Variational approach to the asymptotic mean-value property for the p-Laplacian on Carnot groups

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Abstract

Let $1 \leq p \leq \infty$. We provide an asymptotic characterization of continuous viscosity solutions u of the normalized p-Laplacian $\Delta^N_{p\,\mathbb{G}} u = 0$ in any Carnot group $\mathbb{G}$.

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1 Introduction

The study of mean-value properties of solutions of elliptic PDEs has a long and fruitful history. For harmonic functions in the Euclidean setting, the study goes back to Gauss, Koebe, Volterra, and Zaremba, to mention just a few, see also [1] for recent results in Carnot groups and [23] for an interesting survey on the topic. In the last decade there has been a growing interest in studying a generalized mean-value property originating in [20] and [21], called the asymptotic mean-value property (amv-property for short). It allows to characterize solutions to harmonic, p-harmonic and more general equations of elliptic and parabolic types. Related are applications of p-harmonic functions in statistical Tug-of-War games, see for instance [20] and [24]. The studies in [20] allow, in the simplest case, to weaken the classical characterizations of a harmonic function u in $\mathbb{R}^n$ as follows:

$$u(x) = \fint_{B_\varepsilon(x)} u + o(\varepsilon^2), \quad \text{as } \varepsilon \to 0.$$

It is important from the point of view of our studies below, that the amv-property can be shown to hold for the viscosity solutions to the normalized p-harmonic equation $\Delta^N_p u = 0$ in $\mathbb{R}^n$ for all $1 \leq p \leq \infty$. Namely, in [20] it is proven that $u(x) = \mu^*_p(\varepsilon, u) + o(\varepsilon^2)$, as $\varepsilon \to 0$, where $\mu^*_p(\varepsilon, u)$ is the linear combination of the mean value and the *min-max mean*:

$$\mu^*_p(\varepsilon, u) = \frac{n+2}{n+p} \fint_{B_\varepsilon(x)} u + \frac{1}{2}\frac{p-2}{n+p} \left(\max_{B_\varepsilon(x)} u + \min_{B_\varepsilon(x)} u \right).$$

Similar means characterizing p-harmonic functions have been found in [12, 13], by using the *median* of a function, see also [15]. The results in [12] yield the amv-property for all p but for $n = 2$ only, while results of [15] provide the amv-property for $n \geq 2$. Moreover, the mean-value property for solutions to general elliptic equations with nonsmooth coefficients is studied in [25].

The amv-property has also been investigated beyond the Euclidean setting, see [9] for results in the first Heisenberg group $\mathbb{H}_1$, [17] for the higher order Heisenberg groups $\mathbb{H}_n$ and [10] for the setting of general Carnot groups.

A new approach to the asymptotic mean-value property has been recently proposed in [14] (see also [2] for relations with statistical games). More precisely, in [14], the authors proved that every viscosity solution u to the normalized p-Laplacian in an open set $\Omega \subset \mathbb{R}^n$ for a given $1 \leq p \leq \infty$ (Definition 2.2), can be characterized using an asymptotic mean-value property in terms of the function $\mu_p(\varepsilon, u)(x)$, defined as the unique minimizer of the following variational problem

$$\|u - \mu_p(\varepsilon, u)\|_{L^p(\overline{B_\varepsilon(x)})} = \min_{\lambda \in \mathbb{R}} \|u - \lambda\|_{L^p(\overline{B_\varepsilon(x)})},$$

where $B_\varepsilon(x) \subset \Omega$ denotes the ball centered at x with radius ε. This notion encompasses the median, the mean-value and the min-max mean of a continuous function, see [14] for details.

In the present paper we generalize the results of [14] to the setting of an arbitrary Carnot group. The novelty of our results is threefold: Firstly, we consider the setting of noncommutative metric measure spaces metrically nonequivalent to Euclidean spaces. This shows robustness of the approach in [14] and opens further possible directions of studies in the setting of subriemannian spaces as well as Riemannian manifolds. Secondly, since the geometry of sets in Carnot groups differs from the Euclidean ones due to the complexity of gauge distances and rigidity of the symmetries, the techniques used in our work need to be adjusted accordingly. We comment on these changes throughout the manuscript, see Remark 2 following the proof of Lemma 3.1. Finally, since our computations allow us to obtain the explicit coefficients in the key Lemma 3.1, see Examples 2-4, these computations can be employed to obtain counterparts of results in [2] and [3] for Carnot groups, see also Remark 3 below. Moreover, the proof of our main result can be viewed as a first step towards the parabolic case (see Section 4 in [14]) as below we develop methods which are key in establishing counterparts of parabolic-type results in [14].

Let $\mathbb{G}$ be a Carnot group of step k (Definition 2.1). Denote by $\Delta^N_{p,\mathbb{G}}$ the subelliptic normalized p-Laplacian (see (2) and (3)) and by $\mu_p(\varepsilon, u)$ the generalized median of a function u defined uniquely as in (5). The theorem below states that a viscosity solution of $\Delta^N_{p,\mathbb{G}} u = 0$ can be characterized asymptotically by the minimum $\mu_p(\varepsilon, u)$. This provides one more, intrinsic, way to characterize p-harmonic functions via a variant of the asymptotic mean-value property.

Theorem 1.1. *Let $1 \leq p \leq \infty$ and let $\Omega \subset \mathbb{G}$ be open. For a function $u \in C^0(\Omega)$ the following are equivalent:*

(i) u is a viscosity solution of $\Delta^N_{p,\mathbb{G}} u = 0$ in Ω;

(ii) $u(x) = \mu_p(\varepsilon, u)(x) + o(\varepsilon^2)$ as $\varepsilon \to 0$, in the viscosity sense for every $x \in \Omega$.

In order to prove this theorem we first prove Lemma 3.1, where the asymptotic behavior of minimizers μ_p is described for quadratic polynomials on balls. We illustrate the discussion with examples of the Heisenberg group and Carnot groups of step 2, see Examples 3 and 4 in Section 3. As presented in Remark 1 in Section 3, our results generalize those obtained in the Euclidean setting in [14]. In Remark 2 we compare our Lemma 3.1 to its counterpart in [14], discuss differences between these results and explain difficulties and novelties arising in the setting of general Carnot groups. Furthermore, in Remark 3 we discuss some possible applications of our results.

2 Carnot groups

In what follows, we briefly recall some standard facts on Carnot groups, see [6, 8, 11, 22] for a more detailed treatment.

Definition 2.1. A finite dimensional Lie algebra $\mathfrak{g}$, is said to be stratified of step $k \in \mathbb{N}$, if there exists subspaces $V_1, \ldots, V_k$ of $\mathfrak{g}$ such that:

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_k \text{ and } [V_1, V_i] = V_{i+1} \quad i = 1, \ldots, k-1; \quad [V_1, V_k] = \{0\}.$$

We denote by v_k the dimension of V_k.

A connected and simply connected Lie group $\mathbb{G}$ is a Carnot group if its Lie algebra $\mathfrak{g}$ is finite dimensional and stratified. We also set $h_0 := 0$, $h_i := \sum_{j=1}^{i} v_j$ and $m := h_k$.

Using the exponential map, every Carnot group $\mathbb{G}$ of step k is isomorphic as a Lie group to $(\mathbb{R}^m, \cdot)$ where $\cdot$ is the group operation given by the Baker-Campbell-Hausdorff formula.

For each $x \in \mathbb{G}$ we define left translation by $\tau_x : \mathbb{G} \longrightarrow \mathbb{G}$ by

$$\tau_x(y) := x \cdot y.$$

For each $\lambda > 0$ we define a dilation $\delta_\lambda : \mathbb{G} \longrightarrow \mathbb{G}$ by

$$\delta_\lambda(x) = \delta_\lambda(x_1, \ldots, x_m) := (\lambda^{\sigma_1} x_1, \ldots, \lambda^{\sigma_k} x_m),$$

where $\sigma_i \in \mathbb{N}$ is called the homogeneity of the variable x_i in $\mathbb{G}$ and it is defined by $\sigma_j := i$, whenever $h_{i-1} < j \le h_i$.

We endow $\mathbb{G}$ with a pseudonorm and pseudodistance by defining

$$\begin{aligned} |x|_{\mathbb{G}} &:= |(x^{(1)}, \ldots, x^{(k)})|_{\mathbb{G}} := \Big(\sum_{j=1}^{k} \|x^{(j)}\|^{\frac{2k!}{j}} \Big)^{\frac{1}{2k!}} \qquad (1) \\ \mathrm{d}(x,y) &:= |y^{-1} \cdot x|_{\mathbb{G}}, \end{aligned}$$

where $x^{(j)} := (x_{h_{j-1}+1}, \ldots, x_{h_j})$ and $\|x^{(j)}\|$ denotes the standard Euclidean norm in $\mathbb{R}^{h_j - h_{j-1}}$. We define the pseudoball centered at $x \in \mathbb{G}$ of radius $R > 0$ by

$$B(x,R) = B_R(x) := \{y \in \mathbb{G} : \ |y^{-1} \cdot x|_{\mathbb{G}} < R\}.$$

We illustrate the concept of Carnot groups with the following important examples.

Example 1 (The Heisenberg groups $\mathbb{H}_n$)**.** The n-dimensional Heisenberg group $\mathbb{G} = \mathbb{H}_n$, is the Carnot group with a 2-step Lie algebra and orthonormal basis $\{X_1, \ldots, X_{2n}, Z\}$ such that

$$\mathfrak{g}_1 = \mathrm{Span}\{X_1, \ldots, X_{2n}\}, \quad \mathfrak{g}_2 = \mathrm{Span}\{Z\},$$

and the nontrivial brackets are $[X_i, X_{n+i}] = Z$ for $i = 1, \ldots, n$.

In particular, if $n = 1$, then the Heisenberg group $\mathbb{H}_1$ is often presented using coordinates (z,t), where $z = x + iy \in \mathbb{C}$ and $t \in \mathbb{R}$, and multiplication defined by $(z_1,t_1)(z_2,t_2) = (z_1 + z_2, t_1 + t_2 + 2\,\mathrm{Im}\,(z_1 \bar{z}_2))$. The pseudonorm given by $\|(z,t)\| = (|z|^4 + t^2)^{1/4}$ gives rise to a left invariant distance defined by $d_{\mathbb{H}_1}(p,q) = \|p^{-1}q\|$ which is called the Heisenberg distance. A dilation by $r > 0$ is defined by $\delta_r(z,t) = (rz, r^2 t)$ and the left invariant Haar measure λ is simply the 3-dimensional Lebesgue measure, moreover $\delta_r^* d\lambda = r^4 d\lambda$. It follows that the Hausdorff dimension of the metric measure space $(\mathbb{H}_1, d_{\mathbb{H}_1}, \lambda)$ is 4, and the space is 4-Ahlfors regular, i.e., there exists a positive constant c such that for all balls B with radius r, we have $\frac{1}{c} r^4 \le \mathcal{H}^4(B) \le c r^4$, where $\mathcal{H}^4$ denotes the 4-dimensional Hausdorff measure induced by $d_{\mathbb{H}_1}$.

The following proposition, proved in [6], shows that the Lebesgue measure is the Haar measure on Carnot groups.

Proposition 2.1. *Let $\mathbb{G} = (\mathbb{R}^m, \cdot)$ be a Carnot group. Then the Lebesgue measure on $\mathbb{R}^m$ is invariant with respect to the left and the right translations on $\mathbb{G}$. Precisely, if we denote by $|E|$ the Lebesgue measure of a measurable set $E \subset \mathbb{R}^m$, then for all $x \in \mathbb{G}$ we have that $|x \cdot E| = |E| = |E \cdot x|$. Moreover, for all $\lambda > 0$ it holds $\delta_\lambda(E)| = \lambda^Q |E|$, where $Q := \sum_{j=1}^{m} v_j \sigma_j$.*

A basis $X = \{X_1, \dots, X_m\}$ of $\mathfrak{g}$, is called *the Jacobian basis* if $X_j = J(e_j)$ where $(e_1, \dots, e_m)$ is the canonical basis of $\mathbb{R}^m$ and $J : \mathbb{R}^m \longrightarrow \mathfrak{g}$ is defined by $J(\eta)(x) := \mathcal{J}_{\tau_x}(0) \cdot \eta$, where $\mathcal{J}_{\tau_x}$ denotes the Jacobian matrix of τ_x.

Let us recall the following classical proposition describing the Jacobian basis on Carnot groups, see [6, Corollary 1.3.19] for a proof.

Proposition 2.2. *Let $\mathbb{G} = (\mathbb{R}^m, \cdot)$ be a Carnot group of step $k \in \mathbb{N}$. Then the elements of the Jacobian basis $\{X_1, \dots, X_m\}$ have polynomial coefficients and if $h_{l-1} < j \le h_l$, $1 \le l \le k$, then*

$$X_j(x) = \partial_j + \sum_{i>h_l}^{m} a_i^{(j)}(x)\partial_i,$$

where $a_i^{(j)}(x) = a_i^{(j)}(x_1, \dots, x_{h_{l-1}})$ when $h_{l-1} < i \le h_l$, and $a_i^{(j)}(\delta_\lambda(x)) = \lambda^{\sigma_i - \sigma_j} a_i^{(j)}(x)$.

The following definition is one of the key concepts of the analysis on Carnot groups. Let $X = \{X_1, \dots, X_m\}$ be a Jacobian basis of $\mathbb{G} = (\mathbb{R}^m, \cdot)$. For any function $u \in C^1(\mathbb{R}^m)$, we define its *horizontal gradient* by the formula

$$\nabla_{V_1} u := \sum_{i=1}^{h_1} (X_i u) X_i$$

and the *intrinsic divergence* of u as

$$\operatorname{div}_{V_1} u := \sum_{i=1}^{h_1} X_i u.$$

Moreover, for $2 \le j \le k$, we set $\nabla_{V_j} u := \sum_{h_{j-1} < i \le h_j} (X_i u) X_i$. The horizontal Laplacian $\Delta_{\mathbb{G}} u$ of a function $u : \mathbb{G} \longrightarrow \mathbb{R}$ is defined by the following

$$\Delta_{\mathbb{G}} u := \sum_{i=1}^{h_1} X_i^2 u.$$

A priori, one studies solutions to the Laplace equation under the C^2-regularity assumption. However, as in the Euclidean setting, it is natural to weaken the required degree of regularity and consider weak solutions belonging to the so-called horizontal Sobolev space. For further details we refer to e.g. [7, 19].

The following results describe the Taylor expansion formula in the Carnot groups, see [6, Proposition 20.3.11] .

Proposition 2.3. *Let $\Omega \subset \mathbb{G}$ be an open neighborhood of 0 and let $u \in C^\infty(\Omega)$. Then, the following Taylor formula holds for any point $P = (x^{(1)}, x^{(2)}, \dots, x^{(k)}) \in \Omega$:*

$$u(P) = u(0) + \langle \nabla_{V_1} u(0), x^{(1)} \rangle_{\mathbb{R}^{h_1}} + \langle \nabla_{V_2} u(0), x^{(2)} \rangle_{\mathbb{R}^{h_2}} + \frac{1}{2} \langle D^{2,*}_{V_1} u(0) x^{(1)}, x^{(1)} \rangle_{\mathbb{R}^{h_1}} + o(\|P\|^2)$$

where

$$D^{2,*}_{V_1} u := \left(\frac{(X_i X_j + X_j X_i) u}{2} \right)_{1 \le i,j \le h_1}$$

is the so called symmetrized horizontal Hessian *of u.*

Next, we recall the definition of the main differential operator studied in this work. For $p \in [1, +\infty]$ the *subelliptic normalized p-Laplace operator* is

$$\Delta^N_{p,\mathbb{G}} u := \frac{\operatorname{div}_{V_1}(|\nabla_{V_1} u|^{p-2} \nabla_{V_1} u)}{|\nabla_{V_1} u|^{p-2}} \quad \text{if} \quad 1 \le p < \infty \tag{2}$$

and

$$\Delta^N_{\infty,\mathbb{G}} u := \frac{\left\langle D^{2,*}_{V_1} u \, \frac{\nabla_{V_1} u}{|\nabla_{V_1} u|}, \frac{\nabla_{V_1} u}{|\nabla_{V_1} u|} \right\rangle}{|\nabla_{V_1} u|^2}. \tag{3}$$

Note that for $p = 2$, $\Delta_{2,\mathbb{G}} u = \Delta_{\mathbb{G}} u$ is the so called Kohn-Laplace operator in $\mathbb{G}$. Thus, the p-Laplace operator is the natural generalization of the Laplacian. Furthermore, the ∞-Laplacian can be viewed as a limit of p-Laplacians in the appropriate sense for $p \to \infty$. Among its applications, let us mention best Lipschitz extensions, image processing and mass transport problems, see e.g. the presentation in [20] and references therein.

In the case of the non-renormalized p-Laplacian, notions of a viscosity solution and a weak solution agree for $1 < p < \infty$, see [16] for the result in the Euclidean setting and [4] for the Heisenberg group. Since the normalized p-Laplacian is in the non-divergence form, the concept of viscosity solutions is more handy than weak solutions. Let us now introduce this notion.

Definition 2.2. Fix a value of $p \in [1, \infty]$ and consider the subelliptic normalized p-Laplace equation

$$\Delta^N_{p,\mathbb{G}} u = 0 \qquad \text{in} \qquad \Omega \subset \mathbb{G}. \tag{4}$$

(i) A lower semi-continuous function u, is a viscosity supersolution of (4), if for every $x_0 \in \Omega$, and every $\phi \in C^2(\Omega)$ such that $\nabla_{V_1}\phi(x_0) \neq 0$ and $u - \phi$ has a strict minimum at $x_0 \in \Omega$, we have $\Delta^N_{p,\mathbb{G}}\phi \leq 0$ in Ω.

(ii) A lower semi-continuous function u, is a viscosity subsolution of (4), if for every $x_0 \in \Omega$, and every $\phi \in C^2(\Omega)$ such that $\nabla_{V_1}\phi(x_0) \neq 0$ and $u - \phi$ has a strict maximum at $x_0 \in \Omega$, we have $\Delta^N_{p,\mathbb{G}}\phi \geq 0$ in Ω.

(iii) A continuous function u is a viscosity solution of of (4), if it is both a viscosity supersolution and a viscosity subsolution in Ω.

Fix an open set $\Omega \subset \mathbb{G}$, let $1 \leq p \leq \infty$ and let u be a real-valued continuous function in Ω. For a given $x \in \Omega$, choose $\varepsilon > 0$ so that $\overline{B_\varepsilon(x)} \subset \Omega$, we define the number $\mu_p(\varepsilon, u)(x)$ (or simply $\mu_p(\varepsilon, u)$ if the point x is clear from the context) as the unique real number satisfying

$$\|u - \mu_p(\varepsilon, u)\|_{L^p(\overline{B_\varepsilon(x)})} = \min_{\lambda \in \mathbb{R}} \|u - \lambda\|_{L^p(\overline{B_\varepsilon(x)})}. \tag{5}$$

The following properties of $\mu_p(\varepsilon, u)(x)$ have been proved in [14] for the setting of compact topological spaces X, equipped with a positive Radon measure ν such that $\nu(X) < \infty$. Here we apply results from [14] to $X = \overline{B_\varepsilon(x)} \subset \mathbb{G}$ and ν the Lebesgue measure, cf. Proposition 2.1.

In Theorem 2.1 below, we summarize results proven in Theorems 2.1, 2.4 and 2.5 in [14].

Theorem 2.1. *Let $1 \leq p \leq \infty$ and $u \in C(\overline{B_\varepsilon(x)})$.*

(1) There exists a unique real valued $\mu_p(\varepsilon, u)$ such that

$$\|u - \mu_p(\varepsilon, u)\|_{L^p(\overline{B_\varepsilon(x)})} = \min_{\lambda \in \mathbb{R}} \|u - \lambda\|_{L^p(\overline{B_\varepsilon(x)})}.$$

Furthermore, for $1 \leq p < \infty$, $\mu_p(\varepsilon, u)$ is characterized by the equation

$$\int_{B_\varepsilon(x)} |u(y) - \mu_p(\varepsilon, u)|^{p-2} \left(u(y) - \mu_p(\varepsilon, u)\right) dy = 0, \tag{6}$$

where for $1 \leq p < 2$ we assume that the integrand is zero if $u(y) - \mu_p(\varepsilon, u) = 0$. For $p = \infty$ we have the following equality:

$$\mu_\infty(\varepsilon, u) = \frac{1}{2}\left(\min_{B(x,\varepsilon)} u + \max_{B(x,\varepsilon)} u\right). \tag{7}$$

(2) *If* $1 \le p \le \infty$ *then it follows that*

$$\left| \|u - \mu_p(\varepsilon, u)\|_{L^p(\overline{B_\varepsilon(x)})} - \|v - \mu_p(\varepsilon, v)\|_{L^p(\overline{B_\varepsilon(x)})} \right| \le \|u - v\|_{L^p(\overline{B_\varepsilon(x)})}$$

for any $u, v \in L^p(\overline{B_\varepsilon(x)})$. *Moreover, if* $u_n \to u$ *in* $L^p(\overline{B_\varepsilon(x)})$ *for* $1 \le p \le \infty$ *and* $u_n, u \in C^0(\overline{B_\varepsilon(x)})$ *for* $p = 1$, *then* $\mu_p(\varepsilon, u_n) \to \mu_p(\varepsilon, u)$ *as* $n \to \infty$, *the same is true for any* $p \in [1, \infty]$ *if* $\{u_n\} \subset C^0(\overline{B_\varepsilon(x)})$ *converges uniformly on* $\overline{B_\varepsilon(x)}$ *as* $n \to \infty$.

(3) *Let* u *and* v *be two functions which, in the case* $1 < p \le \infty$, *belong to* $L^p(B_\varepsilon(x))$, *and in the case* $p = 1$, *belong to* $C^0(\overline{B_\varepsilon}(x))$. *If* $u \le v$ *a.e. in* $\overline{B_\varepsilon}(x)$, *then* $\mu_p(\varepsilon, u) \le \mu_p(\varepsilon, v)$.

(4) $\mu_p(\varepsilon, u + c) = \mu_p(\varepsilon, u) + c$ *for every* $c \in \mathbb{R}$.

(5) $\mu_p(\varepsilon, cu) = c\mu_p(\varepsilon, u)$ *for every* $c \in \mathbb{R}$.

The following is [14, Corollary 2.3] in Carnot groups of step k:

Corollary 2.1. *Let* $u \in L^p(B_\varepsilon(x))$, *for* $1 < p \le \infty$, *or in* $C^0(\overline{B_\varepsilon}(x))$ *for* $p = 1$. *Let* $u_\varepsilon(z) = u(x\delta_\varepsilon(z))$ *for* $z \in \overline{B_1}(0)$, *then*

$$\mu_p(\varepsilon, u)(x) = \mu_p(1, u_\varepsilon)(0).$$

Proof. For every $\lambda \in \mathbb{R}$ and $1 \le p < \infty$ it holds:

$$\|u-\lambda\|^p_{L^p(B_\varepsilon(x))} = \int_{B_\varepsilon(x)} |u(\xi)-\lambda|^p \, d\xi = \varepsilon^{\sigma_1+\cdots+\sigma_k} \int_{B_1(0)} |u_\varepsilon(\xi)-\lambda|^p \, d\xi = \varepsilon^{v_1+2v_2+\cdots+kv_k} \|u_\varepsilon-\lambda\|^p_{L^p(B_1(0))}$$

and

$$\|u - \lambda\|_{L^\infty(B_\varepsilon(x))} = \|u_\varepsilon - \lambda\|_{L^\infty(B_1(0))}$$

and the conclusion follows by (1) in Theorem 2.1. □

Next we state carefully what is meant by the statement that the asymptotic expansion of the function u in terms of μ_p holds in the viscosity sense, see (5) and Definition 2.4. First, we need the following auxiliary definition.

Definition 2.3. Let h be a real valued function defined in a neighborhood of zero. We say that

$$h(x) \le o(x^2) \text{ as } x \to 0^+$$

if any of the three equivalent conditions is satisfied:

a) $\limsup_{x \to 0^+} \dfrac{h(x)}{x^2} \le 0,$

b) there exists a nonnegative function $g(x) \ge 0$ such that $h(x) + g(x) = o(x^2)$ as $x \to 0^+$,

c) $\lim_{x \to 0^+} \dfrac{h^+(x)}{x^2} \le 0.$

A similar definition is given for $h(x) \ge o(x^2)$ as $x \to 0^+$ by reversing the inequalities in a) and c), requiring that $g(x) \le 0$ in b) and replacing h^+ by h^- in c)[1].

Let f and g be two real valued functions defined in a neighborhood of $x_0 \in \mathbb{R}$. We say that f *and* g *are asymptotic functions for* $x \to x_0$, if there exists a function h defined in a neighborhood $V_{x_0}$of x_0 such that:

(i) $f(x) = g(x)h(x)$ for all $x \in V_{x_0} \setminus \{x_0\}$.

(ii) $\lim_{x \to x_0} h(x) = 1$.

[1]As usual, we denote by $h^+(x) := \max\{h(x), 0\}$ and $h^-(x) := -\min\{h(x), 0\}$.

If f and g are asymptotic for $x \to x_0$, then we simply write $f \sim g$ as $x \to x_0$.

Definition 2.4. A continuous function u defined in a neighborhood of a point $x \in \mathbb{G}$, satisfies

$$u(x) = \mu_p(\varepsilon, u)(x) + o(\epsilon^2),$$

as $\epsilon \to 0^+$ in the viscosity sense, if the following conditions hold:

(i) for every continuous function ϕ defined in a neighborhood of a point x such that $u - \phi$ has a strict minimum at x with $u(x) = \phi(x)$ and $\nabla_{V_1}\phi(x) \neq 0$, we have

$$\phi(x) \geq \mu_p(\varepsilon, \phi)(x) + o(\epsilon^2), \quad \text{as } \epsilon \to 0^+.$$

(ii) for every continuous function ϕ defined in a neighborhood of a point x such that $u - \phi$ has a strict maximum at x with $u(x) = \phi(x)$ and $\nabla_{V_1}\phi(x) \neq 0$, then

$$\phi(x) \leq \mu_p(\varepsilon, \phi)(x) + o(\epsilon^2), \quad \text{as } \epsilon \to 0^+.$$

3 The proof of Theorem 1.1

In order to prove Theorem 1.1, we need the following key lemma.

Lemma 3.1 (cf. Lemma 3.1 in [14])**.** *Let $\mathbb{G}$ be a Carnot group of step k. Moreover, let $\Omega \subset \mathbb{G}$ be an open set and $x \in \Omega$ be a point such that $B_\varepsilon(x) \subset \Omega$ for all small enough $\varepsilon \leq \varepsilon_0(x)$. Let $1 \leq p \leq \infty$ and $\xi \in \mathbb{R}^{v_1} \setminus \{0\}$, $\eta \in \mathbb{R}^{v_2}$. Let further A be a symmetric $v_1 \times v_1$ matrix with trace* $\mathrm{tr}(A)$*. Moreover, consider the quadratic function $q : B_\varepsilon(x) \to \mathbb{R}$ given by*

$$q(y) = q(x) + \langle \xi, (x^{-1}y)^{(1)} \rangle_{\mathbb{R}^{v_1}} + \langle \eta, (x^{-1}y)^{(2)} \rangle_{\mathbb{R}^{v_2}} + \frac{1}{2}\langle A(x^{-1}y)^{(1)}, (x^{-1}y)^{(1)} \rangle_{\mathbb{R}^{v_1}}, \quad y \in B_\varepsilon(x), \tag{8}$$

where $(x^{-1}y)^{(1)}$ and $(x^{-1}y)^{(2)}$ are the horizontal and the vertical components of $x^{-1}y$, respectively and $\langle \cdot, \cdot \rangle_{\mathbb{R}^{v_1}}$ and $\langle \cdot, \cdot \rangle_{\mathbb{R}^{v_2}}$ denote the Euclidean scalar products on $\mathbb{R}^{v_1}$ and $\mathbb{R}^{v_2}$, respectively. It then follows that

$$\mu_p(\varepsilon, q)(x) = q(x) + \varepsilon^2 c \left(\mathrm{tr}(A) + (p-2)\frac{\langle A\xi, \xi \rangle_{\mathbb{R}^{v_1}}}{|\xi|^2} \right) + o(\varepsilon^2), \tag{9}$$

where

$$c := c(p, v_1, \ldots, v_k) = \frac{1}{2(p+v_1)} \frac{\mathcal{B}\left(\frac{v_k}{2(k-1)!}, \frac{p+\sum_{j=1}^{k-1} jv_j}{2(k-1)!} + 1 \right)}{\mathcal{B}\left(\frac{v_k}{2(k-1)!}, \frac{p-2+\sum_{j=1}^{k-1} jv_j}{2(k-1)!} + 1 \right)} \prod_{j=2}^{k-1} \frac{\mathcal{B}\left(\frac{jv_j}{2k!}, \frac{p+\sum_{i=1}^{j-1} iv_i}{2k!} + 1 \right)}{\mathcal{B}\left(\frac{jv_j}{2k!}, \frac{p-2+\sum_{i=1}^{j-1} iv_i}{2k!} + 1 \right)}$$

and $\mathcal{B}(x, y)$ denotes the Beta function $\mathcal{B}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt$ for all $x, y > 0$. Furthermore, if $u \in C^2(\Omega)$ with $\nabla_{V_1} u(x) \neq 0$, then

$$\mu_p(\varepsilon, u)(x) = u(x) + c\Delta^N_{p,\mathbb{G}} u(x)\varepsilon^2 + o(\varepsilon^2), \quad \text{as} \quad \varepsilon \to 0^+. \tag{10}$$

Remark 1. The formula describing the constant $c(p, v_1, \ldots, v_k)$ is complicated and not easily simplified using the properties of the Beta function.

Before we prove the lemma, let us discuss its assertion in some particular cases:

Example 2 (The Euclidean space $\mathbb{R}^N$)**.** If $\mathbb{G}$ is the Euclidean space $\mathbb{R}^N$ then $c(p, v_1, \ldots, v_k)$ agrees with the constant computed in [14], namely

$$c(p, N) = \frac{1}{2(p+N)}.$$

Example 3 (The Heisenberg group $\mathbb{H}_1$, cf. Example 1)**.** If $\mathbb{G} = \mathbb{H}_1$, then quadratic function q in (8) takes the form:

$$q(y) = q(x) + \langle \xi, (x^{-1}y)^{(1)} \rangle + w(x^{-1}y)^{(2)} + \frac{1}{2}\langle A(x^{-1}y)^{(1)}, (x^{-1}y)^{(1)} \rangle_{\mathbb{R}^2}, \quad y \in B_\varepsilon(x),$$

where $w \in \mathbb{R}$, $\xi \in \mathbb{R}^2 \setminus \{0\}$. Furthermore, the constant $c = c(p)$ appearing in (9) and (10) takes the following form

$$c(p) = \frac{2}{(p+2)(p+4)} \left(\frac{\Gamma\left(\frac{p+6}{4}\right)}{\Gamma\left(\frac{p+4}{4}\right)} \right)^2,$$

where for $t > -1$, $\Gamma(t) = \int_0^\infty x^{t-1} e^{-x}\, dx$ is the Gamma function.

Example 4 (Carnot groups of step 2)**.** Let $\mathbb{G}$ be a Carnot group of step 2, then the quadratic function q in (8) takes the form:

$$q(y) = q(x) + \langle \xi, (x^{-1}y)^{(1)} \rangle_{\mathbb{R}^n} + \langle \eta, (x^{-1}y)^{(2)} \rangle_{\mathbb{R}^k} + \frac{1}{2}\langle A(x^{-1}y)^{(1)}, (x^{-1}y)^{(1)} \rangle_{\mathbb{R}^n}, \quad y \in B_\varepsilon(x),$$

that is $v_1 = n, v_2 = k$, $\xi \in \mathbb{R}^n \setminus \{0\}$ and $\eta \in \mathbb{R}^k$. Moreover, the constant $c = c(p,n,k)$, appearing in (9) and (10), takes the following form

$$c(p,n,k) := \frac{1}{2(n+p)} \frac{\mathcal{B}\left(\frac{k}{2}, \frac{n+p+4}{4}\right)}{\mathcal{B}\left(\frac{k}{2}, \frac{n+p+2}{4}\right)}.$$

In the proof of Lemma 3.1 we employ the following integral formula.

Lemma 3.2. *Let $\alpha_1, \ldots, \alpha_n$ be real numbers such that $\alpha_i > -1$ for $i = 1, \ldots, n$. It then follows that*

$$\int\limits_{T_n} x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n} dx = \frac{1}{2^n} \frac{\prod_{i=1}^n \Gamma(\frac{\alpha_i+1}{2})}{\Gamma(\frac{n+2+\sum \alpha_i}{2})} \tag{11}$$

where $T_n := \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \ldots + x_n^2 < 1,\ x_i \geq 0 \text{ for } i = 1, \ldots, n \right\}$.

Proof of Lemma 3.2. Let $a, b > -1$. Upon applying the change of variables $t = \sin^2 x$, we obtain the following equation:

$$\int_0^{\frac{\pi}{2}} \sin^a x \cos^b x dx = \int_0^1 t^{\frac{a}{2}} (1-t)^{\frac{b}{2}} \frac{1}{2\sqrt{t}\sqrt{1-t}} dt = \frac{1}{2} \int_0^1 t^{\frac{a-1}{2}} (1-t)^{\frac{b-1}{2}} dt = \frac{1}{2} \mathcal{B}\left(\frac{a+1}{2}, \frac{b+1}{2} \right).$$

Now we are in a position to calculate the left-hand side of (11). We apply the spherical coordinates

$$\begin{cases} x_1 &= r\cos\varphi_1 \\ x_2 &= r\sin\varphi_1 \cos\varphi_2 \\ x_3 &= r\sin\varphi_1 \sin\varphi_2 \cos\varphi_3 \\ \vdots & \vdots \\ x_{n-1} &= r\sin\varphi_1 \sin\varphi_2 \cdot \ldots \cdot \cos\varphi_{n-1} \\ x_n &= r\sin\varphi_1 \sin\varphi_2 \ldots \sin\varphi_{n-1} \end{cases}$$

with the Jacobian determinant $|J| = r^{n-1} \sin^{n-2}\varphi_1 \sin^{n-3}\varphi_2 \cdot \ldots \cdot \sin\varphi_{n-2}$ and the spherical coordinates varying as follows: $r \in (0,1)$, $\varphi_i \in (0, \pi/2)$ for $i = 1, \dots, n-2$. The result is

$$\int_{T_n} x_1^{\alpha_1} \cdot \dots \cdot x_n^{\alpha_n} dx = \int_0^1 \int_0^{\frac{\pi}{2}} \dots \int_0^{\frac{\pi}{2}} \Big[r^{\sum_{i=1}^n \alpha_i + n - 1} \cdot \cos^{\alpha_1} \varphi_1 \sin^{\sum_{i=2}^n \alpha_i + n - 2} \varphi_1$$

$$\cdot \cos^{\alpha_2} \varphi_2 \sin^{\sum_{i=3}^n \alpha_i + n - 3} \varphi_2 \cdot \dots \cdot \cos^{\alpha_{n-1}} \varphi_{n-1} \sin^{\alpha_n} \varphi_{n-1} \Big] d\varphi_1 \dots d\varphi_{n-1} dr$$

$$= \frac{1}{n + \sum_{i=1}^n \alpha_i} \frac{1}{2} \mathcal{B}\left(\frac{\sum_{i=2}^n \alpha_i + n - 1}{2}, \frac{\alpha_1 + 1}{2} \right) \frac{1}{2} \mathcal{B}\left(\frac{\sum_{i=3}^n \alpha_i + n - 2}{2}, \frac{\alpha_2 + 1}{2} \right)$$

$$\cdot \dots \cdot \frac{1}{2} \mathcal{B}\left(\frac{\alpha_n + 1}{2}, \frac{\alpha_{n-1} + 1}{2} \right),$$

which is equal to the right-hand side of (11) upon using the well-known formula $\mathcal{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$. □

Proof of Lemma 3.1. In the proof we follow the steps of the proof of Lemma 3.1 in [14]. However, since the setting of Carnot groups differs from the Euclidean one, the computations are to some extent, more demanding and nontrivial.

We begin with computing $\mu_p(\varepsilon, q)$. For $z = (z^{(1)}, \dots, z^{(k)}) \in B := B(0,1)$, we introduce the following functions:

$$q_\varepsilon(z) := q(x\delta_\varepsilon(z)), \quad v_\varepsilon(z) := \frac{q_\varepsilon(z) - q(x)}{\varepsilon} \quad \text{and} \quad v(z) := \langle \xi, (z_1, \dots, z_{v_1}) \rangle_{\mathbb{R}^n} := \langle \xi, z^{(1)} \rangle_{\mathbb{R}^{v_1}}.$$

We know that $\mu_p(\varepsilon, q)(x) = \mu_p(1, q_\varepsilon)(0)$ by Corollary 2.1. Then, by points (4) and (5) of Theorem 2.1, we see that

$$\frac{\mu_p(\varepsilon, q)(x) - q(x)}{\varepsilon} = \mu_p(1, v_\varepsilon)(0).$$

Let us further observe that

$$v_\varepsilon(z) = \langle \xi, z^{(1)} \rangle + \frac{\varepsilon}{2} \langle A z^{(1)}, z^{(1)} \rangle + \varepsilon \langle \eta, z^{(2)} \rangle \tag{12}$$

which shows that v_ε converges uniformly to v as $\varepsilon \to 0$ on $\overline{B}$. We appeal to the second part of claim (2) in Theorem 2.1 to obtain that $\mu_p(1, v_\varepsilon)(0) \to \mu_p(1, v)(0)$ as $\varepsilon \to 0$. Recall that the characterization of $\lambda = \mu_p(1, v)(0)$ given by (6) in Theorem 2.1 states that if $p \in [1, \infty)$, then λ is the unique number such that

$$\int_B |\langle \xi, y^{(1)} \rangle - \lambda|^{p-2} (\langle \xi, y^{(1)} \rangle - \lambda) dy = 0.$$

On the other hand we have

$$\int_B |\langle \xi, y^{(1)} \rangle|^{p-2} (\langle \xi, y^{(1)} \rangle) dy = 0,$$

which follows from the symmetry of the unit ball and the following natural change of variables

$$\Phi(y^{(1)}, y^{(2)}, \dots, y^{(k)}) = (-y^{(1)}, y^{(2)}, \dots, y^{(k)}), \quad |J_\Phi| = 1, \quad \Phi(B) = B.$$

It now follows that $\mu_p(1, v)(0) = \lambda = 0$.

If $p = \infty$, then by (7):

$$\mu_\infty(1, v)(0) = \frac{1}{2} \left(\min_{\overline{B}} \langle \xi, y^{(1)} \rangle + \max_{\overline{B}} \langle \xi, y^{(1)} \rangle \right) = \frac{1}{2} (-|\xi| + |\xi|) = 0.$$

Next, we split the discussion into the cases depending on the value of p. Let us define

$$\gamma_\varepsilon := \frac{\mu_p(\varepsilon, q)(x) - q(x)}{\varepsilon^2}.$$

3.1 Case 1: $1 < p < \infty$.

For the sake of brevity, we introduce a function $f(s) = |s|^{p-2}s$. Then, upon applying (6) to $\mu_p(1, v_\varepsilon)(0) = \varepsilon\gamma_\varepsilon$, we obtain

$$\int_B f(v_\varepsilon(z) - \varepsilon\gamma_\varepsilon)dz = 0.$$

By using (12), this can be transformed to the following expression:

$$\int_B f\left(\langle \xi, z^{(1)}\rangle + \varepsilon\left(\frac{1}{2}\langle Az^{(1)}, z^{(1)}\rangle - \gamma_\varepsilon + \langle \eta, z^{(2)}\rangle\right)\right) dz = 0. \tag{13}$$

Without loss of generality we may assume that $|\xi| = 1$, since otherwise we can consider the quadratic function $\tilde{q} = q/|\xi|$. Let us apply the change of variables $z = (z^{(1)}, z^{(2)}, \dots, z^{(k)}) = (Ry^{(1)}, y^{(2)}, \dots, y^{(k)})$ in (13), where R is a $v_1 \times v_1$ rotation matrix with $R^T\xi = e_1$ and e_1 denotes the first element of the canonical basis of $\mathbb{R}^{\nu_1}$. Set $C = R^TAR$, then (13) reads as

$$\int_B f\left(y_1 + \varepsilon\left(\frac{1}{2}\langle Cy^{(1)}, y^{(1)}\rangle - \gamma_\varepsilon + \langle \eta, y^{(2)}\rangle\right)\right) dy = 0.$$

Since $\int_B f(y_1)dy = 0$, it follows that for all $\varepsilon > 0$, we have:

$$\int_B \frac{1}{\varepsilon}\left(f\left(y_1 + \varepsilon\left(\frac{1}{2}\langle Cy^{(1)}, y^{(1)}\rangle - \gamma_\varepsilon + \langle \eta, y^{(2)}\rangle\right)\right) - f(y_1)\right) dy = 0.$$

Therefore, by the Fundamental Theorem of Calculus, we have:

$$\int_B \left[\int_0^1 f'\left(y_1 + t\varepsilon\left(\frac{1}{2}\langle Cy^{(1)}, y^{(1)}\rangle - \gamma_\varepsilon + \langle \eta, y^{(2)}\rangle\right)\right) dt\right]\left(\frac{1}{2}\langle Cy^{(1)}, y^{(1)}\rangle - \gamma_\varepsilon + \langle \eta, y^{(2)}\rangle\right) dy = 0. \tag{14}$$

Equality (14) implies that γ_ε is a weighted mean value of the function $\frac{1}{2}\langle Cy^{(1)}, y^{(1)}\rangle + \langle \eta, y^{(2)}\rangle$ over B with respect to a weighted Lebesgue measure $w(y)dy$ for

$$w(y) := \int_0^1 f'\left(y_1 + t\varepsilon\left(\frac{1}{2}\langle Cy^{(1)}, y^{(1)}\rangle - \gamma_\varepsilon + \langle \eta, y^{(2)}\rangle\right)\right) dt, \quad y \in B.$$

The weight function w is nonnegative since $f'(s) = (p-1)|s|^{p-2} \geq 0$. Therefore, γ_ε is bounded by $c := \|\frac{1}{2}\langle Cy^{(1)}, y^{(1)}\rangle + \langle \eta, y^{(2)}\rangle\|_{L^\infty(B)}$.

Let us consider any subsequence of (γ_ε) converging to γ_0 as $\varepsilon \to 0^+$, which for the sake of brevity, we also denote by (γ_ε). Let us consider two cases. If $2 \leq p < \infty$, then for all $y \in B$ we obtain

$$\begin{aligned}&\left|\int_0^1 f'\left(y_1 + t\varepsilon\left(\frac{1}{2}\langle Cy^{(1)}, y^{(1)}\rangle - \gamma_\varepsilon + \langle \eta, y^{(2)}\rangle\right)\right) dt\left(\frac{1}{2}\langle Cy^{(1)}, y^{(1)}\rangle - \gamma_\varepsilon + \langle \eta, y^{(2)}\rangle\right)\right| \\ &\leq 2c(p-1)\int_0^1 \left|y_1 + t\varepsilon\left(\frac{1}{2}\langle Cy^{(1)}, y^{(1)}\rangle - \gamma_\varepsilon + \langle \eta, y^{(2)}\rangle\right)\right|^{p-2} dt \leq 2c(p-1)(1+2c\varepsilon).\end{aligned}$$

Therefore, by the dominated convergence theorem the sequence (γ_ε) converges to

$$\gamma_0 := \lim_{\varepsilon\to 0}\gamma_\varepsilon = \frac{\int_B |y_1|^{p-2}\left(\frac{1}{2}\langle Cy^{(1)}, y^{(1)}\rangle + \langle \eta, y^{(2)}\rangle\right)dy}{\int_B |y_1|^{p-2}dy}. \tag{15}$$

Let now $1 < p < 2$. Fix $0 < \theta < 1$ and split the integral (14) into two parts: over the set $G_\theta := B \cap \{|y_1| > \theta\}$ and $F_\theta := B \cap \{|y_1| \leq \theta\}$. Observe that for all $y \in G_\theta$ and for all $\varepsilon > 0$ satisfying $2c\varepsilon < \theta$, we have the following:

$$\begin{aligned}&\left|\int_0^1 f'\left(y_1 + t\varepsilon\left(\frac{1}{2}\langle Cy^{(1)}, y^{(1)}\rangle - \gamma_\varepsilon + \langle \eta, y^{(2)}\rangle\right)\right) dt\left(\frac{1}{2}\langle Cy^{(1)}, y^{(1)}\rangle - \gamma_\varepsilon + \langle \eta, y^{(2)}\rangle\right)\right| \\ &\leq 2c\,||y_1| - 2c\varepsilon|^{p-2}.\end{aligned}$$

Moreover,

$$\lim_{\varepsilon\to 0}\int_{G_\theta} ||y_1|-2c\varepsilon|^{p-2}\,dy=\int_{G_\theta}|y_1|^{p-2}dy<\int_B |y_1|^{p-2}dy, \tag{16}$$

where the inequality holds uniformly for all $\theta\in(0,1)$. Furthermore, the last integral turns out to be finite which can be seen from the explicit calculation below in (17). Hence, by applying Theorem 5.4 in [14] to $X=G_\theta$ with ν being the Lebesgue measure, we obtain the following:

$$\lim_{\varepsilon\to 0}\int_{G_\theta}\int_0^1 f'\left(y_1+t\varepsilon\left(\frac{1}{2}\langle Cy^{(1)},y^{(1)}\rangle-\gamma_\varepsilon+\langle\eta,y^{(2)}\rangle\right)\right)dt\left(\frac{1}{2}\langle Cy^{(1)},y^{(1)}\rangle-\gamma_\varepsilon+\langle\eta,y^{(2)}\rangle\right)dy$$
$$=\int_{G_\theta}(p-1)|y_1|^{p-2}\left(\frac{1}{2}\langle Cy^{(1)},y^{(1)}\rangle+\langle\eta,y^{(2)}\rangle-\gamma_0\right).$$

Observe that here the upper bound in (16) allows us to conclude that the limit as $\theta\to 0^+$ is finite. We now focus on the part of the integral in (14) involving the set F_θ. Since $|F_\theta|=\int_{F_\theta}1dy$, then upon writing this integral as in (17), one sees that $|F_\theta|=c(k,v_1,\dots,v_k)\theta$, and so $|F_\theta|\to 0$, as $\theta\to 0^+$. Moreover, it suffices to consider $\theta=2c\varepsilon$ and the related $\int_{F_{2c\varepsilon}}||y_1|-2c\varepsilon|^{p-2}\,dy$. We again appeal to integral (17) and reduce our computations to finding

$$\int_{B_{v_1}(0,R_1)\cap\{|y_1|\le 2c\varepsilon\}}(2c\varepsilon-|y_1|)^{p-2}\,dy^{(1)}.$$

However, direct computation shows that this integral is of order ε^{p-1}, which then allows us to let $\varepsilon\to 0^+$, and in turn conclude (15).

In order to approach the proof of (9), we first need to compute integrals in (15). We begin with computing the denominator of (15). Once this is completed, the computation of the numerator will be more straightforward. We write

$$I=\int_B|y_1|^{p-2}dy=\int\limits_{B_{v_k}(0,1)}\int\limits_{B_{v_{k-1}}(0,R_{k-1})}\cdots\int\limits_{B_{v_2}(0,R_2)}\int\limits_{B_{v_1}(0,R_1)}|y_1|^{p-2}dy^{(1)}dy^{(2)}\dots dy^{(k-1)}dy^{(k)}, \tag{17}$$

where for $j=1,\dots,k$, $B_{v_j}(0,R_j)$ denotes the Euclidean ball in $\mathbb{R}^{v_j}$ centered at 0 with radius $R_k=1$. Furthermore, each radius $R_j>0$ is a function depending on the variables $y^{(i)}$ with $i>j$, with the following property:

$$\begin{aligned}
R_{k-1}=&R_{k-1}(y^{(k)})=\left(1-\|y^{(k)}\|^{\frac{2k!}{k}}\right)^{\frac{k-1}{2k!}}\\
R_{k-2}=&R_{k-2}(y^{(k)},y^{(k-1)})=\left(1-\|y^{(k)}\|^{\frac{2k!}{k}}-\|y^{(k-1)}\|^{\frac{2k!}{k-1}}\right)^{\frac{k-2}{2k!}}\\
&\vdots\\
R_j=&R_j(y^{(k)}\dots,y^{(j+1)})=\left(1-\|y^{(k)}\|^{\frac{2k!}{k}}-\dots-\|y^{(j+1)}\|^{\frac{2k!}{j+1}}\right)^{\frac{j}{2k!}}\\
&\vdots\\
R_2=&R_2(y^{(k)},\dots,y^{(3)})=\left(1-\sum_{i=3}^k\|y^{(i)}\|^{\frac{2k!}{i}}\right)^{\frac{2}{2k!}}\\
R_1=&R_1(y^{(k)},\dots,y^{(2)})=\left(1-\sum_{i=2}^k\|y^{(i)}\|^{\frac{2k!}{i}}\right)^{\frac{1}{2k!}}.
\end{aligned}$$

Upon applying the scaling change of variables, followed by Lemma 3.2 with $\alpha_1 = p-2$ and $\alpha_i = 0$ for $i = 2, \ldots, v_1$, we obtain the following equality:

$$\int_{B_{v_1}(0,R_1)} |y_1|^{p-2} dy^{(1)} = R_1^{v_1+p-2} \int_{B_{v_1}(0,1)} |y_1|^{p-2} dy^{(1)} = R_1^{v_1+p-2} 2^{v_1} \int_{T_{v_1}} y_1^{p-2} dy^{(1)}$$
$$= R_1^{v_1+p-2} \frac{\Gamma\left(\frac{p-1}{2}\right)\Gamma\left(\frac{1}{2}\right)^{v_1-1}}{\Gamma\left(\frac{v_1+p}{2}\right)}. \tag{18}$$

Using (18) in I, we see that

$$I = \frac{\Gamma\left(\frac{p-1}{2}\right)\Gamma\left(\frac{1}{2}\right)^{v_1-1}}{\Gamma\left(\frac{v_1+p}{2}\right)} \int_{B_{v_k}(0,1)} \cdots \int_{B_{v_2}(0,R_2)} R_1^{v_1+p-2} dy^{(2)} \ldots dy^{(k)}. \tag{19}$$

Since $R_1^{v_1+p-2}$ is a radial function with respect to $y^{(2)}, \ldots, y^{(k)}$, in particular with respect to $y^{(2)}$, we use the spherical coordinates together with the observation that $R_1 = \left(R_2^{\frac{2k!}{2}} - \|y^{(2)}\|^{\frac{2k!}{2}}\right)^{\frac{1}{2k!}}$ to obtain the following:

$$\begin{aligned}
\int_{B_{v_2}(0,R_2)} R_1^{v_1+p-2} dy^{(2)} &= \frac{2\sqrt{\pi}^{v_2}}{\Gamma\left(\frac{v_2}{2}\right)} \int_0^{R_2} \left(R_2^{\frac{2k!}{2}} - r^{\frac{2k!}{2}}\right)^{\frac{v_1+p-2}{2k!}} r^{v_2-1} dr \\
&= \frac{2\sqrt{\pi}^{v_2}}{\Gamma\left(\frac{v_2}{2}\right)} \int_0^1 R_2^{\frac{v_1+p-2}{2}} (1 - s^{\frac{2k!}{2}})^{\frac{v_1+p-2}{2k!}} R_2^{v_2-1} s^{v_2-1} R_2 ds \qquad (R_2 s := r) \\
&= \frac{2\sqrt{\pi}^{v_2}}{\Gamma\left(\frac{v_2}{2}\right)} R_2^{\frac{2v_2+v_1+p-2}{2}} \int_0^1 (1 - s^{\frac{2k!}{2}})^{\frac{v_1+p-2}{2k!}} s^{v_2-1} ds \\
&= \frac{2\sqrt{\pi}^{v_2}}{\Gamma\left(\frac{v_2}{2}\right)} R_2^{\frac{2v_2+v_1+p-2}{2}} \frac{2}{2k!} \int_0^1 (1-t)^{\frac{v_1+p-2}{2k!}} t^{\frac{2(v_2-1)}{2k!}} t^{\frac{2}{2k!}-1} dt \qquad (t := s^{\frac{2k!}{2}}) \\
&= \frac{2\sqrt{\pi}^{v_2}}{\Gamma\left(\frac{v_2}{2}\right)} R_2^{\frac{2v_2+v_1+p-2}{2}} \frac{2}{2k!} \int_0^1 (1-t)^{\frac{v_1+p-2}{2k!}} t^{\frac{2v_2}{2k!}-1} dt \\
&= \frac{4\sqrt{\pi}^{v_2}}{2k!\Gamma\left(\frac{v_2}{2}\right)} R_2^{\frac{2v_2+v_1+p-2}{2}} \mathcal{B}\left(\frac{2v_2}{2k!}, \frac{v_1+2k!+p-2}{2k!}\right).
\end{aligned}$$

In summarise, we now have

$$I = \frac{4\Gamma\left(\frac{p-1}{2}\right)\sqrt{\pi}^{v_1+v_2-1}}{2k!\Gamma\left(\frac{v_1+p}{2}\right)\Gamma\left(\frac{v_2}{2}\right)} \mathcal{B}\left(\frac{2v_2}{2k!}, \frac{v_1+2k!+p-2}{2k!}\right) \int_{B_{v_k}(0,1)} \cdots \int_{B_{v_3}(0,R_3)} R_2^{\frac{2v_2+v_1+p-2}{2}} dy^{(3)} \ldots dy^{(k)}.$$

In order to complete the computation of the iterated integral I, we need to proceed similarly to the previous case. As it turns out, the key step is to calculate the following integral:

$$\int_{B_{v_j}(0,R_j)} R_{j-1}^{\theta_j} dy^{(j)} \tag{20}$$

where $\theta_j > 0$ is defined inductively for $j = 2, 3, \ldots, k-1$. From the previous computations we see that $\theta_2 = v_1 + p - 2$ and $\theta_3 = \frac{2v_2+v_1+p-2}{2}$.

Let us observe, that from the construction of R_j, it follows that

$$R_{j-1} = \left(R_j^{\frac{2k!}{j}} - \|y^{(j)}\|^{\frac{2k!}{j}}\right)^{\frac{j-1}{2k!}}.$$

Hence

$$\int_{B_{v_j}(0,R_j)} R_{j-1}^{\theta_j} dy^{(j)} = \int_{B_{v_j}(0,R_j)} \left(R_j^{\frac{2k!}{j}} - \|y^{(j)}\|^{\frac{2k!}{j}}\right)^{\frac{(j-1)\theta_j}{2k!}} dy^{(j)} = \frac{2\sqrt{\pi}^{v_j}}{\Gamma\left(\frac{v_j}{2}\right)} \int_0^{R_j} \left(R_j^{\frac{2k!}{j}} - r^{\frac{2k!}{j}}\right)^{\frac{(j-1)\theta_j}{2k!}} r^{v_j-1} dr,$$

which again follows by the integrand being radial. We apply the change of variables $R_j s := r$ to obtain

$$\begin{aligned}
\int_0^{R_j} \left(R_j^{\frac{2k!}{j}} - r^{\frac{2k!}{j}}\right)^{\frac{(j-1)\theta_j}{2k!}} r^{v_j-1} dr &= \int_0^1 \left(R_j^{\frac{2k!}{j}} - R_j^{\frac{2k!}{j}} s^{\frac{2k!}{j}}\right)^{\frac{(j-1)\theta_j}{2k!}} R_j^{v_j-1} s^{v_j-1} R_j ds \\
&= R_j^{\frac{(j-1)\theta_j + j v_j}{j}} \int_0^1 (1 - s^{\frac{2k!}{j}})^{\frac{(j-1)\theta_j}{2k!}} s^{v_j-1} ds \\
&= R_j^{\frac{(j-1)\theta_j + j v_j}{j}} \int_0^1 (1-t)^{\frac{(j-1)\theta_j}{2k!}} t^{\frac{j(v_j-1)}{2k!}} \frac{j}{2k!} t^{\frac{j-2k!}{2k!}} dt \qquad (t := s^{\frac{2k!}{j}}) \\
&= \frac{j}{2k!} R_j^{\frac{(j-1)\theta_j + j v_j}{j}} \int_0^1 (1-t)^{\frac{(j-1)\theta_j}{2k!}} t^{\frac{j v_j - 2k!}{2k!}} dt \\
&= \frac{j}{2k!} R_j^{\frac{(j-1)\theta_j + j v_j}{j}} \mathcal{B}\left(\frac{j v_j}{2k!}, \frac{(j-1)\theta_j}{2k!} + 1\right).
\end{aligned}$$

Therefore θ_j is defined by the following recursive formula

$$\theta_2 = v_1 + p - 2 \qquad \text{and} \qquad \theta_{j+1} = v_j + \frac{j-1}{j}\theta_j, \quad j = 2, \dots, k-1,$$

which leads to the following explicit formula:

$$\theta_{j+1} = \frac{p - 2 + \sum_{i=1}^{j} i v_i}{j}. \tag{21}$$

Indeed, observe that

$$\frac{j-1}{j} \cdot \frac{p - 2 + \sum_{i=1}^{j-1} i v_i}{j-1} + v_j = \frac{p - 2 + \sum_{i=1}^{j} i v_i}{j}.$$

Now we are in a position to complete the calculation of the integral I, cf. (17) and (19):

$$\begin{aligned}
I &= \frac{\Gamma\left(\frac{p-1}{2}\right) \sqrt{\pi}^{v_1-1}}{\Gamma\left(\frac{v_1+p}{2}\right)} \int\limits_{B_{v_k}(0,1)} \cdots \int\limits_{B_{v_2}(0,R_2)} R_1^{v_1+p-2} dy^{(2)} \dots dy^{(k)} \\
&= \frac{\Gamma\left(\frac{p-1}{2}\right) \sqrt{\pi}^{v_1-1}}{\Gamma\left(\frac{v_1+p}{2}\right)} \frac{4\sqrt{\pi}^{v_2}}{2k!\Gamma\left(\frac{v_2}{2}\right)} \mathcal{B}\left(\frac{2v_2}{2k!}, \frac{v_1 + 2k! + p - 2}{2k!}\right) \int\limits_{B_{v_k}(0,1)} \cdots \int\limits_{B_{v_3}(0,R_3)} R_2^{\frac{2v_2+v_1+p-2}{2}} dy^{(3)} \dots dy^{(k)}.
\end{aligned}$$

Each inner integral of $R_{j-1}^{\theta_j}$ gives rise to the multiplicative constant

$$\frac{\sqrt{\pi}^{v_j}}{\Gamma\left(\frac{v_j}{2}\right)} \frac{j}{k!} \mathcal{B}\left(\frac{j v_j}{2k!}, \frac{(j-1)\theta_j}{2k!} + 1\right)$$

in the value of the iterated integral. Therefore, we end up with

$$I = \frac{\Gamma\left(\frac{p-1}{2}\right) \sqrt{\pi}^{-1+\sum_{j=1}^{k-1} v_j} (k-1)!}{(k!)^{k-1} \Gamma\left(\frac{v_1+p}{2}\right) \prod_{j=2}^{k-1} \Gamma\left(\frac{v_j}{2}\right)} \prod_{j=2}^{k-1} \mathcal{B}\left(\frac{j v_j}{2k!}, \frac{(j-1)\theta_j}{2k!} + 1\right) \int_{B_{v_k}(0,1)} R_{k-1}^{\theta_k} dy^{(k)}.$$

Recall, that $\theta_k = \frac{p-2+\sum_{j=1}^{k-1} jv_j}{k-1}$, $R_k = (1-\|y^{(k)}\|^{\frac{2k!}{k}})^{\frac{k-1}{2k!}}$ and compute

$$\begin{aligned}\int_{B_{v_k}(0,1)} R_{k-1}^{\theta_k} dy^{(k)} &= \int_{B_{v_k}(0,1)} (1-\|y^{(k)}\|^{\frac{2k!}{k}})^{\frac{\theta_k(k-1)}{2k!}} dy^{(k)} \\ &= \frac{2\sqrt{\pi}^{v_k}}{\Gamma\left(\frac{v_k}{2}\right)} \int_0^1 (1-r^{\frac{2k!}{k}})^{\frac{\theta_k(k-1)}{2k!}} r^{v_k-1} dr \qquad (s := r^{\frac{2k!}{k}}) \\ &= \frac{2\sqrt{\pi}^{v_k}}{\Gamma\left(\frac{v_k}{2}\right)} \frac{1}{2(k-1)!} \int_0^1 (1-s)^{\frac{\theta_k(k-1)}{2k!}} s^{\frac{v_k-1}{2(k-1)!}} s^{\frac{1}{2(k-1)!}-1} ds \\ &= \frac{\sqrt{\pi}^{v_k}}{\Gamma\left(\frac{v_k}{2}\right)(k-1)!} \int_0^1 (1-s)^{\frac{\theta_k(k-1)}{2k!}} s^{\frac{v_k-2(k-1)!}{2(k-1)!}} ds \\ &= \frac{\sqrt{\pi}^{v_k}}{\Gamma\left(\frac{v_k}{2}\right)(k-1)!} \mathcal{B}\left(\frac{v_k}{2(k-1)!}, \frac{\theta_k(k-1)}{2(k-1)!}+1\right).\end{aligned}$$

Hence we arrive at

$$I = \frac{\Gamma\left(\frac{p-1}{2}\right)\sqrt{\pi}^{-1+\sum_{i=1}^k v_i}}{(k!)^{k-1}\Gamma\left(\frac{v_1+p}{2}\right)\prod_{i=2}^k \Gamma\left(\frac{v_i}{2}\right)} \mathcal{B}\left(\frac{v_k}{2(k-1)!}, \frac{\theta_k}{2(k-2)!}+1\right) \prod_{i=2}^{k-1} \mathcal{B}\left(\frac{jv_j}{2k!}, \frac{(j-1)\theta_j}{2k!}+1\right). \quad (22)$$

Next we consider the integral in the numerator of (15), namely

$$J := \int_B |y_1|^{p-2} \left(\frac{1}{2}\langle Cy^{(1)}, y^{(1)}\rangle + \langle \eta, y^{(2)}\rangle\right) dy.$$

We note that $\int_B \langle \eta, y^{(2)}\rangle |y_1|^{p-2} = 0$, which follows by applying the change of variables

$$\psi(y^{(1)}, y^{(2)}, y^{(3)}, \dots, y^{(k)}) = (y^{(1)}, -y^{(2)}, y^{(3)}, \dots, y^{(k)}),$$

with $|J\psi| = 1$ and $\psi(B) = B$, resulting in the value of the integral being invariant under multiplication by -1. Let us denote the coefficients of matrix C as follows: $C = [c_{ij}]_{i,j=1,\dots,v_1}$, then

$$2J = \underbrace{c_{11}\int_B |y_1|^p dy}_{J_1} + \underbrace{\sum_{i\neq j} c_{ij} \int_B |y_1|^{p-2} y_i y_j dy}_{J_2} + \underbrace{\sum_{i=2}^{v_1} c_{ii} \int_B |y_1|^{p-2} y_i^2 dy}_{J_3}. \quad (23)$$

Observe, that by the symmetry of B, every integral term of the sum J_2 vanishes. We will handle J_1 and J_3 analogously to I. First, for $i = 2, \dots, v_1$ we compute the following integrals

$$\int_{B_{v_1}(0,R_1)} |y_1|^{p-2} y_i^2 dy^{(1)} = R_1^{v_1+p} \frac{\Gamma\left(\frac{p-1}{2}\right)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)^{v_1-2}}{\Gamma\left(\frac{p+v_1+2}{2}\right)} = R_1^{v_1+p} \frac{\sqrt{\pi}^{v_1-1}\Gamma\left(\frac{p-1}{2}\right)}{2\Gamma\left(\frac{p+v_1+2}{2}\right)}, \quad (24)$$

where again we use Lemma 3.2 and the familiar property $\Gamma(1+s) = s\Gamma(s)$ with $s = \frac{1}{2}$ (cf. computations at (18)).

Notice that the calculations summarised in (22) work for an arbitrary $p > 1$. More precisely, the integrals J_1 and J_3 over the ball B, can be expressed in the same way as in (17), the multiplicative constants arising from the computation of integrals (20) will be the same but with the exponents θ_j replaced by the exponents θ'_j defined by the following formula (cf. definition of θ_j in (21)):

$$\theta'_j = \frac{p + \sum_{i=1}^{j-1} iv_i}{j-1}.$$

Therefore, by using (24) and calculations analogous to those between formula (20) and (22) we arrive at

$$J_3 = \sum_{i=2}^{v_1} c_{ii} \frac{\sqrt{\pi}^{-1+\sum_{j=1}^{k-1} v_j} \Gamma\left(\frac{p-1}{2}\right)(k-1)!}{2(k!)^{k-1}\Gamma\left(\frac{p+v_1+2}{2}\right)\prod_{j=2}^{k-1}\Gamma\left(\frac{v_j}{2}\right)} \prod_{j=2}^{k-1} \mathcal{B}\left(\frac{jv_j}{2k!}, \frac{(j-1)\theta'_j}{2k!}+1\right) \int\limits_{B_{v_k}(0,1)} (1-\|y^{(k)}\|^{\frac{2k!}{k}})^{\frac{\theta'_k(k-1)}{2k!}} dy^{(k)}$$

$$= \sum_{i=2}^{v_1} c_{ii} \frac{\sqrt{\pi}^{-1+\sum_{j=1}^{k} v_j} \Gamma\left(\frac{p-1}{2}\right)}{2(k!)^{k-1}\Gamma\left(\frac{p+v_1+2}{2}\right)\prod_{j=2}^{k}\Gamma\left(\frac{v_j}{2}\right)} \mathcal{B}\left(\frac{v_k}{2(k-1)!}, \frac{\theta'_k}{2(k-2)!}+1\right) \prod_{j=2}^{k-1} \mathcal{B}\left(\frac{jv_j}{2k!}, \frac{(j-1)\theta'_j}{2k!}+1\right).$$

Moreover, in order to compute J_1, we proceed computationally the same way we did for for (17) with the power p instead of $p-2$, and obtain (22) with p now corresponding to $p+2$:

$$J_1 = c_{11} \frac{\Gamma\left(\frac{p+1}{2}\right)\sqrt{\pi}^{-1+\sum_{j=1}^{k} v_j}}{(k!)^{k-1}\Gamma\left(\frac{v_1+p+2}{2}\right)\prod_{j=2}^{k}\Gamma\left(\frac{v_j}{2}\right)} \mathcal{B}\left(\frac{v_k}{2(k-1)!}, \frac{\theta'_k}{2(k-2)!}+1\right) \prod_{j=2}^{k-1} \mathcal{B}\left(\frac{jv_j}{2k!}, \frac{(j-1)\theta'_j}{2k!}+1\right).$$

We collect the above calculations to arrive at

$$\begin{aligned} J = \frac{J_1+J_3}{2} &= \frac{\sqrt{\pi}^{-1+\sum_{j=1}^{k} v_j}}{2(k!)^{k-1}\Gamma\left(\frac{v_1+p+2}{2}\right)\prod_{j=2}^{k}\Gamma\left(\frac{v_j}{2}\right)} \mathcal{B}\left(\frac{v_k}{2(k-1)!}, \frac{\theta'_k}{2(k-2)!}+1\right) \\ &\times \left(c_{11}\Gamma\left(\frac{p+1}{2}\right) + \sum_{i=1}^{v_1} \frac{1}{2} c_{ii}\Gamma\left(\frac{p-1}{2}\right)\right) \prod_{j=2}^{k-1} \mathcal{B}\left(\frac{jv_j}{2k!}, \frac{(j-1)\theta'_j}{2k!}+1\right) \\ &= \frac{\Gamma\left(\frac{p-1}{2}\right)\sqrt{\pi}^{-1+\sum_{j=1}^{k} v_j}}{4(k!)^{k-1}\Gamma\left(\frac{v_1+p+2}{2}\right)\prod_{j=2}^{k}\Gamma\left(\frac{v_j}{2}\right)} \mathcal{B}\left(\frac{v_k}{2(k-1)!}, \frac{\theta'_k}{2(k-2)!}+1\right) \\ &\times \left(c_{11}(p-1) + \sum_{i=2}^{v_1} c_{ii}\right) \prod_{j=2}^{k-1} \mathcal{B}\left(\frac{jv_j}{2k!}, \frac{(j-1)\theta'_j}{2k!}+1\right), \end{aligned}$$

where we again use the familiar property of the Γ function as in (24). It now follows that

$$\begin{aligned} \gamma_0 = \frac{J}{I} &= \frac{\Gamma\left(\frac{p+v_1}{2}\right)}{4\Gamma\left(\frac{p+2+v_1}{2}\right)} \frac{\mathcal{B}\left(\frac{v_k}{2(k-1)!}, \frac{\theta'_k}{2(k-2)!}+1\right)}{\mathcal{B}\left(\frac{v_k}{2(k-1)!}, \frac{\theta_k}{2(k-2)!}+1\right)} \left(c_{11}(p-1) + \sum_{i=2}^{v_1} c_{ii}\right) \prod_{j=2}^{k-1} \frac{\mathcal{B}\left(\frac{jv_j}{2k!}, \frac{(j-1)\theta'_j}{2k!}+1\right)}{\mathcal{B}\left(\frac{jv_j}{2k!}, \frac{(j-1)\theta_j}{2k!}+1\right)} \\ &= \frac{1}{2(p+v_1)} \frac{\mathcal{B}\left(\frac{v_k}{2(k-1)!}, \frac{\theta'_k}{2(k-2)!}+1\right)}{\mathcal{B}\left(\frac{v_k}{2(k-1)!}, \frac{\theta_k}{2(k-2)!}+1\right)} \prod_{j=2}^{k-1} \frac{\mathcal{B}\left(\frac{jv_j}{2k!}, \frac{(j-1)\theta'_j}{2k!}+1\right)}{\mathcal{B}\left(\frac{jv_j}{2k!}, \frac{(j-1)\theta_j}{2k!}+1\right)} \left(c_{11}(p-1) + \sum_{i=2}^{v_1} c_{ii}\right) \\ &= c(p, v_1, \ldots, v_k) \cdot \left(c_{11}(p-1) + \sum_{i=2}^{v_1} c_{ii}\right), \end{aligned}$$

where the constant $c(p, v_1, \ldots, v_k)$ is defined with the above equality (see also Remark 1 and Examples 2-4 in Section 3 for further discussion about this constant).

In order to arrive at assertion (9), we express the constants c_{11} and $\mathrm{tr}(C)$ in terms of the matrix A and the vector ξ. Recall that $C = R^T AR$ and $R^T\xi = e_1$, which imply that

$$c_{11} = \langle Ce_1, e_1\rangle = \langle CR^T\xi, R^T\xi\rangle = \langle R(R^TAR)R^T\xi, \xi\rangle = \langle A\xi, \xi\rangle,$$

moreover, the orthogonality of R implies that $\mathrm{tr}(C) = \mathrm{tr}(R^TAR) = \mathrm{tr}(A)$. Therefore, we can conclude that

$$\gamma_0 = c(p, v_1, \ldots, v_k)(\langle A\xi, \xi\rangle(p-2) + \mathrm{tr}(A)),$$

which upon substituting ξ with $\xi/|\xi|$, proves the assertion (9).

We now consider the second assertion of the lemma, namely the asymptotic formula (10) for $\mu_p(\varepsilon, u)$ and $u \in C^2(\Omega)$. Suppose $\varepsilon > 0$ is chosen so that $\overline{B_\varepsilon(x)} \subset \Omega$. Consider the function $q(y)$ as in (8), with

$$q(x) = u(x), \quad \xi = \nabla_{V_1} u(x), \quad A = \nabla^2_{V_1} u(x), \quad \text{and} \quad \eta = 2\nabla_{V_2} u(x).$$

Notice that with this notation (and by the assumption $\xi \neq 0$), it holds that

$$\Delta^N_{p,\mathbb{G}} u(x) = \operatorname{tr}(A) + (p-2)\frac{\langle A\xi, \xi\rangle}{|\xi|^2}.$$

Set $u_\varepsilon(z) = u(x\delta_\varepsilon(z))$ and $q_\varepsilon(z) = q(x\delta_\varepsilon(z))$. Since $u \in C^2(\Omega)$, it follows that for all $t > 0$, there exists $\varepsilon(t) > 0$ such that for every $z \in \overline{B}$ and all $\varepsilon \in (0, \varepsilon(t))$ it holds $|u_\varepsilon(z) - q_\varepsilon(z)| < t\varepsilon^2$. Furthermore, by claims (4) and (5) of Theorem 2.1 we have $\mu_p(\varepsilon, q \pm t\varepsilon^2)(x) = \mu_p(\varepsilon, q)(x) \pm t\varepsilon^2$. These observations together with Corollary 2.1 and Part (3) of Theorem 2.1 allow us to obtain the following estimates:

$$\frac{\mu_p(\varepsilon, q) - u(x)}{\varepsilon^2} - t \leq \frac{\mu_p(\varepsilon, u) - u(x)}{\varepsilon^2} \leq \frac{\mu_p(\varepsilon, q) - u(x)}{\varepsilon^2} + t.$$

Applying (9) we obtain

$$\begin{aligned} c(p, v_1, \ldots, v_k)\Delta^N_{p,\mathbb{G}} u(x) - t &\leq \liminf_{\varepsilon \to 0} \frac{\mu_p(\varepsilon, q) - u(x)}{\varepsilon^2} \\ &\leq \limsup_{\varepsilon \to 0} \frac{\mu_p(\varepsilon, q) - u(x)}{\varepsilon^2} \leq c(p, v_1, \ldots, v_k)\Delta^N_{p,\mathbb{G}} u(x) + t, \end{aligned}$$

which implies the assertion (10) for $1 < p < \infty$.

3.2 Case 2: $p = \infty$.

We need to demonstrate that the expression

$$\begin{aligned} \gamma_\varepsilon &= \frac{\mu_\infty(\varepsilon, q) - q(x)}{\varepsilon^2} \\ &= \frac{1}{2\varepsilon}\left(\min_{y \in \overline{B}} \left[\langle \xi, y^{(1)}\rangle + \varepsilon \left(\langle \eta, y^{(2)}\rangle + \frac{1}{2}\langle Ay^{(1)}, y^{(1)}\rangle \right)\right] \right. \\ &\quad \left. + \max_{y \in \overline{B}} \left[\langle \xi, y^{(1)}\rangle + \varepsilon \left(\langle \eta, y^{(2)}\rangle + \frac{1}{2}\langle Ay^{(1)}, y^{(1)}\rangle \right)\right] \right) \end{aligned}$$

has a limit as $\varepsilon \to 0$.

Let us define a function $g : \mathbb{G} \to \mathbb{R}$ by setting $g(y) = \langle \xi, y^{(1)}\rangle + \langle \eta, y^{(2)}\rangle + \frac{1}{2}\langle Ay^{(1)}, y^{(1)}\rangle$. Observe further, that the change of variables $y = \delta_{1/\varepsilon}(z)$ implies the following equalities:

$$\min_{y \in \overline{B_1(0)}} \left[\langle \xi, y^{(1)}\rangle + \varepsilon \left(\langle \eta, y^{(2)}\rangle + \frac{1}{2}\langle Ay^{(1)}, y^{(1)}\rangle \right)\right] = \frac{1}{\varepsilon} \min_{z \in \overline{B_\varepsilon(0)}} g(z),$$

and

$$\max_{y \in \overline{B_1(0)}} \left[\langle \xi, y^{(1)}\rangle + \varepsilon \left(\langle \eta, y^{(2)}\rangle + \frac{1}{2}\langle Ay^{(1)}, y^{(1)}\rangle \right)\right] = \frac{1}{\varepsilon} \max_{z \in \overline{B_\varepsilon(0)}} g(z),$$

and it follows that

$$\gamma_\varepsilon = \frac{1}{2\varepsilon^2}\left(\min_{z \in \overline{B_\varepsilon(0)}} g(z) + \max_{z \in \overline{B_\varepsilon(0)}} g(z) \right).$$

Next we note that $\nabla_{V_1} g(0) = \xi \neq 0$, thus we can apply Lemma 1.5 and 1.6 in [10], and affirm that for all small enough ε, there exist points $P_{\varepsilon,M} = (y^{(1)}_{\varepsilon,M}, \dots, y^{(k)}_{\varepsilon,M})$ and $P_{\varepsilon,m} = (y^{(1)}_{\varepsilon,m}, \dots, y^{(k)}_{\varepsilon,m})$ in $\partial B_\varepsilon(0)$ with the following properties:

$$\max_{B_\varepsilon(0)} g = g(P_{\varepsilon,M}) \qquad \text{and} \qquad \min_{B_\varepsilon(0)} g = g(P_{\varepsilon,m}).$$

In terms of the expression we have the following estimate

$$\frac{1}{2\varepsilon^2}\left(g(P_{\varepsilon,m}) + g(-P_{\varepsilon,m})\right) \leq \gamma_\varepsilon \leq \frac{1}{2\varepsilon^2}\left(g(P_{\varepsilon,M}) + g(-P_{\varepsilon,M})\right). \tag{25}$$

Moreover, by applying again [10, Lemma 1.6], we have that

$$\lim_{\varepsilon\to 0} \frac{y^{(1)}_{\varepsilon,M}}{\varepsilon} = \frac{\xi}{|\xi|} \qquad \text{and} \qquad \lim_{\varepsilon\to 0} \frac{y^{(1)}_{\varepsilon,m}}{\varepsilon} = -\frac{\xi}{|\xi|}, \tag{26}$$

which implies

$$\begin{aligned}\frac{1}{2\varepsilon^2}(g(P_{\varepsilon,M}) + g(-P_{\varepsilon,M})) &= \frac{1}{4\varepsilon^2}\left(\langle A y^{(1)}_{\varepsilon,M}, y^{(1)}_{\varepsilon,M}\rangle + \langle A - y^{(1)}_{\varepsilon,M}, -y^{(1)}_{\varepsilon,M}\rangle\right)\\ &= \frac{1}{2}\langle A\frac{y^{(1)}_{\varepsilon,M}}{\varepsilon}, \frac{y^{(1)}_{\varepsilon,M}}{\varepsilon}\rangle \xrightarrow{\varepsilon\to 0} \frac{1}{2}\frac{\langle A\xi,\xi\rangle}{|\xi|^2}.\end{aligned}$$

We treat the left-hand side of (25) similarly to conclude that

$$\mu_\infty(\varepsilon, q) = q(x) + \frac{\varepsilon^2}{2}\frac{\langle A\xi,\xi\rangle}{|\xi|^2} + o(\varepsilon^2).$$

Upon repeating the reasoning similar to the one for $\Delta^N_{p,\mathbb{G}}$, we obtain that asymptotic formula (10) holds for $\Delta^N_{\infty,\mathbb{G}}$ as well.

3.3 Case 3: $p = 1$.

Recall, that by the discussion at the beginning of the proof of Lemma 3.1 (cf. (12)), the unique number γ_ε is defined with the following equation

$$\begin{aligned}&|\{z \in B : \langle \xi, z^{(1)}\rangle + \varepsilon\left(\frac{1}{2}\langle A z^{(1)}, z^{(1)}\rangle + \langle \eta, z^{(2)}\rangle\right) < \varepsilon\gamma_\varepsilon\}|\\ &= |\{z \in B : \langle \xi, z^{(1)}\rangle + \varepsilon\left(\frac{1}{2}\langle A z^{(1)}, z^{(1)}\rangle + \langle \eta, z^{(2)}\rangle\right) > \varepsilon\gamma_\varepsilon\}|.\end{aligned}$$

We apply the same change of variables via the matrix R, as described in the paragraph following formula (13) (for the sake of simplicity we still use the variable z) and divide both inequalities by ε to arrive at

$$|\{z \in B : \frac{z_1}{\varepsilon} + \frac{1}{2}\langle C z^{(1)}, z^{(1)}\rangle + \langle \eta, z^{(2)}\rangle < \gamma_\varepsilon\}| = |\{z \in B : \frac{z_1}{\varepsilon} + \frac{1}{2}\langle C z^{(1)}, z^{(1)}\rangle + \langle \eta, z^{(2)}\rangle > \gamma_\varepsilon\}|. \tag{27}$$

We again we assume that $|\xi| = 1$ and let $C = R^T A R$, where R denotes the rotation matrix as defined in the discussion following (13). Equation (27) means that for each fixed $\varepsilon > 0$, γ_ε is the median $\mu_1(1, h) =: \mu_1(h)$ of the function $h : \overline{B} \to \mathbb{R}$ defined with the following formula

$$h(z) := \frac{z_1}{\varepsilon} + \frac{1}{2}\langle C z^{(1)}, z^{(1)}\rangle + \langle \eta, z^{(2)}\rangle.$$

Denote by $c' := \|\frac{1}{2}\langle Cz^{(1)}, z^{(1)}\|_{L^\infty(B)} < \infty$. Let us observe, that by monotonicity of μ_1 and property (4) in Theorem 2.1, we obtain the following estimates

$$\begin{aligned} \gamma_\varepsilon &= \mu_1\left(\frac{z_1}{\varepsilon} + \frac{1}{2}\langle Cz^{(1)}, z^{(1)}\rangle + \langle \eta, z^{(2)}\rangle\right) \\ &\le \mu_1\left(\frac{z_1}{\varepsilon} + \langle \eta, z^{(2)}\rangle + c'\right) \\ &= \mu_1\left(\frac{z_1}{\varepsilon} + \langle \eta, z^{(2)}\rangle\right) + c', \end{aligned} \tag{28}$$

and

$$\begin{aligned} \gamma_\varepsilon &= \mu_1\left(\frac{z_1}{\varepsilon} + \frac{1}{2}\langle Cz^{(1)}, z^{(1)}\rangle + \langle \eta, z^{(2)}\rangle\right) \\ &\ge \mu_1\left(\frac{z_1}{\varepsilon} + \langle \eta, z^{(2)}\rangle - c'\right) \\ &= \mu_1\left(\frac{z_1}{\varepsilon} + \langle \eta, z^{(2)}\rangle\right) - c'. \end{aligned} \tag{29}$$

Let us observe, that for all $\varepsilon > 0$ we have

$$|\{z \in B : \frac{z_1}{\varepsilon} + \langle \eta, z^{(2)}\rangle < 0\}| = |\{z \in B : \frac{z_1}{\varepsilon} + \langle \eta, z^{(2)}\rangle > 0\}|.$$

since the two quantities are equivalent under the change of variables $z \mapsto -z$. It then follows that

$$\mu_1\left(\frac{z_1}{\varepsilon} + \langle \eta, z^{(2)}\rangle\right) = 0,$$

and estimates (28) and (29) reads as $-c' \le \gamma_\varepsilon \le c'$. Hence γ_ε is bounded, and after passing to a subsequence, there exists $\gamma_0 := \lim_{\varepsilon\to 0}\gamma_\varepsilon$.

Now let us apply to both sides of (27) the following change of variables

$$(z_1, z_2, \dots, z_{v_1}, z^{(2)}, \dots, z^{(k)}) \mapsto (\varepsilon z_1, z_2, z_3, \dots, z_{v_1}, z^{(2)}, \dots, z^{(k)}) =: \varepsilon z_1 e_1 + \tilde{z},$$

where $\tilde{z} := (0, z_2, \dots, z_{v_1}, z^{(2)}, \dots, z^{(k)})$. The Jacobian of this transformation is constant, hence it cancels out on both sides and (27) becomes

$$\begin{aligned} &|\{z \in \mathbb{R}^m : |\varepsilon z_1 e_1 + \tilde{z}|_{\mathbb{G}} < 1, \quad z_1 + \left(\frac{1}{2}\langle C(\varepsilon z_1 e_1 + \tilde{z}^{(1)}), (\varepsilon z_1 e_1 + \tilde{z}^{(1)})\rangle + \langle \eta, z^{(2)}\rangle\right) < \gamma_\varepsilon\}| \\ &= |\{z \in \mathbb{R}^m : |\varepsilon z_1 e_1 + \tilde{z}|_{\mathbb{G}} < 1, \quad z_1 + \left(\frac{1}{2}\langle C(\varepsilon z_1 e_1 + \tilde{z}^{(1)}), (\varepsilon z_1 e_1 + \tilde{z}^{(1)})\rangle + \langle \eta, z^{(2)}\rangle\right) > \gamma_\varepsilon\}|. \end{aligned} \tag{30}$$

Let us denote by $\tilde{B} := \{(z_2, \dots, z_{v_1}, z^{(2)}, \dots, z^{(k)}) \in \mathbb{R}^{m-1} : |(0, z_2, \dots, z_{v_1}, z^{(2)}, \dots, z^{(k)})|_{\mathbb{G}} < 1\}$ and consider a function $F : \{z \in \mathbb{R}^m : |\varepsilon z_1 e_1 + \tilde{z}|_{\mathbb{G}} < 1\} \to \mathbb{R}$ defined by

$$F(z) := z_1 + \left(\frac{1}{2}\langle C(\varepsilon z_1 e_1 + \tilde{z}^{(1)}), (\varepsilon z_1 e_1 + \tilde{z}^{(1)})\rangle + \langle \eta, z^{(2)}\rangle\right).$$

For small ε, we are going to represent the common boundary of the sets in (30), i.e., the surface $\{F(z) = \gamma_\varepsilon \ : \ |\varepsilon z_1 e_1 + \tilde{z}|_{\mathbb{G}} < 1\}$, as the graph of a function of the form $\tilde{z} \to g_\varepsilon(\tilde{z})e_1 + \tilde{z}$ where $g_\varepsilon : \tilde{B} \to \mathbb{R}$.

Let us observe, that the derivative F_{z_1} can be estimated from below:

$$F_{z_1}(z) = 1 + \varepsilon^2 c_{11} z_1 + \varepsilon(c_{12}z_2 + c_{13}z_3 \dots + c_{1v_1}z_{v_1}) > \frac{1}{2}$$

for ε sufficiently small. This follows from $|\varepsilon z_1 e_1 + \tilde{z}|_{\mathbb{G}} < 1$ and the fact that

$$-\varepsilon\sum_{i=1}^{v_1}|c_{1i}| \le \varepsilon^2 c_{11} z_1 + \varepsilon(c_{12}z_2 + c_{13}z_3 \dots + c_{1v_1}z_{v_1}) \le \varepsilon\sum_{i=1}^{v_1}|c_{1i}|.$$

Hence for a fixed $\tilde{z} \in \tilde{B}$ the function $z_1 \to F(z_1 e_1 + \tilde{z})$ is monotone increasing and therefore has an inverse $h_{\varepsilon,\tilde{z}}(t)$. It follows that $F(h_{\varepsilon,\tilde{z}}(t)e_1 + \tilde{z}) = t$ and $g_\varepsilon(\tilde{z}) = h_{\varepsilon,\tilde{z}}(\gamma_\varepsilon)$ is a point in the common boundary. Furthermore, let us observe that, possibly after passing to a subsequence, the following limit exists for all $\tilde{z} \in \tilde{B}$

$$g_\varepsilon(\tilde{z}) \to \gamma_0 - \frac{1}{2}\langle C\tilde{z}^{(1)}, \tilde{z}^{(1)}\rangle - \langle \eta, z^{(2)}\rangle \quad \text{as } \varepsilon \to 0^+. \tag{31}$$

Indeed, for all $\tilde{z} \in \tilde{B}$ the equation $F(g_\varepsilon(\tilde{z})e_1 + \tilde{z}) = \gamma_\varepsilon$ equivalently reads:

$$g_\varepsilon(\tilde{z}) + \frac{1}{2}\langle C(\varepsilon g_\varepsilon(\tilde{z})e_1 + \tilde{z}^{(1)}), (\varepsilon g_\varepsilon(\tilde{z})e_1 + \tilde{z}^{(1)})\rangle + \langle \eta, z^{(2)}\rangle = \gamma_\varepsilon.$$

From this we get that

$$g_\varepsilon(\tilde{z}) + \frac{1}{2}\left(\varepsilon^2 c_{11} g_\varepsilon^2(\tilde{z}) + 2\varepsilon \sum_{i=2}^{v_1} c_{1i} g_\varepsilon(\tilde{z}) z_i + \langle C\tilde{z}^{(1)}, \tilde{z}^{(1)}\rangle\right) + \langle \eta, z^{(2)}\rangle = \gamma_\varepsilon,$$

which for fixed $\tilde{z}$ and $c_{11} \neq 0$ is the following quadratic equation in $g_\varepsilon(\tilde{z})$:

$$g_\varepsilon^2(\tilde{z})\frac{\varepsilon^2 c_{11}}{2} + g_\varepsilon(\tilde{z})\left(1 + 2\varepsilon \sum_{i=2}^{v_1} c_{1i} z_i\right) + \frac{1}{2}\langle C\tilde{z}^{(1)}, \tilde{z}^{(1)}\rangle + \langle \eta, z^{(2)}\rangle - \gamma_\varepsilon = 0.$$

Therefore $g_\varepsilon(\tilde{z})$ has to be either equal to

$$g_\varepsilon(\tilde{z}) = \frac{-1 - 2\varepsilon\sum_{i=2}^{v_1} c_{1i} z_i + \sqrt{\left(1 + 2\varepsilon\sum_{i=2}^{v_1} c_{1i} z_i\right)^2 - 2\varepsilon^2 c_{11}\left(\frac{1}{2}\langle C\tilde{z}^{(1)}, \tilde{z}^{(1)}\rangle + \langle \eta, z^{(2)}\rangle - \gamma_\varepsilon\right)}}{\varepsilon^2 c_{11}},$$

or equal to

$$g_\varepsilon(\tilde{z}) = \frac{-1 - 2\varepsilon\sum_{i=2}^{v_1} c_{1i} z_i - \sqrt{\left(1 + 2\varepsilon\sum_{i=2}^{v_1} c_{1i} z_i\right)^2 - 2\varepsilon^2 c_{11}\left(\frac{1}{2}\langle C\tilde{z}^{(1)}, \tilde{z}^{(1)}\rangle + \langle \eta, z^{(2)}\rangle - \gamma_\varepsilon\right)}}{\varepsilon^2 c_{11}}.$$

Observe, that the second solution is of order ε^{-2} and therefore violates the condition $|\varepsilon g_\varepsilon(\tilde{z})e_1 + \tilde{z}|_{\mathbb{G}} < 1$ for $\varepsilon \to 0^+$. We consider the first solution, which after cancellation reads

$$g_\varepsilon(\tilde{z}) = \frac{2\left(\gamma_\varepsilon - \frac{1}{2}\langle C\tilde{z}^{(1)}, \tilde{z}^{(1)}\rangle - \langle \eta, z^{(2)}\rangle\right)}{\sqrt{\left(1 + 2\varepsilon\sum_{i=2}^{v_1} c_{1i} z_i\right)^2 - 2\varepsilon^2 c_{11}\left(\frac{1}{2}\langle C\tilde{z}^{(1)}, \tilde{z}^{(1)}\rangle + \langle \eta, z^{(2)}\rangle - \gamma_\varepsilon\right)} + 1 + 2\varepsilon\sum_{i=2}^{v_1} c_{1i} z_i}.$$

If $c_{11} = 0$ then

$$g_\varepsilon(\tilde{z})\left(1 + 2\varepsilon\sum_{i=2}^{v_1} c_{1i} z_i\right) = \gamma_\varepsilon - \frac{1}{2}\langle C\tilde{z}^{(1)}, \tilde{z}^{(1)}\rangle - \langle \eta, z^{(2)}\rangle.$$

Therefore, we conclude (31).

Thus, we can represent the measures of the sets appearing in (30) as integrals, and obtain the following

$$\int_{\tilde{B}} \left[\min\{g_\varepsilon(\tilde{z}), G_\varepsilon(\tilde{z})\} + G_\varepsilon(\tilde{z})\right] d\tilde{z} = \int_{\tilde{B}} \left[G_\varepsilon(\tilde{z}) - \max\{g_\varepsilon(\tilde{z}), G_\varepsilon(\tilde{z})\}\right] d\tilde{z}, \tag{32}$$

where

$$G_\varepsilon(\tilde{z}) := \frac{1}{\varepsilon}\sqrt{\left(1 - \sum_{j=2}^{k} \|z^{(j)}\|^{\frac{2k!}{j}}\right)^{\frac{1}{k!}} - (z_2^2 + \ldots + z_{v_1}^2)}.$$

The function G_ε is the non-negative solution z_1 to the equation $|\varepsilon z_1 e_1 + \tilde{z}|_{\mathbb{G}} = 1$ describing the boundary of B. Observe, that (32) is equivalent to

$$\int_{\tilde{B}} \left[\min\{g_\varepsilon(\tilde{z}), G_\varepsilon(\tilde{z})\} + \max\{g_\varepsilon(\tilde{z}), G_\varepsilon(\tilde{z})\}\right] d\tilde{z} = 0. \tag{33}$$

Applying the dominated convergence theorem to the case $\varepsilon \to 0^+$ in (33) gives the following

$$\int_{\tilde{B}} \left(\gamma_0 - \frac{1}{2}\langle C\tilde{z}^{(1)}, \tilde{z}^{(1)}\rangle - \langle \eta, z^{(2)}\rangle\right) d\tilde{z} = 0. \tag{34}$$

The symmetry of $\tilde{B}$ shows that $\int_{\tilde{B}} \langle \eta, z^{(2)}\rangle = 0$ and so (34) becomes

$$\gamma_0 = \frac{1}{2} ⨍_{\tilde{B}} \langle C\tilde{z}^{(1)}, \tilde{z}^{(1)}\rangle d\tilde{z}.$$

Due to symmetries of $\tilde{B}$ the right-hand side can be written as

$$\gamma_0 = \frac{1}{2}\sum_{i=2}^{v_1} c_{ii} ⨍_{\tilde{B}} z_i^2 d\tilde{z}.$$

Observe, that the calculation of the above integrals is essentially covered by the calculations for I defined in (17) and J_3 defined in (23) with $p = 2$. Indeed, the set $\tilde{B}$ has the same structure as B, since $\tilde{B}$ is defined with the inequality $|\tilde{z}|_{\mathbb{G}} < 1$. More precisely, the expression $|\tilde{z}|_{\mathbb{G}}$ can be interpreted in the following way: define $y^{(1)} := \tilde{z}^{(1)} = (z_2, z_3, \ldots, z_{v_1})$ and $y^{(i)} := \tilde{z}^{(i)}$ for $i = 2, 3, \ldots, k$, $\tilde{v_1} = v_1 - 1$, $\tilde{v_i} = v_i$ for $i = 2, \ldots, k$ and $|y|_{\tilde{\mathbb{G}}} = |\tilde{z}|_{\mathbb{G}}$. Following the calculations below (17) and (23) we obtain

$$\tilde{\theta}_{j+1} = \frac{\sum_{i=i}^{j} i\tilde{v_i}}{j} = \frac{-1 + \sum_{i=i}^{j} iv_i}{j} \quad \text{and} \quad \tilde{\theta}'_{j+1} = \frac{2 + \sum_{i=1}^{j} i\tilde{v_i}}{j} = \frac{1 + \sum_{i=1}^{j} iv_i}{j}$$

for $j = 1, \ldots, k-1$. We sum up these observations with

$$\gamma_0 = \sum_{i=2}^{v_1} c_{ii} \frac{\sqrt{\pi}^{-2+\sum_{j=1}^{k} v_j}\Gamma\left(\frac{2-1}{2}\right)}{4(k!)^{k-1}\Gamma\left(\frac{2+v_1+1}{2}\right)\prod_{j=2}^{k}\Gamma\left(\frac{v_j}{2}\right)} \mathcal{B}\left(\frac{v_k}{2(k-1)!}, \frac{\tilde{\theta}'_k}{2(k-2)!} + 1\right) \prod_{j=2}^{k-1} \mathcal{B}\left(\frac{jv_j}{2k!}, \frac{(j-1)\tilde{\theta}'_j}{2k!} + 1\right)$$
$$/ \frac{\Gamma\left(\frac{2-1}{2}\right)\sqrt{\pi}^{-2+\sum_{i=1}^{k} v_i}}{(k!)^{k-1}\Gamma\left(\frac{v_1+1}{2}\right)\prod_{i=2}^{k}\Gamma\left(\frac{v_i}{2}\right)} \mathcal{B}\left(\frac{v_k}{2(k-1)!}, \frac{\tilde{\theta}_k}{2(k-2)!} + 1\right) \prod_{i=2}^{k-1} \mathcal{B}\left(\frac{jv_j}{2k!}, \frac{(j-1)\tilde{\theta}_j}{2k!} + 1\right),$$

which reduces to

$$\begin{aligned}\gamma_0 &= \frac{1}{2(v_1+1)} \frac{\mathcal{B}\left(\frac{v_k}{2(k-1)!}, \frac{\tilde{\theta}'_k}{2(k-2)!} + 1\right)}{\mathcal{B}\left(\frac{v_k}{2(k-1)!}, \frac{\tilde{\theta}_k}{2(k-2)!} + 1\right)} \prod_{i=2}^{k-1} \frac{\mathcal{B}\left(\frac{jv_j}{2k!}, \frac{(j-1)\tilde{\theta}'_j}{2k!} + 1\right)}{\mathcal{B}\left(\frac{jv_j}{2k!}, \frac{(j-1)\tilde{\theta}_j}{2k!} + 1\right)} \sum_{i=2}^{v_1} c_{ii} \\ &= c(z, v_1, \ldots, v_k)\sum_{i=2}^{v_1} c_{ii} = c(z, v_1, \ldots, v_k)\left(\operatorname{tr}(\mathrm{A}) - \frac{\langle \mathrm{A}\xi, \xi\rangle}{|\xi|^2}\right).\end{aligned}$$

The last equality follows from the same argument used in the case $1 < p < \infty$, and the same reasoning allows us to conclude (9) and (10) for $p = 1$ as well. Thus, the proof of Lemma 3.1 is completed for all $1 \leq p \leq \infty$. □

Let us comment about the differences between the above lemma and [14, Lemma 3.1].

Remark 2. (1) The quadratic polynomial q in formula (8) is defined for any Carnot group of step k and differs from the original one studied in $\mathbb{R}^n$, cf. [14, Lemma 3.1]. The formula for q reflects the dependence of q on the first two layers of $\mathbb{G}$.

(2) The geometry of gauge balls in Carnot groups is far from Euclidean and non-trivial in comparision: balls are flattened at the characteristic points (at poles) and possess less symmetry than balls in $\mathbb{R}^n$. The k-step stratification of $\mathbb{G}$ allows for the integrals I (see (17)), J_1 and J_3 (see the discussion following formula (23)) to be expressed in a straightforward manner as multiple integrals. However, evaluating these integrals leads to technically involved computations, cf. (19) and the computations following it. A noticable difference in comparison with [14] is the appearance of the Beta function which is not present in the Euclidean case and can be viewed as consequence of the stratification in the geometry.

(3) Our proof for the case $p = \infty$ differs from the corresponding one in [14], as it requires appealing to results in [10]. Indeed, the geometry of gauge balls in general Carnot groups makes obtaining limits in (26) a subtle and highly nontrivial task, see the proof of Lemma 1.6 in [10] and the discussion following its formulation in [10] on pg. 207.

We are now in position to prove Theorem 1.1.

The proof of Theorem 1.1. Let $B(x) \subset \Omega$ be ball and let us fix $u \in C^0(\Omega)$ and $\phi \in C^2(B(x))$ with $\nabla_{V_1}\phi(x) \neq 0$. The asymptotic formula (10) implies that

$$\phi(x) = \mu_p(\varepsilon, \phi)(x) - c(p, v_1, \dots, v_k)\Delta^N_{p,\mathbb{G}}\phi(x)\varepsilon^2 + o(\varepsilon^2), \quad \text{as} \quad \varepsilon \to 0. \tag{35}$$

Suppose that u is a viscosity solution, in the sense of Definition 2.2, to the equation $\Delta^N_{p,\mathbb{G}}u = 0$ in Ω. Thus, in particular, u satisfies parts (i) and (ii) of Definition 2.2. Since u is a viscosity supersolution of $\Delta^N_{p,\mathbb{G}} = 0$ in Ω, then at point x, for ϕ as above such that $u - \phi$ has a strict minimum at x and $u(x) = \phi(x)$, it holds that $\Delta^N_{p,\mathbb{G}}\phi(x) \leq 0$. Therefore, from (35) we obtain

$$\phi(x) \geq \mu_p(\varepsilon, u)(x) + o(\varepsilon^2), \quad \text{as} \quad \varepsilon \to 0,$$

which proves that ϕ at x satisfies part (i) of Definition 2.4. By using the fact that u is also a viscosity subsolution (and so u satisfies part (ii) of Definition 2.2) we show that inequality in part (ii) of Definition 2.4 holds as well. This proves that $u(x) = \mu_p(\varepsilon, u)(x) + o(\varepsilon^2)$ as $\varepsilon \to 0$ in the viscosity sense.

Now we will prove the converse. Suppose, that $u(x) = \mu_p(\varepsilon, u)(x) + o(\varepsilon^2)$ as $\varepsilon \to 0$ in the viscosity sense. If $u - \phi$ attains a strict minimum at x, then by Definition 2.4, it follows that $\phi(x) \geq \mu_p(\varepsilon, \phi)(x) + o(\varepsilon^2)$ as $\varepsilon \to 0$. Using this result in (35), we get

$$\Delta^N_{p,\mathbb{G}}\phi(x) = \frac{\mu_p(\varepsilon, \phi)(x) - \phi(x)}{c(p, v_1, \dots, v_k)\varepsilon^2} + o(1) \leq o(1),$$

as $\varepsilon \to 0$, and hence $\Delta^N_{p,\mathbb{G}}\phi(x) \leq 0$. We apply a similar reasoning in the case $u - \phi$ has a strict maximum at x. This proves, that u is a viscosity solution of $\Delta^N_{p,\mathbb{G}}u = 0$ in Ω. □

Remark 3. Mean value formulas similar to the ones proved in the present paper have been used in [18] to study random walks and random tug of war in the Heisenberg group. In [18], the authors implemented the approach of Peres-Sheffield [24] to provide a game-theoretical interpretation of the p-Laplacian in the Heisenberg group, they also characterized its viscosity solutions via an asymptotic mean value expansion similar to the one proved in [20]. We expect that the result proved in the present paper could be used to generalize [18] to general Carnot groups.

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