and $\sigma = 0.1$, and find constants $C = C(n, p)$ and $\tau = \tau(n, p)$ satisfying the thesis of the theorem.

Let $\mathcal{Q}_g \subset \mathcal{Q}$ denote the set of cubes q such that $\mathcal{H}^{n-1}(J_v \cap q'') \le \tau\delta^{n-1}$, let $\mathcal{Q}_b := \mathcal{Q} \setminus \mathcal{Q}_g$, and $\tilde{\omega} := \bigcup_{q\in\mathcal{Q}_b} q$. Since for $q \in \mathcal{Q}_b$ one has $\mathcal{H}^{n-1}(J_v \cap q'') > \tau\delta^{n-1}$, there can be only a finite number of such cubes. Moreover, thanks to (5.4) we have that

$$\begin{cases}\mathcal{H}^{n-1}(\partial^*\tilde{\omega}) \le \frac{c(n)}{\tau}\alpha(1 + (1 + 3c)\mathcal{H}^{n-1}(J)),\\ \mathcal{L}^n(\tilde{\omega}) \le \frac{c(n)}{\tau}\,\delta\,\alpha(1 + (1 + 3c)\mathcal{H}^{n-1}(J)).\end{cases} \tag{5.5}$$

Now, let $q \in \mathcal{Q}_g$. By Theorem 3.2 there exist $w_q \in GSBD^p(q'') \cap W^{1,p}(q'; \mathbb{R}^n)$ and $\omega_q \subset q''$, with $w_q = v$ in $q'' \setminus \omega_q$, and

$$\int_{q''} |e(w_q)|^p dx \le 2 \int_{q''} |e(v)|^p dx,$$

$$\int_{\omega_q} |e(w_q)|^p dx \le 2 \int_{\omega_q} |e(v)|^p dx, \tag{5.6}$$

$$\mathcal{H}^{n-1}(\partial^*\omega_q) \le C\mathcal{H}^{n-1}(J_v \cap q''), \tag{5.7}$$

where (5.6) follows by Remark 3.5.

Possibly reducing τ, we may assume that if $q' \cap \Gamma_i \ne \emptyset$ for some $i = 1, \dots, N$, then $\mathcal{H}^{n-1}(\Gamma_i \cap q'') \ge \tau\delta^{n-1}$ (see point a) in Step 1), so that $q \notin \mathcal{Q}_g$. It then follows that for any $q \in \mathcal{Q}_g$, when $q' \subset B_i$ for some i (or more precisely $q' \subset B_i^{\pm}$, since $q \in \mathcal{Q}_g$ is such that q' does not intersect Γ_i), then $w_q = v$ in q'.

We now 'glue' the functions w_q in order to find a global $W^{1,p}_{\mathrm{loc}}$ function as in the claim of the theorem. To do so, we introduce a cut-off function $\psi \in C_c^\infty((-9/16, 9/16)^n; [0, 1])$ with $\eta = 1$ on $[-1/2, 1/2]^n$. Then for each $q \in \mathcal{Q}_g$, with center c_q, we define $\psi_q(x) := \psi((x - c_q)/\delta) \in C_c^\infty(q'; [0, 1])$, so that $\psi_q = 1$ on q. We then let, for $x \in G := \bigcup_{q\in\mathcal{Q}_g} q$, $\varphi_q(x) := \psi_q(x)/(\sum_{\hat{q}\in\mathcal{Q}_g} \psi_{\hat{q}}(x)) \in [0, 1]$, and

$$w(x) := \begin{cases}\sum_{q\in\mathcal{Q}_g} w_q(x)\varphi_q(x) & \text{if } x \in G\,,\\ 0 & \text{if } x \in \tilde{\omega}\,,\\ v(x) & \text{if } x \in \mathbb{R}^n \setminus (G \cup \tilde{\omega}).\end{cases} \tag{5.8}$$

By construction we have that $w \in W^{1,p}(B_R(0) \setminus (\Gamma \cup \overline{\tilde{\omega}}); \mathbb{R}^n)$ for any $R > 0$. Indeed, we observe that $\mathbb{R}^n \setminus (G \cup \tilde{\omega}) \subset \bigcup_i B_i$, and hence (by the definition of v), in this set the function w is Sobolev outside Γ. Moreover, w does not jump on the intersection between the boundaries of G and $\mathbb{R}^n \setminus (G \cup \tilde{\omega})$. Indeed, if $q \in \mathcal{Q}_g$ is any cube touching the set $\mathbb{R}^n \setminus (G \cup \tilde{\omega})$, then it has to be that $q \in \mathcal{Q}_2$ and $q \subset B_i$ for some i (and therefore, as observed before, $w_q = v$ in q').

Let $\omega_G := \bigcup_{q\in\mathcal{Q}_g} \omega_q$ and $\hat{\omega} := \omega_B \cup \omega_G$. Then, $w = u$ in $\mathbb{R}^n \setminus (\tilde{\omega} \cup \hat{\omega})$, since $w = v$ outside $\omega_G \cup \tilde{\omega}$, and $v = u$ outside ω_B. Hence $e(w) = e(u)$ in that set.

Step 3.1: Traces of w on Γ. We now compare the traces of w and of u on the two sides of Γ. We have already observed that $w = u$ in $\mathbb{R}^n \setminus (\tilde{\omega} \cup \hat{\omega})$, where $\hat{\omega} = \omega_B \cup \omega_G$.

Note that, since $q'' \cap B_i' = \emptyset$ for every $q \in \mathcal{Q}$ and for every $i = 1, \dots, N$, $(\overline{\tilde{\omega}} \cup \overline{\omega_G}) \cap \bigcup_i B_i' = \emptyset$. Hence the exceptional sets $\tilde{\omega}$ and ω_G affect the traces of w only on a subset of Γ of small (in terms of α) $\mathcal{H}^{n-1}$-measure, by (5.3). For the set ω_B we observe that, by the definition of v, for every $i = 1, \dots, N$, $v^{\pm}(x) = u^{\pm}(x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma_i \setminus (\partial^*\omega_i^+ \cup \partial^*\omega_i^-)$. Hence

$$v^{\pm}(x) = u^{\pm}(x) \text{ for } \mathcal{H}^{n-1}\text{- a.e. } x \in \Gamma \setminus \bigcup_{i=1}^{N} (\partial^*\omega_i^+ \cup \partial^*\omega_i^-).$$

By the estimates of $\mathcal{H}^{n-1}(\partial^*\omega_i^{\pm})$ at the end of Step 2 it follows that the traces of u and v on the two sides of Γ can only differ on a small (in terms of α) portion of Γ.

In conclusion, $w^{\pm} = u^{\pm}$ in Γ, up to a set of small (in terms of α) $\mathcal{H}^{n-1}$ measure. More precisely,

$$w^{\pm}(x) = u^{\pm}(x) \text{ for } \mathcal{H}^{n-1}\text{- a.e. } x \in \Gamma \setminus \left(\overline{\tilde{\omega}} \cup \hat{\omega}^{(1)} \cup \partial^*\hat{\omega}\right).$$

Step 3.2: Estimate of $\hat{\omega}$. We now estimate the exceptional set $\hat{\omega}$, both in perimeter and in volume. Note that, by (5.7) and (5.4), $\mathcal{H}^{n-1}(\partial^*\omega_G) \leq C\sum_{q\in\mathcal{Q}_g}\mathcal{H}^{n-1}(J_v \cap q'') \leq c\alpha(1+\mathcal{H}^{n-1}(J))$, for some constant $c = c(n,p)$. We also remark that for each q, one has $\mathcal{L}^n(\omega_q) \leq c\delta\mathcal{H}^{n-1}(\partial^*\omega_q)$ for a dimensional constant c, so in particular

$$\mathcal{L}^n(\omega_G) \leq c\delta\alpha(1+\mathcal{H}^{n-1}(J)), \tag{5.9}$$

with $c = c(n,p)$.

Combining these estimates with the bounds on $\mathcal{H}^{n-1}(\partial^*\omega_B)$ and on $\mathcal{L}^n(\omega_B)$ at the end of Step 2, we have that

$$\mathcal{H}^{n-1}(\partial^*\hat{\omega}) + \mathcal{L}^n(\hat{\omega}) \leq c\alpha(1+\mathcal{H}^{n-1}(J)), \tag{5.10}$$

for a constant $c = c(n,p)$.

Step 3.3: L^p-estimate of $e(w)$. We start by estimating $\int_{\omega_G}|e(w)|^p dx$. From (5.8) we have, for $x \in G$,

$$e(w)(x) = \sum_{q\in\mathcal{Q}_g}\big(e(w_q)(x)\varphi_q(x) + w_q(x)\odot\nabla\varphi_q(x)\big). \tag{5.11}$$

Note that, since the cubes q' have finite overlap, the sum in the right-hand side of (5.11) is done, at each point, over a uniformly bounded number of terms, depending on the dimension.

We estimate the L^p norm of the two terms of the sum in (5.11) separately. For the first term we have that

$$\begin{aligned}\int_{\omega_G}\Big|\sum_{q\in\mathcal{Q}_g}e(w_q)(x)\varphi_q(x)\Big|^p dx &\leq \sum_{\hat{q}\in\mathcal{Q}_g}\int_{\omega_{\hat{q}}}\Big|\sum_{q\in\mathcal{Q}_g}e(w_q)(x)\varphi_q(x)\Big|^p dx\\ &\leq c\sum_{\hat{q}\in\mathcal{Q}_g}\sum_{\substack{q\in\mathcal{Q}_g\\ q'\cap\hat{q}'\neq\emptyset}}\int_{\omega_{\hat{q}}\cap q'}|e(w_q)(x)|^p dx.\end{aligned} \tag{5.12}$$

For fixed $\hat{q}$ and q with $q'\cap\hat{q}'\neq\emptyset$ we estimate

$$\begin{aligned}\int_{\omega_{\hat{q}}\cap q'}|e(w_q)(x)|^p dx &= \int_{(\omega_{\hat{q}}\cap\omega_q)\cap q'}|e(w_q)(x)|^p dx + \int_{(\omega_{\hat{q}}\setminus\omega_q)\cap q'}|e(v)(x)|^p dx\\ &\leq \int_{\omega_q}|e(w_q)(x)|^p dx + \int_{\omega_{\hat{q}}}|e(v)(x)|^p dx\\ &\leq 2\int_{\omega_q}|e(v)(x)|^p dx + \int_{\omega_{\hat{q}}}|e(v)(x)|^p dx,\end{aligned}$$

where in the last step we used (5.6). Since the cubes q' have finite overlap, from (5.12) we conclude that

$$\int_{\omega_G}\Big|\sum_{q\in\mathcal{Q}_g}e(w_q)(x)\varphi_q(x)\Big|^p dx \leq c\int_{\omega_G}|e(v)|^p\, dx. \tag{5.13}$$

Therefore,

$$\int_{\omega_G}\Big|\sum_{q\in\mathcal{Q}_g} e(w_q)(x)\varphi_q(x)\Big|^p dx \le c\int_{\omega_G}|e(v)|^p\,dx$$
$$= c\Big(\int_{\omega_G\setminus\cup_i B_i}|e(v)|^p\,dx+\sum_{i=1}^N\int_{\omega_G\cap B_i}|e(v)|^p dx\Big)$$
$$\le c\Big(\int_{\omega_G}|e(u)|^p dx+\sum_{i=1}^N\int_{B_i}|e(u)|^p dx\Big), \qquad (5.14)$$

where in the last inequality we have used the definition of v, and in particular the fact that $v=u$ outside $\cup_i B_i$, and the estimate of $e(v_i^\pm)$ in terms of $e(u)$ (see Step 2).

We now estimate the second term of the sum in (5.11). For $x\in G$ we define $\mathcal{Q}_g^x:=\{q\in\mathcal{Q}_g:\varphi_q(x)>0\}$, and denote $N_{\mathcal{Q}}:=\#\mathcal{Q}_g^x$ (which, as already observed, is uniformly bounded by a quantity depending only on the dimension, namely 2^n). Using that $\sum_{q\in\mathcal{Q}_g^x}\nabla\varphi_q(x)=0$, one has

$$\sum_{q\in\mathcal{Q}_g} w_q(x)\odot\nabla\varphi_q(x)=\sum_{q\in\mathcal{Q}_g^x}\Big(w_q(x)-\tfrac{1}{N_{\mathcal{Q}}}\sum_{\hat q\in\mathcal{Q}_g^x} w_{\hat q}(x)\Big)\odot\nabla\varphi_q(x)$$
$$=\frac{1}{N_{\mathcal{Q}}}\sum_{q,\hat q\in\mathcal{Q}_g^x}(w_q(x)-w_{\hat q}(x))\odot\nabla\varphi_q(x).$$

Since $q,\hat q\in\mathcal{Q}_g^x\Rightarrow x\in q'\cap\hat q'$, to bound the L^p norm of the above expression, it is enough to estimate

$$\int_{q'\cap\hat q'}|w_q-w_{\hat q}|^p|\nabla\varphi_q|^p dx \qquad (5.15)$$

for any pair of neighbouring cubes $q,\hat q\in\mathcal{Q}_g^x$. Note that $w_q-w_{\hat q}\in W^{1,p}(q'\cap\hat q';\mathbb{R}^n)$ and $w_q-w_{\hat q}=0$ in $(q'\cap\hat q')\setminus(\omega_q\cup\omega_{\hat q})$, since both functions coincide with v. Moreover, since $\mathcal{L}^n(q'\cap\hat q')\ge\delta^n/8^n$ and $\mathcal{L}^n(\omega_q\cup\omega_{\hat q})\le C\tau^{n/(n-1)}\delta^n$ for some dimensional constant C, provided τ is chosen small enough one can ensure that

$$\mathcal{L}^n\big(\{x\in q'\cap\hat q': w_q-w_{\hat q}=0\}\big)\ge\frac12\mathcal{L}^n(q'\cap\hat q').$$

One can then easily deduce from Lemma 4.3 that, for some constant c (depending on p and on the dimension):

$$\int_{q'\cap\hat q'}|w_q-w_{\hat q}|^p dx\le c\delta^p\int_{q'\cap\hat q'}|e(w_q-w_{\hat q})|^p dx\le c\delta^p\int_{\omega_q\cup\omega_{\hat q}}|e(w_q-w_{\hat q})|^p dx.$$

Since $|\nabla\varphi_q|\le C/\delta$ in each cube, we can estimate (5.15) as

$$\int_{q'\cap\hat q'}|w_q-w_{\hat q}|^p|\nabla\varphi_q|^p dx\le c\int_{(q'\cap\hat q')\cap(\omega_q\cup\omega_{\hat q})}|e(w_q-w_{\hat q})|^p dx$$
$$\le c\left(\int_{q'\cap(\omega_q\cup\omega_{\hat q})}|e(w_q)|^p dx+\int_{\hat q'\cap(\omega_q\cup\omega_{\hat q})}|e(w_{\hat q})|^p dx\right)$$
$$\le c\int_{\omega_q\cup\omega_{\hat q}}|e(v)|^p dx.$$

Hence we have that

$$\int_{\omega_G}\Big|\sum_{q\in\mathcal{Q}_g} w_q(x)\odot\nabla\varphi_q(x)\Big|^p dx\le c\int_{\omega_G}|e(v)|^p\,dx$$

which, together with (5.14), gives, from (5.11),

$$\int_{\omega_G} |e(w)|^p dx \le c \int_{\omega_G \cup \left(\bigcup_{i=1}^N B_i\right)} |e(u)|^p dx. \tag{5.16}$$

Finally, we estimate $\int_{\mathbb{R}^n \setminus (\Gamma \cup \tilde{\omega})} |e(w)|^p dx$. We have

$$\begin{aligned}\int_{\mathbb{R}^n \setminus (\Gamma \cup \tilde{\omega})} |e(w)|^p dx &\le \int_{\mathbb{R}^n \setminus (\Gamma \cup \tilde{\omega} \cup \hat{\omega})} |e(w)|^p dx + \int_{\left(\mathbb{R}^n \setminus (\Gamma \cup \tilde{\omega})\right) \cap \hat{\omega}} |e(w)|^p dx \\ &\le \int_{\mathbb{R}^n} |e(u)|^p dx + \int_{(\mathbb{R}^n \setminus \tilde{\omega}) \cap \omega_G} |e(w)|^p dx + \int_{\left(\mathbb{R}^n \setminus (\tilde{\omega} \cup \omega_G)\right) \cap \omega_B} |e(w)|^p dx, \quad (5.17)\end{aligned}$$

since $w = u$ in $\mathbb{R}^n \setminus (\tilde{\omega} \cup \hat{\omega})$. Using that $w = v$ outside $\tilde{\omega} \cup \omega_G$ we have

$$\int_{\left(\mathbb{R}^n \setminus (\tilde{\omega} \cup \omega_G)\right) \cap \omega_B} |e(w)|^p dx = \int_{\left(\mathbb{R}^n \setminus (\tilde{\omega} \cup \omega_G)\right) \cap \omega_B} |e(v)|^p dx \le c \sum_{i=1}^N \int_{B_i} |e(u)|^p dx, \tag{5.18}$$

where the last inequality follows from the definition of ω_B, the definition of v in the balls B_i, and the properties of $v_i^{\pm}$, for $i = 1, \ldots, N$.

In conclusion, from (5.17), (5.16) and (5.18) it follows that

$$\int_{\mathbb{R}^n \setminus (\Gamma \cup \tilde{\omega})} |e(w)|^p dx \le \int_{\mathbb{R}^n} |e(u)|^p dx + c \int_{\omega_G \cup \left(\bigcup_{i=1}^N B_i\right)} |e(u)|^p dx.$$

As a consequence, if we recall point e) of the construction of the B_i's and (5.9) above, if $\rho > 0$ and $\delta > 0$ are chosen small enough, one can ensure that

$$\int_{\mathbb{R}^n \setminus (\Gamma \cup \tilde{\omega})} |e(w)|^p dx \le (1 + \alpha) \int_{\mathbb{R}^n} |e(u)|^p dx. \tag{5.19}$$

By choosing α sufficiently small in (5.2), (5.5), (5.10) and (5.19) the conclusion follows. □

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