PLANAR INCIDENCES AND GEOMETRIC INEQUALITIES IN THE HEISENBERG GROUP

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ABSTRACT. We prove that if $P, \mathcal{L}$ are finite sets of δ-separated points and lines in $\mathbb{R}^2$, the number of δ-incidences between P and $\mathcal{L}$ is no larger than a constant times

$$|P|^{2/3}|\mathcal{L}|^{2/3} \cdot \delta^{-1/3}.$$

We apply the bound to obtain the following variant of the *Loomis-Whitney inequality* in the Heisenberg group:

$$|K| \lesssim |\pi_x(K)|^{2/3} \cdot |\pi_y(K)|^{2/3}, \qquad K \subset \mathbb{H}.$$

Here π_x and π_y are the *vertical projections* to the xt- and yt-planes, respectively, and $|\cdot|$ refers to natural Haar measure on either $\mathbb{H}$, or one of the planes. Finally, as a corollary of the Loomis-Whitney inequality, we deduce that

$$\|f\|_{4/3} \lesssim \sqrt{\|Xf\|\|Yf\|}, \qquad f \in BV(\mathbb{H}),$$

where X, Y are the standard horizontal vector fields in $\mathbb{H}$. This is a sharper version of the classical geometric Sobolev inequality $\|f\|_{4/3} \lesssim \|\nabla_{\mathbb{H}}f\|$ for $f \in BV(\mathbb{H})$.

CONTENTS

1. INTRODUCTION

The *Loomis-Whitney* inequality in $\mathbb{R}^n$ bounds the volume of a set $K \subset \mathbb{R}^n$ by the areas of its coordinate projections:

$$|K| \leqslant \prod_{j=1}^{n} |\pi_j(K)|^{1/(n-1)}, \tag{1.1}$$

where $\pi_j(x_1, \ldots, x_n) = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$. Here $|A|$ refers to d-dimensional Lebesgue measure in $\mathbb{R}^d$ whenever $A \subset \mathbb{R}^d$. The same notation will also refer to cardinality, but the appropriate meaning should always be clear from the context. The inequality (1.1) is due to Loomis and Whitney [29] from 1949. The starting point of the present paper was to find an analogue of (1.1) in Heisenberg groups. It turns out that already in the first group $\mathbb{H} = (\mathbb{R}^3, \cdot)$, this leads to an interesting incidence geometric problem, which we are able to fully resolve. In higher groups, the question will remain open.

1.1. **A bound on δ-incidences in the plane.** Before setting up the Heisenberg notation, we discuss the more elementary incidence geometry problem – in $\mathbb{R}^2$. Let $P \subset Q_0 := [-1, 1]^2$ be a finite set, and let $\mathcal{L}$ be a finite collection of lines $\ell_{(a,b)} := \{y = ax + b\} \subset \mathbb{R}^2$, with $(a, b) \in Q_0$; we denote the collection of all such lines by $\mathcal{Q}_0$. We fix a "scale" $0 < \delta < 1$, and assume that all the points in P and lines in $\mathcal{L}$ are δ-separated. Two points $p, q \in \mathbb{R}^2$ are called δ-separated if $|p - q| \geqslant \delta$. Two lines $\ell_p, \ell_q \in \mathcal{Q}_0$ are called δ-separated if $p, q \in Q_0$ are δ-separated. We say that a point $p \in \mathbb{R}^2$ is δ-*incident* to a line $\ell \subset \mathbb{R}^2$ if p lies in the δ-*neighborhood* $\ell(\delta)$ of ℓ. We also write

$$\mathcal{I}_\delta(P, \mathcal{L}) := \{(p, \ell) \in P \times \mathcal{L} : p \text{ is } \delta\text{-incident to } \ell\}.$$

Here is the first main result of the paper:

Theorem 1.2. *Let $P \subset Q_0$ and $\mathcal{L} \subset \mathcal{Q}_0$ be δ-separated. Then,*

$$|\mathcal{I}_\delta(P, \mathcal{L})| \lesssim |P|^{2/3} |\mathcal{L}|^{2/3} \cdot \delta^{-1/3}.$$

Here $|\cdot|$ refers to cardinality on both sides of the inequality. The implicit constant is absolute.

This estimate is a close relative of the Szemerédi-Trotter incidence bound [35] which, in our notation, says that $|\mathcal{I}_0(P, \mathcal{L})| \lesssim |P|^{2/3} |\mathcal{L}|^{2/3} + |P| + |\mathcal{L}|$ (without any hypotheses on the separation of P or $\mathcal{L}$). The Szemerédi-Trotter bound for $|\mathcal{I}_0|$ is typically much better than the one in Theorem 1.2 for $|\mathcal{I}_\delta|$, but this is to be expected. In fact, the bound in Theorem 1.2 cannot be improved, unless one assumes stronger separation from either P or $\mathcal{L}$. The simplest non-trivial sharpness example is perhaps given by letting P be a δ-packing in a tube of dimensions $\delta^{1/2} \times \delta$. All of the points in P are δ-incident to a δ-separated family of lines of cardinality $\sim \delta^{-1/2}$, giving $|\mathcal{I}_\delta(P, \mathcal{L})| \sim \delta^{-1}$. This matches the upper bound in Theorem 1.2. More generally, sharpness examples are given by letting P be a δ-packing in a rectangle $R = [0, r] \times [0, s] \subset Q_0$, with $\delta \leqslant s \leqslant r \leqslant 1$, and letting $\mathcal{L} \subset \mathcal{Q}_0$ be a δ-packing of lines meeting R.

We will infer Theorem 1.2 from an estimate for the number of *k-rich points* relative to an ϵ-separated line family, with $\epsilon \geqslant \delta$, see Theorem 2.4. To prove Theorem 1.2, only the case $\epsilon = \delta$ of Theorem 2.4 is needed; we decided to include the general case $\epsilon \geqslant \delta$ since the same question has been recently studied by Guth, Solomon, and Wang [18]. We will comment on the differences between the results in Remark 2.6. Other related

results in the plane are contained in [24, 28, 31, 34]. In higher dimensions, the problem of bounding the number of δ-incidences between points and lines is at the heart of Kakeya and restriction problems, see [5, 16, 19, 20, 25, 26, 27] for a few recent papers.

1.2. **Loomis-Whitney and Gagliardo-Nirenberg-Sobolev inequalities in $\mathbb{H}$.** We then move to the Heisenberg group, although we postpone most of the precise definitions to Section 3. In brief, the first Heisenberg group $\mathbb{H}$ is $\mathbb{R}^3$ equipped with a non-commutative group law "$\cdot$" which makes it a nilpotent Lie group. The "vertical" planes in $\mathbb{R}^3$ containing the t-axis are subgroups of $\mathbb{H}$ – known as the *vertical subgroups*. To a vertical subgroup $\mathbb{W} \subset \mathbb{H}$, we associate the *complementary horizontal subgroup* $\mathbb{L}$ which, as a subset of $\mathbb{R}^3$, is just the orthogonal complement of $\mathbb{W}$, a line in the xy-plane. For subsets of $\mathbb{H} \cong \mathbb{R}^3$, the notation $|\cdot|$ will refer to Lebesgue measure on $\mathbb{R}^3$, and for subsets of a vertical plane $\mathbb{R}^2 \cong \mathbb{W} \subset \mathbb{H}$, the notation $|\cdot|$ will refer to Lebesgue measure in $\mathbb{R}^2$. All integrations on $\mathbb{H}$ or $\mathbb{W}$ will be performed with respect to these measures. Up to multiplicative constants, they could also be defined as the 4-and 3-dimensional Hausdorff measures, respectively, relative to a natural metric on $\mathbb{H}$. So, our measures coincide with canonical "intrinsic" objects in $\mathbb{H}$.

Fixing a pair $(\mathbb{W}, \mathbb{L})$, as above, every point $p \in \mathbb{H}$ can be uniquely decomposed as $p = w \cdot v$, where $w \in \mathbb{W}$ and $v \in \mathbb{L}$. This operation gives rise to the *vertical* and *horizontal projections*

$$p \mapsto \pi_{\mathbb{W}}(p) := w \quad \text{and} \quad p \mapsto \pi_{\mathbb{L}}(p) := v.$$

The vertical projections, in particular, play a significant role in the geometric measure theory of Heisenberg groups – as do orthogonal projections in $\mathbb{R}^n$ – so they have been actively investigated in recent years, see [1, 2, 10, 11, 22, 23]. The vertical projections are non-linear maps, but their *fibres* $\pi_{\mathbb{W}}^{-1}\{w\}$ are nevertheless lines. In fact, the fibres of $\pi_{\mathbb{W}}$ are precisely the left translates of the line $\mathbb{L}$, that is, $\pi_{\mathbb{W}}^{-1}\{w\} = w \cdot \mathbb{L}$ for $w \in \mathbb{W}$.

With this introduction in mind, we are interested in proving a variant of the Loomis-Whitney inequality (1.1) for subsets of $\mathbb{H}$ in terms of the vertical projections $\pi_{\mathbb{W}}$. In $\mathbb{R}^n$, the inequality makes a reference to the n coordinate projections. These are, now, best viewed as the projections whose fibres are translates of lines parallel to the coordinate axes. In $\mathbb{H}$, it seems natural to fix a basis for the xy-plane, say $e_1 = (1, 0, 0)$ and $e_2 = (0, 1, 0)$, and consider the two vertical projections $\pi_1 := \pi_{\mathbb{W}_1}$ and $\pi_2 := \pi_{\mathbb{W}_2}$ whose fibres are left translates of $\mathbb{L}_1 := \mathrm{span}(e_1)$ and $\mathbb{L}_2 := \mathrm{span}(e_2)$. The exact formulae are

$$\pi_1(x, y, t) = (0, y, t + \tfrac{xy}{2}) \quad \text{and} \quad \pi_2(x, y, t) = (x, 0, t - \tfrac{xy}{2}).$$

With this notation, we prove the following variant of the Loomis-Whitney inequality:

Theorem 1.3. *Let $K \subset \mathbb{R}^3$ (or $K \subset \mathbb{H}$) be Lebesgue measurable. Then*

$$|K| \lesssim |\pi_1(K)|^{2/3} \cdot |\pi_2(K)|^{2/3}. \tag{1.4}$$

Theorem 1.3 will be derived as a corollary of Theorem 1.2. It is easy to see that the exponents in (1.4) are sharp by considering rectangles of the form $[-r, r] \times [-r, r] \times [-r^2, r^2]$. Besides the difference in the definition of projections, there is another obvious difference between (the case $n = 3$ of) the standard Loomis-Whitney inequality (1.1), and (1.4): the former bounds the volume of K in terms of three projections, and the latter in terms of only two projections. One might therefore ask: is there a version of (1.1) for two orthogonal projections $\mathbb{R}^3 \to \mathbb{R}^2$ – and does it look like (1.4)? The answer is negative.

This is a very special case of [4, Theorem 1.13], but perhaps it is illustrative to see an explicit computation:

Example 1.5. *Consider the two coordinate projections $\tilde{\pi}_1, \tilde{\pi}_2$ in $\mathbb{R}^3$ to the xt- and yt-planes. If $K = [0,1]^2 \times [0,\delta]$, then $|K| = \delta$, and also $|\tilde{\pi}_1(K)| = \delta = |\tilde{\pi}_2(K)|$. So, for $\delta > 0$ small, an inequality of the form*

$$|K| \lesssim |\tilde{\pi}_1(K)|^\lambda \cdot |\tilde{\pi}_2(K)|^\lambda \tag{1.6}$$

can only hold for $\lambda \leqslant \frac{1}{2}$. On the other hand, if $K_R = [0,R]^3$, with $R \gg 1$, then $|K_R| = R^3$ and $|\tilde{\pi}_1(K_R)| = R^2 = |\tilde{\pi}_2(K_R)|$, so (1.6) can only hold for $\lambda \geqslant \frac{3}{4}$. The latter example naturally does not contradict (1.4): note that $|\pi_j(K_R)| \sim R^3$ for $R \gg 1$.

We also mention that Theorem 1.3 is related to *Brascamp-Lieb inequalities*, but, to the best of our knowledge, does not follow from existing results. We direct the reader to e.g. [3, 4, 7] and the references therein. Euclidean Loomis-Whitney and Brascamp-Lieb inequalities can be proven by the technique of *heat flow monotonicity*, see [4]. The same approach has been attempted in Carnot groups by Bramati [6], but there seems to be a gap in the argument, which has been confirmed with the author. More precisely, the exponents appearing in the proof of [6, Theorem 3.2.3] have not been chosen consistently.

In $\mathbb{R}^n$, it is well-known that the Loomis-Whitney inequality implies the *Gagliardo-Nirenberg-Sobolev inequality*

$$\|f\|_{n/(n-1)} \leqslant \prod_{j=1}^{n} \|\partial_j f\|_1^{1/n}, \qquad f \in C_c^1(\mathbb{R}^n). \tag{1.7}$$

Similarly, we obtain an $\mathbb{H}$-analogue of (1.7) as a corollary of Theorem 1.3:

Theorem 1.8. *Let $f \in BV(\mathbb{H})$. Then,*

$$\|f\|_{4/3} \lesssim \sqrt{\|Xf\|\|Yf\|}. \tag{1.9}$$

Here

$$X = \partial_x - \frac{y}{2}\partial_t \quad \text{and} \quad Y = \partial_y + \frac{x}{2}\partial_t$$

are the standard left-invariant "horizontal" vector fields in $\mathbb{H}$, and $BV(\mathbb{H})$ refers to functions $f \in L^1(\mathbb{H})$ whose distributional X and Y derivatives are signed Radon measures with finite total variation, denoted $\|\cdot\|$. Theorem 1.8 presents a sharper version of the well-known "geometric" Sobolev inequality

$$\|f\|_{4/3} \lesssim \|\nabla_{\mathbb{H}} f\|, \qquad f \in BV(\mathbb{H}), \tag{1.10}$$

proven by Pansu [32] as a corollary of the isoperimetric inequality in $\mathbb{H}$. Here $\nabla_{\mathbb{H}} f = (Xf, Yf)$. Versions of geometric Sobolev inequalities and isoperimetric inequalities were obtained in a more general framework by several authors, for instance in [9, 15]. A proof of (1.10), using the fundamental solution of the sub-Laplace operator $\triangle_{\mathbb{H}}$, is discussed in [8, Section 5.3], following the approach of [9]. On the other hand, since Theorem 1.8 is derived from Theorem 1.3, which in turn is a corollary of Theorem 1.2, our proof of the inequality (1.9), and hence (1.10), uses nothing but plane geometry!

It seems plausible that a version of Theorem 1.3 could also hold in higher dimensional Heisenberg groups, but we are not currently able to prove it:

Question 1. *Let $K \subset \mathbb{H}^n \cong \mathbb{R}^{2n+1}$ be Lebesgue measurable, and let $\pi_1, \ldots, \pi_{2n}$ be the vertical projections to the planes perpendicular to the $2n$ standard unit vectors $e_j \in \mathbb{R}^{2n} \times \{0\} \subset \mathbb{R}^{2n+1}$. Then,*

$$|K| \lesssim \prod_{j=1}^{2n} |\pi_j(K)|^{(n+1)/[n(2n+1)]}.$$

2. AN INCIDENCE ESTIMATE IN THE PLANE

Definition 2.1 (A metric on lines). Let $\mathcal{Q}_0$ be the set of lines in $\mathbb{R}^2$ whose slope does not exceed $45°$, and which intersect the y-axis in $\{0\} \times [-1, 1]$:

$$\mathcal{Q}_0 := \{\ell_{(a,b)} := \{(x, y) \in \mathbb{R}^2 : y = ax + b\} : |a|, |b| \leqslant 1\}.$$

For $\ell_{(a,b)}, \ell_{(c,d)} \in \mathcal{Q}_0$, write

$$d(\ell_{(a,b)}, \ell_{(c,d)}) := |(a, b) - (c, d)|.$$

Let $0 < \delta \leqslant 1$. A point $p \in \mathbb{R}^2$ is δ-*incident* to a line $\ell \subset \mathbb{R}^2$ if $p \in \ell(\delta)$. The parameter $\delta > 0$ will be fixed in this section, and the δ-incidence of p and ℓ will be denoted $p \sim \ell$. For another parameter $\epsilon \in [\delta, 1]$, we say that two lines $\ell_1, \ell_2 \in \mathcal{Q}_0$ are called ϵ-*separated* if $d(\ell_1, \ell_2) \geqslant \epsilon$. The point here is that we will only ever consider δ-incidences between points and lines, but sometimes the results can be improved by assuming that the lines are ϵ-separated, and not just δ-separated.

We record a fairly obvious lemma:

Lemma 2.2. *Let $0 < \delta \leqslant \epsilon < 1$. Let $\mathcal{L} \subset \mathcal{Q}_0$ be an ϵ-separated family of lines, all δ-incident to a common point $p \in Q_0 := [-1, 1]^2$. Then $|\mathcal{L}| \leqslant A\epsilon^{-1}$, where $A \geqslant 1$ is an absolute constant.*

Proof. Write $p = (x_0, y_0) \in Q_0$. For every $\ell = \ell_{(a,b)} \in \mathcal{L}$, there exists $x_\ell \in \mathbb{R}$ such that $|x_\ell - x_0| + |y_0 - (ax_\ell + b)| \lesssim \delta$. Since $|a| \leqslant 1$, it follows that $|b - (-x_0a + y_0)| \lesssim \delta$ and hence (a, b) also lies at distance $\lesssim \delta$ from the line $\{y = -x_0x + y_0\}$. Noting that $\epsilon \geqslant \delta$, there can be at most $\lesssim \epsilon^{-1}$ such ϵ-separated choices of (a, b), as claimed. $\qquad\square$

Our restriction to the lines in $\mathcal{Q}_0$ is purely a matter of convenience; it allows us to define the metric d in a neat way, which (i) corresponds to the "geometric intuition" of what the δ-separation of lines should mean, and (ii) behaves well under point-line-duality.

For a (finite) set $P \subset \mathbb{R}^2$, and a (finite) family of lines $\mathcal{L}$ in $\mathbb{R}^2$, we write

$$\mathcal{I}(P, \mathcal{L}) := \{(p, \ell) : p \sim \ell\} = \{(p, \ell) : p \in \ell(\delta)\}.$$

Here is the main result of this section:

Theorem 2.3. *Let $P \subset Q_0 := [-1, 1]^2$ be a δ-separated set, and let $\mathcal{L} \subset \mathcal{Q}_0$ be a δ-separated family of lines. Then,*

$$|\mathcal{I}(P, \mathcal{L})| \lesssim |P|^{2/3}|\mathcal{L}|^{2/3} \cdot \delta^{-1/3}.$$

Theorem 2.3 will be derived as a corollary of the following reformulation, which bounds the number of *k-rich points* for a given ϵ-separated line family $\mathcal{L}$ in $\mathbb{R}^2$. For Theorem 2.3, we will only need the case $\epsilon = \delta$, but proving the more general statement presents no additional challenges. Given a line family $\mathcal{L}$, and an integer $k \geqslant 1$, a point $p \in \mathbb{R}^2$ is is called *k-rich* (relative to $\mathcal{L}$) if $p \sim \ell$ for $\geqslant k$ distinct $\ell \in \mathcal{L}$.

Theorem 2.4. *Let $0 < \delta \leqslant \epsilon \leqslant 1$. Let $\mathcal{L} \subset \mathcal{Q}_0$ be an ϵ-separated family of lines, and let $P \subset Q_0$ be a δ-separated set of k-rich points (relative to $\mathcal{L}$) with $k \geqslant 2$. Then,*

$$|P| \lesssim \frac{|\mathcal{L}|^2}{k^3} \cdot \epsilon^{-1}. \tag{2.5}$$

Remark 2.6. First, we mention that the definition of "$p \sim \ell$" could also be modified by requiring that $p \in \ell(C\delta)$, where $C \geqslant 1$ is a fixed constant. Then both Theorems 2.3 and 2.4 would continue to hold, with the same proofs but with a worse constant, depending only on C.

Second, the bound (2.5) should be compared with the next classical estimate, which follows from the Szemerédi-Trotter incidence theorem [35]: given a family of lines $\mathcal{L}$ in $\mathbb{R}^2$, the set of points in $\mathbb{R}^2$ contained on $\geqslant k$ lines has cardinality

$$\lesssim \frac{|\mathcal{L}|^2}{k^3} + \frac{|\mathcal{L}|}{k}. \tag{2.7}$$

It seems suspicious that (2.5) is completely missing the second term in (2.7), which is indeed necessary: think of "k-stars", where $|\mathcal{L}|/k$ points, each, lie on k lines in $\mathcal{L}$. Since such a construction is possible in the context of Theorem 2.4, it has to be the case that

$$\frac{|\mathcal{L}|}{k} \lesssim \frac{|\mathcal{L}|^2}{k^3} \cdot \epsilon^{-1}. \tag{2.8}$$

This is true: since the lines in $\mathcal{L}$ are ϵ-separated, and $\epsilon \geqslant \delta$, no point in Q_0 can be δ-incident to more than $\min\{|\mathcal{L}|, A\epsilon^{-1}\}$ lines in $\mathcal{L}$. Thus, we may assume in proving Theorem 2.4 that $k \leqslant \min\{|\mathcal{L}|, A\epsilon^{-1}\} \lesssim \sqrt{|\mathcal{L}|\epsilon^{-1}}$. This bound is equivalent to (2.8).

Third, the bound (2.5) should be compared with the recent work of Guth, Solomon, and Wang [18, Theorem 1.1]. Under the hypotheses and terminology of Theorem 2.4, the authors in [18] prove that the number of δ-separated k-rich points relative to $\mathcal{L}$ is $\lessapprox |\mathcal{L}|^2/k^3$ if *a priori* $k \gg \delta\epsilon^{-2}$. Thus, for $0 < \epsilon \ll 1$, the upper bound in [18] is much stronger than (2.5), but it is only applicable for sufficiently large values of k. To prove Theorem 2.3, we also need information about small values of k. For the case $\epsilon = \delta$ in particular, [18, Theorem 1.1] does not seem to contain any information, since if $k \gg \delta\epsilon^{-2} = \delta^{-1}$, the set of k-rich points is always empty by Lemma 2.2.

In the proof of Theorem 2.4, we will employ the following polynomial cell decomposition lemma of Guth and Katz [17, Theorem 4.1]:

Lemma 2.9. *Let $P \subset \mathbb{R}^2$ be a finite set, and let $D \geqslant 1$ be an integer. Then, there exists a polynomial $p \colon \mathbb{R}^2 \to \mathbb{R}$ of degree $\deg p \leqslant D$ such that the following holds. Writing*

$$Z := \{x \in \mathbb{R}^2 : p(x) = 0\},$$

the complement $\mathbb{R}^2 \setminus Z$ is the union of $\lesssim D^2$ open cells O such that $|O \cap P| \lesssim |P|/D^2$.

We are then ready to prove Theorem 2.4:

Proof of Theorem 2.4. While proving Theorem 2.4, we may assume that

$$2 \leqslant k \leqslant \min\{|\mathcal{L}|, A\epsilon^{-1}\}, \tag{2.10}$$

where A is the constant from Lemma 2.2, see Remark 2.6. In addition to (2.10), we may also assume that either (a) $k \geqslant A_0$, or (b) $\epsilon \geqslant A_0\delta$, where $A_0 \geqslant 1$ is an absolute constant of our choosing. Indeed, if both $\epsilon \leqslant A_0\delta$ and $k \leqslant A_0$, the right hand side of (2.5) is

$\geqslant A_0^{-4}|\mathcal{L}|^2 \cdot \delta^{-1} \gtrsim |\mathcal{L}| \cdot \delta^{-1}$. But clearly the number of k-rich points is no larger than the number of 1-rich points, which is $\lesssim |\mathcal{L}| \cdot \delta^{-1}$, recalling that P is δ-separated. In both cases (a) and (b) we can infer the following geometric observation, which will be useful later in the argument:

Lemma 2.11. *The following holds if $A_0 \geqslant 1$ is large enough, and either (a) $\epsilon \geqslant \delta$ and $k \geqslant A_0$ or (b) $\epsilon \geqslant A_0\delta$ and $k \geqslant 2$. If $\mathcal{L}(p) \subset \mathcal{L}$ is an ϵ-separated set of lines which are all δ-incident to a common point $p \in Q_0$, with $N := |\mathcal{L}(p)| \geqslant k$, then there are subsets $\mathcal{L}_1(p), \mathcal{L}_2(p) \subset \mathcal{L}(p)$ of cardinalities $|\mathcal{L}_1(p)| \sim N \sim |\mathcal{L}_2(p)|$ such that $\angle(\ell_1, \ell_2) \gtrsim N\epsilon$ for all $(\ell_1, \ell_2) \in \mathcal{L}_1(p) \times \mathcal{L}_2(p)$.*

We omit the easy proof. We will assume that either (a) or (b) holds, so we have the conclusion of Lemma 2.11. To prove Theorem 2.4, we fix k as in (2.10) (and also with $k \geqslant A_0$ in case (a) holds), and make a counter assumption: there exists an ϵ-separated line family $\mathcal{L}$, and a δ-separated set $P \subset Q_0$ of cardinality

$$|P| \geqslant C \frac{|\mathcal{L}|^2}{k^3} \cdot \epsilon^{-1} \tag{2.12}$$

such that every point in P is $\geqslant k$-rich (relative to $\mathcal{L}$). Here $C \geqslant 1$ is some large absolute constant to be determined later. We apply the cell decomposition lemma, Lemma 2.9, with $D := \lfloor C_{\deg}|\mathcal{L}|/k \rfloor \geqslant 1$, where $C_{\deg} \geqslant 1$ is another constant satisfying

$$1 \ll C_{\deg} \ll \sqrt{C}.$$

The precise requirements will become clear during the proof. We obtain a polynomial $p \colon \mathbb{R}^2 \to \mathbb{R}$ of degree $\deg p \leqslant C_{\deg}|\mathcal{L}|/k$, and a collection of "cells" $\mathcal{O}$, with $|\mathcal{O}| \lesssim |\mathcal{L}|^2/k^2$, such that

$$|O \cap P| \lesssim \frac{|P|}{D^2} \sim \frac{k^2|P|}{C_{\deg}^2|\mathcal{L}|^2}, \qquad O \in \mathcal{O}. \tag{2.13}$$

We split the set P into two parts: the points "well inside" the cells $O \in \mathcal{O}$, and the part "close" to $Z = \{p = 0\}$. The plan is to show that both parts have cardinality $< |P|/2$, which gives a contradiction, and completes the proof. Precisely, we write

$$O' := O \setminus Z(\delta), \qquad O \in \mathcal{O}.$$

We then write $P = P_1 \cup P_2$, where

$$P_1 := P \cap \bigcup_{O \in \mathcal{O}} O' \quad \text{and} \quad P_2 := P \cap Z(\delta).$$

2.0.1. *Proof that $|P_1| < |P|/2$.* We start with the following observation: if a line $\ell \subset \mathbb{R}^2$ (from $\mathcal{L}$ if desired) is δ-incident to a point in $p \in O'$, then $\ell \cap O \neq \varnothing$. It follows: *every line $\ell \subset \mathbb{R}^2$ is δ-incident to a point in at most $\deg p + 1$ sets O'.* Indeed, if ℓ violated this, then it would intersect $> \deg p + 1$ distinct cells $O \in \mathcal{O}$, and hence cross Z in $\geqslant \deg p + 1$ distinct points. By Bézout's theorem, this would force $\ell \subset Z$, and hence $\ell(\delta) \subset Z(\delta)$. In particular, ℓ could not be δ-incident to any points in any of the cells $O' \subset \mathbb{R}^2 \setminus Z(\delta)$. We learned this argument from [16, Lemma 3.2].

We infer the following useful corollary of the previous observation. For $O \in \mathcal{O}$ fixed, we write $\mathcal{L}_O$ for the subset of $\mathcal{L}$ which are δ-incident to at least one point in O'. Then,

$$\sum_{O \in \mathcal{O}} |\mathcal{L}_O| = \sum_{\ell \in \mathcal{L}} |\{O \in \mathcal{O} : \ell \in \mathcal{L}_O\}| \leqslant [\deg p + 1] \cdot |\mathcal{L}|. \tag{2.14}$$

We are now ready to estimate the number of points in P_1. We will first use the k-richness of the points in $P_1 \subset P$, and then the trivial bound $|\mathcal{I}(P', \mathcal{L}')| \leqslant |P'||\mathcal{L}'|$:

$$
\begin{aligned}
k|P_1| \leqslant |\mathcal{I}(P_1, \mathcal{L})| &= \sum_{O \in \mathcal{O}} |\mathcal{I}(P \cap O', \mathcal{L}_O)| \\
&\stackrel{(2.13)}{\lesssim} \frac{k^2|P|}{C_{\mathrm{deg}}^2 |\mathcal{L}|^2} \sum_{O \in \mathcal{O}} |\mathcal{L}_O| \\
&\stackrel{(2.14)}{\leqslant} \frac{k^2|P|}{C_{\mathrm{deg}}^2 |\mathcal{L}|^2} \cdot [\deg p + 1] \cdot |\mathcal{L}| \lesssim \frac{k|P|}{C_{\mathrm{deg}}},
\end{aligned}
$$

recalling that $\deg p \leqslant C_{\mathrm{deg}}|\mathcal{L}|/k$. If $C_{\mathrm{deg}} \geqslant 1$ was chosen large enough, this shows that $|P_1| < |P|/2$, as desired.

2.0.2. *Proof that $|P_2| < |P|/2$.* The argument here follows rather closely [30, §5.2]. There are certain troublesome points $P_{2,\mathrm{bad}} \subset P_2$ whose cardinality we bound first: they are the points $p \in P_2$ such that $B(p, 2\delta)$ contains a component of Z. By Harnack's curve theorem [21], the number of components $Z_1, \ldots, Z_N$ of Z is bounded by

$$
N \lesssim [\deg p]^2 \lesssim \frac{C_{\mathrm{deg}}^2 |\mathcal{L}|^2}{k^2} \leqslant A \frac{C_{\mathrm{deg}}^2 |\mathcal{L}|^2}{k^3} \cdot \epsilon^{-1} \leqslant A \frac{C_{\mathrm{deg}}^2 |P|}{C},
$$

recalling from (2.10) that $k \leqslant A\epsilon^{-1}$, and then applying the counter assumption (2.12). Since the points in P are δ-separated, and the 2δ-neighbourhood of every point in $P_{2,\mathrm{bad}}$ contains one of the N components of Z, we conclude that $|P_{2,\mathrm{bad}}| \lesssim N \lesssim C_{\mathrm{deg}}^2 |P|/C$. Choosing $C \geqslant 1$ large enough, and then $C_{\mathrm{deg}} \ll \sqrt{C}$, we find that $|P_{2,\mathrm{bad}}| < |P|/4$. To conclude the proof, it remains to prove that $|P_{2,\mathrm{good}}| < |P|/4$, where $P_{2,\mathrm{good}} := P_2 \setminus P_{2,\mathrm{bad}}$.
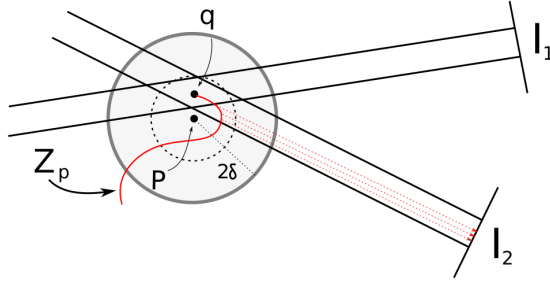

FIGURE 1. $Z \cap B(p, 2\delta)$ must have a large projection in one of two directions with a positive angle.

We make a geometric observation, depicted in Figure 1. Fix $p \in P_{2,\mathrm{good}}$. Since $p \in Z(\delta)$, and $B(p, 2\delta)$ contains no component of Z, we infer that some component Z_p of Z contains a point $q \in B(p, \delta)$, and also intersects $\mathbb{R}^2 \setminus B(p, 2\delta)$. We spend a moment studying the orthogonal projections of $Z_p \cap B(p, 2\delta)$ to lines through the origin. Fix two such lines L_1, L_2 with angle $\angle(L_1, L_2) =: \alpha \geqslant \delta$, and let $\pi_j \colon \mathbb{R}^2 \to L_j$ be the orthogonal projection.

Evidently $\pi_j(q) \in \pi_j(Z_p \cap B(p, 2\delta))$. Let Π_j be the (possibly degenerate) component interval of $\pi_j(Z_p \cap B(p, 2\delta))$ containing $\pi_j(q)$. We claim that

$$\max\{|\Pi_1|, |\Pi_2|\} \geqslant c\alpha\delta, \tag{2.15}$$

where $c > 0$ is a suitable absolute constant. Assume to the contrary that $\max\{|\Pi_1|, |\Pi_2|\} < c\alpha\delta$. This implies that there are points $x_j^1, x_j^2 \in L_1$ at distance $< c\alpha\delta$ from $\pi_j(q)$ which are not in $\pi_j(Z_p \cap B(p, 2\delta))$. Write $I_j := [x_j^1, x_j^2] \subset L_j$, $j \in \{1, 2\}$, and note that

$$Q := \pi_1^{-1}(I_1) \cap \pi_2^{-1}(I_2)$$

is a rectangular box of diameter $\lesssim (c\alpha\delta)/\alpha = c\delta$. Since $q \in Q \cap B(p, \delta)$, choosing $c > 0$ sufficiently small allows us to conclude that $Q \subset B(p, 2\delta)$. Since $x_j^i \notin \pi_j(Z_p \cap B(p, 2\delta))$ for $i, j \in \{1, 2\}$, we have

$$[Z_p \cap B(p, 2\delta)] \cap \pi_j^{-1}(x_j^i) = \varnothing.$$

The boundary of Q is contained in the union of the (four) lines $\pi_j^{-1}(x_j^i)$, $i, j \in \{1, 2\}$, so we infer that

$$[Z_p \cap B(p, 2\delta)] \cap \partial Q = \varnothing. \tag{2.16}$$

However, Z_p is a connected set meeting both Q (at q) and $\mathbb{R}^2 \setminus Q$ (recalling that $Q \subset B(p, 2\delta)$), so $Z_p \cap \partial Q \neq \varnothing$. Using again that $Q \subset B(p, 2\delta)$, this violates (2.16), and proves (2.15). Now that (2.15) has been proven, we can relax it a bit by eliminating the reference to the special component Z_p: we have shown that if $p \in P_{2,\text{good}}$, and L_1, L_2 are two lines through the origin with $\angle(L_1, L_2) = \alpha \geqslant \delta$, then

$$\max\{|\pi_{L_1}(Z \cap B(p, 2\delta))|, |\pi_{L_2}(Z \cap B(p, 2\delta))|\} \geqslant c\alpha\delta. \tag{2.17}$$

We apply this as follows: let $\ell_1, \ell_2 \in \mathcal{L}$ be two lines δ-incident to p with $\angle(\ell_1, \ell_2) =: \alpha \geqslant \delta$, and let π_1, π_2 be the orthogonal projections to $L_1 := \ell_1^\perp$ and $L_2 := \ell_2^\perp$, respectively. It follows from (2.17), and $B(p, 2\delta) \subset \ell_i(4\delta)$, that

$$\max\{|\pi_1([Z \cap B(p, 2\delta)] \cap \ell_1(4\delta))|, |\pi_2([Z \cap B(p, 2\delta)] \cap \ell_2(4\delta))|\} \geqslant c\alpha\delta. \tag{2.18}$$

To exploit this information, recall that $p \in P_{2,\text{good}} \subset P$ is $\geqslant k$-rich, with $k \geqslant 2$, and the lines in $\mathcal{L}$ are ϵ-separated, with $\epsilon \geqslant \delta$. So, using Lemma 2.11, we may isolate two collections $\mathcal{L}_1(p)$ and $\mathcal{L}_2(p)$ of $\gtrsim k$ lines in $\mathcal{L}$, all δ-incident to p, such that $\angle(\ell_1, \ell_2) \gtrsim k\epsilon$ for all pairs $(\ell_1, \ell_2) \in \mathcal{L}_1(p) \times \mathcal{L}_2(p)$. By (2.18), the following holds for either $\mathcal{L}_1(p)$ or $\mathcal{L}_2(p)$:

$$|\pi_{\ell^\perp}([Z \cap B(p, 2\delta)] \cap \ell(4\delta))| \geqslant ck\epsilon\delta, \qquad \ell \in \mathcal{L}_j(p), \tag{2.19}$$

where $c > 0$ might be a bit smaller than in (2.18). Motivated by this observation, we say that $(p, \ell) \in P_{2,\text{good}} \times \mathcal{L}$ is a *good incidence* if $\ell \sim p$, and (2.19) holds. Since $|\mathcal{L}_j(p)| \gtrsim k$ for all $p \in P_{2,\text{good}}$ and $j \in \{1, 2\}$, we see that

$$\sum_{\ell \in \mathcal{L}} |\{p \in P_{2,\text{good}} : (p, \ell) \text{ is a good incidence}\}|$$

$$= \sum_{p \in P_{2,\text{good}}} |\{\ell \in \mathcal{L} : (p, \ell) \text{ is a good incidence}\}| \gtrsim |P_{2,\text{good}}| \cdot k.$$

Averaging over $\ell \in \mathcal{L}$, we find a line $\ell_0 \in \mathcal{L}$ such that

$$|\{p \in P_{2,\text{good}} : (p, \ell_0) \text{ is a good incidence}\}| \gtrsim \frac{|P_{2,\text{good}}| \cdot k}{|\mathcal{L}|}. \tag{2.20}$$

We will now conclude the proof by inferring, from (2.20), that (Lebesgue) positively many lines inside $\ell_0(4\delta)$ are contained in Z. Since however Z has null measure (and *a fortiori* can contain at most $\deg p$ distinct parallel lines), we will have reached a contradiction. So, consider a random line $\ell' \subset \ell_0(4\delta)$; more precisely, let $\mathbb{P}$ be the uniform distribution on $\pi_{\ell_0^\perp}(\ell_0(4\delta)) \cong [0, 8\delta]$, and pick

$$\ell' = \pi_{\ell_0^\perp}^{-1}\{t\} \subset \ell(4\delta)$$

according to $t \sim \mathbb{P}$. Whenever (p, ℓ_0) is a good incidence, (2.19) implies that the probability of ℓ' hitting $Z \cap \bar{B}(p, 2\delta)$ is $\gtrsim k\epsilon$. The balls $\bar{B}(p, 2\delta)$ have bounded overlap as p varies, so (2.20) implies that the expected number "$\mathbb{E}$" of intersections between $\ell' \subset \ell_0(4\delta)$ and Z is

$$\mathbb{E} \gtrsim \frac{|P_{2,\text{good}}| \cdot k^2\epsilon}{|\mathcal{L}|}.$$

On the other hand, $\mathbb{E} \leqslant \deg p \leqslant C_{\deg}|\mathcal{L}|/k$: any line with $> \deg p$ intersections with Z is contained in Z by Bézout's theorem, and this cannot happen for a set of lines with positive probability (or even strictly more than $\deg p$ choices of ℓ'). So, we infer that

$$\frac{|P_{2,\text{good}}| \cdot k^2\epsilon}{|\mathcal{L}|} \lesssim \mathbb{E} \leqslant \frac{C_{\deg}|\mathcal{L}|}{k},$$

which can be rearranged to

$$|P_{2,\text{good}}| \lesssim \frac{C_{\deg}|\mathcal{L}|^2}{k^3} \cdot \epsilon^{-1} \leqslant \frac{C_{\deg}|P|}{C} \leqslant \frac{|P|}{\sqrt{C}},$$

using the counter assumption (2.12) in the end, and also recalling that $C_{\deg} \leqslant \sqrt{C}$. Choosing $C \geqslant 1$ large enough, we infer that $|P_{2,\text{good}}| < |P|/4$, as desired. Since we have now shown that

$$|P| \leqslant |P_1| + |P_{2,\text{bad}}| + |P_{2,\text{good}}| < \frac{|P|}{2} + \frac{|P|}{4} + \frac{|P|}{4} < |P|,$$

a contradiction (starting from (2.12)) has been reached, and the proof of Theorem 2.4 is complete. □

We then quickly derive Theorem 2.3:

Proof of Theorem 2.3. Fix a δ-separated set $P \subset Q_0$, and a δ-separated set of lines $\mathcal{L} \subset \mathcal{Q}_0$. There is no loss of generality assuming that $|\mathcal{L}| \geqslant |P|$: if this fails to begin with, one may apply *point-line duality* to exchange the roles of P and $\mathcal{L}$ and obtain a new set of points, $P_\mathcal{L}$, and a new family of lines, $\mathcal{L}_P$. This is a standard trick, so we only sketch the details: one associates to every $(a, b) \in P$ the line $\tilde{\ell}_{(a,b)} = \{y = -ax + b\}$, and to every line $\ell_{(c,d)} = \{y = cx + d\} \in \mathcal{L}$ the point $(c, d) \in \mathbb{R}^2$. Then, it is clear that (a, b) lies on $\ell_{(c,d)}$ if and only if (c, d) lies on $\tilde{\ell}_{(a,b)}$. Also, the δ-separated set $P \subset Q_0$ gets mapped to a δ-separated set of lines in $\mathcal{Q}_0$, and vice versa, by our definition of "δ-separation". With a little work, one can also check that if (a, b) is δ-incident to $\ell_{(c,d)}$, then (c, d) is $C\delta$-incident to $\tilde{\ell}_{(a,b)}$. Therefore, with suitable choices of constants in the definitions, one has $|\mathcal{I}(P, \mathcal{L})| \lesssim |\mathcal{I}(P_\mathcal{L}, \mathcal{L}_P)|$. But if $|\mathcal{L}| < |P|$, then $|P_\mathcal{L}| < |\mathcal{L}_P|$, and we have arrived at a situation where the number of lines exceeds the number of points, as desired.

So, we assume that $|\mathcal{L}| \geqslant |P|$, and in particular $[|\mathcal{L}|^{2/3}/|P|^{1/3}] \cdot \delta^{-1/3} \geqslant 1$. For $j \geqslant 1$, let

$$P_j := \{p \in P : p \text{ is } k\text{-rich for some } 2^{j-1} \leqslant k < 2^j\}.$$

The set P_1 consists of the 1-rich points in P, and for these we apply the trivial bound $|\mathcal{I}(P_1, \mathcal{L})| \leqslant |P|$. For $j \geqslant 2$, we apply Theorem 2.4 as follows:

$$|\mathcal{I}(P, \mathcal{L})| \lesssim |P| + \sum_{j \geqslant 2} 2^j |P_j| \lesssim \sum_{2^j \leqslant [|\mathcal{L}|^{2/3}/|P|^{1/3}] \cdot \delta^{-1/3}} 2^j |P|$$

$$+ \sum_{2^j > [|\mathcal{L}|^{2/3}/|P|^{1/3}] \cdot \delta^{-1/3}} 2^j \cdot \frac{|\mathcal{L}|^2}{2^{3j}} \cdot \delta^{-1}.$$

One readily verifies that both sums above are comparable to $|P|^{2/3}|\mathcal{L}|^{2/3} \cdot \delta^{-1/3}$, and also $|P| \leqslant |P|^{2/3}|\mathcal{L}|^{2/3} \cdot \delta^{-1/3}$ since we assumed $|P| \leqslant |\mathcal{L}|$. This concludes the proof. $\qquad\square$

3. LOOMIS-WHITNEY INEQUALITY IN THE HEISENBERG GROUP

In this section, we deduce the Loomis-Whitney inequality in Theorem 1.3 from the planar incidence bound established in the previous section. We begin by introducing the Heisenberg concepts and notation carefully. The *first Heisenberg group* $\mathbb{H}$ is the group $(\mathbb{R}^3, \cdot)$ with the group product

$$(x, y, t) \cdot (x', y', t') := (x + x', y + y', t + t' + \tfrac{1}{2}(xy' - yx')). \tag{3.1}$$

The *Heisenberg dilation* δ_λ with constant $\lambda > 0$ is the group isomorphism

$$\delta_\lambda : \mathbb{H} \to \mathbb{H}, \quad \delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t).$$

In geometric measure theory of the sub-Riemannian Heisenberg group [33], an important role is played by *Heisenberg projections* that are adapted to the group and dilation structure of $\mathbb{H}$ and that map onto homogeneous subgroups of $\mathbb{H}$. In the present paper, we only consider two projections associated to two "coordinate" planes introduced below.

Let $\mathbb{W}_x := \{(x, 0, t) : (x, t) \in \mathbb{R}^2\} \subset \mathbb{H}$ and $\mathbb{W}_y = \{(0, y, t) : (y, t) \in \mathbb{R}^2\} \subset \mathbb{H}$ be the *vertical subgroups* of $\mathbb{H}$ containing the x-axis and y-axis, respectively. Write also $\mathbb{L}_x := \{(x, 0, 0) : x \in \mathbb{R}\}$ and $\mathbb{L}_y := \{(0, y, 0) : y \in \mathbb{R}\}$, so

- $\mathbb{L}_x$ is a complementary *horizontal subgroup* of $\mathbb{W}_y$, and
- $\mathbb{L}_y$ is a complementary *horizontal subgroup* of $\mathbb{W}_x$.

This means, for example, that every point $p \in \mathbb{H}$ has a unique decomposition $p = w_x \cdot l_y$, where $w_x \in \mathbb{W}_x$ and $l_y \in \mathbb{L}_y$. Similarly, there is also a unique decomposition $p = w_y \cdot l_x$, where $w_y \in \mathbb{W}_y$ and $l_x \in \mathbb{L}_x$. These decompositions give rise to the *vertical projections*

$$p \mapsto w_x =: \pi_x(p) \in \mathbb{W}_x \quad \text{and} \quad p \mapsto w_y =: \pi_y(p) \in \mathbb{W}_y.$$

It is immediate from the definition that the fibres of the projections π_x and π_y left cosets of $\mathbb{L}_y$ and $\mathbb{L}_x$, respectively:

$$\pi_x^{-1}\{w\} = w \cdot \mathbb{L}_y \quad \text{and} \quad \pi_x^{-1}\{w\} = w \cdot \mathbb{L}_x.$$

Using the group product in (3.1), it is also easy to write down explicit expressions for π_x and π_y:

$$\pi_y(x, y, t) = (0, y, t + \tfrac{xy}{2}) \quad \text{and} \quad \pi_x(x, y, t) = (x, 0, t - \tfrac{xy}{2}).$$

If the reader is not comfortable with the Heisenberg group, he can simply identify both $\mathbb{W}_x$ and $\mathbb{W}_y$ with $\mathbb{R}^2$, and consider the maps $(x, y, t) \mapsto (y, t + (xy)/2))$ and $(x, y, t) \mapsto$

$(x, t - (xy)/2)$ without paying attention to their origin. It is clear that π_x and π_y are smooth, and hence locally Lipschitz with respect to the Euclidean metric in $\mathbb{R}^3$. The vertical projections are, in fact, **not** Lipschitz with respect to the *Korányi distance* $d(p, q) = \|q^{-1} \cdot p\|$, but all the metric concepts which we use in this section (balls, neighborhoods etc.) will be defined using the Euclidean distance.

We recall the statement of Theorem 1.3:

Theorem 3.2. *Let $K \subset \mathbb{H}$ be Lebesgue measurable. Then,*

$$|K| \lesssim |\pi_x(K)|^{2/3} \cdot |\pi_y(K)|^{2/3}. \tag{3.3}$$

Remark 3.4. On the left hand side of (3.3), the notation "$| \cdot |$" refers to either Lebesgue measure on $\mathbb{R}^3$ or $\mathcal{H}_d^4$ (which are the same, up to a multiplicative constant). Similarly, on the right hand side of (3.3), the notation "$| \cdot |$" can either refer to Lebesgue measure on $\mathbb{R}^2$, or $\mathcal{H}_d^3$ restricted to a vertical subgroup; these measures, again, coincide up to a constant. Below, the notation "$| \cdot |$" may also refer to cardinality, but the meaning should always be clear from the context.

3.1. **Reduction to a planar incidence problem.** We start with a few geometric observations which will be used in the proof of Theorem 3.2. Fix a "scale" parameter $0 < \delta < 1$. We write $Q_0 := [-1, 1]^3 \subset \mathbb{R}^3$. The first lemma records that the "tubes" $\pi_x^{-1}(B(p_x, \delta))$ and $\pi_y^{-1}(B(p_y, \delta))$ are fairly close to Euclidean δ-tubes inside the bounded set Q_0:

Lemma 3.5. *There is an absolute constant $A_1 \geqslant 1$ such that the following holds. Let $w_x \in \mathbb{W}_x$, $w_y \in \mathbb{W}_y$, and write $B_x := B(w_x, \delta) \cap \mathbb{W}_x$ and $B_y := B(w_y, \delta) \cap \mathbb{W}_y$. Then*

$$\pi_x^{-1}(B_x) \cap Q_0 \subset [w_x \cdot \mathbb{L}_y](A_1\delta) \quad \text{and} \quad \pi_y^{-1}(B_y) \cap Q_0 \subset [w_y \cdot \mathbb{L}_x](A_1\delta).$$

In other words, the intersection of $\pi_x^{-1}(B_x)$ with Q_0 is contained in the Euclidean $A_1\delta$-neighbourhood of the horizontal line $p_x \cdot \mathbb{L}_y$, and analogously if the roles of x and y are reverted.

Corollary 3.6. *Let $w_x \in \mathbb{W}_x \cap Q_0$, $w_y \in \mathbb{W}_y \cap Q_0$, and consider $T_x = [w_x \cdot \mathbb{L}_y](A_1\delta)$ and $T_y = [w_y \cdot \mathbb{L}_x](A_1\delta)$. Then,*

$$|T_x \cap T_y| \lesssim \delta^3.$$

Proof. The horizontal lines $w_x \cdot \mathbb{L}_y$ and $w_y \cdot \mathbb{L}_x$ hit Q_0, so they are quantitatively non-vertical; their angles with the t-axis are uniformly bounded from below. This implies that the intersection $T_x \cap T_y$ is fairly transversal, and the upper bound follows. $\qquad\square$

Lemma 3.7. *There exists a constant $A \geqslant 1$ such that the following holds. If $w_x = (a, 0, b) \in \mathbb{W}_x$, then*

$$\ell := \pi_y(w_x \cdot \mathbb{L}_y) = \{(0, y, ay + b) : y \in \mathbb{R}\}$$

and

$$\pi_y \left(\pi_x^{-1}(B(w_x, \delta)) \cap Q_0 \right) \subseteq \ell(A\delta).$$

Proof. An easy computation shows for arbitrary $(x, 0, t) \in \mathbb{W}_x$ and $y \in \mathbb{R}$ that

$$\pi_y((x, 0, t) \cdot (0, y, 0)) = \pi_y \left(x, y, t + \tfrac{xy}{2} \right) = (0, y, xy + t). \tag{3.8}$$

This establishes the first claim with $(x, t) = (a, b)$ and $y \in \mathbb{R}$.

The second part of the lemma follows from Lemma 3.5 since vertical projections are locally Lipschitz with respect to the Euclidean metric. Alternatively, one can use again (3.8) and let (x, t) range in a δ-disk centered at (a, b). $\qquad\square$

Proposition 3.9. *Let P_x and P_y be δ-separated sets in $\mathbb{W}_x \cap Q_0$ and $\mathbb{W}_y \cap Q_0$, respectively. Set*

$$\mathcal{L}_y := \{\pi_y(w_x \cdot \mathbb{L}_y) : w_x \in P_x\}.$$

Then $\mathcal{L}_y$ is a δ-separated set of lines in $\mathcal{Q}_0$. Moreover, if $w_x \in P_x$, $w_y \in P_y$, and

$$\pi_x^{-1}(B(w_x, \delta)) \cap \pi_y^{-1}(B(w_y, \delta)) \cap Q_0 \neq \varnothing \tag{3.10}$$

then w_y is $(1 + A)\delta$-incident to $\pi_y(w_x \cdot \mathbb{L}_y)$.

Proof. We first observe that $\mathcal{L}_y$ is a δ-separated set of lines. Indeed, if $w_x, w_x' \in P_x$ are distinct, then Lemma 3.7 shows that $\pi_y(w_x \cdot \mathbb{L}_y)$ and $\pi_y(w_x' \cdot \mathbb{L}_y)$ are two lines in $\mathbb{W}_y \cong \mathbb{R}^2$ of the form

$$\ell := \{(y, ay + b) : y \in \mathbb{R}\} \quad \text{and} \quad \ell' := \{(y, a'y + b') : y \in \mathbb{R}\}$$

with $d(\ell, \ell') = |(a, b) - (a', b')| \geqslant \delta$, and $|a|, |a'|, |b|, |b'| \leqslant 1$ (the latter condition ensures that $\ell, \ell' \in \mathcal{Q}_0$, recalling we only defined the metric d on $\mathcal{Q}_0$).

Next we assume that (3.10) holds. Using Lemma 3.7, this implies that

$$[\pi_y(w_x \cdot \mathbb{L}_y)](A\delta) \cap B(w_y, \delta) \supset \pi_y\left(\pi_x^{-1}(B(w_x, \delta)) \cap Q_0\right) \cap B(w_y, \delta) \neq \varnothing.$$

We infer that $w_y \in [\pi_y(w_x \cdot \mathbb{L}_y)]([1 + A]\delta)$, as claimed. □

Proof of Theorem 3.2. First, we may assume that K is compact, by the inner regularity of Lebesgue measure. Then, we may assume that $K \subset \frac{1}{2}Q_0$, since both sides of (3.3) scale in the same way with respect to the Heisenberg dilations δ_r. Indeed, since the Jacobian determinant of δ_r is r^4, we have $|\delta_r K| = r^4 |K|$. On the other hand, dilations commute with vertical projections, and the maps $\delta_r|_{\mathbb{W}_x}$ and $\delta_r|_{\mathbb{W}_y}$ have Jacobian determinant r^3, so

$$|\pi_x(\delta_r K)|^{2/3} \cdot |\pi_y(\delta_r K)|^{2/3} = |\delta_r(\pi_x(K))|^{2/3} \cdot |\delta_r(\pi_y(K))|^{2/3} = r^4 |\pi_x(K)|^{2/3} \cdot |\pi_y(K)|^{2/3}.$$

Thus, we may and will assume that $K \subset \frac{1}{2}Q_0$, which implies that $\pi_x(K) \subset \mathbb{W}_x \cap Q_0$ and $\pi_y(K) \subset \mathbb{W}_y \cap Q_0$. Since $\pi_x(K)$ and $\pi_y(K)$ are bounded, there exist finite maximal δ-separated subsets $P_x \subset \pi_x(K)$ and $P_y \subset \pi_y(K)$ for any "scale" parameter $0 < \delta < 1$. Fix $\varepsilon > 0$. Then for all $\delta > 0$ small enough (depending on K and ε), we have

$$\delta^2[\operatorname{card} P_x] \lesssim |\pi_x(K)| + \varepsilon \quad \text{and} \quad \delta^2[\operatorname{card} P_y] \lesssim |\pi_y(K)| + \varepsilon. \tag{3.11}$$

To improve clarity, we exceptionally use the notation "card" for cardinality within this proof. The parameter ε is used here only to handle the case where $|\pi_x(K)| = 0$ or $|\pi_y(K)| = 0$. Now, it suffices to prove for δ as in (3.11) that

$$|K| \lesssim [\operatorname{card} P_x]^{2/3}[\operatorname{card} P_y]^{2/3} \cdot \delta^{8/3}. \tag{3.12}$$

This will yield

$$|K| \lesssim (|\pi_x(K)| + \varepsilon)^{2/3} \cdot (|\pi_y(K)| + \varepsilon)^{2/3},$$

and the theorem follows by letting $\varepsilon \to 0$.

We will establish (3.12) as a corollary of Theorem 2.3. In order to relate (3.12) to a set $\mathcal{I}(P, \mathcal{L})$ of incidences, we first recall that

$$\pi_x(K) \subset \bigcup_{w_x \in P_x} B(w_x, \delta) \quad \text{and} \quad \pi_y(K) \subset \bigcup_{w_y \in P_y} B(w_y, \delta),$$

and hence
$$K \subseteq \bigcup_{(w_x, w_y) \in P_x \times P_y} \pi_x^{-1}(B(w_x, \delta)) \cap \pi_y^{-1}(B(w_y, \delta)) \cap Q_0.$$

It follows from Lemma 3.5 and Corollary 3.6 that

$$|K| \lesssim \delta^3 \operatorname{card}\{(w_x, w_y) \in P_x \times P_y : \pi_x^{-1}(B(w_x, \delta)) \cap \pi_y^{-1}(B(w_y, \delta)) \cap Q_0 \neq \varnothing\}. \quad (3.13)$$

To control the cardinality that appears on the right, we use Proposition 3.9. It allows us to deduce from (3.13) that

$$|K| \lesssim \delta^3 \operatorname{card} \mathcal{I}_{(1+A)\delta}(P_y, \mathcal{L}_y),$$

where $\mathcal{I}_{(1+A)\delta}(P_y, \mathcal{L}_y)$ is the set of $(1 + A)\delta$-incidences between the points in P_y and the lines in $\mathcal{L}_y := \{\pi_y(w_x \cdot \mathbb{L}_y) : w_x \in P_x\} \subset \mathcal{Q}_0$. Since $\operatorname{card} \mathcal{L}_y = \operatorname{card} P_x$, the proof of (3.12), and hence Theorem 3.2, is then reduced to showing

$$\operatorname{card} \mathcal{I}_{(1+A)\delta}(P_y, \mathcal{L}_y) \lesssim [\operatorname{card} \mathcal{L}_y]^{\frac{2}{3}} [\operatorname{card} P_y]^{\frac{2}{3}} \cdot \delta^{-\frac{1}{3}}. \quad (3.14)$$

But since P_y consists of δ-separated points, and $\mathcal{L}_y$ of δ-separated lines, (3.14) follows immediately from the incidence bound in Theorem 2.3 (as pointed out in Remark 2.6, the theorem remains valid for $C\delta$-incidences, and now we use this with $C = 1 + A$.) $\qquad \square$

4. Applications of the Loomis-Whitney inequality in the Heisenberg group

In this section, we derive the Gagliardo-Nirenberg-Sobolev inequality, Theorem 1.8, from the Loomis-Whitney inequality, Theorem 1.3. The arguments presented in this section are very standard, and we claim no originality. As a corollary of Theorem 1.8, we obtain the isoperimetric inequality in $\mathbb{H}$ (with a non-optimal constant). At the end of the section, we also show how the Loomis-Whitney inequality can be used, directly, to infer a variant of the isoperimetric inequality, without passing through the Sobolev inequality.

We start by recalling the statement of Theorem 1.8:

Theorem 4.1. *Let $f \in BV(\mathbb{H})$. Then,*

$$\|f\|_{4/3} \lesssim \sqrt{\|Xf\| \|Yf\|}. \quad (4.2)$$

Recall that $f \in BV(\mathbb{H})$ if $f \in L^1(\mathbb{H})$, and the distributional derivatives Xf, Yf are finite signed Radon measures. Smooth compactly supported functions are dense in $BV(\mathbb{H})$ in the sense that if $f \in BV(\mathbb{H})$, then there exists a sequence $\{\varphi_j\}_{j \in \mathbb{N}} \subset C_c^\infty(\mathbb{R}^3)$ such that $\varphi_j \to f$ almost everywhere (and in $L^1(\mathbb{H})$ if desired), and $\|Z\varphi_j\| \to \|Zf\|$ for $Z \in \{X, Y\}$. For a reference, see [13, Theorem 2.2.2]. With this approximation in hand, it suffices to prove Theorem 4.1 for, say, $f \in C_c^1(\mathbb{R}^3)$. The following lemma contains most of the proof:

Lemma 4.3. *Let $f \in C_c^1(\mathbb{R}^3)$, and write*

$$F_k := \{p \in \mathbb{R}^3 : 2^{k-1} \leqslant |f(p)| \leqslant 2^k\}, \qquad k \in \mathbb{Z}. \quad (4.4)$$

Then,

$$|\pi_x(F_k)| \leqslant 2^{-k+2} \int_{F_{k-1}} |Yf| \quad \text{and} \quad |\pi_y(F_k)| \leqslant 2^{-k+2} \int_{F_{k-1}} |Xf|. \quad (4.5)$$

Proof. By symmetry, it suffices to prove the first inequality in (4.5). Let $w = (x, 0, t) \in \pi_x(F_k)$, and fix $p = w \cdot (0, y, 0) \in F_k$ such that $\pi_x(p) = w$. In particular, $|f(p)| \geqslant 2^{k-1}$. Recall the notation $\mathbb{L}_y = \{(0, y, 0) : y \in \mathbb{R}\}$. Since f is compactly supported, we may pick another point $p' \in w \cdot \mathbb{L}_y$ such that $f(p') = 0$. Since $|f|$ is continuous, we infer that there is a non-degenerate line segment I on the line $w \cdot \mathbb{L}_y$ such that $2^{k-2} \leqslant |f(q)| \leqslant 2^{k-1}$ for all $q \in I$ (hence $I \subset F_{k-1}$), and $|f|$ takes the values 2^{k-2} and 2^{k-1}, respectively, at the endpoints $q_i = w \cdot (0, y_i, 0)$ of I, $i \in \{1, 2\}$. Define $\gamma(y) := w \cdot (0, y, 0) = (x, y, t + \tfrac{1}{2}xy)$. With this notation,

$$2^{k-2} \leqslant |f(q_1) - f(q_2)| \leqslant \int_{y_1}^{y_2} |(f \circ \gamma)'(y)| \, dy \leqslant \int_{\{y : (x, y, t + \frac{1}{2}xy) \in F_{k-1}\}} |Yf(x, y, t + \tfrac{1}{2}xy)| \, dy.$$

Writing $\Phi(x, y, t) := (x, 0, t) \cdot (0, y, 0) = (x, y, t + \tfrac{1}{2}xy)$, and integrating over $(x, t) \cong (x, 0, t) \in \pi_x(F_k) \subset \mathbb{W}_x$, it follows that

$$\int_{\pi_x(F_k)} \left[\int_{\{y : \Phi(x, y, t) \in F_{k-1}\}} |Yf(\Phi(x, y, t))| \, dy \right] dx \, dt \geqslant 2^{k-2} |\pi_x(F_k)|. \tag{4.6}$$

Finally, we note that $J_\Phi = \det D\Phi \equiv 1$. Therefore, using Fubini's theorem, and performing a change of variables to the left hand side of (4.6), we see that

$$2^{k-2} |\pi_x(F_k)| \leqslant \int_{\{(x, y, t) \in \mathbb{R}^3 : \Phi(x, y, t) \in F_{k-1}\}} |Yf(\Phi(x, y, t))| \, dx \, dy \, dt$$

$$= \int_{F_{k-1}} |Yf(x, y, t)| \, dx \, dy \, dt.$$

This completes the proof. $\qquad\square$

We are then prepared to prove Theorem 4.1:

Proof of Theorem 4.1. Fix $f \in C_c^1(\mathbb{R}^3)$, and define the sets F_k, $k \in \mathbb{Z}$, as in (4.4). Using first Theorem 3.2, then Lemma 4.3, then Cauchy-Schwarz, and finally the embedding $\ell^1 \hookrightarrow \ell^{4/3}$, we estimate as follows:

$$\int |f|^{4/3} \sim \sum_{k \in \mathbb{Z}} 2^{4k/3} |F_k| \leqslant \sum_{k \in \mathbb{Z}} 2^{4k/3} |\pi_x(F_k)|^{2/3} |\pi_y(F_k)|^{2/3}$$

$$\lesssim \sum_{k \in \mathbb{Z}} \left(\int_{F_{k-1}} |Xf| \right)^{2/3} \left(\int_{F_{k-1}} |Yf| \right)^{2/3}$$

$$\leqslant \left[\sum_{k \in \mathbb{Z}} \left(\int_{F_{k-1}} |Xf| \right)^{4/3} \right]^{1/2} \left[\sum_{k \in \mathbb{Z}} \left(\int_{F_{k-1}} |Yf| \right)^{4/3} \right]^{1/2}$$

$$\leqslant \left[\sum_{k \in \mathbb{Z}} \int_{F_{k-1}} |Xf| \right]^{2/3} \left[\sum_{k \in \mathbb{Z}} \int_{F_{k-1}} |Yf| \right]^{2/3} = \|Xf\|_1^{2/3} \|Yf\|_1^{2/3}.$$

Raising both sides to the power $3/4$ completes the proof. $\qquad\square$

We conclude the paper by discussing isoperimetric inequalities. A measurable set $E \subset \mathbb{H}$ has *finite horizontal perimeter* if $\chi_E \in BV(\mathbb{H})$. Here χ_E is the characteristic function of E. Note that our definition of $BV(\mathbb{H})$ implies, in particular, that $|E| < \infty$. We follow common practice, and write $P_\mathbb{H}(E) := \|\nabla_\mathbb{H} \chi_E\|$. For more information on sets of finite

horizontal perimeter, see [12]. Now, applying Theorem 4.1 to $f = \chi_E$, we infer Pansu's isoperimetric inequality (with a non-optimal constant):

Theorem 4.7 (Pansu). *There exists a constant $C > 0$ such that*

$$|E|^{\frac{3}{4}} \leqslant CP_{\mathbb{H}}(E) \tag{4.8}$$

for any measurable set $E \subset \mathbb{H}$ of finite horizontal perimeter.

We remark that the *a priori* assumption $|E| < \infty$ is critical here; for example the theorem evidently fails for $E = \mathbb{H}$, for which $|E| = \infty$ but $\|\nabla_{\mathbb{H}} \chi_E\| = 0$. We conclude the paper by deducing a weaker version of (4.8) (even) more directly from the Loomis-Whitney inequality. Namely, we claim that

$$|E|^{\frac{3}{4}} \leqslant C\mathcal{H}_d^3(\partial E) \tag{4.9}$$

for any bounded measurable set $E \subset \mathbb{H}$. This inequality is, in general, weaker than (4.8): at least for open sets $E \subset \mathbb{H}$, the property $\mathcal{H}_d^3(\partial E) < \infty$ implies that $P_{\mathbb{H}}(E) < \infty$, and then $P_{\mathbb{H}}(E) \lesssim \mathcal{H}_d^3(\partial E)$, see [14, Theorem 4.18]. However, if E is a bounded open set with C^1 boundary, then $\mathcal{H}_d^3(\partial E) \sim P_{\mathbb{H}}(E)$, see [12, Corollary 7.7].

To prove (4.9), we need the following auxiliary result, see [10, Lemma 3.4]:

Lemma 4.10. *There exists a constant $C > 0$ such that the following holds. Let $\mathbb{W} \subset \mathbb{H}$ be a vertical subgroup. Then,*

$$|\pi_{\mathbb{W}}(A)| \leqslant C\mathcal{H}_d^3(A), \qquad A \subset \mathbb{H}. \tag{4.11}$$

Proof of (4.9). Let $E \subset \mathbb{H}$ be bounded and measurable. We first claim that

$$\pi_x(E) \subseteq \pi_x(\partial E), \tag{4.12}$$

$$\pi_y(E) \subseteq \pi_y(\partial E) \tag{4.13}$$

We prove only (4.12) since (4.13) follows similarly. Let $w \in \pi_x(E)$ and consider $\pi_x^{-1}\{w\} = w \cdot \mathbb{L}_y$ where $\mathbb{L}_y = \{(0, y, 0) \ : \ y \in \mathbb{R}\}$ is as in Section 3. By definition there exists $y_1 \in \mathbb{R}$ such that $w \cdot (0, y_1, 0) \in E$ and since E is bounded there also exists $y_2 \in \mathbb{R}$ such that $w \cdot (0, y_2, 0) \in \mathbb{H} \setminus \overline{E}$. Since $w \cdot \mathbb{L}_y$ is connected, there finally exists $y_3 \in \mathbb{R}$ such that $w \cdot (0, y_3, 0) \in \partial E$ which immediately implies (4.12). Using Theorem 3.2, (4.12), and (4.13) we get

$$|E| \lesssim |\pi_x(\partial E)|^{\frac{2}{3}} |\pi_y(\partial E)|^{\frac{2}{3}}.$$

Now the isoperimetric inequality (4.9) follows using Lemma 4.11. $\square$

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