Sequences of L^∞-optimal control problems: Γ-convergence and Hamilton-Jacobi equations.

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Abstract

We consider the sequence of optimal control problems having as state equation $y'(t) = a_n(t,y) + b_n(t,u)$ ($t \in (0,T]$, $y(0) = x$) and cost functional $J_n(y,u) = \operatorname{ess\,sup}_{t\in[0,T]} f_n(t,y(t),u(t))$. We prove a Γ-convergence result and we study the entailed properties on the stability for the related Hamilton-Jacobi equations.

1 Introduction

In the last years people become interested in the problem of minimizing the maximum of a function instead of his average value or, more generally, his integral. From an application point of view this approach covers a wide range of problems. For example, one could be interested in designing a support structure wanting to minimize the maximum pointwise stress instead of the average stress. Moreover, in designing a feedback control thermostat to control the heat in a room, one observes that it is more important to minimize the uniform distance to the desired temperature than the average square distance. Thus, it become natural to consider optimal control problems having an L^∞-functional as cost. The related Hamilton-Jacobi equation has been first studied by Barron and Ishii in 1989, [2], using the theory of viscosity solution for discontinuous Hamiltonians. As related works we recall, e.g., [3], [4], [5] and [6] while for a purely variational approach to the L^∞-functionals we refer, for example, to [1], [4], [7], [9] and [15].

In this paper we study sequences of L^∞-optimal control problems having as state equation

$$\begin{cases} y'(t) = a_n(t,y) + b_n(t,u) & t \in (0,T] \\ y(0) = x, \end{cases} \tag{1.1}$$

and as cost functional the supremal functional

$$J_n(y,u) = \operatorname*{ess\,sup}_{t\in[0,T]} f_n(t,y(t),u(t)). \tag{1.2}$$

Our aim is twofold. First, we prove a Γ-convergence result for the above sequence and then we study the entailed stability result for the related Hamilton-Jacobi equations.

The Γ-convergence theorem is based on the result obtained for sequences of supremal functionals in [9] and on the general theory of Γ-limits for optimal control problems studied, e.g., in [10] and [11].

The main result (Theorem 2.4) shows that assuming, beside regular assumptions, the following convergence hypotheses:

a) $\lim_{n\to\infty} a_n(\cdot,y) = a(\cdot,y)$ weakly$L^1(0,T;\mathbb{R}^N)$, $\forall y \in \mathbb{R}^N$,

b) $\lim_{n\to\infty} g_n^{\bar{c}}(\cdot,y,\lambda,q,p) = \phi(\cdot,y,\lambda,q,p)$ weakly$^*L^\infty(0,T)$ $\forall(y,\lambda,q,p) \in \mathbb{R}^N \times \Lambda \times \mathbb{R}^M \times \mathbb{R}^N$, (where $g_n^{\bar{c}}(t,y,\lambda,q,p) := \sup_{(u,v)}\{q\cdot u + p\cdot v : (f_n(t,y,u) + \chi_{\{v=b_n(t,u)\}}(t,u,v)) \le \lambda\}$, and $\Lambda \subseteq \mathbb{R}$),

the Γ-limit optimal control problem is given by the following formulation

$$\min_{(y,u)\in Y\times\mathcal{U}} \left\{ \operatorname*{ess\,sup}_{t\in[0,T]} \phi^\gamma(t,y(t),u(t),y'(t) - a(t,y(t))) \right\}.$$

(Where $\phi^\gamma(t,y,u,v) := \sup_{(q,p)} \sup\{\lambda : \phi(t,y,\lambda,q,p) < q\cdot u + p\cdot v\}$.)

Note that the result is not trivial because, also in the simpler case of linear dependence on the control, the Γ-limit can be a purely variational problem. (See Examples 3.1 and 3.2 in Section 3.)

Since the Γ-convergence imply the convergence of minima, if one looks at the related Hamilton -Jacobi equations, the pointwise convergence of the value functions arises as direct consequence. The proof of this fact and more remarks on the convergence of the Hamiltonians entailed by the Γ-convergence are indeed the second goal of this work.

More precisely, in Corollary 4.4, we show that under the convergence assumptions a) and b) we can prove a stability result for the Hamilton-Jacobi equation

$$\begin{cases} (V_n)_t(t,x) + H_n(t,x,V_n,D_xV_n) = 0 & \text{in } (0,T)\times\mathbb{R}^N, \\ \min_u f_n(T,x,u) = V_n(T,x) & \text{in } \mathbb{R}^N, \end{cases}$$

where the Hamiltonian H_n is

$$H_n(t,x,\lambda,p) := \min_u \{(a_n(t,x) + b_n(t,u))\cdot p \ : \ f_n(t,x,u) \le \lambda\}.$$

The interest of this part is due to the fact that the convergence on the Hamiltonians entailed by the Γ-convergence is weaker than the standard hypotheses under which the stability result for discontinuous Hamiltonian are proved. (See, for instance, the pioneering work by Hshii in 1984, [13].)

Our paper is organized as follows. In Section 2 we precisely state the regularity assumptions on our optimal control problems and we prove the Γ-convergence results (Theorem 2.4). Section 3 is devoted to the easier case of linear dependence on the control u in b_n, (i.e. $b_n(t,u) = b_n(t)\cdot u$) and two examples are given in full details. In Section 4 we apply the theory of viscosity solution for Hamilton-Jacobi equations developed in [2] to our optimal control problems (1.1)-(1.2) and we prove the stability results (Theorem 4.3 and Corollary 4.4). We conclude by complete computing an example.

Notations. Fix the real interval $(0,T)$, we will denote the usual Lebesque and Sobolev spaces of functions defined in $(0,T)$ with values on $\mathbb{R}^M$ ($\mathbb{R}$) by $L^p(0,T;\mathbb{R}^M)$ ($L^p(0,T)$) and $W^{1,p}(0,T;\mathbb{R}^M)$ ($W^{1,p}(0,T)$), respectively. Moreover, let us denote by $BUC([0,T]\times\mathbb{R}^N\times U;\mathbb{R}^N)$, ($BUC([0,T]\times\mathbb{R}^N\times U)$) the class of all bounded and uniformly continuous functions on $[0,T]\times\mathbb{R}^N\times U$ with values in $\mathbb{R}^N$ ($\mathbb{R}$).

Given a function $v(t,x) : (0,T)\times\mathbb{R}^N \to \mathbb{R}$ we respectively denote by v_t and D_xv the partial derivative w.r.to t and the gradient w.r.to the x variable.

If $f : \mathbb{R}\to\mathbb{R}$, we set $f(x+0) := \lim_{\varepsilon\to 0^+} f(x+\varepsilon)$ and $f(x-0) := \lim_{\varepsilon\to 0^+} f(x-\varepsilon)$.

By $\chi_C(x)$ we mean the function that takes value 0 on the set C and $+\infty$ otherwise. Finally, it is understood that the minimum over the empty set is $+\infty$.

2 The Γ-convergence result

In this section we describe the optimal control problems and we will state and prove our general Γ-convergence result. Fix $n \in \mathbb{N}$ and $T > 0$, we minimize the supremal functional

$$J_n(y, u) := \operatorname*{ess\,sup}_{t \in [0,T]} f_n(t, y(t), u(t)) \tag{2.1}$$

where y is the solution of the state equation

$$\begin{cases} y'(t) = a_n(t, y(t)) + b_n(t, u(t)) & t \in (0, T] \\ y(0) = x \end{cases} \tag{2.2}$$

and the control u lives in $L^\infty(0, T; \mathbb{R}^N)$. On the data we assume the following regularity.

Hypothesis **(HΓ)**:

- For each $n \in \mathbb{N}$, $a_n : [0, T] \times \mathbb{R}^N \to \mathbb{R}^N$ is a t-measurable function. Moreover there exists $\eta_n(t) \in L^1(0, T)$ such that

$$|a_n(t, z) - a_n(t, y)| \leq \eta_n(t)|z - y| \quad \forall z, y \in \mathbb{R}^N, \ \ \forall t \in [0, T]. \tag{2.3}$$

- For each $n \in \mathbb{N}$, $b_n : [0, T] \times \mathbb{R}^M \to \mathbb{R}^N$ and $b_n(\cdot, u) \in L^\infty(0, T; \mathbb{R}^N)$ for all $u \in \mathbb{R}^M$.
- For each $n \in \mathbb{N}$, the supremand function $f_n : [0, T] \times \mathbb{R}^N \times \mathbb{R}^M \to [0, \infty]$ is a measurable function and $f(t, \cdot, \cdot)$ is lower semicontinuous (l.s.c. for short) on $\mathbb{R}^N \times \mathbb{R}^M$ for a.e. $t \in [0, T]$.
- There exists a function $\alpha : \mathbb{R} \to [0, +\infty]$ such that

$$\lim_{|w| \to \infty} \alpha(|w|) = +\infty$$

  and

$$f_n(t, y, u) \geq \alpha(|u|), \quad \forall t \in [0, T], \forall y \in \mathbb{R}^N, \ \ \forall u \in \mathbb{R}^M.$$

In order to study the Γ-convergence of our optimal control problems we need to write them in a more abstract framework. Let Y be the topological space $W^{1,1}(0, T; \mathbb{R}^N)$ endowed with the L^∞ topology, and $\mathcal{U}$ be $L^\infty(0, T; \mathbb{R}^M)$ endowed with the weakly* topology. Set

$$F_n(y, u) := J_n(y, u) + \chi_{A_n}(y, u) \tag{2.4}$$

where $A_n := \{(y, u) \in Y \times \mathcal{U} \ \text{ such that } \ (2.2) \text{ is fulfilled}\}$. The optimal control problem we are studying can be written now in the following variational form,

$$(\mathcal{P}_n) \qquad \min_{(y,u) \in Y \times \mathcal{U}} F_n(y, u).$$

Definition 2.1 *We will say that the sequence* $(\mathcal{P}_n)_n$ *of optimal control problems*

$$(\mathcal{P}_n) \qquad \min_{(y,u) \in Y \times \mathcal{U}} F_n(y, u)$$

Γ-converge *to the optimal control problem* $(\mathcal{P})$ *given by*

$$(\mathcal{P}) \qquad \min_{(y,u) \in Y \times \mathcal{U}} F(y, u)$$

if $F(y,u) = \Gamma(\mathbb{N}, \mathcal{U}^-, Y^-) \lim_{n\to\infty} F_n(y,u)$. *We recall that, by definition,*

$$\Gamma(\mathbb{N}, \mathcal{U}^-, Y^-) \lim_{n\to\infty} F_n(y,u) := \lim_{n\to\infty} \inf_{(u_n)_n\to u} \inf_{(y_n)_n\to y} F_n(y_n,u_n). \tag{2.5}$$

Definition (2.5) was given for the first time by Buttazzo and Dal Maso in [10] and generalizes the standard definition of Γ-limit. The main reason why we look for the Γ-limit $(\mathcal{P})$ is the entailed convergence of minimum point and values. More precisely, the following general result holds.

Proposition 2.2 *Let* Y *and* $\mathcal{U}$ *be two topological spaces and let* $F_n : Y \times \mathcal{U} \to \overline{\mathbb{R}}$ *be a sequence of functions. Let* (y_n, u_n) *be a minimum point for* F_n*, or simply a pair such that*

$$\lim_{n\to\infty} F_n(y_n,u_n) = \lim_{n\to\infty} [\inf_{Y\times\mathcal{U}} F_n].$$

Assume that $((y_n,u_n))_n$ *converges to* (y,u) *in* $Y \times \mathcal{U}$ *and there exists*

$$F(y,u) = \Gamma(\mathbb{N}, \mathcal{U}^-, Y^-) \lim_{n\to\infty} F_n(y_n,u_n).$$

Then, we have
(i) (y,u) *is a minimum point for* F *on* $Y \times \mathcal{U}$*;*
(ii) $\lim_{n\to\infty} [\inf_{Y\times\mathcal{U}} F_n] = \min_{Y\times\mathcal{U}} F$.

(For the proof see [10], Proposition 2.1, page 388.) For more references on the general theory of Γ-convergence see, e.g., [8] and [12].

Our aim is now to see under which convergence hypotheses on the sequences $(a_n)_n$ $(b_n)_n$ and $(f_n)_n$ we are able to explicitly obtain the Γ-limit $F(y,u) = \Gamma(\mathbb{N}, \mathcal{U}^-, Y^-) \lim_{n\to\infty} F_n(y,u)$.

To this goal we will also use the Γ-convergence theory for supremal functionals developed in [9]. The latest is mainly based on the duality theory for level convex functions studied, e.g., in [4] and [14], therefore we need the following definition.

Definition 2.3 *Let* W *be a Banach space and* Z *its dual space endowed with its weakly* topology. We will denote the duality product between* Z *and* W *by* $\langle \cdot, \cdot \rangle$.

(i) *Given* $\varphi : Z \to \mathbb{R} \cup \{-\infty, +\infty\}$*, we set* $\varphi^c(\lambda, w) := \sup_z\{\langle z, w\rangle : \varphi(z) < \lambda\}$ *for any* $(\lambda, w) \in \mathbb{R} \times W$.

(ii) *Given* $\varphi : Z \to \mathbb{R} \cup \{-\infty, +\infty\}$*, we set* $\varphi^{\bar{c}}(\lambda, w) := \sup_z\{\langle z, w\rangle : \varphi(z) \leq \lambda\}$ *for any* $(\lambda, w) \in \mathbb{R} \times W$.

(iii) *Given* $\psi : \mathbb{R} \times W \to \mathbb{R} \cup \{-\infty, +\infty\}$*, we set* $\psi^\gamma(z) := \sup_{w\in W} \sup_\lambda\{\lambda : \psi(\lambda, w) < \langle z, w\rangle\}$ *for any* $z \in Z$.

Following Volle in [14], we refer to φ^c, ψ^γ as the *conjugate functions* of φ and ψ respectively, and to $\varphi^{c\gamma}$, $\psi^{\gamma c}$ as the *biconjugate functions* of φ and ψ respectively. For simplicity we still call $\varphi^{\bar{c}}$ as a conjugate function of φ.

Let us now introduce a new variable v in $\mathcal{V} := L^\infty(0,T;\mathbb{R}^N)$ endowed with the weakly* topology, and define a sequence of functionals $G_n : Y \times \mathcal{U} \times \mathcal{V} \to [0, +\infty]$ as follows

$$G_n(y,u,v) := J_n(y,u) + \chi_{\{v(t)=b_n(t,u(t))\}}(u,v). \tag{2.6}$$

Note that defining $g_n : [0,T] \times \mathbb{R}^N \times \mathbb{R}^M \times \mathbb{R}^N \to [0,+\infty]$ as

$$g_n(t,y,u,v) := f_n(t,y,u) + \chi_{\{v=b_n(t,u)\}}(t,u,v), \tag{2.7}$$

the functional G is still a supremal functional, precisely

$$G_n(y,u,v) = \operatorname*{ess\,sup}_{t\in[0,T]} \; g_n(t,y(t),u(t),v(t)).$$

Notation. In what follows we will always consider the conjugate operation with respect to the couple (u,v) with all the remaining variable fixed. In particular, we will always denote by

- $g_n^{\bar{c}}(t,y,\lambda,q,p) : [0,T]\times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^M \times \mathbb{R}^N \to \mathbb{R}\cup\{+\infty\}$ (g_n^c), the conjugate function of $g_n(t,y,\cdot,\cdot)$ for each fixed $t\in[0,T]$ and $y\in\mathbb{R}^N$,
- $G_n^{\bar{c}}(y,\lambda,q,p) \; : \; Y\times\mathbb{R}\times L^1(0,T;\mathbb{R}^M)\times L^1(0,T;\mathbb{R}^N) \to \mathbb{R}\cup\{+\infty\}$ (G_n^c) the conjugate functional of $G_n(y,\cdot,\cdot)$ for each fixed $y\in Y$,
- $\phi^\gamma(t,y,u,v) : [0,T]\times\mathbb{R}^N\times\mathbb{R}^M\times\mathbb{R}^N \to \mathbb{R}\cup\{+\infty\}$ the conjugate function of $\phi(t,y,\lambda,q,p)$ with respect to the triple (q,p,λ) at $t\in[0,T]$ and $y\in\mathbb{R}^N$ fixed.

We need to add the following assumption on the cost functional.

(HJ) There exist a function $\Psi:\mathcal{U}\to\mathbb{R}$ bounded on the $\mathcal{U}$-bounded sets, and a function $\omega : Y\times Y\to\mathbb{R}$ with $\lim_{z\to y}\omega(z,y)=0$, $(\forall y\in Y)$, such that

$$J_n(y,u)\le J_n(z,u)+\Psi(u)\omega(y,z) \quad \forall\, y,z\in Y,\;\forall\, u\in\mathcal{U}.$$

Theorem 2.4 *Assume* **(HΓ)** *and* **(HJ)**. *If the following hold,*

(Han) $\lim_{n\to\infty} a_n(\cdot,y) = a(\cdot,y)$ *weakly*$L^1(0,T;\mathbb{R}^N)$, *for every* $y\in\mathbb{R}^N$.

(Hgn) *Set*

$$\Lambda := \{\lambda\in\mathbb{R} \text{ such that } \exists n_0\in\mathbb{N} \text{ such that } G_n^{\bar{c}}(y,\lambda,\cdot,\cdot) \text{ is proper } \forall n\ge n_0\}.$$

There exists a function $\phi:[0,T]\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^M\times\mathbb{R}^N\to\mathbb{R}\cup\{+\infty\}$ *such that*

$$\lim_{n\to\infty} g_n^{\bar{c}}(\cdot,y,\lambda,q,p) = \phi(\cdot,y,\lambda,q,p) \;\; \text{weakly}^* L^\infty(0,T) \quad \forall (y,\lambda,q,p)\in\mathbb{R}^N\times\Lambda\times\mathbb{R}^M\times\mathbb{R}^N,$$

and

$$\phi(\cdot,y,\lambda,\cdot,\cdot) = -\infty \quad \forall y\in\mathbb{R}^N,\;\forall\lambda\in\mathbb{R}\setminus\Lambda.$$

Then, the sequence of optimal control problems $(\mathcal{P}_n)_n$ Γ*-converges to the optimal control problem*

$$(\mathcal{P}) \qquad \min_{(y,u)\in Y\times\mathcal{U}} \left\{ \operatorname*{ess\,sup}_{t\in[0,T]} \phi^\gamma(t,y(t),u(t),y'(t)-a(t,y(t))) \right\}.$$

Proof. By Definition 2.1 the thesis is

$$F(y,u) := \Gamma(\mathbb{N},\mathcal{U}^-,Y^-)\lim_{n\to\infty} F_n(y,u) = \operatorname*{ess\,sup}_{t\in[0,T]} \phi^\gamma(t,y(t),u(t),y'(t)-a(t,y(t))). \tag{2.8}$$

where $F_n(y,u)=J_n(y,u)+\chi_{A_n}(y,u)$ (see (2.4)).

We first observe that setting $C_n := \{(y,v)\in Y\times\mathcal{V} \; : \; y'(t)=a_n(t,y)+v\;\forall t\in(0,T], y(0)=x\}$, definition (2.6) implies $F_n(y,u)=G_n(y,u,v)+\chi_{C_n}(y,v)$. Thus, [10, Proposition 2.2] allows us to compute two separate Γ-limits as follows

$$F(y,u) = \Gamma(\mathbb{N},(\mathcal{U}\times\mathcal{V})^-,Y^-)\lim_{n\to\infty} G_n(y,u,v) + \Gamma(\mathbb{N},\mathcal{V},Y^-)\lim_{n\to\infty}\chi_{C_n}(y,v).$$

By [10, Theorem 3.4] **(Han)** entails

$$\Gamma(\mathbb{N}, \mathcal{V}, Y^-) \lim_{n\to\infty} \chi_{C_n}(y,v) = \chi_C(y,v)$$

where $C := \{(y,v) \in Y \times \mathcal{V} \,:\, y'(t) = a(t,y) + v(t) \;\; \forall t \in (0,T], y(0) = x\}$.

To represent $\Gamma(\mathbb{N}, (\mathcal{U}\times\mathcal{V})^-, Y^-) \lim_{n\to\infty} G_n(y,u,v)$ we first note that **(HJ)** implies

$$\Gamma(\mathbb{N}, (\mathcal{U}\times\mathcal{V})^-, Y^-) \lim_{n\to\infty} G_n(y,u,v) = \Gamma(\mathbb{N}, (\mathcal{U}\times\mathcal{V})^-) \lim_{n\to\infty} G_n(y,u,v).$$

So, by **(Hgn)**, we can directly apply the general representation result Theorem 3.12 (and Remark 3.14) in [9], and conclude that

$$\Gamma(\mathbb{N}, (\mathcal{U}\times\mathcal{V})^-, Y^-) \lim_{n\to\infty} G_n(y,u,v) = \operatorname*{ess\,sup}_{t\in[0,T]} \phi^\gamma(t, y(t), u(t), v(t)).$$

Summarizing, we have

$$F(y,u) = \operatorname*{ess\,sup}_{t\in[0,T]} \left(\phi^\gamma(t,y(t),u(t),v(t)) + \chi_C(y(t),v(t)) \right)$$

which gives us (2.8) and concludes the proof. □

Remark 2.5 Notice that, by Theorem 3.12 in [9], Theorem 2.4 can be proved assuming, instead of **(Hgn)**,

(Hgn1) set

$$\Lambda := \{\lambda \in \mathbb{R} \text{ such that } \exists n_0 \in \mathbb{N} \text{ such that } G_n^c(y,\lambda,\cdot,\cdot) \text{ is proper } \forall n \geq n_0\}.$$

There exists a function $\phi : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^M \times \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ such that

$$\lim_{n\to\infty} g_n^c(\cdot,y,\lambda,q,p) = \phi(\cdot,y,\lambda,q,p) \text{ weakly}^* L^\infty(0,T) \quad \forall (y,\lambda,q,p) \in \mathbb{R}^N \times \Lambda \times \mathbb{R}^M \times \mathbb{R}^N,$$

and

$$\phi(\cdot,y,\lambda,\cdot,\cdot) = -\infty \quad \forall y \in \mathbb{R}^N, \;\; \forall \lambda \in \mathbb{R} \setminus \Lambda.$$

We choose to state the result using $g_n^{\bar{c}}$ instead of g_n^c because of the link with the definition of Hamiltonian for these problems that we will specify in Remark 4.1 below.

3 The linear case: examples

We consider now the important case of linear dependence on the control in the state equation, i.e. $b_n(t,u) = b_n(t)\cdot u$ with $b_n \in L^\infty(0,T;\mathbb{R}^M)$. The main advantage is that now the convergence hypothesis **(Hgn)** (or **(Hgn1)**) can be directly stated on the sequences $(f_n)_n$ and $(b_n)_n$. Indeed, a calculation gives us

$$g_n^{\bar{c}}(t,y,\lambda,q,p) = f_n^{\bar{c}}(t,y,\lambda,q+b_n(t)p) \quad \text{ and } \quad g_n^c(t,y,\lambda,q,p) = f_n^c(t,y,\lambda,q+b_n(t)p),$$

so hypothesis **(Hgn)** can be replaced by the following one.

(Hfn) Set

$$\Lambda := \{\lambda \in \mathbb{R} \text{ such that } \exists n_0 \in \mathbb{N} \text{ such that } G_n^{\bar{c}}(y,\lambda,\cdot,\cdot) \text{ is proper } \forall n \geq n_0\}.$$

There exists a function $\phi : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^M \times \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ such that

$$\lim_{n\to\infty} f_n^{\bar{c}}(\cdot,y,\lambda,q+b_n(\cdot)p) = \phi(\cdot,y,\lambda,q,p) \text{ weakly}^* L^\infty(0,T) \forall (y,\lambda,q,p) \in \mathbb{R}^N\times\Lambda\times\mathbb{R}^M\times\mathbb{R}^N,$$

and

$$\phi(\cdot,y,\lambda,\cdot,\cdot) = -\infty \quad \forall y \in \mathbb{R}^N, \;\; \forall \lambda \in \mathbb{R} \setminus \Lambda.$$

We want to describe now two examples in which a complete calculation of the Γ-limit can be done. The interesting fact is that the Γ-limit is a purely variational problem. Indeed, although the cost functional will not depend on n, the limit problem will not simply be an optimal control problem having the same cost functional and as state equation $y'(t) = a(t,y) + b(t)u(t)$ with b the limit of $(b_n)_n$. (Note that this already happens in the case of integral functionals with weakly converging input operators as showed, for example, in [11]).

In both cases the state equation will be

$$\begin{cases} y'(t) = a(t,y) + b_n(t)u(t) & t \in (0,1] \\ y(0) = x, \end{cases} \tag{3.1}$$

where $a(t,y)$ is a given function fulfilling (2.3) and we will construct the sequence $(b_n)_n$ as follows. Fix $b(t) \in L^\infty(0,1)$, $\alpha > 0$, $\beta > 0$ such that $b(t) > \beta$ for all $t \in [0,1]$, for each $n \in \mathbb{N}$ we define

$$b_n(t) := \begin{cases} b(t) + \alpha & \text{if} \quad \dfrac{2k}{n} \leq t \leq (\dfrac{2k+1}{n} \wedge 1), \\ \\ b(t) - \beta & \text{if} \quad \dfrac{2k+1}{n} \leq t \leq (\dfrac{2k+2}{n} \wedge 1), \end{cases}$$

for $k = 0, \ldots, h-1$ if $n = 2h$ and $k = 0, \ldots, h$ if $n = 2h+1$.

Example 3.1

We consider the cost functional

$$J(y,u) = \operatorname*{ess\,sup}_{t\in[0,1]} \left(\psi(t,y(t)) + c(t)|u(t)|\right)$$

where $\psi(t,y)$ is a continuous positive function on $[0,1] \times \mathbb{R}^N$, $\psi(t,\cdot)$ is Lipschitz on $\mathbb{R}^N$ for all $t \in [0,1]$, and $c(t) : [0,1] \to [c, \infty[$, $c > 0$, is continuous.

By calculation we get

$$f^{\bar{c}}(t,y,\lambda,q + b_n(t)p) = \begin{cases} \frac{(\lambda - \psi(t,y))}{c(t)} |q + b_n(t)p| & \text{if } \lambda \geq \psi(t,y) \\ -\infty & \text{if } \lambda < \psi(t,y), \end{cases}$$

thus, hypothesis **(Hfn)** is fulfilled with

$$\phi(t,y,\lambda,q,p) := \begin{cases} \frac{(\lambda - \psi(t,y))}{c(t)} \left(\frac{1}{2}|q + (b(t)+\alpha)p| + \frac{1}{2}|q + (b(t)-\beta)p| \right) & \text{if } \lambda \geq \psi(t,y) \\ -\infty & \text{if } \lambda < \psi(t,y). \end{cases}$$

Moreover **(HΓ)**, **(HJ)**, **(Han)** can be easily verified, thus Theorem 2.4 applies and we obtain as limit problem

$$\min_{(y,u)\in Y\times\mathcal{U}} \left\{ \operatorname*{ess\,sup}_{t\in[0,1]} \left(\psi(t,y) + \frac{2c(t)}{\alpha+\beta} \left(|y' - a(t,y) - (b(t)+\alpha)u| \vee |y' - a(t,y) - (b(t)-\beta)u| \right) \right) \right\}.$$

Example 3.2

The cost functional we consider now is

$$J(y,u) = \operatorname*{ess\,sup}_{t\in[0,1]} \left\{ \chi_{[-\delta(t),\delta(t)]}(u(t)) + |y(t)| \right\}$$

where $\delta(t) > 0$ is a measurable function. Hypotheses **(HΓ)**, **(HJ)**, **(Han)** are satisfied and calculating we have

$$f^{\bar{c}}(t,y,\lambda,q+b_n(t)p) = \begin{cases} |q+b_n(t)p|\delta(t) & \text{if } \lambda \geq |y| \\ -\infty & \text{if } \lambda < |y| \end{cases}$$

so that, choosing

$$\phi(t,y,\lambda,q,p) = \begin{cases} (\frac{1}{2}|q+(b(t)+\alpha)p| + \frac{1}{2}|q+(b(t)-\beta)p|)\delta(t) & \text{if } \lambda \geq |y| \\ -\infty & \text{if } \lambda < |y|, \end{cases}$$

hypothesis **(Hfn)** is fulfilled too. The limit problem is

$$\min_{Y\times\mathcal{U}} \left\{ \operatorname*{ess\,sup}_{t\in[0,1]} \chi_{[-\delta(t),\delta(t)]} \left(\frac{2}{\alpha+\beta} \left(|y' - a(t,y) - (b(t)+\alpha)u| \vee |y' - a(t,y) - (b(t)-\beta)u| \right) \right) + |y| \right\}.$$

We consider now an interesting slight modification,

$$J(y,u) = \operatorname*{ess\,sup}_{t\in[0,1]} \left\{ \chi_{[-\delta(t,y),\delta(t,y)]}(u(t)) \right\}$$

where $\delta(t,y) > 0$ is a t-measurable function such that:

(i) for each fixed $y \in Y$, and each $y_n \to y$ in Y one has

$$\lim_{n\to\infty} \chi_{[-\delta(t,y_n),\delta(t,y_n)]}(u) = \chi_{[-\delta(t,y),\delta(t,y)]}(u), \ \forall u \in \mathcal{U},$$

(ii) fix $y \in Y$, for each fixed $u \in \mathcal{U}$, and each $u_n \to u$ in $\mathcal{U}$ one has

$$\lim_{n\to\infty} \chi_{[-\delta(t,y),\delta(t,y)]}(u_n) = \chi_{[-\delta(t,y),\delta(t,y)]}(u).$$

Note that now $J(y,u)$ does not fulfill hypothesis **(HJ)**, but one can directly check that (i) and (ii) imply $\Gamma(\mathbb{N},(\mathcal{U}\times\mathcal{V})^-,Y^-)\lim_{n\to\infty} G_n(y,u,v) = \Gamma(\mathbb{N},(\mathcal{U}\times\mathcal{V})^-)\lim_{n\to\infty} G_n(y,u,v)$. Therefore Theorem 2.4 still applies. We obtain

$$f^{\bar{c}}(t,y,\lambda,q+b_n(t)p) = \begin{cases} |q+b_n(t)p|\delta(t,y) & \text{if } \lambda \geq 0 \\ -\infty & \text{if } \lambda < 0 \end{cases}$$

so **(Hfn)** is fulfilled with

$$\phi(t,y,\lambda,q,p) = \begin{cases} (\frac{1}{2}|q+(b(t)+\alpha)p| + \frac{1}{2}|q+(b(t)-\beta)p|)\delta(t,y) & \text{if } \lambda \geq 0 \\ -\infty & \text{if } \lambda < 0, \end{cases}$$

and the limit problem is

$$\min_{Y\times\mathcal{U}} \left\{ \operatorname*{ess\,sup}_{t\in[0,1]} \chi_{[-\delta(t,y),\delta(t,y)]} \left(\frac{2}{\alpha+\beta} \left(|y' - a(t,y) - (b(t)+\alpha)u| \vee |y' - a(t,y) - (b(t)-\beta)u| \right) \right) \right\} \tag{3.2}$$

This example is interesting because the cost function is so degenerate that if we consider an equivalent problem with integral cost functional and calculate the Γ-limit we get the "same" limit problem. More precisely, let us consider the state equation (3.1) and the integral cost functional

$$J(y,u) = \int_0^1 \chi_{[-\delta(t,y),\delta(t,y)]}(u(t))dt.$$

If we compute the Γ-limit following the technique exploited in [11] we obtain as limit problem

$$\min_{Y\times\mathcal{U}} \left\{ \int_0^1 \chi_{[-\delta(t,y),\delta(t,y)]} \left(\frac{2}{\alpha+\beta} \left(|y' - a(t,y) - (b(t)+\alpha)u| \vee |y' - a(t,y) - (b(t)-\beta)u| \right) \right) dt \right\}$$

which is, of course, equivalent to (3.2).

4 On the Hamilton-Jacobi equations

We are interested now in studying the stability properties for the Hamilton-Jacobi equation related to the optimal control problems (2.1)-(2.2) entailed by the Γ-convergence result (Theorem 2.4) and the implied properties (Proposition 2.2).

To this aim we first have to briefly resume the theory of viscosity solution for the Hamilton-Jacobi equations related to optimal control problems in L^∞ developed by Barron-Ishii in [2]. Moreover, in order to directly apply their results, we will add some regularity assumption on our data.

From now on we will refer to this set of assumptions as assumption **(HHJ)**.

First, the optimal control will now take value in a compact set, more precisely if U is a compact subset of $\mathbb{R}^M$, the controls u will live in $\tilde{\mathcal{U}} := L^\infty([0,T];U)$ and similarly if V is a compact subset of $\mathbb{R}^N$, the auxiliary function v will be in $\tilde{\mathcal{V}} := L^\infty([0,T];V)$. We will assume:

- For each $n \in \mathbb{N}$, $a_n + b_n \in BUC([0,T] \times \mathbb{R}^N \times U; \mathbb{R}^N)$.
- For each $n \in \mathbb{N}$, there exists a constant $M > 0$ such that $|a_n(t,y) - a_n(t,z)| \leq M|y - z|$, for any y, z in $\mathbb{R}^N$ and for all $t \in [0,T]$.
- For each $n \in \mathbb{N}$, $f_n \in BUC([0,T] \times \mathbb{R}^N \times U)$.

Fix $n \in \mathbb{N}$, for each initial datum $(\tau, x) \in [0,T] \times \mathbb{R}^N$ we define the value function

$$V_n(\tau, x) := \inf_{u \in \tilde{\mathcal{U}}} \left\{ \operatorname*{ess\,sup}_{\tau \leq t \leq T} f_n(t, y_n^{\tau,x}(t), u(t)) \right\}$$

where $y_n^{\tau,x}$ is the solution of the state equation with initial datum x at time τ, i.e.

$$\begin{cases} y_n'(t) = a_n(t, y_n(t)) + b_n(t, u(t)) & t \in (\tau, T] \\ y_n(\tau) = x. \end{cases}$$

Fix $n \in \mathbb{N}$, under the set of assumptions **(HHJ)** Barron and Ishii proved in [2, Proposition 1.5] that V_n is in $BUC([0,T] \times \mathbb{R}^N)$ and $V_n(T,x) = \min_{u \in U} f_n(T,x,u)$. Moreover, they showed that V_n is the unique viscosity solution of the Hamilton-Jacobi equation

$$(V_n)_t(t,x) + H_n(t,x,V_n,D_x V_n) = 0 \quad \text{in} \quad (0,T) \times \mathbb{R}^N,$$

with terminal condition

$$\min_{u \in U} f_n(T,x,u) = V_n(T,x) \quad \text{in} \quad \mathbb{R}^N,$$

and Hamiltonian $H_n : (0,T) \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ defined as follows

$$H_n(t,x,\lambda,p) := \min_{u \in U} \{ (a_n(t,x) + b_n(t,u)) \cdot p \ : \ f_n(t,x,u) \leq \lambda \} .$$

(For the proof see Theorem 3.2 and Theorem 4.2 in [2].)

Remark 4.1 Note that we can write an explicit relationship between the Hamiltonian H_n and the function $g_n^{\bar{c}}$ appearing in hypothesis **(Hgn)**. Indeed, by definition of g_n, see (2.7), and Definition 2.3 (ii), we have

$$H_n(t,x,\lambda,p) = p \cdot a_n(t,x) - g_n^{\bar{c}}(t,x,\lambda,0,-p). \tag{4.1}$$

Where, of course, in the definition of $g_n^{\bar{c}}$ we are now considering $U \times V$ instead of $\mathbb{R}^M \times \mathbb{R}^N$.

By the sake of completeness, we recall here the definition of viscosity solution for discontinuous Hamiltonians.

Definition 4.2 *Let G be a function on $(0,T)\times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$, possibly taking the values $+\infty$ or $-\infty$. A continuous function u on $(0,T)\times\mathbb{R}^N$ is a viscosity solution of*

$$u_t(t,x) + G(t,x,u(t,x),D_x u(t,x)) = 0 \quad \textit{in } (0,T)\times\mathbb{R}^n,$$

if:
(i) for any $\phi \in C^1((0,T)\times\mathbb{R}^N)$ so that $u-\phi$ has a minimum at $(s,y)\in(0,T)\times\mathbb{R}^N$ we have

$$\phi_t(s,y) + G_*(s,y,\phi(s,y),D_x\phi(s,y)) \leq 0,$$

(ii) for any $\phi \in C^1((0,T)\times\mathbb{R}^N)$ so that $u-\phi$ has a maximum at $(s,y)\in(0,T)\times\mathbb{R}^N$ we have

$$\phi_t(s,y) + G^*(s,y,\phi(s,y),D_x\phi(s,y)) \geq 0.$$

Here G_* and G^* respectively denotes the lower and upper semicontinuous envelope of G with respect to all variables.

As a direct consequence of the Γ-convergence result we obtain the following convergence property for the related sequence of value functions.

Theorem 4.3 *If* **(HΓ)**, **(HHJ)**, **(Han)** *and* **(Hgn)** *holds, then $V_n(\tau,x)\to V(\tau,x)$ pointwise in $[0,T]\times\mathbb{R}^N$, where V is the unique viscosity solution of*

$$V_t(t,x) + H(t,x,V,D_xV) = 0 \quad \textit{in } (0,T)\times\mathbb{R}^N \tag{4.2}$$

with terminal condition

$$\min_{(u,v)\in U\times V} \phi^\gamma(T,x,u,v) = V(T,x) \quad \textit{in } \mathbb{R}^N \tag{4.3}$$

and Hamiltonian given by

$$H(t,x,\lambda,p) = p\cdot a(t,x) - \phi^{\gamma\bar{c}}(t,x,\lambda,0,-p). \tag{4.4}$$

Proof. By Theorem 2.4 and Proposition 2.2, we have $V_n(\tau,x)\to V(\tau,x)$ pointwise in $(0,T)\times\mathbb{R}^N$, where V is the value function related to the Γ-limit problem $(\mathcal{P})$. More precisely,

$$V(\tau,x) := \min_{(u,v)\in\tilde{\mathcal{U}}\times\tilde{\mathcal{V}}} \left\{ \operatorname*{ess\,sup}_{t\in[\tau,T]} \phi^\gamma(t,y^{\tau,x}(t),u(t),v(t)) \right\}$$

where $y^{\tau,x}$ is the solution of

$$\begin{cases} y'(t) = a(t,y(t)) + v(t) & t\in(\tau,T] \\ y(\tau) = x. \end{cases}$$

So by applying again the Barron -Ishii results (ϕ^γ, $a+v$ still verify **(HHJ)**) we can conclude that V is the unique continuous viscosity solution of the Hamilton-Jacobi equation (4.2) with terminal condition (4.3) and Hamiltonian

$$H(t,x,\lambda,p) = \min_{(u,v)\in U\times V} \{(a(t,x)+v)\cdot p \,:\, \phi^\gamma(t,x,u,v)\leq\lambda\}.$$

By Definition 2.3 (ii) we have indeed $H(t,x,\lambda,p) = p \cdot a(t,x) - \phi^{\gamma\bar{c}}(t,x,\lambda,0,-p)$ and this end the proof. $\square$

If we think at this result as a stability result, one is naturally leaded to check if V is the viscosity solution of the "limit Hamilton-Jacobi equation". More precisely, if we define,

$$\tilde{H}(t,x,\lambda,p) := p \cdot a(t,x) - \phi(t,x,\lambda,0,-p),$$

by observing that hypothesis **(Hgn)** imply

$$H_n(\cdot,x,\lambda,p) \to \tilde{H}(\cdot,x,\lambda,p) \quad \text{weakly}^* L^\infty(0,T), \quad \forall (x,\lambda,p) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N, \tag{4.5}$$

it is natural to ask ourself if V is also a viscosity solution of

$$-V_t(t,x) + \tilde{H}(t,x,V,D_xV) = 0 \quad \text{in } (0,T) \times \mathbb{R}^N$$

with (4.3) as terminal condition. The answer is yes. This is due to the fact that, although in general $H(t,x,\lambda,p) \neq \tilde{H}(t,x,\lambda,p)$ (see example 4.6 below), their lower and upper semincontinuous envelope coincide. More precisely the result is the following.

Corollary 4.4 *Under the assumption of Theorem 4.3, $V_n(\tau,x) \to V(\tau,x)$ pointwise in $[0,T] \times \mathbb{R}^N$, where V is the unique viscosity solution of*

$$V_t(t,x) + \tilde{H}(t,x,V,D_xV) = 0 \quad \text{in } (0,T) \times \mathbb{R}^N \tag{4.6}$$

with terminal condition

$$\min_{(u,v)\in U\times V} \phi^\gamma(T,x,u,v) = V(T,x) \quad \text{in } \mathbb{R}^N$$

and Hamiltonian given by

$$\tilde{H}(t,x,\lambda,p) := p \cdot a(t,x) - \phi(t,x,\lambda,0,-p). \tag{4.7}$$

Proof. By Theorem 4.3 we already know that V is a viscosity solution of (4.2) with terminal condition (4.3), so by definition of viscosity solution for discontinuous Hamiltonian (Definition 4.2) we only have to prove that the lower and upper semicontinuous envelope of H and $\tilde{H}$ coincide. Our thesis in then the following:

$$H_*(t,x,\lambda,p) = \tilde{H}_*(t,x,\lambda,p), \quad \text{and} \quad H^*(t,x,\lambda,p) = \tilde{H}^*(t,x,\lambda,p). \tag{4.8}$$

Let us first prove that

$$H(t,x,\lambda+0,p) = \tilde{H}(t,x,\lambda+0,p) \quad \text{and} \quad H(t,x,\lambda-0,p) = \tilde{H}(t,x,\lambda-0,p). \tag{4.9}$$

By definition of H and $\tilde{H}$ this means that we have to prove the following

$$\phi(t,x,\lambda+0,0,-p) = \phi^{\gamma\bar{c}}(t,x,\lambda+0,0,-p) \text{ and } \phi(t,x,\lambda-0,0,-p) = \phi^{\gamma\bar{c}}(t,x,\lambda-0,0,-p). \tag{4.10}$$

By [2, Proposition 2.1] we have that $g_n^{\bar{c}}(t,x,\cdot,q,p)$ is a non decreasing function and $g_n^{\bar{c}}(t,x,\lambda,\cdot,\cdot)$ is convex in $\mathbb{R}^M \times \mathbb{R}^N$ (or identically $-\infty$), so by **(Hgn)** $\phi(t,x,\lambda,q,p)$ enjoys the same properties. Moreover, by [14, Theorem 3.5] and [9, Proposition 2.7] we have $\phi(t,x,\lambda,q,p) =$

$\phi^{\gamma c}(t,x,\lambda,q,p) = \phi^{\gamma \bar{c}}(t,x,\lambda,q,p)$ for each λ such that $\phi(t,x,\cdot,q,p)$ is a continuous function. Therefore, by monotonicity, (4.10) and thus (4.9) follow.

To prove (4.8) we first observe that by definition we always have $\phi^{\gamma \bar{c}} \geq \phi$, thus $H \leq \tilde{H}$. It follows

$$H_*(t,x,\lambda,p) \leq \tilde{H}_*(t,x,\lambda,p) \quad \text{and} \quad H^*(t,x,\lambda,p) \leq \tilde{H}^*(t,x,\lambda,p).$$

Moreover, for each $\varepsilon > 0$, $\tilde{H}_*(t,x,\lambda+\varepsilon,p) \leq \tilde{H}(t,x,\lambda+\varepsilon,p)$, so by letting $\varepsilon \to 0$,

$$\tilde{H}_*(t,x,\lambda,p) \leq \tilde{H}(t,x,\lambda+0,p) = H(t,x,\lambda+0,p) = H_*(t,x,\lambda,p)$$

where we used (4.9) and [2, Proposition 2.5]. This complete the proof for the lower semicontinuous envelope.

Again by [2, Proposition 2.5], (4.9) and the monotonicity of $\tilde{H}$, we have

$$H^*(t,x,\lambda,p) = H(t,x,\lambda-0,p) = \tilde{H}(t,x,\lambda-0,p) \geq \tilde{H}(t,x,\lambda,p),$$

thus $H^*(t,x,\lambda,p) \geq \tilde{H}^*(t,x,\lambda,p)$, and this complete our proof. □

Remark 4.5 Of course assumption **(Hgn)** is different from the usual assumption one would like to have in a stability result for Hamilton-Jacobi equations. In one sense is weaker because of the very weak convergence required in the t-variable, in the other is stronger because the latest is required for each fixed $(y,\lambda,q,p) \in \mathbb{R}^N \times \Lambda \times \mathbb{R}^M \times \mathbb{R}^N$. Indeed, the question if the result still hold if we only ask H_n tends to H as in (4.5) is an open problem also in the easier integral case. This, in a wider contest, is the subject of a work in preparation with G. Buttazzo.

We end this section by showing an easy example in which H is indeed different from $\tilde{H}$.

Note that in the simpler case of relaxation this is not true. Indeed, the relaxation result for supremal functionals was proved by Barron and Liu ([4, Proposition 6.4]) by showing that V and the relaxed value function solve two Hamilton-Jacobi equations having the same Hamiltonian and thus, by uniqueness, coincide.

Example 4.6 Let us consider the following state equation

$$\begin{cases} y'(t) = bu & t \in (\tau,1] \\ y(\tau) = x, \end{cases}$$

where $b > 0$ and $u \in [-10,10]$. As cost function we choose

$$J(y,u) = \operatorname*{ess\,sup}_{\tau \leq t \leq 1} (|y^{\tau,x}(t)| + l_n(u(t)))$$

where

$$l_n(u) = \begin{cases} |u| + \frac{1}{n} & \text{if } |u| > 1 \\ \frac{1}{n}|u| + 1 & \text{if } |u| \leq 1. \end{cases}$$

One easily obtain $V_n(\tau,x) = |x| + 1$ for all $n \in \mathbb{N}$ and

$$H_n(t,x,\lambda,p) = \begin{cases} -|p|b(\lambda - |x| - \frac{1}{n}) & \text{if} \quad \lambda - |x| > 1 + \frac{1}{n} \\ -|p|b(\lambda - |x| - 1)n & \text{if} \quad 1 \leq \lambda - |x| \leq 1 + \frac{1}{n} \\ +\infty & \text{if} \quad \lambda - |x| < 1. \end{cases}$$

Thus one can directly check that V_n is a viscosity solution of

$$(V_n)_t(t,x) + H_n(t,x,V_n,D_xV_n) = 0 \quad \text{in} \quad (0,T)\times \mathbb{R},$$

with terminal condition

$$\min_{u\in U}(|x| + l_n(u)) = V_n(T,x) \quad \text{in} \quad \mathbb{R}^N.$$

Letting $n\to\infty$ we obtain that

$$H_n(t,x,\lambda,p) \to \tilde{H}(t,x,\lambda,p) = \begin{cases} -|p|b(\lambda-|x|) & \text{if} \quad \lambda-|x| > 1 \\ 0 & \text{if} \quad \lambda-|x| = 1 \\ +\infty & \text{if} \quad \lambda-|x| < 1, \end{cases}$$

while, calculating, we get

$$H(t,x,\lambda,p) = \begin{cases} -|p|b(\lambda-|x|) & \text{if} \quad \lambda-|x|\geq 1 \\ +\infty & \text{if} \quad \lambda-|x| < 1. \end{cases}$$

It is now easy to verify that $H(t,x,1+|x|,p) \neq \tilde{H}(t,x,1+|x|,p)$ but $H^*(t,x,\lambda,p) = \tilde{H}^*(t,x,\lambda,p)$ and $H_*(t,x,\lambda,p) = \tilde{H}_*(t,x,\lambda,p)$. Moreover, one can directly check that V is a viscosity solution of both Hamilton-Jacobi equations (4.2) and (4.6).

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