COLLAPSING AND THE CONVEX HULL PROPERTY IN A SOAP FILM CAPILLARITY MODEL

DARREN KING, FRANCESCO MAGGI, AND SALVATORE STUVARD

Abstract. Soap films hanging from a wire frame are studied in the framework of capillarity theory. Minimizers in the corresponding variational problem are known to consist of positive volume regions with boundaries of constant mean curvature/pressure, possibly connected by "collapsed" minimal surfaces. We prove here that collapsing only occurs if the mean curvature/pressure of the bulky regions is negative, and that, when this last property holds, the whole soap film lies in the convex hull of its boundary wire frame.

Keywords: convex hull property, minimal surfaces, constant mean curvature surfaces, Plateau's problem.

AMS Math Subject Classification (2010): 49Q05 (primary), 53A10, 49Q20.

1. Introduction

We continue the analysis, started in [KMS20a], of the variational model for soap films spanning a wire frame introduced in [MSS19]. In this **soap film capillarity model**, soap films are described as three-dimensional regions of small volume, rather than as two-dimensional surfaces with vanishing mean curvature, i.e. as minimal surfaces. In [KMS20a] we have proved the existence of *generalized* minimizers in the soap film capillarity model. The term generalized indicates the possibility for minimizing sequences of three-dimensional regions to locally collapse onto two-dimensional surfaces. Correspondingly, a generalized minimizer consists: of a three-dimensional set enclosing the prescribed small volume of liquid, with boundary of constant mean curvature λ – where the value of λ is proportional to the pressure of the soap film; and, possibly, of a two-dimensional surface with zero mean curvature, whose area has to be counted twice in computing the energy of the minimizer; see Figure 1.1. When this second possibility occurs, we speak of **collapsed minimizers**. When collapsing does not occur, generalized minimizers are just regular minimizers, in the sense that they correspond to three-dimensional regions belonging to the competition class. In this paper we prove two related results concerning important geometric properties of generalized minimizers, that can be roughly stated as follows:

- (i) if collapsing occurs, then the constant mean curvature/pressure λ must be non-positive (Theorem 2.9);
- (ii) if λ is non-positive, then the generalized minimizer is contained into the convex hull of the boundary wire frame (Theorem 2.10); this convex hull property is of course a basic property of minimal surfaces, therefore the interest of establishing it in this setting.

Theorem 2.9 is proved by comparing (through a technically delicate argument) a collapsed minimizer with competitors obtained by slightly de-collapsing its collapsed region (with a net increase of volume), followed by slightly deflating the bulky part of the minimizer (to restore the enclosed volume); see Figure 3.2 below. The proof of Theorem 2.10 is an adaptation to our context of the classical argument used to prove the convex hull property on stationary varifolds.

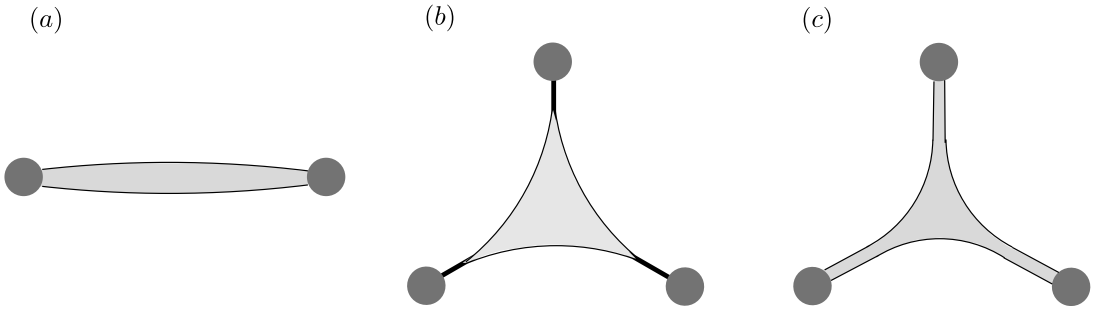

FIGURE 1.1. Generalized minimizers in the soap film capillarity model in the "planar case", where the "boundary wire frame" reduces to finitely many small disks (depicted in dark grey). We minimize the length of the boundary of two-dimensional regions, depicted in light gray, enclosing a given (small) volume ε and spanning the boundary disks. (a) When the boundary consists of two disks, and ε is small enough, we have a non-collapsed minimizing region bounded by two almost flat circular arcs of curvature $\lambda = \mathrm{O}(\varepsilon)$. (b) When the boundary consists of three disks, and ε is small enough, we have a collapsed minimizer given by a combination of a two-dimensional region bounded by circular arcs of negative curvature $\lambda = -\mathrm{O}(1/\sqrt{\varepsilon})$, and of three segments (depicted by thick lines) whose length has to be counted with double multiplicity to compute the minimizing energy. Collapsing corresponds with the situation, depicted in (c), where minimizing sequences consist of two-dimensional regions with opposite parts of their boundaries becoming increasingly closer to each other.

The paper is organized as follows. In section 2 we formally introduce the soap film capillarity model and state our main results (together with some necessary background results proved in [KMS20a]). Sections 3 and 4 contain, respectively, the proofs of Theorem 2.9 and Theorem 2.10.

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2. Statements

2.1. **Notation.** The ambient space we will be working in is Euclidean space $\mathbb{R}^{n+1}$ with $n \geq 2$. For $A \subset \mathbb{R}^{n+1}$, $\mathrm{cl}\,(A)$ is the topological closure of A in $\mathbb{R}^{n+1}$, $\mathrm{conv}(A)$ is its convex hull, and $I_\delta(A), U_\delta(A)$ are its closed and open δ-tubular neighborhoods, respectively. $B_r(x)$ is the open ball centered at $x \in \mathbb{R}^{n+1}$ with radius $r > 0$. If A is (Borel) measurable, $|A|$ and $\mathcal{H}^s(A)$ denote its Lebesgue and s-dimensional Hausdorff measure, respectively. We will adopt standard terminology in Geometric Measure Theory, for which we refer the reader to [Sim83, AFP00, Mag12]. In particular, given an integer $0 \leq k \leq n+1$, a Borel measurable set $M \subset \mathbb{R}^{n+1}$ is **countably k-rectifiable** if it can be covered, up to an $\mathcal{H}^k$-negligible set, by countably many Lipschitz images of $\mathbb{R}^k$; it is **(locally) $\mathcal{H}^k$-rectifiable** if it is countably k-rectifiable and, in addition, the $\mathcal{H}^k$ measure of M is (locally) finite. A Borel set $E \subset \mathbb{R}^{n+1}$ is of **locally finite perimeter** if there exists an $\mathbb{R}^{n+1}$-valued Radon measure μ_E on $\mathbb{R}^{n+1}$ such that $\langle \mu_E, X \rangle = \int_E \mathrm{div}\,(X)\,dx$ for all vector fields $X \in C^1_c(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$, and of **finite perimeter** if $P(E) := |\mu_E|(\mathbb{R}^{n+1}) < \infty$. More generally, one can consider, for any Borel set $F \subset \mathbb{R}^{n+1}$, the quantity $|\mu_E|(F)$, which is called the **relative perimeter** $P(E; F)$ of E in F. The **reduced boundary** of a set E of finite perimeter is the set $\partial^* E$ of points $x \in \mathrm{spt}\,|\mu_E|$ such that $(|\mu_E|(B_r(x)))^{-1}\,\mu_E(B_r(x)) \to \nu_E(x)$ for some vector $\nu_E(x) \in \mathbb{S}^n$ as $r \to 0^+$. By De Giorgi's structure theorem, if E has finite perimeter then

$\partial^* E$ is $\mathcal{H}^n$-rectifiable, and the Gauss-Green measure μ_E and its total variation $|\mu_E|$ satisfy $\mu_E = \nu_E \, \mathcal{H}^n \llcorner \partial^* E$ and $|\mu_E| = \mathcal{H}^n \llcorner \partial^* E$, respectively.

2.2. **The soap film capillarity model.** Next, we recall the precise formulation of the variational problem introduced in [KMS20a], and we outline the theory developed in there. We fix a compact set $W \subset \mathbb{R}^{n+1}$ (the "wire frame"), and we denote the region accessible by the soap film as

$$\Omega := \mathbb{R}^{n+1} \setminus W \, .$$

The model scenario we have in mind is the physical case when $n+1=3$, and $W = I_\delta(\Gamma)$ is the closed δ-neighborhood of a closed Jordan curve $\Gamma \subset \mathbb{R}^3$; nonetheless, admissible choice of W will be more general than that. Following the Harrison-Pugh formulation of Plateau's problem [HP16, HP17] as extended in [DLGM17], we introduce a **spanning class** $\mathcal{C}$, that is, a non-empty family of smooth embeddings of $\mathbb{S}^1$ into Ω which is closed by homotopy in Ω, in the sense that if $\gamma \in \mathcal{C}$ and $\tilde\gamma$ is smooth and homotopically equivalent to γ in Ω [1] then $\tilde\gamma \in \mathcal{C}$. A set S is **$\mathcal{C}$-spanning** W if $S \cap \gamma \neq \emptyset$ for all $\gamma \in \mathcal{C}$. The (homotopic) **Plateau's problem defined by** $(W, \mathcal{C})$ is then

$$\ell := \inf \{ \mathcal{H}^n(S) \, : \, S \in \mathcal{S} \} \, , \tag{2.1}$$

where

$$\mathcal{S} := \{ S \subset \Omega \, : \, S \text{ is relatively closed in } \Omega \text{ and } S \text{ is } \mathcal{C}\text{-spanning } W \} \, . \tag{2.2}$$

The capillarity approximation (2.3) of the Plateau's problem (2.1) has been studied in [KMS20a] under the following set of assumptions on W and $\mathcal{C}$:

Assumption 2.1. The compact set W and the spanning class $\mathcal{C}$ are such that the following holds:

- (A1) Plateau's problem ℓ defined in (2.1) satisfies $\ell < \infty$; in particular, by [HP16, DLGM17], there exists a relatively compact, $\mathcal{H}^n$-rectifiable set $S \subset \Omega$ such that $\mathcal{H}^n(S) = \ell$ [2];
- (A2) $\partial W = \partial \Omega$ is a C^2-regular hypersurface in $\mathbb{R}^{n+1}$;
- (A3) there exists $\tau_0 > 0$ such that, for every $\tau < \tau_0$, $\mathbb{R}^{n+1} \setminus I_\tau(W)$ is connected;
- (A4) there exist $\eta_0 > 0$ and a minimizer S in ℓ such that $\gamma \setminus I_{\eta_0}(S) \neq \emptyset$ for every $\gamma \in \mathcal{C}$.

The conditions in Assumption 2.1 seem very reasonable towards the development of a theory of soap films, and are definitely valid in a reasonably large class of initial conditions. In fact, as a by-product of a technical result contained in the present paper, see Lemma 3.2 below, one can see that all the results from [KMS20a] (and thus all the results of the present paper) still hold without the need of assuming (A4). This point is explained in detail in Section 5 below.

Next, we can define the capillarity problem $\psi(\varepsilon)$ at volume $\varepsilon > 0$ as

$$\psi(\varepsilon) := \inf \{ \mathcal{H}^n(\Omega \cap \partial E) : \; E \in \mathcal{E}, \, |E| = \varepsilon, \, \Omega \cap \partial E \text{ is } \mathcal{C}\text{-spanning } W \} \, , \tag{2.3}$$

where the competition class $\mathcal{E}$ is given by

$$\mathcal{E} := \{ E \subset \Omega \, : \, E \text{ is an open set and } \partial E \text{ is } \mathcal{H}^n\text{-rectifiable} \} \, . \tag{2.4}$$

We explicitly observe that each $E \in \mathcal{E}$ is an open set of finite perimeter, and that $P(E;\Omega) = \mathcal{H}^n(\Omega \cap \partial^* E) \le \mathcal{H}^n(\Omega \cap \partial E)$. We also define the class

$$\begin{aligned} \mathcal{K} := \Big\{ (K,E) \, : E \subset \Omega \text{ is open with } \Omega \cap \mathrm{cl}\,(\partial^* E) = \Omega \cap \partial E \subset K \, , \\ K \in \mathcal{S} \text{ and } K \text{ is } \mathcal{H}^n\text{-rectifiable} \Big\} \, . \end{aligned} \tag{2.5}$$

[1] This means that there exists a continuous map $f \colon \, [0,1] \times \mathbb{S}^1 \to \Omega$ such that $f(0,\cdot) = \gamma$ and $f(1,\cdot) = \tilde\gamma$.

[2] In addition, when $n = 2$, every such minimizer S is an Almgren-minimizer in Ω, and therefore satisfies Plateau's laws away from W thanks to [Alm76, Tay76]. This result will not be needed in the sequel, but it is important because it establishes the physical relevance of the model.

For $(K,E)\in\mathcal{K}$, its relaxed energy is given by

$$\mathcal{F}(K,E):=\mathcal{H}^n(\Omega\cap\partial^*E)+2\,\mathcal{H}^n(K\setminus\partial^*E)\,. \tag{2.6}$$

We are now in the position to recall the main results from [KMS20a], which lay the groundwork for the present analysis.

Theorem 2.2 (Existence of generalized minimizers, see [KMS20a, Theorem 1.4])**.** *Let W and $\mathcal{C}$ satisfy Assumption 2.1, and let $\varepsilon>0$. If $\{E_j\}_{j=1}^\infty$ is a minimizing sequence for $\psi(\varepsilon)$, then there exists a pair $(K,E)\in\mathcal{K}$ with $|E|=\varepsilon$ such that, up to possibly extracting subsequences, and up to possible modifications of each E_j outside a large ball containing W (with both operations resulting in defining a new minimizing sequence for $\psi(\varepsilon)$, still denoted by $\{E_j\}_j$), we have that*

$$\begin{aligned}&E_j\to E \text{ in } L^1(\Omega)\,,\\&\mathcal{H}^n\llcorner(\Omega\cap\partial E_j)\overset{*}{\rightharpoonup}\theta\,\mathcal{H}^n\llcorner K \qquad \text{as Radon measures in } \Omega\end{aligned} \tag{2.7}$$

as $j\to\infty$, for an upper semicontinuous multiplicity function $\theta:K\to\mathbb{R}$ satisfying

$$\theta=2\ \mathcal{H}^n\text{-a.e. on } K\setminus\partial^*E,\qquad \theta=1 \text{ on } \Omega\cap\partial^*E\,. \tag{2.8}$$

Moreover, $\psi(\varepsilon)=\mathcal{F}(K,E)$ and, for a suitable constant C, $\psi(\varepsilon)\le 2\,\ell+C\,\varepsilon^{n/(n+1)}$.

Definition 2.3. A pair $(K,E)\in\mathcal{K}$ with $|E|=\varepsilon$ is a **generalized minimizer** for the capillarity problem $\psi(\varepsilon)$ if:

(a) there exists a minimizing sequence $\{E_j\}_{j=1}^\infty$ of sets $E_j\in\mathcal{E}$ such that (2.7) holds for an upper semicontinuous function θ as in (2.8);
(b) $\mathcal{F}(K,E)=\psi(\varepsilon)$.

Theorem 2.4 (Euler-Lagrange equation for generalized minimizers, see [KMS20a, Theorem 1.6])**.** *If (K,E) is a generalized minimizer of $\psi(\varepsilon)$ and $f:\Omega\to\Omega$ is a diffeomorphism such that $|f(E)|=|E|$, then*

$$\mathcal{F}(K,E)\le\mathcal{F}(f(K),f(E))\,. \tag{2.9}$$

In particular:

(i) *there exists $\lambda\in\mathbb{R}$ such that*

$$\lambda\int_{\partial^*E}X\cdot\nu_E\,d\mathcal{H}^n=\int_{\partial^*E}\operatorname{div}^K X\,d\mathcal{H}^n+2\int_{K\setminus\partial^*E}\operatorname{div}^K X\,d\mathcal{H}^n \tag{2.10}$$

for every $X\in C^1_c(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ with $X\cdot\nu_\Omega=0$ on $\partial\Omega$, where div^K denotes the tangential divergence operator along K;

(ii) *there exists $\Sigma\subset K$, closed and with empty interior in K, such that $K\setminus\Sigma$ is a smooth hypersurface, $K\setminus(\Sigma\cup\partial E)$ is a smooth embedded minimal hypersurface, $\mathcal{H}^n(\Sigma\setminus\partial E)=0$, $\Omega\cap(\partial E\setminus\partial^*E)\subset\Sigma$ has empty interior in K, and $\Omega\cap\partial^*E$ is a smooth embedded hypersurface with constant scalar (w.r.t. ν_E) mean curvature λ.*

Remark 2.5. The conclusions about the regularity properties of the set K achieved in Theorem 2.4(ii) are a straightforward consequence of the Euler-Lagrange equation (2.10) and of Allard's regularity theorem for varifolds with bounded generalized mean curvature. A more refined analysis, which crucially relies on the structure of the variational problem $\psi(\varepsilon)$, was carried out in [KMS20b]. A fundamental outcome is that, if one still denotes Σ the singular set appearing in Theorem 2.4(ii), the set $\Sigma\setminus\mathrm{cl}(E)$ is *empty* in all dimensions $n\le 6$ (thus, in particular, in the physical dimension $n=2$), so that $K\setminus\mathrm{cl}(E)$ is a smooth (in fact, analytic) stable minimal hypersurface of $\Omega\setminus\mathrm{cl}(E)$ in such cases; see [KMS20b, Theorem 1.5].

2.3. **Main results.** We start making precise the notion of collapsing.

Definition 2.6. A generalized minimizer $(K, E) \in \mathcal{K}$ of $\psi(\varepsilon)$ is **collapsed** if $K \setminus \partial E \neq \emptyset$. It is **exteriorly collapsed** if $K \setminus \mathrm{cl}\,(E) \neq \emptyset$.

Theorem 2.7 (Convex hull property)**.** *If $(K, E) \in \mathcal{K}$ is an exteriorly collapsed generalized minimizer of $\psi(\varepsilon)$, then $K \subset \mathrm{conv}(W)$.*

Remark 2.8. Theorem 2.7 can be regarded as an extension to the capillarity model of the classical convex hull property valid in the context of (generalized) minimal surfaces. It is worth noticing that the assumption of exterior collapsing is necessary in this setting. It is easy to construct examples of non-collapsed minimizers of $\psi(\varepsilon)$ for which the convex hull property fails: for instance, in the situation of Figure 1.1(a), as soon as the volume parameter ε is slightly increased, it is clear that part of the corresponding minimizer lies outside of the convex hull of the boundary data.

Theorem 2.7 will be proved in two steps, which are of independent interest, and for this reason we record them in two separate statements. First, we show that exterior collapsing enforces a sign condition on the multiplier λ appearing in the Euler–Lagrange equation (2.10). Then, we establish the validity of the convex hull property for a solution to (2.10) in the regime $\lambda \leq 0$.

Theorem 2.9. *Let $(K, E) \in \mathcal{K}$ be an exteriorly collapsed generalized minimizer of $\psi(\varepsilon)$. Then, the Lagrange multiplier λ in the Euler-Lagrange equation* (2.10) *satisfies $\lambda \leq 0$.*

Theorem 2.10. *Suppose that a pair $(K, E) \in \mathcal{K}$ satisfies the identity* (2.10) *with $\lambda \leq 0$. Then, K is contained in the convex hull $\mathrm{conv}(W)$. Moreover, if $\lambda < 0$, then $K \subset \mathrm{conv}(W \cap \mathrm{cl}\,(K))$.*

Theorem 2.7 is then an immediate corollary of Theorems 2.9 and 2.10. Observe that the validity of the strict inequality $\lambda < 0$ produces a stronger version of the convex hull property compared to the classical result for minimal surfaces. The proof of Theorem 2.10 is obtained by adapting the argument typically used to establish the convex hull property for stationary varifolds (roughly, the case $\lambda = 0$ of Theorem 2.10), see [Sim83, Theorem 19.2]. Proving Theorem 2.9 is more challenging, and is based on the following geometric idea. Given an exteriorly collapsed generalized minimizer (K, E), we define a one-parameter family of competitors $\{(K_t, E_t)\}_{t>0}$ with $(K_t, E_t) \in \mathcal{K}$ and $|E_t| = \varepsilon$ by first adding some positive volume t near a point in the collapsed region $K \setminus \mathrm{cl}\,(E)$, and then restoring the volume constraint by "locally pushing inwards" E at a point in $\partial^* E$; see Figure 3.2 below. Since $K \setminus \mathrm{cl}\,(E)$ and $\partial^* E$ have, respectively, 0 and λ mean curvature, we find $\mathcal{F}(K_t, E_t) = \mathcal{F}(K, E) - \lambda\, t + \mathrm{O}(t^2)$, so that $\lambda \leq 0$ follows by letting $t \to 0^+$, *provided* we can show that $\mathcal{F}(K, E) \leq \mathcal{F}(K_t, E_t)$. This inequality requires a dedicated argument. Indeed, we only know that (K, E) minimizes the relaxed energy $\mathcal{F}$ with respect to its diffeomorphic images, and in fact K_t cannot be represented as the image of K through a map, let alone through a diffeomorphism. To prove $\mathcal{F}(K, E) \leq \mathcal{F}(K_t, E_t)$, we will instead approximate (K_t, E_t) by a sequence of open sets F_j in $\mathcal{E}$ having volumes $|F_j|$ converging to ε as $j \to \infty$. Since $\mathcal{F}(K, E) = \psi(\varepsilon)$, and $\psi(\cdot)$ is lower semicontinuous on $(0, \infty)$, we will obtain the desired inequality if we are able to enforce that the $\mathcal{H}^n$ measure of the boundaries ∂F_j in Ω is not larger than $\mathcal{F}(K_t, E_t)$ for large j. This construction is the main technical difficulty of this note, and it exploits in a crucial way the regularity properties of K as described in Theorem 2.4. The details are discussed in Lemma 3.2.

3. Proof of Theorem 2.9

We start with a simple lemma on orientability, which allows to strengthen conclusion (ii) in Theorem 2.4 from "there exists $\Sigma \subset K$, closed and with empty interior in K, such that $K \setminus \Sigma$ is a smooth hypersurface" into "there exists $\Sigma \subset K$, closed and with empty interior in K, such that $K \setminus \Sigma$ is a smooth **orientable** hypersurface". We do not claim that the set Σ resulting from this change still satisfies $\mathcal{H}^n(\Sigma \setminus \partial E) = 0$.

Lemma 3.1. *If M is a smooth hypersurface in $\mathbb{R}^{n+1}$, then there exists a meager closed set $J \subset M$ such that a smooth unit normal vector field to M can be defined on $M \setminus J$.*

Proof. Let $\mathcal{U}$ denote the family of the open sets $A \subset M$ such that a smooth unit normal vector field to M can be defined on A. Let $\mathcal{U}^*$ be a non-empty subset of $\mathcal{U}$ which is totally ordered by set inclusion, and set

$$A^* := \bigcup \{A : A \in \mathcal{U}^*\}.$$

Let $\{A_j\}_{j \in \mathbb{N}} \subset \mathcal{U}^*$ be such that

$$A^* = \bigcup_{j \in \mathbb{N}} A_j .$$

Since $\mathcal{U}^*$ is totally ordered by set inclusion, we can assume without loss of generality that $A_j \subset A_{j+1}$. By exploiting this monotonicity property we easily prove that $A^* \in \mathcal{U}$, and therefore that $\mathcal{U}^*$ admits an upper bound in the ordering of $\mathcal{U}$. By Zorn's lemma, $\mathcal{U}$ admits a maximal element A with respect to set inclusion. The set $J = M \setminus A$ is closed in M. Should J have non-empty interior, we could find $r > 0$ and $p \in J$ such that $B_r(p) \cap M \subset J$. Up to decrease r, we can entail $B_r(p) \cap M \in \mathcal{U}$, and then that $A \cup (B_r(p) \cap M) \in \mathcal{U}$, against the maximality of A in $\mathcal{U}$. □

Next we show that any $(K, E) \in \mathcal{K}$ such that K is a smooth orientable hypersurface outside of a meager closed set can be approximated in energy by sets $F \in \mathcal{E}$.

Lemma 3.2. *Let $(K, E) \in \mathcal{K}$, that is, let K be $\mathcal{H}^n$-rectifiable, relatively closed in Ω, and $\mathcal{C}$-spanning W, and let $E \subset \Omega$ be open with $\Omega \cap \mathrm{cl}\,(\partial^* E) = \Omega \cap \partial E \subset K$. Let $\Sigma \subset K$ be a closed set with empty interior relatively to K such that $K \setminus \Sigma$ is a smooth hypersurface in Ω and such that there exists $\nu \in C^\infty(K \setminus \Sigma; \mathbb{S}^n)$ with $\nu(x)^\perp = T_x(K \setminus \Sigma)$ for every $x \in K \setminus \Sigma$. Let*

$$M_0 := (K \setminus \Sigma) \setminus \mathrm{cl}\,(E), \qquad M_1 := (K \setminus \Sigma) \cap E, \qquad M := M_0 \cup M_1 = K \setminus (\Sigma \cup \partial E).$$

For every $x \in M$, let $\rho(x) > 0$ be such that $\{x + t\,\rho(x)\,\nu(x) : x \in M \text{ and } |t| < 1\}$ is a tubular neighborhood of M in $\mathbb{R}^{n+1}$ (see e.g. [Lee03, Theorem 6.24]*). Also, let $\|A_M\|(x)$ be the maximal principal curvature (in absolute value) of M at x. Define then a positive function $u : M \to (0, \eta]$ by setting*

$$u(x) := \min\Big\{\eta, \frac{\mathrm{dist}(x, \Sigma \cup \partial E \cup W)}{2}, \delta\,\rho(x), \frac{\delta}{\|A_M\|(x)}\Big\}, \qquad \eta, \delta \in (0,1),$$

where $\eta, \delta \in (0, 1)$, and let

$$\begin{aligned} A_0 &:= \Big\{x + t\,u(x)\,\nu(x) : x \in M_0, 0 < t < 1\Big\}, \\ A_1 &:= \Big\{x + t\,u(x)\,\nu(x) : x \in M_1, 0 < t < 1\Big\}, \\ F &:= A_0 \cup \big(E \setminus \mathrm{cl}\,(A_1)\big); \end{aligned}$$

see Figure 3.1. Then $F \subset \Omega$ is open, ∂F is $\mathcal{H}^n$-rectifiable, $K \subset \Omega \cap \partial F$ (in particular, $\Omega \cap \partial F$ is $\mathcal{C}$-spanning W), and

$$\limsup_{\delta \to 0^+} \limsup_{\eta \to 0^+} \mathcal{H}^n(\Omega \cap \partial F) \le \mathcal{F}(K, E). \tag{3.1}$$

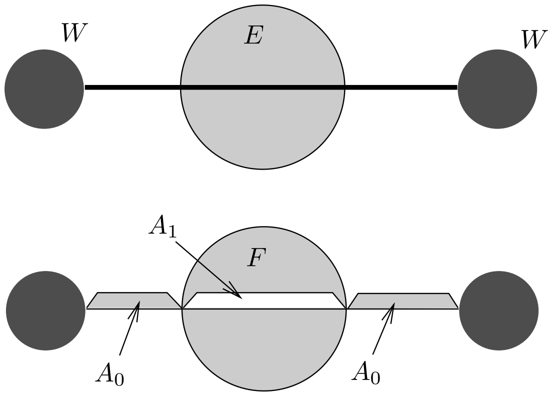

FIGURE 3.1. The construction in Lemma 3.2. The part of K outside $\partial^* E$ is denoted by a bold line to recall that in computing $\mathcal{F}(K,E)$ it is counted with multiplicity 2. Notice that, in principle, $K\setminus\partial^* E$ could intersect E.

Proof. Step one: In this step we prove that

$$F \text{ is open with } F\subset\Omega\,, \tag{3.2}$$

$$K\cup\Big\{x+u(x)\,\nu(x): x\in M\Big\}=\Omega\cap\partial F\,. \tag{3.3}$$

Since M_0 and M_1 are relatively open in M and u is positive on M, it is easily seen that A_0 and A_1 are open, and thus that F is open. Let us define a map $\Phi: M\times\mathbb{R}\to\mathbb{R}^{n+1}$ by setting $\Phi(x,t)=x+t\,u(x)\,\nu(x)$, so that

$$A_k=\Phi(M_k\times(0,1))\,,\qquad \Phi(M_k\times\{0,1\})\subset\partial A_k\,,\qquad k=0,1\,. \tag{3.4}$$

Since $M\subset\Omega$ and $u(x)<\mathrm{dist}(x,W)$ for every $x\in M$, we deduce that

$$\Phi(M\times[0,1])\subset\Omega\,. \tag{3.5}$$

In particular, $F\subset\Omega$ and (3.2) is proved. Next we prove that

$$\Omega\cap\partial F\ \subset\ K\cup\Big\{x+u(x)\,\nu(x): x\in M\Big\}\,. \tag{3.6}$$

Since the boundary of the union and of the intersection of two sets is contained in the union of the boundaries, and since the boundary of a set coincides with the boundary of its complement, the inclusion $\partial\mathrm{cl}\,(A_1)\subset\partial A_1$ gives

$$\begin{aligned}\Omega\cap\partial F\ &\subset\ \Omega\cap\Big(\partial A_0\cup\partial[E\setminus\mathrm{cl}\,(A_1)]\Big)\subset\Omega\cap\Big(\partial A_0\cup\partial E\cup\partial[\mathbb{R}^{n+1}\setminus\mathrm{cl}\,(A_1)]\Big)\\ &=\ \Omega\cap\Big(\partial A_0\cup\partial E\cup\partial\mathrm{cl}\,(A_1)\Big)\subset\Omega\cap\big(\partial E\cup\partial A_0\cup\partial A_1\big)\,.\end{aligned}$$

Hence (3.6) follows from $\Omega\cap\partial E\subset K$, and the fact that, for $k=0,1$,

$$\begin{aligned}\Omega\cap\partial A_k\ &\subset\ K\cup\Phi(M_k\times\{0,1\})\\ &\subset\ K\cup\big\{x+u(x)\,\nu(x): x\in M_k\big\}\,.\end{aligned}$$

This proves (3.6), so that the proof of (3.3) is completed by showing that

$$M\cup\Big\{x+u(x)\,\nu(x): x\in M\Big\}\ \subset\ \Omega\cap\partial F\,, \tag{3.7}$$

$$\Sigma\setminus\partial E\ \subset\ \Omega\cap\partial F\,, \tag{3.8}$$

$$\Omega\cap\partial E\ \subset\ \Omega\cap\partial F\,. \tag{3.9}$$

Proof of (3.7): Since $M_0\cap\mathrm{cl}\,(E)=\emptyset$, $M_1\subset E$, and $u(x)<\mathrm{dist}(x,\partial E)$ for every $x\in M$, by (3.4) we find

$$\Phi\big(M_0\times[0,1]\big)\cap\mathrm{cl}\,(E)=\emptyset\,,\qquad \Phi\big(M_1\times[0,1]\big)\subset E\,. \tag{3.10}$$

By (3.4) and (3.10) we find $A_0 \cap \mathrm{cl}\,(E) = \emptyset$ and $A_1 \subset E$, so that

$$\Big((\partial A_0) \setminus \mathrm{cl}\,(E)\Big) \cup \Big(E \cap \partial A_1\Big) \subset \partial F\,.$$

Again by (3.4) and (3.10) we have

$$\Phi(M_0 \times \{0,1\}) \subset \partial A_0 \setminus \mathrm{cl}\,(E)\,, \qquad \Phi(M_1 \times \{0,1\}) \subset E \cap \partial A_1\,, \tag{3.11}$$

and (3.7) follows by (3.5) and (3.11). *Proof of* (3.8): Since $M_1 = (K \setminus \Sigma) \cap E$ and Σ has empty interior in K, we find that $\mathrm{cl}\,(M_1) \cap E = K \cap E$. At the same time, $M \subset \Omega \cap \partial F$ gives $M_1 \cap E \subset E \cap \partial F$ and thus $\mathrm{cl}\,(M_1) \cap E \subset E \cap \partial F$: hence,

$$\Sigma \cap E \ \subset \ K \cap E \ = \ \mathrm{cl}\,(M_1) \cap E \ \subset \ \Omega \cap \partial F\,;$$

similarly, $M_0 = (K \setminus \Sigma) \setminus \mathrm{cl}\,(E)$ implies $\mathrm{cl}\,(M_0) \setminus \mathrm{cl}\,(E) = K \setminus \mathrm{cl}\,(E)$, while $M \subset \Omega \cap \partial F$ gives $\mathrm{cl}\,(M_0) \setminus \mathrm{cl}\,(E) \subset (\partial F) \setminus \mathrm{cl}\,(E)$, hence

$$\Sigma \setminus \mathrm{cl}\,(E) \ \subset \ K \setminus \mathrm{cl}\,(E) \ \subset \ (\partial F) \setminus \mathrm{cl}\,(E)\,,$$

which combined with $\Sigma \subset K \subset \Omega$ gives $\Sigma \setminus \mathrm{cl}\,(E) \subset \Omega \cap \partial F$. *Proof of* (3.9): since F and E coincide in the complement of $\mathrm{cl}\,(A_0) \cup \mathrm{cl}\,(A_1)$, we have

$$\Omega \cap \partial E \setminus \big(\mathrm{cl}\,(A_0) \cup \mathrm{cl}\,(A_1)\big) \ = \ \Omega \cap \partial F \setminus \big(\mathrm{cl}\,(A_0) \cup \mathrm{cl}\,(A_1)\big) \ \subset \ \Omega \cap \partial F\,.$$

Let $y \in \Omega \cap \partial E \cap \mathrm{cl}\,(A_1)$: by (3.10), $y \notin \Phi(M_1 \times [0,1])$ while $A_1 = \Phi(M_1 \times (0,1))$, so that y is in the closure of M_1, and thus of M, relatively to K. In particular, $y \in \Omega \cap \mathrm{cl}\,(M) \subset \Omega \cap \partial F$ thanks to $M \subset \Omega \cap \partial F$. Similarly, we can show that $\Omega \cap \partial E \cap \mathrm{cl}\,(A_0) \subset \Omega \cap \partial F$ and thus prove (3.9).

Step two: By (3.2) and (3.3) we immediately deduce all the conclusions except (3.1). To prove (3.1) we first notice that thanks to (3.3)

$$\mathcal{H}^n(\Omega \cap \partial F) \le \mathcal{H}^n(K) + \mathcal{H}^n\Big(\big\{x + u(x)\,\nu(x) : x \in M\big\}\Big)\,. \tag{3.12}$$

Since $\mathrm{dist}(x, \Sigma \cup \partial E \cup W) > 0$, $\rho(x) > 0$, and $\|A_M\|(x) < \infty$ for every $x \in M$, we find that the sets

$$M_\eta = \big\{x \in M : u(x) = \eta\big\} = \Big\{x \in M : \mathrm{dist}(x, \Sigma \cup \partial E \cup W) \ge 2\eta\,, \rho(x) \ge \frac{\eta}{\delta}\,, \|A_M\|(x) \le \frac{\delta}{\eta}\Big\}$$

are increasingly converging to M as $\eta \to 0^+$. Moreover, $x \mapsto x + u(x)\,\nu(x) = x + \eta\,\nu(x)$ is smooth on M_η, so that the area formula gives

$$\begin{aligned} \mathcal{H}^n\Big(\big\{x + u(x)\,\nu(x) : x \in M_\eta\big\}\Big) &= \int_{M_\eta} \prod_{i=1}^n |1 + \eta\,\kappa_i| \\ &\le (1+\delta)^n\,\mathcal{H}^n(M_\eta) \le (1+\delta)^n\,\mathcal{H}^n(M)\,, \end{aligned} \tag{3.13}$$

where κ_i are the principal curvatures of M with respect to ν. In the limit as $\eta \to 0^+$, the sets $\Phi(M_\eta \times \{1\}) = \{x + u(x)\,\nu(x) : x \in M_\eta\}$ are increasingly converging to $\Phi(M \times \{1\}) = \{x + u(x)\,\nu(x) : x \in M\}$, so that (3.12) and (3.13) yield

$$\limsup_{\eta \to 0^+} \mathcal{H}^n(\Omega \cap \partial F) \le \mathcal{H}^n(K) + (1+\delta)^n\,\mathcal{H}^n(M)\,. \tag{3.14}$$

Finally, (3.1) follows from (3.14) once we observe that $M = K \setminus (\Sigma \cup \partial E) \subset K \setminus \partial^* E$, so that

$$\begin{aligned} \mathcal{H}^n(K) + \mathcal{H}^n(M) &= \mathcal{H}^n(\Omega \cap \partial^* E) + \mathcal{H}^n(K \setminus \partial^* E) + \mathcal{H}^n(M) \\ &\le \mathcal{H}^n(\Omega \cap \partial^* E) + 2\,\mathcal{H}^n(K \setminus \partial^* E) = \mathcal{F}(K, E)\,, \end{aligned}$$

as required. □

Proof of Theorem 2.9. Let $(K,E)\in\mathcal{K}$ be a generalized minimizer of $\psi(\varepsilon)$ satisfying the exterior collapsing condition $K\setminus \mathrm{cl}\,(E)\neq\emptyset$. The goal is to show that the Lagrange multiplier λ appearing in (2.10) must be negative. We introduce the notation

$$
\begin{aligned}
Q_r^\nu(x) &:= \Big\{y\in\mathbb{R}^{n+1}\,:\,|(x-y)\cdot\nu|<r\,,\Big|(x-y)-[(x-y)\cdot\nu]\,\nu\Big|<r\Big\}\,, && (3.15)\\
D_r^\nu(x) &:= \big\{y\in\mathbb{R}^{n+1}\,:\,|(x-y)\cdot\nu|=0\,,|x-y|<r\big\}\,, && (3.16)
\end{aligned}
$$

for the cylinder $Q_r^\nu(x)$ with axis along $\nu\in\mathbb{S}^n$, center at x, radius r and height $2\,r$, and for its midsection $D_r^\nu(x)$.

First recall from [KMS20a, Formula (3.24)] that the measure $\mathcal{H}^n\llcorner K$ satisfies a uniform lower density estimate, in the sense that there is a constant $c_0(n)>0$ such that if $x\in K$ then $\mathcal{H}^n(K\cap B_r(x))\geq c_0\,r^n$ for every $B_r(x)\subset\subset\Omega$. The above estimate applied with $x\in K\setminus\mathrm{cl}\,(E)$ and $0<r<\min\{\mathrm{dist}(x,\partial\Omega),\mathrm{dist}(x,\mathrm{cl}\,(E))\}$ implies that $\mathcal{H}^n(K\setminus\mathrm{cl}\,(E))>0$. By Theorem 2.4-(ii), there exists $B_{2\,r_1}(x_1)\subset\subset\Omega\setminus\mathrm{cl}\,(E)$ with $x_1\in K$ such that $K\cap B_{2\,r_1}(x_1)$ is a smooth embedded minimal surface. Let us set

$$
Q_1=Q_{r_1}^{\nu_1}(x_1)\,,\qquad D_1=D_{r_1}^{\nu_1}(x_1)\,,
$$

where ν_1 is a unit normal to K at x_1, and observe that $Q_1\subset B_{2\,r_1}(x_1)$. Upon further decreasing the value of r_1, there exists a smooth solution to the minimal surfaces equation $u_1:\mathrm{cl}\,(D_1)\to\mathbb{R}$ such that

$$
K\cap\mathrm{cl}\,(Q_1)=\big\{z+u_1(z)\,\nu_1:z\in\mathrm{cl}\,(D_1)\big\}\,,\qquad \max_{\mathrm{cl}\,(D_1)}|u_1|\leq\frac{r_1}{2}\,. \tag{3.17}
$$

Next we pick a smooth function $v_1:\mathrm{cl}\,(D_1)\to\mathbb{R}$ with

$$
v_1=0\quad\text{on }\partial D_1\,,\qquad v_1>0\quad\text{on }D_1\,,\qquad \int_{D_1}v_1=1\,, \tag{3.18}
$$

and for $t>0$ we define an open set G_1^t by

$$
G_1^t=\Big\{z+h\,\nu_1:z\in D_1\,,u_1(z)<h<u_1(z)+t\,v_1(z)\Big\}\,. \tag{3.19}
$$

For t sufficiently small (depending only on r_1 and on the choice of v_1) we have that $G_1^t\subset Q_1$ with

$$
\partial G_1^t\cap\partial Q_1=K\cap\partial Q_1=\big\{z+u_1(z)\,\nu_1:z\in\partial D_1\big\}\,, \tag{3.20}
$$

and

$$
K\cap\mathrm{cl}\,(Q_1)\subset\partial G_1^t\,. \tag{3.21}
$$

Moreover we easily see that

$$
|G_1^t|=t\,,\qquad \mathcal{H}^n(\partial G_1^t)=\mathcal{H}^n(Q_1\cap\partial G_1^t)=2\,\mathcal{H}^n(K\cap Q_1)+\mathrm{O}(t^2)\quad\text{as }t\to0^+\,, \tag{3.22}
$$

where we have used $\int_{D_1}v_1=1$, $v_1=0$ on ∂D_1, and the fact that u_1 solves the minimal surfaces equation. Next, we perform an analogous construction at a point $x_2\in\Omega\cap\partial^*E$, taking once again advantage of Theorem 2.4(ii). More precisely, if we let ν_2 denote the exterior unit normal vector to ∂^*E at x_2, we find a cylinder $Q_2=Q_{r_2}^{\nu_2}(x_2)$ with mid-section $D_2=D_{r_2}^{\nu_2}(x_2)$ and with $\mathrm{dist}(Q_1,Q_2)>0$, and a smooth function $u_2:\mathrm{cl}\,(D_2)\to\mathbb{R}$ with

$$
\begin{aligned}
E\cap\mathrm{cl}\,(Q_2) &= \Big\{z+h\,\nu_2:z\in\mathrm{cl}\,(D_2)\,,-r_2\leq h<u_2(z)\Big\}\,, && (3.23)\\
K\cap\mathrm{cl}\,(Q_2) &= \partial E\cap\mathrm{cl}\,(Q_2)=\Big\{z+u_2(z)\,\nu_2:z\in\mathrm{cl}\,(D_2)\Big\}\,, && (3.24)
\end{aligned}
$$

and

$$
-\mathrm{div}\Big(\frac{\nabla u_2}{\sqrt{1+|\nabla u_2|^2}}\Big)=\lambda\quad\text{on }D_2\,,\qquad \max_{\mathrm{cl}\,(D_2)}|u_2|\leq\frac{r_2}{2}\,. \tag{3.25}
$$

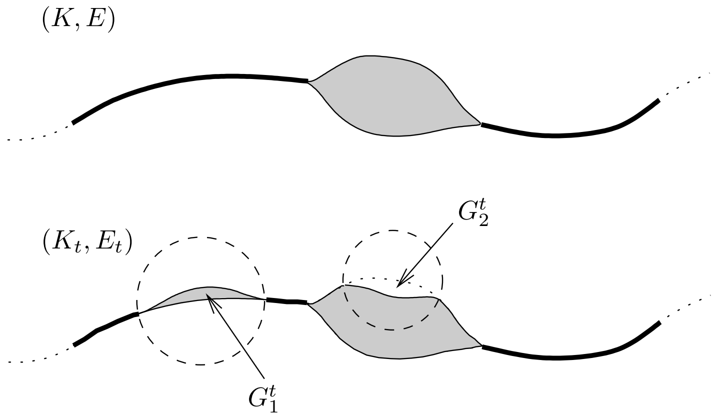

FIGURE 3.2. The competitors (K_t, E_t) used in proving that if exterior collapsing occurs for (K, E), then the Lagrange multplier λ is non-positive. Multiplicity two regions are depicted in bold, the sets E and E_t in gray. The competitor (K_t, E_t) is obtained by adding a volume t near a point of $K \setminus \mathrm{cl}\,(E)$ by bulging one of the two available sheets, at an area cost of $\mathrm{O}(t^2)$ (as $K \setminus \mathrm{cl}\,(E)$ is minimal); and by restoring the total volume by pushing inwards E at a point in $\partial^* E$, at an area cost of $-\lambda\, t + \mathrm{O}(t^2)$.

We choose a smooth function $v_2 : \mathrm{cl}\,(D_2) \to \mathbb{R}$ with

$$v_2 = 0 \quad \text{on } \partial D_2\,, \qquad v_2 > 0 \quad \text{on } D_2\,, \qquad \int_{D_2} v_2 = 1\,, \tag{3.26}$$

and then define an open set G_2^t by setting

$$G_2^t = \Big\{ z + h\,\nu_2 : z \in D_2\,, u_2(z) - t\,v_2(z) < h < u_2(z) \Big\}\,. \tag{3.27}$$

For t small enough (depending only on r_2 and on the choice of v_2) we have that $G_2^t \subset E \cap Q_2$, with

$$\partial G_2^t \cap \partial Q_2 = K \cap \partial Q_2 = \big\{ z + u_2(z)\,\nu_2 : z \in \partial D_2 \big\}\,. \tag{3.28}$$

Furthermore, if we let Y denote the closed set

$$Y = \{ z + (u_2(z) - t v_2(z))\,\nu_2 : \; z \in \mathrm{cl}\,(D_2) \}\,, \tag{3.29}$$

it is easily seen that for $t < t_0$

$$|G_2^t| = t\,, \qquad \mathcal{H}^n(Y) = \mathcal{H}^n(Y \cap Q_2) = \mathcal{H}^n(\partial E \cap Q_2) - \lambda\, t + \mathrm{O}(t^2)\,, \tag{3.30}$$

where we have used $\int_{D_2} v_2 = 1$, $v_2 = 0$ on ∂D_2, and (3.25).

Now set

$$K_t \;:=\; \Big(K \setminus \big(Q_1 \cup Q_2 \big) \Big) \cup \partial G_1^t \cup Y\,, \tag{3.31}$$

$$E_t \;:=\; \Big(E \setminus \mathrm{cl}\,(G_2^t) \Big) \cup G_1^t\,; \tag{3.32}$$

see Figure 3.2. We claim that the following holds:

$$K_t \setminus Q_2 \supset K \setminus Q_2\,, \tag{3.33}$$

$$\partial E_t \cap \Omega \setminus (\mathrm{cl}\,(Q_1) \cup \mathrm{cl}\,(Q_2)) = \partial E \cap \Omega \setminus (\mathrm{cl}\,(Q_1) \cup \mathrm{cl}\,(Q_2))\,, \tag{3.34}$$

$$\partial E_t = \partial \left[E \setminus \mathrm{cl}\,(G_2^t) \right] \cup \partial G_1^t\,, \tag{3.35}$$

$$\partial E_t \cap \mathrm{cl}\,(Q_1) = \partial G_1^t\,, \tag{3.36}$$

$$\partial E_t \cap \mathrm{cl}\,(Q_2) = Y \tag{3.37}$$

The inclusion in (3.33) follows from $K\setminus Q_2 \subset K\setminus(Q_1\cup Q_2)\cup\partial G_1^t$; (3.34) is a consequence of $E_t \setminus (\mathrm{cl}\,(Q_1)\cup\mathrm{cl}\,(Q_2)) = E\setminus(\mathrm{cl}\,(Q_1)\cup\mathrm{cl}\,(Q_2))$ together with the observation that $\Omega\setminus(\mathrm{cl}\,(Q_1)\cup\mathrm{cl}\,(Q_2))$ is an open set; to prove (3.35), it suffices to observe that $\partial\left[E\setminus\mathrm{cl}\,(G_2^t)\right]\subset\mathrm{cl}\,(E)$ whereas $\partial G_1^t\subset\mathrm{cl}\,(Q_1)\subset B_{2r_1}(x_1)\subset\subset\Omega\setminus\mathrm{cl}\,(E)$; (3.36) then follows immediately from (3.35). To prove $\partial E_t\cap\mathrm{cl}\,(Q_2)\subset Y$ (the other inclusion being trivial), we proceed as follows. First, we deduce from (3.35) that $\partial E_t\cap\mathrm{cl}\,(Q_2)=\partial\left[E\setminus\mathrm{cl}\,(G_2^t)\right]\cap\mathrm{cl}\,(Q_2)$. Then, we notice that $\partial\left[E\setminus\mathrm{cl}\,(G_2^t)\right]\cap\partial Q_2\subset K\cap\partial Q_2\subset Y$. Finally, suppose that $x\in\partial\left[E\setminus\mathrm{cl}\,(G_2^t)\right]\cap Q_2$, so that there exists a sequence $\{x_j\}_{j=1}^\infty$ such that $x_j\in E\setminus\mathrm{cl}\,(G_2^t)\cap Q_2$ and $x_j\to x$. In particular, we have $x_j=z_j+h_j\,\nu_2$, where $z_j\in D_2$ and $-r_2<h_j<u_2(z_j)-tv_2(z_j)$. By compactness, and using the continuity of the functions u_2 and v_2, we have that, possibly along a (not relabeled) subsequence, $z_j\to z_\infty\in\mathrm{cl}\,(D_2)$, and $h_j\to h_\infty\in[-r_2,u_2(z_\infty)-tv_2(z_\infty)]$, so that $x=z_\infty+h_\infty\,\nu_2$. But then it has to be $h_\infty=u_2(z_\infty)-t\,v_2(z_\infty)$, otherwise $x\in E\setminus\mathrm{cl}\,(G_2^t)\subset E_t$. This shows that $x\in Y$, thus completing the proof of (3.37).

Next, we claim that $(K_t,E_t)\in\mathcal{K}$, and that

$$|E_t|=|E|=\varepsilon\,,\qquad \mathcal{F}(K_t,E_t)=\mathcal{F}(K,E)-\lambda\,t+\mathrm{O}(t^2)\,. \tag{3.38}$$

First, it is clear that $E_t\subset\Omega$ is open, and that $K_t\subset\Omega$ is a relatively closed and $\mathcal{H}^n$-rectifiable set in Ω. Moreover, K_t is $\mathcal{C}$-spanning W. To see this, first observe that by (3.33) any curve $\gamma\in\mathcal{C}$ with $\gamma\cap(K\setminus Q_2)\neq\emptyset$ must intersect K_t. If, on the other hand, $\gamma\cap(K\setminus Q_2)=\emptyset$, then necessarily $\gamma\cap K\cap Q_2\neq\emptyset$ because K is $\mathcal{C}$-spanning W. In turn, this implies that $\gamma\cap\partial E\cap\mathrm{cl}\,(Q_2)\neq\emptyset$, and thus also $\gamma\cap Y\neq\emptyset$ as a consequence of [KMS20a, Lemma 2.3] since Y is a diffeomorphic image of $\partial E\cap\mathrm{cl}\,(Q_2)$. Finally, $\Omega\cap\partial E_t\subset K_t$ follows immediately from (3.34), (3.36), and (3.37). The volume identity in (3.38) is deduced from the volume identities in (3.22) and (3.30) given that G_1^t and E are disjoint. We can then proceed with the proof of the second equation in (3.38). Using the analogous of (3.34) for the reduced boundary together with (3.36) and (3.37), and then applying (3.22) and (3.30) we obtain

$$\begin{aligned}\mathcal{H}^n(\Omega\cap\partial^*E_t)&=\mathcal{H}^n(\Omega\cap\partial^*E\setminus(\mathrm{cl}\,(Q_1)\cup\mathrm{cl}\,(Q_2)))+\mathcal{H}^n(\partial G_1^t)+\mathcal{H}^n(Y)\\&=\mathcal{H}^n(\Omega\cap\partial^*E)+2\,\mathcal{H}^n((K\setminus\partial^*E)\cap Q_1)-\lambda\,t+\mathrm{O}(t^2)\,,\end{aligned} \tag{3.39}$$

whereas

$$2\,\mathcal{H}^n(K_t\setminus\partial^*E_t)=2\,\mathcal{H}^n((K\setminus\partial^*E)\setminus Q_1)\,. \tag{3.40}$$

The second part of (3.38) is then obtained by summing (3.39) and (3.40).

Finally, we claim that there exists a closed set $\Sigma_t\subset K_t$ with empty interior relatively to K_t and such that $K_t\setminus\Sigma_t$ is a smooth orientable hypersurface in Ω. Indeed, in the construction of K_t from K, we may have increased Σ, at most, by adding to it the closed sets $\{z+u_k(z)\,\nu_k:\,z\in\partial D_k\}$, which have definitely empty interiors relatively to K_t.

Therefore we can apply Lemma 3.2 to (K_t,E_t) to find a sequence $\{F_j\}_j\subset\mathcal{E}$ such that $\Omega\cap\partial F_j$ is $\mathcal{C}$-spanning W, with

$$F_j\to E_t\quad\text{in } L^1(\mathbb{R}^{n+1})\,,\qquad \limsup_{j\to\infty}\mathcal{H}^n(\Omega\cap\partial F_j)\le\mathcal{F}(K_t,E_t)\,. \tag{3.41}$$

Since $|F_j|\to|E_t|=\varepsilon$ as $j\to\infty$ and ψ is lower semicontinuous on $(0,\infty)$ (see [KMS20a, Theorem 1.9]), we conclude that

$$\begin{aligned}\mathcal{F}(K,E)&=\psi(\varepsilon)\le\liminf_{j\to\infty}\psi\big(|F_j|\big)\le\limsup_{j\to\infty}\mathcal{H}^n(\Omega\cap\partial F_j)\\&\le\mathcal{F}(K_t,E_t)=\mathcal{F}(K,E)-\lambda\,t+\mathrm{O}(t^2)\,,\end{aligned}$$

thanks to (3.38). By letting $t\to0^+$ we find that it must be $\lambda\le0$, thus completing the proof. □

4. Proof of Theorem 2.10

Proof of Theorem 2.10. Let $(K, E) \in \mathcal{K}$ be such that

$$\lambda \int_{\partial^* E} X \cdot \nu_E \, d\mathcal{H}^n = \int_{\partial^* E} \operatorname{div}^K X \, d\mathcal{H}^n + 2 \int_{K \setminus \partial^* E} \operatorname{div}^K X \, d\mathcal{H}^n \,, \tag{4.1}$$

with $\lambda \leq 0$ for every $X \in C^1_c(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$ with $X \cdot \nu_\Omega = 0$ on $\partial\Omega$. We then prove that $K \subset \operatorname{conv}(W)$ if $\lambda = 0$, and $K \subset \operatorname{conv}(W \cap \operatorname{cl}(K))$ if $\lambda < 0$. The first claim is classical: indeed, if (4.1) holds with $\lambda = 0$ then the varifold V supported on K with multiplicity $\theta = 1$ on $\Omega \cap \partial^* E$ and $\theta = 2$ on $K \setminus \partial^* E$ is stationary in $\Omega = \mathbb{R}^{n+1} \setminus W$. The result is then a straightforward consequence of [Sim83, Theorem 19.2]. We are left with the case $\lambda < 0$. In order to ease the notation, we set $Z := \operatorname{conv}(W \cap \operatorname{cl}(K))$, and, denoting $u(x) := \operatorname{dist}(x, Z)$, we consider the test field

$$X(x) := \chi(x) \, \gamma(u(x)) \, \nabla u(x) \,, \tag{4.2}$$

where γ is a non-negative smooth function on $[0, \infty)$ with $\gamma = 0$ on an interval $[0, 2\eta)$ and $\gamma' \geq 0$ everywhere, and χ is a smooth cut-off function with $0 \leq \chi \leq 1$ and

$$\chi(x) = \begin{cases} 1 & \text{on } I_\sigma(K \setminus U_\eta(Z)) \\ 0 & \text{on } I_\sigma(W) \cup (\mathbb{R}^{n+1} \setminus B_R(0)) \,. \end{cases}$$

Here $0 < \sigma \ll \eta$, and $B_R(0)$ is a large ball containing $K \cup W$. Observe that the function χ is well-defined. Indeed, the definition of Z implies that the set $K \setminus U_\eta(Z)$ is closed in $\mathbb{R}^{n+1}$, so that $\operatorname{dist}(K \setminus U_\eta(Z), W) \geq 3\,\sigma > 0$, and thus the closed sets $I_\sigma(K \setminus U_\eta(Z))$ and $I_\sigma(W)$ are disjoint. Since $X = 0$ both in a neighborhood of W and outside of $B_R(0)$, X is admissible in (4.1). Furthermore,

$$X(x) = \gamma(u(x)) \, \nabla u(x) \qquad \text{in a neighborhood of } K \,. \tag{4.3}$$

Hence, by $|\nabla u| = 1$ we compute

$$\begin{aligned} \nabla X &= \gamma'(u) \, \nabla u \otimes \nabla u + \gamma(u) \, \nabla^2 u && \text{in a neighborhood of } K \,, \\ \operatorname{div} X &= \gamma'(u) + \gamma(u) \, \Delta u && \text{in a neighborhood of } K \,, \\ \operatorname{div}^K X &= \gamma'(u) \, (1 - (\nabla u \cdot \nu)^2) + \gamma(u) \left(\Delta u - \nabla^2 u[\nu, \nu] \right) && \mathcal{H}^n\text{-a.e. on } K \,, \end{aligned}$$

where $\nu(x)$ is a unit normal vector to K at x, for every $x \in K$ such that the approximate tangent plane $T_x K$ exists. Since u is convex (distance from a convex set) we have $\Delta u \geq 0$, $\Delta u - \nabla^2 u[\nu, \nu] \geq 0$, and thus $\operatorname{div}^K X \geq 0$ $\mathcal{H}^n$-a.e. on K. By [Mag12, Chapter 16], for a.e. $\eta > 0$, $E \setminus I_\eta(Z)$ is a set of finite perimeter with

$$\partial^*(E \setminus I_\eta(Z)) = \big((\partial^* E) \setminus I_\eta(Z) \big) \cup \big(E \cap \partial^* I_\eta(Z) \big) \qquad \text{modulo } \mathcal{H}^n \,,$$

and

$$\begin{aligned} \nu_{E \setminus I_\eta(Z)} &= \nu_E \,, && \mathcal{H}^n\text{-a.e. on } (\partial^* E) \setminus I_\eta(Z) \,, \\ \nu_{E \setminus I_\eta(Z)} &= -\nabla u \,, && \mathcal{H}^n\text{-a.e. on } E \cap \partial^* I_\eta(Z) \,. \end{aligned}$$

By (4.1), $\operatorname{div}^K X \geq 0$, and by applying the divergence theorem on $E \setminus I_\eta(Z)$ we find that

$$\begin{aligned} 0 \;&\leq\; \lambda \int_{\partial^* E} X \cdot \nu_E = \lambda \int_{(\partial^* E) \setminus I_\eta(Z)} (\gamma(u) \, \nabla u) \cdot \nu_E \\ &=\; \lambda \Big\{ \int_{E \setminus I_\eta(Z)} \operatorname{div}(\gamma(u) \, \nabla u) - \int_{E \cap \partial^* I_\eta(Z)} (\gamma(u) \, \nabla u) \cdot (-\nabla u) \Big\} \\ &=\; \lambda \Big\{ \int_{E \setminus I_\eta(Z)} \gamma'(u) + \gamma(u) \, \Delta u + \int_{E \cap \partial^* I_\eta(Z)} \gamma(u) \Big\} \end{aligned}$$

Now we use the condition $\lambda<0$. We have

$$\int_{E\setminus I_\eta(Z)}\gamma'(u)+\gamma(u)\,\Delta u+\int_{E\cap\partial^* I_\eta(Z)}\gamma(u)=0\,,$$

which implies $|E\setminus I_\eta(Z)|=0$ by the arbitrariness of γ, and thus $E\subset Z$ by the arbitrariness of η. Applying again (4.1) we now find

$$0=\int_{\partial^* E}\mathrm{div}^K X+2\int_{K\setminus\partial^* E}\mathrm{div}^K X=2\int_{K\setminus\partial^* E}\mathrm{div}^K X$$

which now gives $\mathcal{H}^n(K\setminus I_\eta(Z))=0$ for every $\eta>0$. Thus $K\subset Z$, as claimed. □

5. Removing assumption (A4)

In this final section we show that all the results in [KMS20a] and in the present paper hold without the need of assuming (A4) from Assumption 2.1. We notice that (A4) corresponds to (1.12) in [KMS20a].

Theorem 5.1. *Theorem 1.4, Theorem 1.6 and Theorem 1.9 from* [KMS20a] *and Theorem 2.6, Theorem 2.8 and Theorem 2.9 from this paper hold under the sole assumption that W and $\mathcal{C}$ satisfy the conditions (A1), (A2) and (A3) stated in Assumption 2.1.*

Proof. As noticed in the introductory remarks to the proof of Theorem 1.4 from [KMS20a], see section 3 of that paper, assumption (A4) (equivalently, [KMS20a, (1.12)]) is only used in step one of [KMS20a, Proof of Theorem 1.4] to show that

$$\psi(\varepsilon)\le 2\,\ell+C\,\varepsilon^{n/(n+1)}\,. \tag{5.1}$$

Indeed, (5.1) is proved in [KMS20a] by considering a minimizer S of ℓ, and then by using as competitors in $\psi(\varepsilon)$ the open sets, corresponding to a sequence $\eta_j\to 0^+$, obtained by first taking open η_j-neighborhoods F_j of S in Ω (contributing in the limit $j\to\infty$ to the factor $2\,\ell$ in (5.1)), and then by adding to these neighborhoods some disjoint balls of volume $\varepsilon-|F_j|$ (whose energy contributions are controlled by $C\,\varepsilon^{n/(n+1)}$). The role of assumption (A4) is ensuring that the boundaries $\Omega\cap\partial F_j$ are $\mathcal{C}$-spanning W, and thus that these open set are admissible competitors for $\psi(\varepsilon)$.

We can avoid this difficulty if, rather than working with η-neighborhoods of S, we exploit Lemma 3.2 to work with "unilateral" open neighborhoods of S, which still contain S in their boundary, and thus are automatically $\mathcal{C}$-spanning. More precisely, let us recall that if S is a minimizer of ℓ, then there exists an $\mathcal{H}^n$-negligible and closed subset Σ^* of S such that $S\setminus\Sigma^*$ is a smooth hypersurface (indeed, S is an Almgren minimizer, and therefore it is $\mathcal{H}^n$-a.e. everywhere smooth by the main result in [Alm76]). By Lemma 3.1, we can find a closed meager subset Σ of S (with $\Sigma^*\subset\Sigma$) with the property that $S\setminus\Sigma$ is a smooth orientable hypersurface. Therefore we can apply Lemma 3.2 with

$$K=S\,,\qquad E=\emptyset\,,$$

to find, for every $\eta,\delta\in(0,1)$, an open subset F of Ω such that ∂F is $\mathcal{H}^n$-rectifiable, $S\subset\Omega\cap\partial F$, and

$$\limsup_{\delta\to 0^+}\,\limsup_{\eta\to 0^+}\,\mathcal{H}^n\big(\Omega\cap\partial F\big)\le\mathcal{F}(S,\emptyset)=2\,\ell\,.$$

Let $\{F_j\}$ correspond to $\delta_j\to 0^+$ and $\eta_j\to 0^+$ so that $\limsup_j\mathcal{H}^n(\Omega\cap\partial F_j)\le 2\,\ell$, and notice that, by construction, $|F_j|\to 0^+$. We can thus define $E_j=F_j\cup B_{r_j}(p)$ where r_j is such that $|B_{r_j}(p)|=\varepsilon-|F_j|$ and where p is such that $\mathrm{cl}\,[B_{r_j}(p)]$ is disjoint from $W\cup\mathrm{cl}\,(F_j)$: the resulting sets are competitors for $\psi(\varepsilon)$, and their existence implies the validity of (5.1). □

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