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Refined Gauss–Green formulas and evolution problems for Radon measures

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To my family

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Abstract

In this PhD thesis, we present some developments in the theory of sets of finite perimeter, weak integration by parts formulas and systems of coupled evolution equations for nonnegative Radon measures.

First, we introduce a characterization of the perimeter of a measurable set in $\mathbb{R}^n$ via a family of functionals originating from a BMO-type seminorm. This result comes from a joint work with Luigi Ambrosio and is based on a previous paper by Ambrosio, Bourgain, Brezis and Figalli. In this paper, the authors considered functionals depending on a BMO-type seminorm and disjoint coverings of cubes with arbitrary orientations, and proved the convergence to a multiple of the perimeter. We modify their approach by using, instead of cubes, covering families made by translations of a given open connected bounded set with Lipschitz boundary. We show that the new functionals converge to an anisotropic surface measure, which is indeed a multiple of the perimeter if we allow for isotropic coverings (*e.g.* balls). This result underlines that the particular geometry of the covering sets is not essential.

We then present the proof of a one-sided interior approximation for sets of finite perimeter, which was introduced in a paper of Chen, Torres and Ziemer. The original proof contained a gap, which was corrected in a joint work with Monica Torres. Given a set of finite perimeter E, the key idea for the approximation consists in taking the superlevel sets above $1/2$ (respectively, below) of the mollification of the characteristic function of E. Then, we have that, asymptotically, the boundary of the approximating sets has negligible intersection with the measure theoretic interior (respectively, exterior) of E with respect to the $(n-1)$-dimensional Hausdorff measure.

The main motivation for the study of this finer type of approximation was the aim to establish Gauss–Green formulas for sets of finite perimeter and divergence-measure fields; that is, L^p-summable vector fields whose divergence is a Radon measure. Exploiting an alternative approach, we lay out a direct proof of generalized versions of the Gauss–Green formulas, which relies solely on the Leibniz rule for essentially bounded divergence-measure fields and scalar essentially bounded BV functions. In addition, we show some recent refinements. In particular, we provide a new Leibniz rule for L^p-summable divergence-measure fields and scalar essentially bounded Sobolev functions with gradient in $L^{p'}$ and we derive Green's identities on sets of finite perimeter. This part is based on joint works with Kevin R. Payne and with Gui-Qiang Chen and Monica Torres.

Due to the robustness of the Euclidean theory of divergence-measure fields, we can extend it to some non-Euclidean context. In particular, based on a joint work with Valentino Magnani, we develop a theory of divergence-measure fields in noncommutative stratified nilpotent Lie groups. Thanks to some nontrivial approximation arguments, we prove a Leibniz rule for essentially bounded horizontal divergence-measure fields and essentially bounded scalar function of bounded h-variation. As a consequence, we achieve the existence of normal traces and the related Gauss–Green theorem on sets of finite h-perimeter. Despite the fact that the Euclidean theory of normal traces relies heavily on De Giorgi's blow-up theorem, which does not hold in general stratified groups, we are able to provide alternative proofs for the locality of the normal traces and other related results.

Finally, we present a work in progress concerning the study of dislocations in crystals and their connection with evolution equations for signed measures, based on a current research project with Luigi Ambrosio, Mark A. Peletier and Oliver Tse. Starting from previous works of Ambrosio, Mainini and Serfaty, we consider couples of nonnegative measures instead of signed measures. Then, we employ techniques from the theory of optimal transport in order to represent the evolution equations as the gradient flows of a given energy with respect to a suitable distance among couples of nonnegative measures. To this purpose, we study a version of a Hellinger-Kantorovich distance introduced by Liero, Mielke and Savaré. In particular, we prove

the existence of (weakly) continuous minimizing curves of measures which realize this distance and investigate its alternative representation as infimum of some action functional. Future research shall go in the direction of analyzing further properties of this Hellinger-Kantorovich distance, such as its dual representation, with the final aim to apply the classical methods of minimizing movements to prove the existence of solutions satisfying a certain type of energy dissipation equality.

Contents

Introduction

Geometric Measure Theory is the branch of Analysis which studies the fine properties of weakly regular functions and nonsmooth surfaces, generalizing techniques from differential geometry through measure theoretic arguments. More specifically, it deals with the generalizations and the weaker versions of many classical problems and statements from Geometry and Analysis, such as the isoperimetric and Plateau's problem, Stokes' theorem and the Gauss–Green formula. In addition, tools and concepts from Geometric Measure Theory play a relevant role in many other fields where weak convergence results and vector fields are involved. One of the main examples of such applications is the theory of gradient flows, which deals with the study of differential systems where the velocity of the solution is given by the gradient of a given functional. In particular, such theory has been exploited to derive existence and regularity results for solutions of differential equations which could be seen as a gradient flows of some functionals acting on suitable spaces of finite Radon (or probability) measures.

This thesis is devoted to some of the topics from Geometric Measure Theory, to which I dedicated my research during my years of PhD: different characterization of sets of finite perimeter and their smooth approximation (in the Euclidean and the stratified groups framework), the Gauss–Green and integration by parts formulas under weak regularity assumptions and some evolution problems for signed measures.

We provide now an outline of the thesis, while we shall give a more detailed overview of these topics in the following subsections.

- In Chapter 1 we set some notations and introduce basic tools and results from Geometric Measure Theory which shall be used in the subsequent chapters. In particular, we recall relevant preliminary notions on the theory of functions with bounded variation (BV) and sets of finite perimeter in stratified groups, while providing certain new results on smooth approximation for BV functions, based on the first sections of [51], a joint work with Valentino Magnani.

- Chapter 2 contains a collection of results on the theory of sets with finite perimeter in the Euclidean space and in stratified groups. In particular, in Section 2.2, based on a joint work with Luigi Ambrosio [6], we deal with a characterization of the perimeter of sets in $\mathbb{R}^n$ through functionals arising from seminorms of the bounded mean oscillation type. In Section 2.3, based on [55], in collaboration with Monica Torres, we provide an improvement to the standard approximation results for Euclidean sets of finite perimeter. In Section 2.4 we investigate the existence and uniqueness of weak* limit of mollifications of characteristic functions of sets with finite h-perimeter in stratified groups. These results are part of the preliminaries of [51], in collaboration with Valentino Magnani.

- In Chapter 3 we give an exposition on the main features of the Euclidean theory of divergence-measure fields in relation with the generalization of Gauss–Green and integration by parts formulas. This chapter contains material from [52], in collaboration with Kevin R. Payne, and [40], in collaboration with Gui-Qiang Chen and Monica Torres.

- Chapter 4 is devoted to the extension of the notion of divergence-measure field to the framework of stratified groups and the subsequent derivation of related Gauss–Green formulas on sets of finite h-perimeter. While the fundamental steps of this derivation are similar to the Euclidean case, thanks to the robustness of the theory, we stress the fact that there are some new difficulties which require different techniques and methods of proof to be overcome. These results were presented in [51], in collaboration with Valentino Magnani.

- Finally, in Chapter 5 we describe the state of the art of an ongoing research project with Luigi Ambrosio, Mark A. Peletier and Oliver Tse, related to the study of a certain type of evolution problems in divergence form for couples of nonnegative Radon measures. The interest on these type of systems of equations comes from the theory of dislocation in crystals. The aim of the investigation is to represent these systems as gradient flows of a suitable free energy with respect to a distance between couples of nonnegative Radon measures, exploiting ideas from the theory of optimal transport and gradient flows.

The characterization of the perimeter of sets through BMO-type seminorms

The notion of *set of finite perimeter* is at the core of Geometric Measure Theory. Broadly speaking, it extends the idea of manifold with smooth boundary, in this way providing a suitable space in which is possible to study the existence of a solution to Plateau's problem on minimal surfaces with a prescribed boundary and other similar geometric variational problems. We say that a Lebesgue measurable set E has (locally) finite perimeter in $\mathbb{R}^n$ if the total variation of its characteristic function $V(\chi_E, \Omega)$ is finite on any bounded open set Ω; that is,

$$V(\chi_E, \Omega) := \sup \left\{ \int_\Omega \chi_E \, \mathrm{div}\phi \, dx : \phi \in C_c^\infty(\Omega; \mathbb{R}^n), \|\phi\|_{L^\infty(\Omega;\mathbb{R}^n)} \le 1 \right\} < \infty. \tag{I.1}$$

We define the *perimeter* of E in Ω to be the total variation of χ_E, $\mathsf{P}(E, \Omega) := V(\chi_E, \Omega)$. Thanks to Riesz's Representation Theorem, it is possible to show that this definition is equivalent to ask the existence of a (locally finite) vector valued Radon measure $D\chi_E$ such that the following weak version of the Gauss–Green formula holds

$$\int_E \mathrm{div}\phi \, dx = - \int_{\mathbb{R}^n} \phi \cdot dD\chi_E, \text{ for any } \phi \in C_c^1(\mathbb{R}^n; \mathbb{R}^n). \tag{I.2}$$

As an immediate consequence, we see that $|D\chi_E|(\Omega) = \mathsf{P}(E, \Omega)$ for any bounded open set Ω.

The first seminal idea of looking for a new notion of orientable surface suitable for satisfying extensions of the Gauss–Green theorem is due to Caccioppoli [36, 37], who defined the sets of finite perimeter through an approximation procedure via polyhedral sets. It was De Giorgi [63, 64] who fully accomplished this program and named this family of sets after Caccioppoli. De Giorgi actually gave a definition different from (I.1) and (I.2), since he employed the heat kernel and the properties of its gradient. Nevertheless, he proved that his definition included the one of Caccioppoli, by a compactness argument, and that it is equivalent to the validity of (I.2). In addition, he proved a remarkable representation formula for the measure $D\chi_E$; that is,

$$D\chi_E = \nu_E \, \mathscr{H}^{n-1} \llcorner \mathscr{F} E, \tag{I.3}$$

where ν_E is the *measure theoretic unit interior normal*, $\mathscr{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure and $\mathscr{F} E$ is the *reduced boundary* (see Definition 1.1.7 and Theorem 1.1.10). For an account on the early stages of the theory of sets of finite perimeter, we refer to [3], while in Chapter 1 the main results on this theory are recalled, together with the notion of functions of bounded variation (BV), which is strongly tied to it.

The standard definition of a set of finite perimeter (I.2) is of distributional nature: it basically means that, if E is a set of (locally) finite perimeter in $\mathbb{R}^n$, then each distributional partial derivative of its characteristic function is a distribution of order 0; that is, a (locally) finite Radon measure. Even the smoothing procedure of De Giorgi implied the use of a differential operator.

It is therefore of interest to consider new possible ways of defining sets of finite perimeter without employing any weak differentiation. The classical approach of Federer [71, 72] was based on the definition of the *measure theoretic boundary* of a set E,

$$\partial^* E := \left\{ x \in \mathbb{R}^n : \limsup_{r \to 0} \frac{|B(x,r) \cap E|}{|B(x,r)|} > 0, \liminf_{r \to 0} \frac{|B(x,r) \cap E|}{|B(x,r)|} < 1 \right\},$$

which is the set of points with Lebesgue density in E neither 1 nor 0. Equivalently, $\partial^* E = \mathbb{R}^n \setminus (E^1 \cup E^0)$, where

$$E^1 := \left\{ x \in \mathbb{R}^n : \lim_{r \to 0} \frac{|B(x,r) \cap E|}{|B(x,r)|} = 1 \right\} \text{ and } E^0 := \left\{ x \in \mathbb{R}^n : \lim_{r \to 0} \frac{|B(x,r) \cap E|}{|B(x,r)|} = 0 \right\}$$

are the *measure theoretic interior and exterior* of E, respectively. A famous and deep theorem by Federer states that a Lebesgue measurable set E is of (locally) finite perimeter in $\mathbb{R}^n$ if and only if $\partial^* E$ has (locally) finite $(n-1)$-dimensional Hausdorff measure; that is, $\mathscr{H}^{n-1}(K) < \infty$ for any compact set $K \subset \partial^* E$.

Recently, a different approach has been investigated, based on the approximation of Sobolev and BV norms by some family of functionals. In particular, [5] Ambrosio, Bourgain, Brezis and Figalli considered the following functionals

$$\mathsf{I}_\varepsilon(f) = \varepsilon^{n-1} \sup_{\mathcal{G}_\varepsilon} \sum_{Q' \in \mathcal{G}_\varepsilon} \fint_{Q'} \left| f(x) - \fint_{Q'} f \right| dx, \tag{I.4}$$

for any measurable function f, where $\mathcal{G}_\varepsilon$ is any disjoint collection of ε-cubes Q' with arbitrary orientation. The authors then focused on the case $f = \chi_A$; that is, the characteristic function of a measurable set A, and proved that

$$\lim_{\varepsilon \to 0} \mathsf{I}_\varepsilon(\chi_A) = \frac{1}{2} \mathsf{P}(A). \tag{I.5}$$

In Chapter 2, Section 2.2 (based on [6], a joint work with Luigi Ambrosio), we modify their approach by using, instead of cubes, covering families made by translations of a given open bounded connected set C with Lipschitz boundary. Hence, we define

$$H^C_\varepsilon(A) := \varepsilon^{n-1} \sup_{\mathcal{H}_\varepsilon} \sum_{C' \in \mathcal{H}_\varepsilon} \fint_{C'} \left| \chi_A(x) - \fint_{C'} \chi_A \right| dx,$$

where $\mathcal{H}_\varepsilon$ is any disjoint family of translations C' of the set εC with no bounds on cardinality. In order to show the convergence, the key idea is to define suitable localized versions $H^C_\varepsilon(A, \Omega)$ of the functionals, by considering only coverings inside a given open set Ω. Then we consider $H^C_\varepsilon(S_\nu, Q_\nu)$, where $\nu \in \mathbb{S}^{n-1}$, $S_\nu := \{x \in \mathbb{R}^n : \ x \cdot \nu \geq 0\}$ and Q_ν is a unit cube centered in the origin having one face orthogonal to ν and bisected by the hyperplane ∂S_ν. Then, we prove that the limit as $\varepsilon \to 0$ of this functionals is well defined and depends only on ν, so that we can define

$$\varphi^C(\nu) := \lim_{\varepsilon \to 0} H^C_\varepsilon(S_\nu, Q_\nu).$$

The subsequent steps consist in proving suitable density estimates for the functionals

$$H^C_+(A, \Omega) := \limsup_{\varepsilon \to 0} H^C_\varepsilon(A, \Omega), \quad H^C_-(A, \Omega) := \liminf_{\varepsilon \to 0} H^C_\varepsilon(A, \Omega),$$

from which we deduce that, for any set with finite perimeter E,

$$\liminf_{r\to 0}\frac{H_-^C(E,Q_{\nu_E(x)}(x,r))}{\mathsf{P}(E,Q_{\nu_E(x)}(x,r))}=\limsup_{r\to 0}\frac{H_+^C(E,Q_{\nu_E(x)}(x,r))}{\mathsf{P}(E,Q_{\nu_E(x)}(x,r))}=\varphi^C(\nu_E(x))\text{ for }\mathscr{H}^{n-1}\text{-a.e. }x\in\mathscr{F}E,$$

where $Q_\nu(x,r)$ is a cube of side length r centered in x and with one face orthogonal to ν. Adapting the classical proofs of the differentiation theorem for Radon measures, we are able to achieve a generalization of (I.5), by showing that, for any set of finite perimeter A, we have

$$\lim_{\varepsilon\to 0}H_\varepsilon^C(A)=\int_{\mathscr{F}A}\varphi^C(\nu_A(x))\,d\mathscr{H}^{n-1}(x). \tag{I.6}$$

On the other hand, if A is measurable and $\mathsf{P}(A)=\infty$, using a comparison argument with $\mathsf{I}_\varepsilon(\chi_A)$ we obtain

$$\lim_{\varepsilon\to 0}H_\varepsilon^C(A)=+\infty.$$

This result means that the functionals H_ε^C converge to some type of anisotropic surface measure. In addition, we can prove that, if C is a ball, then φ^C is constant, so that the surface measure at the right hand side of (I.6) reduces to a multiple of the perimeter.

We remark that (I.6) raises the question whether the limit functional is indeed an anisotropic perimeter. It is well known that $\int_{\mathscr{F}A}\varphi^C(\nu_A)\,d\mathscr{H}^{n-1}$ is lower semicontinuous with respect to the convergence in measure if and only if φ^C is the restriction to the unit sphere of a positively 1-homogeneous and convex function $\widetilde{\varphi}^C$. The problem is nontrivial since we are able to prove that, if C is the unit square $(0,1)^2$ in $\mathbb{R}^2$, then $\widetilde{\varphi}^C$ is not convex ([6, Section 4]). In particular, the convexity of C is not a sufficient condition to obtain the convexity of $\widetilde{\varphi}^C$. However, φ^C is constant in the case C is a ball, and so it is trivially convex in this case. Therefore, a future research in this field is to look for conditions under which we can ensure the convexity of $\widetilde{\varphi}^C$.

Smooth approximations of sets of finite perimeter

Somehow in the spirit of the original definition of Caccioppoli, it is well known that any set E with finite perimeter in $\mathbb{R}^n$, for $n\geq 2$, can be approximated by a sequence of smooth sets E_k in the sense that

$$|E_k\Delta E|\to 0\text{ and }\mathsf{P}(E_k)\to\mathsf{P}(E). \tag{I.7}$$

As showed in [11, Theorem 3.42], these sets may be constructed by taking the convolution of χ_E against a standard mollifier ρ, and considering suitable superlevel sets of $\rho_{\varepsilon_k}*\chi_E$, for some nonnegative sequence $\varepsilon_k\to 0$. Then, exploiting the coarea formula for BV functions and Sard's theorem, we deduce the existence of some $t\in(0,1)$ such that $\{\rho_{\varepsilon_k}*\chi_E>t\}$ is a smooth set for any k.

In Section 2.3, we consider a refinement of this approximation result, by exploiting the tangential properties of the reduced boundary of sets of finite perimeter. It is a classical result due to De Giorgi (Theorem 1.1.10) that in any point x of the reduced boundary of a set E with finite perimeter in $\mathbb{R}^n$ we have the following blow-ups:

$$\frac{E-x}{\varepsilon}\to H_{\nu_E}^+(x):=\{y\in\mathbb{R}^n:y\cdot\nu_E(x)\geq 0\}\ \text{ in }\ L^1_{\mathrm{loc}}(\mathbb{R}^n)\ \text{ as }\ \varepsilon\to 0 \tag{I.8}$$

and

$$\frac{(\mathbb{R}^n\setminus E)-x}{\varepsilon}\to H_{\nu_E}^-(x):=\{y\in\mathbb{R}^n:y\cdot\nu_E(x)\leq 0\}\ \text{ in }\ L^1_{\mathrm{loc}}(\mathbb{R}^n)\ \text{ as }\ \varepsilon\to 0. \tag{I.9}$$

Roughly speaking, this means that on small scales the set $E\cap B(x,r)$ is asymptotically equal to the half ball centered in x and bisected by the hyperplane

$$\{x+y:y\cdot\nu_E(x)=0\}.$$

A simple consequence of (I.8) and (I.9) is that $(\rho_\varepsilon * \chi_E)(x) \to 1/2$ for any $x \in \mathscr{F}E$ and any standard mollifier ρ (Lemma 1.1.16). This suggests that it is possible to distinguish between the approximating superlevel sets $\{\rho_{\varepsilon_k} * \chi_E > t\}$ according to whether $t < 1/2$ or $t > 1/2$.

Indeed, we show that, in the first case, the difference between the level sets and the measure theoretic interior of E is asymptotically vanishing with respect to the $\mathscr{H}^{n-1}$-measure; in the latter, we obtain the same result for the measure theoretic exterior. Therefore, we call this type of approximation "one-sided" since it provides different *interior* and *exterior* approximations of the set (see Theorem 2.3.4). In addition, for this one-sided approximation the first limit in (I.7) holds when substituting the Lebesgue measure with any Radon measure μ absolutely continuous with respect to the Hausdorff measure $\mathscr{H}^{n-1}$. More rigorously, we prove that, if E is a bounded set of finite perimeter in $\mathbb{R}^n$ and μ is a Radon measure such that $|\mu| \ll \mathscr{H}^{n-1}$, there exist two sequences $\{E_{k;i}\}, \{E_{k;e}\}$ of sets with smooth boundary such that

$$|\mu|(E_{k;i}\Delta E^1) \to 0 \;\text{ and }\; \mathsf{P}(E_{k,i}) \to \mathsf{P}(E), \tag{I.10}$$

$$|\mu|(E_{k;e}\Delta(E^1 \cup \mathscr{F}E)) \to 0 \;\text{ and }\; \mathsf{P}(E_{k,e}) \to \mathsf{P}(E), \tag{I.11}$$

and

$$\mathscr{H}^{n-1}(\partial E_{k,i} \setminus E^1) \to 0 \;\text{ and }\; \mathscr{H}^{n-1}(\partial E_{k,e} \setminus E^0) \to 0. \tag{I.12}$$

Generalizations of the Gauss–Green formula in the Euclidean setting

The Gauss–Green formula, or divergence theorem, plays a ubiquitous role in Mathematical Analysis, Mathematical Physics and Continuum Physics, since it provides a way to establish energy identities and energy inequalities for PDEs, to derive the governing PDEs from basic physical principles and to rigorously justify balance or conservation laws for classes of subbodies of a given body. Thus, of particular importance is the search for extending the validity of such formulas to vector fields of low regularity and for general classes of subdomains.

The classical statement of the Gauss–Green formula requires a vector field $F \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$ and an open set E such that ∂E is a C^1 smooth $(n-1)$-dimensional manifold, in order to conclude that

$$\int_E \mathrm{div} F\, dx = -\int_{\partial E} F \cdot \nu_E \, d\mathscr{H}^{n-1}, \tag{I.13}$$

where ν_E is the unit interior normal to ∂E. Such assumptions are clearly too strong for many practical purposes, since, for instance, open sets with Lipschitz boundary would not be allowed, and thus integration by parts on cubes would not be possible. As we mentioned above, the first relevant generalization of (I.13) is strongly tied to the notion of set of finite perimeter, and it is due to De Giorgi [63, 64] and Federer [71, 72]. Indeed, if we exploit (I.2) and (I.3), we immediately obtain that, given any set E with locally finite perimeter in $\mathbb{R}^n$, we have

$$\int_E \mathrm{div} F\, dx = -\int_{\mathscr{F}E} F \cdot \nu_E \, d\mathscr{H}^{n-1}, \tag{I.14}$$

for any $F \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$. While we can argue that the assumption of being a set with (locally) finite perimeter is optimal for the integration domain, it is clear that we can still weaken the regularity hypotheses on the vector field. In 1967 Vol'pert [156,157] obtained an extension of the Gauss–Green theorem for essentially bounded vector fields of bounded variations. The space of *functions of bounded variation* on $\mathbb{R}^n$, $BV(\mathbb{R}^n)$, is the set of L^1-functions whose distributional gradient Du is a finite vector valued Radon measure. Thus, it can be seen as the natural extension of the Sobolev space $W^{1,1}(\mathbb{R}^n)$. The notion of BV function was first considered by C. Jordan in 1881 in the one-dimensional case [102], in order to deal with convergence criteria of Fourier series. The initial definition was based on a pointwise notion of total variation, and could not be easily extended to many variables. It was Fichera [74] and De Giorgi [63] who gave the modern distributional definition, exploiting Schwartz's theory of distributions. In particular, De Giorgi proved that a set E has locally finite perimeter in $\mathbb{R}^n$ if and only if the characteristic function of E is of locally bounded variation.

Exploiting the fine properties of BV functions, Vol'pert proved integration by parts formulas for essentially bounded functions with bounded variations on sets of finite perimeter. More precisely, he showed that, if $u \in BV(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and E is a bounded set of finite perimeter in $\mathbb{R}^n$, then for any $j \in \{1, \dots, n\}$ we have

$$(Du)_j(E^1) = -\int_{\mathscr{F}E} u_{\nu_E} (\nu_E)_j \, d\mathscr{H}^{n-1}, \tag{I.15}$$

$$(Du)_j(E^1 \cup \mathscr{F}E) = -\int_{\mathscr{F}E} u_{-\nu_E} (\nu_E)_j \, d\mathscr{H}^{n-1}, \tag{I.16}$$

where $(Du)_j$ and $(\nu_E)_j$ are the j-th components of Du and ν_E, respectively, and $u_{\pm\nu_E}$ are the exterior and interior traces of u on $\mathscr{F}E$; that is, the approximate limits of u at $x \in \mathscr{F}E$ restricted to the half spaces $\{y \in \mathbb{R}^n : (y - x) \cdot (\pm\nu_E(x)) \geq 0\}$. The existence of such traces is a consequence of the fact that any BV function u admits a *precise representative* u^* which is well defined for $\mathscr{H}^{n-1}$-a.e. $x \in \mathbb{R}^n$. It is then clear that we can apply (I.15) and (I.16) to the j-th component of a vector field $F \in BV(\mathbb{R}^n; \mathbb{R}^n) \cap L^\infty(\mathbb{R}^n; \mathbb{R}^n)$, for any $j \in \{1, \dots, n\}$, and then sum up the resulting idenities, obtaining in such a way the following Gauss–Green formulas, which extend (I.14):

$$\mathrm{div}F(E^1) = -\int_{\mathscr{F}E} F_{\nu_E} \cdot \nu_E \, d\mathscr{H}^{n-1}, \tag{I.17}$$

$$\mathrm{div}F(E^1 \cup \mathscr{F}E) = -\int_{\mathscr{F}E} F_{-\nu_E} \cdot \nu_E \, d\mathscr{H}^{n-1}. \tag{I.18}$$

It is easy to notice that not all the partial derivative of the vector field F actually need to be Radon measures, since only the divergence of F appears in the left hand sides of (I.17) and (I.18). This simple observation leads to the idea that the Gauss–Green formulas may hold also for a larger family of vector fields, for which only the distributional divergence is a Radon measure. Thus, it seems natural to define the space of (p-summable) *divergence-measure fields*, $\mathcal{DM}^p(\mathbb{R}^n)$, for any $p \in [1, \infty]$, to be the set of L^p-summable vector fields whose divergence is a finite Radon measure on $\mathbb{R}^n$. It is clear that divergence-measure fields generalize the vector fields of bounded variation, and indeed they were studied in the last two decades in order to achieve Gauss–Green and integration by parts formulas with lower regularity assumptions on both the vector fields and the integration domains. After having been first introduced by Anzellotti in [23], divergence-measure fields proved to be very important in applications, as in hyperbolic conservations laws, in the theory of contact interactions in Continuum Physics, and in the study of 1-Laplace, minimal surface and prescribed mean curvature type equations (we refer for instance to [41, 104, 107, 134, 138]).

In Chapter 3 we present an approach to the proof of the Gauss–Green formula based on the adaptation of the techniques already developed for BV functions in Vol'pert's monograph [157]. We start with a short exposition on the general properties of $\mathcal{DM}^p$-fields, such as the absolute continuity properties of the divergence-measure and the product rules. In particular, we prove a product rule not present in the literature to the best of our knowledge (Theorem 3.2.3): if $p, p' \in [1, \infty]$, $\frac{1}{p} + \frac{1}{p'} = 1$, $F \in \mathcal{DM}^p_{\mathrm{loc}}(\mathbb{R}^n)$ and $g \in W^{1,p'}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$, then $gF \in \mathcal{DM}^p_{\mathrm{loc}}(\mathbb{R}^n)$ and

$$\mathrm{div}(gF) = g^* \mathrm{div}F + F \cdot \nabla g \, \mathscr{L}^n, \tag{I.19}$$

where g^* is a suitable representative of g. Then, we focus ourselves on the case $p = \infty$, in which we exploit an already established product rule for essentially bounded divergence-measure fields and essentially bounded scalar functions with bounded variation: if $F \in \mathcal{DM}^{\infty}_{\rm loc}(\mathbb{R}^n)$ and $g \in BV_{\rm loc}(\mathbb{R}^n) \cap L^{\infty}_{\rm loc}(\mathbb{R}^n)$, then $gF \in \mathcal{DM}^{\infty}_{\rm loc}(\mathbb{R}^n)$ and

$$\mathrm{div}(gF) = g^* \mathrm{div} F + (F, Dg), \tag{I.20}$$

where g^* is the precise representative of g (see Definition 1.1.14) and (F, Dg) is the (unique) pairing measure between F and the weak gradient of g (see Lemma 1.1.3). In particular, the measure (F, Dg) satisfies $|(F, Dg)| \le \|F\|_{L^{\infty}(\mathbb{R}^n;\mathbb{R}^n)} |Dg|$. We stress the fact that, for $p = \infty$, we have $|\mathrm{div} F| \ll \mathscr{H}^{n-1}$, so that the expression $g^* \mathrm{div} F$ is meaningful. The starting point of our method to obtain generalized Gauss-Green formulas is to apply (I.20) to the case $g = \chi_E$, for some set E with locally finite perimeter, in order to derive suitable identities between Radon measures. Thanks to some algebraic manipulations, we obtain

$$\mathrm{div}(\chi_E F) = \chi_{E^1} \mathrm{div} F + 2(\chi_E F, D\chi_E), \tag{I.21}$$

$$\mathrm{div}(\chi_E F) = \chi_{E^1 \cup \mathscr{F}_H E} \mathrm{div} F + 2(\chi_{\mathbb{R}^n \setminus E} F, D\chi_E). \tag{I.22}$$

Then, we define the generalized *interior and exterior normal traces* of F on $\mathscr{F}E$, which we denote by $(\mathcal{F}_i \cdot \nu_E)$ and $(\mathcal{F}_e \cdot \nu_E)$, as the densities of the pairing measures $(\chi_E F, D\chi_E)$ and $(\chi_{\mathbb{R}^n \setminus E} F, D\chi_E)$ with respect to the perimeter measure $|D\chi_E|$:

$$2(\chi_E F, D\chi_E) = (\mathcal{F}_i \cdot \nu_E)|D\chi_E|, \tag{I.23}$$

$$2(\chi_{\mathbb{R}^n \setminus E} F, D\chi_E) = (\mathcal{F}_e \cdot \nu_E)|D\chi_E|. \tag{I.24}$$

As an immediate consequence, (I.21) and (I.22) can be rewritten as

$$\mathrm{div}(\chi_E F) = \chi_{E^1} \mathrm{div} F + (\mathcal{F}_i \cdot \nu_E)|D\chi_E|, \tag{I.25}$$

$$\mathrm{div}(\chi_E F) = \chi_{E^1 \cup \mathscr{F}_H E} \mathrm{div} F + (\mathcal{F}_e \cdot \nu_E)|D\chi_E|. \tag{I.26}$$

In addition, the following trace estimates hold (Theorem 3.3.5):

$$\|\mathcal{F}_i \cdot \nu_E\|_{L^{\infty}(\mathscr{F}E;\mathscr{H}^{n-1})} \le \|F\|_{L^{\infty}(E;\mathbb{R}^n)} \quad \text{and} \quad \|\mathcal{F}_e \cdot \nu_E\|_{L^{\infty}(\mathscr{F}E;\mathscr{H}^{n-1})} \le \|F\|_{L^{\infty}(\mathbb{R}^n \setminus E;\mathbb{R}^n)}.$$

The next key observation is that, for any $G \in \mathcal{DM}^{\infty}(\mathbb{R}^n)$ compactly supported, we have $\mathrm{div} G(\mathbb{R}^n) = 0$. Hence, for $F \in \mathcal{DM}^{\infty}(\mathbb{R}^n)$ and E is a bounded set of finite perimeter in $\mathbb{R}^n$, it is enough to evaluate (I.25) and (I.26) on $\mathbb{R}^n$ in order to obtain the following Gauss–Green formulas:

$$\mathrm{div} F(E^1) = -\int_{\mathscr{F}E} \mathcal{F}_i \cdot \nu_E \, d\mathscr{H}^{n-1} \quad \text{and} \quad \mathrm{div} F(E^1 \cup \mathscr{F}E) = -\int_{\mathscr{F}E} \mathcal{F}_e \cdot \nu_E \, d\mathscr{H}^{n-1}.$$

As a consequence, exploiting the product rule (I.19) for $p = \infty$, we derive integration by parts formulas for $F \in \mathcal{DM}^{\infty}_{\rm loc}(\mathbb{R}^n)$, $\varphi \in L^{\infty}_{\rm loc}(\mathbb{R}^n)$ with $\nabla \varphi \in L^1_{\rm loc}(\mathbb{R}^n;\mathbb{R}^n)$ and a set of locally finite perimeter E such that $\mathrm{supp}(\chi_E \varphi)$ is compact:

$$\int_{E^1} \varphi^* \, d\mathrm{div} F + \int_E F \cdot \nabla \varphi \, dx = -\int_{\mathscr{F}E} \varphi^* (\mathcal{F}_i \cdot \nu_E) \, d\mathscr{H}^{n-1} \tag{I.27}$$

and

$$\int_{E^1 \cup \mathscr{F}E} \varphi^* \, d\mathrm{div} F + \int_E F \cdot \nabla \varphi \, dx = -\int_{\mathscr{F}E} \varphi^* (\mathcal{F}_e \cdot \nu_E) \, d\mathscr{H}^{n-1}, \tag{I.28}$$

where $(\mathcal{F}_i \cdot \nu_E), (\mathcal{F}_e \cdot \nu_E) \in L^{\infty}_{\rm loc}(\mathscr{F}E; \mathscr{H}^{n-1})$ are defined as in (I.23) and (I.24), since those identities can be clearly localized to any bounded open set. In addition, we show that this notion of generalized normal traces is consistent with the case of continuous vector fields F, for which we have

$$(\mathcal{F}_i \cdot \nu_E)(x) = (\mathcal{F}_e \cdot \nu_E)(x) = F(x) \cdot \nu_E(x) \quad \text{for } \mathscr{H}^{n-1}\text{-a.e. } x \in \mathscr{F}E.$$

Then, as an improvement of the preexisting results in the literature (for instance, [8, Proposition 3.2]), we show the following locality properties of the normal traces: if $F \in \mathcal{DM}^{\infty}_{\text{loc}}(\mathbb{R}^n)$ and E_1, E_2 are sets of locally finite perimeter in $\mathbb{R}^n$ such that $\mathscr{H}^{n-1}(\mathscr{F}E_1 \cap \mathscr{F}E_2) \neq 0$, then we have

$$\mathcal{F}_i \cdot \nu_{E_1} = \mathcal{F}_i \cdot \nu_{E_2} \quad \text{and} \quad \mathcal{F}_e \cdot \nu_{E_1} = \mathcal{F}_e \cdot \nu_{E_2} \tag{I.29}$$

for $\mathscr{H}^{n-1}$-a.e. $x \in \{y \in \mathscr{F}E_1 \cap \mathscr{F}E_2 : \nu_{E_1}(y) = \nu_{E_2}(y)\}$, and

$$\mathcal{F}_i \cdot \nu_{E_1} = -\mathcal{F}_e \cdot \nu_{E_2} \quad \text{and} \quad \mathcal{F}_e \cdot \nu_{E_1} = -\mathcal{F}_i \cdot \nu_{E_2} \tag{I.30}$$

for $\mathscr{H}^{n-1}$-a.e. $x \in \{y \in \mathscr{F}E_1 \cap \mathscr{F}E_2 : \nu_{E_1}(y) = -\nu_{E_2}(y)\}$. Finally, exploiting (I.27) and (I.28) in the case $F = \nabla u$ for $u \in \text{Lip}_{\text{loc}}(\mathbb{R}^n)$ with $\Delta u \in \mathcal{M}_{\text{loc}}(\mathbb{R}^n)$, we derive generalized versions of the Green's identities (Theorem 3.5.1 and Corollary 3.5.2).

The theory of divergence-measure fields in stratified groups

In the past decades, Geometric Measure Theory experienced a number of generalizations and extensions to the more general settings of abstract metric measure spaces [1, 2, 14, 15, 123]. In particular, great attention was given to the theory of functions of bounded variation and sets of finite perimeter in stratified groups, starting from the pioneering work of Franchi, Serapioni and Serra Cassano in the Heisenberg group [79]. A *stratified (or Carnot) group* can be seen as a linear space $\mathbb{G}$ equipped with an analytic group operation such that its Lie algebra Lie($\mathbb{G}$) is stratified, and with left invariant *horizontal vector fields* $X_1, \ldots, X_{\mathrm{m}}$, that determine the directions along which it is possible to differentiate. Given a function f differentiable in this sense, $f \in C^1_H(\mathbb{G})$, we denote by $\nabla_H f$ its *horizontal gradient*; that is,

$$\nabla_H f := \sum_{j=1}^{\mathrm{m}} (X_j f) X_j.$$

Analogously, if $\varphi : \mathbb{G} \to H\mathbb{G}$ is a suitably regular horizontal section, we define its divergence as

$$\text{div}\varphi := \sum_{j=1}^{\mathrm{m}} X_j \varphi_j.$$

It is then clearly possible to consider corresponding horizontal distributional derivatives, and this leads to the definition of the functions of bounded h-variation: $f \in BV_H(\mathbb{G})$, if $f \in L^1(\mathbb{G})$ and

$$|D_H f|(\mathbb{G}) := \sup \left\{ \int_{\mathbb{G}} f \text{div}\phi \, dx : \phi \in C^1_c(H\mathbb{G}), |\phi| \leq 1 \right\} < \infty.$$

Analogously to the Euclidean case, we say that a measurable set $E \subset \mathbb{G}$ is of *locally finite h-perimeter* in $\mathbb{G}$ (or is a locally *h-Caccioppoli set*) if $\chi_E \in BV_{H,\text{loc}}(\mathbb{G})$; that is, for any bounded open set U, we have

$$\mathsf{P}(E, U) := |D_H \chi_E|(U) < \infty,$$

where $\mathsf{P}(E, U)$ is the *h-perimeter of E in U*. Many of the classical results from the Euclidean BV theory proved to be true in the context of stratified groups. However, the extension of the notion of rectifiability resulted to be problematic. Remarkably, De Giorgi's Blow-up Theorem (Theorem 1.1.10) may be false in general, if the step of nilpotence of the group ι is strictly larger than 2, as showed by a counterexample in the Engel group [80].

The Euclidean theory of divergence-measure fields presented in Chapter 3 proves to be sufficiently robust to be extended to some non-Euclidean contexts, such as noncommutative stratified nilpotent Lie groups. In Chapter 4 we lay down the foundations for such a theory. We define the *divergence-measure horizontal fields* as L^p-summable sections of the horizontal subbundle $H\mathbb{G}$, such that their distributional divergence is a finite Radon measure. In other words, we say that $F \in \mathcal{DM}^p(H\mathbb{G})$, for some $1 \le p \le \infty$, if $F : \mathbb{G} \to H\mathbb{G}$, $|F| \in L^p(\mathbb{G})$ and there exists $\mathrm{div} F \in \mathcal{M}(\mathbb{G})$ such that

$$\int_{\mathbb{G}} \langle F, \nabla_H \varphi \rangle \, dx = - \int_{\mathbb{G}} \varphi \, d\mathrm{div} F$$

for any $\varphi \in C^1_c(\mathbb{G})$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product associated to the left invariant Riemannian metric of $\mathbb{G}$. We observe that $\mathcal{DM}^p(H\mathbb{G})$ contains divergence-measure horizontal fields that are not BV even with respect to the group structure. In analogy with the Euclidean context, we can prove absolute continuity properties of the divergence measure with respect to the spherical Hausdorff measure. In particular, if $p = \infty$, we have $|\mathrm{div} F| \ll \mathcal{S}^{Q-1}$, where Q is the homogeneous dimension of the group $\mathbb{G}$.

Thanks to some nontrivial approximation arguments, we derive a Leibniz rule for essentially bounded horizontal divergence-measure fields and essentially bounded scalar function of bounded h-variation. In particular, we show that, if $F \in \mathcal{DM}^\infty(H\mathbb{G})$ and $g \in L^\infty(\mathbb{G})$ with $|D_H g|(\mathbb{G}) < +\infty$, then $gF \in \mathcal{DM}^\infty(H\mathbb{G})$. Then, if we take a mollifier ρ (which we may choose to be only continuous with compact support), for any infinitesimal sequence $\tilde{\varepsilon}_k > 0$, there exists a subsequence ε_k such that $(\rho_{\varepsilon_k} * g) \overset{*}{\rightharpoonup} \tilde{g}$ in $L^\infty(\Omega; |\mathrm{div} F|)$ and $\langle F, \nabla_H(\rho_{\varepsilon_k} * g) \rangle \, \mu \rightharpoonup (F, D_H g)$ in $\mathcal{M}(\Omega)$. In addition, the following formula holds

$$\mathrm{div}(gF) = \tilde{g} \, \mathrm{div} F + (F, D_H g), \tag{I.31}$$

where the measure $(F, D_H g)$ satisfies

$$|(F, D_H g)| \le \|F\|_{L^\infty(\Omega)} |D_H g|. \tag{I.32}$$

We stress the fact that, due to the noncommutativity of the group operation, it is essential to employ a convolution with the mollifier on the left. Following the same techniques introduced in the Euclidean case, the product rule (I.31) is the starting point of the derivation of the Gauss–Green formulas.

It is however important to stress the fact that a priori we cannot ensure the uniqueness of $\tilde{g}$ and of the pairing $(F, D_H g)$, as they may both depend on the approximating sequence. Despite this, a totally unexpected fact occurs, since in the case $g = \chi_E$ and E has finite h-perimeter, it is possible to prove that the limit $\widetilde{\chi_E}$ is uniquely determined, regardless of the choice of the mollifying sequence $\rho_{\varepsilon_k} * \chi_E$. This seems rather surprising, since we have no rectifiability result for the reduced boundary of a set with finite h-perimeter in arbitrary stratified groups. The proof of this fact relies mainly on some refinements of the Leibniz rule in the case $g = \chi_E$, on the absolute continuity $|\mathrm{div} F| \ll \mathcal{S}^{Q-1}$ and on the fact that the weak* limit of $\rho_\varepsilon * \chi_E$ in $L^\infty(\mathbb{G}; |D_H \chi_E|)$ is precisely $1/2$, for any set $E \subset \mathbb{G}$ of finite h-perimeter and any symmetric mollifier ρ (Proposition 2.4.2). In particular, we are able to prove that there exists a unique $|\mathrm{div} F|$-measurable subset

$$E^{1,F} \subset \Omega \setminus \mathscr{F}_H E,$$

up to $|\mathrm{div} F|$-negligible sets, such that

$$\widetilde{\chi_E}(x) = \chi_{E^{1,F}}(x) + \frac{1}{2} \chi_{\mathscr{F}_H E}(x) \quad \text{ for } |\mathrm{div} F|\text{-a.e. } x \in \mathbb{G}.$$

We call $E^{1,F}$ the *measure theoretic interior of E with respect to F*. As an immediate consequence, we deduce that the pairing $(F, D_H \chi_E)$ is unique, since, by (I.31), we have

$$\mathrm{div}(\chi_E F) = \widetilde{\chi_E} \mathrm{div} F + (F, D_H \chi_E).$$

Thanks to some algebraic manipulations, we can deduce the uniqueness also of the pairings $(\chi_E F, D_H\chi_E)$ and $(\chi_{\mathbb{G}\setminus E} F, D_H\chi_E)$, and this implies the uniqueness of the interior and exterior normal traces $\langle \mathcal{F}_i, \nu_E\rangle$ and $\langle \mathcal{F}_e, \nu_E\rangle$, which we define as in the Euclidean case:

$$\begin{aligned} 2(\chi_E F, D_H\chi_E) &= \langle \mathcal{F}_i, \nu_E\rangle \, |D_H\chi_E|, \\ 2(\chi_{\Omega\setminus E} F, D_H\chi_E) &= \langle \mathcal{F}_e, \nu_E\rangle \, |D_H\chi_E|. \end{aligned}$$

Rather unexpectedly, we obtain the locality properties of these normal traces without any blow-up technique related to rectifiability of the reduced boundary. In fact, as in the Euclidean case, the normal traces of a divergence-measure horizontal section F only depend on the orientation of the reduced boundary. In particular, in the case the divergence-measure field F is continuous we have

$$\langle \mathcal{F}_i, \nu_E\rangle (x) = \langle \mathcal{F}_e, \nu_E\rangle (x) = \langle F(x), \nu_E(x)\rangle \quad \text{for} \quad |D_H\chi_E|\text{-a.e. } x \in \mathscr{F}_H E.$$

In addition, thanks to (I.32), these normal traces belong to $L^\infty(\mathscr{F}_H E; |D_H\chi_E|)$, and the following refined estimates on the L^∞-norms hold:

$$\| \langle \mathcal{F}_i, \nu_E\rangle \|_{L^\infty(\mathscr{F}_H E; |D_H\chi_E|)} \le \|F\|_{L^\infty(E)}, \quad \text{and} \quad \| \langle \mathcal{F}_e, \nu_E\rangle \|_{L^\infty(\mathscr{F}_H E; |D_H\chi_E|)} \le \|F\|_{L^\infty(\mathbb{G}\setminus E)}.$$

Finally, we obtain the following Leibniz rules:

$$\mathrm{div}(\chi_E F) = \chi_{E^{1,F}} \mathrm{div} F + \langle \mathcal{F}_i, \nu_E\rangle \, |D_H\chi_E|, \tag{I.33}$$

$$\mathrm{div}(\chi_E F) = \chi_{E^{1,F} \cup \mathscr{F}_H E} \mathrm{div} F + \langle \mathcal{F}_e, \nu_E\rangle \, |D_H\chi_E|. \tag{I.34}$$

As a consequence, we achieve the related Gauss–Green and integration by parts formulas for $\mathcal{DM}^\infty$-fields on sets of finite h-perimeter. Indeed, it is enough to observe that, if $G \in \mathcal{DM}^\infty(H\mathbb{G})$ has compact support, then $\mathrm{div} G(\mathbb{G}) = 0$. Then, if we take E to be a bounded set with finite h-perimeter in $\mathbb{G}$ and we evaluate (4.1.5) and (4.1.6) on $\mathbb{G}$, we obtain the following general versions of the Gauss–Green formulas in stratified groups:

$$\mathrm{div} F(E^{1,F}) = - \int_{\mathscr{F}_H E} \langle \mathcal{F}_i, \nu_E\rangle \, d|D_H\chi_E|, \tag{I.35}$$

$$\mathrm{div} F(E^{1,F} \cup \mathscr{F}_H E) = - \int_{\mathscr{F}_H E} \langle \mathcal{F}_e, \nu_E\rangle \, d|D_H\chi_E|. \tag{I.36}$$

As a simple consequence of (I.33), we deduce that $E^{1,F}$, up to $|\mathrm{div} F|$-negligible sets, can be seen as the Borel set in $\Omega \setminus \mathscr{F}_H E$ satisfying

$$\mathrm{div}(\chi_E F) \llcorner \Omega \setminus \mathscr{F}_H E = \mathrm{div} F \llcorner E^{1,F}. \tag{I.37}$$

However, an explicit and geometric characterization of $E^{1,F}$ is still an open problem. Even more interesting would be to prove (or disprove) the existence of a unique Borel set $E^{1,*}$ satisfying (I.37) for any $F \in \mathcal{DM}^\infty(H\Omega)$, as it happens in the Euclidean context, where $E^{1,*} = E^1$, the measure theoretic interior. Nevertheless, under some assumptions, involving either the regularity of E or of the field F, the set $E^{1,F}$ can be properly determined. This immediately yields different versions of Gauss–Green and integration by parts formulas.

If we assume $|\mathrm{div} F| \ll \mu$, where μ is the Haar measure of the group, then we obtain $|\mathrm{div} F|(E^{1,F} \Delta E) = 0$ and the existence of a unique normal trace $\langle \mathcal{F}, \nu_E\rangle \in L^\infty(\Omega; |D_H\chi_E|)$ such that there holds

$$\mathrm{div} F(E) = - \int_{\mathscr{F}_H E} \langle \mathcal{F}, \nu_E\rangle \, d|D_H\chi_E|. \tag{I.38}$$

This means that, in the case F is Lipschitz or Sobolev regular, then $E^{1,F}$ coincides with E, up to $|\mathrm{div} F|$-negligible sets. It is also worth to point out that (I.38) holds also for sets whose boundary is not rectifiable in the Euclidean sense (Example 4.5.2).

Another important case we considered is the one in which the set $E \subset \mathbb{G}$ has finite perimeter in the Euclidean sense. First, we prove that the group pairing $(F, D_H \chi_E)$ is actually equal to the Euclidean pairing $(F, D\chi_E)$, and that we have $E^{1,F} = E^1_{|\cdot|}$, up to a $|\mathrm{div}F|$-negligible set, where $E^1_{|\cdot|}$ is the Euclidean measure theoretic interior of E; that is, the set of points with density 1 with respect to the balls defined using the Euclidean distance in the group. Thanks to this result and to the Leibniz rule, we obtain the following integration by parts formulas: for any $F \in \mathcal{DM}^\infty_{\mathrm{loc}}(H\mathbb{G})$, any set of locally finite h-perimeter E and any $\varphi \in C(\mathbb{G})$ with $\nabla_H \varphi \in L^1_{\mathrm{loc}}(H\mathbb{G})$ such that $\mathrm{supp}(\varphi\chi_E)$ is compact, we have

$$\int_{E^1_{|\cdot|}} \varphi \, d\mathrm{div}F + \int_E \langle F, \nabla_H \varphi \rangle \, dx = - \int_{\mathscr{F}_H E} \varphi \, \langle \mathcal{F}_i, \nu_E \rangle \, d|D_H \chi_E|,$$

$$\int_{E^1_{|\cdot|} \cup \mathscr{F}_H E} \varphi \, d\mathrm{div}F + \int_E \langle F, \nabla_H \varphi \rangle \, dx = - \int_{\mathscr{F}_H E} \varphi \, \langle \mathcal{F}_e, \nu_E \rangle \, d|D_H \chi_E|.$$

As a consequence of our results, we derive very general versions of Green's identities in stratified groups. In particular, in Theorem 4.5.3 such formulas are extended to sets of h-finite perimeter and C^1_H-scalar functions with sub-Laplacian measure which is absolutely continuous with respect to the Haar measure of the group. Instead, in Theorem 4.1.6 the domain of integration is assumed to be a set with Euclidean finite perimeter, while the sub-Laplacians are measures.

Evolution equations for Radon measures related to the dynamics of dislocations

The theory of optimal transport provides a way to give a notion of gradient flows on the space of probability measures with finite second moment endowed with the L^2-Wasserstein distance, W_2. The main advantage of this is the possibility to represent some evolution equations for Radon measures as gradient flows of a given free energy with respect to the W_2 distance. Then, it is possible to apply the *minimizing movement* scheme to obtain existence of solutions satisfying some energy dissipation inequality. We refer to the monograph [13] for a full account of this theory.

In Chapter 5 we investigate the possibility of applying this method to systems of evolution equations for couples of nonnegative measures (μ_1, μ_2) of the following form

$$\begin{cases} \frac{d}{dt}\mu_1 & = \mathrm{div}(\mu_1 \nabla(V * \mu)) - \sigma \\ \frac{d}{dt}\mu_2 & = -\mathrm{div}(\mu_2 \nabla(V * \mu)) - \sigma \end{cases} \tag{I.39}$$

for $\mu = \mu_1 - \mu_2$, some interaction potential V, and some (possibly nonlinear) dissipation term σ depending on μ_1 and μ_2. The interest of (I.39) lies in its close relations with the continuum models of dislocations in crystals, such as the Groma-Balogh equations introduced in [93, 94].

Our purpose is to see these evolution equations as the gradient flows of an energy of the form

$$\Phi(\mu_1, \mu_2) := \frac{1}{2} \int_{\mathbb{R}^n} (V * \mu) \, d\mu + \mu_1(\mathbb{R}^n) + \mu_2(\mathbb{R}^n),$$

with respect to a suitable distance among couples of nonnegative measures. In Section 5.3 we outline the definition of a family of Hellinger-Kantorovich distances analyzed by Liero, Mielke and Savaré in [108, 109], which appears to be useful to this aim. In particular, we focused on the following definition:

$$\begin{aligned}\mathrm{D}^2_{\mathbb{K}}((\nu_1,\nu_2),(\mu_1,\mu_2)) := \inf\Bigg\{ &\int_0^1 \left(\int_{\mathbb{R}^n} |v_1|^2\, d\rho_{1,t} + |v_2|^2\, d\rho_{2,t} + \frac{|\xi|^2}{2}\, d\mathbf{f}(\rho_{1,t},\rho_{2,t})\right) dt,\\ &\frac{d}{dt}\begin{pmatrix}\rho_1\\ \rho_2\end{pmatrix} = -\mathrm{div}\begin{pmatrix} v_1\rho_1\\ v_2\rho_2\end{pmatrix} + \frac{\xi}{2}\mathbf{f}(\rho_1,\rho_2)\begin{pmatrix}1\\1\end{pmatrix}, \qquad \text{(I.40)}\\ &(\rho_1,\rho_2)\in C([0,1];\mathcal{M}_+(\mathbb{R}^n)\times\mathcal{M}_+(\mathbb{R}^n)),\\ &\rho_{i,0}=\nu_i, \rho_{i,1}=\mu_i, i=1,2\Bigg\}, \qquad \text{(I.41)}\end{aligned}$$

where $\mathbf{f} : \mathcal{M}_+(\mathbb{R}^n)\times\mathcal{M}_+(\mathbb{R}^n) \to \mathcal{M}_+(\mathbb{R}^n)$ is a measure-valued map derived from a suitable concave nonnegative function f.

Section 5.4 is devoted to the study of the properties of $\mathrm{D}_{\mathbb{K}}$. In particular, we give an alternative representation of $\mathrm{D}_{\mathbb{K}}$ in terms of the minimization of a certain action functional $\mathcal{A}$. In this way we prove that, if

$$\mathrm{D}_{\mathbb{K}}((\nu_1,\nu_2),(\mu_1,\mu_2)) < \infty,$$

than there exist (weakly) continuous curves $(\rho_1,\rho_2)\in C([0,1];\mathcal{M}_+(\mathbb{R}^n)\times\mathcal{M}_+(\mathbb{R}^n))$, vector fields $v_1, v_2 \in L^2((0,1);L^2(\mathbb{R}^n;\rho_i))$ and a scalar reaction term $\xi\in L^2((0,1);L^2(\mathbb{R}^n;\mathbf{f}(\rho_1,\rho_2)))$ satisfying (I.40), (I.41) and

$$\mathrm{D}^2_{\mathbb{K}}((\nu_1,\nu_2),(\mu_1,\mu_2)) = \int_0^1 \left(\int_{\mathbb{R}^n} |v_1|^2\, d\rho_{1,t} + |v_2|^2\, d\rho_{2,t} + \frac{|\xi|^2}{2}\, d\mathbf{f}(\rho_{1,t},\rho_{2,t})\right) dt.$$

Then, we show that $\mathrm{D}_{\mathbb{K}}$ is indeed an (extended) distance on $\mathcal{M}_+(\mathbb{R}^n)\times\mathcal{M}_+(\mathbb{R}^n)$, and we find a necessary and sufficient condition under which $\mathrm{D}_{\mathbb{K}}((\nu_1,\nu_2),(\mu_1,\mu_2))<\infty$, namely,

$$\nu_1(\mathbb{R}^n)-\nu_2(\mathbb{R}^n) = \mu_1(\mathbb{R}^n)-\mu_2(\mathbb{R}^n).$$

Finally, we also prove that

$$\mathrm{D}_{\mathbb{K}}((\mu_1,\mu_2),(\mu_1^k,\mu_2^k)) \to 0 \text{ as } k\to+\infty$$

implies $\mu_i^k \rightharpoonup \mu_i$ and $\mu_i^k(\mathbb{R}^n)\to\mu_i(\mathbb{R}^n)$ for $i=1,2$. However, as showed in Example 5.4.17, the convergence with respect to $\mathrm{D}_{\mathbb{K}}$ does not imply the convergence of the total mass of the couples of Radon measures (μ_1^k,μ_2^k). Then, in Section 5.5 we describe the state of the art of our investigations on the first variation of $\mathrm{D}_{\mathbb{K}}$ under different types of perturbations. The final aim would be to obtain Euler-Lagrange equations for the distance $\mathrm{D}_{\mathbb{K}}$. Future research shall go in the direction of analyzing further properties of the distance $\mathrm{D}_{\mathbb{K}}$, as its dual representation, for instance. Our final aim is to apply the minimizing movements scheme to obtain the existence of solutions to

$$\begin{cases}\frac{d}{dt}\mu_1 &= \mathrm{div}(\mu_1\nabla(V*\mu)) - \mathbf{f}(\mu_1,\mu_2)\\ \frac{d}{dt}\mu_2 &= -\mathrm{div}(\mu_2\nabla(V*\mu)) - \mathbf{f}(\mu_1,\mu_2),\end{cases}$$

satisfying some type of energy dissipation inequality. We conclude our current exposition with Section 5.6, where we prove that the local (or descending) slope of the self energy

$$\Phi_{\mathrm{self}}(\mu_1,\mu_2) := \mu_1(\mathbb{R}^n)+\mu_2(\mathbb{R}^n)$$

is

$$\sqrt{2\int_{\mathbb{R}^n} d\mathbf{f}(\mu_1,\mu_2)},$$

which is not lower semicontinuous with respect to the distance $\mathrm{D}_{\mathbb{K}}$. This seems to represent an issue in the application of the classical results from the theory of gradient flows.

Other works

We conclude this introduction with a brief summary of other relevant research projects developed during my years of PhD.

A distributional approach to fractional Sobolev spaces and fractional variation: existence of blow-up and asymptotics

In the last decades, fractional Sobolev spaces have been receiving increasing attention ([67]), and, in particular, the theory of sets with finite fractional perimeter has been deeply studied, with a focus on minimal fractional surfaces ([56, Section 7]). However, differently from the standard Sobolev space $W^{1,p}(\mathbb{R}^n)$, the space $W^{\alpha,p}(\mathbb{R}^n)$ does not seem to have a clear distributional nature.

In the past few years, several authors ([137, 142, 143, 147]), looking for a good notion of fractional differential operator, considered the following fractional gradient:

$$\nabla^\alpha u(x) := \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(y-x)(u(y)-u(x))}{|y-x|^{n+\alpha+1}}\, dy,$$

where

$$\mu_{n,\alpha} := 2^\alpha \pi^{-\frac{n}{2}} \frac{\Gamma\left(\frac{n+\alpha+1}{2}\right)}{\Gamma\left(\frac{1-\alpha}{2}\right)}$$

is a multiplicative renormalizing constant. In a similar way, one can define the associated fractional divergence

$$\operatorname{div}^\alpha \varphi(x) := \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(y-x)\cdot(\varphi(y)-\varphi(x))}{|y-x|^{n+\alpha+1}}\, dy,$$

so that the operators ∇^α and $\operatorname{div}^\alpha$ are dual, in the sense that

$$\int_{\mathbb{R}^n} u \operatorname{div}^\alpha \varphi \, dx = -\int_{\mathbb{R}^n} \varphi \cdot \nabla^\alpha u \, dx \tag{I.42}$$

for all $u \in C_c^\infty(\mathbb{R}^n)$ and $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$.

This is the starting point of [54], a joint work with G. Stefani, which concerns a new distributional characterization of the notion of sets of finite fractional perimeter, and consequently the study of a new space of functions of fractional bounded variation.

Indeed, thanks to (I.42), we can define

$$BV^\alpha(\mathbb{R}^n) := \left\{ u \in L^1(\mathbb{R}^n) : |D^\alpha u|(\mathbb{R}^n) < +\infty \right\},$$

where

$$|D^\alpha u|(\mathbb{R}^n) = \sup\left\{ \int_{\mathbb{R}^n} u \operatorname{div}^\alpha \varphi \, dx : \varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n),\ \|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \le 1 \right\}.$$

In addition, we say that a measurable set E is a set with finite fractional Caccioppoli α-perimeter in an open set Ω if

$$|D^\alpha \chi_E|(\Omega) = \sup\left\{ \int_E \operatorname{div}^\alpha \varphi \, dx : \varphi \in C_c^\infty(\Omega;\mathbb{R}^n),\ \|\varphi\|_{L^\infty(\Omega;\mathbb{R}^n)} \le 1 \right\} < \infty,$$

and we say that E has locally finite Caccioppoli α-perimeter in $\mathbb{R}^n$ if $|D^\alpha \chi_E|(\Omega) < \infty$ for any open bounded set Ω.

In perfect analogy with the classical space of functions of bounded variation $BV(\mathbb{R}^n)$, in [54] we prove that $BV^\alpha(\mathbb{R}^n)$ is a Banach space and its norm is lower semicontinuous with respect to L^1-convergence; that $u \in L^1(\mathbb{R}^n)$ belongs to $BV^\alpha(\mathbb{R}^n)$ if and only if there exists a vector valued finite Radon measure $D^\alpha u$ such that

$$\int_{\mathbb{R}^n} u \operatorname{div}^\alpha \varphi \, dx = - \int_{\mathbb{R}^n} \varphi \cdot \, dD^\alpha u \tag{I.43}$$

for any $\varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$.

In addition, we show that any uniformly bounded sequence in $BV^\alpha(\mathbb{R}^n)$ admits limit points in $L^1(\mathbb{R}^n)$ with respect the L^1_{loc}-convergence.

Then, exploiting again (I.42) and arguing similarly to the classical case, it seems natural to define the *weak fractional α-gradient* of a function $u \in L^p(\mathbb{R}^n)$, for $p \in [1, +\infty]$, as the function $\nabla^\alpha_w u \in L^1_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n)$ satisfying

$$\int_{\mathbb{R}^n} u \operatorname{div}^\alpha \varphi \, dx = - \int_{\mathbb{R}^n} \nabla^\alpha_w u \cdot \varphi \, dx$$

for any $\varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$. For $\alpha \in (0,1)$ and $p \in [1, +\infty]$, we can define the *distributional fractional Sobolev space*

$$S^{\alpha,p}(\mathbb{R}^n) := \left\{ u \in L^p(\mathbb{R}^n) : \exists \, \nabla^\alpha_w u \in L^p(\mathbb{R}^n; \mathbb{R}^n) \right\}, \tag{I.44}$$

naturally endowed with the norm

$$\|u\|_{S^{\alpha,p}(\mathbb{R}^n)} := \|u\|_{L^p(\mathbb{R}^n)} + \|\nabla^\alpha_w u\|_{L^p(\mathbb{R}^n; \mathbb{R}^n)}. \tag{I.45}$$

It is clearly interesting to make a comparison between the distributional fractional Sobolev spaces $S^{\alpha,p}(\mathbb{R}^n)$ and the well-known *fractional Sobolev space* $W^{\alpha,p}(\mathbb{R}^n)$, which, for $\alpha \in (0,1)$ and $p \in [1, \infty)$, is defined as

$$W^{\alpha,p}(\mathbb{R}^n) := \left\{ u \in L^p(\mathbb{R}^n) : [u]_{W^{\alpha,p}(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+p\alpha}} \, dx \, dy \right)^{\frac{1}{p}} < +\infty \right\},$$

endowed with the norm

$$\|f\|_{W^{\alpha,p}(\mathbb{R}^n)} := \|u\|_{L^p(\mathbb{R}^n)} + [u]_{W^{\alpha,p}(\mathbb{R}^n)} \qquad \forall u \in W^{\alpha,p}(\mathbb{R}^n),$$

while, for $p = \infty$, $W^{\alpha,\infty}(\mathbb{R}^n) := C^{0,\alpha}_{\mathrm{b}}(\mathbb{R}^n)$, the space of bounded α-Hölder continuous functions.

In [54], we focus on the case $p = 1$, and we show that the inclusions

$$W^{\alpha,1}(\mathbb{R}^n) \subset S^{\alpha,1}(\mathbb{R}^n) \subset BV^\alpha(\mathbb{R}^n)$$

are all continuous and strict.

As for the sets with finite fractional Caccioppoli α-perimeter, we show that indeed they include the family of sets with standard finite fractional α-perimeter; that is, we have

$$|D^\alpha \chi_E|(\Omega) \le \mu_{n,\alpha} P_\alpha(E; \Omega)$$

for any open set $\Omega \subset \mathbb{R}^n$, where

$$P_\alpha(E; \Omega) := \int_\Omega \int_\Omega \frac{|\chi_E(x) - \chi_E(y)|}{|x-y|^{n+\alpha}} \, dx \, dy + 2 \int_\Omega \int_{\mathbb{R}^n \setminus \Omega} \frac{|\chi_E(x) - \chi_E(y)|}{|x-y|^{n+\alpha}} \, dx \, dy.$$

Employing the notion of weak fractional gradient to define a natural analogue of De Giorgi's reduced boundary, the *fractional reduced boundary $\mathscr{F}^\alpha E$*, as the set of points satisfying

$$x \in \operatorname{supp}(D^\alpha \chi_E) \qquad \text{and} \qquad \exists \lim_{r \to 0} \frac{D^\alpha \chi_E(B(x,r))}{|D^\alpha \chi_E|(B(x,r))} \in \mathbb{S}^{n-1}.$$

We also let

$$\nu_E^\alpha \colon \mathscr{F}^\alpha E \to \mathbb{S}^{n-1}, \qquad \nu_E^\alpha(x) := \lim_{r\to 0} \frac{D^\alpha \chi_E(B(x,r))}{|D^\alpha \chi_E|(B(x,r))}, \quad x \in \mathscr{F}^\alpha E,$$

be the *measure theoretic unit interior fractional normal* to E.

Then, following an approach similar to the one presented in [69, Section 5.7], we derive density estimates for $|D^\alpha \chi_E|(B(x,r))$ for any $x \in \mathscr{F}^\alpha E$, thanks to which we prove that

$$|D^\alpha \chi_E| \ll \mathscr{H}^{n-\alpha} \llcorner \mathscr{F}^\alpha E,$$

where $\mathscr{H}^{n-\alpha}$ is the $(n-\alpha)$-dimensional Hausdorff measure. In addition, exploiting a suitable compactness result we are able to show the following result on $\mathrm{Tan}(E,x)$; that is, the set of all *tangent sets of E at x*, i.e. the set of all limit points in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$-topology of the family $\left\{ \frac{E-x}{r} : r > 0 \right\}$ as $r \to 0$.

Theorem I.1. *Let $\alpha \in (0,1)$. Let E be a set with locally finite fractional Caccioppoli α-perimeter in $\mathbb{R}^n$. For any $x \in \mathscr{F}^\alpha E$ we have $\mathrm{Tan}(E,x) \neq \varnothing$. In addition, if $F \in \mathrm{Tan}(E,x)$, then F is a set of locally finite fractional Caccioppoli α-perimeter such that $\nu_F^\alpha(y) = \nu_E^\alpha(x)$ for $|D^\alpha \chi_F|$-a.e. $y \in \mathscr{F}^\alpha F$.*

Hence, we obtain a first partial extension of De Giorgi's Blow-up Theorem for sets of finite fractional Caccioppoli perimeter, by proving existence of blow-ups on points of the fractional reduced boundary.

This new distributional approach provides a tool to deal with a large variety of classical results in the context of functions with fractional bounded variation. We list here the principal directions of future research:

- achieve a better characterization of the blow-ups, and possibly their uniqueness;
- prove a Structure Theorem for $\mathscr{F}^\alpha E$ in the spirit of De Giorgi's Theorem;
- develop a calibration theory for sets of finite fractional Caccioppoli α-perimeter as a useful tool for the study of fractional minimal surfaces;
- consider the asymptotics as $\alpha \to \beta$ for $\beta \in [0,1]$, and in particular as $\alpha \to 1^-$, in which case it is of interest to investigate the Γ-convergence of $|D^\alpha \chi_E|$ to the classical perimeter;
- extend the Gauss–Green and integration by parts formulas to sets of finite fractional Caccioppoli α-perimeter;
- give a good definition of BV^α functions on a general open set.

Indeed, the study of the asymptotics as $\alpha \to \beta^-$, for any $\beta \in (0,1]$ is the core of the forthcoming work [53], in collaboration with G. Stefani, while the case $\alpha \to 0^+$ shall be treated in [34], in collaboration with M. Calzi, E. Brué and G. Stefani. We outline here the key aspects of these future developments.

It is well-known that, for any $p \in [1,\infty)$ and $n \geq 1$, there exists a constant $C_{n,p} > 0$ such that

$$\lim_{\alpha\to 1^-} (1-\alpha)\, [f]^p_{W^{\alpha,p}(\mathbb{R}^n)} = C_{n,p}\, \|\nabla f\|^p_{L^p(\mathbb{R}^n;\mathbb{R}^n)} \tag{I.46}$$

for any $f \in W^{1,p}(\mathbb{R}^n)$ (see [29]). In [53], we improve (I.46) by showing the following asymptotic behaviours.

- If $p \in (1, \infty)$, then $W^{1,p}(\mathbb{R}^n) \subset S^{\alpha,p}(\mathbb{R}^n)$ for any $\alpha \in (0,1)$ and, for any $f \in W^{1,p}(\mathbb{R}^n)$,

$$\lim_{\alpha \to 1^-} \|\nabla_w^\alpha f - \nabla_w f\|_{L^p(\mathbb{R}^n;\,\mathbb{R}^n)} = 0. \tag{I.47}$$

- If $p = 1$, then $BV(\mathbb{R}^n) \subset BV^\alpha(\mathbb{R}^n)$ for any $\alpha \in (0,1)$ and, for any $f \in BV(\mathbb{R}^n)$,

$$D^\alpha f \rightharpoonup Df \text{ in } \mathcal{M}(\mathbb{R}^n;\mathbb{R}^n),\ \ |D^\alpha f| \rightharpoonup |Df| \text{ in } \mathcal{M}(\mathbb{R}^n) \text{ as } \alpha \to 1^- \tag{I.48}$$

  and

$$\lim_{\alpha \to 1^-} |D^\alpha f|(\mathbb{R}^n) = |Df|(\mathbb{R}^n). \tag{I.49}$$

- If $p = \infty$, then $W^{1,\infty}(\mathbb{R}^n) \subset S^{\alpha,\infty}(\mathbb{R}^n)$ for any $\alpha \in (0,1)$ and, for any $f \in W^{1,\infty}(\mathbb{R}^n)$,

$$\nabla_w^\alpha f \overset{*}{\rightharpoonup} \nabla_w f \quad \text{in } L^\infty(\mathbb{R}^n;\mathbb{R}^n) \text{ as } \alpha \to 1^- \tag{I.50}$$

  and

$$\|\nabla_w f\|_{L^\infty(\mathbb{R}^n;\,\mathbb{R}^n)} \le \liminf_{\alpha \to 1^-} \|\nabla_w^\alpha f\|_{L^\infty(\mathbb{R}^n;\,\mathbb{R}^n)}. \tag{I.51}$$

It is interesting to notice that no renormalising factor is required in the limits (I.47) – (I.51), contrarily to what happened for the standard fractional Sobolev seminorm, since it is not difficult to show that

$$\mu_{n,\alpha} \sim \frac{1-\alpha}{\omega_n} \quad \text{as } \alpha \to 1^-.$$

In addition, [53] contains an extension of the result of Γ-convergence in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ of the fractional α-perimeter P_α to the standard De Giorgi's perimeter P as $\alpha \to 1^-$ (see [9]). More precisely, [9, Theorem 2] states that, if $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary, then there exists a constant $c_n > 0$ such that

$$\Gamma(L^1_{\mathrm{loc}}) - \lim_{\alpha \to 1^-} (1-\alpha)\, P_\alpha(E;\Omega) = c_n P(E;\Omega) \tag{I.52}$$

for any measurable sets $E \subset \mathbb{R}^n$. We refer the interested reader to [31,61] for complete treatment of the subject of Γ-convergence.

Our counterpart of (I.52) for the fractional α-variation as $\alpha \to 1^-$ is the following: if $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary, then

$$\Gamma(L^1_{\mathrm{loc}}) - \lim_{\alpha \to 1^-} |D^\alpha \chi_E|(\Omega) = P(E;\Omega) \tag{I.53}$$

for any measurable set $E \subset \mathbb{R}^n$. In addition, it is interesting to notice that our approach allows to prove that Γ-convergence holds true also at the level of functions. Indeed, if $f \in BV(\mathbb{R}^n)$ and $\Omega \subset \mathbb{R}^n$ is an open set such that either Ω is bounded with Lipschitz boundary or $\Omega = \mathbb{R}^n$, then

$$\Gamma(L^1) - \lim_{\alpha \to 1^-} |D^\alpha f|(\Omega) = |Df|(\Omega). \tag{I.54}$$

It is relevant to mention that, as a byproduct of the techniques developed for the asymptotic study of the fractional α-variation as $\alpha \to 1^-$, we are also able to characterise the behaviour of the fractional α-variation as $\alpha \to \beta^-$, for any given $\beta \in (0,1)$. On the one hand, if $f \in BV^\beta(\mathbb{R}^n)$, then

$$D^\alpha f \rightharpoonup D^\beta f \text{ in } \mathcal{M}(\mathbb{R}^n;\mathbb{R}^n),\ \ |D^\alpha f| \rightharpoonup |D^\beta f| \text{ in } \mathcal{M}(\mathbb{R}^n) \text{ as } \alpha \to \beta^-$$

and, moreover,

$$\lim_{\alpha \to \beta^-} |D^\alpha f|(\mathbb{R}^n) = |D^\beta f|(\mathbb{R}^n).$$

On the other hand, if $f \in BV^{\beta}(\mathbb{R}^n)$ and $\Omega \subset \mathbb{R}^n$ is an open set such that either Ω is bounded and $|D^{\beta} f|(\partial\Omega) = 0$ or $\Omega = \mathbb{R}^n$, then

$$\Gamma(L^1) - \lim_{\alpha \to \beta^-} |D^{\alpha} f|(\Omega) = |D^{\beta} f|(\Omega).$$

As for the asymptotics as $\alpha \to 0^+$, it was proved in [120, 121] that for any $p \in [1, +\infty)$, there exists a constant $\widetilde{C}_{n,p} > 0$ such that

$$\lim_{\alpha \to 0^+} \alpha \, [f]^p_{W^{\alpha,p}(\mathbb{R}^n)} = \widetilde{C}_{n,p} \, \|f\|^p_{L^p(\mathbb{R}^n)} \tag{I.55}$$

for any $f \in \bigcup_{\alpha \in (0,1)} W^{\alpha,p}(\mathbb{R}^n)$. Starting from (I.55), in [34] we study what happens to the fractional α-variation as $\alpha \to 0^+$. Note that

$$\lim_{\alpha \to 0^+} \mu_{n,\alpha} = \pi^{-\frac{n}{2}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} =: \mu_{n,0},$$

so there is no renormalization factor as $\alpha \to 0^+$.

At least formally, as $\alpha \to 0^+$ the fractional α-gradient is converging to the operator

$$\nabla^0 u(x) := \mu_{n,0} \int_{\mathbb{R}^n} \frac{(y-x)(u(y)-u(x))}{|y-x|^{n+1}} \, dy.$$

The operator ∇^0 is well defined on $C_c^{\infty}(\mathbb{R}^n)$ and, actually, coincides with the well-known vector-valued *Riesz transform* Rf, see [92, Section 5.1.4] and [148, Chapter 3]. Similarly, the fractional α-divergence is formally converging to the operator

$$\operatorname{div}^0 \varphi(x) := \mu_{n,0} \int_{\mathbb{R}^n} \frac{(y-x) \cdot (\varphi(y)-\varphi(x))}{|y-x|^{n+1}} \, dy, \tag{I.56}$$

which is well defined for any $\varphi \in C_c^{\infty}(\mathbb{R}^n; \mathbb{R}^n)$.

In perfect analogy with what we did before, it seems natural to introduce the space $BV^0(\mathbb{R}^n)$ as the space of functions $u \in L^1(\mathbb{R}^n)$ such that

$$|D^0 u|(\mathbb{R}^n) := \sup\left\{ \int_{\mathbb{R}^n} u \operatorname{div}^0 \varphi \, dx : \varphi \in C_c^{\infty}(\mathbb{R}^n; \mathbb{R}^n), \ \|\varphi\|_{L^{\infty}(\mathbb{R}^n; \mathbb{R}^n)} \le 1 \right\} < \infty.$$

Surprisingly (and differently from the fractional α-variation, recall [54, Section 3.10]), it turns out that $|D^0 u| \ll \mathscr{L}^n$ for all $f \in BV^0(\mathbb{R}^n)$. More precisely, one can actually prove that $BV^0(\mathbb{R}^n) = H^1(\mathbb{R}^n)$, in the sense that $u \in BV^0(\mathbb{R}^n)$ if and only if $u \in H^1(\mathbb{R}^n)$, with

$$D^0 u = Ru \mathscr{L}^n \text{ in } \mathcal{M}(\mathbb{R}^n; \mathbb{R}^n),$$

where

$$H^1(\mathbb{R}^n) := \left\{ u \in L^1(\mathbb{R}^n) : Ru \in L^1(\mathbb{R}^n; \mathbb{R}^n) \right\}$$

is the (real) *Hardy space*, see [150, Chapter III] for the precise definition. Thus, it would be interesting to understand for which functions $u \in L^1(\mathbb{R}^n)$ the fractional α-gradient $\nabla^{\alpha} u$ tends (in a suitable sense) to the Riesz transform Ru as $\alpha \to 0^+$.

On BV functions and essentially bounded divergence-measure fields in metric spaces

In [35], a work in collaboration with Vito Buffa and Michele Miranda Jr., we give another extension of the theory of divergence-measure fields and generalized Gauss–Green formulas in the context of complete and separable metric measure spaces $(\mathbb{X}, d, \mu)$ equipped with a nonnegative Radon measure μ finite on bounded sets. In order to deal with "vector fields" on metric measure spaces, one needs to refer to some differential structure of the ambient space, in terms of which the usual differential objects of the "smooth" analysis and geometry find a consistent and equivalent counterpart.

Following the definitions of tangent and cotangent module given by Gigli ([90, 91]), we first give a notion of functions of bounded variation and sets of finite perimeter in terms of suitable vector fields. Then, we extend the concept of divergence-measure field. We say that X is an L^p-summable divergence-measure field, and we write $X \in \mathcal{DM}^p(\mathbb{X})$, if it belongs to the tangent module $L^p(T\mathbb{X})$ and there exists a finite Radon measure $\operatorname{div}(X)$ which satisfies

$$-\int_{\mathbb{X}} \phi \, d\operatorname{div}(X) = \int_{\mathbb{X}} \mathrm{d}\phi(X) \, d\mu$$

for any $\phi \in \operatorname{Lip}(\mathbb{X})$ with bounded support, where $\mathrm{d}\phi$ stands for the differential of ϕ, seen as an element of the cotangent module $L^{p'}(T^*\mathbb{X})$, for $p' = p/(p-1)$. While this definition and some basic properties do not require any other assumption on the metric space, we need to ask $\mathbb{X}$ to be locally compact in order to derive Gauss–Green formula on *regular domains*.

Inspired by [117], we say that an open set of finite perimeter $\Omega \subset \mathbb{X}$ is a regular domain if the upper inner Minkowski content of its boundary satisfies

$$\mathfrak{M}_i^*(\partial\Omega) := \limsup_{t \to 0} \frac{\mu(\Omega \setminus \Omega_t)}{t} = |D\chi_\Omega|(\mathbb{X}),$$

where, for $t > 0$,

$$\Omega_t := \{x \in \Omega;\ \operatorname{dist}(x, \Omega^c) \geq t\}.$$

This property allows us to construct a good family of smooth functions approximating χ_Ω so that we obtain the following result.

Theorem I.2. *Let $\mathbb{X}$ be locally compact, $X \in \mathcal{DM}^\infty(\mathbb{X})$ and $\Omega \subset \mathbb{X}$ be a regular domain. Then there exists a function $(X \cdot \nu_\Omega)^-_{\partial\Omega} \in L^\infty(\partial\Omega; |D\chi_\Omega|)$ such that*

$$\int_\Omega \varphi \, d\operatorname{div}(X) + \int_\Omega \mathrm{d}\varphi(X) \, d\mu = -\int_{\partial\Omega} \varphi \, (X \cdot \nu_\Omega)^-_{\partial\Omega} \, d|D\chi_\Omega|, \tag{I.57}$$

for any $\varphi \in \operatorname{Lip_b}(\mathbb{X})$ such that $\operatorname{supp}(\varphi\chi_\Omega)$ is a bounded set. In addition, we have the following estimate:

$$\|(X \cdot \nu_\Omega)^-_{\partial\Omega}\|_{L^\infty(\partial\Omega;|D\chi_\Omega|)} \leq \||X|\|_{L^\infty(\Omega)}.$$

As customary, we call the function $(X \cdot \nu_\Omega)^-_{\partial\Omega}$ the *interior normal trace* of X on $\partial\Omega$.

Aiming to integration by parts formulas on sets of finite perimeter, we need to require additional structural assumptions on the metric measure space: in particular, in the second part of [35] we focus ourselves on locally compact $\mathsf{RCD}(K, \infty)$ metric measure spaces. Following idea analogous to those developed in [51, 52] (see also Chapters 3 and 4), we first obtain a Leibniz rule for the divergence of the product of a field in $X \in \mathcal{DM}^\infty(\mathbb{X})$ and a scalar function in $f \in BV(\mathbb{X}) \cap L^\infty(\mathbb{X})$, and then we exploit it to gain the Gauss–Green and the integration by parts formula on sets of finite perimeter. In this setting, we employ the heat semigroup h_t in order to regularize the bounded scalar BV function in the proof of the Leibniz rule, and so we strongly relied on the Bakry-Emery curvature-dimension condition and its related contraction estimate. Even though the heat semigroup is not a local operator, as the mollification instead was in the Euclidean spaces and the stratified groups, we are able to obtain similar convergence results. In particular, we can define the *pairing* between Df and X as any (possibly not unique) accumulation point $\boldsymbol{D}f(X)$ of the family of measures $\mathrm{dh}_t f(X)\mu$ in $\mathcal{M}(\mathbb{X})$ is absolutely continuous with respect to the total variation measure $|Df|$. This fact plays a fundamental role in the definition of the normal traces of a divergence-measure field. Indeed, we set the *interior and exterior distributional normal traces* of $X \in \mathcal{DM}^\infty(\mathbb{X})$ on the boundary ∂E of a set of finite perimeter $E \subset \mathbb{X}$ are given as the functions $\langle X, \nu_E \rangle^-, \langle X, \nu_E \rangle^+_{\partial E} \in L^\infty(\mathbb{X}; |D\mathbf{1}_E|)$ such that

$$2\boldsymbol{D}\chi_E(\chi_E X) = \langle X, \nu_E \rangle^-_{\partial E} |D\chi_E|,$$
$$2\boldsymbol{D}\chi_E(\chi_{E^c} X) = \langle X, \nu_E \rangle^+_{\partial E} |D\chi_E|.$$

Due to the non-uniqueness of the pairing, a priori, we cannot ensure the uniqueness of these normal traces either. However, assuming to have fixed a sequence $t_j \to 0$ such that

$$\mathrm{dh}_{t_j}(\chi_E)(\chi_E X)\mu \rightharpoonup \boldsymbol{D}\chi_E(\chi_E X) \ \text{ and } \ \mathrm{dh}_{t_j}(\chi_E)(\chi_{E^c} X)\mu \rightharpoonup \boldsymbol{D}\chi_E(\chi_{E^c} X),$$

we are able to carry on our analysis in an analogous way as in [51] (see also Chapter 4), with the additional difficulty given by the fact that we cannot characterize in general the weak* accumulation points $\widehat{\mathbf{1}_E}$ of $\mathrm{h}_{t_j}\mathbf{1}_E$ in $L^\infty(\mathbb{X}; |D\mathbf{1}_E|)$, and so we cannot achieve uniqueness of the normal traces. Nevertheless, we obtain general Gauss–Green and integration by parts formulas.

We remark that the issue of the dependence on the approximating sequence $\mathrm{h}_{t_j}\chi_E$ can be solved under the additional assumption that $|\mathrm{div}(X)| \ll \mu$. In this case, the interior and exterior distributional normal traces of $X \in \mathcal{DM}^\infty(\mathbb{X})$ on the boundary of the set of finite perimeter E are uniquely determined and coincide, so that the unique normal trace, denoted by $\langle X, \nu_E \rangle_{\partial E}$, satisfies

$$\int_E \varphi \, d\mathrm{div}(X) + \int_E \mathrm{d}\varphi(X)\, d\mu = -\int_{\partial E} \varphi \, \langle X, \nu_E \rangle_{\partial E} \, d|D\mathbf{1}_E| \tag{I.58}$$

for any $\varphi \in \mathrm{Lip_b}(\mathbb{X})$ such that $\mathrm{supp}(\mathbf{1}_E \varphi)$ is bounded.

Finer entropy estimates for systems of evolution equations for measures

In [7], a current research project with Luigi Ambrosio, Mark A. Peletier and Oliver Tse, we consider some refinements of entropy estimates related to systems of evolution equations for Radon measures, starting from previous works of Ambrosio, Mainini and Serfaty ([18,21,115]). The aim of this research is to obtain the existence of solutions for less regular initial data.

In the framework of [115] we consider couples $(\mu_1, \mu_2) \in \mathcal{M}^2_\alpha(\mathbb{R}^2) \times \mathcal{M}^2_\beta(\mathbb{R}^2)$, where

$$\mathcal{M}^2_\alpha(\mathbb{R}^2) := \left\{ \mu \in \mathcal{M}^+(\mathbb{R}^2) : \mu(\mathbb{R}^2) = \alpha, \int_{\mathbb{R}^2} |x|^2 \, d\mu(x) < \infty \right\}.$$

Given an initial datum

$$(\mu_1^0, \mu_2^0) \in \mathcal{M}^2_\alpha(\mathbb{R}^2) \times \mathcal{M}^2_\beta(\mathbb{R}^2),$$

for some $\alpha, \beta \geq 0$, we want to find a couple of measures $(\mu_1(t), \mu_2(t))$ which is a solution to

$$\begin{cases} \frac{d}{dt}\mu_1(t) - \mathrm{div}(\nabla h_{\mu(t)} \mu_1(t)) & = 0 \\ \frac{d}{dt}\mu_2(t) + \mathrm{div}(\nabla h_{\mu(t)} \mu_2(t)) & = 0 \end{cases} \text{in } \mathcal{D}'((0, +\infty) \times \mathbb{R}^2), \tag{I.59}$$

where $\mu(t) = \mu_1(t) - \mu_2(t)$ and $h_{\mu(t)}$ is the solution, for any $t > 0$, to

$$-\Delta h_{\mu(t)} = \mu(t) \text{ in } \mathbb{R}^2. \tag{I.60}$$

We take as free energy the Dirichlet energy associated to (I.60),

$$\Phi(\mu) = \frac{1}{2} \int_{\mathbb{R}^2} h_\mu \, d\mu.$$

The key idea of [115] is to represent (I.59) as the gradient flow of Φ with respect to the 2-Wasserstein distance between couples of measures in $(\mu_1, \mu_2), (\nu_1, \nu_2) \in \mathcal{M}^2_\alpha(\mathbb{R}^2) \times \mathcal{M}^2_\beta(\mathbb{R}^2)$, given by

$$\sqrt{W_2^2(\mu_1, \nu_1) + W_2^2(\mu_2, \nu_2)}.$$

In this way, it is possible to build a solution by applying the minimizing movement scheme. To this purpose, we choose initial data $(\mu_1^0, \mu_2^0) \in \mathcal{M}^2_\alpha(\mathbb{R}^2) \times \mathcal{M}^2_\beta(\mathbb{R}^2)$, and a time step $\tau > 0$, and we look for $\mu_{1,\tau}$ and $\mu_{2,\tau}$ which are solution to

$$\min_{(\nu_1,\nu_2) \in \mathcal{M}^2_\alpha(\mathbb{R}^2) \times \mathcal{M}^2_\beta(\mathbb{R}^2)} \Phi(\nu_1 - \nu_2) + \frac{1}{2\tau}(W_2^2(\nu_1, \mu_1^0) + W_2^2(\nu_2, \mu_2^0)). \tag{I.61}$$

Then, we employ $(\mu_{1,\tau}, \mu_{2,\tau})$ as initial data and we look for $(\mu^2_{1,\tau}, \mu^2_{2,\tau})$ which solves (I.61) for this new initial data; and proceeding in this way we construct by iteration a sequence $(\mu^k_{1,\tau}, \mu^k_{2,\tau})$. The method employed in [115] is based on deriving the Euler-Lagrange equations for the minimizers $\mu_{1,\tau}, \mu_{2,\tau}$ and exploiting them to get some entropy estimates; that is, a bound of the form

$$\int_{\mathbb{R}^2} \varphi(\mu_{1,\tau}) + \varphi(\mu_{2,\tau}) \leq \int_{\mathbb{R}^2} \varphi(\mu_1^0) + \varphi(\mu_2^0), \tag{I.62}$$

for an entropy function φ satisfying certain properties. In particular, this means that, if $\varphi(x) \approx x^p$ as $x \to +\infty$ and $\mu_i^0 \in L^p(\mathbb{R}^2), i = 1, 2$, then the minimizers $\mu_{1,\tau}, \mu_{2,\tau}$ are also L^p-summable. In [115] it is proved (using an argument from [18]) that, if $p \geq 4$, then the piecewise constant interpolation $(\bar{\mu}_{1,\tau}(t), \bar{\mu}_{2,\tau}(t)) := (\mu_{1,\tau}^{\lceil t/\tau \rceil}, \mu_{2,\tau}^{\lceil t/\tau \rceil})$ admits a narrow limit as $\tau \to 0$, up to a subsequence, which is a solution of of (I.59) and belongs to $L^p(\mathbb{R}^2) \times L^p(\mathbb{R}^2)$.

The starting point of our approach is to give a slightly different definition of an entropy function. We say that a nondecrasing, C^1-differentiable and piecewise C^2-differentiable function $\varphi : [0, +\infty) \to [0, \infty)$ is an entropy function if it satisfies

1. $\varphi(0) = 0$,
2. $\varphi'(0) = \lim_{x \to 0} \dfrac{\varphi(x)}{x} \in \mathbb{R}$,
3. $2x^2\varphi''(x) \geq x\varphi'(x) - \varphi(x)$, for any x where φ is twice differentiable,
4. $\lim_{x \to +\infty} \dfrac{\varphi(x)}{x} = +\infty$.

Then, we associate to each entropy function φ a dissipation function ψ; that is, a convex function $\psi : [0, +\infty) \to \mathbb{R}$ satisfying $\psi'(x) = x\varphi'(x) - \varphi(x)$. Thanks to some rather technical approximation arguments, we are able to refine the method of proof of [115] (see also [18]), obtaining the following stronger version of (I.62):

$$\int_{\mathbb{R}^2} \varphi(\mu_{1,\tau}) + \varphi(\mu_{2,\tau}) \leq \int_{\mathbb{R}^2} \varphi(\mu_1^0) + \varphi(\mu_2^0) - \tau \int_{\mathbb{R}^2} (\psi'(\mu_{1,\tau}) - \psi'(\mu_{2,\tau}))\mu_\tau, \tag{I.63}$$

where $\mu_\tau := \mu_{1,\tau} - \mu_{2,\tau}$. Given initial data satisfying $\int_{\mathbb{R}^2} \varphi(\mu_i^0) < \infty$ for $i = 1, 2$ and under some additional assumptions on φ and ψ, we show that any solution $(\mu_1(t), \mu_2(t))$ of (5.2.12), built as narrow limit of the piecewise constant interpolation $(\bar{\mu}_1(t), \bar{\mu}_2(t))$, up to a subsequence, satisfies

$$\int_{\mathbb{R}^2} \varphi(\mu_1(t)) + \varphi(\mu_2(t)) - \int_{\mathbb{R}^2} \varphi(\mu_1^0) + \varphi(\mu_2^0) \leq - \int_0^t \int_{\mathbb{R}^2} |\mu(r)| \psi'(|\mu(r)|) \, dr, \tag{I.64}$$

where $\mu(r) = \mu_1(r) - \mu_2(r)$.

In particular, choosing $\varphi(x) = x^p$, $p > 1$, we obtain the following improvement of the existence result given in [115].

Proposition I.3. *Let $(\mu_1^0, \mu_2^0) \in L^p(\mathbb{R}^2) \times L^p(\mathbb{R}^2)$, for some $p > 1$, and let $(\mu_1(t), \mu_2(t))$ be a solution of* (I.59)*, built as narrow limit of the piecewise constant interpolation $(\bar{\mu}_1(t), \bar{\mu}_2(t))$, up to a subsequence $\tau_k \to 0$. Then, we have*

$$\int_{\mathbb{R}^2} (\mu_1(t))^p + (\mu_2(t))^p \, dx - \int_{\mathbb{R}^2} (\mu_1^0)^p + (\mu_2^0)^p \, dx \leq -(p-1) \int_0^t \int_{\mathbb{R}^2} |\mu(r)|^{p+1} \, dx \, dr, \tag{I.65}$$

and so $\mu_i \in L^\infty(0, T; L^p(\mathbb{R}^2))$, $i = 1, 2$; while $\mu \in L^{p+1}(0, T; L^{p+1}(\mathbb{R}^2))$, for any $T > 0$.

In the case of a logarithmic entropy function

$$\varphi(x) = (1 + x) \log(1 + x)$$

we show that there exists a solution to (I.59) also for initial data in the Orlicz space $L \log L(\mathbb{R}^2)$. We remark that it is possible to obtain analogous results also in the framework of [18].

Chapter 1

Preliminaries

In this chapter we introduce some basic notions and tools of Geometric Measure Theory in the Euclidean and the stratified groups frameworks. In particular, in the latter context we also present in Section 1.3 some new smoothing results for BV functions.

1.1 BV and capacity theory in the Euclidean space

This section is devoted to recalling definitions and well known results from the Euclidean theory of functions of bounded variation and of capacity.

We start by setting some notation. Unless otherwise stated, Ω is an open subset of $\mathbb{R}^n$ and $\subset$ is equivalent to $\subseteq$. We denote by A^c the complement of A and by $A \Delta B := (A \setminus B) \cup (B \setminus A)$ the symmetric difference of the sets A, B. We denote by $E \Subset \Omega$ a set E whose closure, $\overline{E}$, is compact and contained in Ω, by E° the interior of the set E and by ∂E its topological boundary.

We denote by $\mathscr{L}^n$ and $\mathscr{H}^\alpha$ the Lebesgue and α-dimensional Hausdorff measures on $\mathbb{R}^n$, where $\alpha \geq 0$. Unless otherwise stated, a measurable set is a $\mathscr{L}^n$-measurable set. For any measurable set $E \subset \mathbb{R}^n$, we denote by $|E|$ the $\mathscr{L}^n$-measure of E, while, when applied to a function with values in $\mathbb{R}^m$, $| \cdot |$ is the Euclidean norm. $B(x, r)$ is the open ball with center in x and radius $r > 0$ and $\omega_n = |B(0, 1)|$. The unit sphere in $\mathbb{R}^n$ is denoted by $\mathbb{S}^{n-1}$ and we recall that $\mathscr{H}^{n-1}(\mathbb{S}^{n-1}) = n\omega_n$. We denote by $\mathcal{B}(\Omega)$ the Borel σ-algebra generated by the open subsets of $(\Omega, | \cdot |)$ which is a locally compact and separable metric space. We also use the standard notation $\mu \llcorner A$ for the restriction of a measure μ to the set A and $\mu \ll \nu$ to indicate that the measure μ is absolutely continuous with respect to the measure ν.

For $k \in \mathbb{N}_0 \cup \{+\infty\}$ and $m \in \mathbb{N}$ we denote by $C^k_c(\Omega; \mathbb{R}^m) := \{\phi \in C^k(\Omega; \mathbb{R}^m),\ \mathrm{supp}(\phi) \Subset \Omega\}$ the space of C^k functions compactly supported in Ω which will be endowed with the sup norm

$$\|\phi\|_{L^\infty(\Omega;\mathbb{R}^m)} = \sup_{x \in \Omega} |\phi(x)|.$$

We denote by $\mathrm{Lip}(\Omega)$, $\mathrm{Lip}_{\mathrm{loc}}(\Omega)$ and $\mathrm{Lip}_c(\Omega)$ the spaces of Lipschitz, locally Lipschitz and Lipschitz functions with compact support in Ω, respectively.

As it is customary, the space of signed Radon measures on Ω is denoted by $\mathcal{M}_{\mathrm{loc}}(\Omega)$ and the space of $\mathbb{R}^m$-vector valued Radon measures by $\mathcal{M}_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$. In addition, if $\mu \in \mathcal{M}_{\mathrm{loc}}(\Omega)$ and its total variation of $|\mu|$ is finite on Ω, then μ is a finite signed Radon measure on Ω and we write $\mu \in \mathcal{M}(\Omega)$; if μ is nonnegative, then $\mu = |\mu|$ and we write $\mu \in \mathcal{M}_{+,\mathrm{loc}}(\Omega)$ (or $\mu \in \mathcal{M}_+(\Omega)$ if $\mu(\Omega) < \infty$). Analogously, we say that μ is a finite $\mathbb{R}^m$-vector valued Radon measure on Ω, and we write $\mu \in \mathcal{M}(\Omega; \mathbb{R}^m)$, if $\mu \in \mathcal{M}_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$ and $|\mu|(\Omega) < \infty$.

We introduce now the notion of local weak* convergence for Radon measures. The Riesz representation theorem shows that the space $\mathcal{M}_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$ can be identified with the dual of $C_c(\Omega; \mathbb{R}^m)$, for any $m \geq 1$. Hence, we can give the following definition.

Definition 1.1.1 (Local weak* convergence)**.** We say that a sequence of Radon measures $\nu_k \in \mathcal{M}_{\mathrm{loc}}(\Omega)$ *locally weakly* converges in* Ω *to* $\nu \in \mathcal{M}_{\mathrm{loc}}(\Omega)$, and we write $\nu_k \rightharpoonup \nu$, if for every $\phi \in C_c(\Omega)$ we have

$$\int_\Omega \phi \, d\nu_k \to \int_\Omega \phi \, d\nu \quad \text{as} \quad k \to +\infty. \tag{1.1.1}$$

Analogously, given $\nu_k, \nu \in \mathcal{M}_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$, we have the local weak* convergence $\nu_k \rightharpoonup \nu$, if for every $\phi \in C_c(\Omega; \mathbb{R}^m)$ we have

$$\int_\Omega \phi \, d\nu_k \to \int_\Omega \phi \, d\nu \quad \text{as} \quad k \to +\infty. \tag{1.1.2}$$

Remark 1.1.2. In the sequel, the local weak* convergence above will also refer to measures $\nu_\varepsilon \in \mathcal{M}(\Omega^\varepsilon)$ defined on a family of increasing open sets $\Omega^\varepsilon \subset \Omega$ as ε decreases, such that $\bigcup_{\varepsilon>0} \Omega^\varepsilon = \Omega$ and for every compact set $K \subset \Omega$ there exists $\varepsilon' > 0$ such that $K \subset \Omega_{\varepsilon'}$. This type of local weak* convergence does not make a substantial difference compared to the standard one, so we will not use a different symbol.

For instance, the local weak* convergence of (1.3.18) refers to a family of measures that are not defined on all of Ω for every fixed $\varepsilon > 0$.

We introduce now the notion of pairing between an essentially bounded vector field and a finite vector valued Radon measure.

Lemma 1.1.3. *Let $F \in L^\infty(\Omega; \mathbb{R}^n)$ and $\nu \in \mathcal{M}(\Omega; \mathbb{R}^n)$. Let $\rho \in C_c(B(0,1))$ be a nonnegative mollifier satisfying $\rho(-x) = \rho(x)$ and $\int_{B(0,1)} \rho \, dx = 1$. Then, the measures $F \cdot (\rho_\varepsilon * \nu)\mathscr{L}^n$ satisfy the estimate*

$$\int_\Omega |F \cdot (\rho_\varepsilon * \nu)| \, dx \le \|F\|_{L^\infty(\Omega;\mathbb{R}^n)} \, |\nu|(\Omega) \tag{1.1.3}$$

for $\varepsilon > 0$ and any weak limit point $(F, \nu) \in \mathcal{M}(\Omega)$ satisfies $|(F, \nu)| \le \|F\|_{L^\infty(\Omega;\mathbb{R}^n)} |\nu|$.*

We call any weak* limit (F, ν) constructed as in Lemma 1.1.3 the *pairing measure* between the vector field F and the vector valued Radon mesure ν.

Proof. Let $\phi \in C_c(\Omega)$. It is not difficult to see that

$$\begin{aligned}\int_\Omega \phi(x) F(x) \cdot (\rho_\varepsilon * \nu)(x) \, dx &= \int_\Omega \int_\Omega \phi(x) \rho_\varepsilon(x - y) F(x) \cdot \, d\nu(y) \, dx \\ &= \int_\Omega (\rho_\varepsilon * (\phi F))(y) \cdot \, d\nu(y).\end{aligned}$$

This implies that

$$\left| \int_\Omega \phi F \cdot (\rho_\varepsilon * \nu) \, dx \right| \le \|\rho_\varepsilon * (\phi F)\|_{L^\infty(\Omega;\mathbb{R}^n)} |\nu|(\Omega) \le \|\phi\|_{L^\infty(\Omega)} \|F\|_{L^\infty(\Omega;\mathbb{R}^n)} |\nu|(\Omega),$$

and therefore the sequence $F \cdot (\rho_\varepsilon * \nu)\mathscr{L}^n$ satisfies (1.1.3). This means that there exists a weakly* converging subsequence $F \cdot (\rho_{\varepsilon_k} * \nu)\mathscr{L}^n$, whose limit we denote by (F, ν). Hence, for any $\phi \in C_c(\Omega)$ we obtain

$$\begin{aligned}\left| \int_\Omega \phi \, d(F, \nu) \right| &= \lim_{\varepsilon_k \to 0} \left| \int_\Omega \phi \, F \cdot (\rho_\varepsilon * \nu) \, dx \right| \le \lim_{\varepsilon_k \to 0} \|F\|_{L^\infty(\Omega;\mathbb{R}^n)} \int_\Omega |\phi| |\rho_{\varepsilon_k} * \nu| \, dx \\ &\le \lim_{\varepsilon_k \to 0} \|F\|_{L^\infty(\Omega;\mathbb{R}^n)} \int_\Omega |\phi| (\rho_\varepsilon * |\nu|) \, dx = \|F\|_{L^\infty(\Omega;\mathbb{R}^n)} \int_\Omega |\phi| \, d|\nu|,\end{aligned}$$

since $(\rho_\varepsilon * |\nu|)\mu \rightharpoonup |\nu|$ by Remark 1.2.12. This concludes our proof. $\square$

Remark 1.1.4. We stress the fact that the pairing (F,ν) introduced in Lemma 1.1.3 is not unique, a priori. However, if $\nu \ll \mathscr{L}^n$, we have $\nu = G\,\mathscr{L}^n$, for some vector field $G \in L^1(\Omega;\mathbb{R}^n)$, and so it is easy to see that

$$\lim_{\varepsilon\to 0}\int_\Omega \phi\, F\cdot(\rho_\varepsilon * \nu)\,dx = \lim_{\varepsilon\to 0}\int_\Omega \phi\, F\cdot(\rho_\varepsilon * G)\,dx = \int_\Omega \phi F\cdot G\,dx,$$

for any $\phi \in C_c(\Omega)$. This implies $(F,\nu) = F\cdot G\mathscr{L}^n$, which yields the uniqueness of the pairing in the case of absolutely continuous measures.

1.1.1 Functions of bounded variations and sets of finite perimeter

In this section we recall some basic definitions and results in the theory of functions of bounded variation and sets of finite perimeter, known as Caccioppoli sets. In particular, we will make use of elements in the structure theory of sets of finite perimeter as developed by De Giorgi [64] and Federer [71] (see also the manuscript of Federer [72]). We follow mainly the treatment of the monographs [11, 69, 111].

Definition 1.1.5. Let $\Omega \subset \mathbb{R}^n$ be open.

a) A function $u \in L^1(\Omega)$ is said to be of *bounded variation in* Ω if the distributional gradient Du is a finite $\mathbb{R}^n$-vector valued Radon measure on Ω; that is,

$$|Du|(\Omega) := \sup\left\{\int_\Omega u\,\mathrm{div}\phi\,dx : \phi\in C^1_c(\Omega;\mathbb{R}^n), |\phi|\le 1\right\} < \infty. \tag{1.1.4}$$

The space of all such functions is denoted by $BV(\Omega)$. Analogously, we say that u is of *locally of bounded variation in* Ω if, for every open set $W \Subset \Omega$, we have $u\in BV(W)$; the space of all such functions is denoted by $BV_{\mathrm{loc}}(\Omega)$.

b) A measurable set $E\subset\Omega$ is said to be a *set of locally finite perimeter in* Ω (or is a locally *Caccioppoli set*) if $\chi_E \in BV_{\mathrm{loc}}(\Omega)$. For any open set $U \Subset \Omega$, we denote the *perimeter of* E in U by

$$\mathsf{P}(E,U) := |D\chi_E|(U).$$

We say that E is a set of *finite perimeter* in Ω if $|D\chi_E|$ is a finite Radon measure on Ω.

Thanks to the Radon-Nikodým theorem, for any $u\in BV(\Omega)$ we have the following decomposition of the weak gradient:

$$Du = D^{\mathrm{a}}u + D^{\mathrm{s}}u = \nabla u\mathscr{L}^n + D^{\mathrm{s}}u,$$

where $D^{\mathrm{a}}u$ denotes the absolutely continuous part of Du, with density ∇u, and $D^{\mathrm{s}}u$ the singular part.

It is not difficult to see that $W^{1,1}(\Omega)\subset BV(\Omega)$, since it corresponds to the case $(Du)^{\mathrm{s}} = 0$, and indeed some properties of the Sobolev space extend to the functions of bounded variation. In particular, the Poincaré inequality holds also in BV, and it implies the linear form of the relative relative isoperimetric inequality for sets of finite perimeter (see for instance [11, Theorem 3.44]).

Theorem 1.1.6. *Let Ω be an open bounded connected set with Lipschitz boundary. Then there exists a constant $\kappa = \kappa_\Omega > 0$ such that*

$$\|u - u_\Omega\|_{L^1(\Omega)} \le \kappa |Du|(\Omega)$$

for any $u\in BV(\Omega)$. In particular, if E is a set of finite perimeter in Ω and $\gamma = \kappa/|\Omega|$, we have

$$\frac{|\Omega\cap E||\Omega\setminus E|}{|\Omega|^2} \le \gamma\mathsf{P}(E,\Omega). \tag{1.1.5}$$

From the definition, if E is a set of locally finite perimeter in Ω, then $D\chi_E$ is an $\mathbb{R}^n$-vector valued Radon measure on Ω whose total variation is $|D\chi_E|$. By the polar decomposition of measures ([11, Corollary 1.29]), one can write $D\chi_E = \nu_E|D\chi_E|$, where ν_E is a $|D\chi_E|$-measurable function such that $|\nu_E(x)| = 1$ for $|D\chi_E|$-a.e. $x \in \Omega$.

Important examples of sets of finite perimeter in Ω are open bounded sets $U \Subset \Omega$ such that $\mathscr{H}^{n-1}(\partial U) < \infty$ or ∂U is Lipschitz. In this second case, it is possible to show that

$$|D\chi_U| = \mathscr{H}^{n-1} \llcorner \partial U, \tag{1.1.6}$$

as is known from the work of Federer (see [11, Proposition 3.62], for example).

While (1.1.6) says that $|D\chi_U|$ is concentrated on the topological boundary of a bounded Lipschitz domain U, this does not happen in general. Indeed, the topological boundary of a bounded set of finite perimeter E can be very irregular, including the possibility of having positive Lebesgue measure. On the other hand, De Giorgi [64] discovered a suitable subset of ∂E of finite $\mathscr{H}^{n-1}$-measure on which $|D\chi_E|$ is concentrated if E has finite perimeter in Ω.

Definition 1.1.7. Let E be a measurable subset of $\mathbb{R}^n$ and let Ω be the largest open subset for which E is of locally finite perimeter in Ω. The *reduced boundary* of E, denoted by $\mathscr{F}E$, is defined as the set of all $x \in \operatorname{supp}(|D\chi_E|) \cap \Omega$ such that the limit

$$\nu_E(x) := \lim_{r\to 0} \frac{D\chi_E(B(x,r))}{|D\chi_E|(B(x,r))} \tag{1.1.7}$$

exists in $\mathbb{R}^n$ and satisfies

$$|\nu_E(x)| = 1. \tag{1.1.8}$$

The function $\nu_E : \mathscr{F}E \to \mathbb{S}^{n-1}$ is called the *measure theoretic unit interior normal* to E.

A precise justification for calling ν_E a generalized interior normal comes from *De Giorgi's blow-up analysis* of E around a point of $\mathscr{F}E$ illustrated in Theorem 1.1.10 below. First, we need to recall the definitions of rectifiable set and approximated tangent space.

Definition 1.1.8. Let $k \in [0, n]$ be an integer and let $S \subset \mathbb{R}^n$ be a $\mathscr{H}^k$-measurable set. We say that S is countably k-rectifiable if there exist countably many Lipschitz functions $f_i : \mathbb{R}^k \to \mathbb{R}^n$ such that

$$S \subset \bigcup_i f_i(\mathbb{R}^k).$$

Definition 1.1.9. Let $k \in [0, n]$ be an integer, μ be a Radon measure in Ω and $x \in \Omega$. We say that the approximate tangent space of μ is a k-plane π with multiplicity $\theta \in \mathbb{R}$ at x, and we write

$$\operatorname{Tan}^k(\mu, x) = \theta \mathscr{H}^k \llcorner \pi,$$

if $r^{-k}\mu_{x,r}$ locally weak* converges to $\theta\mathscr{H}^k \llcorner \pi$ in $\mathbb{R}^n$ as $r \to 0$; that is,

$$\lim_{r\to 0} \frac{1}{r^k} \int_\Omega \phi\left(\frac{y-x}{r}\right) d\mu(y) = \int_\pi \phi(y)\, d\mathscr{H}^k(y)$$

for any $\phi \in C_c(\mathbb{R}^n)$.

Theorem 1.1.10. *Let E be a set of locally finite perimeter in $\mathbb{R}^n$. Then $\mathscr{F}E$ is countably $(n-1)$-rectifiable and we have*

$$|D\chi_E| = \mathscr{H}^{n-1} \llcorner \mathscr{F}E. \tag{1.1.9}$$

In addition, for any $x \in \mathscr{F}E$,

$$\operatorname{Tan}^{n-1}(|D\chi_E|, x) = \mathscr{H}^{n-1} \llcorner \nu_E^\perp(x), \tag{1.1.10}$$

and the following convergence results hold:

$$\frac{E-x}{\varepsilon} \to H^+_{\nu_E}(x) := \{y \in \mathbb{R}^n : y \cdot \nu_E(x) \geq 0\} \quad in \quad L^1_{\mathrm{loc}}(\mathbb{R}^n) \quad as \quad \varepsilon \to 0 \tag{1.1.11}$$

and

$$\frac{(\mathbb{R}^n \setminus E)-x}{\varepsilon} \to H^-_{\nu_E}(x) := \{y \in \mathbb{R}^n : y \cdot \nu_E(x) \leq 0\} \quad in \quad L^1_{\mathrm{loc}}(\mathbb{R}^n) \quad as \quad \varepsilon \to 0. \tag{1.1.12}$$

For the proof of this result we refer to [11, Theorem 3.59].

Remark 1.1.11. Thanks to Whitney's extension theorem ([69, Theorem 6.10]), it is actually possible to show that $\mathscr{H}^k$-almost all of a k-rectifiable set can be covered by a sequence of C^1 k-graphs. In particular, if E is a set of finite perimeter, then there exist a sequence of C^1 hypersurfaces Γ_i whose union covers $\mathscr{H}^{n-1}$-almost all of $\mathscr{F}E$ and such that $\nu_E|_{\Gamma_i}$ is the interior normal of the subgraph of Γ_i.

One of the many consequences of Theorem 1.1.10 is the extension of the classical Gauss–Green formula to the sets of finite perimeter: indeed, thanks to the definition of weak gradient, the polar decomposition and (1.1.9), one can see that

$$\int_E \mathrm{div}\phi \, dx = -\int_{\mathscr{F}E} \phi \cdot \nu_E \, d\mathscr{H}^{n-1}, \tag{1.1.13}$$

for any $\phi \in C^1_c(\Omega; \mathbb{R}^n)$.

Crucial to the calculus on sets of finite perimeter E in Ω is Federer's structure theorem. For any measurable set $E \subset \Omega$ and for any $\alpha \in [0,1]$ define the subsets

$$E^\alpha := \{x \in \Omega : \theta(E,x) = \alpha\}, \tag{1.1.14}$$

where

$$\theta(E,x) := \lim_{r \to 0} \frac{|B(x,r) \cap E|}{|B(x,r)|}, \tag{1.1.15}$$

is the *Lebesgue density* of x in E. One calls E^1 and E^0 the *measure theoretic interior* and *exterior* of E in Ω, respectively, while $\partial^* E := \Omega \setminus (E^0 \cup E^1)$ is called the *measure theoretic boundary of* E in Ω.

Theorem 1.1.12 (Federer's structure theorem)**.** *If E has finite perimeter in Ω, then we have*

$$\mathscr{F}E \subset E^{1/2} \subset \partial^* E \tag{1.1.16}$$

and there exists a subset $\mathcal{N}_E$ with $\mathscr{H}^{n-1}(\mathcal{N}_E) = 0$ such that

$$\Omega = E^1 \cup \mathscr{F}E \cup E^0 \cup \mathcal{N}_E. \tag{1.1.17}$$

For a proof, we refer to [11, Theorem 3.61].

Remark 1.1.13. An easy consequence of Theorem 1.1.12 is that $\mathscr{H}^{n-1}(\partial^* E \setminus \mathscr{F}E) = 0$, so that we can integrate indifferently over $\mathscr{F}E$ or $\partial^* E$ with respect to the Hausdorff measure $\mathscr{H}^{n-1}$. In addition, E has density $0, 1/2$ or 1 in Ω at $\mathscr{H}^{n-1}$-a.e. $x \in E$.

As for the fine properties of general BV functions, we recall a well-known result on the existence of the precise representative, for which we refer for instance to [11, Corollary 3.80].

Definition 1.1.14. Let $u \in L^1_{\rm loc}(\Omega)$. The *precise representative* u^* of u is defined by

$$u^*(x) := \begin{cases} \lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y)\, dy & \text{if this limit exists} \\ 0 & \text{otherwise} \end{cases} \tag{1.1.18}$$

Theorem 1.1.15. *If $u \in BV(\Omega)$, then*

$$u^*(x) = \lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y)\, dy$$

for $\mathscr{H}^{n-1}$-a.e. $x \in \Omega$. In addition, if we set $u_\varepsilon := \rho_\varepsilon * u$ in $\Omega^\varepsilon := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$, for any nonnegative radially symmetric mollifier $\rho \in C^\infty_c(B(0,1))$ with $\int_{B(0,1)} \rho\, dx = 1$, we have

$$u_\varepsilon(x) \to u^*(x) \quad \text{for } \mathscr{H}^{n-1}\text{-a.e. } \; x \in \Omega. \tag{1.1.19}$$

We conclude this subsection with the needed properties of mollifying characteristic functions of sets of finite perimeter.

Lemma 1.1.16. *Let $E \subset \Omega$ be a set of locally finite perimeter in Ω and $\rho \in C^\infty_c(B(0,1))$ be a nonnegative radially symmetric mollifier such that $\int_{B(0,1)} \rho\, dx = 1$. Then, the following results hold:*

1. *there is a set $\mathcal{N}$ with $\mathscr{H}^{n-1}(\mathcal{N}) = 0$ such that, for all $x \in \Omega \setminus \mathcal{N}$, $(\rho_\varepsilon * \chi_E)(x) \to \chi_E^*(x)$ where*
$$\chi_E^*(x) = \begin{cases} 1 & \text{if } x \in E^1 \\ \frac{1}{2} & \text{if } x \in \mathscr{F}E\,; \\ 0 & \text{if } x \in E^0 \end{cases} \tag{1.1.20}$$

2. *$\rho_\varepsilon * \chi_E \in C^\infty(\Omega^\varepsilon)$ and $\nabla(\rho_\varepsilon * \chi_E)(x) = (\rho_\varepsilon * D\chi_E)(x)$ for any $x \in \Omega^\varepsilon$;*

3. *one has the following weak* limits in $\mathcal{M}_{\rm loc}(\Omega; \mathbb{R}^n)$:*
   (a) *$\nabla(\rho_\varepsilon * \chi_E) \rightharpoonup D\chi_E$;*
   (b) *$\chi_E \nabla(\rho_\varepsilon * \chi_E) \rightharpoonup (1/2) D\chi_E$;*
   (c) *$\chi_{\Omega\setminus E} \nabla(\rho_\varepsilon * \chi_E) \rightharpoonup (1/2) D\chi_E$;*

Proof. For the pointwise convergence of point (1), we notice that, since the pointwise convergerce is a local property, we may assume without loss of generality that E is a set of finite perimeter in Ω. Theorem 1.1.15 implies that, for $\mathscr{H}^{n-1}$-a.e. $x \in \Omega$, $(\rho_\varepsilon * \chi_E)(x) \to \chi_E^*(x)$ and

$$\chi_E^*(x) = \lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} \chi_E(y) dy = \theta(E, x),$$

where $\theta(E,x)$ is the Lebesgue density defined in (1.1.15). It follows that $\chi_E^*(x) = 1, 0$ if $x \in E^1, E^0$ respectively. Moreover, by (1.1.16), we see that $\chi_E^*(x) = \frac{1}{2}$ if $x \in \mathscr{F}E$. Finally, thanks to Theorem 1.1.12, we conclude that $\theta(E,x)$ is well defined $\mathscr{H}^{n-1}$-a.e. in Ω.

The smoothness of $\rho_\varepsilon * \chi_E$ is a well known property of the mollification. In order to prove the commutation of point (2), let $\phi \in C^1_c(\Omega^\varepsilon; \mathbb{R}^n)$. Then, it is easy to see that

$$\begin{aligned} \int_\Omega \phi \cdot \nabla(\rho_\varepsilon * \chi_E)\, dx &= -\int_\Omega (\rho_\varepsilon * \chi_E) \operatorname{div}\phi\, dx = -\int_\Omega \chi_E \operatorname{div}(\rho_\varepsilon * \phi)\, dx \\ &= \int_\Omega (\rho_\varepsilon * \phi) \cdot \, dD\chi_E = \int_\Omega \phi \cdot (\rho_\varepsilon * D\chi_E)\, dx. \end{aligned}$$

Since ϕ is arbitrary, the result is proved.

For the weak* limit (a) of point (3), since $(\rho_\varepsilon * \chi_E) \to \chi_E$ in $L^1_{\mathrm{loc}}(\Omega)$, one has

$$\int_\Omega \nabla(\rho_\varepsilon * \chi_E) \cdot \phi \, dx = -\int_\Omega (\rho_\varepsilon * \chi_E) \mathrm{div}\phi \, dx \to -\int_\Omega \chi_E \mathrm{div}\phi \, dx = \int_\Omega \phi \cdot dD\chi_E$$

for each $\phi \in C^1_c(\Omega; \mathbb{R}^n)$. Consequently, one has the limit (a) in the sense of $\mathbb{R}^n$-vector valued Radon measures, by the density of $C^1_c(\Omega; \mathbb{R}^n)$ in $C_c(\Omega; \mathbb{R}^n)$ with respect to the supremum norm.

In order to show limit (b), consider $\phi \in C^1_c(\Omega; \mathbb{R}^n)$ and notice that

$$\begin{aligned}\int_\Omega \phi \chi_E \cdot \nabla(\rho_\varepsilon * \chi_E) \, dx &= \int_\Omega \chi_E \mathrm{div}((\rho_\varepsilon * \chi_E)\phi) \, dx - \int_\Omega \chi_E (\rho_\varepsilon * \chi_E) \mathrm{div}\phi \, dx \\ &= -\int_\Omega \phi (\rho_\varepsilon * \chi_E) \cdot dD\chi_E - \int_\Omega \chi_E (\rho_\varepsilon * \chi_E) \mathrm{div}\phi \, dx.\end{aligned}$$

Now, let $\varepsilon \to 0$ and apply Lebesgue's dominated convergence theorem to the measures $|D\chi_E|$ and $\mathscr{L}^n$ and use point (1) in order to obtain

$$\begin{aligned}\lim_{\varepsilon \to 0} \int_\Omega \phi \chi_E \cdot \nabla(\rho_\varepsilon * \chi_E) \, dx &= -\int_\Omega \phi \chi_E^* \cdot dD\chi_E - \int_\Omega \chi_E^2 \mathrm{div}\phi \, dx \\ &= -\int_\Omega \frac{1}{2}\phi \cdot dD\chi_E - \int_\Omega \chi_E \mathrm{div}\phi \, dx \\ &= -\int_\Omega \frac{1}{2}\phi \cdot dD\chi_E + \int_\Omega \phi \cdot dD\chi_E\end{aligned}$$

since $\chi_E^* = \frac{1}{2}$ on $\mathscr{F}E$ and $|D\chi_E|(\Omega \setminus \mathscr{F}E) = 0$. Therefore, by the density of $C^1_c(\Omega; \mathbb{R}^n)$ in $C_c(\Omega; \mathbb{R}^n)$ with respect to the supremum norm, the claim (b) follows.

Finally, for the limit (c), observe that

$$\chi_{\Omega \setminus E} \nabla(\rho_\varepsilon * \chi_E) = \nabla(\rho_\varepsilon * \chi_E) - \chi_E \nabla(\rho_\varepsilon * \chi_E) \rightharpoonup \left(1 - \frac{1}{2}\right) D\chi_E$$

as $\varepsilon \to 0$ by combining the limits (a) and (b). □

1.1.2 Some notions of capacity theory

As is well known, the notion of capacity is very useful in the study of the fine properties of Sobolev functions and for Sobolev type inequalities for functions of bounded variation. We recall here a few well-known results which will play a key role in the proof of the Leibniz rule for p-summable divergence-measure fields in Chapter 3. The brief exposition here borrows from the monographs of [69, 101, 116, 119].

Definition 1.1.17. For $1 \le p \le n$ and a compact subset K of Ω we define the *p-capacity* of K relative to Ω as

$$\mathrm{cap}_p(K, \Omega) := \inf \left\{ \int_\Omega |\nabla \phi|^p \, dx : \phi \in C^\infty_c(\Omega), \phi \ge 1 \, \mathrm{on} \, K \right\}.$$

If $U \subset \Omega$ is open, we set

$$\mathrm{cap}_p(U, \Omega) := \sup\{\mathrm{cap}_p(K, \Omega) : K \subset U \text{ compact}\}$$

and, for an arbitrary set $A \subset \Omega$,

$$\mathrm{cap}_p(A, \Omega) := \inf\{\mathrm{cap}_p(U, \Omega) : A \subset U \subset \Omega, \, U \text{ open}\}.$$

If $\Omega = \mathbb{R}^n$, we write $\mathrm{cap}_p(A, \mathbb{R}^n) = \mathrm{cap}_p(A)$, for any set A.

It is possible to show that, for any compact subset K of Ω, Definition 1.1.17 is equivalent to

$$\mathrm{cap}_p(K,\Omega) := \inf\left\{\int_\Omega |\nabla\phi|^p\,dx : \phi\in C_c^\infty(\Omega), 0\le\phi\le 1, \{\phi=1\}^\circ\supset K\right\},$$

by an approximation argument one finds in [119, § 2.2.1, point (ii)].
It is also easy to see that

1. $\mathrm{cap}_p(A_1,\Omega)\le \mathrm{cap}_p(A_2,\Omega)$ if $A_1\subset A_2\subset\Omega$,
2. $\mathrm{cap}_p(A,\Omega_2)\le \mathrm{cap}_p(A,\Omega_1)$ if $\Omega_1\subset\Omega_2$, hence $\mathrm{cap}_p(A)\le\mathrm{cap}_p(A,\Omega)$.

Definition 1.1.18. For $1\le p\le n$ and for a set E we define the *p-Sobolev capacity* as

$$\mathbf{C}_p(E) := \inf\left\{\int_{\mathbb{R}^n} |\nabla\phi|^p + |\phi|^p\,dx : \phi\in C_c^\infty(\mathbb{R}^n), \{\phi\ge 1\}^\circ\supset E\right\}.$$

It is clear that $\mathbf{C}_p(E_1)\le\mathbf{C}_p(E_2)$ for any $E_1\subset E_2$.

We also have $\mathrm{cap}_p(E)\le\mathbf{C}_p(E)$ for any set E: indeed, $\mathrm{cap}_p(K)\le\mathbf{C}_p(K)$ for any compact K, hence $\mathrm{cap}_p(U)\le\mathbf{C}_p(U)$ for any open set U, which easily implies the inequality for a general set.

Following the notation of [101], we say that a set E in $\mathbb{R}^n$ has zero p-capacity if

$$\mathrm{cap}_p(E\cap\Omega,\Omega)=0\ \ \forall\Omega \text{ open}.$$

Lemma 1.1.19. *If $1\le p<n$ and if $\mathrm{cap}_p(E,\Omega)=0$ for an open set $\Omega\supset E$, then $\mathrm{cap}_p(E,\Omega')=0$ for any bounded open set Ω' satisfying $E\subset\Omega'\Subset\Omega$.*

Proof. By the definition of capacity, it is enough to prove the statement for a compact $K\subset\Omega$. Let Ω' be such that $K\subset\Omega'\Subset\Omega$. We take $\phi\in C_c^\infty(\Omega), 0\le\phi\le 1, \{\phi=1\}^\circ\supset K$ and $\psi\in C_c^\infty(\Omega'), 0\le\psi\le 1, \{\psi=1\}^\circ\supset K$, then $\phi\psi\in C_c^\infty(\Omega'), 0\le\phi\psi\le 1, \{\phi\psi=1\}^\circ\supset K$. Thus

$$\begin{aligned}\mathrm{cap}_p(K,\Omega') &\le \int_{\Omega'}|\nabla(\phi\psi)|^p\,dx \le 2^p\left(\int_\Omega|\nabla\phi|^p\,dx + \|\nabla\psi\|^p_{L^\infty(\Omega;\mathbb{R}^n)}\int_{\Omega'}|\phi|^p\,dx\right)\\ &\le 2^p\left(\int_\Omega|\nabla\phi|^p\,dx + \|\nabla\psi\|^p_{L^\infty(\Omega;\mathbb{R}^n)}|\Omega'|^{\frac{p}{n}}\|\phi\|^p_{L^{p^*}(\Omega)}\right)\le C(\nabla\psi,\Omega',p)\int_\Omega|\nabla\phi|^p\,dx,\end{aligned}$$

by Gagliardo-Nirenberg-Sobolev inequality. Passing to the infimum in ϕ, we obtain the desired result, since $\mathrm{cap}_p(K,\Omega)=0$. □

Lemma 1.1.20. *If $1\le p<n$, E is a bounded set and there exists an open set $\Omega\supset E$ such that $\mathrm{cap}_p(E,\Omega)=0$, then E has zero p-capacity.*

Proof. By Lemma 1.1.19, we can assume Ω to be bounded. Then the result follows from [101, Lemma 2.9] in the case $p>1$, while the case $p=1$ can be proved easily with a similar argument. □

Lemma 1.1.21. *For $1\le p\le n$, $\mathbf{C}_p(E)=0$ if and only if E has zero p-capacity; that is, $\mathrm{cap}_p(E\cap\Omega,\Omega)=0$ for any open set Ω.*

Proof. See [101, Corollary 2.39], the case $p=1$ follows easily by the same techniques. □

We recall now two important results on the relations between the Hausdorff measures, the capacities and the Sobolev spaces.

Theorem 1.1.22. *If $1<p\le n$ and E is a Borel set such that $\mathscr{H}^{n-p}(E)<\infty$, then $\mathbf{C}_p(E)=0$. For $p=1$, we have that $\mathscr{H}^{n-1}(E)=0$ if and only if $\mathbf{C}_1(E)=0$.*

Proof. For the first part of the statement, we refer to [116, Theorem 2.52]. The second part follows from [96, Theorem 4.4, Theorem 5.1]. □

In what follows, we denote by ρ a smooth radially symmetric mollifier $\rho \in C_c^\infty(B(0,1))$, with $\rho \geq 0$ and $\int_{B(0,1)} \rho \, dx = 1$, and we set $\rho_\varepsilon(x) := \varepsilon^{-n}\rho(x/\varepsilon)$.

Definition 1.1.23. A function u is called *p-quasicontinuous* on Ω if for any $\varepsilon > 0$ there exists an open set V with $\mathbf{C}_p(V) < \varepsilon$ such that u restricted to $\Omega \setminus V$ is bounded and continuous. We say that a property holds *p-quasi everywhere* (or at *p-quasi every point*) if it holds expect for a set of zero Sobolev capacity.

Theorem 1.1.24. *Let $p \in [1, n]$ and $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, then u^* is a p-quasicontinuous representative of u and*

$$\lim_{r\to 0} \fint_{B(x,r)} |u(y) - u^*(x)| \, dy = 0 \tag{1.1.21}$$

at p-quasi every $x \in \Omega$; that is, there exists a set Z with $\mathbf{C}_p(Z) = 0$ such that (1.1.21) *holds for any $x \in \Omega \setminus Z$. In particular, $(u * \rho_\varepsilon)(x) \to u^*(x)$ at p-quasi every $x \in \Omega$.*

Proof. For the proof of the case $p \in (1, n]$ we refer to [116, Theorem 2.55], where it is assumed $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$, however this result is clearly of local nature, hence it is valid also for $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$. For $p = 1$ we refer to [69, Theorem 4.19], observing again that this statement is local and that clearly $\mathbf{C}_1(Z) = 0$ implies $\mathrm{cap}_1(Z) = 0$.

Then, (1.1.21) implies easily that

$$|(u * \rho_\varepsilon)(x) - u^*(x)| \leq \|\rho\|_{L^\infty(B(0,1))}\omega_n \fint_{B(x,\varepsilon)} |u(y) - u^*(x)| \, dy \to 0.$$

□

1.2 Differentiation and intrinsic convolution in stratified groups

[1] In this section we recall the main features of the stratified groups, also well known as Carnot groups. For a general theory on these groups we refer for instance to [75,76,100]. In particular, we are focusing ourselves on the notion of differentiation and intrinsic convolution. At the end of the section we provide an original approximation results for intrinsic Lipschitz functions.

1.2.1 Basic facts on stratified groups

A *stratified group* can be seen as a linear space $\mathbb{G}$ equipped with an analytic group operation such that its Lie algebra $\mathrm{Lie}(\mathbb{G})$ is *stratified*. This assumption on $\mathrm{Lie}(\mathbb{G})$ corresponds to the following conditions:

$$\mathrm{Lie}(\mathbb{G}) = \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_\iota, \qquad [\mathcal{V}_1, \mathcal{V}_j] = \mathcal{V}_{j+1}$$

for all integers $j \geq 0$ and $\mathcal{V}_j = \{0\}$ for all $j > \iota$ with $\mathcal{V}_\iota \neq \{0\}$. The integer ι is the step of nilpotence of $\mathbb{G}$. The tangent space $T_0\mathbb{G}$ can be canonically identified with $\mathrm{Lie}(\mathbb{G})$ by associating to each $v \in T_0\mathbb{G}$ the unique left invariant vector field $X \in \mathrm{Lie}(\mathbb{G})$ such that $X(0) = v$. This allows for transferring the Lie algebra structure from $\mathrm{Lie}(\mathbb{G})$ to $T_0\mathbb{G}$. We can further simplify the structure of $\mathbb{G}$ by identifying it with $T_0\mathbb{G}$, hence having a Lie product on $\mathbb{G}$, that yields

[1] This section is based on a joint work with Valentino Magnani [51].

the group operation by the Baker-Campbell-Hausdorff formula. This identification also gives a graded structure to $\mathbb{G}$, obtaining the subspaces H^j of $\mathbb{G}$ from the subspaces

$$\{v \in T_0\mathbb{G} : v = X(0),\ X \in \mathcal{V}_j\},$$

therefore getting $\mathbb{G} = H^1 \oplus \cdots \oplus H^\iota$. By these assumptions the exponential mapping

$$\exp : \mathrm{Lie}(\mathbb{G}) \to \mathbb{G}$$

is somehow the "identity mapping" $\exp X = X(0)$. It is clearly a bianalytic diffeomorphism. We will denote by q the dimension of $\mathbb{G}$, seen as a vector space. Those dilations that are compatible with the algebraic structure of $\mathbb{G}$ are defined as linear mappings $\delta_r : \mathbb{G} \to \mathbb{G}$ such that $\delta_r(p) = r^i p$ for each $p \in H^i$, $r > 0$ and $i = 1, \ldots, \iota$.

1.2.2 Metric structure, distances and graded coordinates

We may use a graded basis to introduce a natural scalar product on a stratified group $\mathbb{G}$. We then define the unique scalar product on $\mathbb{G}$ such that the graded basis is orthonormal.

We will denote by $|\cdot|$ the associated Euclidean norm, that exactly becomes the Euclidean norm with respect to the corresponding graded coordinates.

On the other hand, the previous identification of $\mathbb{G}$ with $T_0\mathbb{G}$ yields a scalar product on $T_0\mathbb{G}$, that defines by left translations a left invariant Riemannian metric on $\mathbb{G}$. By a slight abuse of notation, we use the symbols $|\cdot|$ and $\langle\cdot,\cdot\rangle$ to denote the norm arising from this left invariant Riemannian metric and its corresponding scalar product. By $\langle\cdot,\cdot\rangle_{\mathbb{R}^{\mathrm{q}}}$ we will denote the Euclidean scalar product, that makes the fixed graded basis $(e_1, \ldots, e_{\mathrm{q}})$ orthonormal.

Notice that the basis $(X_1, \ldots, X_{\mathrm{q}})$ of $\mathrm{Lie}(\mathbb{G})$ associated to our graded basis is clearly orthonormal with respect to the same left invariant Riemannian metric.

A *homogeneous distance*

$$d : \mathbb{G} \times \mathbb{G} \to [0, +\infty)$$

on a stratified group $\mathbb{G}$ is a continuous and left invariant distance with

$$d(\delta_r(p), \delta_r(q)) = r\, d(p, q)$$

for all $p, q \in \mathbb{G}$ and $r > 0$. We define the open balls as

$$B(p, r) = \Big\{q \in \mathbb{G} : d(q, p) < r\Big\}.$$

The corresponding *homogeneous norm* will be denoted by $\|x\| = d(x, 0)$ for all $x \in \mathbb{G}$. It is worth to compare d with our fixed Euclidean norm on $\mathbb{G}$, getting

$$C^{-1}|x - y| \le d(x, y) \le C|x - y|^{1/\iota} \tag{1.2.1}$$

on compact sets of $\mathbb{G}$. A homogeneous distance also defines a Hausdorff measure $\mathscr{H}^\alpha$ and a spherical measure $\mathscr{S}^\alpha$. As it is customary, we set for $\delta > 0$ and $A \subset \mathbb{G}$:

$$\mathscr{H}^\alpha_\delta(A) := \inf\left\{\sum_{j\in J}\left(\frac{\mathrm{diam}A_j}{2}\right)^\alpha \ :\ \mathrm{diam}A_j < \delta,\ A \subset \bigcup_{j\in J} A_j\right\},$$

$$\mathscr{S}^\alpha_\delta(A) := \inf\left\{\sum_{j\in J} r_j^\alpha\ :\ 2r_j < \delta,\ A \subset \bigcup_{j\in J} B(x_j, r_j)\right\}$$

and we take the following suprema

$$\mathscr{H}^\alpha(A) := \sup_{\delta>0}\mathscr{H}^\alpha_\delta(A) \quad \text{and} \quad \mathscr{S}^\alpha(A) := \sup_{\delta>0}\mathscr{S}^\alpha_\delta(A).$$

It will be useful to introduce the right invariant distance $d^{\mathcal{R}}$ associated to d as follows

$$d^{\mathcal{R}}(x,y) := \|xy^{-1}\| = d(xy^{-1},0) = d(x^{-1},y^{-1}). \tag{1.2.2}$$

It is not difficult to check that $d^{\mathcal{R}}$ is a continuous and right invariant distance, that is also homogeneous, namely

$$d^{\mathcal{R}}(\delta_r x, \delta_r y) = r d^{\mathcal{R}}(x,y)$$

for $r>0$ and $x,y \in \mathbb{G}$. The local estimates (1.2.1) also show that $d^{\mathcal{R}}$ defines the same topology of both d and the Euclidean norm $|\cdot|$. The metric balls associated to $d^{\mathcal{R}}$ are

$$B^{\mathcal{R}}(p,r) = \Big\{ q \in \mathbb{G} : d^{\mathcal{R}}(q,p) < r \Big\}. \tag{1.2.3}$$

We notice that

$$B^{\mathcal{R}}(0,1) = B(0,1), \tag{1.2.4}$$

being $d^{\mathcal{R}}(x,0) = d(x^{-1},0) = d(0,x)$ for all $x \in \mathbb{G}$.

A basis $(e_1,\ldots,e_{\mathrm{q}})$ of $\mathbb{G}$ that respects the grading of $\mathbb{G}$ has the property that

$$(e_{\mathrm{m}_{j-1}+1}, e_{\mathrm{m}_{j-1}+2}, \ldots, e_{\mathrm{m}_j})$$

is a basis of H^j for each $j=1,\ldots,\iota$, where $\mathrm{m}_j = \sum_{i=1}^{j} \dim H^i$ for every $j=1,\ldots,\iota$, $\mathrm{m}_0 = 0$ and $\mathrm{m} = \mathrm{m}_1$. The basis $(e_1,\ldots,e_{\mathrm{q}})$ is then called *graded basis* of $\mathbb{G}$. Such basis provides the corresponding *graded coordinates* $x = (x_1,\ldots,x_{\mathrm{q}}) \in \mathbb{R}^{\mathrm{q}}$, that give the unique element of $\mathbb{G}$ that satisfies

$$p = \sum_{j=1}^{\mathrm{q}} x_j e_j \in \mathbb{G}.$$

We define a graded basis $(X_1,\ldots,X_{\mathrm{q}})$ of $\mathrm{Lie}(\mathbb{G})$ defining $X_j \in \mathrm{Lie}(\mathbb{G})$ as the unique left invariant vector field with $X_j(0) = e_j$ and $j=1,\ldots,\mathrm{q}$.

We assign *degree* i to each left invariant vector field of $\mathcal{V}_i$. In different terms, for each $j \in \{1,\ldots,\mathrm{q}\}$ we define the integer function d_j on $\{1,\ldots,\iota\}$ such that

$$\mathrm{m}_{d_j-1} < j \le \mathrm{m}_{d_j}.$$

The previous definitions allow to represent any left invariant vector field X_j as follows

$$X_j = \partial_{x_j} + \sum_{i:d_i>d_j}^{\mathrm{q}} a_j^i \partial_{x_i}, \tag{1.2.5}$$

where $j=1,\ldots,\mathrm{q}$ and a_j^i are suitable polynomials. The vector fields $X_1, X_2, \ldots, X_{\mathrm{m}}$ of degree one, are the so-called *horizontal left invariant vector fields* and constitute the horizontal left invariant frame of $\mathbb{G}$.

Using graded coordinates, the dilation of $x \in \mathbb{R}^{\mathrm{q}}$ is given by

$$\delta_r(x) = \sum_{j=1}^{\mathrm{q}} r^{d_j} x_j e_j.$$

Through the identification of $\mathbb{G}$ with $T_0\mathbb{G}$, it is also possible to write explicitly the group product in the graded coordinates. In the sequel, an auxiliary scalar product on $\mathbb{G}$ is fixed such that our fixed graded basis is orthonormal. The restriction of this scalar product to V_1 can be translated to the so-called *horizontal fibers*

$$H_p\mathbb{G} = \{X(p) \in T_p\mathbb{G} : X \in \mathcal{V}_1\}$$

as p varies in $\mathbb{G}$, hence defining a left invariant sub-Riemannian metric g on $\mathbb{G}$. We denote by $H\mathbb{G}$ the *horizontal subbundle* of $\mathbb{G}$, whose fibers are $H_x\mathbb{G}$.

The Hausdorff dimension Q of the stratified group $\mathbb{G}$ with respect to any homogeneous distance is given by the formula

$$Q = \sum_{i=1}^{\iota} i \dim(H^i).$$

We fix a Haar measure μ on $\mathbb{G}$, that with respect to our graded coordinates becomes the standard q-dimensional Lebesgue measure $\mathscr{L}^{\mathrm{q}}$. Because of this identification, we shall write dx instead of $d\mu(x)$ in the integrals. This measure defines the corresponding Lebesgue spaces $L^p(A)$ and $L^p_{\mathrm{loc}}(A)$ for any measurable set $A \subset \mathbb{G}$. The L^p-norm will be denoted using the same symbols we will use for horizontal vector fields in Definition 1.3.1.

For any measurable set $E \subset \mathbb{G}$, we have $\mu(xE) = \mu(E)$ for any $x \in \mathbb{G}$ and

$$\mu(\delta_\lambda E) = \lambda^Q \mu(E) \quad \text{for any } \lambda > 0.$$

Since $B(p,r) = p\, \delta_r B(0,1)$ and $B^{\mathcal{R}}(p,r) = \delta_r(B(0,1))p$, we get

$$\mu(B(p,r)) = r^Q \mu(B(0,1)) \quad \text{and} \quad \mu(B^{\mathcal{R}}(p,r)) = r^Q \mu(B(0,1)) \tag{1.2.6}$$

due to the left and right invariance of the Haar measure μ. The previous formulas show the existence of constants $c_1, c_2 > 0$ such that

$$\mathscr{H}^Q = c_1\, \mathscr{H}^Q_{\mathcal{R}} = c_2\, \mu, \tag{1.2.7}$$

where $\mathscr{H}^Q$ and $\mathscr{H}^Q_{\mathcal{R}}$ are the Hausdorff measures with respect to d and $d^{\mathcal{R}}$, respectively. In particular, (1.2.6) shows that μ is doubling with respect to both d and $d^{\mathcal{R}}$, hence the Lebesgue differentiation theorem holds with respect to μ and both distances d and $d^{\mathcal{R}}$.

Theorem 1.2.1. *Given $f \in L^1_{\mathrm{loc}}(\mathbb{G})$, we have*

$$\lim_{r\to 0} \fint_{B(x,r)} |f(y) - f(x)|\, dy = 0 \quad \textit{and} \quad \lim_{r\to 0} \fint_{B^{\mathcal{R}}(x,r)} |f(y) - f(x)|\, dy = 0,$$

for μ-a.e. $x \in \mathbb{G}$.

For a general proof of the previous theorem in metric measure spaces equipped with a doubling measure, we refer for instance to [22, Theorem 5.2.3].

1.2.3 Differentiability, local convolution and smoothing

The group structure and the intrinsic dilations naturally give a notion of "differential" and of "differentiability" made by the corresponding operations. A map $L : \mathbb{G} \to \mathbb{R}$ is a homogeneous homomorphism, in short, a h-homomorphism if it is a Lie group homomorphism such that $L \circ \delta_r = r\, L$. It can be proved that $L : \mathbb{G} \to \mathbb{R}$ is a h-homomorphism if and only if there exists $(a_1, \ldots, a_{m_1}) \in \mathbb{R}^{m_1}$ such that $L(x) = \sum_{j=1}^{m_1} a_j x_j$ with respect to our fixed graded coordinates. If not otherwise stated, in the following we denote by Ω an open set in $\mathbb{G}$.

Definition 1.2.2 (Differentiability)**.** We say that $f : \Omega \to \mathbb{R}$ is differentiable at $x_0 \in \Omega$ if there is an h-homomorphism $L : \mathbb{G} \to \mathbb{R}$ such that

$$\lim_{x\to x_0} \frac{f(x) - f(x_0) - L(x_0^{-1}x)}{d(x, x_0)} = 0.$$

If f is differentiable, then L is unique and we denote it simply by $df(x_0)$.

A weaker notion of differentiability, that holds for Sobolev and BV functions on groups is the following.

Definition 1.2.3 (Approximate differentiability)**.** We say that $f : \Omega \to \mathbb{R}$ is approximately differentiable at $x_0 \in \Omega$ if there is an h-homomorphism $L : \mathbb{G} \to \mathbb{R}$ such that

$$\lim_{r\to 0^+} \fint_{B(x_0,r)} \frac{|f(x) - f(x_0) - L(x_0^{-1}x)|}{r}\, dx = 0.$$

The function L is uniquely defined and it is called the approximate differential of f at x_0. The unique vector defining L with respect to the scalar product is denoted by $\nabla_H f(x_0)$.

Remark 1.2.4. When $\mathbb{G}$ is the Euclidean space, the simplest stratified group, Definition 1.2.2 and Definition 1.2.3 yield the standard notions of differentiability and approximate differentiability in Euclidean spaces.

We denote by $C^1_H(\Omega)$ the linear space of real-valued functions $f : \Omega \to \mathbb{R}$ such that the pointwise partial derivatives $X_1 f, \ldots, X_{\mathrm{m}} f$ are continuous in Ω. For any $f \in C^1_H(\Omega)$ we introduce the *horizontal gradient*

$$\nabla_H f := \sum_{j=1}^{\mathrm{m}} (X_j f) X_j, \tag{1.2.8}$$

whose components $X_j f$ are continuous functions in Ω. Taylor's inequality [76, Theorem 1.41] simply leads us to the everywhere differentiability of f and to the formula

$$df(x)(v) = \langle \nabla_H f(x), v\rangle = \sum_{j=1}^{\mathrm{m}} v_j X_j f(x)$$

for any $x \in \Omega$ and $v = \sum_{j=1}^{\mathrm{q}} v_j e_j \in \mathbb{G}$.

We denote by $\mathrm{Lip}(\Omega)$, $\mathrm{Lip}_{\mathrm{loc}}(\Omega)$ and $\mathrm{Lip}_c(\Omega)$ the spaces of Euclidean Lipschitz, locally Lipschitz and Lipschitz functions with compact support in Ω, respectively. Analogously, we can define the space of Lipschitz functions with respect to any homogeneous distance of the stratified group, $\mathrm{Lip}_H(\Omega)$. It is well known that $\mathrm{Lip}_{\mathrm{loc}}(\Omega) \subset \mathrm{Lip}_{H,\mathrm{loc}}(\Omega)$, due to the local estimate (1.2.1). In addition, analogously to the Euclidean Rademacher's theorem, a differentiation theorem for Lipschitz functions holds in stratified groups, and it was proved by Pansu, [130].

Theorem 1.2.5 (Pansu-Rademacher)**.** *If $f \in \mathrm{Lip}_{H,\mathrm{loc}}(\Omega)$, then f is differentiable μ-almost everywhere.*

This result follows also from a Rademacher's type theorem by Monti and Serra Cassano, proved in more general Carathéodory spaces, [124, Theorem 3.2].

Remark 1.2.6. From the standard Leibniz rule, if $f, g \in \mathrm{Lip}_{H,\mathrm{loc}}(\Omega)$, the definition of differentiability joined with Theorem 1.2.5 gives

$$\nabla_H(fg)(x) = f(x)\nabla_H g(x) + g(x)\nabla_H f(x) \quad \text{for } \mu\text{-a.e. } x.$$

The Haar measure on stratified groups allows for defining the convolution with respect to the group operation.

Definition 1.2.7 (Convolution)**.** For $f, g \in L^1(\mathbb{G})$, we define the *convolution* of f with g by the integral

$$(f * g)(x) := \int_{\mathbb{G}} f(y) g(y^{-1}x)\, dy = \int_{\mathbb{G}} f(xy^{-1}) g(y)\, dy,$$

that is well defined for μ-a.e. $x \in \mathbb{G}$, see for instance [76, Proposition 1.18].

Due to the noncommutativity of the group operation, one may clearly expect that $g * f$ differs from $f * g$, in general. This difference appears especially when we wish to localize the convolution. In the sequel, Ω denotes an open set, if not otherwise stated. For every $\varepsilon > 0$, two possibly empty open subsets of Ω are defined as follows

$$\Omega_\varepsilon^{\mathcal{R}} = \left\{x \in \mathbb{G} : \operatorname{dist}^{\mathcal{R}}(x, \Omega^c) > \varepsilon\right\} \quad \text{and} \quad \Omega_\varepsilon = \{x \in \mathbb{G} : \operatorname{dist}(x, \Omega^c) > \varepsilon\}, \tag{1.2.9}$$

where we have defined the distance functions

$$\operatorname{dist}^{\mathcal{R}}(x, A) = \inf\left\{d^{\mathcal{R}}(x, y) : y \in A\right\} \quad \text{and} \quad \operatorname{dist}(x, A) = \inf\{d(x, y) : y \in A\}$$

for an arbitrary subset $A \subset \mathbb{G}$. We finally define the open set

$$A^{\mathcal{R},\varepsilon} = \left\{x \in \mathbb{G} : \operatorname{dist}^{\mathcal{R}}(x, A) < \varepsilon\right\}.$$

Definition 1.2.8 (Mollification)**.** Given a function $\rho \in C_c(B(0,1))$, we set

$$\rho_\varepsilon(x) := \varepsilon^{-Q}\rho(\delta_{1/\varepsilon}(x))$$

for $\varepsilon > 0$. If $f \in L^1(\Omega)$ and $x \in \mathbb{G}$, we define

$$\rho_\varepsilon * f(x) = \int_\Omega \rho_\varepsilon(xy^{-1}) f(y)\, dy. \tag{1.2.10}$$

If we restrict the domain of this convolution considering $x \in \Omega_\varepsilon^{\mathcal{R}}$, then we can allow for $f \in L^1_{\mathrm{loc}}(\Omega)$ and we have

$$\rho_\varepsilon * f(x) = \int_{B^{\mathcal{R}}(x,\varepsilon)} \rho_\varepsilon(xy^{-1}) f(y)\, dy, \tag{1.2.11}$$

which is well posed since the map $y \to \rho_\varepsilon(xy^{-1})$ has compact support inside $B^{\mathcal{R}}(x, \varepsilon) \subset \Omega$. In addition, under these assumptions, a simple change of variables also yields

$$\rho_\varepsilon * f(x) = \int_{B(0,\varepsilon)} \rho_\varepsilon(y)\, f(y^{-1}x)\, dy = \int_{B(0,1)} \rho(y)\, f\left((\delta_\varepsilon y^{-1})x\right)\, dy. \tag{1.2.12}$$

Due to the noncommutativity, a different convolution may also be introduced

$$f * \rho_\varepsilon(x) = \int_\Omega f(y)\rho_\varepsilon(y^{-1}x)\, dy = \int_{B(x,\varepsilon)} \rho_\varepsilon(y^{-1}x) f(y)\, dy, \tag{1.2.13}$$

where the first integral makes sense for all $x \in \mathbb{G}$ and the second one only for $x \in \Omega_\varepsilon$.

It is not difficult to show that the mollified functions $\rho_\varepsilon * f$ and $f * \rho_\varepsilon$ enjoy many standard properties. For instance, $\rho_\varepsilon * f$ converges to f in $L^1_{\mathrm{loc}}(\Omega)$, whenever $f \in L^1_{\mathrm{loc}}(\Omega)$.

We may also define the convolution between a (signed) Radon measure and a continuous function. As it is customary, we denote by $\mathcal{M}_{\mathrm{loc}}(\Omega)$ the space of signed Radon measures on Ω, and by $\mathcal{M}(\Omega)$ the space of finite signed Radon measures on Ω.

Definition 1.2.9 (Local convolution of measures)**.** Let us consider two open sets $\Omega, U \subset \mathbb{G}$ and define the new open set $O = U(\Omega^{-1}) \subset \mathbb{G}$. Let $f \in C(O)$ and $\nu \in \mathcal{M}(\Omega)$. Then the *convolution between f and ν* is given by

$$(f * \nu)(x) := \int_\Omega f(xy^{-1})\, d\nu(y), \tag{1.2.14}$$

with the additional assumption that $\Omega \ni y \mapsto f(xy^{-1})$ is $|\nu|$-integrable for every $x \in U$. Thus, $f * \nu$ is well defined in U. If $\rho \in C_c(B(0,1))$, for any $x \in \Omega_\varepsilon^{\mathcal{R}}$ we may represent the convolution as follows

$$(\rho_\varepsilon * \nu)(x) = \int_\Omega \rho_\varepsilon(xy^{-1})\, d\nu(y) = \int_{B^{\mathcal{R}}(x,\varepsilon)} \rho_\varepsilon(xy^{-1})\, d\nu(y). \tag{1.2.15}$$

The first integral makes sense for all $x \in \mathbb{G}$, being ρ continuously extendable by zero outside $B(0,1)$. In addition, (1.2.15) is well posed also for $\nu \in \mathcal{M}_{\mathrm{loc}}(\Omega)$, if $x \in \Omega_{2\varepsilon}^{\mathcal{R}}$. The function $\rho_\varepsilon * \nu$ is the *mollification* of ν.

Definition 1.2.10 (Local weak* convergence)**.** We say that a family of Radon measures $\nu_\varepsilon \in \mathcal{M}(\Omega)$ *locally weakly* converges to* $\nu \in \mathcal{M}(\Omega)$, if for every $\phi \in C_c(\Omega)$ we have

$$\int_\Omega \phi \, d\nu_\varepsilon \to \int_\Omega \phi \, d\nu \quad \text{as} \quad \varepsilon \to 0^+ \tag{1.2.16}$$

and in this case we will use the symbols $\nu_\varepsilon \rightharpoonup \nu$ as $\varepsilon \to 0^+$.

Remark 1.2.11. In the sequel, the local weak* convergence above will also refer to measures $\nu_\varepsilon \in \mathcal{M}(\Omega^\varepsilon)$ defined on a family of increasing open sets $\Omega^\varepsilon \subset \Omega$ as ε decreases, such that $\bigcup_{\varepsilon>0} \Omega^\varepsilon = \Omega$ and for every compact set $K \subset \Omega$ there exists $\varepsilon' > 0$ such that $K \subset \Omega_{\varepsilon'}$. This type of local weak* convergence does not make a substantial difference compared to the standard one, so we will not use a different symbol.

For instance, the local weak* convergence of (1.3.18) refers to a family of measures that are not defined on all of Ω for every fixed $\varepsilon > 0$. We stress that this distinction is important, since our mollifier ρ is assumed to be only continuous.

Remark 1.2.12. For any measure $\nu \in \mathcal{M}(\Omega)$ and any mollifier $\rho \in C_c(B(0,1))$ satisfying $\rho(x) = \rho(x^{-1})$ and $\int_{B(0,1)} \rho \, dx = 1$, we observe that $\rho_\varepsilon * \nu \in C(\mathbb{G})$ and we have the local weak* convergence of measures

$$(\rho_\varepsilon * \nu)\mu \rightharpoonup \nu \tag{1.2.17}$$

in Ω, as $\varepsilon \to 0^+$. Indeed, let $\phi \in C_c(\Omega)$ and let $\varepsilon > 0$ small enough, such that $\operatorname{supp} \phi \subset U$ and $U \subset \Omega^{\mathcal{R}}_\varepsilon$ is an open set. Then we have

$$\begin{aligned}
\int_\Omega \phi(x)(\rho_\varepsilon * \nu)(x) \, dx &= \int_U \phi(x) \left(\int_{B^{\mathcal{R}}(x,\varepsilon)} \rho_\varepsilon(xy^{-1}) \, d\nu(y) \right) dx \\
&= \int_{U^{\mathcal{R},\varepsilon}} \left(\int_U \rho_\varepsilon(yx^{-1}) \phi(x) \, dx \right) d\nu(y) \\
&= \int_\Omega (\rho_\varepsilon * \phi)(y) \, d\nu(y) \to \int_\Omega \phi(y) \, d\nu(y),
\end{aligned}$$

since $U^{\mathcal{R},\varepsilon} \subset (\Omega^{\mathcal{R}}_\varepsilon)^{\mathcal{R},\varepsilon} \subset \Omega$ and $\rho_\varepsilon * \phi \to \phi$ uniformly on compact subsets of Ω. The previous equalities also show that

$$\int_\Omega \phi(x)(\rho_\varepsilon * \nu)(x) \, dx = \int_\Omega (\rho_\varepsilon * \phi)(y) \, d\nu(y), \tag{1.2.18}$$

whenever $\rho \in C_c(B(0,1))$, $\nu \in \mathcal{M}(\Omega)$ and $\phi \in C_c(\Omega)$ such that $\operatorname{supp}(\phi) \subset \Omega^{\mathcal{R}}_\varepsilon$.

Remark 1.2.13. The previous arguments also show that $\rho_\varepsilon * f$, for $f \in L^1(\Omega)$, enjoys all properties of the convolution in Remark 1.2.12. The same is true for $\rho_\varepsilon * \nu$, if we consider

$$\nu = (u_1, \ldots, u_{\mathrm{m}})\beta, \tag{1.2.19}$$

where $u_1, \ldots, u_{\mathrm{m}} : \Omega \to \overline{\mathbb{R}}$ belong to $L^1(\Omega; \beta)$ and $\beta \in \mathcal{M}^+(\Omega)$.

Proposition 1.2.14. *Let $\Omega^{\mathcal{R}}_\varepsilon \neq \emptyset$ for some $\varepsilon > 0$. The following statements hold.*

1. *If $f \in C^1_H(\Omega)$, $\rho \in C_c(B(0,1))$ and $\int_{B(0,1)} \rho \, dx = 1$, then $\rho_\varepsilon * f \in C^1_H(\Omega^{\mathcal{R}}_\varepsilon)$,*

$$X_j(\rho_\varepsilon * f) = \rho_\varepsilon * X_j f \quad \text{in } \Omega^{\mathcal{R}}_\varepsilon, \tag{1.2.20}$$

and both $\rho_\varepsilon * f$ and $\nabla_H(\rho_\varepsilon * f)$ uniformly converge to f and $\nabla_H f$ on compact subsets of Ω. In addition, if $f \in L^1(\Omega)$ and $\rho \in C^k_c(B(0,1))$ for some $k \geq 1$, then $\rho_\varepsilon * f \in C^k(\mathbb{G})$.

2. *If $f \in L^\infty(\Omega)$ and $\rho \in \mathrm{Lip}_c(B(0,1))$, then $\rho_\varepsilon * f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{G})$.*

Proof. Let $f \in C^1_H(\Omega)$ and $\rho \in C_c(B(0,1))$. By the estimate of [76, Theorem 1.41] and Lebesgue's dominated convergence, we have

$$\begin{aligned} X_j(\rho_\varepsilon * f)(x) &= \lim_{t\to 0} \int_{B(0,\varepsilon)} \rho_\varepsilon(y)\, \frac{f(y^{-1}x(te_j)) - f(y^{-1}x)}{t} dy \\ &= \int_{B(0,\varepsilon)} \rho_\varepsilon(y)\, \lim_{t\to 0} \frac{f(y^{-1}x(te_j)) - f(y^{-1}x)}{t} dy \\ &= \int_{B(0,\varepsilon)} \rho_\varepsilon(y)(X_j f)(y^{-1}x)\, dy = (\rho_\varepsilon * X_j f)(x) \end{aligned}$$

for any $x \in \Omega^{\mathcal{R}}_\varepsilon$, due to the left invariance of X_j. By the condition $\int_{B(0,1)} \rho\, dx = 1$, the uniform convergence follows from the continuity of both f and $\nabla_H f$, along with the standard properties of the convolution. The second point can be proved in a similar way, by differentiating the mollifier ρ_ε. Here we only add that this differentiation is possible at every point of $\mathbb{G}$, being $\rho_\varepsilon * f$ defined on the whole group.

If $f \in L^\infty(\Omega)$ and $\rho \in \mathrm{Lip}_c(B(0,1))$, then it is easy to notice that the mollification $\rho_\varepsilon * f$ as in (1.2.10) is well posed and belongs to $L^\infty(\mathbb{G})$. Hence, for each compact set $K \subset \mathbb{G}$ and any $x, y \in K$, we have

$$\begin{aligned} |f^\varepsilon(x) - f^\varepsilon(y)| &\le \int_\Omega |f(z)|\, |\rho_\varepsilon(xz^{-1}) - \rho_\varepsilon(yz^{-1})|\, dz \\ &= \varepsilon^{-Q} \int_V |f(z)| \left|\rho\left(\delta_{1/\varepsilon}(xz^{-1})\right) - \rho\left(\delta_{1/\varepsilon}(yz^{-1})\right)\right| dz \\ &\le \|f\|_{L^\infty(\Omega)}\, 2\, \mu(B(0,1))\, L\, C_\varepsilon |x-y|, \end{aligned}$$

where $V = \overline{B^{\mathcal{R}}(x,\varepsilon) \cup B^{\mathcal{R}}(y,\varepsilon)} \subset \Omega$, $L > 0$ is the Lipschitz constant of ρ and $C_\varepsilon > 0$ is the supremum of all Lipschitz constants $L_{\varepsilon,z}$ of $K \ni x \mapsto \delta_{1/\varepsilon}(xz^{-1})$ as z varies in V. Due to this fact, we have $L_\varepsilon < +\infty$. □

The next density theorem follows from the choice of suitable mollified functions.

Theorem 1.2.15. *If $g \in \mathrm{Lip}_{H,\mathrm{loc}}(\Omega)$, then there exists a sequence $(g_k)_k$ in $C^\infty(\Omega)$ with the following properties*

1. *$g_k \to g$ uniformly on compact subsets of Ω;*
2. *$\||\nabla_H g_k|\|_{L^\infty(U)}$ is bounded for each $U \Subset \Omega$ and k sufficiently large;*
3. *$\nabla_H g_k \to \nabla_H g$ μ-a.e. in Ω.*

If $g \in \mathrm{Lip}_{H,c}(\Omega)$, then we can choose all g_k to have compact support in Ω.

Proof. We consider $\rho \in C^\infty_c(B(0,1))$ satisfying $\rho \ge 0$ and $\int_{B(0,1)} \rho\, dy = 1$. Then we define

$$g_k(x) = (\rho_{\varepsilon_k} * g)(x) = \int_\Omega \rho_{\varepsilon_k}(xy^{-1}) g(y)\, dy$$

for a positive sequence ε_k converging to zero, where $\rho_\varepsilon(x) = \varepsilon^{-Q}\rho(x/\varepsilon)$. In this way, we have $g_k \in C^\infty(\Omega)$ and Proposition 1.2.14 implies the uniform convergence on compact subsets of Ω. For the subsequent claims, we may consider an open set $U \Subset \Omega$ and take k sufficiently large such that $U \Subset \Omega^{\mathcal{R}}_{\varepsilon_k}$. For every fixed $x \in U$, formula (1.2.12) yields

$$g_k(x) = \int_{B(0,\varepsilon_k)} \rho_{\varepsilon_k}(y) g(y^{-1}x) dy,$$

therefore the following equalities hold:

$$
\begin{aligned}
&\left(g_k(xh) - g_k(x) - \left\langle \int_{B(0,\varepsilon_k)} \rho_{\varepsilon_k}(y) \nabla_H g(y^{-1}x) dy, h \right\rangle\right) \|h\|^{-1} \\
&= \left(g_k(xh) - g_k(x) - \int_{B(0,\varepsilon_k)} \rho_{\varepsilon_k}(y) \left\langle \nabla_H g(y^{-1}x), h \right\rangle dy\right) \|h\|^{-1} \\
&= \int_{B(0,\varepsilon_k)} \rho_{\varepsilon_k}(y) \left(\frac{g(y^{-1}xh) - g(y^{-1}x) - \langle \nabla_H g(y^{-1}x), h\rangle}{\|h\|}\right) dy
\end{aligned}
$$

for h sufficiently small. The difference quotient in the last integral is uniformly bounded with respect to y and h, due to the Lipschitz continuity of g. The a.e. differentiability of g, by Theorem 1.2.5, joined with Lebesgue's dominated convergence show that

$$
\nabla_H g_k(x) = \int_{B(0,\varepsilon_k)} \rho_{\varepsilon_k}(y) \nabla_H g(y^{-1}x) dy.
$$

The local Lipschitz continuity of g provides local boundedness for $\nabla_H g$, hence the previous formula immediately establishes the second property. By a change of variables, we get

$$
\nabla_H g_k(x) - \nabla_H g(x) = \int_{B^{\mathcal{R}}(x,\varepsilon_k)} \rho_{\varepsilon_k}(xz^{-1}) \left(\nabla_H g(z) - \nabla_H g(x)\right) dy.
$$

From this, it follows that

$$
|\nabla_H g_k(x) - \nabla_H g(x)| \leq \|\rho\|_{L^\infty(\mathbb{G})} \frac{1}{\varepsilon_k^Q} \int_{B^{\mathcal{R}}(x,\varepsilon_k)} |\nabla_H g(z) - \nabla_H g(x)| \, dy,
$$

and now we can conclude by Theorem 1.2.1. Finally, if g has compact support, it is clear that also $\operatorname{supp}(\rho_\varepsilon * g)$ is compact in Ω, for ε small enough. □

1.3 Basic notions of Geometric Measure Theory in stratified groups

[2] In this section we present some basic notions on BV functions and sets of finite perimeter in stratified groups. In particular, some new smoothing arguments for BV functions are presented. Additional results and references on these topics can be found for instance in [139].

We also introduce the important concept of *horizontal vector field*, that will be fundamental in Chapter 4, in connection with Leibniz formulas and the Gauss–Green theorem in stratified groups. To this purpose, we prove the well posedness of the pairings between essentially bounded horizontal vector fields and weak horizontal gradients of BV functions (see Lemma 1.3.6).

1.3.1 BV functions and horizontal vector fields in stratified groups

Let $\Omega \subset \mathbb{G}$ be an open set and denote by $H\Omega$ the restriction of the *horizontal subbundle* $H\mathbb{G}$ to the open set Ω, whose *horizontal fibers* $H_p\mathbb{G}$ are restricted to all points $p \in \Omega$.

Definition 1.3.1 (Horizontal vector fields)**.** Any measurable section $F : \Omega \to H\Omega$ of $H\Omega$ is called a *measurable horizontal vector field in* Ω. We denote by $|F|$ the measurable function $x \to |F(x)|$, where $|\cdot|$ denotes the fixed graded invariant Riemannian norm.

[2]This section is based on a joint work with Valentino Magnani [51].

The L^p-norm of a measurable horizontal vector field F in Ω is defined as follows:

$$\|F\|_{L^p(\Omega)} := \left(\int_\Omega |F(x)|^p \, dx \right)^{1/p} \quad \text{if } 1 \le p < \infty, \tag{1.3.1}$$

$$\|F\|_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_{x \in \Omega} |F(x)| \quad \text{if } p = \infty. \tag{1.3.2}$$

We say that a measurable horizontal vector field F is a *p-summable horizontal field* if $|F| \in L^p(\Omega)$. For $1 \le p \le \infty$, we denote by $L^p(H\Omega)$ the space of p-summable horizontal fields endowed with the norms defined either in (1.3.1) or (1.3.2). A measurable horizontal vector field F in Ω is *locally p-summable* if for any open subset $W \Subset \Omega$, we have $F \in L^p(HW)$. The space of all such vector fields is denoted by $L^p_{\text{loc}}(H\Omega)$.

For $k \in \mathbb{N} \setminus \{0\}$, the linear space of all C^k smooth sections of Ω is denoted by $C^k(H\Omega)$ and its elements will be called *horizontal vector fields of class C^k*. Considering the subclass of all C^k smooth horizontal vector fields with compact support in Ω yields the space $C^k_c(H\Omega)$. When $k = 0$ the integer k is omitted and the corresponding space of vector fields will include those with continuous coefficients.

It is easy to observe that, for all $f \in C^1_H(\Omega)$, the horizontal gradient $\nabla_H f$, given by (1.2.8), automatically defines a continuous horizontal vector field in Ω.

Definition 1.3.2. We say that a function $f : \Omega \to \mathbb{R}$ is a function of *bounded h-variation*, or simply a *BV function*, and write $f \in BV_H(\Omega)$, if $f \in L^1(\Omega)$ and

$$|D_H f|(\Omega) := \sup \left\{ \int_\Omega f \operatorname{div}\phi \, dx : \phi \in C^1_c(H\Omega), |\phi| \le 1 \right\} < \infty. \tag{1.3.3}$$

We denote by $BV_{H,\text{loc}}(\Omega)$ the space of functions in $BV_H(U)$ for any open set $U \Subset \Omega$.

Remark 1.3.3. In the case $\mathbb{G}$ is commutative and equipped with the Euclidean metric, the previous notion of BV function coincides with the classical one.

Due to the standard Riesz representation theorem, it is possible to show that when $f \in BV_H(\Omega)$ the total variation of its distributional horizontal grandient $|D_H f|$ is a nonnegative Radon measure on Ω. In addition, there exists a $|D_H f|$-measurable horizontal vector field $\sigma_f : \Omega \to H\Omega$ in Ω such that $|\sigma_f(x)| = 1$ for $|D_H f|$-a.e. $x \in \Omega$, and

$$\int_\Omega f \operatorname{div}\phi \, dx = - \int_\Omega \langle \phi, \sigma_f \rangle \; d|D_H f|, \tag{1.3.4}$$

for all $\phi \in C^1_c(H\Omega)$. In fact, these conditions are equivalent to the finiteness of (1.3.3).

Remark 1.3.4. Using Theorem 1.2.15 one can actually see that in (1.3.4) the horizontal vector field ϕ can be taken with coefficients in $\text{Lip}_{H,c}(\Omega)$.

The integration by parts formula (1.3.4) allows us to think of $D_H f$ as a kind of "measure with values in $H\Omega$", even though the horizontal tangent spaces of $H\Omega$ may have different frames, in principle.

Definition 1.3.5 (Measures in $H\Omega$)**.** Let $\gamma \in \mathcal{M}(\Omega)$ be a measure and let $\alpha : \Omega \to H\Omega$ be a horizontal vector field such that $|\alpha| \in L^\infty_{\text{loc}}(\Omega, \gamma)$. We define the *vector measure $\alpha\gamma$ in $H\Omega$* as the linear operator

$$C_c(H\Omega) \ni \phi \longrightarrow \int_\Omega \langle \phi, \alpha \rangle \; d\gamma =: \int_\Omega \langle \phi, d(\alpha\gamma) \rangle \,,$$

bounded on $C_c(HU)$ for any open set $U \Subset \Omega$ with respect to the L^∞-topology.

According to the previous definition, $\sigma_f |D_H f|$ is a vector measure on $H\Omega$, that will be also denoted by $D_H f$. When a horizontal frame $(X_1, \ldots, X_{\mathrm{m}})$ is fixed, we can represent σ_f by the $|D_H f|$-measurable functions $\sigma_f^1, \ldots, \sigma_f^{\mathrm{m}} : \Omega \to \mathbb{R}$ such that

$$\sigma_f = \sum_{j=1}^{\mathrm{m}} \sigma_f^j X_j.$$

Thanks to this representation, $D_H f$ can be naturally identified with the vector valued Radon measure

$$(\sigma_f^1, \ldots, \sigma_f^{\mathrm{m}})|D_H f|. \tag{1.3.5}$$

For each $j = 1, \ldots, \mathrm{m}$, we define the scalar measures

$$D_{X_j} f = \sigma_f^j |D_H f|, \tag{1.3.6}$$

that represent the distributional derivatives of a BV function as Radon measures. In view of the Radon-Nikodým theorem, we have the decomposition

$$D_H f = D_H^{\mathrm{a}} f + D_H^{\mathrm{s}} f$$

where $D_H^{\mathrm{a}} f$ denotes the absolutely continuous part of $D_H f$ with respect to the Haar measure of the group and $D_H^{\mathrm{s}} f$ the singular part.

Any BV function is approximately differentiable a.e. and in addition the approximate differential coincides a.e. with the vector density of $D_H^{\mathrm{a}} f$, see [17, Theorem 2.2]. As a result, we are entitled to denote $X_j f \in L^1(\Omega)$ as the unique measurable function such that

$$D_{X_j}^{\mathrm{a}} f = X_j f \, \mu. \tag{1.3.7}$$

Thus, to a BV function f we can assign a unique horizontal vector field $\nabla_H f \in L^1(H\Omega)$ whose components are defined in (1.3.7) and by definition we have

$$D_H^{\mathrm{a}} f = \nabla_H f \, \mu.$$

As a result, we have the decomposition of measures

$$D_H f = \nabla_H f \, \mu + D_H^{\mathrm{s}} f. \tag{1.3.8}$$

In the previous formula, $\nabla_H f$ is uniquely defined, up to μ-negligible sets, and it coincides a.e. with the approximate differential of f, see Definition 1.2.3.

The vector measure $D_H f$ in $H\Omega$ enjoys some standard properties of vector measures, as those mentioned in Remark 1.2.13. The mollification of $D_H f$ is the vector field

$$\rho_\varepsilon * D_H f(x) := \sum_{j=1}^{\mathrm{m}} \Big(\rho_\varepsilon * (\sigma_f^j \, |D_H f|)\Big)(x) X_j(x) = \sum_{j=1}^{\mathrm{m}} \Big(\rho_\varepsilon * (D_{X_j} f)\Big)(x) X_j(x). \tag{1.3.9}$$

We state now a technical lemma concerning an extension of the Euclidean notion of pairing introduced in Lemma 1.1.3.

Lemma 1.3.6. *Let $F \in L^\infty(H\Omega)$, $\gamma \in \mathcal{M}(\Omega)$ and $\alpha : \Omega \to H\Omega$ be a γ-measurable horizontal section such that $|\alpha(x)| = 1$ for γ-a.e. $x \in \Omega$. Let $\nu := \alpha\gamma$ be the corresponding vector measure in $H\Omega$ and let $\rho \in C_c(B(0,1))$ be a nonnegative mollifier satisfying $\rho(x) = \rho(x^{-1})$ and $\int_{B(0,1)} \rho \, dx = 1$. Then the measures $\langle F, (\rho_\varepsilon * \nu)\rangle \, \mu$ satisfy the estimate*

$$\int_\Omega |\langle F, (\rho_\varepsilon * \nu)\rangle| \, dx \le \|F\|_{L^\infty(\Omega)} \, |\nu|(\Omega) \tag{1.3.10}$$

for $\varepsilon > 0$ and any weak limit point $(F, \nu) \in \mathcal{M}(\Omega)$ satisfies $|(F, \nu)| \le \|F\|_{L^\infty(\Omega)} |\nu|$.*

Proof. For any $\phi \in C_c(\Omega)$, denoting by $K \subset \Omega$ its support, we have

$$\begin{aligned}\int_\Omega \phi(x)\left\langle F(x), (\rho_\varepsilon * \nu)(x)\right\rangle dx &= \int_K \phi(x)\left\langle F(x), \int_{K^{\mathcal{R},\varepsilon}} \rho_\varepsilon(xy^{-1})\alpha(y)\right\rangle d\gamma(y)\,dx \\ &= \int_{K^{\mathcal{R},\varepsilon}}\int_\Omega \phi(x)\left\langle F(x), \alpha(y)\right\rangle \rho_\varepsilon(yx^{-1})\,dx\,d\gamma(y) \\ &= \int_\Omega \left\langle (\rho_\varepsilon * (\phi F))(y), \alpha(y)\right\rangle d\gamma(y).\end{aligned}$$

This implies that

$$\left|\int_\Omega \phi(x)\left\langle F(x), (\rho_\varepsilon * \nu)(x)\right\rangle dx\right| \le \|\rho_\varepsilon * (\phi F)\|_{L^\infty(\Omega)}|\nu|(\Omega) \le \|\phi\|_{L^\infty(\Omega)}\|F\|_{L^\infty(\Omega)}|\nu|(\Omega),$$

therefore the sequence $\langle F, (\rho_\varepsilon * \nu)\rangle\,\mu$ satisfies (1.3.10). Let now $\langle F, (\rho_{\varepsilon_k} * \nu)\rangle\,\mu$ be a weakly converging subsequence, whose limit we denote by (F, ν). Then, by definition of weak* limit, for any $\phi \in C_c(\Omega)$ we obtain

$$\begin{aligned}\left|\int_\Omega \phi\, d(F,\nu)\right| = \lim_{\varepsilon_k \to 0}\left|\int_\Omega \phi\left\langle F, (\rho_\varepsilon * \nu)\right\rangle dx\right| &\le \lim_{\varepsilon_k\to 0}\|F\|_{L^\infty(\Omega)}\int_\Omega |\phi||\rho_{\varepsilon_k} * \nu|\,dx \\ \le \lim_{\varepsilon_k\to 0}\|F\|_{L^\infty(\Omega)}\int_\Omega |\phi|(\rho_\varepsilon * |\nu|)\,dx &= \|F\|_{L^\infty(\Omega)}\int_\Omega |\phi|\,d|\nu|,\end{aligned}$$

since $(\rho_\varepsilon * |\nu|)\mu \rightharpoonup |\nu|$ by Remark 1.2.12. This concludes our proof. □

Remark 1.3.7. We stress the fact that the pairing measure (F, ν) is not unique in general, unless $|\nu| \ll \mu$. Indeed, in the absolutely continuous case, we can write $\nu = G\mu$, for some $G \in L^1(H\Omega)$ and we have $\rho_\varepsilon * G \to G$ in $L^1(H\Omega)$. Hence, it is clear that

$$\langle F, (\rho_\varepsilon * \nu)\rangle\,\mu \rightharpoonup \langle F, G\rangle\,\mu \quad \text{in } \mathcal{M}(\Omega),$$

and so $(F, v) = \langle F, G\rangle\,\mu$.

We give now the definition of a weak notion of divergence for nonsmooth horizontal fields, which we shall need in the sequel.

Definition 1.3.8 (Distributional divergence)**.** The *divergence* of a measurable horizontal vector field $F \in L^1_{\text{loc}}(H\Omega)$ is defined as the following distribution

$$C_c^\infty(\Omega) \ni \phi \mapsto -\int_\Omega \langle F, \nabla_H \phi\rangle\, dx. \tag{1.3.11}$$

We denote this distribution by $\mathrm{div}F$. The same symbol will denote the measurable function defining the distribution, whenever it exists.

Remark 1.3.9. Due to Theorem 1.2.15, we can extend (1.3.11) to test functions ϕ in $\mathrm{Lip}_{H,c}(\Omega)$.

The representation of left invariant vector fields (1.2.5) gives

$$F = \sum_{j=1}^{\mathrm{m}} F_j X_j = \sum_{j=1}^{\mathrm{m}} F_j \partial_{x_j} + \sum_{i=\mathrm{m}+1}^{\mathrm{q}}\Big(\sum_{j=1}^{\mathrm{m}} F_j a_j^i\Big)\partial_{x_i} = \sum_{i=1}^{\mathrm{q}} f_i \partial_{x_i}, \tag{1.3.12}$$

therefore we have an analytic expression for the components f_i of F and we may observe that the distributional divergence (1.3.11) corresponds to the standard divergence

$$(\mathrm{div}F)(\phi) = -\int_\Omega \langle F, \nabla_H\phi\rangle\, dx = -\int_\Omega \langle F, \nabla\phi\rangle_{\mathbb{R}^{\mathrm{q}}}\, dx, \tag{1.3.13}$$

where ∇ denotes the Euclidean gradient in the fixed graded coordinates.

Let us consider a horizontal vector field $F = \sum_{j=1}^{\mathrm{m}} F_j X_j$ of class C^1_H, namely $F_j \in C^1_H(\Omega)$ for every $j = 1, \dots, \mathrm{m}$. It is easy to notice that its distributional divergence coincides with its pointwise divergence. Indeed, for $\phi \in C^1_c(\Omega)$ we have

$$-\int_\Omega \langle F, \nabla_H \phi\rangle \, dx = -\int_\Omega \sum_{j=1}^{\mathrm{m}} X_j(F_j\phi) + \int_\Omega \phi \sum_{j=1}^{\mathrm{m}} X_j F_j \, dx = \int_\Omega \phi \sum_{j=1}^{\mathrm{m}} X_j F_j \, dx.$$

The last equality follows by approximation, using Theorem 1.2.15, the divergence theorem for C^1 smooth functions and the fact that $\mathrm{div} X_j = 0$. For this reason, in the sequel we will not use a different notation to distinguish between the distributional divergence and the pointwise divergence.

The following lemma will play an important role in the sequel. It tells us that a mollifier that is only continuous turns a BV function into a C^1_H function.

Lemma 1.3.10. *If $f \in BV_{H,\mathrm{loc}}(\Omega)$, $\varepsilon > 0$ is such that $\Omega^{\mathcal{R}}_{2\varepsilon} \neq \emptyset$, $\rho \in C_c(B(0,1))$ is nonnegative such that $\rho(x) = \rho(x^{-1})$ and $\int_{B(0,1)} \rho = 1$, then $\rho_\varepsilon * f \in C^1_H(\Omega^{\mathcal{R}}_{2\varepsilon}) \cap C(\mathbb{G})$ and*

$$\nabla_H(\rho_\varepsilon * f) = (\rho_\varepsilon * D_H f) \quad \text{on} \quad \Omega^{\mathcal{R}}_{2\varepsilon}. \tag{1.3.14}$$

Proof. Let $\phi \in C^1_c(H\Omega^{\mathcal{R}}_{2\varepsilon})$. In particular, this means $\phi \in C^1_c(H\mathbb{G})$, so that (1.2.20) implies

$$(\rho_\varepsilon * \mathrm{div}\phi)(y) = \mathrm{div}(\rho_\varepsilon * \phi)(y) \tag{1.3.15}$$

for any $y \in \mathbb{G}$. Arguing similarly as in the proof of (1.2.18) and observing that $(\Omega^{\mathcal{R}}_{2\varepsilon})^{\mathcal{R},\varepsilon} \subset \Omega^{\mathcal{R}}_{\varepsilon}$, we get the following equalities, where the second one is a consequence of (1.3.15):

$$\begin{aligned}
\int_{\Omega^{\mathcal{R}}_{2\varepsilon}} (\rho_\varepsilon * f)(x)\, \mathrm{div}\phi(x)\, dx &= \int_{\Omega^{\mathcal{R}}_{\varepsilon}} f(y)\, (\rho_\varepsilon * \mathrm{div}\phi)(y)\, dy = \int_{\Omega^{\mathcal{R}}_{\varepsilon}} f(y)\, \mathrm{div}(\rho_\varepsilon * \phi)(y)\, dy \\
&= -\int_{\Omega^{\mathcal{R}}_{\varepsilon}} \langle (\rho_\varepsilon * \phi)(y), \sigma_f(y)\rangle \; d|D_H f|(y) \\
&= -\int_{\Omega^{\mathcal{R}}_{\varepsilon}} \int_{\Omega^{\mathcal{R}}_{2\varepsilon}} \rho_\varepsilon(yx^{-1})\, \langle \phi(x), \sigma_f(y)\rangle \; dx\, d|D_H f|(y) \\
&= -\int_{\Omega^{\mathcal{R}}_{2\varepsilon}} \int_{\Omega^{\mathcal{R}}_{\varepsilon}} \rho_\varepsilon(xy^{-1})\, \langle \phi(x), \sigma_f(y)\rangle \; d|D_H f|(y)\, dx \\
&= -\int_{\Omega^{\mathcal{R}}_{2\varepsilon}} \langle \phi(x), (\rho_\varepsilon * D_H f)(x)\rangle \; dx.
\end{aligned}$$

The standard density of $C^1_c(\Omega^{\mathcal{R}}_{2\varepsilon})$ in $C_c(\Omega^{\mathcal{R}}_{2\varepsilon})$, shows that $\rho_\varepsilon * f \in BV_{H,\mathrm{loc}}(\Omega^{\mathcal{R}}_{2\varepsilon})$ and proves the following formula

$$D_H(\rho_\varepsilon * f) = (\rho_\varepsilon * D_H f)\mu \quad \text{on} \quad \Omega^{\mathcal{R}}_{2\varepsilon}. \tag{1.3.16}$$

By Remark 1.2.12 and Remark 1.2.13, it follows that both $\rho_\varepsilon * f$ and $\rho_\varepsilon * D_H f$ are continuous, therefore $\rho_\varepsilon * f \in C^1_H(\Omega^{\mathcal{R}}_{2\varepsilon})$ and formula (1.3.14) follows. □

Taking into account (1.3.6), formula (1.3.14) can be written in components as follows

$$X_j(\rho_\varepsilon * f)(x) = (\rho_\varepsilon * D_{X_j} f)(x) \quad \text{for every } x \in \Omega^{\mathcal{R}}_{2\varepsilon}. \tag{1.3.17}$$

Theorem 1.3.11. *Let $f \in BV_{H,\mathrm{loc}}(\Omega)$ be such that $|D_H f|(\Omega) < +\infty$ and let $\rho \in C_c(B(0,1))$ with $\rho \geq 0$, $\int_{B(0,1)} \rho\, dx = 1$ and $\rho(x) = \rho(x^{-1})$. Then $\rho_\varepsilon * f \in C^1_H(\Omega^{\mathcal{R}}_{2\varepsilon})$ and we have*

$$\nabla_H(\rho_\varepsilon * f) \rightharpoonup D_H f \quad \text{and} \quad |\rho_\varepsilon * D_H f| \rightharpoonup |D_H f|, \tag{1.3.18}$$

$$|\nabla_H(\rho_\varepsilon * f)|\, \mu \leq (\rho_\varepsilon * |D_H f|)\, \mu \quad \text{on} \quad \Omega^{\mathcal{R}}_{2\varepsilon} \tag{1.3.19}$$

for every $\varepsilon > 0$ such that $\Omega^{\mathcal{R}}_{2\varepsilon} \neq \emptyset$. Finally, the following estimate holds

$$|\nabla_H(\rho_\varepsilon * f)|(\Omega^{\mathcal{R}}_{2\varepsilon}) \leq |D_H f|(\Omega). \tag{1.3.20}$$

Proof. The C^1_H smoothness of $\rho_\varepsilon * f$ and $\nabla_H(\rho_\varepsilon * f) = \rho_\varepsilon * D_H f$ on $\Omega^{\mathcal{R}}_{2\varepsilon}$ follow from Lemma 1.3.10. As a result, by (1.2.17) and taking into account (1.3.6) we obtain the local weak* convergence $X_j(\rho_\varepsilon * f) \rightharpoonup D_{X_j} f$ for any $j = 1, \ldots, \mathrm{m}$. This proves the first convergence of (1.3.18).

To prove (1.3.19), we consider $\phi \in C_c(H\Omega^{\mathcal{R}}_{2\varepsilon})$, therefore

$$\begin{aligned}
\left| \int_{\Omega^{\mathcal{R}}_{2\varepsilon}} \langle \phi(x), \nabla_H(\rho_\varepsilon * f)(x) \rangle \, dx \right| &= \left| \sum_{j=1}^{\mathrm{m}} \int_{\Omega^{\mathcal{R}}_{2\varepsilon}} \phi_j(x) X_j(\rho_\varepsilon * f)(x) \, dx \right| \\
&= \left| \int_\Omega \sum_{j=1}^{\mathrm{m}} (\rho_\varepsilon * \phi_j)(y) \, dD_{X_j} f(y) \right| \\
&= \left| \int_\Omega \langle (\rho_\varepsilon * \phi)(y), \sigma_f \rangle \, d|D_H f|(y) \right| .
\end{aligned}$$

The second equality follows from (1.3.17) joined with (1.2.18) and the last equality is a consequence of (1.3.6). As a result, applying again (1.2.18), we get

$$\begin{aligned}
\left| \int_{\Omega^{\mathcal{R}}_{2\varepsilon}} \langle \phi(x), \nabla_H(\rho_\varepsilon * f)(x) \rangle \, dx \right| &\le \int_\Omega (\rho_\varepsilon * |\phi|)(y) \, d|D_H f|(y) \\
&= \int_{\Omega^{\mathcal{R}}_{2\varepsilon}} |\phi(x)| \, (\rho_\varepsilon * |D_H f|)(x) \, dx.
\end{aligned} \tag{1.3.21}$$

By taking the supremum among all $\phi \in C_c(HU)$ with $\|\phi\|_{L^\infty(U)} \le 1$ and $U \subset \Omega^{\mathcal{R}}_{2\varepsilon}$ open set, we are immediately lead to (1.3.19). From the first inequality of (1.3.21), we also get

$$\left| \int_{\Omega^{\mathcal{R}}_{2\varepsilon}} \langle \phi(x), \nabla_H(\rho_\varepsilon * f)(x) \rangle \, dx \right| \le \|\phi\|_{L^\infty(U)} |D_H f|(\Omega),$$

whenever $\phi \in C_c(H\Omega^{\mathcal{R}}_{2\varepsilon})$. This immediately proves (1.3.20).

Finally, we are left to show the second local weak* convergence of (1.3.18). We fix an open set $U \Subset \Omega$ and notice that, by (1.2.17), we have

$$\rho_\varepsilon * |D_H f| \rightharpoonup |D_H f| \quad \text{in} \quad U. \tag{1.3.22}$$

In addition, by (1.3.20) and (1.3.17) we know that

$$\limsup_{\varepsilon \to 0} |\nabla_H(\rho_\varepsilon * f)|(U) \le \limsup_{\varepsilon \to 0} |\rho_\varepsilon * D_H f|(\Omega^{\mathcal{R}}_{2\varepsilon}) \le |D_H f|(\Omega),$$

hence there exists a weakly* converging sequence $|\nabla_H(\rho_{\varepsilon_k} * f)| \, \mu$ with limit ν in U. By virtue of [11, Proposition 1.62] with (1.3.18), we have $|D_H f| \le \nu$ in U. Therefore, taking nonnegative test functions $\varphi \in C_c(U)$ and using (1.3.19), we get

$$\int_U \varphi \, |\nabla_H(\rho_\varepsilon * f)| \, dx \le \int_U \varphi \, (\rho_\varepsilon * |D_H f|) \, dx$$

for $\varepsilon > 0$ sufficiently small, depending on U. Passing to the limit as $\varepsilon \to 0$, due to (1.3.22) we get the opposite inequality $\nu \le |D_H f|$ in U, therefore establishing the second local weak* convergence of (1.3.18). □

Remark 1.3.12. In the assumptions of Theorem 1.3.11, the first local weak* convergence of (1.3.18) joined with the lower semicontinuity of the total variation with respect to the weak* convergence of measures imply that

$$\liminf_{\varepsilon \to 0} |\nabla_H(\rho_\varepsilon * f)|(U) \ge |D_H f|(U)$$

for every open set $U \Subset \Omega$. If in addition $\rho \in C^1_c(B(0,1))$, and then $\rho_\varepsilon * f \in C^1(\mathbb{G})$ by Proposition 1.2.14, the previous inequality immediately gives

$$\liminf_{\varepsilon \to 0} |\nabla_H(\rho_\varepsilon * f)|(\Omega) \ge |D_H f|(\Omega).$$

1.3.2 Sets of finite perimeter in stratified groups

Functions of bounded h-variation, introduced in the previous section, naturally yield sets of finite h-perimeter as soon as we consider their characteristic functions.

Definition 1.3.13 (Sets of finite h-perimeter)**.** A measurable set $E \subset \mathbb{G}$ is of *locally finite h-perimeter* in Ω (or is a locally *h-Caccioppoli set*) if $\chi_E \in BV_{H,\mathrm{loc}}(\Omega)$. In this case, for any open set $U \Subset \Omega$, we denote the *h-perimeter of* E in U by

$$\mathsf{P}(E, U) := |D_H \chi_E|(U).$$

We say that E is a set of *finite h-perimeter* in Ω if $|D_H\chi_E|$ is a finite Radon measure on Ω. The *measure theoretic unit interior h-normal* of E in Ω is the $|D_H\chi_E|$-measurable horizontal section $\nu_E := \sigma_{\chi_E}$.

We can define two relevant subsets of the topological boundary of a set of locally finite h-perimter E: the reduced boundary $\mathscr{F}_H E$ and the measure theoretic boundary $\partial_H^* E$.

Definition 1.3.14 (Reduced boundary)**.** If $E \subset \mathbb{G}$ is a set of locally finite h-perimeter, we say that x belongs to the *reduced boundary* if

1. $|D_H\chi_E|(B(x,r)) > 0$ for any $r > 0$;
2. there exists $\lim_{r\to 0} ⨍_{B(x,r)} \nu_E \, d|D_H\chi_E|$;
3. $\left| \lim_{r\to 0} ⨍_{B(x,r)} \nu_E \, d|D_H\chi_E| \right| = 1.$

The reduced boundary is denoted by $\mathscr{F}_H E$.

Definition 1.3.15 (Measure theoretic boundary)**.** Given a measurable set $E \subset \mathbb{G}$, we say that $x \in \partial_H^* E$, if the following two conditions hold:

$$\limsup_{r\to 0} \frac{\mu(B(x,r)\cap E)}{r^Q} > 0 \quad \text{and} \quad \limsup_{r\to 0} \frac{\mu(B(x,r)\setminus E)}{r^Q} > 0.$$

The Lebesgue differentiation of Theorem 1.2.1 immediately shows that

$$\mu(\partial_H^* E) = 0. \tag{1.3.23}$$

However, a deeper differentiability result shows that indeed $\partial_H^* E$ is σ-finite with respect to the h-perimeter measure. Indeed, a general result on the integral representation of the perimeter measure holds in doubling metric measure spaces which admit a Poincaré inequality [2].

The following result restates [1, Theorem 4.2] in the special case of stratified groups, that are special instances of Ahlfors regular metric spaces equipped with a Poincaré inequality.

Theorem 1.3.16. *Given a set of finite h-perimeter E in $\mathbb{G}$, there exists $\gamma \in (0,1)$ such that the measure $\mathsf{P}(E,\cdot)$ is concentrated on the set $\Sigma_\gamma \subset \partial_H^* E$ defined as*

$$\Sigma_\gamma = \left\{ x : \limsup_{r\to 0} \min\left\{ \frac{\mu(E\cap B(x,r))}{\mu(B(x,r))}, \frac{\mu(B(x,r)\setminus E)}{\mu(B(x,r))} \right\} \geq \gamma \right\}.$$

Moreover, $\mathscr{S}^{Q-1}(\partial_H^* E \setminus \Sigma_\gamma) = 0$, $\mathscr{S}^{Q-1}(\partial_H^* E) < \infty$ and there exists $\alpha > 0$, independent of E, and a Borel function $\theta_E : \mathbb{G} \to [\alpha, +\infty)$ such that

$$\mathsf{P}(E, B) = \int_{B\cap \partial_H^* E} \theta_E \, d\mathscr{S}^{Q-1} \tag{1.3.24}$$

for any Borel set $B \subset \mathbb{G}$. Finally, the perimeter measure is asymptotically doubling, i.e., for $\mathsf{P}(E,\cdot)$-a.e. $x \in \mathbb{G}$ we have $\limsup_{r\to 0} \dfrac{\mathsf{P}(E, B(x,2r))}{\mathsf{P}(E, B(x,r))} < \infty.$

Lemma 1.3.17. *If $E \subset \mathbb{G}$ is a set of locally finite h-perimeter, then*

$$\mathscr{F}_H E \subset \partial_H^* E \quad \textit{and} \quad \mathscr{H}^{Q-1}(\partial_H^* E \setminus \mathscr{F}_H E) = 0. \tag{1.3.25}$$

Proof. The lower estimates of [80] joined with the invariance of reduced boundary and perimeter measure when passing to the complement of E immediately give the inclusion of (1.3.25). By Theorem 1.3.16, the perimeter measure $\mathsf{P}(E,\cdot) = |D\chi_E|(\cdot)$ is a.e. asymptotically doubling, therefore the following differentiation property holds:

$$\lim_{r\to 0} \fint_{B(x,r)} \nu_E \, d|D_H\chi_E| = \nu_E(x) \text{ for } |D_H\chi_E|\text{-a.e. } x,$$

according to [72, Sections 2.8.17 and 2.9.6]. This implies that $|D_H\chi_E|$-a.e. x belongs to $\mathscr{F}_H E$; that is, $|D_H\chi_E|(\mathbb{G} \setminus \mathscr{F}_H E) = 0$. Moreover, (1.3.24) yields $|D_H\chi_E|(B) \geq \alpha \mathscr{S}^{Q-1}(B \cap \partial_H^* E)$ on Borel sets $B \subset \mathbb{G}$. This inequality also extends to $|D_H\chi_E|$-measurable sets, hence taking $B = \mathbb{G} \setminus \mathscr{F}_H E$, we obtain $\mathscr{S}^{Q-1}(\partial_H^* E \setminus \mathscr{F}_H E) = 0$. Since $\mathscr{H}^{Q-1}$ and $\mathscr{S}^{Q-1}$ have the same negligible sets, the equality of (1.3.25) follows. □

Remark 1.3.18. The previous lemma joined with (1.2.7) and (1.3.23) shows that

$$\mu(\mathscr{F}_H E) = 0. \tag{1.3.26}$$

In addition, (1.3.24) and (1.3.25) imply that, for any Borel set B, $|D_H\chi_E|(B) = 0$ if and only if $\mathscr{S}^{Q-1}(B \cap \mathscr{F}_H E) = 0$; that is, the measures $|D_H\chi_E|$ and $\mathscr{S}^{Q-1} \llcorner \mathscr{F}_H E$ have the same negligible sets. In particular, $|D_H\chi_E| \geq \alpha \mathscr{S}^{Q-1} \llcorner \mathscr{F}_H E$.

1.3.3 Precise representatives and mollifications

As in the Euclidean setting, we can introduce the notion of precise representative of a locally summable function in a stratified group. However, due to the noncommutativity of the group, our choice of mollifying functions by putting the mollifier on the left, requires us to consider averages on balls associated to a right invariant distance. Therefore, it does not seem straightforward to recover in stratified groups a result on the existence $\mathscr{S}^{Q-1}$-a.e. of the pointwise limit of mollifications of BV functions in the spirit of Theorem 1.1.15. Nevertheless, we can still ensure a convergence result on a suitable family of points.

Definition 1.3.19 (Precise representative)**.** Assume $u \in L^1_{\mathrm{loc}}(\mathbb{G})$. Then

$$u^{*,\mathcal{R}}(x) := \begin{cases} \lim_{r\to 0} \fint_{B^{\mathcal{R}}(x,r)} u(y)\, dy & \text{if the limit exists} \\ 0 & \text{otherwise} \end{cases} \tag{1.3.27}$$

is the *precise representative* of u on the balls with respect to the right invariant distance. We denote by $C_u^{\mathcal{R}}$ the set of points such that the limit in (1.3.27) exists.

It is clear that, by Theorem 1.2.1, all Lebesgue points of u belong to $C_u^{\mathcal{R}}$. Given a measurable set $E \subset \Omega$, one can consider its points with density $\alpha \in [0,1]$ with respect to the right invariant distance

$$E^{\alpha,\mathcal{R}} := \left\{ x \in \mathbb{G} : \lim_{r\to 0} \frac{\mu(E \cap B^{\mathcal{R}}(x,r))}{\mu(B^{\mathcal{R}}(x,r))} = \alpha \right\},$$

and hence define

$$\partial_H^{*,\mathcal{R}} E = \Omega \setminus (E^{1,\mathcal{R}} \cup E^{0,\mathcal{R}}). \tag{1.3.28}$$

Then, if we set $C^{\mathcal{R}}_{\chi_E} = C^{\mathcal{R}}_E$, we clearly have

$$C^{\mathcal{R}}_E = \bigcup_{\alpha\in[0,1]} E^{\alpha,\mathcal{R}}$$

and

$$\chi_E^{*,\mathcal{R}} = \chi_{E^{1,\mathcal{R}}} \ \text{ in } \ \Omega \setminus \partial_H^{*,\mathcal{R}} E. \tag{1.3.29}$$

We state now a simple result which relates the pointwise limit of the mollification of a function $f \in L^1_{\rm loc}(\mathbb{G})$ and the precise representative of f on right invariant balls.

Proposition 1.3.20. *Let $\eta \in \mathrm{Lip}([0,1])$ with $\eta \geq 0$ and $\eta(1) = 0$, and $\rho(x) = \eta(d(x,0))$ for all $x \in \mathbb{G}$ such that $\int_{B(0,1)} \rho(x)\,dx = 1$. If $f \in L^1_{\rm loc}(\Omega)$ and $x \in C^{\mathcal{R}}_f$, then we have*

$$(\rho_\varepsilon * f)(x) \to f^{*,\mathcal{R}}(x) \qquad \textit{as} \qquad \varepsilon \to 0.$$

Proof. Let $x \in C^{\mathcal{R}}_f$ and $\varepsilon > 0$ be sufficiently small, so that $B^{\mathcal{R}}(x,\varepsilon) \subset \Omega$. We assume first that η is strictly decreasing. By Cavalieri's formula, we have

$$\begin{aligned}
(\rho_\varepsilon * f)(x) &= \int_{B^{\mathcal{R}}(x,\varepsilon)} \varepsilon^{-Q} \rho(\delta_{1/\varepsilon}(xy^{-1})) f(y)\,dy \\
&= \int_0^{+\infty} \int_{\{y\in B(x,\varepsilon):\, \rho(\delta_{1/\varepsilon}(xy^{-1}))>t\}} f(y)\varepsilon^{-Q}\,dy\,dt \\
\left(t = \eta\left(\frac{r}{\varepsilon}\right)\right) \qquad &= -\int_0^\varepsilon \frac{1}{\varepsilon}\eta'\left(\frac{r}{\varepsilon}\right)\frac{1}{\varepsilon^Q}\int_{B^{\mathcal{R}}(x,r)} f(y)\,dy\,dr \\
(r = s\varepsilon) \qquad &= -\int_0^1 \eta'(s)\mu(B(0,1))s^Q \fint_{B^{\mathcal{R}}(x,s\varepsilon)} f(y)\,dy\,ds.
\end{aligned}$$

The last equalities have been obtained from the standard area formula for one-dimensional Lipschitz functions. Now, we use the existence of the limit of the averages of f on the balls $B^{\mathcal{R}}(x, s\varepsilon)$. This also implies that these averages are uniformly bounded with respect to ε sufficiently small. Thus, by Lebesgue's dominated convergence we obtain

$$(\rho_\varepsilon * f)(x) \to -\mu(B(0,1))f^{*,\mathcal{R}}(x)\int_0^1 \eta'(s)s^Q\,ds.$$

We observe that the constant $C_{\eta,Q} := -\mu(B(0,1))\int_0^1 \eta'(s)s^Q\,ds$ is independent from f. In addition, if we take $f \equiv 1$, we clearly have $(\rho_\varepsilon * f) \equiv 1$ on $\Omega^{\mathcal{R}}_\varepsilon$. Thus, we can conlude that

$$-\mu(B(0,1))\int_0^1 \eta'(s)s^Q\,ds = 1,$$

and the statement follows. We use now the well known fact that any Lipschitz continuous function in one variable can be written as the difference of two strictly decreasing functions to write $\eta = \eta_1 - \eta_2$, with $\eta_i \in \mathrm{Lip}([0,1])$, strictly decreasing and satisfying $\eta_1(1) = \eta_2(1)$. We can now repeat the above argument and so we obtain

$$\begin{aligned}
(\rho_\varepsilon * f)(x) &\to -\mu(B(0,1))f^{*,\mathcal{R}}(x)\int_0^1 (\eta_1'(s) - \eta_2'(s))s^Q\,ds \\
&= -f^{*,\mathcal{R}}(x)\mu(B(0,1))\int_0^1 \eta'(s)s^Q\,ds = f^{*,\mathcal{R}}(x),
\end{aligned}$$

for any $x \in C^{\mathcal{R}}_f$. □

Remark 1.3.21. We point out that the previous result also holds in the Euclidean case, corresponding to a commutative group $\mathbb{G}$. It is then easy to see that the hypothesis that ρ is radially symmetric cannot be removed. Indeed, we may consider $f = \chi_E$, where $E = (0,1)^2$ and $\mathbb{G} = \mathbb{R}^2$, with $x = 0$. Clearly, $\chi_E^{*,\mathcal{R}}(0) = 1/4$. If we choose

$$\rho \in C_c^\infty(B(0,1) \cap (-1,0)^2), \ \rho \geq 0, \quad \text{with} \quad \int_{B(0,1)} \rho(y)\, dy = 1,$$

then we have

$$(\rho_\varepsilon * \chi_E)(0) = \int_{B(0,\varepsilon)\cap E} \rho_\varepsilon(-y)\, dy = \int_{B(0,\varepsilon)\cap(-1,0)^2} \rho_\varepsilon(y)\, dy = \int_{B(0,1)\cap(-1/\varepsilon,0)^2} \rho(y)\, dy = 1,$$

for any $\varepsilon \in (0,1]$.

Chapter 2

New contributions to the classical theory of sets of finite perimeter

2.1 Introduction

In this chapter we describe some new results in the theory of sets of finite perimeter in the Euclidean and stratified groups frameworks. We start in Section 2.2 by considering a recent characterization of the Euclidean perimeter through a family of functionals related to BMO-type seminorms. This alternative approach provides a way to define sets of finite perimeter without employing the theory of distributions, as done classically. We investigate an anisotropic version ot these functionals and prove a convergence result to a certain surface measure, related to the perimeter. In Section 2.3 we describe a way to improve the standard result on the approximation of sets of finite perimeter by smooth sets in $\mathbb{R}^n$: in particular, we construct two sequences of open sets with smooth boundaries approximating a given set of finite perimeter from the interior and from the exterior in a suitable measure theoretic sense. Finally, Section 2.4 is devoted to the study of weak* limit of mollifications of characteristic functions of sets with finite h-perimeter in stratified groups. It is relevant to notice that we determine the limit even in the absence of any rectifiability result analogous to De Giorgi's theorem (Theorem 1.1.10).

2.2 Anisotropic surface measures as limits of volume fractions

[1] The literature on approximation of Sobolev and BV norms, and on the characterizations of the corresponding spaces in terms of these approximations, is by now very wide, see in particular [29] for the case of Sobolev spaces, [129] and the more recent papers [32, 33] which deal with non-local approximations, in the sense of Γ-convergence of (a multiple of) the total variation norm, with intriguing connection to problems considered in image processing. Still in connection with non-local functionals, it is worth to mention the paper [38] which gave origin to the theory of nonlocal minimal surfaces.

Somehow in the same vein, motivated by [30], Ambrosio, Bourgain, Brezis and Figalli recently studied in [4] and [5] a new characterization of the perimeter of a set in $\mathbb{R}^n$ by considering the following functionals originating from a BMO-type seminorm

$$\mathsf{I}_\varepsilon(f) = \varepsilon^{n-1} \sup_{\mathcal{G}_\varepsilon} \sum_{Q' \in \mathcal{G}_\varepsilon} \fint_{Q'} |f(x) - \fint_{Q'} f| \, dx, \tag{2.2.1}$$

[1]This section is based on a joint work with Luigi Ambrosio [6], and on [50].

where $\mathcal{G}_\varepsilon$ is any disjoint collection of ε-cubes Q' with arbitrary orientation and cardinality not exceeding ε^{1-n}.
In particular, they studied the case $f = \chi_A$; that is, the characteristic function of a measurable set A, and proved that

$$\lim_{\varepsilon\to 0} \mathsf{I}_\varepsilon(\chi_A) = \frac{1}{2}\min\{1, \mathsf{P}(A)\}. \tag{2.2.2}$$

This theme has been further investigated in [84], for BV functions, see also [85] for a variant of this construction leading to Sobolev norms and spaces.

In this section we study more in detail the structure of the optimization problem in (2.2.1). We remove the upper bound on cardinality that seems to be very special of the case of cubes, at least if one is willing to get a precise formula as (2.2.2) and not only upper and lower bounds on I_ε. With this simplification, we prove that the existence of the limit and the emergence of a surface measure are general phenomena. In particular we prove that, for some dimensional constant $\xi = \xi(n)$, one has

$$\lim_{\varepsilon\to 0} H^B_\varepsilon(\chi_A) = \xi \mathsf{P}(A), \tag{2.2.3}$$

where H^B_ε is defined as (2.2.1), without the bound on cardinality and using disjoint ε-balls.
More generally, if C is a bounded connected open set containing the origin with Lipschitz boundary and if we define

$$H^C_\varepsilon(A) := \varepsilon^{n-1} \sup_{\mathcal{H}_\varepsilon} \sum_{C'\in\mathcal{H}_\varepsilon} ⨍_{C'} \left|\chi_A(x) - ⨍_{C'} \chi_A\right| dx, \tag{2.2.4}$$

where $\mathcal{H}_\varepsilon$ is any disjoint family of translations C' of the set εC with no bounds on cardinality, we are able to prove the following result.

Theorem 2.2.1. *There exists $\varphi^C : \mathbb{S}^{n-1} \to (0, +\infty)$, bounded and lower semicontinuous, such that, for any set of finite perimeter A, one has*

$$\lim_{\varepsilon\to 0} H^C_\varepsilon(A) = \int_{\mathscr{F}A} \varphi^C(\nu_A(x))\, d\mathscr{H}^{n-1}(x), \tag{2.2.5}$$

where $\mathscr{F}A$ and ν_A are respectively the reduced boundary of A and the measure theoretic unit interior normal to $\mathscr{F}A$. Moreover, if A is measurable and $\mathsf{P}(A) = \infty$, one has

$$\lim_{\varepsilon\to 0} H^C_\varepsilon(A) = +\infty. \tag{2.2.6}$$

The right hand side of (2.2.5) can be seen as an anisotropic version of the perimeter, $\mathsf{P}_\varphi(A)$. This result, while shows that the particular geometry of the covering sets is not essential, raises indeed some open questions. The most important is maybe the following one:

$$\text{Is the function } \tilde\varphi^C(p) := \begin{cases} |p|\varphi^C\Big(\dfrac{p}{|p|}\Big) & \text{if } p \neq 0 \\ 0 & \text{if } p = 0 \end{cases} \quad \text{convex?}$$

This question is natural, in view of the fact that the anisotropic perimeter

$$A \to \int_{\mathscr{F}A} \varphi(\nu_A)\, d\mathscr{H}^{n-1}$$

is lower semicontinuous w.r.t. the convergence in measure if and only if φ is the restriction to the unit sphere of a positively 1-homogeneous and convex function. The problem is nontrivial since we were able to prove that, if C is the unit square $(0, 1)^2$ in $\mathbb{R}^2$, then $\tilde\varphi^C$ is not convex, as it is shown in Section 2.2.5. In particular, the convexity of C is not a sufficient condition to obtain $\tilde{\varphi}^C$ convex.

Finally, we wish to mention that in the recent paper [70] a new version of Theorem 2.2.1 for SBV fuctions has been proved, under some additional regularity assumptions on the covering set C.

The section is organized as follows: in Section 2.2.1 we define suitable localized versions $H_\varepsilon(A,\Omega)$ of our functionals and we provide the proof of (2.2.6), by a simple comparison argument based on the results of [5]. Then, in Section 2.2.2 we consider a set A of finite perimeter in Ω and study the properties of the functionals $H_\varepsilon(A,\cdot)$ and $H_\pm(A,\cdot)$; the latter arise by taking the lim sup and the lim inf w.r.t. the scale parameter ε. Thanks to symmetry and superadditivity arguments, in Section 2.2.3 we show that $H_+ = H_-$ when both are evaluated in halfspaces S_ν and in cubical domains Q_ν with faces parallel or orthogonal to the normal to the halfspace. Eventually, in Section 2.2.4 we use covering theorems as well as the fine properties of sets of finite perimeter to extend the result to general sets of finite perimeter and general domains. We conclude with Section 2.2.5, where we discuss examples and variants of our result.

2.2.1 Convergence in the case $\mathsf{P}(A) = \infty$

In this section we introduce some useful tools and we show that (2.2.6) follows easily from the results of [5] and comparison arguments.

In order to prove Theorem 2.2.1, we define a localized version of H_ε: for any measurable set A and any open set Ω we set

$$H_\varepsilon^C(A,\Omega) := \varepsilon^{n-1} \sup_{\mathcal{H}_\varepsilon} \sum_{C' \in \mathcal{H}_\varepsilon} \fint_{C'} \left|\chi_A(x) - \fint_{C'} \chi_A\right| dx, \tag{2.2.7}$$

where the supremum runs among all disjoint families $\mathcal{H}_\varepsilon$ made with translations of the set εC in Ω. Since

$$\fint_{C'} \left|\chi_A(x) - \fint_{C'} \chi_A\right| dx = \fint_{C'} \fint_{C'} |\chi_A(x) - \chi_A(y)|\, dxdy = 2\frac{|C' \cap A||C' \setminus A|}{|C'|^2}, \tag{2.2.8}$$

we have the following equivalent definition

$$H_\varepsilon^C(A,\Omega) := \varepsilon^{n-1} \sup_{\mathcal{H}_\varepsilon} \sum_{C' \in \mathcal{H}_\varepsilon} 2\frac{|C' \cap A||C' \setminus A|}{|C'|^2}, \tag{2.2.9}$$

which we are going to use mostly.

It is not difficult to compare H_ε^C to H_ε^D when $D \subset C$ and D is an open set containing the origin[2]. Indeed, it is clear that that for any measurable set A one has

$$\frac{|D \cap A||D \setminus A|}{|D|^2} \leq \frac{|C|^2}{|D|^2} \frac{|C \cap A||C \setminus A|}{|C|^2}, \tag{2.2.10}$$

and that the same holds for any translated and dilated copies of C and D. Now, for any disjoint family $\mathcal{H}_{\varepsilon,D}$ of translations of εD we can find a family $\mathcal{H}_{\varepsilon,C}$ of translations of εC such that for any $D_j \in \mathcal{H}_{\varepsilon,D}$ there exists $C_j \in \mathcal{H}_{\varepsilon,C}$ with $D_j \subset C_j$. Even though the family $\mathcal{H}_{\varepsilon,C}$ is not disjoint in general, it is easily seen, using the inclusions

$$B(x_j,\lambda\varepsilon) \subset D_j \subset C_j \subset B(x_j,\varepsilon) \qquad \text{for some } x_j \in \mathbb{R}^n$$

(where $\lambda > 0$ satisfies $B(0,\lambda) \subset D$), that it has bounded overlap. More precisely, there exists $\theta = \theta(n,\lambda) > 0$ such that for any fixed j we have $\#\{k : \; B(x_k,\varepsilon) \cap B(x_j,\varepsilon) \neq \emptyset\} \leq \theta$ and so

[2]Without loss of generality, we can always assume $0 \in D \subset C$.

the same property holds if we replace the balls by the corresponding sets C_j. Therefore, the family $\mathcal{H}_{\varepsilon,C}$ can be seen as the union of at most θ disjoint subfamilies $\mathcal{H}_{\varepsilon,C,i}$. This argument yields

$$\varepsilon^{n-1}\sum_{D'\in\mathcal{H}_{\varepsilon,D}} 2\frac{|D'\cap A||D'\setminus A|}{|D'|^2} \le \frac{|C|^2}{|D|^2}\sum_{i=1}^{\theta}\varepsilon^{n-1}\sum_{C'\in\mathcal{H}_{\varepsilon,C,i}} 2\frac{|C'\cap A||C'\setminus A|}{|C'|^2} \le \frac{|C|^2}{|D|^2}\theta H_\varepsilon^C(A,\Omega),$$

and, taking the supremum over the families $\mathcal{H}_{\varepsilon,D}$, we obtain

$$H_\varepsilon^D(A,\Omega) \le \frac{|C|^2}{|D|^2}\theta H_\varepsilon^C(A,\Omega), \qquad H_\pm^D(A,\Omega) \le \frac{|C|^2}{|D|^2}\theta H_\pm^C(A,\Omega). \tag{2.2.11}$$

In addition, we notice that, for any rotation R we have $H_\varepsilon^{R(C)}(R(A),R(\Omega)) = H_\varepsilon^C(A,\Omega)$ and $H_\pm^{R(C)}(R(A),R(\Omega)) = H_\pm^C(A,\Omega)$.

Since in the following the set C will be mostly fixed, we drop the superscript C from H_ε^C, $H_\pm^C$.

We pass now to the proof of (2.2.6).

Proof of (2.2.6). Let

$$\mathsf{I}_\varepsilon(\chi_A,\Omega) := \varepsilon^{n-1}\sup_{\mathcal{F}_\varepsilon}\sum_{Q'\in\mathcal{F}_\varepsilon} 2\frac{|Q'\cap A||Q'\setminus A|}{|Q'|^2},$$

where $\mathcal{F}_\varepsilon$ denotes a collection of disjoint open cubes $Q'\subset\Omega$ with side length ε and arbitrary orientation. In [5] it was shown that, for any Borel set A, one has

$$\lim_{\varepsilon\to 0}\mathsf{I}_\varepsilon(\chi_A,\mathbb{R}^n) = \frac{1}{2}\mathsf{P}(A). \tag{2.2.12}$$

For later purposes, we recall also a local version of (2.2.12) which is proved in [5] in order to get the global version, namely

$$\liminf_{\varepsilon\to 0}\mathsf{I}_\varepsilon(\chi_A,\Omega) \ge \frac{1}{2}\mathsf{P}(A,\Omega) \qquad \text{for any open set } \Omega\subset\mathbb{R}^n. \tag{2.2.13}$$

Arguing as in the proof of (2.2.11), we observe that for any cube Q' with arbitrary orientation and side length $2\varepsilon/\sqrt{n}$, we can find an open ε-ball $B'\supset Q'$. Hence, for any collection $\mathcal{F}_{2\varepsilon/\sqrt{n}}$ of disjoint cubes Q' with arbitrary orientation and side length $2\varepsilon/\sqrt{n}$, we find a family $\mathcal{G}_{\varepsilon,B}$ of ε-balls with bounded overlap; that is, there exists $\theta_n>0$ such that for any fixed $B'\in\mathcal{G}_{\varepsilon,B}$ we have $\#\{B''\in\mathcal{G}_{\varepsilon,B} : B''\cap B'\neq\emptyset\}\le\theta_n$.
Then, if we denote by H_ε^B the functional where we take a covering with ε-balls, we get

$$\theta_n H_\varepsilon^B(A,\Omega) \ge \frac{4^n}{n^n\omega_n^2}\mathsf{I}_{2\varepsilon/\sqrt{n}}(\chi_A,\Omega) \tag{2.2.14}$$

If $\mathsf{P}(A)=+\infty$, inequalities (2.2.12) and (2.2.14) clearly give

$$\liminf_{\varepsilon\to 0} H_\varepsilon^B(A) \ge \liminf_{\varepsilon\to 0}\frac{4^n}{n^n\omega_n^2\theta_n}\mathsf{I}_{\frac{2}{\sqrt{n}}\varepsilon}(\chi_A) = +\infty.$$

The case of a general open bounded connected set with Lipschitz boundary C containing the origin follows immediately by (2.2.11), since $C\supset B(0,\lambda)$ for some $\lambda=\lambda(C)>0$. $\square$

2.2.2 First properties of H_ε

We start by stressing the fact that we require C to be an open bounded connected set with Lipschitz boundary in order to employ the relative isoperimetric inequality (Theorem 1.1.6) to obtain a bound for the volume fraction. Indeed, thanks to (1.1.5), we know that there exists a constant $\gamma = \gamma(C)$ such that

$$\frac{|C \cap E||C \setminus E|}{|C|^2} \leq \gamma \mathsf{P}(E, C), \tag{2.2.15}$$

for any measurable set E. By scaling, it follows that if $C' = \varepsilon C$ we have

$$\frac{|C' \cap E||C' \setminus E|}{|C'|^2} \leq \varepsilon^{1-n} \gamma \mathsf{P}(E, C'),$$

for any measurable set E.

We notice that it is convenient to define the following set functions

$$H_+^C(A, \Omega) := \limsup_{\varepsilon \to 0} H_\varepsilon^C(A, \Omega), \tag{2.2.16}$$

$$H_-^C(A, \Omega) := \liminf_{\varepsilon \to 0} H_\varepsilon^C(A, \Omega). \tag{2.2.17}$$

Clearly, we have $H_-^C(A, \Omega) \leq H_+^C(A, \Omega)$. In order to show the existence of the limit in the case of a set of finite perimeter A, we need to prove the converse inequality $H_-(A, \Omega) \geq H_+(A, \Omega)$.

The following scaling properties will be useful:

$$H_\varepsilon^{\lambda C}(A, \Omega) = \lambda^{1-n} H_{\varepsilon\lambda}^C(A, \Omega), \qquad H_\pm^{\lambda C}(A, \Omega) = \lambda^{1-n} H_\pm^C(A, \Omega). \tag{2.2.18}$$

In the sequel, we also often assume with no loss of generality that $\mathrm{diam}(C) = 1$. Indeed, if we set $\tilde{C} := C/\mathrm{diam}(C)$, then (2.2.18) with $\lambda = \mathrm{diam}(C)$, so that $C = \lambda \tilde{C}$, implies

$$H_\varepsilon^C(A, \Omega) = \mathrm{diam}(C)^{1-n} H_{\varepsilon \mathrm{diam}(C)}^{\tilde{C}}(A, \Omega), \qquad H_\pm^C(A, \Omega) = \mathrm{diam}(C)^{1-n} H_\pm^{\tilde{C}}(A, \Omega).$$

We show now some elementary properties of the functionals H_ε and $H_\pm$, omitting the proof of the simplest ones and assuming the normalization $\mathrm{diam}(C) = 1$.

1. Translation invariance: for any $\tau \in \mathbb{R}^n$, we have $H_\varepsilon(A + \tau, \Omega + \tau) = H_\varepsilon(A, \Omega)$; taking limits, one has also $H_\pm(A + \tau, \Omega + \tau) = H_\pm(A, \Omega)$;

2. Monotonicity: $H_\varepsilon(A, \cdot)$ and $H_\pm(A, \cdot)$ are increasing set functions on the class of open sets in $\mathbb{R}^n$;

3. Homogeneity: for any $t > 0$, $H_{t\varepsilon}(tA, t\Omega) = t^{n-1} H_\varepsilon(A, \Omega)$. Indeed, $tC' \subset t\Omega$ if and only if $C' \subset \Omega$, and
$$\frac{|tC' \cap tA||tC' \setminus tA|}{|tC'|^2} = \frac{|C' \cap A||C' \setminus A|}{|C'|^2}.$$
It follows immediately that
$$H_\pm(tA, t\Omega) = t^{n-1} H_\pm(A, \Omega). \tag{2.2.19}$$

4. Superadditivity of H_-: it is easy to see that
$$H_\varepsilon(A, \Omega_1 \cup \Omega_2) = H_\varepsilon(A, \Omega_1) + H_\varepsilon(A, \Omega_2) \tag{2.2.20}$$
whenever $\Omega_1 \cap \Omega_2 = \emptyset$. From (2.2.20) we get
$$H_-(A, \Omega_1 \cup \Omega_2) \geq H_-(A, \Omega_1) + H_-(A, \Omega_2). \tag{2.2.21}$$

5. Almost subadditivity of H_+:

$$H_\varepsilon(A, \Omega_1 \cup \Omega_2) \le H_\varepsilon(A, I_\varepsilon(\Omega_1)) + H_\varepsilon(A, I_\varepsilon(\Omega_2)), \tag{2.2.22}$$

for any open set Ω_1, Ω_2, where $I_t(\Omega) := \{x \in \mathbb{R}^n : \operatorname{dist}(x, \Omega) < t\}$. Indeed, if $C' \subset \Omega_1 \cup \Omega_2$, then it must be contained in the ε-neighborhood of one of the two open sets, since $\operatorname{diam}(C') = \varepsilon \operatorname{diam}(C) = \varepsilon$. From (2.2.22) we get

$$H_+(A, \Omega_1 \cup \Omega_2) \le H_+(A, W_1) + H_+(A, W_2), \tag{2.2.23}$$

for any open sets $W_i \supset I_\delta(\Omega_i)$, $i = 1, 2$, for some $\delta > 0$.

6. Upper bound for H_+: using (2.2.15), we see that

$$H_\varepsilon(A, \Omega) \le 2\gamma \mathsf{P}(A, \Omega)$$

and so

$$H_+(A, \Omega) \le 2\gamma \mathsf{P}(A, \Omega). \tag{2.2.24}$$

2.2.3 Lower and upper densities of $H_\pm$

We set

$$\varphi_+(\nu) := H_+(S_\nu, Q_\nu),$$
$$\varphi_-(\nu) := H_-(S_\nu, Q_\nu),$$

where $\nu \in \mathbb{S}^{n-1}$, $S_\nu := \{x \in \mathbb{R}^n : \ x \cdot \nu \ge 0\}$ and Q_ν an open unit cube centered in the origin having one face orthogonal to ν and bisected by the hyperplane ∂S_ν. Due to the translation invariance, this definition does not actually depend on the choice of the origin, since we could take any hyperplane $\{(x - x_0) \cdot \nu \ge 0\}$ and cubes centered in x_0.

While in $\mathbb{R}^2$ there exists only one unit cube centered in origin, bisected by the hyperplane ∂S_ν and with one face orthogonal to $\nu \in \mathbb{S}^1$, if instead $n \ge 3$, given any such cube Q_ν, $R(Q_\nu)$ satisfies the same conditions for any rotation R such that $R\nu = \nu$. Therefore, in $\mathbb{R}^n$, for $n \ge 2$, there are $n - 2$ degrees of freedom in the choice of Q_ν, as noticed in [70]. Hence, if $n \ge 3$, a priori $\varphi_+(\nu)$ and $\varphi_-(\nu)$ depend also on the choice of the unit cube Q_ν. However, as showed in Lemma 2.2.3 below, such dependence is illusory, and so, with a little abuse of notation, we may omit to indicate it.

It is obvious that $\varphi_-(\nu) \le \varphi_+(\nu)$. We collect in the next proposition a few elementary properties of $\varphi_\pm$ (more refined estimates in some special cases will be given in Section 2.2.5) and then we prove that these two functions coincide.

Proposition 2.2.2. *We have the following upper and lower bounds for $\varphi_\pm$:*

1. *$\varphi_+ \le 2\gamma$, where γ is the same constant in* (2.2.24)*;*

2. *$\varphi_- \ge \lambda^{n+1} \dfrac{2^{2n-1}}{|C|^2 \theta n^n \theta_n}$, where $\lambda = \lambda(C) := \sup\{r > 0 : B(0, r) \subset C\}$, $\theta = \theta(n, \lambda)$ and θ_n are defined in the proofs of* (2.2.11) *and* (2.2.6)*, respectively.*

In addition, $\varphi_- = \varphi_+$ and φ_- is lower semicontinuous.

Proof. The inequality $\varphi_+ \le 2\gamma$ is easy, since by (2.2.15) we have

$$H_+(S_\nu, Q_\nu) \le 2\gamma \mathsf{P}(S_\nu, Q_\nu)$$

and $\mathsf{P}(S_\nu, Q_\nu) = 1$, by the definition of S_ν and Q_ν.
As for the lower bound on $\varphi_-(\nu)$, it can be obtained as follows: first we take $r > 0$ such that $B(0, r) \subset C$, then we apply (2.2.11), (2.2.18) and eventually (2.2.14) to get

$$\begin{aligned} H_\varepsilon^C(S_\nu, Q_\nu) &\geq \frac{|B(0,r)|^2}{|C|^2\theta} H_\varepsilon^{B(0,r)}(S_\nu, Q_\nu) \\ &= r^{1-n}\frac{|B(0,r)|^2}{|C|^2\theta} H_{\varepsilon r}^{B}(S_\nu, Q_\nu) \\ &\geq r^{n+1}\frac{|B|^2}{|C|^2\theta}\frac{4^n}{n^n\omega_n^2\theta_n}\mathsf{I}_{2\varepsilon\lambda/\sqrt{n}}(S_\nu, Q_\nu) = r^{n+1}\frac{1}{|C|^2\theta}\frac{4^n}{n^n\theta_n}\mathsf{I}_{2\varepsilon\lambda/\sqrt{n}}(S_\nu, Q_\nu). \end{aligned}$$

Now we let $\varepsilon \to 0$, using (2.2.13) with $A = S_\nu$ and $\Omega = Q_\nu$, and finally we take the supremum over $r > 0$ such that $B(0, r) \subset C$.
Finally, homogeneity implies that

$$\varphi_-(\nu) = H_-(S_\nu, Q_\nu) = \liminf_{\varepsilon\to 0} \varepsilon^{n-1} H_1(S_\nu, (1/\varepsilon)Q_\nu),$$

since $(1/\varepsilon)S_\nu = S_\nu$ for any $\varepsilon > 0$.
We observe that $(1/\varepsilon)Q_\nu$ contains the union of at least $\lfloor (t/\varepsilon)\rfloor^{n-1}$ open disjoint cubes of side length $1/t$, for any $t > \varepsilon$, which are translations of $(1/t)Q_\nu$ centered in points of ∂S_ν. Clearly, $H_\varepsilon(S_\nu, x+Q_\nu) = H_\varepsilon(S_\nu, Q_\nu)$ for any $x \in \partial S_\nu$. Hence, the monotonicity in the second argument, the additivity of H_ε and the homogeneity imply

$$\begin{aligned} \varphi_-(\nu) \geq \liminf_{\varepsilon\to 0} \varepsilon^{n-1} \lfloor (t/\varepsilon)\rfloor^{n-1} H_1(S_\nu, (1/t)Q_\nu) &= t^{n-1} H_{t(1/t)}((1/t)S_\nu, (1/t)Q_\nu) \\ &= H_t(S_\nu, Q_\nu), \end{aligned}$$

which implies $\varphi_-(\nu) \geq \sup_{t>0} H_t(S_\nu, Q_\nu)$.
On the other hand, it is clear that

$$\varphi_-(\nu) \leq \varphi_+(\nu) = \limsup_{\varepsilon\to 0} H_\varepsilon(S_\nu, Q_\nu) = \lim_{\varepsilon\to 0}\sup_{0<s<\varepsilon} H_s(S_\nu, Q_\nu) \leq \sup_{s>0} H_s(S_\nu, Q_\nu)$$

from which we deduce that $\varphi_-(\nu) = \varphi_+(\nu) = \sup_{t>0} H_t(S_\nu, Q_\nu)$.
As a byproduct, we also obtain that φ_- is lower semicontinuous in ν, being the supremum with respect to the parameter t of the supremum over the families $\mathcal{H}_t$ of translations of tC of the quantities

$$t^{n-1}\sum_{C'\in\mathcal{H}_t} 2\frac{|C'\cap S_\nu||C'\setminus S_\nu|}{|C'|^2},$$

which are continuous functions of ν.

$\square$

We define $\varphi(\nu) := \lim_{\varepsilon\to 0} H_\varepsilon(S_\nu, Q_\nu)$, since Proposition 2.2.2 showed the existence of the limit, and we proceed to prove that it does not depend on the choice of Q_ν in the family of unit cubes centered in the origin, bisected by ∂S_ν with one face orthogonal to ν, even in the case $n \geq 3$.

Lemma 2.2.3. *Let $\nu \in \mathbb{S}^{n-1}$, and Q_ν, Q'_ν be two cubes centered in the origin, bisected by ∂S_ν and with one face orthogonal to ν. Then, we have*

$$H_-(S_\nu, Q_\nu) = H_+(S_\nu, Q_\nu) = H_+(S_\nu, Q'_\nu) = H_-(S_\nu, Q'_\nu).$$

Proof. Clearly,
$$Q'_\nu = (Q'_\nu \setminus I_\delta(\partial S_\nu)) \cup (I_\delta(\partial S_\nu) \cap Q'_\nu)$$
for any $\delta > 0$, and we notice that
$$H_\varepsilon(S_\nu, Q'_\nu \setminus I_\delta(\partial S_\nu)) = 0$$
for any $\varepsilon, \delta > 0$, since, if $C' \subset Q'_\nu \setminus I_\delta(\partial S_\nu)$, then
$$|C' \cap S_\nu||C' \setminus S_\nu| = 0.$$
By (2.2.20), we have
$$H_\varepsilon(S_\nu, Q'_\nu) = H_\varepsilon(S_\nu, I_\delta(\partial S_\nu) \cap Q'_\nu)$$
for any $\delta > 0$, which implies
$$H_+(S_\nu, Q'_\nu) = H_+(S_\nu, I_\delta(\partial S_\nu) \cap Q'_\nu). \tag{2.2.25}$$
Let now $r \in (0,1)$. We can cover $(\partial S_\nu) \cap Q'_\nu$ with a disjoint family $\{P_{j,r}\}_{j=1}^{m_r}$ of translations of $(\partial S_\nu) \cap rQ_\nu$ up to a closed set Γ_r such that $\mathscr{H}^{n-1}(\Gamma_r) \to 0$ as $r \to 0$. Clearly, $m_r r^{n-1} \leq 1$. If we associate to each $P_{j,r}$ the translation of rQ_ν which generates it, we obtain a family of cubes $\{Q_{j,r}\}_{j=1}^{m_r}$, $Q_{j,r} = x_j + rQ_\nu$ for some $x_j \in (\partial S_\nu) \cap Q'_\nu)$, such that
$$I_r(\partial S_\nu) \cap Q'_\nu = \bigcup_{j=1}^{m_r} Q_{j,r} \cup U_r,$$
for some open set U_r satisfying $\mathscr{H}^{n-1}((U_r \cap \partial S_\nu)\Delta\Gamma_r) = 0$. Thanks to (2.2.25), (2.2.23), the translation invariance, (2.2.19) and (2.2.24), for any $r, \delta > 0$ we have
$$\begin{aligned} H_+(S_\nu, Q'_\nu) = H_+(S_\nu, I_r(\partial S_\nu) \cap Q'_\nu) &\leq \sum_{j=1}^{m_r} H_+(S_\nu, x_j + (1+\delta)rQ_\nu) + H_+(S_\nu, I_\delta(U_r)) \\ &\leq m_r(1+\delta)^{n-1}r^{n-1}H_+(S_\nu, Q_\nu) + 2\gamma \mathsf{P}(S_\nu, I_\delta(U_r)) \\ &\leq (1+\delta)^{n-1}H_+(S_\nu, Q_\nu) + 2\gamma\mathscr{H}^{n-1}(I_\delta(\Gamma_r) \cap \partial S_\nu). \end{aligned}$$
Hence, if we send $\delta \to 0$, we obtain
$$H_+(S_\nu, Q'_\nu) \leq H_+(S_\nu, Q_\nu) + 2\gamma\mathscr{H}^{n-1}(\Gamma_r),$$
from which we immediately get $H_+(S_\nu, Q'_\nu) \leq H_+(S_\nu, Q_\nu)$, since $r > 0$ is arbitrary. Exchanging now the role of Q_ν and Q'_ν, we obtain the reverse inequality and we conclude that
$$H_+(S_\nu, Q'_\nu) = H_+(S_\nu, Q_\nu).$$
Finally, Propositon (2.2.2) implies that $H_-(S_\nu, Q) = H_+(S_\nu, Q)$ for $Q \in \{Q_\nu, Q'_\nu\}$, and this ends the proof. □

We wish now to prove that the upper and lower densities of $H_\pm$ coincide with φ. To this purpose, we need a modulus of continuity for $E \to H_\varepsilon(E, \Omega)$ similar to the one shown in [5, Lemma 3.6]. We recall that for any E, F sets of finite perimeter in Ω we have
$$\mathscr{H}^{n-1}(\partial^*(E\Delta F) \cap \Omega) = \mathscr{H}^{n-1}((\partial^* E \Delta \partial^* F) \cap \Omega), \tag{2.2.26}$$
see for instance [5, Section 2].

Lemma 2.2.4. *For any E, F sets of finite perimeter in Ω and any $\varepsilon > 0$ we have*

$$H_\varepsilon(F,\Omega) \le H_\varepsilon(E,\Omega) + 4\gamma\mathscr{H}^{n-1}((\mathscr{F}E\Delta\mathscr{F}F)\cap\Omega). \tag{2.2.27}$$

In particular one has

$$H_\pm(F,\Omega) \le H_\pm(E,\Omega) + 4\gamma\mathscr{H}^{n-1}((\mathscr{F}E\Delta\mathscr{F}F)\cap\Omega). \tag{2.2.28}$$

Proof. For any C' and any measurable set $L \subset C'$ we have the relative isoperimetric inequality (2.2.15) and, combining it with the inequality $\min\{t, 1-t\} \le 2t(1-t)$ for any $t \in [0,1]$, we obtain also

$$\min\{|L|, |C' \setminus L|\} \le 2\gamma|C|\varepsilon\mathsf{P}(L, C'). \tag{2.2.29}$$

Let now $\mathcal{H}_\varepsilon$ be a disjoint family of translations of εC in Ω.
For any $C' \in \mathcal{H}_\varepsilon$, we have

$$\fint_{C'}\fint_{C'} |\chi_F(x)-\chi_F(y)|\,dx\,dy \le \fint_{C'}\fint_{C'} |\chi_E(x)-\chi_E(y)|\,dx\,dy + \frac{2}{|C|}\varepsilon^{-n}|C'\cap(F\Delta E)|. \tag{2.2.30}$$

Indeed,

$$\begin{aligned}
&\fint_{C'}\fint_{C'} |\chi_F(x)-\chi_E(x)-\chi_F(y)+\chi_E(y)|\,dx\,dy\\
&= \frac{2}{|C'|^2}(2|C'\cap(F\setminus E)||C'\cap(E\setminus F)| + |C'\setminus(F\Delta E)||C'\cap(F\Delta E)|)\\
&= \frac{2}{|C'|^2}(2|C'\cap(F\setminus E)||C'\cap(E\setminus F)| + |C'||C'\cap(F\Delta E)| - (|C'\cap(E\setminus F)| + |C'\cap(F\setminus E)|)^2)\\
&\le \frac{2}{|C|}\varepsilon^{-n}|C'\cap(F\Delta E)|.
\end{aligned}$$

Since $\chi_{E^c}(x) - \chi_{E^c}(y) = \chi_E(y) - \chi_E(x)$, then we have also

$$\fint_{C'}\fint_{C'} |\chi_F(x)-\chi_F(y)|\,dx\,dy \le \fint_{C'}\fint_{C'} |\chi_E(x)-\chi_E(y)|\,dx\,dy + \frac{2}{|C|}\varepsilon^{-n}|C'\cap(F\Delta E^c)|. \tag{2.2.31}$$

It is clear that $F\Delta E = \Omega\setminus(F\Delta E^c)$, hence we can apply (2.2.29) to $L = C'\cap(F\Delta E)$.
Therefore, by (2.2.8), we obtain

$$\varepsilon^{n-1}\sum_{C'\in\mathcal{H}_\varepsilon}\fint_{C'}|\chi_F(x) - \fint_{C'}\chi_F|\,dx \le H_\varepsilon(E,\Omega) + 4\gamma\sum_{C'\in\mathcal{H}_\varepsilon}\mathsf{P}(F\Delta E, C').$$

Since $\sum_{C'\in\mathcal{H}_\varepsilon}\mathsf{P}(F\Delta E, C') \le \mathsf{P}(F\Delta E,\Omega) = \mathscr{H}^{n-1}((\mathscr{F}E\Delta\mathscr{F}F)\cap\Omega)$ by (2.2.26), we can pass to the supremum at the left hand side and we get (2.2.27). □

Let now $x \in \partial S_\nu$. If we denote by $Q_\nu(x,r)$ the cube of side length r centered in $x \in \mathbb{R}^n$ and with one face orthogonal to ν, by homogeneity we have

$$\lim_{r\to 0}\frac{H_\pm(S_\nu, Q_\nu(x,r))}{r^{n-1}} = H_\pm(S_\nu, Q_\nu(x,1)) = \varphi(\nu). \tag{2.2.32}$$

Theorem 2.2.5. *Let E be a set of finite perimeter and ν_E be its measure theoretic unit interior normal. Then, for $\mathscr{H}^{n-1}$-a.e. $x \in \mathscr{F}E$, we have*

$$\liminf_{r\to 0}\frac{H_-(E, Q_{\nu_E(x)}(x,r))}{\mathsf{P}(E, Q_{\nu_E(x)}(x,r))} \ge \varphi(\nu_E(x)). \tag{2.2.33}$$

Proof. By our previous remarks, the result holds if E is the half-space $\{y: (y-x)\cdot\nu_E(x)\geq 0\}$. Indeed, $\mathsf{P}(E, Q_{\nu_E(x)}(x,r)) = r^{n-1}$, so that (2.2.32) implies (2.2.33).

If E is a set of finite perimeter, for any $x \in \mathscr{F}E$ there exists the measure theoretic unit interior normal $\nu_E(x)$ and the approximate tangent space to the measure $|D\chi_E|$ is $\nu_E^{\perp}(x)$, namely (1.1.10) holds. This implies that

$$\lim_{r\to 0}\frac{\mathscr{H}^{n-1}(\mathscr{F}E\cap Q_{\nu_E(x)}(x,r))}{r^{n-1}} = \mathscr{H}^{n-1}((\nu_E^{\perp}(x))\cap Q_{\nu_E(x)}(x,1)) = 1.$$

Therefore, since $\mathsf{P}(E,\cdot) = \mathscr{H}^{n-1}\llcorner\mathscr{F}E$, by (1.1.9), we deduce that, for all $x \in \mathscr{F}E$ one has

$$\mathsf{P}(E, Q_{\nu_E(x)}(x,r)) = r^{n-1} + o(r^{n-1}). \tag{2.2.34}$$

If F is the subgraph of a C^1 function in a neighborhood of x, then $(F-x)/\rho$ is bi-Lipschitz equivalent to the half-space $S_{\nu_F(x)}$ in $Q_{\nu_F(x)}(0,1)$, with bi-Lipschitz constants converging to 1 as $\rho\to 0$. Hence, we can use a C^1 deformation map Φ with bi-Lipschitz constant close to 1 near to x to transform any disjoint family C_i' admissible for F into a disjoint family $D_i = \Phi(C_i)$; we can then find $C_i'' \subset D_i \subset C_i'''$ translated and scaled copies of C_i' whose diameters satisfy $\operatorname{diam}(C_i')/\operatorname{diam}(C_i)\sim 1$, $\operatorname{diam}(C_i''')/\operatorname{diam}(C_i)\sim 1$. Summing up, for $r>0$ small enough there exists a nonnegative modulus of continuity $\omega(r)$ satisfying

$$(1-\omega(r))|C''\cap S_{\nu_F(x)}| \leq |C'\cap F| \leq (1+\omega(r))|C'''\cap S_{\nu_F(x)}|$$

for $0<\rho<r$ and any translated copy C' of ρC contained in $Q_{\nu_F(x)}(x,r)$. We can choose the modulus of continuity in such a way that similar inequalities hold with the roles of F and $S_{\nu_F(x)}$ reversed. Hence, we have

$$\frac{|C'\cap F||C'\setminus F|}{|C'|^2} \leq (1+\omega(r))^2\frac{|C'''\cap S_{\nu_F(x)}||C'''\setminus S_{\nu_F(x)}|}{|C'''|^2}, \tag{2.2.35}$$

and

$$\frac{|C'\cap F||C'\setminus F|}{|C'|^2} \geq (1-\omega(r))^2\frac{|C''\cap S_{\nu_F(x)}||C''\setminus S_{\nu_F(x)}|}{|C''|^2}. \tag{2.2.36}$$

In particular, (2.2.36) and (2.2.32) imply

$$H_-(F, Q_{\nu_E(x)}(x,r)) \geq \varphi(\nu_F(x))r^{n-1} + o(r^{n-1}). \tag{2.2.37}$$

Now, in order to obtain (2.2.37) also for E, we are going to use the rectifiability of $\mathscr{F}E$ (Theorem 1.1.10 and Remark 1.1.11) and apply Lemma 2.2.4 to E and to the subgraph of one of the C^1 hypersurfaces Γ_i whose union covers $\mathscr{H}^{n-1}$-almost all of $\mathscr{F}E$ and such that $\nu_E|_{\Gamma_i}$ is the interior normal of the subgraph of Γ_i. Indeed, we fix i and observe that for $\mathscr{H}^{n-1}$-a.e. $x\in\Gamma_i\cap\mathscr{F}E$ one has

$$\mathscr{H}^{n-1}((\Gamma_i\Delta\mathscr{F}E)\cap B(x,r)) = o(r^{n-1}),$$

arguing as in the proof of [5, Lemma 3.7] and using the density properties of the Hausdorff measure (see [11, Theorem 2.56, Eq. (2.41)]). It follows easily that we have also

$$\mathscr{H}^{n-1}((\Gamma_i\Delta\mathscr{F}E)\cap Q_{\nu_E(x)}(x,r)) = o(r^{n-1})$$

for $\mathscr{H}^{n-1}$-a.e. $x\in\Gamma_i\cap\mathscr{F}E$. Now we use (2.2.28) choosing $\Omega = Q_{\nu_E(x)}(x,r)$ and F to be the subgraph of Γ_i inside $Q_{\nu_E(x)}(x,r)$, obtaining

$$H_-(F, Q_{\nu_E(x)}(x,r)) \leq H_-(E, Q_{\nu_E(x)}(x,r)) + 4\gamma\mathscr{H}^{n-1}((\Gamma_i\Delta\mathscr{F}E)\cap Q_{\nu_E(x)}(x,r)).$$

Since Γ_i is a C^1 hypersurface, we have (2.2.37) for F, with $\nu_F(x) = \nu_E(x)$. Since i is arbitrary this implies (2.2.37) for $\mathscr{H}^{n-1}$-a.e. $x\in\mathscr{F}E$.

Combining (2.2.34) and (2.2.37), we get the desired resut. $\square$

Theorem 2.2.6. *For any Borel set $B \subset \mathscr{F}E$ and $t > 0$, we have that*

$$\liminf_{r \to 0} \frac{H_-(E, Q_{\nu_E(x)}(x,r))}{\mathsf{P}(E, Q_{\nu_E(x)}(x,r))} \geq t \tag{2.2.38}$$

for all $x \in B$ implies $H_-(E,U) \geq t\mathscr{H}^{n-1}(B)$ for any open set $U \supset B$.

Proof. Without loss of generality, let $U \supset B$ be a bounded open set, since $\mathscr{H}^{n-1} \llcorner B$ is inner regular. For a given $\delta \in (0,1)$, we consider the family $\mathcal{F}$ of all the closed cubes inside U centered in the points $x \in B$ with one face oriented as $\nu_E(x)$, such that, if we denote their interior by $Q_{\nu_E(x)}(x,r)$, we have $H_-(E, Q_{\nu_E(x)}(x,r)) \geq t(1-\delta)\mathsf{P}(E, Q_{\nu_E(x)}(x,r))$ and $|D\chi_E|(\partial Q_{\nu_E(x)}(x,r)) = 0$.

In this way, since this family covers B finely, we can apply the version of Vitali theorem for cubes (see [128, Theorem 5.13]) and find a disjoint countable subfamily $\{\overline{Q_j}\}$ which covers $\mathscr{H}^{n-1}$-almost all of B. It is also clear that $\mathsf{P}(E, \overline{Q_j}) = \mathsf{P}(E, Q_j)$, hence we can use an open covering. Therefore, the superadditivity of $H_-(E,\cdot)$ implies

$$\begin{aligned} t\mathscr{H}^{n-1}(B) &\leq t\mathsf{P}(E, \bigcup_j Q_j) = t\sum_j \mathsf{P}(E, Q_j) \leq (1-\delta)^{-1} \sum_j H_-(E, Q_j) \\ &\leq (1-\delta)^{-1} H_-(E, \bigcup_j Q_j) \leq (1-\delta)^{-1} H_-(E, U). \end{aligned}$$

Letting $\delta \to 0$, we prove the theorem. □

We can now extend the result of Theorem 2.2.5 to $H_+(E,\cdot)$ using similar techniques.

Theorem 2.2.7. *Let E be a set of finite perimeter and ν_E be its measure theoretic unit interior normal. Then, for $\mathscr{H}^{n-1}$-a.e. $x \in \mathscr{F}E$, we have*

$$\limsup_{r \to 0} \frac{H_+(E, Q_{\nu_E(x)}(x,r))}{\mathsf{P}(E, Q_{\nu_E(x)}(x,r))} \leq \varphi(\nu_E(x)). \tag{2.2.39}$$

Proof. In the beginning of the proof of Theorem 2.2.5 we showed that $\mathsf{P}(E, Q_{\nu_E(x)}(x,r)) = r^{n-1} + o(r^{n-1})$ for all $x \in \mathscr{F}E$.
By (2.2.32), (2.2.39) holds if E is a half space S_ν. Then we need to use estimate (2.2.35) in order to prove the inequality in the case that E is a subgraph of a C^1 function in a neighborhood of x.
Finally, we switch the roles of F and E in (2.2.28) and we repeat the steps of the last part of the proof of Theorem 2.2.5 to obtain

$$H_+(E, Q_{\nu_E(x)}(x,r)) \leq \varphi(\nu_E(x))r^{n-1} + o(r^{n-1})$$

for $\mathscr{H}^{n-1}$-a.e. $x \in \mathscr{F}E$.
Combining these results, we obtain (2.2.39). □

In order to prove the upper estimate for H_+, we need to consider the inner regularization of the nondecreasing set functions $H_+(E,\cdot)$ defined on the open sets of $\mathbb{R}^n$.

Definition 2.2.8. Let $\mathcal{A}$ be the family of open sets in $\mathbb{R}^n$ and let $\alpha : \mathcal{A} \to [0, +\infty]$ be a nondecreasing set function. The inner regular envelope of α is the function $\alpha^* : \mathcal{A} \to \overline{\mathbb{R}}$ defined by

$$\alpha^*(A) := \sup\{\alpha(A') : \ A' \Subset A\}.$$

It is not hard to show (see for instance [60]) that α^* is the largest inner regular function smaller than α (namely $\alpha^*(A) = \sup\{\alpha^*(A') : A' \Subset A\}$). Recall also that any inner regular and subadditive function α is σ-subadditive, namely

$$\alpha(A) \leq \sum_{i=0}^{\infty} \alpha(A_i) \qquad \text{whenever } A \subset \bigcup_{i=0}^{\infty} A_i.$$

The proof of this statement can be adapted also to the case when α is weakly subadditive as our set function H_+, this leads to the following result.

Proposition 2.2.9. *$H_+^*(E, \cdot)$ is σ-subadditive and*

$$H_+^*(E, \Omega) \leq 2\gamma \mathsf{P}(E, \Omega). \tag{2.2.40}$$

Proof. Given open sets Ω_i, $i = 1, 2$, let $0 < t < H_+^*(E, \Omega_1 \cup \Omega_2)$. Then there exists $W \Subset \Omega_1 \cup \Omega_2$ such that $H_+(E, W) \geq t$. By [60, Lemma 14.20], there exist open sets Ω_i', $i = 1, 2$, such that $W \Subset \Omega_1' \cup \Omega_2'$ and $\Omega_i' \Subset \Omega_i$, $i = 1, 2$. Hence, we can find open sets W_i such that $\Omega_i' \Subset W_i \Subset \Omega_i$, $i = 1, 2$, and, by (2.2.23), we obtain

$$t \leq H_+(E, W) \leq H_+(E, W_1) + H_+(E, W_2) \leq H_+^*(E, \Omega_1) + H_+^*(E, \Omega_2).$$

Since $t < H_+^*(E, \Omega_1 \cup \Omega_2)$ is arbitrary, this proves the subadditivity.

Since $H_+^*(E, \cdot)$ is inner regular and subadditive the σ-subadditivity follows. The last statement follows by (2.2.24) and the inner regularity of $\mathscr{H}^{n-1} \llcorner \mathscr{F}E$. $\square$

We are now able to show the same result of Theorem 2.2.6 for H_+.

Theorem 2.2.10. *For any Borel set $B \subset \mathscr{F}E$ and $t > 0$, we have that*

$$\limsup_{r \to 0} \frac{H_+(E, Q_{\nu_E(x)}(x, r))}{\mathsf{P}(E, Q_{\nu_E(x)}(x, r))} \leq t \tag{2.2.41}$$

for all $x \in B$ implies $H_+^*(E, U) \leq t\mathsf{P}(E, U) + 2\gamma \mathsf{P}(E, U \setminus B)$ for any open set $U \supset B$.

Proof. Since H_+^* is inner regular, we may assume $U \supset B$ to be a bounded open set without loss of generality. We fix $\delta \in (0, 1)$ and we consider the family $\mathcal{F}$ of all the closed cubes inside U centered in the points $x \in B$ with one face oriented as $\nu_E(x)$, such that, if we denote their interior by $Q_{\nu_E(x)}(x, r)$, we have

$$H_+^*(E, Q_{\nu_E(x)}(x, r)) \leq H_+(E, Q_{\nu_E(x)}(x, r)) \leq (1 + \delta)t\mathsf{P}(E, Q_{\nu_E(x)}(x, r)),$$

and $|D\chi_E|(\partial Q_{\nu_E(x)}(x, r)) = 0$.

As in the proof of Theorem 2.2.6, we can apply the version of Vitali theorem for cubes (see [128, Theorem 5.13]) and find a disjoint countable subfamily $\{\overline{Q_j}\}$ which covers $\mathscr{H}^{n-1}$-almost all B. It is also clear that, since $\mathsf{P}(E, \overline{Q_j}) = \mathsf{P}(E, Q_j)$, then we have

$$\mathscr{H}^{n-1}(B \setminus \bigcup_j Q_j) = 0. \tag{2.2.42}$$

Therefore the subadditivity of $H_+^*(E, \cdot)$ and (2.2.40) imply

$$\begin{aligned}
H_+^*(E, U) &\leq H_+^*(E, \bigcup_{j=1}^{N} Q_j) + H_+^*(E, U \setminus \bigcup_{j=1}^{N} (1 - \delta)\overline{Q_j}) \\
&\leq (1 + \delta)t \sum_{j=1}^{N} \mathsf{P}(E, Q_j) + 2\gamma \mathsf{P}(E, U \setminus \bigcup_{j=1}^{N} (1 - \delta)\overline{Q_j}) \\
&\leq (1 + \delta)t\mathsf{P}(E, U) + 2\gamma \mathsf{P}(E, U \setminus \bigcup_{j=1}^{N} (1 - \delta)\overline{Q_j}).
\end{aligned}$$

Letting first $\delta \to 0$ and then $N \to +\infty$, we obtain

$$
\begin{aligned}
H_+^*(E,U) &\le t\mathsf{P}(E,U) + 2\gamma\mathsf{P}(E,U\setminus\bigcup_j Q_j) \\
&= t\mathsf{P}(E,U) + 2\gamma\mathsf{P}(E,U\setminus(B\cup\bigcup_j Q_j)) + 2\gamma\mathsf{P}(E,B\setminus\bigcup_j Q_j) \\
&\le t\mathsf{P}(E,U) + 2\gamma\mathsf{P}(E,U\setminus B),
\end{aligned}
$$

because of (2.2.42). □

Remark 2.2.11. We notice that, by combining Theorems 2.2.5 and 2.2.7, for $\mathscr{H}^{n-1}$-a.e. $x \in \mathscr{F}E$ we obtain

$$
\varphi(\nu_E(x)) \le \liminf_{r\to 0} \frac{H_-(E,Q_{\nu_E(x)}(x,r))}{\mathsf{P}(E,Q_{\nu_E(x)}(x,r))} \le \limsup_{r\to 0} \frac{H_+(E,Q_{\nu_E(x)}(x,r))}{\mathsf{P}(E,Q_{\nu_E(x)}(x,r))} \le \varphi(\nu_E(x)),
$$

which yields the following equalities:

$$
\liminf_{r\to 0} \frac{H_-(E,Q_{\nu_E(x)}(x,r))}{\mathsf{P}(E,Q_{\nu_E(x)}(x,r))} = \limsup_{r\to 0} \frac{H_+(E,Q_{\nu_E(x)}(x,r))}{\mathsf{P}(E,Q_{\nu_E(x)}(x,r))} = \varphi(\nu_E(x)). \tag{2.2.43}
$$

2.2.4 Final estimates

Now we use the results of the previous section to adapt the classical results concerning differentiation of Radon measures to the nondecreasing set functions $H_\pm(E,\cdot)$.

Theorem 2.2.12. *For any set of finite perimeter E in $\mathbb{R}^n$ one has*

$$
H_+(E) = H_-(E) = \int_{\mathscr{F}E} \varphi(\nu_E(x))\, d\mathscr{H}^{n-1}(x). \tag{2.2.44}
$$

Proof. We consider first the case of H_-. Then, fixed $t > 1$, we define the Borel sets

$$
D_k := \{x \in \mathscr{F}E : \varphi(\nu_E(x)) \in (t^k, t^{k+1}]\}
$$

for $k \in \mathbb{Z}$.

For any $\varepsilon_k > 0$ we can find compact sets $K_k \subset D_k$ such that

$$
\mathscr{H}^{n-1}(D_k \setminus K_k) < \varepsilon_k. \tag{2.2.45}
$$

Since this family of compact sets is disjoint, it is then clear that

$$
\min_{-J\le k\ne k'\le J} \operatorname{dist}(K_k, K_{k'}) > 0. \qquad \forall J \in \mathbb{N}.
$$

Hence, for any J, we can find a disjoint family of open sets $U_k \supset K_k$, for $-J \le k \le J$. By the superadditivity of H_-, Theorem 2.2.6 and (2.2.43), we get

$$
\begin{aligned}
H_-(E) &\ge H_-(E, \bigcup_{-J\le k\le J} U_k) \ge \sum_{-J\le k\le J} H_-(E,U_k) \\
&\ge \sum_{-J\le k\le J} t^k \mathscr{H}^{n-1}(K_k) \\
&\ge \sum_{-J\le k\le J} t^{-1} \int_{K_k} \varphi(\nu_E)\, d\mathscr{H}^{n-1} \\
&= t^{-1} \int_{\bigcup_{-J\le k\le J} K_k} \varphi(\nu_E)\, d\mathscr{H}^{n-1}
\end{aligned} \tag{2.2.46}
$$

for any $J \in \mathbb{N}$. Since the measure $\mathscr{H}^{n-1} \llcorner \mathscr{F}E$ is regular and ε_k are arbitrary, we can pass to the supremum to get

$$H_-(E) \geq t^{-1} \int_{\bigcup_{-J \leq k \leq J} D_k} \varphi(\nu_E)\, d\mathscr{H}^{n-1}$$

Finally, we pass to the supremum over J and then send $t \to 1$ to get

$$H_-(E) \geq \int_{\mathscr{F}E} \varphi(\nu_E)\, d\mathscr{H}^{n-1}. \tag{2.2.47}$$

Now we deal with H_+. Fixed $t > 1$, we define the Borel sets D_k as above. For $\varepsilon > 0$, we can therefore find open sets $U_k \supset D_k$ with

$$\sum_k t^{k+1} \mathsf{P}(E, U_k \setminus D_k) < \varepsilon, \quad \sum_k 2\gamma \mathsf{P}(E, U_k \setminus D_k) < \varepsilon.$$

Since $\bigcup_k U_k$ covers $\mathscr{H}^{n-1}$-almost all of $\mathscr{F}E$ we can cover $\mathbb{R}^n \setminus \bigcup_k U_k$ with an open set U_0 with $\mathsf{P}(E, U_0)$ arbitrarily small and use the σ-subadditivity of H_+^* and (2.2.40) to get

$$H_+^*(E, \mathbb{R}^n) \leq \sum_k H_+^*(E, U_k).$$

Now, using Theorem 2.2.10, we estimate

$$\begin{aligned} H_+^*(E, \mathbb{R}^n) &\leq \sum_k H_+^*(E, U_k) \\ &\leq \sum_k t^{k+1} \mathsf{P}(E, U_k) + 2\gamma \mathsf{P}(E, U_k \setminus D_k) \\ &\leq \sum_k t^{k+1} \mathsf{P}(E, D_k) + 2\varepsilon \\ &\leq \sum_k t \int_{(\mathscr{F}E) \cap D_k} \varphi(\nu_E)\, d\mathscr{H}^{n-1} + 2\varepsilon \\ &\leq t \int_{\mathscr{F}E} \varphi(\nu_E)\, d\mathscr{H}^{n-1} + 2\varepsilon. \end{aligned} \tag{2.2.48}$$

Now we let $\varepsilon \downarrow 0$ and $t \downarrow 1$ to get

$$H_+^*(E, \mathbb{R}^n) \leq \int_{\mathscr{F}E} \varphi(\nu_E)\, d\mathscr{H}^{n-1}.$$

We show now that $H_+^*(E, \mathbb{R}^n) = H_+(E, \mathbb{R}^n)$. Indeed, we need only to show $H_+(E, \mathbb{R}^n) \leq H_+^*(E, \mathbb{R}^n)$. Fix $W \Subset \mathbb{R}^n$ open and let Ω such that $W \Subset \Omega$; by (2.2.23) we have

$$H_+(E, \mathbb{R}^n) \leq H_+(E, \Omega) + H_+(E, \mathbb{R}^n \setminus \overline{W}),$$

since we can take $\tilde{\Omega}$ and $\tilde{W}$ such that $W \Subset \tilde{W} \Subset \tilde{\Omega} \Subset \Omega$ and write $\mathbb{R}^n = \tilde{\Omega} \cup (\mathbb{R}^n \setminus \overline{\tilde{W}})$. By (2.2.24), we have

$$H_+(E, \mathbb{R}^n) \leq H_+^*(E, \mathbb{R}^n) + 2\gamma \mathsf{P}(E, \mathbb{R}^n \setminus \overline{W}),$$

which implies $H_+(E, \mathbb{R}^n) \leq H_+^*(E, \mathbb{R}^n)$, since W is arbitrary. In this way we obtain the inequality

$$H_+(E) \leq \int_{\mathscr{F}E} \varphi(\nu_E)\, d\mathscr{H}^{n-1}. \tag{2.2.49}$$

Combining (2.2.47) and (2.2.49), we prove the theorem. □

Remark 2.2.13 (A local version of Theorem 2.2.12)**.** By similar arguments one can prove that $\mathsf{P}(E,\mathbb{R}^n)<\infty$ implies that the family

$$\mathcal{R}:=\left\{A\subset\mathbb{R}^n:\ A \text{ open, } H_{\pm}(E,A)=\int_{A\cap\mathscr{F}E}\varphi(\nu_E)\,d\mathscr{H}^{n-1}\right\}$$

is rich, namely the set $\{i\in[0,1]:\ A_i\notin\mathcal{R}\}$ is at most countable whenever the family $\{A_i\}_{i\in[0,1]}$ satisfies $A_i\Subset A_j$ for $i<j$.
Indeed, since the density arguments are local, one need just to start with $H_-(E,A)$ in (2.2.46) and with $H_+^*(E,A)$ in (2.2.48) and to estimate in a finer way. Then, we recall that $H_+^*(E,\cdot)=H_+(E,\cdot)$ on a rich family of open sets. More specifically, one can use (2.2.40) and an argument similar to the last part of the proof of Theorem 2.2.12 to prove that any open set $A\subset\mathbb{R}^n$ such that $|D\chi_E|(\partial A)=0$ belongs to this family.

2.2.5 Some examples

In this section we discuss a few examples and estimates of the function φ. We also introduce a variant of the functionals H_ε in which we allow for dilations ηC, for any $\eta\in(0,\varepsilon]$ (i.e. the size of the sets in the family need not be the same).

Covering with balls

If we choose the set C to be the unit ball $B(0,1)$, it is easy to see that the function φ is a constant ξ_n depending only on the space dimension. Indeed, in this case the functionals H_ε and $H_\pm$ are rotationally invariant.

We are also able to estimate ξ_n, see (2.2.53) below. A result due to Cianchi ([48]) shows that we have the following sharp form of the relative linear isoperimetric inequality in the unit ball B:

$$\frac{|E\cap B||B\setminus E|}{|B|^2}\le\frac{1}{4\omega_{n-1}}\mathsf{P}(E,B)\qquad\text{for any measurable set } E.$$

This inequality clearly gives us the upper bound

$$\xi_n=H_+^B(S_\nu,Q_\nu)\le\frac{1}{2\omega_{n-1}}\mathsf{P}(S_\nu,Q_\nu)=\frac{1}{2\omega_{n-1}}.\tag{2.2.50}$$

On the other hand, the derivation of a lower bound is related to the well-known Kepler's problem (see for instance [97, 151]). This problem, also called "packing problem", consists in looking for the best way to place finite unions of disjoint open balls with the same (small) radius inside a unit cube in $\mathbb{R}^n$ in order to cover as much volume as possible. As the radius tends to 0, this problem identifies the best fraction $\rho_n\in(0,1]$ of volume covered. Kepler's problem is highly non trivial, since only in 1998 Hales ([97,98]) was able to prove that in three dimensions the best packing is the face centered cubic lattice (which is the one used to pack oranges and cannon balls), and that $\rho_3=\frac{\pi}{3\sqrt{2}}$, as Kepler conjectured. In two dimensions the best packing is the exagonal lattice and therefore $\rho_2=\frac{\pi}{2\sqrt{3}}$, as it was proved by Thue in 1890 ([73, 152]). In dimensions higher than 3 the problem is still essentially open, even though the best packing constant has been recently determined in dimension 8 and 24 ([49, 154]). Nevertheless, it is not difficult to prove the existence of the constant ρ_n by standard subadditivity arguments.

Our aim is to give a lower estimate of the number of disjoint ε-balls which can stay inside Q_ν and are bisected by ∂S_ν. Thus, it is clear that this problem is related to the one of looking for the optimal fraction $\rho_n\in(0,1]$ of the volume of the n-dimensional unit cube covered by finite unions of disjoint balls with the same radii. We claim that we have

$$\xi_n\ge\frac{\rho_{n-1}}{2\omega_{n-1}}.\tag{2.2.51}$$

Indeed, we can cover $\partial S_\nu \cap Q_\nu$ with a number N_ε of $(n-1)$-dimensional ε-balls satisfying

$$N_\varepsilon \sim \rho_{n-1}\frac{1}{\omega_{n-1}\varepsilon^{n-1}}. \tag{2.2.52}$$

Such $(n-1)$-dimensional ε-balls can be seen as the sections $\partial S_\nu \cap B'$ for some disjoint n-dimensional ε-balls B' which are bisected by the hyperplane ∂S_ν and lie inside the cube Q_ν. Therefore, we get

$$\xi_n = H^B_-(S_\nu, Q_\nu) \geq \liminf_{\varepsilon\to 0} \varepsilon^{n-1}\frac{1}{2}N_\varepsilon = \frac{\rho_{n-1}}{2\omega_{n-1}}.$$

Combining (2.2.50) and (2.2.51), we obtain

$$\frac{\rho_{n-1}}{2\omega_{n-1}} \leq \xi_n \leq \frac{1}{2\omega_{n-1}}. \tag{2.2.53}$$

In particular, it is easy to see that $\rho_1 = 1$, since the ball centered in the origin of radius r coincides with the cube, being the interval $(-r, r)$. Therefore, we conclude that $\xi_2 = 1/(2\omega_1) = 1/4$.

We notice that we can use the above arguments to estimate φ also in the case when C is the spherical shell $B(0,1) \setminus \overline{B(0,r)}$, for some $r \in (0,1)$.
Indeed, it is clear that φ is a constant $\xi_{n,r}$ depending only on the interior radius and the space dimension, due to the rotational invariance. If we choose the arrangement of disjoint copies of εC which are bisected by ∂S_ν and cover the maximum fraction of surface area, then their number will be the same N_ε as in (2.2.52): in fact, $C \cap (\partial S_\nu)$ occupies the same surface area as $B(0,1) \cap (\partial S_\nu)$. Hence, we have

$$\xi_{n,r} \geq \frac{\rho_{n-1}}{2\omega_{n-1}}.$$

On the other hand, it is clear $C \subset B(0,1)$ and that any disjoint family of translations of εC generates a disjoint family of full ε-balls. Hence, the inequalities (2.2.10) and (2.2.50) imply

$$\xi_{n,r} \leq \frac{|B(0,1)|^2}{|B(0,1) \setminus \overline{B(0,r)}|^2}\frac{1}{2\omega_{n-1}} = \frac{1}{(1-r^n)^2 2\omega_{n-1}}.$$

Anisotropic coverings

We present now three examples of φ^C, for anisotropic sets C. For simplicity, we shall restrict ourself to dimension $n = 2$ for the actual calculations.

The unit square

Let us consider at first C be the unit cube $Q = (0,1)^n$ in $\mathbb{R}^n$.
In order to evaluate $\varphi^Q(\nu)$, we want to maximize in $x \in \mathbb{R}^n$ for any fixed unit vector ν the function

$$f(x,\nu) := \begin{cases} \frac{|(x+Q)\cap S_\nu||(x+Q)\setminus S_\nu|}{\mathsf{P}(S_\nu,(x+Q))} & \text{if } \mathsf{P}(S_\nu,(x+Q)) > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2.54}$$

We define then

$$g(\nu) := \sup_{x\in\mathbb{R}^n} f(x,\nu). \tag{2.2.55}$$

We observe that g is well defined and that the supremum is a maximum.
Indeed, for any fixed $\nu \in \mathbb{S}^{n-1}$, $f(x,\nu)$ is continuous in x. Clearly, $f(x,\nu) = 0$ if $x \notin \{y : -\sqrt{n} \leq y\cdot\nu \leq \sqrt{n}\}$, and, if $v\cdot\nu = 0$, then $f(x+v,\nu) = f(x,\nu)$. Thus, by simmetry, we can restrict ourselves to a compact set K_ν containing the origin inside the stripe

$\{y : -\sqrt{n} \le y \cdot \nu \le \sqrt{n}\}$ such that $f(K_\nu, \nu) = f(\mathbb{R}^n, \nu)$. Next, we notice that if we have a sequence $y_k \to x$ with $\mathsf{P}(S_\nu, (y_k + Q)) > 0$ for any k and $\mathsf{P}(S_\nu, (y_k + Q)) \to 0$ as $y_k \to x$, then $\min\{|(y_k + Q) \cap S_\nu|, |(y_k + Q) \setminus S_\nu|\} = o(\mathsf{P}(S_\nu, (y_k + Q)))$, because one of the two parts of $(y_k + Q)$ reduces to a simplex, for k large enough, and so its volume is proportional to the product of the basis area, $\mathsf{P}(S_\nu, (y_k + Q))$, and the relative height, which is going to zero. Hence, $\sup_{x\in\mathbb{R}^n} f(x, \nu) = \max_{x\in K_\nu} f(x, \nu)$.

By the definition of φ^Q, it follows that $\varphi^Q(\nu) \le 2g(\nu)$, since

$$\begin{aligned}\varphi^Q(\nu) &= \lim_{\varepsilon\to 0} \varepsilon^{n-1} \sup_{\mathcal{H}_\varepsilon} \sum_{Q'\in\mathcal{H}_\varepsilon} 2\frac{|Q' \cap S_\nu||Q' \setminus S_\nu|}{|Q'|^2} \\ &\le \lim_{\varepsilon\to 0} \sup_{\mathcal{H}_\varepsilon} 2g(\nu)\mathsf{P}\left(S_\nu, \bigcup_{Q'\in\mathcal{H}_\varepsilon} Q'\right) \le 2g(\nu)\mathsf{P}(S_\nu, Q_\nu) = 2g(\nu).\end{aligned}$$

On the other hand, by symmetry, there exists $\tau \ge 0$ such that $g(\nu) = f(\pm\tau\nu + tv, \nu)$, for any $v \in \mathbb{S}^{n-1}$ orthogonal to ν and $t > 0$. Then, for any ε, we can choose the disjoint family $\mathcal{G}_\varepsilon$ of translations of εQ inside Q_ν which corresponds to a subset of $\{\pm\tau\nu + tv : v \in \mathbb{S}^{n-1}, v \cdot \nu = 0, t > 0\}$ and which covers ∂S_ν up to a set of $\mathscr{H}^{n-1}$-measure going to zero as $\varepsilon \to 0$. The existence of such a family of translation for any fixed $\varepsilon > 0$ follows easily from the fact that one can cover $\mathbb{R}^n$ with a tessellation of open disjoint cubes, up to a Lebesgue negligible set. For such a sequence of families we obtain

$$\varphi^Q(\nu) \ge \lim_{\varepsilon\to 0} 2g(\nu)\mathsf{P}\left(S_\nu, \bigcup_{Q'\in\mathcal{G}_\varepsilon} Q'\right) = 2g(\nu).$$

Thus, we conclude that $\varphi^Q(\nu) = 2g(\nu)$.

We consider now the case $n = 2$. By the symmetries of the problem, we can redefine the function f as

$$f(q, m) := \frac{|Q \cap S^q_m||Q \setminus S^q_m|}{\mathsf{P}(S^q_m, Q)}, \tag{2.2.56}$$

where $Q = (0, 1) \times (0, 1)$, $S^q_m := \{(x, y) \in \mathbb{R}^2 : y \ge mx + q\}$, $q \in [-m, 1]$, $m = -(\nu_1/\nu_2) \in [0, +\infty)$. It is enough now to distinguish between the cases $0 \le m \le 1$ and $m \ge 1$.

If $0 \le m \le 1$, then we need only to consider $q \in [0, 1]$. The line $\{y = mx + q\}$ intersects the edges of Q in the points $A = (0, q)$ and

$$B = \begin{cases} (1, m + q) & \text{if } 0 \le q \le 1 - m, \\ \left(\frac{1-q}{m}, 1\right) & \text{if } 1 - m \le q \le 1. \end{cases}$$

Hence, we have

$$f(q, m) = \frac{1}{\sqrt{1 + m^2}} \begin{cases} \left(q + \frac{m}{2}\right) - \left(q + \frac{m}{2}\right)^2 & \text{if } 0 \le q \le 1 - m, \\ \frac{1-q}{2} - \frac{(1-q)^3}{4m} & \text{if } 1 - m \le q \le 1. \end{cases} \tag{2.2.57}$$

It is easy to see that, for any fixed m, the partial derivative in q is

$$\frac{\partial f}{\partial q}(q, m) = \frac{1}{\sqrt{1 + m^2}} \begin{cases} 1 - 2q - m & \text{if } 0 < q < 1 - m, \\ -\frac{1}{2} + \frac{3}{4m}(1 - q)^2 & \text{if } 1 - m < q < 1. \end{cases}$$

Hence, $\frac{\partial f}{\partial q} \geq 0$ if and only if

$$\begin{cases} q \leq \frac{1-m}{2} & \text{if } 0 < q < 1-m, \\ q \leq 1 - \sqrt{\frac{2m}{3}} & \text{if } 1-m < q < 1, \end{cases}$$

which means that

$$\max_{q\in[0,1-m]} f(q,m) = \frac{1}{4\sqrt{1+m^2}}$$

and

$$\max_{q\in[1-m,1]} f(q,m) = \frac{1}{\sqrt{1+m^2}} \begin{cases} \frac{1}{3}\sqrt{\frac{2m}{3}} & \text{if } \frac{2}{3} < m \leq 1, \\ \frac{m}{2}\left(1 - \frac{m}{2}\right) & \text{if } 0 \leq m \leq \frac{2}{3}. \end{cases}$$

Since

$$\frac{m}{2}\left(1 - \frac{m}{2}\right) \leq \frac{1}{4}$$

for any $m \in [0,1]$, and

$$\frac{1}{3}\sqrt{\frac{2m}{3}} \leq \frac{1}{4}$$

only for $0 \leq m \leq (27/32)$, it follows that

$$\max_{q\in[0,1]} f(q,m) = \frac{1}{\sqrt{1+m^2}} \begin{cases} \frac{1}{3}\sqrt{\frac{2m}{3}} & \text{if } \frac{27}{32} \leq m \leq 1, \\ \frac{1}{4} & \text{if } 0 \leq m \leq \frac{27}{32}. \end{cases} \tag{2.2.58}$$

If $m > 1$, we need only to consider $q \in [1-m, 1]$ and the intersections are

$$A = \begin{cases} (0,q) & \text{if } 0 \leq q \leq 1, \\ \left(-\frac{q}{m}, 0\right) & \text{if } 1-m \leq q \leq 0, \end{cases}$$

and $B = \left(\dfrac{1-q}{m}, 1\right)$. Hence, we have

$$f(q,m) = \frac{1}{\sqrt{1+m^2}} \begin{cases} \frac{1-q}{2} - \frac{(1-q)^3}{4m} & \text{if } 0 \leq q \leq 1, \\ \left(\frac{1}{2} - q\right) - \frac{1}{m}\left(\frac{1}{2} - q\right)^2 & \text{if } 1-m \leq q \leq 0. \end{cases} \tag{2.2.59}$$

We have that, for any fixed m, the partial derivative in q is

$$\frac{\partial f}{\partial q}(q,m) = \frac{1}{\sqrt{1+m^2}} \begin{cases} -\frac{1}{2} + \frac{3}{4m}(1-q)^2 & \text{if } 0 < q < 1, \\ -1 + \frac{2}{m}\left(\frac{1}{2} - q\right) & \text{if } 1-m < q < 0. \end{cases}$$

Hence, $\frac{\partial f}{\partial q} \geq 0$ if and only if

$$\begin{cases} q \leq 1 - \sqrt{\frac{2m}{3}} & \text{if } 0 < q < 1, \\ q \leq \frac{1-m}{2} & \text{if } 1-m < q < 0, \end{cases}$$

which means that

$$\max_{q\in[1-m,0]} f(q,m) = \frac{m}{4\sqrt{1+m^2}}$$

and

$$\max_{q\in[0,1]} f(q,m) = \frac{1}{\sqrt{1+m^2}} \begin{cases} \frac{1}{3}\sqrt{\frac{2m}{3}} & \text{if } 1 < m < \frac{3}{2}, \\ \frac{2m-1}{4m} & \text{if } m \geq \frac{3}{2}. \end{cases}$$

Since

$$\frac{2m-1}{m} \leq m$$

for any $m > 1$, and

$$\frac{1}{3}\sqrt{\frac{2m}{3}} \leq \frac{m}{4}$$

only for $1 < m \leq (32/27)$, it follows that

$$\max_{q\in[1-m,1]} f(q,m) = \frac{1}{\sqrt{1+m^2}} \begin{cases} \frac{1}{3}\sqrt{\frac{2m}{3}} & \text{if } 1 < m \leq \frac{32}{27}, \\ \frac{m}{4} & \text{if } m \geq \frac{32}{27}. \end{cases} \tag{2.2.60}$$

Because of the symmetry of the cube, we can conclude that

$$g(\nu) = \begin{cases} \frac{|\nu_2|}{4} & \text{if } |\nu_1| \leq \frac{27}{32}|\nu_2|, \\ \frac{1}{3}\sqrt{\frac{2}{3}|\nu_1||\nu_2|} & \text{if } \frac{27}{32}|\nu_2| \leq |\nu_1| \leq \frac{32}{27}|\nu_2|, \\ \frac{|\nu_1|}{4} & \text{if } |\nu_1| \geq \frac{32}{27}|\nu_2|, \end{cases}$$

which means

$$g(\nu) = \begin{cases} \frac{1}{3}\sqrt{\frac{2}{3}|\nu_1||\nu_2|} & \text{if } \frac{27}{32}|\nu_2| \leq |\nu_1| \leq \frac{32}{27}|\nu_2|, \\ \frac{\|\nu\|_\infty}{4} & \text{if } |\nu_1| \leq \frac{27}{32}|\nu_2| \text{ or } |\nu_1| \geq \frac{32}{27}|\nu_2|, \end{cases} \tag{2.2.61}$$

and so

$$\varphi^Q(\nu) = \begin{cases} \frac{2}{3}\sqrt{\frac{2}{3}|\nu_1||\nu_2|} & \text{if } \frac{27}{32}|\nu_2| \leq |\nu_1| \leq \frac{32}{27}|\nu_2|, \\ \frac{\|\nu\|_\infty}{2} & \text{if } |\nu_1| \leq \frac{27}{32}|\nu_2| \text{ or } |\nu_1| \geq \frac{32}{27}|\nu_2|. \end{cases}$$

It is clear that its 1-homogeneous extension $\Phi^Q(x,y) := \sqrt{x^2+y^2}\varphi^Q\left(\frac{(x,y)}{\sqrt{x^2+y^2}}\right)$ is indeed $\varphi^Q(x)$ and that it is not convex in the region $\{\frac{27}{32}|y| \leq |x| \leq \frac{32}{27}|y|\}$.

We also notice that $\max_{\nu\in\mathbb{S}^1} \varphi^Q(\nu) = (1/2)$, coherently with the results of [5] in the isotropic case.

The rectangles

Let us now deal with anisotropic coverings made with rectangles. Let $R = \prod_{i=1}^n (0,a_i)$, $a_i > 0$, then we can argue as before to show that

$$\varphi^R(\nu) = 2\sup_{x\in\mathbb{R}^n} \frac{|(x+R)\cap S_\nu||(x+R)\setminus S_\nu|}{|R|^2 \mathsf{P}(R_\nu,(x+R))}.$$

If $n = 2$, we can work with $R_\lambda = (0,1)\times(0,\lambda)$, $\lambda > 0$, since for a generic rectangle $R = (0,a)\times(0,b)$, we have $R = aR_\lambda$ if $\lambda = (b/a)$, and so $\varphi^R(\nu) = (1/a)\varphi^{R_\lambda}(\nu)$.
In order to deal with the explicit calculation, we can proceed in a similar way as before, by considering the function

$$f_\lambda(q,m) := \frac{|R_\lambda \cap S^q_m||R_\lambda \setminus S^q_m|}{|R_\lambda|^2 \mathsf{P}(S^q_m, R_\lambda)}, \tag{2.2.62}$$

where $S_m^q := \{(x,y) \in \mathbb{R}^2 : y \geq mx + q\}$, $q \in [-m, \lambda]$, $m = -(\nu_1/\nu_2) \in [0, +\infty)$, and dividing in the two cases $0 \leq m \leq \lambda$ and $m \geq \lambda$. Then, it is not difficult to show that we have

$$g^{R_\lambda}(\nu) = \begin{cases} \frac{|\nu_2|}{4} & \text{if } |\nu_1| \leq \frac{27}{32}\lambda|\nu_2|, \\ \frac{1}{3}\sqrt{\frac{2}{3\lambda}|\nu_1||\nu_2|} & \text{if } \frac{27}{32}\lambda|\nu_2| \leq |\nu_1| \leq \frac{32}{27}\lambda|\nu_2|, \\ \frac{|\nu_1|}{4\lambda} & \text{if } |\nu_1| \geq \frac{32}{27}\lambda|\nu_2|. \end{cases} \tag{2.2.63}$$

Since $\varphi^{R_\lambda}(\nu) = 2g^{R_\lambda}(\nu)$, then neither the 1-homogeneous extension of this function is convex. In conclusion, for the rectangle $R = (0,a) \times (0,b)$ we have

$$\varphi^{R}(\nu) = \begin{cases} \frac{|\nu_2|}{2a} & \text{if } a|\nu_1| \leq \frac{27}{32}b|\nu_2|, \\ \frac{2}{3}\sqrt{\frac{2}{3ab}|\nu_1||\nu_2|} & \text{if } \frac{27}{32}b|\nu_2| \leq a|\nu_1| \leq \frac{32}{27}b|\nu_2|, \\ \frac{|\nu_1|}{2b} & \text{if } a|\nu_1| \geq \frac{32}{27}b|\nu_2|. \end{cases}$$

It is also easy to see that

$$\max_{\nu \in \mathbb{S}^1} \varphi^R(\nu) = \frac{1}{2\min\{a,b\}}, \tag{2.2.64}$$

which gives the value of the constant $\xi(R)$ in the case of isotropic coverings with rectangles, when arbitrary rotations are allowed.

The ellipse

As a last example, let C be the ellipse $E = \{(x,y) : (x/a)^2 + (y/b)^2 < 1\}$, for some a, $b > 0$. In order to estimate φ from below, we choose the arrangement of copies of εE such that each one is bisected by the line ∂S_ν and the contiguous copies are tangent in the intersection between their boundaries and ∂S_ν. Hence, we need to evaluate the length of the segment intersected by a copy of εE on the line $\partial S_\nu = \{(x,y) \cdot \nu = 0\}$.
If $m = -\nu_1/\nu_2$, then $\partial S_\nu = \{y = mx\}$ and the intesections with ∂E are $\pm \dfrac{ab}{\sqrt{b^2 + m^2 a^2}}(1,m)$. Therefore[3], the length of the segment is $2\varepsilon ab\sqrt{(1+m^2)/(b^2+m^2a^2)}$. Since the copies of εE need to cover the unitary segment $(\partial S_\nu) \cap Q_\nu$, we obtain

$$\varphi^E(\nu) \geq \lim_{\varepsilon \to 0} \frac{1}{2}\varepsilon \left\lfloor \frac{\sqrt{b^2 + m^2 a^2}}{\varepsilon 2ab\sqrt{1+m^2}} \right\rfloor = \frac{1}{4ab}\sqrt{b^2\nu_2^2 + a^2\nu_1^2}.$$

In particular, if $a = b = 1$, we get $\varphi^B(\nu) \geq (1/4)$, coherently with (2.2.53).

2.2.6 Variants

Isotropic coverings

If we redefine H_ε in an isotropic way; that is, allowing for any orientation of the sets C' in the covering, we clearly get the rotational invariance for the modified functionals $H_\varepsilon^{\mathrm{iso}}$ and so the associated function φ^{iso} is a constant $\xi(C)$. This was done in [5] with C equal to the unit cube Q and it is not difficult to show that $\xi(Q) = 1/2$, as Ambrosio, Brezis, Bourgain and Figalli proved. Indeed, by the relative isoperimetric inequality in the unit cube

$$|E|(1 - |E|) \leq \frac{1}{4}\mathsf{P}(E, Q) \tag{2.2.65}$$

[3] If $\nu_2 = 0$, the length is $2\varepsilon b$.

for any measurable set $E \subset Q$ (see [5, Eq. (2.2)]), we have that

$$\frac{|Q' \cap S_\nu||Q' \setminus S_\nu|}{|Q'|^2} \le \frac{1}{4}\varepsilon^{1-n}\mathsf{P}(S_\nu, Q')$$

for any ε-cube Q'. This gives $H_+^{\mathrm{iso},Q}(S_\nu, Q_\nu) \le \frac{1}{2}\mathsf{P}(S_\nu, Q_\nu) = \frac{1}{2}$.
On the other hand, we can take the ε-cubes with one face oriented as ν, bisected by ∂S_ν and whose intersection with it gives the canonical partition of $\partial S_\nu \cap Q_\nu$ in order to obtain $H_\varepsilon^{\mathrm{iso},Q}(S_\nu, Q_\nu) \ge \frac{1}{2}\varepsilon^{n-1}\lfloor \varepsilon^{1-n} \rfloor$. This gives the result, coherently with [5].
Actually, using (2.2.13) and (2.2.65) we have immediately

$$\lim_{\varepsilon \to 0} H_\varepsilon^{\mathrm{iso},Q}(A, \Omega) = \frac{1}{2}\mathsf{P}(A, \Omega) \tag{2.2.66}$$

for any measurable set A and any open sets Ω.
It is also possible to show that we obtain a similar result if C is the pluri-rectangle $R = \prod_{j=1}^n (-a_j/2, a_j/2)$, for $a_j > 0$.
Indeed, we can take the copies of εR having one face oriented as ν, bisected by ∂S_ν and whose intersection with it gives a partition of $\partial S_\nu \cap Q_\nu$ with the largest cardinality; that is, at least $\lfloor \varepsilon^{1-n}/m \rfloor$, where

$$m := \min_{i=1,\dots,n} \prod_{j \ne i} a_j.$$

Thus, we obtain the lower bound $H_\varepsilon^{\mathrm{iso},R}(S_\nu, Q_\nu) \ge \frac{1}{2}\varepsilon^{n-1}\lfloor \varepsilon^{1-n}/m \rfloor$ and so $\xi(R) \ge \frac{1}{2m}$.
As for the upper bound, by (2.2.64) in the following subsection, we have $\xi(R) = 1/(2m) = 1/(2\min\{a_1, a_2\})$ if $n = 2$.

We notice that in these isotropic cases the result of Theorem 2.2.1 for sets of finite perimeter follows directly from Theorems 2.2.6 and 2.2.10 with $B = \mathscr{F}E$. Indeed, these theorems still hold true since $H_\varepsilon^{\mathrm{iso}}$ has the same properties of H_ε.
Then, if we take $t = \xi(C)$, Theorem 2.2.6 implies $H_-^{\mathrm{iso}}(E, U) \ge \xi(C)\mathscr{H}^{n-1}(\mathscr{F}E)$ for any open set $U \supset \mathscr{F}E$: it follows immediately that $H_-^{\mathrm{iso}}(E) \ge \xi(C)\mathsf{P}(E)$.
On the other hand, if we take an open set U containing $\mathscr{F}E$ and an open set $W \Subset U$, the subadditivity of $H_+^{\mathrm{iso},*}$ gives

$$H_+^{\mathrm{iso},*}(E) \le H_+^{\mathrm{iso},*}(E, U) + H_+^{\mathrm{iso},*}(E, \mathbb{R}^n \setminus \overline{W}) \le H_+^{\mathrm{iso},*}(E, U) + 2\gamma\mathsf{P}(E, \mathbb{R}^n \setminus \overline{W})$$

and clearly $\mathsf{P}(E, \mathbb{R}^n \setminus \overline{W}) \downarrow 0$ as $\overline{W} \uparrow U$.
Now, Theorem 2.2.10 yields $H_+^{\mathrm{iso},*}(E) \le \xi(C)\mathsf{P}(E, U) = \xi(C)\mathsf{P}(E)$. It suffices now to repeat the same argument at the end of the proof of Theorem 2.2.12 in order to obtain $H_+^{\mathrm{iso}}(E) \le \xi(C)\mathsf{P}(E)$, which concludes the proof.

Infinitesimal coverings

One may define a family of functionals similar to H_ε allowing for different dilations of the set C under a fixed level $\varepsilon > 0$. More specifically, we set

$$\tilde{H}_\varepsilon(A, \Omega) := \sup_{\mathcal{G}_\varepsilon} \sum_{C' \in \mathcal{G}_\varepsilon} 2(\varepsilon(C'))^{n-1}\frac{|C' \cap A||C' \setminus A|}{|C'|^2}, \tag{2.2.67}$$

where $C' = \varepsilon(C')(C + a)$, for some translation vector a, and $\mathcal{G}_\varepsilon$ is a disjoint family inside Ω of translations of the set ηC, for any $\eta \in (0, \varepsilon]$.

It is clear that $H_\varepsilon(A, \Omega) \le \tilde{H}_\varepsilon(A, \Omega)$, and so (2.2.6) follows for $\tilde{H}_\varepsilon$ in the case $\mathsf{P}(A) = +\infty$. We can define the functionals $\tilde{H}_\pm$ as liminf and limsup of $\tilde{H}_\varepsilon$. It is also not difficult to see that $\tilde{H}_\varepsilon$ and $\tilde{H}_\pm$ satisfy the same elementary properties of H_ε shown in Section 2.2.2. For instance, the homogeneity $\tilde{H}_{t\varepsilon}(tA, t\Omega) = t^{n-1}\tilde{H}_\varepsilon(A, \Omega)$ follows from the fact that each set $C'' \in \mathcal{G}_{t\varepsilon}$ can be seen as $C'' = tC'$, with $C' = \varepsilon(C')C$, for $\varepsilon(C') \leq \varepsilon$, and so $C' \in \mathcal{G}_\varepsilon$.

Since these functionals satisfy the same properties of H_ε and $H_\pm$, we can define the functions $\tilde{\varphi}_\pm(\nu) := \tilde{H}_\pm(S_\nu, Q_\nu)$ and show an analogous version of Proposition 2.2.2 for them. Then, one may follow the same steps in order to prove Theorem 2.2.1 for $\tilde{H}_\varepsilon$ in the rectifiable case. Thus, we obtain that for any set of finite perimeter A

$$\lim_{\varepsilon \to 0} \tilde{H}_\varepsilon(A) = \int_{\mathscr{F}A} \tilde{\varphi}(\nu_A)\, d\mathscr{H}^{n-1}.$$

Let us now consider the case in which the set C is the unit ball. Then $\tilde{\varphi}$ is a constant, since $\tilde{H}_\varepsilon$ is rotation invariant, and $\tilde{\varphi} \equiv 1/(2\omega_{n-1})$, since arbitrarily small radii are allowed. Indeed, the upper estimate is given by (2.2.50).

On the other hand, we notice that we can find a lower bound by considering only the family of balls which are bisected by the hyperplane ∂S_ν. For any fixed $\varepsilon > 0$, we can apply Vitali-Besicovitch Theorem ([11, Theorem 2.19]) to the measure $\mu = \mathscr{H}^{n-1} \llcorner Q''$, where Q'' is a unit cube in $\mathbb{R}^{n-1}$ and to a fine cover of balls with radii smaller than ε, in order to find a disjoint family $\mathcal{F}_{\varepsilon,(n-1)}$ of $(n-1)$-dimensional balls with radii smaller than ε such that

$$\mathscr{H}^{n-1}\left(Q'' \setminus \bigcup_{B'' \in \mathcal{F}_{\varepsilon,(n-1)}} B''\right) = 0. \tag{2.2.68}$$

Hence, we can take the family $\mathcal{F}_\varepsilon$ of n-dimensional balls bisected by $(\partial S_\nu) \cap Q_\nu$ and whose intersections with it generate the family $\mathcal{F}_{\varepsilon,(n-1)}$. Then, we use (2.2.68) to obtain

$$\begin{aligned}
\tilde{\varphi}(\nu) &\geq \lim_{\varepsilon \to 0} \frac{1}{2} \sum_{B' \in \mathcal{F}_\varepsilon} \varepsilon^{n-1} \\
&= \lim_{\varepsilon \to 0} \frac{1}{2\omega_{n-1}} \sum_{B'' \in \mathcal{F}_{\varepsilon,(n-1)}} \omega_{n-1}\varepsilon^{n-1} \\
&= \frac{1}{2\omega_{n-1}} \mathscr{H}^{n-1}((\partial S_\nu) \cap Q_\nu) = \frac{1}{2\omega_{n-1}}.
\end{aligned}$$

Finally, we observe that if we redefine $\tilde{H}_\varepsilon$ allowing for the possibility to rotate the sets C' in the covering, we obtain that $\tilde{\varphi}$ is a constant, as it happens for H_ε.

In particular, if we take C to be the unit cube Q as in [5], then, by (2.2.66) and (2.2.65), we have

$$\frac{1}{2}\mathsf{P}(A, \Omega) = \lim_{\varepsilon \to 0} H_\varepsilon^{\mathrm{iso},Q}(A, \Omega) \leq \lim_{\varepsilon \to 0} \tilde{H}_\varepsilon^{\mathrm{iso},Q}(A, \Omega) \leq \frac{1}{2}\mathsf{P}(A, \Omega),$$

which gives $\lim_{\varepsilon \to 0} \tilde{H}_\varepsilon^{\mathrm{iso},Q}(A, \Omega) = \frac{1}{2}\mathsf{P}(A, \Omega)$ for any measurable set A and open set Ω.

2.3 One-sided approximation of sets of finite perimeter

[4] It is a classical result in Geometric Measure Theory that a set of finite perimeter E in $\mathbb{R}^n$, for $n \geq 2$, can be approximated with smooth sets E_k such that

$$|E_k \Delta E| \to 0 \text{ and } \mathsf{P}(E_k) \to \mathsf{P}(E). \tag{2.3.1}$$

Suitable approximating smooth sets (see for instance [11, Theorem 3.42] and [111, Theorem 13.8]) are the superlevel sets of the convolutions of χ_E, which can be chosen for $\mathscr{L}^1$-a.e. $t \in (0, 1)$.

[4] This section is based on a joint work with Monica Torres [55].

In this section, we introduce a one-sided approximation which refines the classical result in the sense that it distinguishes between the superlevel sets for $\mathscr{L}^1$-a.e. $t \in \left(\frac{1}{2}, 1\right)$ from the ones corresponding to $\mathscr{L}^1$-a.e. $t \in \left(0, \frac{1}{2}\right)$, thus providing an *interior* and an *exterior* approximation of the set respectively (see Theorem 2.3.4). Indeed, in the first case, the difference between the level sets and the measure theoretic interior is asymptotically vanishing with respect to the $\mathscr{H}^{n-1}$-measure; in the latter, we obtain the same result for the measure theoretic exterior. In addition, the one-sided approximation allows to extend the first limit in (2.3.1) from the Lebesgue measure to any Radon measure μ absolutely continuous with respect to the Hausdorff measure $\mathscr{H}^{n-1}$. All in all, we shall prove that, if E is a bounded set of finite perimeter in $\mathbb{R}^n$ and μ is a Radon measure such that $|\mu| \ll \mathscr{H}^{n-1}$, there exist two sequences $\{E_{k;i}\}, \{E_{k;e}\}$ of sets with smooth boundary such that

$$|\mu|(E_{k;i} \Delta E^1) \to 0 \quad \text{and} \quad \mathsf{P}(E_{k,i}) \to \mathsf{P}(E), \tag{2.3.2}$$

$$|\mu|(E_{k;e} \Delta (E^1 \cup \mathscr{F}E)) \to 0 \quad \text{and} \quad \mathsf{P}(E_{k,e}) \to \mathsf{P}(E), \tag{2.3.3}$$

and

$$\mathscr{H}^{n-1}(\partial E_{k,i} \setminus E^1) \to 0 \quad \text{and} \quad \mathscr{H}^{n-1}(\partial E_{k,e} \setminus E^0) \to 0. \tag{2.3.4}$$

2.3.1 The approximation of E with respect to measures $\mu \ll \mathscr{H}^{n-1}$

In this section we will work in $\mathbb{R}^n$, for $n \geq 2$. Let $\rho \in C_c^\infty(B(0,1))$ be a smooth nonnegative radially symmetric mollifier. We denote the mollification of χ_E by $\chi_{E;\varepsilon_k}(x) := (\chi_E * \rho_{\varepsilon_k})(x)$ for some positive sequence $\varepsilon_k \to 0$. We define, for $t \in (0,1)$,

$$A_{k;t} := \{\chi_{E;\varepsilon_k} > t\}. \tag{2.3.5}$$

By Sard's theorem ([111, Lemma 13.15]), we know that, since $\chi_{E;\varepsilon_k} \in C^\infty(\mathbb{R}^n)$, $\mathscr{L}^1$-a.e. $t \in (0,1)$ is not the image of a critical point for $\chi_{E;\varepsilon_k}$. Hence, $A_{k;t}$ has a smooth boundary for these good values of t. Thus, for each k there exists a set $Z_k \subset (0,1)$, with $\mathscr{L}^1(Z_k) = 0$, which is the set of values of t for which $A_{k;t}$ has not a smooth boundary. If we set $Z := \bigcup_{k=1}^{+\infty} Z_k$, then $\mathscr{L}^1(Z) = 0$ and, for each $t \in (0,1) \setminus Z$ and for each k, $A_{k;t}$ has a smooth boundary.

It is clear that the convergence $\mathsf{P}(A_{k;t}) \to \mathsf{P}(E)$ for $\mathscr{L}^1$-a.e. $t \in (0,1)$ follows in the same way as in the classical proof of (2.3.1) (for which we refer to [11, Theorem 3.42]).

As for the first two limit in (2.3.2) and (2.3.2), they are an immediate consequence of the following result.

Theorem 2.3.1. *Let μ be a Radon measure such that $|\mu| \ll \mathscr{H}^{n-1}$ and E be a bounded set of finite perimeter in $\mathbb{R}^n$. Then:*

1. $|\mu|(E^1 \Delta A_{k;t}) \to 0$ *for any* $t \in \left(\frac{1}{2}, 1\right)$;
2. $|\mu|((E^1 \cup \mathscr{F}E) \Delta A_{k;t}) \to 0$ *for any* $t \in \left(0, \frac{1}{2}\right)$.

Proof. By Lemma 1.1.16, we have

$$\chi_{E;\varepsilon_k} \to \chi_E^*(x) \text{ for } \mathscr{H}^{n-1}\text{-a.e. } x. \tag{2.3.6}$$

We also notice that $\{0 < |\chi_{E;\varepsilon_k} - \chi_E^*| \leq 1\} \subset E_\delta := \{x \in \mathbb{R}^n : \operatorname{dist}(x, E) \leq \delta\}$, for any k if $\delta > \max \varepsilon_k$. Therefore, since E_δ is bounded, χ_{E_δ} is a summable majorant of $|\chi_{E;\varepsilon_k} - \chi_E^*|$ with respect to $|\mu|$, and so we can apply Lebesgue's dominated convergence theorem with respect to the measure $|\mu|$, obtaining $\|\chi_{E;\varepsilon_k} - \chi_E^*\|_{L^1(\mathbb{R}^n;|\mu|)} \to 0$. Then, we observe that, if $\frac{1}{2} < t < 1$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} |\chi_{E;\varepsilon_k}(x) - \chi_E^*(x)|d|\mu| &\geq \int_{A_{k;t}\setminus E^1} (\chi_{E;\varepsilon_k}(x) - \chi_E^*(x))d|\mu| + \int_{E^1\setminus A_{k;t}} (\chi_E^*(x) - \chi_{E;\varepsilon_k}(x))d|\mu| \\ &\geq (t - \frac{1}{2})|\mu|(A_{k;t} \setminus E^1) + (1-t)|\mu|(E^1 \setminus A_{k;t}) \\ &\geq \min\left\{t - \frac{1}{2}, 1 - t\right\} |\mu|(A_{k;t}\Delta E^1).\end{aligned}$$

Thus, for any $\frac{1}{2} < t < 1$ we obtain

$$|\mu|(A_{k;t}\Delta E^1) \leq \frac{\|\chi_{E;\varepsilon_k} - \chi_E^*\|_{L^1(\mathbb{R}^n;|\mu|)}}{\min\left\{t - \frac{1}{2}, 1 - t\right\}},$$

which implies point 1. Analogously, if $0 < t < \frac{1}{2}$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} |\chi_{E;\varepsilon_k}(x) - \chi_E^*(x)|d|\mu| &\geq \int_{A_{k;t}\setminus(E^1\cup\mathscr{F}E)} (\chi_{E;\varepsilon_k}(x) - \chi_E^*(x))d|\mu| + \\ &+ \int_{(E^1\cup\mathscr{F}E)\setminus A_{k;t}} (\chi_E^*(x) - \chi_{E;\varepsilon_k}(x))d|\mu| \\ &\geq t|\mu|(A_{k;t} \setminus (E^1 \cup \mathscr{F}E)) + \left(\frac{1}{2} - t\right) |\mu|((E^1 \cup \mathscr{F}E) \setminus A_{k;t}) \\ &\geq \min\left\{t, \frac{1}{2} - t\right\} |\mu|(A_{k;t}\Delta(E^1 \cup \mathscr{F}E)).\end{aligned}$$

Thus, for any $0 < t < \frac{1}{2}$,

$$|\mu|(A_{k;t}\Delta(E^1 \cup \mathscr{F}E)) \leq \frac{\|\chi_{E;\varepsilon_k} - \chi_E^*\|_{L^1(\mathbb{R}^n;|\mu|)}}{\min\left\{t, \frac{1}{2} - t\right\}},$$

which gives point 2. □

Therefore, Theorem 2.3.1 implies that we can choose the approximating sets $E_{k;i}$ and $E_{k;e}$ to be $A_{k;t}$ for any $t \in \left(\frac{1}{2}, 1\right) \setminus Z$ and $t \in \left(0, \frac{1}{2}\right) \setminus Z$, respectively.

Remark 2.3.2. If we choose $\mu = \mathscr{H}^{n-1} \llcorner \mathscr{F}E$, we obtain from Theorem 2.3.1:

1. $\mathscr{H}^{n-1}(\mathscr{F}E \cap A_{k;t}) \to 0$ for any $t \in \left(\frac{1}{2}, 1\right)$;
2. $\mathscr{H}^{n-1}(\mathscr{F}E \cap (\mathbb{R}^n \setminus A_{k;t})) \to 0$ for any $t \in \left(0, \frac{1}{2}\right)$.

Indeed, this is clear from the following identities:

$$\mathscr{F}E \cap (E^1\Delta A_{k;t}) = \mathscr{F}E \cap [(E^1 \setminus A_{k;t}) \cup (A_{k;t} \setminus E^1)] = \mathscr{F}E \cap A_{k;t},$$

$$\mathscr{F}E \cap (E\Delta A_{k;t}) = \mathscr{F}E \cap [(E \setminus A_{k;t}) \cup (A_{k;t} \setminus E)] = \mathscr{F}E \cap (\mathbb{R}^n \setminus A_{k;t}).$$

In particular, this implies also

$$\mathscr{H}^{n-1}(\mathscr{F}E \cap \partial A_{k;t}) \to 0 \ \text{ for any } t \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right),$$

Indeed, $\partial A_{k;t} \subset A_{k;s}$ for $\frac{1}{2} < s < t < 1$ and $\partial A_{k;t} \subset \mathbb{R}^n \setminus A_{k;s}$ for $0 < t \leq s < \frac{1}{2}$.

2.3.2 The one-sided convergence of boundaries

We prove now that indeed the sets $A_{k;t}$ satisfy the first limit in (2.3.4) for $\mathscr{L}^1$-a.e. $t \in \left(\frac{1}{2}, 1\right)$ and for a suitable sequence $\varepsilon_k \to 0$, independent of t. First, we recall the classical coarea formula, for which we refer to [69, Theorem 3.10].

Theorem 2.3.3. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz. Then, for any Borel-measurable set A, we have*

$$\int_A |\nabla u| \, dx = \int_{\mathbb{R}} \mathscr{H}^{n-1}(A \cap u^{-1}(t)) \, dt. \tag{2.3.7}$$

Theorem 2.3.4. *Let E be a set of finite perimeter in $\mathbb{R}^n$. There exists a sequence ε_k converging to 0 such that, if $\chi_{E;\varepsilon_k} := \chi_E * \rho_{\varepsilon_k}$ and $A_{k;t} := \{\chi_{E;\varepsilon_k} > t\}$, we have*

$$\lim_{k \to +\infty} \mathscr{H}^{n-1}(\partial A_{k;t} \setminus E^1) = 0 \tag{2.3.8}$$

for $\mathscr{L}^1$-a.e. $t \in \left(\frac{1}{2}, 1\right)$.

Proof. We take $s > \frac{1}{2}$ and a sequence ε_k, with $\varepsilon_k \to 0$. By the coarea formula (2.3.7), we have

$$\begin{aligned} \int_{A_{k;s} \setminus E^1} |\nabla \chi_{E;\varepsilon_k}| \, dx &= \int_0^1 \mathscr{H}^{n-1}(\chi_{E;\varepsilon_k}^{-1}(t) \cap (A_{k;s} \setminus E^1)) \, dt \\ &= \int_s^1 \mathscr{H}^{n-1}(\partial A_{k;t} \setminus E^1) \, dt, \end{aligned} \tag{2.3.9}$$

since, for $t \le s$,

$$\chi_{E;\varepsilon_k}^{-1}(t) \cap (A_{k;s} \setminus E^1) = \emptyset,$$

while, for $t > s$,

$$\chi_{E;\varepsilon_k}^{-1}(t) \cap (A_{k;s} \setminus E^1) = \chi_{E;\varepsilon_k}^{-1}(t) \setminus E^1 = \partial A_{k;t} \setminus E^1.$$

We claim that

$$\|\nabla \chi_{E;\varepsilon_k}\|_{L^1(A_{k;s} \setminus E^1; \mathbb{R}^n)} \to 0. \tag{2.3.10}$$

In order to prove the claim, we observe that $\nabla \chi_{E;\varepsilon_k} = (D\chi_E * \rho_{\varepsilon_k})$, and so

$$|\nabla \chi_{E;\varepsilon_k}| \le |D\chi_E| * \rho_{\varepsilon_k}. \tag{2.3.11}$$

Since $|E \Delta E^1| = 0$, (2.3.9), (2.3.11) and (1.1.9) yield

$$\begin{aligned} \|\nabla \chi_{E;\varepsilon_k}\|_{L^1(A_{k;s} \setminus E^1; \mathbb{R}^n)} &= \int_{\mathbb{R}^n} |\nabla \chi_{E;\varepsilon_k}| \, \chi_{A_{k;s} \setminus E} \, dx \\ &\le \int_{\mathbb{R}^n} (|D\chi_E| * \rho_{\varepsilon_k}) \, \chi_{A_{k;s} \setminus E} \, dx = \int_{\mathbb{R}^n} (\rho_{\varepsilon_k} * \chi_{A_{k;s} \setminus E}) \, d|D\chi_E| \\ &= \int_{\mathscr{F}E} (\rho_{\varepsilon_k} * \chi_{A_{k;s} \setminus E}) \, d\mathscr{H}^{n-1}. \end{aligned} \tag{2.3.12}$$

Thus, we need to investigate, for any $x \in \mathscr{F}E$, the behaviour of $(\rho_{\varepsilon_k} * \chi_{A_{k;s} \setminus E})(x)$ as $k \to +\infty$. We have

$$\begin{aligned} (\rho_{\varepsilon_k} * \chi_{A_{k;s} \setminus E})(x) &= \int_{\mathbb{R}^n} \varepsilon_k^{-n} \rho\left(\frac{x-y}{\varepsilon_k}\right) \chi_{A_{k;s}}(y) \, \chi_{(\mathbb{R}^n \setminus E)}(y) \, dy \\ &= [y = x + \varepsilon_k z] = \int_{B(0,1)} \rho(z) \, \chi_{A_{k;s}}(x + \varepsilon_k z) \, \chi_{(\mathbb{R}^n \setminus E)}(x + \varepsilon_k z) \, dz. \end{aligned}$$

We observe that $x + \varepsilon_k z \in \mathbb{R}^n \setminus E$ if and only if $z \in \frac{(\mathbb{R}^n \setminus E) - x}{\varepsilon_k}$: hence, by (1.1.12), we have

$$\chi_{(\mathbb{R}^n \setminus E)}(x + \varepsilon_k \cdot) = \chi_{\frac{(\mathbb{R}^n \setminus E) - x}{\varepsilon_k}}(\cdot) \to \chi_{H^-_{\nu_E(x)}}(\cdot) \;\text{ in }\; L^1(B(0,1)) \;\text{ as }\; k \to +\infty.$$

In particular, this means that the L^1 limit of $\chi_{(\mathbb{R}^n\setminus E)}(x+\varepsilon_k z)$ is not $\mathscr{L}^n$-a.e. zero only if $z\cdot\nu_E(x)\le 0$, so that we can restrict the integration domain to $B(0,1)\cap H^-_{\nu_E(x)}$. On the other hand, $x+\varepsilon_k z\in A_{k;s}=\{\chi_{E;\varepsilon_k}>s\}$ if and only if $\chi_{E;\varepsilon_k}(x+\varepsilon_k z)>s$. We see that

$$\begin{aligned}\chi_{E;\varepsilon_k}(x+\varepsilon_k z) &= \int_{\mathbb{R}^n}\rho_{\varepsilon_k}(x+\varepsilon_k z-y)\,\chi_E(y)\,dy\\ &= [y=x+\varepsilon_k z+\varepsilon_k u]=\int_{B(0,1)}\rho(u)\,\chi_E(x+\varepsilon_k(u+z))\,du.\end{aligned}$$

Arguing similarly as before by applying (1.1.11), we obtain $\chi_E(x+\varepsilon_k(z+\cdot))\to\chi_{H^+_{\nu_E(x)}}(z+\cdot)$ in $L^1(B(0,1))$ as $k\to+\infty$, for any $x\in\mathscr{F}E$ and $z\in B(0,1)$. Now, we recall that $z\cdot\nu_E(x)\le 0$, and since we have $\chi_{H^+_{\nu_E(x)}}(z+u)=1$ if and only if $0\le(z+u)\cdot\nu_E(x)$, we conclude that $0\le -z\cdot\nu_E(x)\le u\cdot\nu_E(x)\le 1$; that is, u belongs to the half ball $B(0,1)\cap H^+_{\nu_E}(x)$. This implies that, for any $x\in\mathscr{F}E$ and $z\in B(0,1)\cap H^-_{\nu_E}(x)$,

$$\lim_{k\to+\infty}\chi_{E;\varepsilon_k}(x+\varepsilon_k z):=v(x,z)=\int_{B(0,1)}\rho(u)\,\chi_{H^+_{\nu_E(x)}}(z+u)\,du\le\frac{1}{2}. \tag{2.3.13}$$

Therefore, since $0\le\chi_{A_{k;s}}(x+\varepsilon_k z)\le 1$ and $0\le\chi_{(\mathbb{R}^n\setminus E)}(x+\varepsilon_k z)\le 1$, these calculations yield

$$\begin{aligned}(\rho_{\varepsilon_k}*\chi_{A_{k;s}\setminus E})(x) &= \int_{B(0,1)}\rho(z)\,\chi_{A_{k;s}}(x+\varepsilon_k z)\,\chi_{(\mathbb{R}^n\setminus E)}(x+\varepsilon_k z)\,dz\\ &\to \int_{B(0,1)}\rho(z)\,\chi_{\{v(x,z)>s\}}(z)\,\chi_{H^-_{\nu_E(x)}}(z)\,dz,\end{aligned}\tag{2.3.14}$$

for any $x\in\mathscr{F}E$.

Equation (2.3.13) shows then that the limit in (2.3.14) is identically zero, since

$$\left\{z\in\mathbb{R}^n: v(x,z)>s>\frac{1}{2}\right\}\cap B(0,1)\cap H^-_{\nu_E}(x)=\emptyset,$$

for any $x\in\mathscr{F}E$.

We can now apply to (2.3.12) the Lebesgue dominated convergence theorem with respect to the measure $\mathscr{H}^{n-1}\llcorner\mathscr{F}E$ and the sequence of functions $\rho_{\varepsilon_k}*\chi_{A_{k;s}\setminus E}$ (since the constant 1 is clearly a summable majorant), thus obtaining (2.3.10).

Finally, up to passing to another subsequence (which we shall keep calling ε_k with a little abuse of notation), (2.3.9) and (2.3.10) yield (2.3.8), for $\mathscr{L}^1$-a.e. $t>s$. Since $s>\frac{1}{2}$ is fixed arbitrarily, we can conclude that (2.3.8) is valid for $\mathscr{L}^1$-a.e. $t>\frac{1}{2}$, up to a diagonalization argument. □

Remark 2.3.5. An analogous result holds for the measure theoretic exterior; namely, there exists a sequence ε_k converging to 0 such that, if $\chi_{E;\varepsilon_k}:=\chi_E*\rho_{\varepsilon_k}$ and $A_{k;t}:=\{\chi_{E;\varepsilon_k}>t\}$, we have

$$\lim_{k\to+\infty}\mathscr{H}^{n-1}(\partial A_{k;t}\setminus E^0)=0 \tag{2.3.15}$$

for $\mathscr{L}^1$-a.e. $t\in\left(0,\frac{1}{2}\right)$.

2.4 Weak* limit points of mollified sets of finite h-perimeter

[5] Through this section, let $\mathbb{G}$ be a stratified group, $\Omega\subset\mathbb{G}$ be an open set and E be a set of finite h-perimeter in Ω. Let also $\rho\in C_c(B(0,1))$ be a mollifier satisfying $\rho\ge 0$, $\rho(x)=\rho(x^{-1})$

[5] This section is based on a joint work with Valentino Magnani [51].

and $\int_{B(0,1)} \rho(y)\, dy = 1$. Thanks to Theorem 1.3.11, we know that the right mollification $\rho_\varepsilon * \chi_E$ is well behaved in the sense that $\rho_\varepsilon * \chi_E \in C^1_H(\Omega^{\mathcal{R}}_{2\varepsilon})$ and $\nabla_H(\rho_\varepsilon * \chi_E) \rightharpoonup D_H\chi_E$. It seems natural to ask whether we could obtain convergence results similar to those of Lemma 1.1.16 also in the stratified groups context. However, as pointed out in Section 1.3.3, we cannot expect in general a pointwise convergence $\mathscr{S}^{Q-1}$-a.e. result for the mollification $\rho_\varepsilon * \chi_E$, since we have the convergence to the precise representative defined using balls with respect to the right invariant distance $d^{\mathcal{R}}$, while the spherical Hausdorff measure is constructed using the left invariant distance d. Nevertheless, in this section we prove that the weak* limit points of the family $\rho_\varepsilon * \chi_E$ in $L^\infty(\Omega; |D_H\chi_E|)$ is precisely 1/2. This proposition is proved by a soft argument borrowed from [10, Proposition 4.3]. It should be noted that this weak* convergence does not require any existence of blow-ups, since it holds in any stratified groups, even in those where it is not known whether a De Giorgi's type theorem may hold. As an immediate consequence, in Theorem 2.4.5 we recover in any stratified group the weak* limits

$$
\begin{aligned}
\chi_E(\rho_\varepsilon * D_H\chi_E)\mu &\rightharpoonup \frac{1}{2} D_H\chi_E, \\
\chi_{\Omega\setminus E}(\rho_\varepsilon * D_H\chi_E)\mu &\rightharpoonup \frac{1}{2} D_H\chi_E
\end{aligned}
$$

which are point (2b) and (2c) of Lemma 1.1.16.

We start with a preliminary remark.

Remark 2.4.1. Let $\nu \in \mathcal{M}(\Omega)$ be any nonnegative Radon measure and denote by $\widetilde{\chi_E}$ any weak* cluster point of $\rho_\varepsilon * \chi_E$ in $L^\infty(\Omega; \nu)$. Then the lower semicontinuity of the L^∞-norm gives

$$
\|\widetilde{\chi_E}\|_{L^\infty(\Omega;\nu)} \le \liminf_{\varepsilon_k \to 0} \|(\rho_{\varepsilon_k} * \chi_E)\|_{L^\infty(\Omega,\nu)} \le 1
$$

for some positive sequence of ε_k converging to zero. Considering a nonnegative test function $\psi \in L^1(\Omega; \nu)$, we also have

$$
0 \le \int_\Omega \psi\, (\rho_\varepsilon * \chi_E)\, d\nu \to \int_\Omega \psi \widetilde{\chi_E}\, d\nu,
$$

hence proving that $0 \le \widetilde{\chi_E}(x) \le 1$ for ν-a.e. $x \in \Omega$.

We pass now to the main convergence result, which exhibits a deep similarity with the statement of [10, Proposition 4.3], where the authors study points of density 1/2 and relate them to the representation of perimeters in Wiener spaces.

Proposition 2.4.2. *Let $E \subset \Omega$ be a set of locally finite h-perimeter, $\rho \in C_c(B(0,1))$ be a mollifier satisfying $\rho \ge 0$, $\rho(x) = \rho(x^{-1})$ and $\int_{B(0,1)} \rho(y)\, dy = 1$. It follows that*

$$
D_H((\rho_\varepsilon * \chi_E)\chi_E) = (\rho_\varepsilon * \chi_E) D_H\chi_E + \chi_E(\rho_\varepsilon * D_H\chi_E) \quad \text{in } \mathcal{M}(\Omega^{\mathcal{R}}_{2\varepsilon}) \tag{2.4.1}
$$

for any $\varepsilon > 0$ such that $\Omega^{\mathcal{R}}_{2\varepsilon} \neq \emptyset$ and

$$
\rho_\varepsilon * \chi_E \overset{*}{\rightharpoonup} \frac{1}{2} \quad \text{as} \quad \varepsilon \to 0^+ \quad \text{in } L^\infty(\Omega; |D_H\chi_E|). \tag{2.4.2}
$$

Proof. It suffices to show that for any cluster point $\overline{\chi_E} \in L^\infty(\Omega; |D_H\chi_E|)$ of $\rho_\varepsilon * \chi_E$ as $\varepsilon \to 0$, then we have $\overline{\chi_E} = 1/2$ a.e. with respect to $|D_H\chi_E|$. We consider a positive vanishing sequence ε_k such that $\rho_{\varepsilon_k} * \chi_E \overset{*}{\rightharpoonup} \overline{\chi_E}$ in $L^\infty(\Omega; |D_H\chi_E|)$. Indeed, $\rho_\varepsilon * \chi_E$ is clearly uniformly bounded in $L^\infty(\Omega; |D_H\chi_E|)$, and therefore there exists at least a converging subsequence. We have first to prove (2.4.1). We know that $\rho_\varepsilon * \chi_E \in C^1_H(\Omega^{\mathcal{R}}_{2\varepsilon}) \cap C(\mathbb{G})$ by Lemma 1.3.10. Choosing any $\phi \in C^1_c(H\Omega^{\mathcal{R}}_{2\varepsilon})$ and taking into account (1.3.17), it follows that

$$\int_{\Omega^{\mathcal{R}}_{2\varepsilon}} (\rho_\varepsilon * \chi_E)\chi_E \mathrm{div}\phi \, dx = \int_{\Omega^{\mathcal{R}}_{2\varepsilon}} \chi_E \mathrm{div}(\phi(\rho_\varepsilon * \chi_E)) \, dx - \int_{\Omega^{\mathcal{R}}_{2\varepsilon}} \chi_E \langle \phi, \rho_\varepsilon * D_H\chi_E \rangle \, dx. \tag{2.4.3}$$

By Remark 1.3.4, we get

$$\int_{\Omega^{\mathcal{R}}_{2\varepsilon}} (\rho_\varepsilon * \chi_E)\chi_E \mathrm{div}\phi \, dx = -\int_{\Omega^{\mathcal{R}}_{2\varepsilon}} (\rho_\varepsilon * \chi_E) \langle \phi, D_H\chi_E \rangle - \int_{\Omega^{\mathcal{R}}_{2\varepsilon}} \chi_E \langle \phi, \rho_\varepsilon * D_H\chi_E \rangle \, dx,$$

which implies (2.4.1). Thus, taking into account (1.3.14) and (1.3.19), for any open set $A \Subset \Omega$ such that $A \subset \Omega^{\mathcal{R}}_{2\varepsilon}$, we obtain

$$|D_H((\rho_\varepsilon * \chi_E)\chi_E)|(A) \le \int_A \rho_\varepsilon * \chi_E \, d|D_H\chi_E| + \int_{E\cap A} \rho_\varepsilon * |D_H\chi_E| \, dx. \tag{2.4.4}$$

Now we observe that

$$\begin{aligned} \int_{E\cap A} \rho_\varepsilon * |D_H\chi_E| \, dx &= \int_{\mathbb{G}} \int_\Omega \chi_{E\cap A}(x)\rho_\varepsilon(yx^{-1}) \, d|D_H\chi_E|(y) \, dx \\ &= \int_\Omega (\rho_\varepsilon * \chi_{E\cap A})(y) \, d|D_H\chi_E|(y), \end{aligned} \tag{2.4.5}$$

since $\rho_\varepsilon(xy^{-1}) = \rho_\varepsilon(yx^{-1})$. We notice that $(\rho_\varepsilon * \chi_{E\cap A}) \le (\rho_\varepsilon * \chi_E)$ and

$$(\rho_\varepsilon * \chi_{E\cap A})(x) = 0$$

for any $x \notin A^{\mathcal{R},\varepsilon}$. Taking into account this vanishing property, along with (2.4.4), (2.4.5), (1.2.17) and the lower semicontinuity of the total variation, we let $\varepsilon = \varepsilon_k$ and, for any open set $A \Subset \Omega$, we obtain

$$|D_H\chi_E|(A) \le 2\int_A \overline{\chi_E} \, d|D_H\chi_E|,$$

since $\overline{\chi_E} \ge 0$, as observed in Remark 2.4.1, in the particular case $\nu = |D_H\chi_E|$. This inequality can be refined by noticing that, given any open set $A \subset \Omega$, if we take an increasing sequence of open sets A_j such that $A_j \Subset A_{j+1}$ and $\bigcup_j A_j = A$, the regularity of the Radon measure $|D_H\chi_E|$ yields

$$\begin{aligned} |D_H\chi_E|(A) &= \limsup_{j\to+\infty} |D_H\chi_E|(A_j) \\ &\le 2\limsup_{j\to+\infty} \int_{\overline{A_j}} \overline{\chi_E} \, d|D_H\chi_E| \le 2\int_A \overline{\chi_E} \, d|D_H\chi_E|. \end{aligned} \tag{2.4.6}$$

This means that $\overline{\chi_E}(x) \ge 1/2$ for $|D_H\chi_E|$-a.e. $x \in \Omega$. Finally, we notice that also $\Omega \setminus E$ is a set of locally finite h-perimeter in Ω and the equality

$$\rho_{\varepsilon_k} * \chi_\Omega = \rho_{\varepsilon_k} * \chi_E + \rho_{\varepsilon_k} * \chi_{\Omega\setminus E}$$

yields the weak* convergence of $\rho_{\varepsilon_k} * \chi_{\Omega\setminus E}$ to $1 - \overline{\chi_E}$ in $L^\infty(\Omega; |D_H\chi_E|)$. This implies that $1 - \overline{\chi_E} \ge 1/2$ at $|D_H\chi_E|$-a.e. point of Ω, therefore our claim is achieved. □

Lemma 2.4.3. *If $\gamma \in \mathcal{M}(\Omega)$ is a nonnegative measure and $f_k \overset{*}{\rightharpoonup} f$ in $L^\infty(\Omega;\gamma)$ as $k \to \infty$, then for every $\theta \in L^1(\Omega;\gamma)$, setting $\nu = \theta\gamma$, we have*

$$f_k\nu \rightharpoonup f\nu$$

in the sense of Radon measures on Ω and $f_k \overset{*}{\rightharpoonup} f$ in $L^\infty(\Omega; |\nu|)$.

Proof. For any $\phi \in C_c(\Omega)$, one clearly has $\phi\theta \in L^1(\Omega;\gamma)$ and so we get

$$\int_\Omega \phi f_k \, d\nu = \int_\Omega \phi\theta f_k \, d\gamma \to \int_\Omega \phi\theta f \, d\gamma = \int_\Omega \phi f \, d\nu.$$

We observe that $|\nu| = |\theta|\gamma$, and so, for any $\psi \in L^1(\Omega;|\nu|)$, we have $\psi|\theta| \in L^1(\Omega;\gamma)$. Thus, we obtain

$$\int_\Omega \psi f_k \, d|\nu| = \int_\Omega \psi|\theta| f_k \, d\gamma \to \int_\Omega \psi|\theta| f \, d\gamma = \int_\Omega \psi f \, d|\nu|,$$

concluding the proof. □

Remark 2.4.4. By (2.4.2) and the previous lemma, we notice that

$$(\rho_\varepsilon * \chi_E)\nu \rightharpoonup (1/2)\nu,$$

having $\nu = \theta|D_H\chi_E|$ and $\theta \in L^1(\Omega;|D_H\chi_E|)$.

Lemma 2.4.5. *Let $E \subset \Omega$ be a set of locally finite h-perimeter and $\rho \in C_c(B(0,1))$ be a mollifier satisfying $\rho \geq 0$, $\rho(x) = \rho(x^{-1})$ and $\int_{B(0,1)} \rho(y)\,dy = 1$. Then, we have*

$$\chi_E(\rho_\varepsilon * D_H\chi_E)\mu \rightharpoonup \frac{1}{2} D_H\chi_E, \tag{2.4.7}$$

$$\chi_{\Omega\setminus E}(\rho_\varepsilon * D_H\chi_E)\mu \rightharpoonup \frac{1}{2} D_H\chi_E. \tag{2.4.8}$$

Proof. By (2.4.1) and (1.3.14), we have

$$\chi_E(\rho_\varepsilon * D_H\chi_E)\mu = \chi_E\nabla_H(\rho_\varepsilon * \chi_E)\mu = D_H((\rho_\varepsilon * \chi_E)\chi_E) - (\rho_\varepsilon * \chi_E)D_H\chi_E \quad \text{in } \mathcal{M}(\Omega^{\mathcal{R}}_{2\varepsilon}).$$

Since for any $\phi \in C^1_c(H\Omega)$ we have $\mathrm{supp}(\phi) \subset \Omega^{\mathcal{R}}_{2\varepsilon}$ for $\varepsilon > 0$ sufficiently small, we get

$$\int_\Omega \phi\chi_E\nabla_H(\rho_\varepsilon * \chi_E)\,dx = -\int_\Omega (\rho_\varepsilon * \chi_E)\chi_E \,\mathrm{div}\phi\,dx - \int_\Omega (\rho_\varepsilon * \chi_E)\left\langle \phi, dD_H\chi_E\right\rangle.$$

We pass now to the limit on the right hand side, and, by Remark 2.4.4, we obtain that

$$-\int_\Omega (\rho_\varepsilon * \chi_E)\chi_E \,\mathrm{div}\phi\,dx - \int_\Omega (\rho_\varepsilon * \chi_E)\left\langle \phi, dD_H\chi_E\right\rangle$$

converges to

$$-\int_\Omega \chi_E \,\mathrm{div}\phi\,dx - \int_\Omega \frac{1}{2}\left\langle \phi, dD_H\chi_E\right\rangle$$

as $\varepsilon \to 0^+$. The last limit equals

$$\int_\Omega \frac{1}{2}\left\langle \phi, dD_H\chi_E\right\rangle.$$

Therefore, by the density of $C^1_c(H\Omega)$ in $C_c(H\Omega)$ with respect to the sup norm, we get (2.4.7). We observe that $\chi_{\Omega\setminus E}(\rho_\varepsilon * D_H\chi_E) = (1-\chi_E)\nabla_H(\rho_\varepsilon * D_H\chi_E)$, and so (2.4.8) follows from the first local weak* convergence of (1.3.18) and from (2.4.7). □

Chapter 3

Divergence-measure fields in the Euclidean space

3.1 Introduction

[1] The Gauss–Green formula is of significant importance in pure and applied Mathematics, as it plays a role in PDEs, Geometric Measure Theory and Mathematical Physics. In the last two decades, there have been many efforts in extending this formula to 'nonsmooth domains' and 'weakly regular vector fields'. Through this introduction, we shall illustrate the various versions of the Gauss–Green theorem in the whole space, to avoid unessential technicalities deriving from the restriction to an open set Ω.

The classical Gauss–Green theorem, or divergence theorem, asserts that, for a vector field $F \in C^1_c(\mathbb{R}^n; \mathbb{R}^n)$ and an open set E such that $\overline{E}$ is a C^1 smooth manifold with boundary, there holds

$$\int_E \mathrm{div} F \, dx = - \int_{\partial E} F \cdot \nu_E \, d\mathscr{H}^{n-1}, \tag{3.1.1}$$

where ν_E is the unit interior normal to ∂E and $\mathscr{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. The class of open sets considered above is too restrictive and this motivated the search for a wider class of integration domains for which the Gauss–Green theorem holds true in a suitable weaker form. Such a research was one of the aims which historically led to the theory of functions of bounded variation (BV) and sets of finite perimeter, or Caccioppoli sets. Indeed, an equivalent definition of set of finite perimeter requires the validity of a measure theoretic Gauss–Green formula restricted to compactly supported smooth vector fields.

While it is well known that a set of finite perimeter E may have very irregular topological boundary, even with positive Lebesgue measure, it is possible to consider a particular subset of ∂E, namely, the reduced boundary $\mathscr{F}E$, on which one can define a unit vector ν_E, called measure theoretic unit interior normal. In view of De Giorgi's theorem (Theorem 1.1.10), which shows the rectifiability of the reduced boundary, we know that $|D\chi_E| = \mathscr{H}^{n-1} \llcorner \mathscr{F}E$ and so (1.1.13) implies a first important relaxation of (3.1.1):

$$\int_E \mathrm{div} F \, dx = - \int_{\mathscr{F}E} F \cdot \nu_E \, d\mathscr{H}^{n-1}, \tag{3.1.2}$$

where E is a set of locally finite perimeter in $\mathbb{R}^n$ and $F \in \mathrm{Lip}_c(\mathbb{R}^n; \mathbb{R}^n)$. Even though this result is important because of the large family of integration domains, it is however restricted to a class of integrands with a still relatively strong regularity.

The subsequent generalization of (3.1.2) is due to Vol'pert [156] (we refer also to the classical monograph [157]). Thanks to further developments in the BV theory, he was able to consider

[1]This chapter is based on joint works with Kevin R. Payne [52], and with Gui-Qiang Chen and Monica Torres [40].

vector fields in $F \in BV(\mathbb{R}^n;\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n;\mathbb{R}^n)$ and bounded sets E of finite perimeter in $\mathbb{R}^n$, getting the following formulas

$$\mathrm{div}F(E^1) = -\int_{\mathscr{F}E} F_{\nu_E} \cdot \nu_E \, d\mathscr{H}^{n-1}, \tag{3.1.3}$$

$$\mathrm{div}F(E^1 \cup \mathscr{F}E) = -\int_{\mathscr{F}E} F_{-\nu_E} \cdot \nu_E \, d\mathscr{H}^{n-1}, \tag{3.1.4}$$

where E^1 is the measure theoretic interior of E and $F_{\pm\nu_E}$ are the exterior and interior traces of F on $\mathscr{F}E$; that is, the approximate limits of F at $x \in \mathscr{F}E$ restricted to the half spaces $\{y \in \mathbb{R}^n : (y-x)\cdot(\pm\nu_E(x)) \geq 0\}$. The existence of such traces follows from the fact that any BV function admits a representative which is well defined $\mathscr{H}^{n-1}$-a.e.

Not all distributional partial derivatives of a vector field are required to be Radon measures in (3.1.3) and (3.1.4), since only the divergence appears. Moreover, Gauss–Green formulas with vector fields of lower regularity have proved to be very important in applications, as for instance in hyperbolic conservations laws or in the study of contact interactions in Continuum Physics, [41, 138]. All of these facts finally led to the study of p-summable divergence-measure fields, namely, L^p vector fields on $\mathbb{R}^n$ whose divergence is a Radon measure.

Divergence-measure fields provide a natural way to extend the Gauss–Green formula. The family of L^p summable divergence-measure fields, denoted by $\mathcal{DM}^p$, clearly generalizes the vector fields of bounded variation. It was first introduced by Anzellotti for $p = \infty$ in [23], where he studied different pairings between vector fields and gradients of weakly differentiable functions. Thus, he considered $F \in \mathcal{DM}^\infty$ in order to define pairings between bounded vector fields and vector valued measures given by weak gradients of BV functions. One of the main results is [23, Theorem 1.2], which shows the existence of $L^\infty(\partial\Omega)$ traces of the normal component of $\mathcal{DM}^\infty(\Omega)$ fields on the boundary of open bounded sets Ω with Lipschitz boundary. These traces are referred to as *normal traces* in the literature.

After the works of Anzellotti ([23, 24]), the notion of divergence-measure fields was rediscovered in the early 2000s by many authors with different purposes. Chen and Frid proved generalized Gauss–Green formulas for divergence-measure fields on open bounded sets with Lipschitz deformable boundary (see [41, Theorem 2.2] and [43, Theorem 3.1]), motivated by applications to the theory of systems of conservation laws with the Lax entropy condition. The idea of their proof rests on an approximation argument, which allows to obtain a Gauss–Green formula on a family of Lipschitz open bounded sets approximating the given integration domain. Later, Chen, Torres and Ziemer generalized this method to the case of sets of finite perimeter in order to extend the result in the case $p = \infty$, achieving Gauss–Green formulas for essentially bounded divergence-measure fields and sets of finite perimeter ([45, Theorem 5.2]). Then, Chen and Torres [44] applied this theorem to the study of the trace properties of solutions of nonlinear hyperbolic systems of conservation laws. Further studies, improvements and simplifications of [45] have subsequently appeared in [40, 52, 55].

It is of interest to mention also other methods to prove the Gauss–Green formula, and different applications. Degiovanni, Marzocchi and Musesti in [66] and Schuricht in [138] were interested in the existence of a normal trace under weak regularity hypotheses in order to achieve a representation formula for Cauchy fluxes, contact interactions and forces in the context of the foundations of Continuum Mechanics. As is well explained in [138], the search for a rigorous proof of Cauchy's stress theorem under weak regularity assumptions is a common theme in much of the literature on divergence-measure fields. The Gauss–Green formulas obtained in [66,138] are valid for $F \in \mathcal{DM}^p(\mathbb{R}^n)$ and $p \geq 1$, even though the domains of integration E must be taken from a suitable subalgebra of sets of finite perimeter, related to the vector field F.

Šilhavý in [144] also studied the problem of finding a representation of Cauchy fluxes through traces of suitable divergence-measure fields. He gave a detailed description of generalized Gauss–Green formulas for $\mathcal{DM}^p$-fields with respect to $p \in [1, \infty]$ and suitable hypotheses on concentration of $\mathrm{div}F$. In particular, he provided sufficient conditions under which the interior normal traces (and so also the exterior) can be seen as integrable functions with respect to the measure $\mathscr{H}^{n-1}$ on the reduced boundary of a set of finite perimeter. We should also note that Šilhavý studied the so-called extended divergence-measure fields, already introduced by Chen-Frid in [43], which are vector valued Radon measures whose divergence is still a Radon measure. He proved absolute continuity results and Gauss–Green formulas in [145, 146]. It is also worth to mention the paper by Ambrosio, Crippa and Maniglia [8], where the authors employed techniques similar to the original ones of Anzellotti and studied a class of essentially bounded divergence-measure fields induced by functions of bounded deformation. Their results were motivated by the aim of extending DiPerna-Lions theory of the transport equation to special vector fields with bounded deformation.

In the last decades and more recently, Anzellotti's pairings and Gauss–Green formulas have appeared in several applied and theoretical questions, as the 1-Laplace equation, minimal surface equation, the obstacle problem for the area functional and theories of integration to extend the Gauss–Green theorem. We refer for instance to the works [65,104,106,107,131,134–136]. Recently Anzellotti's pairing theory has been extended in [57], see also [58,59], where the authors have also established integration by parts formulas for essentially bounded divergence-measure fields, sets of finite perimeter and essentially bounded scalar functions of bounded variation.

In the context of unbounded divergence-measure fields, in [52, Example 6.1] it was showed that, for any $p \in [1, \infty)$, there exists $F \in \mathcal{DM}^p_{\mathrm{loc}}(\mathbb{R}^n) \setminus \mathcal{DM}^\infty_{\mathrm{loc}}(\mathbb{R}^n)$ which fails to have locally integrable interior and exterior normal traces on the boundary of a smooth set. Nevertheless, in the joint work with Gui-Qiang Chen and Monica Torres [40] the case of $\mathcal{DM}^p$-fields for $p < \infty$ is considered, new integration by parts formulas are presented and the normal trace functional is studied in relation with the Leibniz rules. In particular, if p and p' are Sobolev conjugate exponents, a new Leibniz rule for $\mathcal{DM}^p$-fields and essentially bounded scalar functions with gradient in $L^{p'}$ is established. Theorem 3.2.3 is a refinement of this result.

In this chapter we shall present the alternative approach developed in [52], which follows the original idea employed by Vol'pert to prove (3.1.3) and (3.1.4). While the statement of the fundamental result (Theorem 3.3.4) is essentially the same as the main result in [45, Theorem 5.2], our proof is much simpler. Indeed, beyond known facts from Geometric Measure Theory concerning sets of finite perimeter and functions of bounded variation, it relies only on the following three ingredients:

1. the absolute continuity property of the divergence of a field $F \in \mathcal{DM}^\infty$: $|\mathrm{div}F| \ll \mathscr{H}^{n-1}$ (Theorem 3.2.2);

2. the Leibniz rule between fields in $\mathcal{DM}^\infty$ and essentially bounded BV scalar functions (Theorem 3.2.4);

3. the divergence theorem in the case of compactly supported divergence-measure fields (Lemma 3.3.3): if F has compact support, then $\mathrm{div}F(\mathbb{R}^n) = 0$.

The Gauss–Green theorem for essentially bounded divergence-measure fields and sets of finite perimeter states that, if $F \in \mathcal{DM}^\infty(\mathbb{R}^n)$ and E is a set of locally finite perimeter in $\mathbb{R}^n$, then there exist *interior and exterior normal traces of F on $\mathscr{F}E$*; that is,

$$(\mathcal{F}_i \cdot \nu_E), (\mathcal{F}_e \cdot \nu_E) \in L^\infty(\mathscr{F}E; \mathscr{H}^{n-1})$$

such that, for any $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n)$, we have

$$\int_{E^1} \varphi \, d\mathrm{div}F + \int_E F \cdot \nabla\varphi \, dx = - \int_{\mathscr{F}E} \varphi \, \mathcal{F}_i \cdot \nu_E \, d\mathscr{H}^{n-1},$$
$$\int_{E^1 \cup \mathscr{F}E} \varphi \, d\mathrm{div}F + \int_E F \cdot \nabla\varphi \, dx = - \int_{\mathscr{F}E} \varphi \, \mathcal{F}_e \cdot \nu_E \, d\mathscr{H}^{n-1}.$$

In addition, the following trace estimates hold:

$$\|\mathcal{F}_i \cdot \nu_E\|_{L^\infty(\mathscr{F}E;\mathscr{H}^{n-1})} \leq \|F\|_{L^\infty(E;\mathbb{R}^n)} \quad \text{and} \quad \|\mathcal{F}_e \cdot \nu_E\|_{L^\infty(\mathscr{F}E;\mathscr{H}^{n-1})} \leq \|F\|_{L^\infty(\mathbb{R}^n \setminus E;\mathbb{R}^n)}. \tag{3.1.5}$$

The chapter is structured in the following way. In Section 3.2, we recall the notion of p-summable divergence-measure fields and their absolute continuity properties, from which general Leibniz rules for any $p \in [1, \infty]$ are derived. Then, Section 3.3 is devoted to the study of the Gauss–Green formulas on sets of finite perimeter in the case $p = \infty$. In addition, we give the definition of the interior and exterior normal traces and we obtain the integration by parts formulas. The properties of the normal traces are further investigated in Section 3.4, where we show their consistency in the case of a continuous vector field and their locality. Finally, in Section 3.5 we apply the integration by parts formula previously obtained to derive general Green's identities for sets of finite perimeter and Lipschitz scalar function whose distributional Laplacian is a Radon measure.

3.2 Product rules for divergence-measure fields

In the rest of the chapter, Ω will denote an open subset of $\mathbb{R}^n$.

Definition 3.2.1. Let $1 \leq p \leq \infty$.

a) A vector field $F \in L^p(\Omega; \mathbb{R}^n)$ is a *divergence-measure field*, and we write $F \in \mathcal{DM}^p(\Omega)$, if the distributional divergence $\mathrm{div}F$ is a real finite Radon measure on Ω.

b) A vector field F is a *locally divergence-measure field*, and we write $F \in \mathcal{DM}^p_{\mathrm{loc}}(\Omega)$, if $F \in \mathcal{DM}^p(W)$ for any open set $W \Subset \Omega$.

In the case $p = \infty$, F is called a *(locally) essentially bounded divergence-measure field.*

It is worth mentioning that, if $F = (F_1, \dots, F_n)$ is a vector field with components $F_j \in BV(\Omega) \cap L^p(\Omega)$, then $F \in \mathcal{DM}^p(\Omega)$; however, cancellations in the singular part of the measure $\mathrm{div}F$ can allow for $\mathcal{DM}^p(\Omega)$ without having components in $BV(\Omega) \cap L^p(\Omega)$.

A first important result concerns the absolute continuity properties of $\mathrm{div}F$ with respect to the Sobolev capacity and the Hausdorff measure, which depends on the Lebesque index p for $F \in \mathcal{D}M^p_{\mathrm{loc}}(\Omega)$. In what follows, we denote by p' the conjugate exponent to p; that is, the real positive number satisfying $\frac{1}{p} + \frac{1}{p'} = 1$.

Theorem 3.2.2. *If $n/(n-1) \leq p \leq \infty$ and $F \in \mathcal{DM}^p_{\mathrm{loc}}(\Omega)$, then $|\mathrm{div}F| \ll \mathbf{C}_{p'}$. In particular, for any Borel set E such that $\mathscr{H}^{n-p'}(E) < \infty$ we have $|\mathrm{div}F|(E) = 0$.*

If $p = \infty$, $|\mathrm{div}F| \ll \mathscr{H}^{n-1}$.

Proof. If $n/(n-1) \leq p < \infty$, the result follows from [132, Theorem 2.8] and Theorem 1.1.22 (we refer also to [144, Theorem 3.2]). If $p = \infty$, then [144, Theorem 3.2] implies $|\mathrm{div}F| \ll \mathscr{H}^{n-1}$, and so, by Theorem 1.1.22, we also obtain $|\mathrm{div}F| \ll \mathbf{C}_1$. $\square$

We notice that, if $1 \leq p < n/(n-1)$, we can always find a field $F \in \mathcal{DM}^p_{\mathrm{loc}}(\Omega)$ such that $\mathrm{div}F = \delta_{x_0}$ for some $x_0 \in \Omega$, namely

$$F(x) := \frac{1}{n\omega_n} \frac{x - x_0}{|x - x_0|^n}.$$

Therefore, we cannot expect to extend the previous result for such values of p.

In addition, Šilhavý proved in [144, Example 3.3 and Proposition 6.1] that Theorem 4.2.7 is sharp also for $p \geq n/(n-1)$ in the sense that for any $\varepsilon > 0$ there exists $F \in \mathcal{DM}^p_{\rm loc}(\Omega)$ such that $|{\rm div}F|$ is not absolutely continuous with respect to $\mathscr{H}^{n-p'+\varepsilon}$.

We prove now a result concerning the product rule between a field $F \in \mathcal{DM}^p_{\rm loc}(\Omega)$ and a scalar function $g \in L^\infty_{\rm loc}(\Omega) \cap W^{1,p'}_{\rm loc}(\Omega)$. This provides an extension to [40, Proposition 3.1] which we could not find in the literature.

Theorem 3.2.3. *Let $p \in [1,\infty]$, $F \in \mathcal{DM}^p_{\rm loc}(\Omega)$ and $g \in L^\infty_{\rm loc}(\Omega)$ with $\nabla g \in L^{p'}_{\rm loc}(\Omega;\mathbb{R}^n)$, then we have $gF \in \mathcal{DM}^p_{\rm loc}(\Omega)$ and*

$$
{\rm div}(gF) = g^* {\rm div}F + F \cdot \nabla g \, \mathscr{L}^n, \tag{3.2.1}
$$

where g^* is the p'-quasicontinuous representative of g if $p \geq n/(n-1)$, and it is the continuous representative of g if $p \in [1, n/(n-1))$.

Proof. It is clear that $gF \in L^p_{\rm loc}(\Omega;\mathbb{R}^n)$.

We consider now the mollification of g, $g_\varepsilon := g * \rho_\varepsilon$, where $\rho \in C^\infty_c(B(0,1))$ is radially simmetric, with $\rho \geq 0$ and $\int_{B(0,1)} \rho \, dx = 1$ and $\rho_\varepsilon(x) := \varepsilon^{-n}\rho(x/\varepsilon)$. Then we obtain

$$
{\rm div}(g_\varepsilon F) = g_\varepsilon {\rm div}F + F \cdot \nabla g_\varepsilon \, \mathscr{L}^n \tag{3.2.2}
$$

in the sense of distributions: indeed, for any $\phi \in C^1_c(\Omega)$ we have

$$
\begin{aligned}
\int_\Omega g_\varepsilon F \cdot \nabla \phi \, dx &= \int_\Omega F \cdot \nabla(g_\varepsilon \phi) \, dx - \int_\Omega \phi F \cdot \nabla g_\varepsilon \, dx \\
&= - \int_\Omega \phi g_\varepsilon \, d{\rm div}F - \int_\Omega \phi F \cdot \nabla g_\varepsilon \, dx.
\end{aligned}
$$

In particular, if we take $\phi \in C^1_c(\Omega')$ for some open set $\Omega' \Subset \Omega$, this implies that

$$
\left| \int_\Omega g_\varepsilon F \cdot \nabla \phi \, dx \right| \leq \|\phi\|_{L^\infty(\Omega')} \left(\|g\|_{L^\infty(\Omega')} |{\rm div}F|(\Omega') + \|F\|_{L^p(\Omega';\mathbb{R}^n)} \|\nabla g\|_{L^{p'}(\Omega';\mathbb{R}^n)} \right).
$$

Therefore, for any fixed $\Omega' \Subset \Omega$, we showed that $\{{\rm div}(g_\varepsilon F)\}_\varepsilon$ is a uniformly bounded sequence in $\mathcal{M}(\Omega')$. However, it is clear that ${\rm div}(g_\varepsilon F) \rightharpoonup {\rm div}(gF)$ in the duality with $C^1_c(\Omega)$, hence we conclude that ${\rm div}(g_\varepsilon F) \rightharpoonup {\rm div}(gF)$ in $\mathcal{M}_{\rm loc}(\Omega)$, by the density of $C^1_c(\Omega)$ in $C_c(\Omega)$.
Now we need to pass to the limit as $\varepsilon \to 0$ also in the right hand side of (3.2.2).
If $p \geq n/(n-1)$, which means $p' \in [1,n]$, we have that $g_\varepsilon(x) \to g^*(x)$ at p'-quasi every $x \in \Omega$, by Theorem 1.1.24 since $g \in W^{1,p'}_{\rm loc}(\Omega)$. This implies that $g_\varepsilon(x) \to g^*(x)$ for $|{\rm div}F|$-a.e. $x \in \Omega$, since $|{\rm div}F| \ll \mathbf{C}_{p'}$ by Theorem 4.2.7. On the other hand, it is clear that $|g_\varepsilon(x)| \leq \|g\|_{L^\infty(\Omega')}$ for any $x \in \Omega' \Subset \Omega$. Thus we can apply Lebesgue theorem with respect to the measure $|{\rm div}F|$ in order to obtain

$$
\int_\Omega \phi g_\varepsilon \, d{\rm div}F \to \int_\Omega \phi g^* \, d{\rm div}F \tag{3.2.3}
$$

for any fixed $\phi \in C_c(\Omega)$, since ${\rm supp}(\phi) \subset \Omega'$ for some $\Omega' \Subset \Omega$.
If instead $p \in [1, n/(n-1))$, which means $p' > n$, then, by Morrey's inequality, g admits a continuous representative, which we shall denote again by g^*, and so $g_\varepsilon(x) \to g^*(x)$ for any $x \in \Omega$. Therefore, we can apply again Lebesgue theorem and obtain (3.2.3).
As for the second term, we notice that, if $p > 1$, and so $p' < \infty$, then $\nabla g_\varepsilon \to \nabla g$ in $L^{p'}_{\rm loc}(\Omega;\mathbb{R}^n)$ and so it follows that

$$
\int_\Omega \phi F \cdot \nabla g_\varepsilon \, dx \to \int_\Omega \phi F \cdot \nabla g \, dx \tag{3.2.4}
$$

for any $\phi \in C_c(\Omega)$.
Finally, we consider the case $p = 1, p' = \infty$: we have that $g \in W^{1,\infty}_{\rm loc}(\Omega)$ and so it is almost everywhere differentiable and the weak gradient coincide with the classical one almost everywhere. Then, for any $\phi \in C_c(\Omega)$ we obtain

$$\int_\Omega \phi(x) F(x) \cdot \nabla g_\varepsilon(x)\, dx = \int_\Omega \nabla g(y) \cdot \int_\Omega F(x) \rho_\varepsilon(x-y)(\phi(x) - \phi(y))\, dx\, dy + \int_\Omega \phi(y) \nabla g(y) \cdot F_\varepsilon(y)\, dy.$$

Since $F_\varepsilon \to F$ in $L^1_{\rm loc}(\Omega; \mathbb{R}^n)$, it follows that

$$\int_\Omega \phi \nabla g \cdot F_\varepsilon\, dy \to \int_\Omega \phi \nabla g \cdot F\, dy.$$

On the other hand, since ϕ is uniformly continuous, for any $\eta > 0$ there exists an $\varepsilon > 0$ such that, if $|x - y| < \varepsilon$, then $|\phi(x) - \phi(y)| < \eta$, and this implies that

$$\left| \int_\Omega \nabla g(y) \cdot \int_\Omega F(x) \rho_\varepsilon(x-y)(\phi(x) - \phi(y))\, dx\, dy \right| \le \eta \|\nabla g\|_{L^\infty(\Omega'; \mathbb{R}^n)} \|F\|_{L^1(\Omega'; \mathbb{R}^n)},$$

if $\operatorname{supp}(\phi) \subset \Omega' \Subset \Omega$. Since ϕ is fixed and η is arbitray, we obtain again (3.2.4). This concludes the proof. □

We notice that, if $p = \infty$, it is possible to extend Theorem 3.2.3 to scalar functions $g \in L^\infty(\Omega) \cap BV_{\rm loc}(\Omega)$. To this purpose, it is necessary to employ the notion of pairing measure between an essentially bounded vector field and a vector valued Radon measure, introduced in Lemma 1.1.3. This case has indeed been widely studied, see for instance [41, Theorem 3.1] and [82, Theorem 2.1]. For the sake of completeness, we recall here its statement in the most general form.

Theorem 3.2.4. *If $F \in \mathcal{DM}^\infty_{\rm loc}(\Omega), g \in L^\infty_{\rm loc}(\Omega)$ and $Dg \in \mathcal{M}_{\rm loc}(\Omega; \mathbb{R}^n)$, then $gF \in \mathcal{DM}^\infty_{\rm loc}(\Omega)$ and there exists a unique pairing meaure (F, Dg) such that*

$$F \cdot (\rho_\varepsilon * Dg)\, \mathscr{L}^n \rightharpoonup (F, Dg) \quad \text{in}\, \mathcal{M}_{\rm loc}(\Omega)$$

for any nonnegative radially symmetric mollifier $\rho \in C^\infty_c(B(0,1))$ satisfying $\int_{B(0,1)} \rho\, dx = 1$. This measure satisfies

$$\operatorname{div}(gF) = g^* \operatorname{div} F + (F, Dg), \tag{3.2.5}$$

where g^* is the precise representative of g given by (1.1.18), and

$$|(F, Dg)| \llcorner \Omega' \le \|F\|_{L^\infty(\Omega'; \mathbb{R}^n)} |Dg| \llcorner \Omega' \tag{3.2.6}$$

for any open set $\Omega' \Subset \Omega$. In addition, we have the decompositon

$$(F, Dg)^{\rm a} = F \cdot \nabla g\, \mathscr{L}^n, \quad (F, Dg)^{\rm s} = (F, D^{\rm s} g),$$

where $(Dg)^{\rm a} = \nabla g \mathscr{L}^n$.

Proof. Since the statement is of local nature, we can restrict ourselves to any open set $\Omega' \Subset \Omega$ without loss of generality, so that we have $F \in \mathcal{DM}^\infty(\Omega')$ and $g \in L^\infty(\Omega') \cap BV(\Omega')$. Then, for the proof under these assumptions we refer to [82, Theorem 2.1]. □

Remark 3.2.5. As an immediate consequence of (3.2.5), we notice that the pairing measure is linear in the first component, for any fixed $g \in L^\infty_{\rm loc}(\Omega)$ with $Dg \in \mathcal{M}_{\rm loc}(\Omega; \mathbb{R}^n)$.

3.3 The Gauss–Green formulas

In this section, we establish versions of the Gauss–Green formula for $\mathcal{DM}^{\infty}_{\rm loc}(\Omega)$ fields on sets of locally finite perimeter in Ω. The method of the proof is analogous to the one Vol'pert used in order to prove his integration by parts theorem and it is based on the product rule recalled in Theorem 3.2.4. The results are similar to those presented in the paper of Chen, Torres and Ziemer [45], but this exposition does not require the theory of the one-sided approximation of sets of finite perimeter by sets with smooth boundary introduced in Section 2.3. Therefore, we do not need to state a preliminary version of the theorem for open sets with smooth boundary. Nevertheless, we will show in Remark 3.4.4 that our approach actually implies the one of [45], thanks to the approximation result of Section 2.3. In addition, our approach can be easily generalized to sets of locally finite perimeter, and employed to obtain integration by parts formulas, Green's identities and the locality properties of the normal traces.

3.3.1 The normal traces

Let $F \in \mathcal{DM}^{\infty}(\Omega)$ and E be a set of finite perimeter in Ω. Then, Theorem 3.2.4 implies that the pairing measures $(\chi_E F, D\chi_E)$ and $(\chi_{\Omega\setminus E} F, D\chi_E)$ are well defined and unique. In addition, thanks to (3.2.6), we clearly have

$$|(\chi_E F, D\chi_E)| \le \|F\|_{L^\infty(E;\mathbb{R}^n)}|D\chi_E| \quad \text{and} \quad |(\chi_{\Omega\setminus E} F, D\chi_E)| \le \|F\|_{L^\infty(\Omega\setminus E;\mathbb{R}^n)}|D\chi_E|. \tag{3.3.1}$$

We may now employ Radon-Nikodým Theorem and define the *interior and exterior normal traces* of F on the boundary of E as the functions $\mathcal{F}_i \cdot \nu_E, \mathcal{F}_e \cdot \nu_E \in L^\infty(\mathscr{F}E; |D\chi_E|)$ satisfying

$$2(\chi_E F, D\chi_E) = \mathcal{F}_i \cdot \nu_E \, |D\chi_E|, \tag{3.3.2}$$
$$2(\chi_{\Omega\setminus E} F, D\chi_E) = \mathcal{F}_e \cdot \nu_E \, |D\chi_E|. \tag{3.3.3}$$

These definitions may be justified in the light of the following result, which is a refinement of the Leibniz rule between a field in $\mathcal{DM}^{\infty}(\Omega)$ and the characteristic function of a set of finite perimeter.

Theorem 3.3.1. *Let $F \in \mathcal{DM}^{\infty}(\Omega)$ and E be a set of finite perimeter in Ω. Then the following formulas hold in $\mathcal{M}(\Omega)$*

$$\mathrm{div}(\chi_E F) = \chi_{E^1}\mathrm{div}F + 2(\chi_E F, D\chi_E), \tag{3.3.4}$$
$$\mathrm{div}(\chi_E F) = \chi_{E^1\cup\mathscr{F}E}\mathrm{div}F + 2(\chi_{\Omega\setminus E} F, D\chi_E), \tag{3.3.5}$$
$$\chi_{\mathscr{F}E}\mathrm{div}F = 2(\chi_E F, D\chi_E) - 2(\chi_{\Omega\setminus E} F, D\chi_E). \tag{3.3.6}$$

Proof. Using the product rule of Theorem 3.2.4, one has

$$\begin{aligned}\mathrm{div}(\chi_E^2 F) &= \mathrm{div}(\chi_E(\chi_E F)) = \chi_E^*\mathrm{div}(\chi_E F) + (\chi_E F, D\chi_E)\\ &= \chi_E^*\left(\chi_E^*\mathrm{div}F + (F, D\chi_E)\right) + (\chi_E F, D\chi_E)\\ &= (\chi_E^*)^2\mathrm{div}F + \chi_E^*(F, D\chi_E) + (\chi_E F, D\chi_E),\end{aligned} \tag{3.3.7}$$

where χ_E^* is the precise representative of χ_E given in formula (1.1.20). On the other hand, one also has

$$\mathrm{div}(\chi_E^2 F) = \mathrm{div}(\chi_E F) = \chi_E^*\mathrm{div}F + (F, D\chi_E) \tag{3.3.8}$$

and combining (3.3.7) with (3.3.8) yields

$$\chi_E^*(1-\chi_E^*)\mathrm{div}F = (\chi_E^*-1)(F, D\chi_E) + (\chi_E F, D\chi_E). \tag{3.3.9}$$

Since $|\mathrm{div}F| \ll \mathscr{H}^{n-1}$, by Theorem 3.2.2, formula (1.1.20) of Lemma 1.1.16 implies that

$$\chi_E^*(1-\chi_E^*)\mathrm{div}F = \frac{1}{4}\chi_{\mathscr{F}E}\mathrm{div}F. \tag{3.3.10}$$

By Theorem 3.2.4, $|(F, D\chi_E)| \ll |D\chi_E|$ and so $\chi_E^*(F, D\chi_E) = \frac{1}{2}(F, D\chi_E)$, since this measure is concentrated on $\mathscr{F}E$. From this fact and (3.3.10) we obtain

$$\frac{1}{2}\chi_{\mathscr{F}E}\mathrm{div}F = 2(\chi_E F, D\chi_E) - (F, D\chi_E). \tag{3.3.11}$$

Thanks to the linearity of the pairing measure (Remark 3.2.5), we have

$$(F, D\chi_E) = (\chi_E F, D\chi_E) + (\chi_{\Omega\setminus E}F, D\chi_E), \tag{3.3.12}$$

and so (3.3.6) easily follows from (3.3.11). Now, we can employ (3.3.8) and (3.3.11) to obtain

$$\begin{aligned}
\mathrm{div}(\chi_E F) &= \chi_{E^1}\mathrm{div}F + \frac{1}{2}\chi_{\mathscr{F}E}\mathrm{div}F + (F, D\chi_E) \\
&= \chi_{E^1}\mathrm{div}F + 2(\chi_E F, D\chi_E) - (F, D\chi_E) + (F, D\chi_E) \\
&= \chi_{E^1}\mathrm{div}F + 2(\chi_E F, D\chi_E),
\end{aligned}$$

which is (3.3.4). On the other hand, if we add and subtract the term $\frac{1}{2}\chi_{\mathscr{F}E}\mathrm{div}F$ in (3.3.8), then (3.3.6) and (3.3.12) yield (3.3.5):

$$\begin{aligned}
\mathrm{div}(\chi_E F) &= \chi_{E^1\cup\mathscr{F}E}\mathrm{div}F - \frac{1}{2}\chi_{\mathscr{F}E}\mathrm{div}F + (F\cdot D\chi_E) \\
&= \chi_{E^1\cup\mathscr{F}E}\mathrm{div}F - (\chi_E F, D\chi_E) + (\chi_{\Omega\setminus E}F, D\chi_E) + (F\cdot D\chi_E) \\
&= \chi_{E^1\cup\mathscr{F}E}\mathrm{div}F + 2(\chi_E F, D\chi_E).
\end{aligned}$$

□

As an immediate consequence, (3.3.2) and (3.3.3) imply that the formulas of Theorem 3.3.1 may be rewritten in terms of the normal traces, instead of the pairing measures.

Corollary 3.3.2. *Let $F \in \mathcal{DM}^\infty(\Omega)$ and E be a set of finite perimeter in Ω. Then the normal traces $\mathcal{F}_i\cdot\nu_E$ and $\mathcal{F}_e\cdot\nu_E$ satisfy the following formulas in $\mathcal{M}(\Omega)$*

$$\mathrm{div}(\chi_E F) = \chi_{E^1}\mathrm{div}F + \mathcal{F}_i\cdot\nu_E|D\chi_E|, \tag{3.3.13}$$
$$\mathrm{div}(\chi_E F) = \chi_{E^1\cup\mathscr{F}E}\mathrm{div}F + \mathcal{F}_e\cdot\nu_E|D\chi_E|, \tag{3.3.14}$$
$$\chi_{\mathscr{F}E}\mathrm{div}F = (\mathcal{F}_i\cdot\nu_E - \mathcal{F}_e\cdot\nu_E)\,|D\chi_E|. \tag{3.3.15}$$

Proof. The result easily follows by combining (3.3.4), (3.3.5) and (3.3.6) with the definitions of interior and exterior normal traces (3.3.2) and (3.3.3). □

We state now the following simple result concerning fields with compact support, which is valid for any $1 \le p \le \infty$ and can be seen as the easy case of the Gauss–Green formula, since there are no boundary terms.

Lemma 3.3.3. *Let $p \in [1, \infty]$. If $F \in \mathcal{DM}^p(\Omega)$ has compact support in Ω, then*

$$\mathrm{div}F(\Omega) = 0.$$

Proof. Since F has compact support, there exists an open set V satisfying $\operatorname{supp}(F) \subset V \Subset \Omega$. Then, we have $\operatorname{div}F = 0$ in $\Omega \setminus \overline{V}$.

Now, if we choose $\varphi \in C_c^\infty(\Omega)$ such that $\varphi \equiv 1$ on a neighborhood of V, we obtain

$$0 = -\int_{\Omega \setminus V} F \cdot \nabla\varphi \, dx = -\int_{\Omega} F \cdot \nabla\varphi \, dx = \int_{\Omega} \varphi \, d\operatorname{div}F = \int_{\overline{V}} \varphi \, d\operatorname{div}F = \operatorname{div}F(\overline{V})$$

and hence $\operatorname{div}F(\Omega) = 0$. □

By combining Lemma 3.3.3 and Corollary 3.3.2, we obtain the Gauss–Green formulas.

Theorem 3.3.4. *Let $F \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$ and let $E \Subset \Omega$ be a set of finite perimeter in Ω. Then, we have*

$$\operatorname{div}F(E^1) = -\int_{\mathscr{F}E} \mathcal{F}_i \cdot \nu_E \, d\mathscr{H}^{n-1} \quad \textit{and} \quad \operatorname{div}F(E^1 \cup \mathscr{F}E) = -\int_{\mathscr{F}E} \mathcal{F}_e \cdot \nu_E \, d\mathscr{H}^{n-1}. \tag{3.3.16}$$

Proof. Without loss of generality, we may assume that $F \in \mathcal{DM}^\infty(\Omega)$. Indeed, it is clear that there exists an open set $\Omega' \Subset \Omega$ such that $F \in \mathcal{DM}^\infty(\Omega')$ and $E \Subset \Omega'$; so that we may work on Ω' instead of Ω.

Since $\chi_E F \in \mathcal{DM}^\infty(\Omega)$ and clearly has compact support in Ω, by Lemma 3.3.3 and (3.3.13) one has

$$0 = \operatorname{div}(\chi_E F)(\Omega) = \operatorname{div}F(E^1) + \int_{\Omega} \mathcal{F}_i \cdot \nu_E \, d|D\chi_E|.$$

Then, thanks to De Giorgi's theorem (Theorem 1.1.10), we have $|D\chi_E| = \mathscr{H}^{n-1} \llcorner \mathscr{F}E$ and the first part of (3.3.16) follows. In an analogous way, Lemma 3.3.3, (3.3.13) and Theorem 1.1.10 yield the second part of (3.3.16). □

By (3.3.1), (3.3.2) and (3.3.3), it is easy to obtain a first rough bound on the L^∞-norm of the normal traces of a divergence-measure field, namely

$$\|\mathcal{F}_i \cdot \nu_E\|_{L^\infty(\mathscr{F}E; \mathscr{H}^{n-1})} \le 2\|F\|_{L^\infty(E;\mathbb{R}^n)} \quad \text{and} \quad \|\mathcal{F}_e \cdot \nu_E\|_{L^\infty(\mathscr{F}E; \mathscr{H}^{n-1})} \le 2\|F\|_{L^\infty(\Omega\setminus E;\mathbb{R}^n)}.$$

However, it is possible to get a refined version of such estimates, as shown in the following theorem.

Theorem 3.3.5. *Let $F \in \mathcal{DM}^\infty(\Omega)$ and let E be a set of finite perimeter in Ω. Then, we have the following estimates on the normal traces:*

$$\|\mathcal{F}_i \cdot \nu_E\|_{L^\infty(\mathscr{F}E; \mathscr{H}^{n-1})} \le \|F\|_{L^\infty(E;\mathbb{R}^n)} \quad \textit{and} \quad \|\mathcal{F}_e \cdot \nu_E\|_{L^\infty(\mathscr{F}E; \mathscr{H}^{n-1})} \le \|F\|_{L^\infty(\Omega\setminus E;\mathbb{R}^n)}. \tag{3.3.17}$$

Proof. By the Lebesgue-Besicovitch differentiation theorem, for $\mathscr{H}^{n-1}$-a.e. $x \in \mathscr{F}E$ one has

$$\mathcal{F}_i \cdot \nu_E(x) = \lim_{r \to 0} \frac{2(\chi_E F, D\chi_E)(B(x,r))}{|D\chi_E|(B(x,r))}.$$

We recall that, by Theorem 3.2.4, for any nonnegative radially symmetric mollifier $\rho \in C_c^\infty(B(0,1))$ satisfying $\int_{B(0,1)} \rho \, dy = 1$, we have

$$\chi_E F \cdot (\rho_\varepsilon * D\chi_E)\mathscr{L}^n \rightharpoonup (\chi_E F, D\chi_E).$$

In addition, by (1.1.3) in Lemma 1.1.3, the family of measures $|\chi_E F \cdot (\rho_\varepsilon * D\chi_E)|\mathscr{L}^n$ is uniformly bounded in $\mathcal{M}(\Omega)$, with the estimate

$$\int_{\Omega} |\chi_E F \cdot (\rho_\varepsilon * D\chi_E)| \, dy \le \|F\|_{L^\infty(E;\mathbb{R}^n)} |D\chi_E|(\Omega).$$

Thus, there exists a weak* converging subsequence, which we label with ε_k, and let the positive measure $\lambda_i \in \mathcal{M}(\Omega)$ be its limit:

$$|\chi_E F \cdot (\rho_{\varepsilon_k} * D\chi_E)|\mathscr{L}^n \rightharpoonup \lambda_i.$$

Then, by [11, Proposition 1.62] we know that $|(\chi_E F, D\chi_E)| \le \lambda_i$. Moreover, we observe that also the sequence $\chi_{\Omega\setminus E}|\rho_{\varepsilon_k} * D\chi_E|$ is uniformly bounded with a similar estimate as above, for any nonnegative sequence $\varepsilon_k \to 0$. So there exists a weak* converging subsequence which we shall not relabel for simplicity of notation and which converge to positive measures $\mu_e \in \mathcal{M}(\Omega)$.

By a standard concentration argument (see [11, Example 1.63]), we can choose a sequence of balls $B(x, r_j) \Subset \Omega$ with $r_j \to 0$ in such a way that

$$|D\chi_E|(\partial B(x, r_j)) = \lambda_i(\partial B(x, r_j)) = \mu_e(\partial B(x, r_j)) = 0.$$

Hence, by [11, Proposition 1.62] and Lemma 1.1.16, we have

$$\begin{aligned}
\lim_{r_j \to 0} \left| \frac{2(\chi_E F, D\chi_E)(B(x, r_j))}{|D\chi_E|(B(x, r_j))} \right| &= \lim_{r_j \to 0} \left| \frac{\lim_{\varepsilon_k \to 0} 2 \int_{B(x,r_j)} \chi_E F \cdot (\rho_{\varepsilon_k} * D\chi_E)\, dy}{\lim_{\varepsilon_k \to 0} \int_{B(x,r_j)} |\nabla(\rho_{\varepsilon_k} * \chi_E)|\, dy} \right| \\
&\le \lim_{r_j \to 0} \frac{2\|F\|_{L^\infty(E;\mathbb{R}^n)} \lim_{\varepsilon_k \to 0} \int_{B(x,r_j)} \chi_E |\rho_{\varepsilon_k} * D\chi_E|\, dy}{\lim_{\varepsilon_k \to 0} \int_{B(x,r_j)} |\nabla(\rho_{\varepsilon_k} * \chi_E)|\, dy} \\
&= 2\|F\|_{L^\infty(E;\mathbb{R}^n)} \lim_{r_j \to 0} \left(1 - \frac{\lim_{\varepsilon_k \to 0} \int_{B(x,r_j)} \chi_{\Omega\setminus E} |\rho_{\varepsilon_k} * D\chi_E|\, dy}{\lim_{\varepsilon_k \to 0} \int_{B(x,r_j)} |\nabla(\rho_{\varepsilon_k} * \chi_E)|\, dy} \right) \\
&\le 2\|F\|_{L^\infty(E;\mathbb{R}^n)} \lim_{r_j \to 0} \left(1 - \frac{\lim_{\varepsilon_k \to 0} |\int_{B(x,r_j)} \chi_{\Omega\setminus E} \nabla(\rho_{\varepsilon_k} * \chi_E)\, dy|}{\lim_{\varepsilon_k \to 0} \int_{B(x,r_j)} |\nabla(\rho_{\varepsilon_k} * \chi_E)|\, dy} \right) \\
&= 2\|F\|_{L^\infty(E;\mathbb{R}^n)} \lim_{r_j \to 0} \left(1 - \frac{1}{2} \frac{|D\chi_E(B(x, r_j))|}{|D\chi_E|(B(x, r_j))} \right) = \|F\|_{L^\infty(E;\mathbb{R}^n)}.
\end{aligned}$$

In the last equality we used the definition of reduced boundary (Definition 1.1.7): if $x \in \mathscr{F}E$, then $|\nu_E|(x) = 1$, $|D\chi_E|(B(x, r)) > 0$ for $r > 0$ and $\nu_E(x) = \lim_{r \to 0} \frac{D\chi_E(B(x,r))}{|D\chi_E|(B(x,r))}$. This implies that

$$\lim_{r \to 0} \frac{|D\chi_E(B(x, r))|}{|D\chi_E|(B(x, r))} = |\nu_E(x)| = 1.$$

The estimate for the exterior normal trace $\mathcal{F}_e \cdot \nu_E$ can be obtained in a similar way. Indeed, Theorem 3.2.4 and Lemma 1.1.3 imply that

$$\chi_{\Omega\setminus E} F \cdot (\rho_\varepsilon * D\chi_E)\mathscr{L}^n \rightharpoonup (\chi_{\Omega\setminus E} F, D\chi_E),$$

and that the family of Radon measures $|\chi_{\Omega\setminus E} F \cdot (\rho_\varepsilon * D\chi_E)|\mathscr{L}^n$ is uniformly bounded in $\mathcal{M}(\Omega)$, with bound

$$\int_\Omega |\chi_{\Omega\setminus E} F \cdot (\rho_\varepsilon * D\chi_E)|\, dy \le \|F\|_{L^\infty(\Omega\setminus E;\mathbb{R}^n)} |D\chi_E|(\Omega).$$

Hence, there exists a weakly* converging subsequence, which we label again with ε_k, whose limit is a positive Radon measure λ_e satisfying $|(\chi_{\Omega\setminus E} F, D\chi_E)| \le \lambda_e$. Analogously, one can show that the sequence $\chi_E |\rho_{\varepsilon_k} * D\chi_E|\mathscr{L}^n$ is uniformly bounded in $\mathcal{M}(\Omega)$, extract a weakly* converging subsequence (not relabeled) and denote its limit by μ_i. Now we can consider balls $B(x, r_j) \Subset \Omega$ which satisfy

$$|D\chi_E|(\partial B(x, r_j)) = \lambda_e(\partial B(x, r_j)) = \mu_i(\partial B(x, r_j)) = 0$$

and we use the inequality

$$\left| \int_{B(x,r)} \chi_{\Omega \setminus E} F \cdot (\rho_{\varepsilon_k} * D\chi_E) \, dy \right| \le \|F\|_{L^\infty(\Omega \setminus E; \mathbb{R}^n)} \int_{B(x,r)} \chi_{\Omega \setminus E} |\rho_{\varepsilon_k} * D\chi_E| \, dy$$

to complete the proof. □

Before proceeding, we would like to formalize a few remarks comparing the case of $\mathcal{DM}^\infty(\Omega)$ and $BV(\Omega; \mathbb{R}^n) \cap L^\infty(\Omega; \mathbb{R}^n)$ fields.

Remark 3.3.6. Since the proof of Theorem 3.3.4 given above relies on the product rule for $F \in \mathcal{DM}^\infty(\Omega)$ and $g \in BV(\Omega) \cap L^\infty(\Omega)$ and on Lemma 3.3.3, then it is not difficult to show that Theorem 3.3.4 is consistent with Vol'pert's Gauss–Green formula for $BV(\Omega; \mathbb{R}^n) \cap L^\infty(\Omega; \mathbb{R}^n)$ fields as given in [157, Chapter 5, Section 1.8]. Indeed, if $F \in BV(\Omega; \mathbb{R}^n) \cap L^\infty(\Omega; \mathbb{R}^n)$, then Theorem 1.1.15 and some straighforward calculations show that

$$(F, Dg) = F^* \cdot Dg.$$

Therefore, we have

$$\mathcal{F}_i \cdot \nu_E = F_{\nu_E} \cdot \nu_E \quad \text{and} \quad \mathcal{F}_e \cdot \nu_E = F_{-\nu_E} \cdot \nu_E,$$

where $F_{\pm \nu_E}(x)$ are the approximate limits of F in $\mathscr{H}^{n-1}$-a.e. $x \in \mathscr{F}E$ restricted to E and $\Omega \setminus E$, respectively. Actually, thanks to De Giorgi's Theorem (Theorem 1.1.10), this is equivalent to say that $F_{\pm \nu_E}$ are the approximate limits of F restricted to

$$\Pi^{\pm}_{\nu_E}(x) := \{y \in \mathbb{R}^n : (y - x) \cdot (\pm \nu_E(x)) \ge 0\};$$

that is, for any $\varepsilon > 0$ one has

$$\lim_{r \to 0} \frac{|\{y \in \mathbb{R}^n : |F(y) - F_{\pm \nu_E}(x)| \ge \varepsilon\} \cap B(x, r) \cap \Pi^{\pm}_{\nu_E}(x)|}{|B(x, r)|} = 0.$$

3.3.2 Integration by parts formulas

In this section we prove general integration by parts formulas for a $\mathcal{DM}^\infty_{\mathrm{loc}}$ vector fields and scalar functions in $W^{1,1}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$ over sets of locally finite perimeter, under some assumptions on the compactness of the supports.

Theorem 3.3.7. *Let $F \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$ and let E be a set of locally finite perimeter in Ω. Then, there exist $(\mathcal{F}_i \cdot \nu_E), (\mathcal{F}_e \cdot \nu_E) \in L^\infty_{\mathrm{loc}}(\mathscr{F}E; \mathscr{H}^{n-1})$ satisfying (3.3.2) and (3.3.3) in any open set $U \Subset \Omega$, and such that the following estimates hold:*

$$\|\mathcal{F}_i \cdot \nu_E\|_{L^\infty(\mathscr{F}E \cap U; \mathscr{H}^{n-1})} \le \|F\|_{L^\infty(E \cap U; \mathbb{R}^n)} \quad \text{and} \quad \|\mathcal{F}_e \cdot \nu_E\|_{L^\infty(\mathscr{F}E \cap U; \mathscr{H}^{n-1})} \le \|F\|_{L^\infty(U \setminus E; \mathbb{R}^n)}. \tag{3.3.18}$$

In addition, for any $\varphi \in L^\infty_{\mathrm{loc}}(\Omega)$ such that $\nabla \varphi \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ and $\mathrm{supp}(\chi_E \varphi) \Subset \Omega$, the following formulas hold:

$$\int_{E^1} \varphi^* \, d\mathrm{div} F + \int_E F \cdot \nabla \varphi \, dx = -\int_{\mathscr{F}E} \varphi^* (\mathcal{F}_i \cdot \nu_E) \, d\mathscr{H}^{n-1} \tag{3.3.19}$$

and

$$\int_{E^1 \cup \mathscr{F}E} \varphi^* \, d\mathrm{div} F + \int_E F \cdot \nabla \varphi \, dx = -\int_{\mathscr{F}E} \varphi^* (\mathcal{F}_e \cdot \nu_E) \, d\mathscr{H}^{n-1}. \tag{3.3.20}$$

Proof. It is clear that for any open set $U \Subset \Omega$ we have $F_{|U} \in \mathcal{DM}^\infty(U)$ and $(\chi_E)_{|U} = \chi_{E\cap U} \in BV(U)$. With a slight abuse of notation, from now on, we will write F instead of $F|_U$. Hence, in the open set U the interior and exterior normal traces of F on $\mathscr{F}E$ are well defined as in (3.3.2) and (3.3.3). Then, (3.3.18) follows easily from the restriction of (3.3.17) to U.

As for the second part of the statement, we see that there exists $U \Subset \Omega$ such that $\mathrm{supp}(\chi_E\varphi) \Subset U$. By applying the Leibniz rules (Theorem 3.2.3 and Corollary 3.3.2) to $\varphi\chi_E F$, we obtain:

$$\mathrm{div}(\varphi\chi_E F) = \varphi^*\chi_{E^1}\mathrm{div}F + \varphi^*\mathcal{F}_i\cdot\nu_E|D\chi_E| + \chi_E F\cdot\nabla\varphi\,\mathscr{L}^n, \tag{3.3.21}$$
$$\mathrm{div}(\varphi\chi_E F) = \varphi^*\chi_{E^1\cup\mathscr{F}E}\mathrm{div}F + \varphi^*\mathcal{F}_e\cdot\nu_E|D\chi_E| + \chi_E F\cdot\nabla\varphi\,\mathscr{L}^n, \tag{3.3.22}$$

as identities between Radon measures in $\mathcal{M}(U)$. Then, it is enough evaluate (3.3.21) and (3.3.22) over U and to apply Lemma 3.3.3 in order to get (3.3.19) and (3.3.20). □

Remark 3.3.8. It is possible to improve the estimates in (3.3.18) on the L^∞-norm of the normal traces. Indeed, if $F \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$ and $E \subset \Omega$ is a set of locally finite perimeter in Ω, we can choose $U = (\mathscr{F}E)_\varepsilon \cap V$, where $(\mathscr{F}E)_\varepsilon = \{x \in \Omega : \mathrm{dist}(x,\mathscr{F}E) < \varepsilon\}$ and $V \Subset \Omega$ is open. Then, we get

$$\|\mathcal{F}_i\cdot\nu_E\|_{L^\infty((\mathscr{F}E)\cap V;\mathscr{H}^{n-1})} \le \inf_{\varepsilon>0}\{\|F\|_{L^\infty(E_\varepsilon;\mathbb{R}^n)}\},$$

where $E_\varepsilon := U \cap E = \{x \in E\cap V : \mathrm{dist}(x,\mathscr{F}E) < \varepsilon\}$. On the other hand, a similar argument yields

$$\|\mathcal{F}_e\cdot\nu_E\|_{L^\infty((\mathscr{F}E)\cap V;\mathscr{H}^{n-1})} \le \inf_{\varepsilon>0}\{\|F\|_{L^\infty(E^\varepsilon;\mathbb{R}^n)}\},$$

where $E^\varepsilon := U \cap (\Omega\setminus E) = \{x \in (\Omega\setminus E)\cap V : \mathrm{dist}(x,\mathscr{F}E) < \varepsilon\}$.

3.4 Consistency of normal traces

3.4.1 The continuous case

Because of (3.3.15), we see that for a general divergence-measure field the measure $\mathrm{div}F$ contains a jump component at the boundary of a set of finite perimeter where the exterior and interior normal traces do not coincide. However, this does not happen if the field F is continuous. The following theorem is similar to [45, Theorem 7.2], however, our proof does not need the preliminary result given by [45, Lemma 7.1] and it is consequently more direct.

Theorem 3.4.1. (Consistency of the normal traces) *Let $F \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega) \cap C(\Omega;\mathbb{R}^n)$. If E is a set of locally finite perimeter in Ω, then*

$$(\mathcal{F}_i\cdot\nu_E)(x) = (\mathcal{F}_e\cdot\nu_E)(x) = F(x)\cdot\nu_E(x) \quad \text{for } \mathscr{H}^{n-1}\text{-a.e. } x \in \mathscr{F}E.$$

In particular, $|\mathrm{div}F|(\mathscr{F}E) = 0$ and, for any $\varphi \in L^\infty_{\mathrm{loc}}(\Omega)$ such that $\nabla\varphi \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ and $\mathrm{supp}(\chi_E\varphi) \Subset \Omega$, we have:

$$\int_{E^1}\varphi^*\,d\mathrm{div}F + \int_E F\cdot\nabla\varphi\,dx = -\int_{\mathscr{F}E}\varphi^* F\cdot\nu_E\,d\mathscr{H}^{n-1}. \tag{3.4.1}$$

Proof. By Theorem 3.3.7 and (3.3.2), one has that $2(\chi_E F, D\chi_E) = (\mathcal{F}_i\cdot\nu_E)\,\mathscr{H}^{n-1}\llcorner\mathscr{F}E$ in $\mathcal{M}(U)$ for any open set $U \Subset \Omega$, and $\mathcal{F}_i\cdot\nu_E \in L^\infty(\mathscr{F}E\cap U;\mathscr{H}^{n-1})$. This means that, for $\mathscr{H}^{n-1}$-a.e. $x \in \mathscr{F}E$, one has

$$(\mathcal{F}_i\cdot\nu_E)(x) = \lim_{r\to 0} 2\frac{(\chi_E F, D\chi_E)(B(x,r))}{|D\chi_E|(B(x,r))}. \tag{3.4.2}$$

In addition, we know that

$$\chi_E F \cdot (\rho_\varepsilon * D\chi_E)\mathscr{L}^n \rightharpoonup (\chi_E F, D\chi_E) \text{ in } \mathcal{M}_{\text{loc}}(\Omega),$$

which means that, $\forall \phi \in C_c(\Omega)$,

$$\int_\Omega \phi \chi_E F \cdot (\rho_\varepsilon * D\chi_E)\, dx \to \int_\Omega \phi\, d(\chi_E F, D\chi_E) \text{ as } \varepsilon \to 0.$$

Observe that $\phi F \in C_c(\Omega; \mathbb{R}^n)$ and, since $\chi_E(\rho_\varepsilon * D\chi_E) \rightharpoonup (1/2)D\chi_E$, by point (3)($b$) in Lemma 1.1.16, one also has

$$\int_\Omega (\phi F) \cdot \chi_E(\rho_\varepsilon * D\chi_E)\, dx \to \int_\Omega (\phi F) \cdot \frac{1}{2} dD\chi_E \text{ as } \varepsilon \to 0.$$

Thus, we conclude that $(\chi_E F, D\chi_E) = \frac{1}{2} F \cdot D\chi_E$ in $\mathcal{M}_{\text{loc}}(\Omega)$, which means that

$$2(\chi_E F, D\chi_E)(B(x,r)) = \int_{B(x,r)} F \cdot dD\chi_E = \int_{B(x,r)} F \cdot \nu_E\, d|D\chi_E|,$$

for any $r > 0$ small enough so that $B(x,r) \Subset \Omega$. Moreover, by the continuity of F, the function $F \cdot \nu_E$ is well defined on $\mathscr{F}E$ and is also in $L^\infty_{\text{loc}}(\mathscr{F}E; \mathscr{H}^{n-1})$. Therefore, from (3.4.2), for $\mathscr{H}^{n-1}$-a.e. $x \in \mathscr{F}E$, one obtains

$$\begin{aligned}(\mathcal{F}_i \cdot \nu_E)(x) &= \lim_{r \to 0} \frac{\int_{B(x,r)} F(y) \cdot \nu_E(y) d|D\chi_E|(y)}{|D\chi_E|(B(x,r))} \\ &= F(x) \cdot \nu_E(x),\end{aligned}$$

by the Lebesgue-Besicovitch differentiation theorem.

Applying the same steps to the measure $2(\chi_{\Omega \setminus E} F, D\chi_E)$ yields that it is equal to $F \cdot D\chi_E$ and hence one also finds that $\mathcal{F}_e \cdot \nu_E$ admits $F \cdot \nu_E$ as representative and hence it coincides with $\mathcal{F}_i \cdot \nu_E$. Finally, (3.3.15) easily implies $|\text{div} F|(\mathscr{F}E) = 0$, and (3.4.1) follows from (3.3.19). $\square$

From this theorem, we see that continuous divergence-measure fields have no jump component in their distributional divergence. However, we remark that $\chi_{\mathscr{F}E}|\text{div}F| = 0$ does not imply a better absolute continuity property of $\text{div}F$ such as $|\text{div}F| \ll \mathscr{H}^{n-t}$ for some $t \in [0, 1)$.

Remark 3.4.2. We note that the L^∞ estimates in Theorem 3.3.7 are sharp. Indeed, given a set of finite perimeter E in Ω, there exists a divergence-measure fields F for which

$$\|\mathcal{F}_i \cdot \nu_E\|_{L^\infty(\mathscr{F}E; \mathscr{H}^{n-1})} = \|\mathcal{F}_e \cdot \nu_E\|_{L^\infty(\mathscr{F}E; \mathscr{H}^{n-1})} = \|F\|_{L^\infty(E; \mathbb{R}^n)} = \|F\|_{L^\infty(\Omega \setminus E; \mathbb{R}^n)}.$$

Indeed, it is enough to select a constant vector field $F \equiv \nu_E(x)$, for some fixed $x \in \mathscr{F}E$, so that $(\mathcal{F}_i \cdot \nu_E)(x) = (\mathcal{F}_e \cdot \nu_E)(x) = 1$.

We conclude this section with a pair of remarks concerning normal traces.

Remark 3.4.3. We observe that in general the normal traces of an essentially bounded (but discontinuous) divergence-measure field on the reduced boundary of a set of finite perimeter do not coincide $\mathscr{H}^{n-1}$-a.e. with the pointwise dot product. However, in [8] it has been shown that, roughly speaking, the normal traces coincide with the classical one on almost every surface. More precisely, let $I \subset \mathbb{R}$ be an open interval and let $\{\Sigma_t\}_{t \in I}$ be a family of oriented hypersurfaces in Ω such that there exists $\Omega' \Subset \Omega$, $\Phi \in C^1(\overline{\Omega'})$ and a family of open set $\Omega_t \Subset \Omega'$, $t \in I$, with $\Phi(\Omega') = I, \{\Phi = t\} = \Sigma_t = \partial \Omega_t$ for any $t \in I$, $|\nabla \Phi| > 0$ in Ω' and Σ_t is oriented by $\nabla \Phi / |\nabla \Phi|$. Then, if $F \in \mathcal{DM}^\infty_{\text{loc}}(\Omega)$, we have

$$\mathcal{F}_i \cdot \nu_{\Omega_t} = \mathcal{F}_e \cdot \nu_{\Omega_t} = F \cdot \nu_{\Omega_t} \quad \mathscr{H}^{n-1}\text{-a.e. on } \Sigma_t, \text{ for } \mathscr{L}^1\text{-a.e. } t \in I.$$

For a proof of this result, see [8, Proposition 3.6] (although in their paper the definition of exterior normal trace is slightly different from ours, they are indeed equivalent by Proposition 3.4.6 below).

We notice that, in particular, this statement applies to any family of balls $\{B(x_0, r)\}_{r\in(0,R)}$ inside Ω: indeed in this case $I = (0, R)$ and $\Phi(x) = |x - x_0|^2$. Thus, for $\mathscr{L}^1$-a.e. $r \in (0, R)$, we have $|\mathrm{div}F|(\partial B(x_0, r)) = 0$,

$$\mathcal{F}_i \cdot \nu_{B(x_0,r)}(x) = \mathcal{F}_e \cdot \nu_{B(x_0,r)}(x) = -F(x) \cdot \frac{(x - x_0)}{|x - x_0|} \quad \mathscr{H}^{n-1}\text{-a.e. } x \in \partial B(x_0, r)$$

and

$$\mathrm{div}F(B(x_0, r)) = \int_{\partial B(x_0,r)} F(x) \cdot \frac{(x - x_0)}{|x - x_0|} \, d\mathscr{H}^{n-1}(x).$$

Remark 3.4.4. We notice that, by combining Theorem 3.3.4 and Remark 3.4.3, one can recover the approximation result of Chen, Torres and Ziemer (as in (i)(b), (i)(g), (ii)(b) and (ii)(g) of [45, Theorem 5.2]); that is, the integrals of the interior and the exterior normal traces over the reduced boundary are the limits of the integrals of the classical normal trace over the boundaries of a suitable family of smooth sets. Indeed, let $F \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$ and let $E \Subset \Omega$ be a set of finite perimeter. Pick a smooth nonnegative radially symmetric mollifier $\rho \in C^\infty_c(B(0, 1))$ and consider the mollification $\chi_{E;\varepsilon_k}(x) := (\chi_E * \rho_{\varepsilon_k})(x)$ of χ_E for some positive sequence $\varepsilon_k \to 0$, as in Section 2.3. For $t \in (0, 1)$, one has $A_{k;t} := \{u_k > t\} \Subset \Omega$ if ε_k is small enough. Since $|\mathrm{div}F| \ll \mathscr{H}^{n-1}$ (by Theorem 3.2.2), we can apply Theorem 2.3.1 to the measure $\mathrm{div}F$ in order to obtain

$$\lim_{k\to+\infty} |\mathrm{div}F|(E^1 \Delta A_{k;t}) = 0 \quad \text{for} \quad t \in \left(\frac{1}{2}, 1\right) \tag{3.4.3}$$

and

$$\lim_{k\to+\infty} |\mathrm{div}F|((E^1 \cup \mathscr{F}E)\Delta A_{k;t}) = 0 \quad \text{for} \quad t \in \left(0, \frac{1}{2}\right). \tag{3.4.4}$$

It is clear that the sets $A_{k;t}$ satisfy the hypothesis of Remark 3.4.3 for any k with $\Phi = \chi_{E;\varepsilon_k}$, and so

$$\mathcal{F}_i \cdot \nu_{A_{k;t}} = \mathcal{F}_e \cdot \nu_{A_{k;t}} = F \cdot \nu_{A_{k;t}} \quad \mathscr{H}^{n-1}\text{-a.e. on } \partial A_{k;t}, \text{ for } \mathscr{L}^1\text{-a.e. } t \in (0, 1).$$

Now, since $A_{k;t}$ has a smooth boundary for $\mathscr{L}^1$-a.e. $t \in (0, 1)$, then for these values of t one has $\mathscr{H}^{n-1}(\partial A_{k;t} \setminus \mathscr{F}A_{k;t}) = 0$ (see for instance [11, Proposition 3.62]), and this implies $\mathscr{H}^{n-1}((A_{k;t})^1 \setminus A_{k;t}) = 0$. Hence, by the Gauss–Green formulas (3.3.16), one has

$$\mathrm{div}F(A_{k;t}) = -\int_{\partial A_{k;t}} F \cdot \nu_{A_{k;t}} \, d\mathscr{H}^{n-1} \tag{3.4.5}$$

for any $t \in (0, 1) \setminus Z_k$, with $\mathscr{L}^1(Z_k) = 0$. Clearly, $Z := \bigcup_k Z_k$ is $\mathscr{L}^1$-negligible, and so (3.4.5) holds for any k and for any $t \in (0, 1) \setminus Z$. Finally, one applies (3.3.16) to the set E and uses (3.4.3) and (3.4.4) to obtain

$$\lim_{k\to+\infty} \int_{\partial A_{k;t}} F \cdot \nu_{A_{k;t}} \, d\mathscr{H}^{n-1} = - \lim_{k\to+\infty} \mathrm{div}F(A_{k;t}) = -\mathrm{div}F(E^1) = \int_{\mathscr{F}E} \mathcal{F}_i \cdot \nu_E \, d\mathscr{H}^{n-1}$$

for $\mathscr{L}^1$-a.e. $t \in (\frac{1}{2}, 1)$, and

$$\lim_{k\to+\infty} \int_{\partial A_{k;t}} F \cdot \nu_{A_{k;t}} \, d\mathscr{H}^{n-1} = - \lim_{k\to+\infty} \mathrm{div}F(A_{k;t}) = -\mathrm{div}F(E^1 \cup \mathscr{F}E) = \int_{\mathscr{F}E} \mathcal{F}_e \cdot \nu_E \, d\mathscr{H}^{n-1}$$

for $\mathscr{L}^1$-a.e. $t \in (0, \frac{1}{2})$, which are the desired approximation results.

3.4.2 Locality properties of the normal traces

In this section, we show that, for sets of locally bounded perimeter, the normal traces are determined by $\mathscr{F}E$ and its orientation, thus generalizing what is known for the case of E open, bounded with C^1 boundary (see [8, Proposition 3.2]). Our treatment begins by considering the normal traces on complementary sets.

If $E \subset \Omega$ has locally finite perimeter in Ω, then it is well known that the complementary set $\Omega \setminus E$ also has locally finite perimeter in Ω, with $D\chi_{\Omega\setminus E} = -D\chi_E$, $\mathscr{F}(\Omega \setminus E) = \mathscr{F}E$, and $\nu_{\Omega\setminus E} = -\nu_E$. Therefore, Theorem 3.3.7 shows that $F \in \mathcal{DM}^{\infty}_{\mathrm{loc}}(\Omega)$ also admits interior and exterior normal traces

$$\mathcal{F}_i \cdot \nu_{\Omega\setminus E}, \mathcal{F}_e \cdot \nu_{\Omega\setminus E} \in L^{\infty}_{\mathrm{loc}}(\mathscr{F}E; \mathscr{H}^{n-1}).$$

One easily obtains the following useful relations for normal traces on the boundary of complementary sets of locally finite perimeter in Ω.

Proposition 3.4.5. *If $F \in \mathcal{DM}^{\infty}_{\mathrm{loc}}(\Omega)$ and $E \subset \Omega$ is a set of locally finite perimeter in Ω, then*

$$\mathcal{F}_e \cdot \nu_E = -\mathcal{F}_i \cdot \nu_{\Omega\setminus E} \ \mathscr{H}^{n-1}\text{-a.e. on } \mathscr{F}E \tag{3.4.6}$$

and

$$\mathcal{F}_e \cdot \nu_{\Omega\setminus E} = -\mathcal{F}_i \cdot \nu_E \ \mathscr{H}^{n-1}\text{-a.e. on } \mathscr{F}E. \tag{3.4.7}$$

Proof. By the definition of normal traces, (3.3.2) and (3.3.3), and Theorem 3.3.7, we have

$$2(\chi_{\Omega\setminus E}F, D\chi_{\Omega\setminus E}) = \mathcal{F}_i \cdot \nu_{\Omega\setminus E}\,|D\chi_E|, \quad 2(\chi_E F, D\chi_{\Omega\setminus E}) = \mathcal{F}_e \cdot \nu_{\Omega\setminus E}\,|D\chi_E|.$$

Then, by the definition of pairing measure, for any nonnegative radially symmetric mollifier $\rho \in C^{\infty}_c(B(0,1))$ we have

$$\begin{aligned} \chi_{\Omega\setminus E}F \cdot (\rho_\varepsilon * D\chi_{\Omega\setminus E})\mathscr{L}^n &\rightharpoonup (\chi_{\Omega\setminus E}F, D\chi_{\Omega\setminus E}), \\ \chi_E F \cdot (\rho_\varepsilon * D\chi_{\Omega\setminus E})\mathscr{L}^n &\rightharpoonup (\chi_E F, D\chi_{\Omega\setminus E}) \end{aligned}$$

On the other hand, it is clear that

$$\begin{aligned} \chi_{\Omega\setminus E}F \cdot (\rho_\varepsilon * D\chi_{\Omega\setminus E})\mathscr{L}^n &= -\chi_{\Omega\setminus E}F \cdot (\rho_\varepsilon * D\chi_E)\mathscr{L}^n \rightharpoonup -(\chi_{\Omega\setminus E}F, D\chi_E), \\ \chi_E F \cdot (\rho_\varepsilon * D\chi_{\Omega\setminus E})\mathscr{L}^n &= -\chi_E F \cdot (\rho_\varepsilon * D\chi_E)\mathscr{L}^n \rightharpoonup -(\chi_E F, D\chi_E). \end{aligned}$$

All in all, we get

$$(\chi_{\Omega\setminus E}F, D\chi_{\Omega\setminus E}) = -(\chi_{\Omega\setminus E}F, D\chi_E) \quad \text{and} \quad (\chi_E F, D\chi_{\Omega\setminus E}) = -(\chi_E F, D\chi_E).$$

Therefore, (3.4.6) and (3.4.7) follow from the definition of $\mathcal{F}_i \cdot \nu_E$ and $\mathcal{F}_e \cdot \nu_E$. □

We consider now the normal traces of F on a common portion of the reduced boundary of two sets of locally finite perimeter. We will show that the traces agree if the measure theoretic unit interior normals are the same, while they have opposite signs if the measure theoretic unit interior normals have opposite orientation. Our proof will adapt that given in [8, Proposition 3.2] for bounded open sets with C^1 boundary.

For the proof, we need to recall a few additional facts from Geometric Measure Theory. First, we recall a consequence of the basic comparison result between a positive Radon measure μ and k-dimensional Hausdorff measures through the use of k-dimensional densities of μ: if $\mu \in \mathcal{M}_{\mathrm{loc}}(\Omega)$ with μ positive and $\mu \llcorner A = 0$ for a Borel set $A \subset \Omega$, then for each $k \geq 0$ one has

$$\mu(B(x,\rho)) = o(\rho^k) \text{ for } \mathscr{H}^k\text{-a.e. } x \in A. \tag{3.4.8}$$

For a proof of this fact, see [11, Theorem 2.56]. Next, we recall elements of the structure of sets of locally finite perimeter given by De Giorgi's blow up construction. By Theorem 1.1.10, if E is a set of locally finite perimeter in Ω, then for any $x \in \mathscr{F}E$ one has

$$\chi_{(E-x)/\rho} \to \chi_{H^+_{\nu_E}(x)} \quad \text{and} \quad \chi_{((\Omega\setminus E)-x)/\rho} \to \chi_{H^-_{\nu_E}(x)} \quad \text{in} \quad L^1(B(0,1)) \text{ as } \rho \to 0^+, \tag{3.4.9}$$

where $H^\pm_{\nu_E}(x) := \{y \in \mathbb{R}^n : \pm y \cdot \nu_E(x) \geq 0\}$. Moreover, by (1.1.10), the hyperplane $H_{\nu_E}(x) := \{y : y \cdot \nu_E(x) = 0\}$ is the approximate tangent space to the measure $\mathscr{H}^{n-1} \llcorner \mathscr{F}E$ at $x \in \mathscr{F}E$ in the sense that for any $\varphi \in C_c(\Omega)$ one has

$$\lim_{\rho \to 0^+} \rho^{-(n-1)} \int_{\mathscr{F}E} \varphi\left(\frac{y-x}{\rho}\right) d\mathscr{H}^{n-1}(y) = \int_{H_{\nu_E}(x)} \varphi(z)\, d\mathscr{H}^{n-1}(z). \tag{3.4.10}$$

Finally, let us consider two sets E_1, E_2 of locally finite perimeter in Ω. Then, for $\mathscr{H}^{n-1}$-a.e. $x \in \mathscr{F}E_1 \cap \mathscr{F}E_2$, we have either $\nu_{E_1}(x) = \nu_{E_2}(x)$ or $\nu_{E_1}(x) = -\nu_{E_2}(x)$. This follows from the locality property of approximate tangent spaces, for which we refer to [11, Proposition 2.85 and Remark 2.87].

Proposition 3.4.6. *Let $F \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$ and let E_1, E_2 be sets of locally finite perimeter in Ω such that $\mathscr{H}^{n-1}(\mathscr{F}E_1 \cap \mathscr{F}E_2) \neq 0$. Then one has*

$$\mathcal{F}_i \cdot \nu_{E_1} = \mathcal{F}_i \cdot \nu_{E_2} \quad \text{and} \quad \mathcal{F}_e \cdot \nu_{E_1} = \mathcal{F}_e \cdot \nu_{E_2} \tag{3.4.11}$$

for $\mathscr{H}^{n-1}$-a.e. $x \in \{y \in \mathscr{F}E_1 \cap \mathscr{F}E_2 : \nu_{E_1}(y) = \nu_{E_2}(y)\}$ and

$$\mathcal{F}_i \cdot \nu_{E_1} = -\mathcal{F}_e \cdot \nu_{E_2} \quad \text{and} \quad \mathcal{F}_e \cdot \nu_{E_1} = -\mathcal{F}_i \cdot \nu_{E_2} \tag{3.4.12}$$

for $\mathscr{H}^{n-1}$-a.e. $x \in \{y \in \mathscr{F}E_1 \cap \mathscr{F}E_2 : \nu_{E_1}(y) = -\nu_{E_2}(y)\}$.

Proof. We begin with the first claim in (3.4.11). For $\mathscr{H}^{n-1}$-a.e. $x \in \mathscr{F}E_1 \cap \mathscr{F}E_2$ such that $\nu_{E_1}(x) = \nu_{E_2}(x)$ one has

$$x \text{ is a Lebesgue point for } \mathcal{F}_i \cdot \nu_{E_j} \text{ with respect to } \mathscr{H}^{n-1} \llcorner \mathscr{F}E_j \text{ for } j = 1, 2 \tag{3.4.13}$$

and

$$|\mathrm{div} F|((E_1^1 \cup E_2^1) \cap B(x,\rho)) = o(\rho^{n-1}). \tag{3.4.14}$$

Indeed, the normal traces are in $L^\infty_{\mathrm{loc}}(\mathscr{F}E; \mathscr{H}^{n-1})$ and so the Lebesgue-Besicovich differentiation theorem gives (3.4.13). For (3.4.14), it suffices to observe that $(E_1^1 \cup E_2^1) \cap \mathscr{F}E_j = \emptyset$ for $j = 1, 2$, and so the property follows from (3.4.8) with $\mu = |\mathrm{div} F| \llcorner (E_1^1 \cup E_2^1)$ and $k = n-1$.

Let $\eta \in C^\infty_c(B(0,1))$ and define $\eta_{x,\rho}(y) := \eta((y-x)/\rho)$ for any $\rho > 0$. It is clear that $\mathrm{supp}(\eta_{x,\rho}) \Subset \Omega$ for ρ small enough. By the integration by parts formula (3.3.19), we have

$$\int_{E_j^1} \eta_{x,\rho}\, d\mathrm{div} F = -\int_{\mathscr{F}E_j} \eta_{x,\rho} (\mathcal{F}_i \cdot \nu_{E_j})\, d\mathscr{H}^{n-1} - \int_{E_j} F \cdot \nabla \eta_{x,\rho}\, dy \tag{3.4.15}$$

for $j = 1, 2$. Using (3.4.14), we see that

$$\left| \int_{E_1^1} \eta_{x,\rho}\, d\mathrm{div} F - \int_{E_2^1} \eta_{x,\rho}\, d\mathrm{div} F \right| \leq |\mathrm{div} F|((E_1^1 \cup E_2^1) \cap B(x,\rho)) = o(\rho^{n-1}). \tag{3.4.16}$$

Since $\nabla \eta_{x,\rho} = (1/\rho)(\nabla \eta)_\rho$, one also has

$$\left| \int_{E_1} F \cdot \nabla \eta_{x,\rho}\, dy - \int_{E_2} F \cdot \nabla \eta_{x,\rho}\, dy \right| \leq \frac{1}{\rho} ||F||_{L^\infty(B(x,1);\mathbb{R}^n)} ||\nabla \eta||_{L^\infty(B(0,1);\mathbb{R}^n)} |(E_1 \Delta E_2) \cap B(x,\rho)|. \tag{3.4.17}$$

Next, observe that

$$\begin{aligned}\rho^{-n}|(E_1\Delta E_2)\cap B(x,\rho)| &= \rho^{-n}\int_{B(x,\rho)}|\chi_{E_1}-\chi_{E_2}|\,dy\\ &= \int_{B(0,1)}|\chi_{E_1}(x+\rho z)-\chi_{E_2}(x+\rho z)|\,dz\\ &= \int_{B(0,1)}|\chi_{\frac{E_1-x}{\rho}}(z)-\chi_{\frac{E_2-x}{\rho}}(z)|\,dz\to 0,\end{aligned}$$

as $\rho\to 0$, where one uses (3.4.9) and the fact that $H^+_{\nu_{E_1}}(x)=H^+_{\nu_{E_2}}(x)$. Hence, (3.4.17) implies

$$\left|\int_{E_1}F\cdot\nabla\eta_{x,\rho}\,dy-\int_{E_2}F\cdot\nabla\eta_{x,\rho}\,dy\right|=o(\rho^{n-1}). \tag{3.4.18}$$

Subtracting (3.4.15) with $j=2$ from (3.4.15) with $j=1$ and using (3.4.16) and (3.4.18), one obtains

$$\int_{\mathscr{F}E_1}\eta_{x,\rho}(\mathcal{F}_i\cdot\nu_{E_1})\,d\mathscr{H}^{n-1}-\int_{\mathscr{F}E_2}\eta_{x,\rho}(\mathcal{F}_i\cdot\nu_{E_2})\,d\mathscr{H}^{n-1}=o(\rho^{n-1}). \tag{3.4.19}$$

On the other hand, since x is a Lebesgue point for $\mathcal{F}_i\cdot\nu_{E_j}$ with respect to $\mathscr{H}^{n-1}\llcorner\mathscr{F}E_j$, one has

$$\begin{aligned}&\left|\int_{\mathscr{F}E_j}\eta_{x,\rho}(\mathcal{F}_i\cdot\nu_{E_j})\,d\mathscr{H}^{n-1}-(\mathcal{F}_i\cdot\nu_{E_j})(x)\int_{\mathscr{F}E_j}\eta_{x,\rho}\,d\mathscr{H}^{n-1}\right| \\ &\le\int_{\mathscr{F}E_j}\eta_{x,\rho}(y)|(\mathcal{F}_i\cdot\nu_{E_j})(y)-(\mathcal{F}_i\cdot\nu_{E_j})(x)|\,d\mathscr{H}^{n-1}(y)=o(\rho^{n-1})\end{aligned} \tag{3.4.20}$$

for $j=1,2$. In addition, (3.4.10) implies that

$$\left|\rho^{-(n-1)}\int_{\mathscr{F}E_j}\eta_{x,\rho}\,d\mathscr{H}^{n-1}-\int_{H_{\nu_{E_j}}(x)}\eta\,d\mathscr{H}^{n-1}\right|=o(1), \tag{3.4.21}$$

for $j=1,2$. Hence, by (3.4.20), (3.4.21) and the triangle inequality, one has

$$\left|\rho^{-(n-1)}\int_{\mathscr{F}E_j}\eta_{x,\rho}(\mathcal{F}_i\cdot\nu_{E_j})\,d\mathscr{H}^{n-1}-(\mathcal{F}_i\cdot\nu_{E_j})(x)\int_{H_{\nu_{E_j}}(x)}\eta\,d\mathscr{H}^{n-1}\right|=o(1).$$

Hence, for $j=1,2$ one has

$$\rho^{-(n-1)}\int_{\mathscr{F}E_j}\eta_{x,\rho}(\mathcal{F}_i\cdot\nu_{E_j})\,d\mathscr{H}^{n-1}\to(\mathcal{F}_i\cdot\nu_{E_j})(x)\int_{H_{\nu_{E_j}}(x)}\eta\,d\mathscr{H}^{n-1}\quad\text{as}\quad\rho\to 0. \tag{3.4.22}$$

Now choose η such that $\eta\ge(1/2)$ on $H_{\nu_{E_j}}(x)\cap B(0,(1/2))$ so that the integral over $H_{\nu_{E_j}}(x)$ is not zero. Recalling that $H_{\nu_{E_1}}(x)=H_{\nu_{E_2}}(x)$, then (3.4.19) and (3.4.22) imply $(\mathcal{F}_i\cdot\nu_{E_1})(x)=(\mathcal{F}_i\cdot\nu_{E_2})(x)$.

As for the other identities, notice that (3.4.6) gives $(\mathcal{F}_e\cdot\nu_{E_j})=-(\mathcal{F}_i\cdot\nu_{\Omega\setminus E_j})$ for $\mathscr{H}^{n-1}$-a.e. $x\in\mathscr{F}E_j$, for $j=1,2$. Moreover, since $\nu_{\Omega\setminus E_j}=-\nu_{E_j}$ $\mathscr{H}^{n-1}$-a.e. on $\mathscr{F}E_j$ and $\nu_{E_1}(x)=\nu_{E_2}(x)$, one has $\nu_{\Omega\setminus E_1}(x)=\nu_{\Omega\setminus E_2}(x)$. Since $\Omega\setminus E_j$ is a set of locally finite perimeter in Ω, one can apply the identity we just proved to obtain $(\mathcal{F}_e\cdot\nu_{E_1})(x)=-(\mathcal{F}_i\cdot\nu_{\Omega\setminus E_1})(x)=-(\mathcal{F}_i\cdot\nu_{\Omega\setminus E_2})(x)=(\mathcal{F}_e\cdot\nu_{E_2})(x)$ for $\mathscr{H}^{n-1}$-a.e. $x\in\{y\in\mathscr{F}E_1\cap\mathscr{F}E_2:\nu_{E_1}(y)=\nu_{E_2}(y)\}$, which is the second claim in (3.4.11). The identities of (3.4.12) follow in an analogous way by using (3.4.6), (3.4.7) and the previous argument applied to E_1 and $\Omega\setminus E_2$. $\square$

3.5 The Green's identities

As an application of the integration by parts formulas, we can generalize the classical Green's identities to Lipschitz functions whose gradients are locally essentially bounded divergence-measure fields.

Theorem 3.5.1. *Let $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ be such that $\Delta u \in \mathcal{M}_{\mathrm{loc}}(\Omega)$, and let $E \subset \Omega$ be a set of locally finite perimeter. Then there exist interior and exterior normal traces of ∇u: $(\nabla u_i \cdot \nu_E), (\nabla u_e \cdot \nu_E) \in L^\infty_{\mathrm{loc}}(\mathscr{F}E; \mathscr{H}^{n-1})$ such that, for any $v \in C(\Omega)$ satisfying $\nabla v \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ and* $\mathrm{supp}(\chi_E v) \Subset \Omega$*,*

$$\int_{E^1} v \, d\Delta u + \int_E \nabla v \cdot \nabla u \, dx = - \int_{\mathscr{F}E} v(\nabla u_i \cdot \nu_E) \, d\mathscr{H}^{n-1}, \tag{3.5.1}$$

$$\int_{E^1 \cup \mathscr{F}E} v \, d\Delta u + \int_E \nabla v \cdot \nabla u \, dx = - \int_{\mathscr{F}E} v(\nabla u_e \cdot \nu_E) \, d\mathscr{H}^{n-1}. \tag{3.5.2}$$

For any open set $U \Subset \Omega$, the following estimates hold:

$$\|\nabla u_i \cdot \nu_E\|_{L^\infty(\mathscr{F}E \cap U; \mathscr{H}^{n-1})} \le \|\nabla u\|_{L^\infty(U \cap E; \mathbb{R}^n)}, \tag{3.5.3}$$

$$\|\nabla u_e \cdot \nu_E\|_{L^\infty(\mathscr{F}E \cap U; \mathscr{H}^{n-1})} \le \|\nabla u\|_{L^\infty(U \setminus E; \mathbb{R}^n)}. \tag{3.5.4}$$

In addition, if $v \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ with $\Delta v \in \mathcal{M}_{\mathrm{loc}}(\Omega)$, and $\mathrm{supp}(\chi_E v), \mathrm{supp}(\chi_E u) \Subset \Omega$*, then the following formulas hold:*

$$\int_{E^1} v \, d\Delta u - u \, d\Delta v = - \int_{\mathscr{F}E} \Big(v(\nabla u_i \cdot \nu_E) - u(\nabla v_i \cdot \nu_E) \Big) \, d\mathscr{H}^{n-1}, \tag{3.5.5}$$

$$\int_{E^1 \cup \mathscr{F}E} v \, d\Delta u - u \, d\Delta v = - \int_{\mathscr{F}E} \Big(v(\nabla u_e \cdot \nu_E) - u(\nabla v_e \cdot \nu_E) \Big) \, d\mathscr{H}^{n-1}. \tag{3.5.6}$$

In particular, if $\mathrm{supp}(\chi_E u) \Subset \Omega$*, then*

$$\int_{E^1} u \, d\Delta u + \int_E |\nabla u|^2 \, dx = - \int_{\mathscr{F}E} u(\nabla u_i \cdot \nu_E) \, d\mathscr{H}^{n-1}, \tag{3.5.7}$$

$$\int_{E^1 \cup \mathscr{F}E} u \, d\Delta u + \int_E |\nabla u|^2 \, dx = - \int_{\mathscr{F}E} u(\nabla u_e \cdot \nu_E) \, d\mathscr{H}^{n-1}. \tag{3.5.8}$$

Proof. Since $\nabla u \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$, the existence of interior and exterior normal traces $\nabla u_i \cdot \nu_E, \nabla u_e \cdot \nu_E \in L^\infty_{\mathrm{loc}}(\mathscr{F}E; \mathscr{H}^{n-1})$ and the estimates (3.5.3) and (3.5.4) follow from Theorem 3.3.7. Analogously, (3.5.1) and (3.5.2) are an immediate consequence of (3.3.19) and (3.3.20), respectively, with $F = \nabla u$ and $\varphi = v$.

In addition, if $\mathrm{supp}(\chi_E u) \Subset \Omega$ and $v \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ with $\Delta v \in \mathcal{M}_{\mathrm{loc}}(\Omega)$, then we can exchange the role of u and v in (3.5.1) and (3.5.2):

$$\int_{E^1} u \, d\Delta v + \int_E \nabla v \cdot \nabla u \, dx = - \int_{\mathscr{F}E} u(\nabla v_i \cdot \nu_E) \, d\mathscr{H}^{n-1}, \tag{3.5.9}$$

$$\int_{E^1 \cup \mathscr{F}E} v \, d\Delta v + \int_E \nabla v \cdot \nabla u \, dx = - \int_{\mathscr{F}E} u(\nabla v_e \cdot \nu_E) \, d\mathscr{H}^{n-1}. \tag{3.5.10}$$

Thus, it suffices to subtract (3.5.9) from (3.5.1) to obtain (3.5.5), and to subtract (3.5.10) from (3.5.2) to obtain (3.5.6). Finally, choosing $u = v$ in (3.5.1) and (3.5.2), we obtain (3.5.7) and (3.5.8), respectively. $\square$

In the case we deal with continuously differentiable functions, thanks to Theorem 3.4.1, we can write the normal traces of the gradient as the classical scalar product with the measure theoretic unit interior normal.

Corollary 3.5.2. *Let $u \in C^1(\Omega)$ satisfy $\Delta u \in \mathcal{M}_{\mathrm{loc}}(\Omega)$ and let E be a set of locally finite perimeter in Ω. Then, for any $v \in C(\Omega)$ satisfying $\nabla v \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ and $\operatorname{supp}(\chi_E v) \Subset \Omega$, we have*

$$\int_{E^1} v \, d\Delta u = - \int_{\mathscr{F}E} v \, \nabla u \cdot \nu_E \, d\mathscr{H}^{n-1} - \int_E \nabla v \cdot \nabla u \, dx. \tag{3.5.11}$$

In addition, if $v \in C^1(\Omega)$ with $\Delta v \in \mathcal{M}_{\mathrm{loc}}(\Omega)$, and $\operatorname{supp}(\chi_E v), \operatorname{supp}(\chi_E u) \Subset \Omega$, then we get

$$\int_{E^1} v \, d\Delta u - u \, d\Delta v = - \int_{\mathscr{F}E} (v \nabla u - u \nabla v) \cdot \nu_E \, d\mathscr{H}^{n-1}. \tag{3.5.12}$$

In particular, if $\operatorname{supp}(\chi_E u) \Subset \Omega$, then

$$\int_{E^1} u \, d\Delta u + \int_E |\nabla u|^2 \, dx = - \int_{\mathscr{F}E} u \nabla u \cdot \nu_E \, d\mathscr{H}^{n-1}. \tag{3.5.13}$$

Proof. We begin by noticing that $\nabla u \in \mathcal{DM}^{\infty}_{\mathrm{loc}}(\Omega) \cap C(\Omega; \mathbb{R}^n)$, and so Theorem 3.4.1 implies that the normal traces of ∇u on $\mathscr{F}E$ coincide with the classical dot product $\nabla u(x) \cdot \nu_E(x)$ for $\mathscr{H}^{n-1}$-a.e. $x \in \mathscr{F}E$. Thus, (4.4.18), (4.4.19) and (3.5.13) follow from (3.5.1), (3.5.5) and (3.5.7). $\square$

Chapter 4

Divergence-measure fields in stratified groups

4.1 Introduction

[1] The Gauss–Green formula has been deeply studied also in a number of different non-Euclidean contexts, see for instance [95, 99, 110]. Related to these results is also the recent study by Züst, on functions of bounded fractional variation, [158]. Other extensions of the Gauss–Green formula appears in the framework of doubling metric spaces satisfying a Poincaré inequality, as in [117]. Through special trace theorems for BV functions in Carnot–Carathéodory spaces, an integration by parts formula has been established also in [155], assuming an intrinsic Lipschitz regularity on the boundary of the domain of integration.

The main objective of this chapter is to establish a Gauss–Green theorem for sets of finite perimeter and divergence-measure vector fields in a family of noncommutative nilpotent Lie groups, called *stratified groups* or *Carnot groups*. Such Lie groups equipped with a suitable *homogeneous distance* represent infinitely many different types of non-Euclidean geometries, with Hausdorff dimension strictly greater than their topological dimension. Notice that commutative stratified Lie groups coincide with normed vector spaces, where our results agree with the classical ones. Stratified groups arise from Harmonic Analysis and PDE, [75, 149], and represent an important class of connected and simply connected real nilpotent Lie groups. They are characterized by a family of dilations, along with a left invariant distance that properly scales with dilations, giving a large class of metric spaces that are not bi-Lipschitz equivalent to each other.

The theory of sets of *finite h-perimeter* in stratified groups has known a wide development in the last two decades, especially in relation to topics like De Giorgi's rectifiability, minimal surfaces and differentiation theorems. We mention for instance some relevant works [1, 16, 26, 46, 62, 80, 81, 105, 112, 113, 118, 124, 125, 127], only to give a small glimpse of the vast and always expanding literature. Some basic facts on the theory of sets of finite perimeter and BV functions hold in this setting, once these notions are properly defined. Indeed, other related notions such as reduced boundary and essential boundary, intrinsic rectifiability and differentiability can be naturally introduced in this setting, see for instance [139] for a recent overview on these topics and further references.

The stratified group $\mathbb{G}$, also called *Carnot group*, is always equipped with left invariant *horizontal vector fields* $X_1, \ldots, X_{\mathrm{m}}$, that determine the directions along which it is possible to differentiate. The corresponding distributional derivatives define functions of bounded h-

[1] This chapter is based on a joint work with Valentino Magnani [51]. However, in order to align it to the rest of the thesis, we adopted the convention that ν_E denotes the measure theoretic unit *interior* normal, resulting in a change of signs from the formulas in [51].

variation (Definition 1.3.2) and sets of finite h-perimeter (Definition 1.3.13). We consider *divergence-measure horizontal fields*, that are L^p-summable sections of the horizontal subbundle $H\Omega$, where Ω is an open set of $\mathbb{G}$ (Definition 4.2.1). Notice that the space of these fields, $\mathcal{DM}^p(H\Omega)$, with $1 \leq p \leq \infty$, contains divergence-measure horizontal fields that are not BV even with respect to the group structure (Example 4.2.3). Nevertheless, horizontal fields in $\mathcal{DM}^\infty(H\Omega)$ satisfy a Leibniz rule when multiplied by a function of bounded h-variation, that might be much less regular than a BV function on Euclidean space, see Theorem 4.1.1 below. The loss of Euclidean regularity can be already seen with sets of finite h-perimeter, that are not necessarily of finite perimeter in Euclidean sense, [79, Example 1]. Sets of finite h-perimeter are in some sense the largest class of measurable sets for which one can expect existence of normal traces and Gauss–Green formulas for divergence-measure horizontal fields.

Among our techniques, a special tool is the smooth approximation result given in Theorem 1.3.11, which provides a number of natural properties that are satisfied by the "correct" mollified function. We obtained it by the noncommutative group convolution (Definition 1.2.8). This is a well known tool in Harmonic Analysis and PDE on homogeneous Lie groups, [76, 150], that has been already used to study perimeters and BV functions on Heisenberg groups, [126, 141]. On the other hand, a number of smooth approximations can be obtained in Carnot-Carathéodory spaces or sub-Riemannian manifolds using the Euclidean convolution, also in relation to Meyers–Serrin theorem and Anzellotti–Giaquinta approximations for Sobolev and BV functions, [12, 77, 78, 86, 87, 155].

One should also notice that the minimal regularity of the mollifier ρ is necessary in order to have Proposition 1.3.20 and its consequences. Indeed, the mollifier ρ_ε can be also built using a homogeneous distance, that in general may not be smooth even outside the origin. Theorem 1.3.11 plays an important role also in the proof of the Leibniz rule of Theorem 4.1.1. The noncommutativity of the group convolution makes necessary a right invariant distance $d^{\mathcal{R}}$ canonically associated to d (1.2.2) and the 'right inner parts' of an open set $\Omega^{\mathcal{R}}_{2\varepsilon}$ (1.2.9), that appear in the statement of Theorem 1.3.11.

Theorem 4.1.1 (Approximation and Leibniz rule)**.** *If $F \in \mathcal{DM}^\infty(H\Omega)$ and $g \in L^\infty(\Omega)$ with $|D_H g|(\Omega) < +\infty$, then $gF \in \mathcal{DM}^\infty(H\Omega)$. If $\rho \in C_c(B(0,1))$ is nonnegative, $\rho(x) = \rho(x^{-1})$ and $\int_{B(0,1)} \rho \, dx = 1$, then for any infinitesimal sequence $\tilde{\varepsilon}_k > 0$, setting $g_\varepsilon := \rho_\varepsilon * g$, there exists a subsequence ε_k such that $g_{\varepsilon_k} \overset{*}{\rightharpoonup} \tilde{g}$ in $L^\infty(\Omega; |\mathrm{div} F|)$ and $\langle F, \nabla_H g_{\varepsilon_k} \rangle \, \mu \rightharpoonup (F, D_H g)$ in $\mathcal{M}(\Omega)$. Moreover, the following formula holds*

$$\mathrm{div}(gF) = \tilde{g} \, \mathrm{div} F + (F, D_H g), \tag{4.1.1}$$

where the measure $(F, D_H g)$ satisfies

$$|(F, D_H g)| \leq \|F\|_{L^\infty(\Omega)} |D_H g|. \tag{4.1.2}$$

Finally, we have the decompositions

$$(F, D_H g)^{\mathrm{a}} \mu = \langle F, \nabla_H g \rangle \, \mu \quad \textit{and} \quad (F, D_H g)^{\mathrm{s}} = (F, D^{\mathrm{s}}_H g), \tag{4.1.3}$$

where $\nabla_H g$ denotes the approximate differential of g.

In the Euclidean setting, this Leibniz rule has been established in [41, Theorem 3.1] and [82, Theorem 2.1]. The product rule (4.1.1) is the starting point of many of our results. For instance, applying this formula to $F \in \mathcal{DM}^\infty(H\Omega)$ and $g = \chi_E$, for a set of finite h-perimeter $E \Subset \Omega$, and using Lemma 4.2.6, one is led to a first embryonic Gauss–Green formula. Here the pairing $(F, D_H \chi_E)$ has still to be related to suitable notions of normal trace. Indeed, the interior and exterior normal traces $\langle \mathcal{F}_i, \nu_E \rangle$ and $\langle \mathcal{F}_e, \nu_E \rangle$, respectively, are defined in Section 4.3 through the notion of pairing measure as follows:

$$
\begin{aligned}
(\chi_E F, D_H \chi_E) &= \frac{1}{2} \langle \mathcal{F}_i, \nu_E \rangle \, |D_H \chi_E|, \\
(\chi_{\Omega \setminus E} F, D_H \chi_E) &= \frac{1}{2} \langle \mathcal{F}_e, \nu_E \rangle \, |D_H \chi_E|.
\end{aligned}
$$

We notice that this definition is well posed, since $(\chi_E F, D_H \chi_E)$ and $(\chi_{\Omega \setminus E} F, D_H \chi_E)$ are absolutely continuous with respect to the perimeter measure thanks to (4.1.2). It is important to stress that the weak assumptions of Theorem 4.1.1 a priori do not ensure the uniqueness of $\tilde{g}$ and of the pairing $(F, D_H g)$. They may both depend on the approximating sequence. A first remark is that at those points where the averaged limit of g exists with respect to $d^{\mathcal{R}}$, the function $\tilde{g}$ can be characterized explicitly (Proposition 4.2.9). However, the appearance of the right invariant distance $d^{\mathcal{R}}$ prevents the use of any intrinsic regularity of the reduced boundary (Definition 1.3.14) for sets of finite h-perimeter.

Despite these difficulties, a rather unexpected fact occurs, since in the case $g = \chi_E$ and E has finite h-perimeter, it is possible to prove that the limit $\widetilde{\chi_E}$ is uniquely determined, along with the normal traces, regardless of the choice of the mollifying sequence $\rho_{\varepsilon_k} * \chi_E$. The surprising aspect is that we have no rectifiability result for the reduced boundary in arbitrary stratified groups. We mainly use functional analytic arguments, the absolute continuity $\mathrm{div} F \ll \mathcal{S}^{Q-1}$ and the important Proposition 2.4.2, that is further discussed below. We summarize these relevant facts by restating here the main results of Theorem 4.3.13.

Theorem 4.1.2 (Uniqueness of traces)**.** *If $F \in \mathcal{DM}^\infty(H\Omega)$ and $E \subset \Omega$ is a set of finite h-perimeter, then there exists a unique $|\mathrm{div} F|$-measurable subset*

$$
E^{1,F} \subset \Omega \setminus \mathscr{F}_H E,
$$

up to $|\mathrm{div} F|$-negligible sets, such that

$$
\widetilde{\chi_E}(x) = \chi_{E^{1,F}}(x) + \frac{1}{2} \chi_{\mathscr{F}_H E}(x) \quad \text{for} \ \ |\mathrm{div} F|\text{-a.e. } x \in \Omega. \tag{4.1.4}
$$

In addition, there exist unique normal traces

$$
\langle \mathcal{F}_i, \nu_E \rangle, \langle \mathcal{F}_e, \nu_E \rangle \in L^\infty(\mathscr{F}_H E; |D_H \chi_E|)
$$

satisfying

$$
\mathrm{div}(\chi_E F) = \chi_{E^{1,F}} \mathrm{div} F + \langle \mathcal{F}_i, \nu_E \rangle \, |D_H \chi_E|, \tag{4.1.5}
$$

$$
\mathrm{div}(\chi_E F) = \chi_{E^{1,F} \cup \mathscr{F}_H E} \mathrm{div} F + \langle \mathcal{F}_e, \nu_E \rangle \, |D_H \chi_E|. \tag{4.1.6}
$$

Equalities (4.1.5) and (4.1.6) immediately lead to general Gauss–Green formulas. Indeed, taking $F \in \mathcal{DM}^\infty(H\Omega)$ and a set of finite h-perimeter $E \Subset \Omega$, it is enough to evaluate (4.1.5) and (4.1.6) on Ω, and then to exploit the fact that $\chi_E F \in \mathcal{DM}^\infty(H\Omega)$, thanks to Theorem 4.1.1, and Lemma 4.2.6. In this way, we obtain the following general versions of the Gauss–Green formulas in stratified groups:

$$
\mathrm{div} F(E^{1,F}) = - \int_{\mathscr{F}_H E} \langle \mathcal{F}_i, \nu_E \rangle \; d|D_H \chi_E|, \tag{4.1.7}
$$

$$
\mathrm{div} F(E^{1,F} \cup \mathscr{F}_H E) = - \int_{\mathscr{F}_H E} \langle \mathcal{F}_e, \nu_E \rangle \; d|D_H \chi_E|. \tag{4.1.8}
$$

We notice that, as a simple consequence of (4.1.5), we can define $E^{1,F}$, up to $|\mathrm{div}F|$-negligible sets, as that Borel set in $\Omega \setminus \mathscr{F}_H E$ satisfying

$$\mathrm{div}(\chi_E F) \llcorner \Omega \setminus \mathscr{F}_H E = \mathrm{div}F \llcorner E^{1,F}. \tag{4.1.9}$$

However, it is still an appealing open question to characterize $E^{1,F}$ explicitly, namely in geometric terms, or even to prove that the set $E^{1,F}$ does not depend on the vector field F, as it happens in the Euclidean context.

Nevertheless, we are able to find different sets of assumptions, involving either the regularity of E or of the field F, for which $E^{1,F}$ can be properly detected. This immediately yields a number of Gauss–Green and integration by parts formulas in the spirit of the well known Euclidean results.

Before discussing Gauss–Green formulas, it is natural to ask whether normal traces have the locality property. Rather unexpectedly, also locality of normal traces is obtained without any blow-up technique related to rectifiability of the reduced boundary. Indeed, the classical proofs in the literature heavily employ the existence of an approximate tangent space at almost every point on the reduced boundary of a set of finite perimeter ([8, Proposition 3.2] and [52, Proposition 4.10]).

In Theorem 4.3.6 we show that the normal traces of a divergence-measure horizontal section F only depend on the orientation of the reduced boundary. It can be seen using the Leibniz rule established in Proposition 4.3.2, the locality of perimeter in stratified groups proved by Ambrosio-Scienza [20] and general arguments of measure theory. Another important tool that somehow allows us to overcome the absence of regularity of the reduced boundary is Proposition 2.4.2, where we prove that the weak* limit of $\rho_\varepsilon * \chi_E$ in $L^\infty(\Omega; |D_H \chi_E|)$ is precisely $1/2$, for any set $E \subset \mathbb{G}$ of finite h-perimeter and any symmetric mollifier ρ. This proposition can be proved by a soft argument borrowed from [10, Proposition 4.3]. It seems quite interesting that this weak* convergence comes from an analogous study in the infinite dimensional setting of Wiener spaces and it does not require any existence of blow-ups.

Proposition 2.4.2, together with Remark 2.4.4 and Lemma 2.4.5, is fundamental in proving the refinements (4.1.5) and (4.1.6) of the Leibniz rule, along with the uniqueness results of Theorem 4.1.2. Furthermore, Proposition 2.4.2 immediately leads to the 'intrinsic blow-up property' (Lemma 2.4.5), that is fundamental to prove the estimates of Proposition 4.3.4 for the normal traces of $F \in \mathcal{DM}^\infty(\Omega)$. We point out that the names of interior and exterior normal traces can be also justified by the same estimates (4.3.18) and (4.3.19). We stress that the proofs of this result in the Euclidean literature rely on De Giorgi's blow-up theorem, see [52, Theorem 3.2], while Proposition 4.3.4, when the group is commutative, provides an alternative proof.

Returning to Gauss–Green formulas, we observe first that when $\mathscr{F}_H E$ is negligible with respect to $|\mathrm{div}F|$ (Theorem 4.4.5), then the interior and exterior normal traces coincide. In particular, we can define the *average normal trace* $\langle \mathcal{F}, \nu_E \rangle$ as the density of the pairing $(F, D_H \chi_E)$ with respect to the h-perimeter measure $|D_H \chi_E|$, according to Definition 4.4.2. Thanks to (4.3.46), it is immediate to observe that

$$\langle \mathcal{F}, \nu_E \rangle = \frac{\langle \mathcal{F}_i, \nu_E \rangle + \langle \mathcal{F}_e, \nu_E \rangle}{2}.$$

As a result, when $|\mathrm{div}F|(\mathscr{F}_H E) = 0$, we have $\langle \mathcal{F}_i, \nu_E \rangle = \langle \mathcal{F}_e, \nu_E \rangle = \langle \mathcal{F}, \nu_E \rangle$, so that there exists a unique normal trace and the Gauss–Green formula (4.4.7) holds.

In case the divergence-measure field F is continuous (Theorem 4.4.7), then the Gauss–Green formula (4.4.9) holds and the normal trace has an explicit representation by the scalar product between the field F and the measure theoretic unit interior h-normal ν_E.

A first important consequence of the previous theorems is a Gauss–Green formula for horizontal fields with divergence-measure absolutely continuous with respect to the Haar measure of the group. Such a result could be also achieved from a modified product rule with additional assumptions on $\mathrm{div}F$, but we have preferred to start from a more general Leibniz rule and then derive some special cases from it.

Theorem 4.1.3. *Let $F \in \mathcal{DM}^\infty(H\Omega)$ such that $|\mathrm{div}F| \ll \mu$ and let $E \Subset \Omega$ be a set of finite h-perimeter. Then there exists a unique normal trace $\langle \mathcal{F}, \nu_E \rangle \in L^\infty(\Omega; |D_H\chi_E|)$ such that there holds*

$$\mathrm{div}F(E) = -\int_{\mathscr{F}_H E} \langle \mathcal{F}, \nu_E \rangle \, d|D_H\chi_E|. \tag{4.1.10}$$

The key point is to show that $E^{1,F}$ can be replaced by E, namely, to prove that their symmetric difference is $|\mathrm{div}F|$-negligible. The Gauss–Green formula (4.1.10) naturally leads to the following integration by parts formula.

Theorem 4.1.4. *Let $F \in \mathcal{DM}^\infty_{\mathrm{loc}}(H\Omega)$ be such that $|\mathrm{div}F| \ll \mu$, and let E be a set of locally finite h-perimeter in Ω. Let $\varphi \in C(\Omega)$ with $\nabla_H\varphi \in L^1_{\mathrm{loc}}(H\Omega)$ such that*

$$\mathrm{supp}(\varphi\chi_E) \Subset \Omega.$$

Then there exists a unique normal trace $\langle \mathcal{F}, \nu_E \rangle \in L^\infty_{\mathrm{loc}}(\Omega; |D_H\chi_E|)$ of F, such that the following formula holds

$$\int_E \varphi \, d\mathrm{div}F + \int_E \langle F, \nabla_H\varphi \rangle \, dx = -\int_{\mathscr{F}_H E} \varphi \, \langle \mathcal{F}, \nu_E \rangle \, d|D_H\chi_E|. \tag{4.1.11}$$

We notice that the assumption $|\mathrm{div}F| \ll \mu$ is very general in the sense that it is satisfied by $F \in W^{1,p}_{H,\mathrm{loc}}(H\Omega)$, for any $1 \le p \le \infty$. Moreover, it clearly implies $|\mathrm{div}F|(\mathscr{F}_H E) = 0$, which means that the divergence-measure is not concentrated on the reduced boundary of E, and thus there is no jump component in the divergence. It is also worth to point out that both (4.1.10) and (4.1.11) hold also for sets whose boundary is not rectifiable in the Euclidean sense (Example 4.5.2).

If we slightly weaken the absolute continuity assumption on $\mathrm{div}F$, requiring instead

$$|\mathrm{div}F|(\partial_H^{*,\mathcal{R}} E) = 0,$$

where $\partial_H^{*,\mathcal{R}} E$ is the measure theoretic boundary of E with respect to the right invariant distance $d^{\mathcal{R}}$ (1.3.28), we are able to prove that $E^{1,F}$ is equivalent to $E^{1,\mathcal{R}}$; that is, the measure theoretic interior with respect to $d^{\mathcal{R}}$. As a consequence, we can derive a modified Gauss–Green formula (Theorem 4.5.6) and related statements.

Finally, other versions of the Gauss–Green theorem and integration by parts formulas can be obtained in the case the set $E \subset \mathbb{G}$ has finite perimeter in the Euclidean sense. Here it is important to investigate the behavior of the Euclidean pairing of a field $F \in \mathcal{DM}^\infty(H\Omega)$ and a function $g \in BV(\Omega) \cap L^\infty(\Omega)$. Let us remark that, even if the family $\mathcal{DM}^p(H\Omega)$ with $1 \le p \le \infty$ is strictly contained in the known space of divergence-measure fields (Section 4.2.1), the known Euclidean results could only prove that the Euclidean pairing measure (F, Dg) is absolutely continuous with respect to the total variation $|Dg|$. This result does not imply the absolute continuity of the pairing with respect to $|D_H g|$, since this measure is absolutely continuous with respect to $|Dg|$ while the opposite may not hold in general.

In Theorem 4.6.3 we refine the classical results on (F, Dg), proving that, up to a restriction to bounded open sets,

$$|(F, Dg)| \le \|F\|_{L^\infty(\Omega)} |D_H g|.$$

For this purpose, we have used the Euclidean convolution to compare the Euclidean pairing with the intrinsic pairing in the stratified group. While no exact commutation rule between the horizontal gradient and the Euclidean convolution holds, it is however possible to use an asymptotic commutator estimate similar to the classical one by Friedrichs [83], see also [86]. Thanks to the above absolute continuity, we can actually prove that, given a set of Euclidean finite perimeter E, the group pairing $(F, D_H\chi_E)$ defined in Theorem 4.1.1 is actually equal to the Euclidean pairing $(F, D\chi_E)$, according to Theorem 4.6.4. An important tool used in the proof of this result is Theorem 4.2.7, which states that $|\mathrm{div}F| \ll \mathscr{S}^{Q-1}$, if $F \in \mathcal{DM}^\infty_{\mathrm{loc}}(H\Omega)$. This property of $\mathrm{div}F$ allows us to show in Theorem 4.6.4 that we have $E^{1,F} = E^1_{|\cdot|}$, up to a $|\mathrm{div}F|$-negligible set, where we denote by $E^1_{|\cdot|}$ the Euclidean measure theoretic interior of E; that is, the set of points with density 1 with respect to the balls defined using the Euclidean distance in the group. These results allow us to prove the following Leibniz rules and integration by parts formulas for sets of Euclidean finite perimeter in stratified groups.

Theorem 4.1.5. *Let $F \in \mathcal{DM}^\infty_{\mathrm{loc}}(H\Omega)$ and $E \subset \Omega$ be a set of Euclidean locally finite perimeter in Ω, then there exist interior and exterior normal traces $\langle \mathcal{F}_i, \nu_E\rangle, \langle \mathcal{F}_e, \nu_E\rangle \in L^\infty_{\mathrm{loc}}(\Omega; |D_H\chi_E|)$ such that, for any open set $U \Subset \Omega$, we have*

$$\mathrm{div}(\chi_E F) = \chi_{E^1_{|\cdot|}}\mathrm{div}F + \langle \mathcal{F}_i, \nu_E\rangle\, |D_H\chi_E|, \tag{4.1.12}$$

$$\mathrm{div}(\chi_E F) = \chi_{E^1_{|\cdot|}\cup\mathscr{F}_H E}\,\mathrm{div}F + \langle \mathcal{F}_e, \nu_E\rangle\, |D_H\chi_E|, \tag{4.1.13}$$

$$\chi_{\mathscr{F}_H E}\,\mathrm{div}F = (\langle \mathcal{F}_i, \nu_E\rangle - \langle \mathcal{F}_e, \nu_E\rangle)\, |D_H\chi_E| \tag{4.1.14}$$

in $\mathcal{M}(U)$. Moreover, we get the trace estimates

$$\|\, \langle \mathcal{F}_i, \nu_E\rangle \,\|_{L^\infty(\mathscr{F}_H E\cap U; |D_H\chi_E|)} \le \|F\|_{L^\infty(E\cap U)},$$
$$\|\, \langle \mathcal{F}_e, \nu_E\rangle \,\|_{L^\infty(\mathscr{F}_H E\cap U; |D_H\chi_E|)} \le \|F\|_{L^\infty(U\setminus E)}.$$

For any $\varphi \in C(\Omega)$ with $\nabla_H\varphi \in L^1_{\mathrm{loc}}(H\Omega)$ such that $\mathrm{supp}(\varphi\chi_E) \Subset \Omega$, we have

$$\int_{E^1_{|\cdot|}} \varphi\, d\mathrm{div}F + \int_E \langle F, \nabla_H\varphi\rangle\, dx = -\int_{\mathscr{F}_H E} \varphi\, \langle \mathcal{F}_i, \nu_E\rangle\, d|D_H\chi_E|, \tag{4.1.15}$$

$$\int_{E^1_{|\cdot|}\cup\mathscr{F}_H E} \varphi\, d\mathrm{div}F + \int_E \langle F, \nabla_H\varphi\rangle\, dx = -\int_{\mathscr{F}_H E} \varphi\, \langle \mathcal{F}_e, \nu_E\rangle\, d|D_H\chi_E|. \tag{4.1.16}$$

Formulas (4.1.11) and (4.1.15) extend Anzellotti's pairings to stratified groups in the case the BV function of the pairing is the characteristic function of a finite h-perimeter set. Indeed, if we take E to be an open bounded set with Euclidean Lipschitz boundary, as in the assumptions of [23, Theorem 1.1], then it is well known that $E^1_{|\cdot|} = E$. Thus, for this choice of E, it is clear that (4.1.11) and (4.1.15) are equivalent to definition of (interior) normal trace of Anzellotti; that is, the pairing between F and $D\chi_E$ (see [23, Definition 1.4]).

Let us point out that Theorem 4.1.5 is new even when seen in Euclidean coordinates, since the measures appearing in the Leibniz rules are in fact absolutely continuous with respect to the h-perimeter.

In the assumptions of Theorem 4.1.5, if $E \Subset \Omega$, taking the test function $\varphi \equiv 1$ in both (4.1.15) and (4.1.16), we get the following general Gauss–Green formulas

$$\mathrm{div}F(E^1_{|\cdot|}) = -\int_{\mathscr{F}_H E} \langle \mathcal{F}_i, \nu_E\rangle\, d|D_H\chi_E|, \tag{4.1.17}$$

$$\mathrm{div}F(E^1_{|\cdot|}\cup\mathscr{F}_H E) = -\int_{\mathscr{F}_H E} \langle \mathcal{F}_e, \nu_E\rangle\, d|D_H\chi_E|. \tag{4.1.18}$$

Analogously, the estimates on the L^∞-norm of the normal traces are similar to those in (3.1.5). When the vector field F is C^1 smooth up to the boundary of a bounded set $E \Subset \Omega$ of Euclidean finite perimeter, then all (4.1.10), (4.1.17) and (4.1.18) boil down to the following one

$$\int_E \mathrm{div} F\, dx = -\int_{\mathscr{F}E} \langle F, \nu_E\rangle\, d|D_H\chi_E| = -\int_{\mathscr{F}E} \left\langle F, N_E^H \right\rangle d|D\chi_E|, \tag{4.1.19}$$

where $N_E^H = \sum_{j=1}^m \langle N_E, X_j\rangle_{\mathbb{R}^\mathrm{q}} X_j$ is the non-normalized interior horizontal normal, $\mathbb{G}$ is linearly identified with $\mathbb{R}^\mathrm{q}$ (Section 1.2.2), $\langle\cdot,\cdot\rangle_{\mathbb{R}^\mathrm{q}}$ denotes the Euclidean scalar product, N_E is the Euclidean measure theoretic interior normal, $|D\chi_E|$ is the Euclidean perimeter and $\mathscr{F}E$ is the Euclidean reduced boundary. In the special case of (4.1.19) the proof is a simple application of the Euclidean theory of sets of finite perimeter, see for instance [47, Remark 2.1].

Equalities of (4.1.19) can be also written using Hausdorff measures, getting

$$\int_E \mathrm{div} F\, dx = -\int_{\mathscr{F}E} \left\langle F, N_E^H \right\rangle d\mathscr{H}_{|\cdot|}^{\mathrm{q}-1} = -\int_{\mathscr{F}E} \langle F, \nu_E\rangle\, d\mathscr{S}^{Q-1}. \tag{4.1.20}$$

The first equality is a consequence of the rectifiability of Euclidean finite perimeter sets [64] and the second one follows from [113], when the homogeneous distance d constructing $\mathscr{S}^{Q-1}$ is suitably symmetric. For instance, when E is bounded, ∂E is piecewise smooth and F is a C^1 smooth vector field on a neighborhood of $\overline{E}$, then (4.1.19) and (4.1.20) hold and the reduced boundary $\mathscr{F}E$ can be replaced by the topological boundary ∂E, coherently with the classical result (3.1.1).

For smooth functions and sufficiently smooth domains, Green's formulas, that are simple consequences of the Gauss-Green theorem, have proved to have a wide range of applications in classical PDE's. In the context of sub-Laplacians in stratified groups these formulas play an important role, [28, 133].

As a consequence of our results, we obtain a very general version of Green's formulas in stratified groups. Precisely in the next theorem, (4.1.21) and (4.1.22) represent the first and the second Green's formulas, where the domain of integration is only assumed to be a set with Euclidean finite perimeter and the sub-Laplacians are measures.

Theorem 4.1.6. *Let $u \in C^1_H(\Omega)$ satisfy $\Delta_H u \in \mathcal{M}_{\mathrm{loc}}(\Omega)$ and let $E \subset \Omega$ be a set of Euclidean locally finite perimeter in Ω. Then for each $v \in C_c(\Omega)$ with $\nabla_H v \in L^1(H\Omega)$ one has*

$$\int_{E^1_{|\cdot|}} v\, d\Delta_H u = -\int_{\mathscr{F}_H E} v\, \langle \nabla_H u, \nu_E\rangle\, d|D_H\chi_E| - \int_E \langle \nabla_H v, \nabla_H u\rangle\, dx. \tag{4.1.21}$$

If $u, v \in C^1_{H,c}(\Omega)$ also satisfy $\Delta_H u, \Delta_H v \in \mathcal{M}(\Omega)$, one has

$$\int_{E^1_{|\cdot|}} v\, d\Delta_H u - u\, d\Delta_H v = \int_{\mathscr{F}_H E} \langle u\nabla_H v - v\nabla_H u, \nu_E\rangle\, d|D_H\chi_E|. \tag{4.1.22}$$

If $E \Subset \Omega$, one can drop the assumption that u and v have compact support in Ω.

These Green's formulas are extended in Theorem 4.5.3 to sets of h-finite perimeter, assuming that the sub-Laplacian is absolutely continuous with respect to the Haar measure of the group.

The chapter is structured as follows: in Section 4.2 we introduce the divergence-measure horizontal fields and we present some of their first properties, including the absolute continuity with respect to Hausdorff spherical measure (Theorem 4.2.7). Then, we prove the Leibniz rule (Theorem 4.1.1) for the essentially bounded case, together with some refinements in special cases. In Section 4.3 the normal traces for essentially bounded horizontal divergence-measure fields on the boundaries of sets with finite h-perimeter are defined, their relation with the Leibniz rules is explored and their locality properties are established (Theorem 4.3.6). In addition, the existence and uniqueness of measure theoretic interior and exterior of a set of finite h-perimeter E with respect to the divergence-measure field F, $E^{1,F}$ and $E^{0,F}$, are established in Theorem 4.3.13. Then, by exploiting the Leibniz rules, we obtain the uniqueness also of the normal traces, thus paving the way for the general Gauss–Green and integration by parts formulas (Theorem 4.4.1 and Theorem 4.4.8), presented in Section 4.4. In this section we also prove that, if the horizontal field is continuous, the interior and exterior normal traces coincide with the scalar product associated with the invariant Riemannian metric. Finally, we consider some special cases. In Section 4.5 we deal with horizontal fields whose divergence is absolutely continuous with respect to the Haar measure of the group, and with a slightly more general assumption on the concentration properties of the divergence measure. Instead, Section 4.6 deals with the case of sets with Euclidean finite perimeter.

4.2 Divergence-measure horizontal fields

In this section we will introduce and study the function spaces of p-summable horizontal sections whose horizontal divergence is a Radon measure. In the sequel, Ω will denote a fixed open set of $\mathbb{G}$.

4.2.1 General properties and Leibniz rules

By a little abuse of notation, for any μ-measurable set E we shall use the symbols $\|F\|_{L^p(E)}$ and $\|F\|_{L^\infty(E)}$ with the same meaning as in (1.3.1) and (1.3.2).

Definition 4.2.1 (Divergence-measure horizontal field)**.** A p-summable divergence-measure horizontal field is a field $F \in L^p(H\Omega)$ whose distributional divergence $\mathrm{div}F$ is a Radon measure on Ω. We denote by $\mathcal{DM}^p(H\Omega)$ the space of all p-summable divergence-measure horizontal fields, where $1 \le p \le \infty$. A measurable section F of $H\Omega$ is a *locally p-summable divergence-measure horizontal field* if, for any open subset $W \Subset \Omega$, we have $F \in \mathcal{DM}^p(HW)$. The space of all such section is denoted by $\mathcal{DM}^p_{\mathrm{loc}}(H\Omega)$.

It is easy to observe that, if $F = \sum_{j=1}^{\mathrm{m}} F_j X_j$ and $F_j \in L^p(\Omega) \cap BV_H(\Omega)$ for all $j = 1, \dots, \mathrm{m}$, then $F \in \mathcal{DM}^p(H\Omega)$. We also notice that, from (1.3.12) and (1.3.13), the divergence-measure horizontal fields forms a subspace of the whole space of divergence-measure fields. Hence, if we denote by $T\Omega$ the tangent bundle of Ω, we have $\mathcal{DM}^p(H\Omega) \subset \mathcal{DM}^p(T\Omega)$, for any $p \in [1, \infty]$, where $\mathcal{DM}^p(T\Omega)$ denotes the classical space of divergence-measure fields with respect to the Euclidean structure fixed on $\mathbb{G}$. Actually $\mathcal{DM}^p(H\Omega)$ is a closed subspace of $\mathcal{DM}^p(T\Omega)$, according to the next remark.

Remark 4.2.2. As in the Euclidean case ([41, Corollary 1.1]), $\mathcal{DM}^p(H\Omega)$ endowed with the following norm

$$\|F\|_{\mathcal{DM}^p(H\Omega)} := \|F\|_{L^p(\Omega)} + |\mathrm{div}F|(\Omega)$$

is a Banach space. Any Cauchy sequence $\{F_k\}$ is clearly a Cauchy sequence in $L^p(H\Omega)$, and so there exists $F \in L^p(H\Omega)$ such that $F_k \to F$ in $L^p(H\Omega)$. Then, the lower semicontinuity of the total variation and the property of the Cauchy sequence yield $F \in \mathcal{DM}^p(H\Omega)$ and $|\mathrm{div}(F - F_k)|(\Omega) \to 0$.

The following example shows that fields of $\mathcal{DM}^p(H\Omega)$ may have components that are not BV functions. It is a simple modification of an example of Chen and Frid, see [42, Example 1.1].

Example 4.2.3. Let $\mathbb{G} = \mathbb{H}^1$, be the first Heisenberg group, equipped with graded coordinates (x, y, z) and horizontal left invariant vector fields $X_1 = \partial_1 - y\partial_3$ and $X_2 = \partial_2 + x\partial_3$. We define the divergence-measure horizontal field

$$F(x, y, z) = \sin\left(\frac{1}{x-y}\right)(X_1 + X_2).$$

It is plain to see that $F \in L^\infty(H\mathbb{H}^1)$, and that

$$\mathrm{div}F = X_1 \sin\left(\frac{1}{x-y}\right) + X_2 \sin\left(\frac{1}{x-y}\right) = 0,$$

in the sense of Radon measures, but the components of F are not BV.

Remark 4.2.4. We notice that, for a given $F \in \mathcal{DM}^p(T\Omega)$, if we denote by F_H its projection on the horizontal subbundle with respect to a fixed left invariant Riemannian metric that makes $X_1, \ldots, X_{\mathrm{q}}$ orthonormal, we may not get $F_H \in \mathcal{DM}^p(H\Omega)$. Let us consider the Heisenberg group $\mathbb{H}^1$ identified with $\mathbb{R}^3$, as in the previous example, along with the vector fields X_1, X_2, and define $X_3 = \partial_3$.

Let us consider the following measurable vector field

$$G(x, y, z) = \sin\left(\frac{1}{x-z}\right)(\partial_1 + \partial_2 + \partial_3).$$

We clearly have $G \in \mathcal{DM}^\infty(T\mathbb{H}^1)$, i.e. G is a divergence-measure field. However, if we consider its projection onto horizontal fibers

$$G_H(x, y, z) = \sin\left(\frac{1}{x-z}\right)(X_1 + X_2),$$

for any $x \neq z$, we have

$$\mathrm{div}\, G_H(x, y, z) = -\frac{1 + x + y}{(x-z)^2} \cos\left(\frac{1}{x-z}\right),$$

which is not a locally summable function in any neighborhood of $\{x = z\}$. This shows that $\mathrm{div}\, G_H \notin \mathcal{M}(\mathbb{H}^1)$.

We show now an easy extension result (see also [52, Remark 2.20]).

Remark 4.2.5. If $1 \leq p \leq \infty$ and $F \in \mathcal{DM}^p(H\Omega)$ has compact support in Ω, then its trivial extension

$$\hat{F}(x) := \begin{cases} F(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \mathbb{G} \setminus \Omega, \end{cases}$$

belongs to $\mathcal{DM}^p(H\mathbb{G})$. Indeed, since $\hat{F} \in L^p(H\mathbb{G})$ and for any $\phi \in C_c^\infty(\mathbb{G})$ and a fixed $\xi \in C_c^\infty(\Omega)$ that equals one on a neighborhood of the support of F, we have

$$\begin{aligned} \int_{\mathbb{G}} \langle \hat{F}, \nabla_H \phi \rangle \, dx &= \int_{\Omega} \langle \hat{F}, \nabla_H \phi \rangle \, dx \\ &= \int_{\Omega} \langle F, \nabla_H (\xi\phi) \rangle \, dx + \int_{\Omega} \langle F, \nabla_H ((1-\xi)\phi) \rangle \, dx \\ &= -\int_{\Omega} \phi \, d(\mathrm{div}F \llcorner \xi) = -\int_{\mathbb{G}} \phi \, d(\mathrm{div}F \llcorner \xi), \end{aligned} \tag{4.2.1}$$

where we denote by $\mathrm{div}F \llcorner \xi$ the signed Radon measure on $\mathbb{G}$ such that

$$\mathrm{div}F \llcorner \xi(E) = \int_{\Omega \cap E} \xi \, d\mathrm{div}F$$

for every relatively compact Borel subset $E \subset \mathbb{G}$. Thus, we have shown that $\hat{F} \in \mathcal{DM}^p(H\mathbb{G})$ and $\mathrm{div}\hat{F} = \mathrm{div}F \llcorner \xi$. The equalities of (4.2.1) imply that the restriction of $\mathrm{div}\hat{F}$ to Ω coincides with $\mathrm{div}F$ and in particular $|\mathrm{div}\hat{F}|(\Omega) = |\mathrm{div}F|(\Omega)$. The same equalities also imply that $|\mathrm{div}\hat{F}|(\mathbb{G} \setminus \Omega) = 0$.

As a consequence, we can prove the following result concerning fields with compact support, which can be seen as the easy case of the Gauss–Green formula, since there are no boundary terms. A similar result has been proved in the Euclidean setting in [52, Lemma 3.1].

Lemma 4.2.6. *If $1 \leq p \leq \infty$ and $F \in \mathcal{DM}^p(H\Omega)$ has compact support in Ω, then*

$$\mathrm{div}F(\Omega) = 0.$$

Proof. Since F has compact support in Ω, the extension $\hat{F}$ defined in Remark 4.2.5 shows that $\hat{F} \in \mathcal{DM}^p(H\mathbb{G})$, $\mathrm{div}\hat{F} = \mathrm{div}F$ as signed Radon measure in Ω and $\mathrm{div}\hat{F}$ is the null measure when restricted to $\mathbb{G} \setminus \Omega$. As a consequence, if $\phi \in C_c^\infty(\mathbb{G})$ is chosen such that $\phi = 1$ on a neighborhood of Ω, then

$$\int_{\mathbb{G}} \phi \, d\mathrm{div}\hat{F} = \int_{\Omega} d\mathrm{div}\hat{F} = \mathrm{div}F(\Omega).$$

By definition of distributional divergence, there holds

$$\int_{\mathbb{G}} \phi \, d\mathrm{div}\hat{F} = -\int_{\Omega} \langle F, \nabla_H \phi \rangle \, dx = 0,$$

since F has support inside Ω and ϕ is constant on this set. This concludes the proof. $\square$

We show now a result concerning the absolute continuity properties of $\mathrm{div}F$ with respect to the $\mathscr{S}^\alpha$-measure, for a suitable α related to the summability exponent p. This is a generalization of a known result in the Euclidean case ([144, Theorem 3.2]).

Theorem 4.2.7. *If $F \in \mathcal{DM}^p_{\mathrm{loc}}(H\Omega)$ and $\frac{Q}{Q-1} \leq p < +\infty$, then $|\mathrm{div}F|(B) = 0$ for any Borel set $B \subset \Omega$ of σ-finite $\mathscr{S}^{Q-p'}$ measure. If $p = \infty$, then $|\mathrm{div}F| \ll \mathscr{S}^{Q-1}$.*

Proof. Let $\frac{Q}{Q-1} \leq p < +\infty$. It suffices to consider a Borel set B such that $\mathscr{S}^{Q-p'}(B) < \infty$. We can use the Hahn decomposition in order to split B into $B_+ \cup B_-$, in such a way that $\pm\mathrm{div}F \llcorner B_\pm \geq 0$, thus reducing ourselves to show that $\mathrm{div}F(K) = 0$ for any compact set $K \subset B_\pm$. Without loss of generality, we consider $K \subset B_+$. Let $\varphi : \mathbb{G} \to [0, 1]$ defined as follows

$$\varphi(x) := \begin{cases} 1 & \text{if}\, d(x, 0) < 1 \\ 2 - d(x, 0) & \text{if}\, 1 \leq d(x, 0) \leq 2 \\ 0 & \text{if}\, d(x, 0) > 2 \end{cases}.$$

It is clear that $\varphi \in \mathrm{Lip}_c(\mathbb{G})$, therefore it is also differentiable μ-a.e. with $|\nabla_H \varphi| \leq L$ for some constant $L > 0$, by Theorem 1.2.5.

We notice that since $\mathscr{S}^{Q-p'}(K) < \infty$, then $\mu(K) = 0$. This implies that for any $\varepsilon > 0$ there exists an open set $U \Subset \Omega$ such that $K \subset U$ and $\|F\|_{L^p(U)} < \varepsilon$, because $|F| \in L^p_{\mathrm{loc}}(\Omega)$. In addition, we can ask that such an U satisfies $|\mathrm{div}F|(U \setminus K) < \varepsilon$, because of the regularity of Radon measures.

It is clear that there exists $\delta > 0$ such that for any $0 < 2r < \delta$ and for any open ball $B(x, r)$ which intersects K we have $B(x, 2r) \subset U$. Then we can select a covering of K (which can be also taken finite by compactness) of such balls $\{B(x_j, r_j)\}_{j \in J}$ and so, by the definition of spherical measure, we have

$$\sum_{j \in J} r_j^{Q-p'} < \mathscr{S}^{Q-p'}(K) + 1,$$

for δ small enough.
We set $\varphi_j(x) := \varphi(\delta_{1/r_j}(x_j^{-1}x))$ and $\psi(x) := \sup\{\varphi_j(x) : j \in J\}$. It is easy to see that $0 \leq \psi \leq 1$, φ_j is supported in $B(x_j, 2r_j)$, $\psi \in \mathrm{Lip}_c(\Omega)$, $\mathrm{supp}(\psi) \subset U$ and $\psi \equiv 1$ on K. Then, by Remark 1.3.9, we have

$$\mathrm{div}F(K) = \int_K \psi \, d\mathrm{div}F = -\int_U \langle F, \nabla_H \psi \rangle \, dx - \int_{U \setminus K} \psi \, d\mathrm{div}F,$$

which implies

$$\mathrm{div}F(K) < \|F\|_{L^p(U)} \|\nabla_H \psi\|_{L^{p'}(U)} + \varepsilon < \varepsilon(\|\nabla_H \psi\|_{L^{p'}(U)} + 1).$$

Since ψ is the maximum of a finite family of functions, we have $\nabla_H \psi(x) = \nabla_H \varphi_j(x)$ for some $j \in J$ and μ-a.e. $x \in \Omega$. Indeed, Theorem 1.2.5 shows that Lipschitz functions are differentiable μ-a.e., moreover, $\psi(x) = \varphi_j(x)$ in the open set $\{\varphi_j > \varphi_i, \forall i \neq j\}$, while $\nabla_H \varphi_j(x) = \nabla_H \varphi_i(x)$ for μ-a.e. x on the set $\{\varphi_j = \varphi_i\}$. Then

$$\begin{aligned}\int_U |\nabla_H \psi|^{p'} \, dx &\leq \sum_{j \in J} \int_U |\nabla_H \varphi_j|^{p'} \, dx = \sum_{j \in J} \int_{B(x_j, 2r_j)} |\nabla_H \varphi_j|^{p'} \, dx \\ &\leq 2^Q \mu(B(0,1)) L^{p'} \sum_{j \in J} r_j^{Q-p'} \leq 2^Q L^{p'} \mu(B(0,1)) (\mathscr{S}^{Q-p'}(K) + 1).\end{aligned}$$

This implies

$$0 \leq \mathrm{div}F(K) \leq \varepsilon(1 + 2^{\frac{Q}{p'}} L \mu(B(0,1))^{\frac{1}{p'}} (\mathscr{S}^{Q-p'}(K) + 1)^{\frac{1}{p'}})$$

and, since ε is arbitrary, we conclude the proof.

In the case $p = \infty$, we proceed similarly by considering a Borel set B such that $\mathscr{S}^{Q-1}(B) = 0$ and a compact subset of $B_{\pm}$. For any $\varepsilon > 0$, there exists an open set U satisfying $K \subset U \Subset \Omega$ and $|\mathrm{div}F|(U \setminus K) < \varepsilon$, as before. Now, there exists a $\delta > 0$ small enough such that we can find a finite open covering $\{B(x_j, r_j)\}_{j \in J}$, $2r_j < \delta$, of K, which satisfies $\sum_{j \in J} r_j^{Q-1} < \varepsilon$, and $B(x_j, 2r_j) \subset U$ whenever $B(x_j, r_j) \cap K \neq \emptyset$.

It is clear that

$$\left| \int_U \langle F, \nabla_H \psi \rangle \, dx \right| \leq \|F\|_{L^\infty(U)} \|\nabla_H \psi\|_{L^1(U)}$$

and that

$$\begin{aligned}\int_U |\nabla_H \psi| \, dx &\leq \sum_{j \in J} \int_\Omega |\nabla_H \varphi_j| \, dx = \sum_{j \in J} \int_{B(x_j, 2r_j)} |\nabla_H \varphi_j| \, dx \\ &\leq 2^Q L \mu(B(0,1)) \sum_{j \in J} r_j^{Q-1} < 2^Q L \mu(B(0,1)) \varepsilon.\end{aligned}$$

Thus, we conclude that

$$\mathrm{div}F(K) = -\int_U \langle F, \nabla_H \psi \rangle \, dx - \int_{U \setminus K} \psi \, d\mathrm{div}F < \varepsilon(1 + 2^Q L \mu(B(0,1)) \|F\|_{L^\infty(U)}),$$

which implies $\mathrm{div}F(K) = 0$, since ε is arbitrary. □

We notice that there is a precise way to compare $\mathscr{S}^{Q-p'}$ and the Euclidean Hausdorff measure $\mathscr{H}_{|\cdot|}^{\mathrm{q}-p'}$ on a stratified group $\mathbb{G}$ of topological dimension q, as shown in [25]. In particular, [25, Proposition 3.1] implies that

$$\mathscr{S}^{Q-p'} \ll \mathscr{H}_{|\cdot|}^{\mathrm{q}-p'}.$$

This shows that Theorem 4.2.7 is coherent with the Euclidean case ([144, Theorem 3.2]), and that the divergence-measure horizontal fields have finer absolute continuity properties than the general ones.

Now we prove a first case of Leibniz rule between an essentially bounded divergence-measure horizontal field and a scalar Lipschitz function, whose gradient is in $L^1(H\Omega)$.

Proposition 4.2.8. *If $F \in \mathcal{DM}^\infty(H\Omega)$ and $g \in L^\infty(\Omega) \cap \mathrm{Lip}_{H,\mathrm{loc}}(\Omega)$, with $\nabla_H g \in L^1(H\Omega)$, then $gF \in \mathcal{DM}^\infty(H\Omega)$ and the following formula holds*

$$\mathrm{div}(gF) = g\mathrm{div}F + \langle F, \nabla_H g\rangle \mu. \tag{4.2.2}$$

Proof. It is clear that $gF \in L^\infty(H\Omega)$. For any $\phi \in C^1_c(\Omega)$ with $\|\phi\|_{L^\infty(\Omega)} \leq 1$, by Remarks 1.2.6 and 1.3.9, we have

$$\begin{aligned}\int_\Omega \langle gF, \nabla_H \phi\rangle \, dx &= \int_\Omega \langle F, \nabla_H(g\phi)\rangle \, dx - \int_\Omega \phi \langle F, \nabla_H g\rangle \, dx \\ &= -\int_\Omega \phi g \, d\mathrm{div}F - \int_\Omega \phi \langle F, \nabla_H g\rangle \, dx,\end{aligned}$$

which clearly implies that $gF \in \mathcal{DM}^\infty(H\Omega)$ and (4.2.2) holds. □

We have now all the tools to establish a general product rule for essentially bounded divergence-measure horizontal fields and BV_H functions, see Theorem 4.1.1. This is one of the main ingredients in the proof of the Gauss–Green formulas.

Proof of Theorem 4.1.1. We notice that (4.2.2) holds for every g_ε with $\varepsilon > 0$ and we also have $|g_\varepsilon(x)| \leq \|g\|_{L^\infty(\Omega)}$ for any $x \in \Omega$. The family $\{g_{\tilde{\varepsilon}_k}\}$ is then equibounded in $L^\infty(\Omega; |\mathrm{div}F|)$ and there exists $\tilde{g} \in L^\infty(\Omega; |\mathrm{div}F|)$ and a subsequence $\varepsilon_k \to 0$ such that $g_{\varepsilon_k} \overset{*}{\rightharpoonup} \tilde{g}$. It follows that

$$\int_\Omega \phi g_{\varepsilon_k} \, d\mathrm{div}F \to \int_\Omega \phi \tilde{g} \, d\mathrm{div}F$$

for any $\phi \in L^1(\Omega; |\mathrm{div}F|)$. In particular, the previous convergence holds for any $\phi \in C_c(\Omega)$, and so $g_{\varepsilon_k}\mathrm{div}F \rightharpoonup \tilde{g}\mathrm{div}F$ in $\mathcal{M}(\Omega)$.

Now we show that $\{\mathrm{div}(g_\varepsilon F)\}$ is uniformly bounded in $\mathcal{M}(\Omega)$: by (4.2.2), we obtain that

$$\begin{aligned}\left|\int_\Omega \phi \, d\mathrm{div}(g_\varepsilon F)\right| &\leq \left|\int_\Omega \phi g_\varepsilon \, d\mathrm{div}F\right| + \left|\int_\Omega \phi \langle F, \nabla_H(g_\varepsilon)\rangle \, dx\right| \\ &\leq \|\phi\|_{L^\infty(\Omega)}\|g\|_{L^\infty(\Omega)}|\mathrm{div}F|(\Omega) + \|F\|_{L^\infty(\Omega)} \int_\Omega |\phi||\nabla_H g_\varepsilon| \, dx.\end{aligned}$$

As a result, considering $\mathrm{supp}(\phi) \subset \Omega^{\mathcal{R}}_{2\varepsilon}$, by (1.3.20) we conclude that

$$|\mathrm{div}(g_\varepsilon F)|(\Omega^{\mathcal{R}}_{2\varepsilon}) \leq \|g\|_{L^\infty(\Omega)}|\mathrm{div}F|(\Omega) + \|F\|_{L^\infty(\Omega)}|D_H g|(\Omega). \tag{4.2.3}$$

We have shown that $\{\mathrm{div}(g_\varepsilon F)\}$ is uniformly bounded in $\mathcal{M}(\Omega')$ for any open set $\Omega' \Subset \Omega$, and so up to extracting a further subsequence that we relabel as ε_k, the sequence $\{\mathrm{div}(g_{\varepsilon_k} F)\}$ is a locally weakly* converging subsequence. However, it is clear that $\mathrm{div}(g_{\varepsilon_k} F)$ weakly* converges to $\mathrm{div}(gF)$ in the sense of distributions, and that $C^\infty_c(\Omega)$ is dense in $C_c(\Omega)$. Therefore, by uniqueness of weak* limits, we conclude that $\mathrm{div}(g_{\varepsilon_k} F) \rightharpoonup \mathrm{div}(gF)$ in $\mathcal{M}(\Omega)$.

Thus, $(F, \nabla_H g_{\varepsilon_k})$ is weakly* convergent, being the difference of two weakly* converging sequences, and taking into account (4.2.2) we get

$$(F, \nabla_H g_{\varepsilon_k}) \rightharpoonup (F, D_H g) := \mathrm{div}(gF) - \tilde{g}\mathrm{div}F. \tag{4.2.4}$$

In relation to $(F, D_H g)$, we first argue as in Lemma 1.3.6. For any $\phi \in C_c(\Omega)$, we have

$$\begin{aligned}\left|\int_\Omega \phi \, d(F, D_H g)\right| &= \lim_{\varepsilon_k \to 0}\left|\int_\Omega \phi \langle F, \nabla_H g_{\varepsilon_k}\rangle \, dx\right| \leq \|F\|_{L^\infty(\Omega)} \limsup_{\varepsilon_k \to 0} \int_\Omega |\phi||\nabla_H g_{\varepsilon_k}| \, dx \\ &\leq \|F\|_{L^\infty(\Omega)} \limsup_{\varepsilon_k \to 0} \int_\Omega |\phi|(\rho_{\varepsilon_k} * |D_H g|) \, dx \\ &= \|F\|_{L^\infty(\Omega)} \lim_{\varepsilon_k \to 0} \int_\Omega (\rho_{\varepsilon_k} * |\phi|) \, d|D_H g| = \|F\|_{L^\infty(\Omega)} \int_\Omega |\phi| \, d|D_H g|,\end{aligned}$$

where the second inequality follows by (1.3.19), since $\mathrm{supp}(\phi) \subset \Omega^{\mathcal{R}}_{2\varepsilon_k}$ for ε_k small enough. The subsequent equality is a consequence of (1.2.18), therefore proving (4.1.2). The decomposition (1.3.8) in our case yields

$$D_H g = \nabla_H g\, \mu + D_H^{\mathrm{s}} g,$$

where $\nabla_H g$ is also characterized as the approximate differential of g, [17, Theorem 2.2]. We aim to show that

$$(F, D_H g)^{\mathrm{a}} \mu = \langle F, \nabla_H g \rangle\, \mu \quad \text{and} \quad (F, D_H g)^{\mathrm{s}} = (F, D_H^{\mathrm{s}} g),$$

for some measure $(F, D_H^{\mathrm{s}} g) \in \mathcal{M}(\Omega)$ that is absolutely continuos with respect to $|D_H^{\mathrm{s}} g|$. Indeed, by (1.3.14) we get $\nabla_H g_\varepsilon = \rho_\varepsilon * D_H g$ on Ω' for every fixed open set $\Omega' \Subset \Omega$ and $\varepsilon > 0$ sufficiently small. On this open set we have

$$\langle F, \nabla_H g_\varepsilon \rangle = \langle F, \rho_\varepsilon * \nabla_H g \rangle + \langle F, \rho_\varepsilon * D_H^{\mathrm{s}} g \rangle\,. \tag{4.2.5}$$

By Lemma 1.3.6 the measures $\langle F, \rho_\varepsilon * D_H^{\mathrm{s}} g \rangle\, \mu$ are uniformly bounded, so that possibly selecting a subsequence of ε_k, denoted by the same symbol, there exists $(F, D_H^{\mathrm{s}} g) \in \mathcal{M}(\Omega)$ such that

$$\langle F, \rho_{\varepsilon_k} * D_H^{\mathrm{s}} g \rangle\, \mu \rightharpoonup (F, D_H^{\mathrm{s}} g)$$

and applying again Lemma 1.3.6 we get

$$|(F, D_H^{\mathrm{s}} g)| \le \|F\|_{L^\infty(\Omega)}\, |D_H^{\mathrm{s}} g|. \tag{4.2.6}$$

Since $\nabla_H g \in L^1(H\Omega)$, we clearly have $\rho_\varepsilon * \nabla_H g \to \nabla_H g$ in $L^1(H\Omega)$, which yields the following weak* convergence

$$\langle F, \rho_\varepsilon * \nabla_H g \rangle\, \mu \rightharpoonup \langle F, \nabla_H g \rangle\, \mu$$

in $\mathcal{M}(\Omega)$. Since (4.2.5) holds on every relatively compact open subset of Ω, we get

$$(F, D_H g) = \langle F, \nabla_H g \rangle\, \mu + (F, D_H^{\mathrm{s}} g). \tag{4.2.7}$$

Due to (4.2.6) the previous sum is made by mutually singular measures, then showing that

$$(F, D_H g)^{\mathrm{a}} \mu = \langle F, \nabla_H g \rangle\, \mu \quad \text{and} \quad (F, D_H g)^{\mathrm{s}} = (F, D_H^{\mathrm{s}} g),$$

hence (4.1.3) holds. □

Proposition 4.2.9. *Let $F \in \mathcal{DM}^\infty(H\Omega)$ and let $g \in L^\infty(\Omega)$ with $|D_H g|(\Omega) < +\infty$. If we define $g_\varepsilon := \rho_\varepsilon * g$ using the mollifier ρ of Proposition 1.3.20, then any weak* limit point $\tilde{g} \in L^\infty(\Omega; |\mathrm{div} F|)$ of some subsequence g_{ε_k} satisfies the property*

$$\tilde{g}(x) = g^{*,\mathcal{R}}(x) \quad \textit{for} \quad |\mathrm{div} F|\textit{-a.e. } x \in C_g^{\mathcal{R}}.$$

In addition, if $g \in L^\infty(\Omega) \cap C(\Omega)$ and $\nabla_H g \in L^1(H\Omega)$, then $\tilde{g}(x) = g(x)$ for $|\mathrm{div} F|$-a.e. $x \in \Omega$ and

$$\mathrm{div}(gF) = g\, \mathrm{div} F + \langle F, \nabla_H g \rangle\, \mu. \tag{4.2.8}$$

Proof. Let $g_{\varepsilon_k} \overset{*}{\rightharpoonup} \tilde{g}$ in $L^\infty(\Omega; |\mathrm{div} F|)$. By Proposition 1.3.20, we know that $g_{\varepsilon_k}(x) \to g^{*,\mathcal{R}}(x)$ for any $x \in C_g^{\mathcal{R}}$. If we choose as test function $\phi = \chi_{C_g^{\mathcal{R}}} \psi$, for some $\psi \in L^1(\Omega; |\mathrm{div} F|)$, we have

$$\int_\Omega \phi g_{\varepsilon_k}\, d\mathrm{div} F = \int_{C_g^{\mathcal{R}}} \psi g_{\varepsilon_k}\, d\mathrm{div} F \to \int_{C_g^{\mathcal{R}}} \psi g^{*,\mathcal{R}}\, d\mathrm{div} F$$

by Lebesgue's theorem with respect to the measure $|\mathrm{div} F|$. Since ψ is arbitrary, this implies $\tilde{g}(x) = g^{*,\mathcal{R}}(x)$ for $|\mathrm{div} F|$-a.e. $x \in C_g^{\mathcal{R}}$. Let now $g \in L^\infty(\Omega) \cap C(\Omega)$ with $\nabla_H g \in L^1(H\Omega)$. It is clear that $\tilde{g}(x) = g(x)$ for $|\mathrm{div} F|$-a.e. $x \in \Omega$, since $g^{*,\mathcal{R}}(x) = g(x)$ for any $x \in \Omega$, being g continuous. In addition, since $D_H g$ has no singular part, (4.1.3) implies immediately (4.2.8). □

Remark 4.2.10. We stress that in Theorem 4.1.1 the pairing term $(F, D_H g)$ depends on the particular sequence g_{ε_k}, and therefore on $\tilde{g}$. In order to obtain uniqueness, one should be able to show that there exists only one accumulation point $\tilde{g}$ of g_ε. For instance, this happens in the Euclidean case $\mathbb{G} = \mathbb{R}^n$, in which $\tilde{g} = g^*(= g^{*,\mathcal{R}})$ $\mathscr{H}^{n-1}$-a.e. However, it is possible to impose some more conditions on the measures $\mathrm{div}F$ and $|D_H g|$ under which $\tilde{g}$ and $(F, D_H g)$ are uniquely determined.

Corollary 4.2.11. *Let $F \in \mathcal{DM}^\infty(H\Omega)$ and let $g \in L^\infty(\Omega)$ with $|D_H g|(\Omega) < +\infty$. Let $\mathrm{div}^{\mathrm{s}}F$ and $D_H^{\mathrm{s}} g$ be the singular parts of the measures $\mathrm{div}F$ and $D_H g$. If we also assume that $|\mathrm{div}^{\mathrm{s}}F|$ and $|D_H^{\mathrm{s}} g|$ are mutually singular measures, then we have*

$$\mathrm{div}(gF) = \Big(g\, \mathrm{div}^{\mathrm{a}}F + \langle F, \nabla_H g\rangle \Big)\mu + \tilde{g}\, \mathrm{div}^{\mathrm{s}}F + (F, D_H^{\mathrm{s}} g), \tag{4.2.9}$$

where $\tilde{g} \in L^\infty(\Omega; |\mathrm{div}F|)$ and $(F, D_H^{\mathrm{s}} g) \in \mathcal{M}(\Omega)$ are defined as in Theorem 4.1.1 and the singular measures $\tilde{g}\, \mathrm{div}^{\mathrm{s}}F$ and $(F, D_H^{\mathrm{s}} g)$ are uniquely determined by g and F. In particular, if $|\mathrm{div}F| \ll \mu$, we have

$$\mathrm{div}(gF) = \Big(g\, \mathrm{div}^{\mathrm{a}}F + \langle F, \nabla_H g\rangle \Big)\mu + (F, D_H^{\mathrm{s}} g). \tag{4.2.10}$$

Proof. It is well known that we can decompose the measures $\mathrm{div}F$ and $(F, D_H g)$ in their absolutely continuous and singular parts. By Theorem 4.1.1, we know that

$$\mathrm{div}(gF) = \Big(\tilde{g}\, \mathrm{div}^{\mathrm{a}}F + \langle F, \nabla_H g\rangle \Big)\mu + \tilde{g}\, \mathrm{div}^{\mathrm{s}}F + (F, D_H^{\mathrm{s}} g).$$

Since g_ε converges to g in $L^1_{\mathrm{loc}}(\Omega)$ for any mollification of g, we clearly obtain $\tilde{g}\, \mathrm{div}^{\mathrm{a}}F\mu = g\, \mathrm{div}^{\mathrm{a}}F\mu$ in the sense of Radon measures. It follows that (4.2.9) holds and clearly

$$\tilde{g}\, \mathrm{div}^{\mathrm{s}}F + (F, D_H^{\mathrm{s}} g) = \mathrm{div}(gF) - \Big(g\, \mathrm{div}^{\mathrm{a}}F + \langle F, \nabla_H g\rangle \Big)\mu.$$

We have shown that the singular measures on the left hand side are uniquely determined by g and F, since the right hand side is uniquely determined and the two measures are also mutually singular. Indeed we have $|(F, D_H^{\mathrm{s}} g)| \le \|F\|_{L^\infty(\Omega)} |D_H^{\mathrm{s}} g|$ by Theorem 4.1.1. To conclude the proof, we observe that the condition $|\mathrm{div}F| \ll \mu$ clearly gives $\mathrm{div}^{\mathrm{s}}F = 0$, so (4.2.10) immediately follows. $\square$

Remark 4.2.12. It is clear that one can obtain (4.2.10) if we have $F \in L^\infty(H\Omega)$ with $\mathrm{div}F \in L^1(\Omega)$ and $g \in L^\infty(\Omega) \cap BV_H(\Omega)$.

Remark 4.2.13. Under no additional assumption on $F \in \mathcal{DM}^\infty(H\Omega)$ and $g \in L^\infty(\Omega)$ with $|D_H g|(\Omega) < +\infty$, we can always decompose the term $\tilde{g}\mathrm{div}F$. Indeed, we have

$$\tilde{g}\, \mathrm{div}F = g\, \mathrm{div}^{\mathrm{a}}F\mu + g^{*,\mathcal{R}}\, \mathrm{div}^{\mathrm{s}}F \llcorner C_g^{\mathcal{R}} + \tilde{g}\, \mathrm{div}^{\mathrm{s}}F \llcorner (\Omega \setminus C_g^{\mathcal{R}}). \tag{4.2.11}$$

Then, it follows that $\tilde{g}\mathrm{div}F$ is uniquely determined by $\mathrm{div}F$ and g if $|\mathrm{div}^{\mathrm{s}}F|(\Omega \setminus C_g^{\mathcal{R}}) = 0$.

4.3 Interior and exterior normal traces

In this section we introduce interior and exterior normal traces for a divergence-measure field. The absence of sufficient regularity for the reduced boundary (Definition 1.3.14) does not guarantee their uniqueness a priori. However, the next section will present different conditions that lead to a unique normal trace and a corresponding Gauss–Green theorem.

Let $F \in \mathcal{DM}^\infty(H\Omega)$ and $E \subset \Omega$ be a set of finite h-perimeter. Let $\rho \in C_c(B(0,1))$ be a nonnegative mollifier satisfying $\rho(x) = \rho(x^{-1})$ and $\int \rho \, dx = 1$, and ε_k be a suitable vanishing sequence such that

$$\begin{aligned} \langle \chi_E F, \rho_{\varepsilon_k} * D_H \chi_E \rangle \, \mu &\rightharpoonup (\chi_E F, D_H \chi_E) \\ \left\langle \chi_{\Omega \setminus E} F, \rho_{\varepsilon_k} * D_H \chi_E \right\rangle \mu &\rightharpoonup (\chi_{\Omega \setminus E} F, D_H \chi_E) \end{aligned} \quad \text{in } \mathcal{M}(\Omega). \tag{4.3.1}$$

The existence of such converging subsequences follows from Lemma 1.3.6, which implies also the estimates

$$\begin{aligned} |(\chi_E F, D_H \chi_E)| &\le \|F\|_{L^\infty(E)} \, |D_H \chi_E| \quad \text{and} \\ |(\chi_{\Omega \setminus E} F, D_H \chi_E)| &\le \|F\|_{L^\infty(\Omega \setminus E)} \, |D_H \chi_E|. \end{aligned} \tag{4.3.2}$$

It is worth to mention that another definition equivalent to (4.3.1) is possible. Employing formula (1.3.14), we obtain

$$\begin{aligned} \langle \chi_E F, \rho_{\varepsilon_k} * D_H \chi_E \rangle &= \langle \chi_E F, \nabla_H (\rho_{\varepsilon_k} * \chi_E) \rangle \\ \left\langle \chi_{\Omega \setminus E} F, \rho_{\varepsilon_k} * D_H \chi_E \right\rangle &= \left\langle \chi_{\Omega \setminus E} F, \nabla_H (\rho_{\varepsilon_k} * \chi_E) \right\rangle \end{aligned} \quad \text{in } \Omega^{\mathcal{R}}_{2\varepsilon_k}. \tag{4.3.3}$$

We point out that the measures at the right hand side in (4.3.3) are not defined on the whole Ω, while this is true for those at the left hand side. However, arguing as in Remark 1.2.11, we can see that the weak* convergence (4.3.1) is equivalent to the weak* convergence

$$\begin{aligned} \langle \chi_E F, \nabla_H (\rho_{\varepsilon_k} * \chi_E) \rangle \, \mu &\rightharpoonup (\chi_E F, D_H \chi_E), \\ \left\langle \chi_{\Omega \setminus E} F, \nabla_H (\rho_{\varepsilon_k} * \chi_E) \right\rangle \mu &\rightharpoonup (\chi_{\Omega \setminus E} F, D_H \chi_E). \end{aligned} \tag{4.3.4}$$

It is important to stress that at the moment the "pairing measures" $(\chi_E F, D_H \chi_E)$ and $(\chi_{\Omega \setminus E} F, D_H \chi_E)$ may depend on the choice of the sequence ε_k and also on the mollifier ρ.

We are now in the position to define the *interior and exterior normal traces* of F on the boundary of E as the functions $\langle \mathcal{F}_i, \nu_E \rangle, \langle \mathcal{F}_e, \nu_E \rangle \in L^\infty(\Omega; |D_H \chi_E|)$ satisfying

$$2(\chi_E F, D_H \chi_E) = \langle \mathcal{F}_i, \nu_E \rangle \, |D_H \chi_E|, \tag{4.3.5}$$

$$2(\chi_{\Omega \setminus E} F, D_H \chi_E) = \langle \mathcal{F}_e, \nu_E \rangle \, |D_H \chi_E|. \tag{4.3.6}$$

Since $\rho_{\varepsilon_k} * \chi_E$ is uniformly bounded, up to extracting a subsequence, we may also assume that

$$\rho_{\varepsilon_k} * \chi_E \overset{*}{\rightharpoonup} \widetilde{\chi_E} \quad \text{in} \quad L^\infty(\Omega; |\mathrm{div} F|). \tag{4.3.7}$$

This allows us to define the sets

$$\widetilde{E}^1 := \{x \in \Omega : \widetilde{\chi_E}(x) = 1\} \quad \text{and} \quad \widetilde{E}^0 := \{x \in \Omega : \widetilde{\chi_E}(x) = 0\} \tag{4.3.8}$$

to be the *measure theoretic interior* and the *measure theoretic exterior of* E, respectively, *with respect to* F *and* $\widetilde{\chi_E}$. We may also define an associated reduced boundary

$$\widetilde{\mathscr{F}_H E} = \mathscr{F}_H E \setminus \left(\widetilde{E}^1 \cup \widetilde{E}^0 \right). \tag{4.3.9}$$

We wish to underline again the fact that these notions heavily depend on $\widetilde{\chi_E}$, which is not unique, a priori, since it depends on the choice of the sequence $\rho_{\varepsilon_k} * \chi_E$.

In the sequel, we will refer to the above sequence ε_k, or possible subsequences, such that (4.3.1) and (4.3.7) hold. Notice that despite this dependence we will provide conditions under which the limit measures of (4.3.1) and the sets of (4.3.8) and (4.3.9) prove to have an intrinsic geometric meaning.

Remark 4.3.1. By (4.3.1), observing that $\rho_{\varepsilon_k} * D_H\chi_E = -\rho_{\varepsilon_k} * D_H\chi_{\Omega\setminus E}$, we also get the following equalities

$$(\chi_E F, D_H\chi_E) = -(\chi_E F, D_H\chi_{\Omega\setminus E}),$$
$$(\chi_{\Omega\setminus E} F, D_H\chi_E) = -(\chi_{\Omega\setminus E} F, D_H\chi_{\Omega\setminus E}).$$

We conclude that the normal traces of F on E and $\Omega\setminus E$ satisfy the following relations

$$\langle \mathcal{F}_i, \nu_E\rangle = -\left\langle \mathcal{F}_e, \nu_{\Omega\setminus E}\right\rangle \quad \text{and} \quad \langle \mathcal{F}_e, \nu_E\rangle = -\left\langle \mathcal{F}_i, \nu_{\Omega\setminus E}\right\rangle.$$

We employ now the Leibniz rule (Theorem 4.1.1) and (4.3.1) to achieve the following result, which is a key step in order to prove the Gauss–Green formulas.

Proposition 4.3.2. *Let $F \in \mathcal{DM}^\infty(H\Omega)$ and $E \subset \Omega$ be a set of finite h-perimeter, then the following formulas hold*

$$\operatorname{div}(\chi_E F) = \widetilde{\chi_E}\operatorname{div}F + (F, D_H\chi_E), \tag{4.3.10}$$
$$\operatorname{div}(\chi_E F) = (\widetilde{\chi_E})^2\operatorname{div}F + \frac{1}{2}(F, D_H\chi_E) + (\chi_E F, D_H\chi_E), \tag{4.3.11}$$
$$\widetilde{\chi_E}(1-\widetilde{\chi_E})\operatorname{div}F = \frac{1}{2}(\chi_E F, D_H\chi_E) - \frac{1}{2}(\chi_{\Omega\setminus E}F, D_H\chi_E) \tag{4.3.12}$$

in the sense of Radon measures on Ω, where $\widetilde{\chi_E} \in L^\infty(\Omega; |\operatorname{div}F|)$ is defined in (4.3.7).

Proof. By Theorem 4.1.1 applied to F and $g = \chi_E$, up to extracting a subsequence, we can assume that the choice of ε_k leads to (4.3.10) in analogy with Theorem 4.1.1. We observe that

$$\operatorname{div}((\rho_{\varepsilon_k} * \chi_E)\chi_E F) \rightharpoonup \operatorname{div}(\chi_E^2 F) = \operatorname{div}(\chi_E F),$$

as measures, since $\chi_E^2 = \chi_E$. By (4.2.2), (4.3.10) and (1.3.14), we get the following identities of measures on $\Omega^{\mathcal{R}}_{2\varepsilon_k}$:

$$\operatorname{div}(F\chi_E(\rho_{\varepsilon_k} * \chi_E)) = (\rho_{\varepsilon_k} * \chi_E)\operatorname{div}(\chi_E F) + \langle \chi_E F, \nabla_H(\rho_{\varepsilon_k} * \chi_E)\rangle\,\mu \tag{4.3.13}$$
$$= (\rho_{\varepsilon_k} * \chi_E)\widetilde{\chi_E}\operatorname{div}F + (\rho_{\varepsilon_k} * \chi_E)(F, D_H\chi_E) \tag{4.3.14}$$
$$+ \langle \chi_E F, \rho_{\varepsilon_k} * D_H\chi_E\rangle\,\mu.$$

Recall that our subsequence ε_k is chosen such that both (4.3.7) and (4.3.1) hold. In view of (2.4.2), we have $(\rho_{\varepsilon_k} * \chi_E) \overset{*}{\rightharpoonup} \frac{1}{2} \in L^\infty(\Omega; |D_H\chi_E|)$. By (4.1.2) we get

$$|(F, D_H\chi_E)| \le \|F\|_{L^\infty(\Omega)}|D_H\chi_E|$$

and we observe that the definition of $(F, D_H\chi_E)$ from Theorem 4.1.1 fits with the definitions (4.3.1), thanks to (4.3.4), getting the obvious identity

$$(F, D_H\chi_E) = (\chi_E F, D_H\chi_E) + (\chi_{\Omega\setminus E}F, D_H\chi_E). \tag{4.3.15}$$

Remark 2.4.4 shows that

$$(\rho_{\varepsilon_k} * \chi_E)(F, D_H\chi_E) \rightharpoonup \frac{1}{2}(F, D_H\chi_E).$$

All in all, by passing to the weak* limits in (4.3.13), we get (4.3.11). Subtracting (4.3.11) from (4.3.10) we have

$$\widetilde{\chi_E}(1-\widetilde{\chi_E})\operatorname{div}F = (\chi_E F, D_H\chi_E) - \frac{1}{2}(F, D_H\chi_E). \tag{4.3.16}$$

From (4.3.16) and (4.3.15) we get (4.3.12). $\square$

Remark 4.3.3. In the assumptions of Proposition 4.3.2, joining (4.3.12), (4.3.5) and (4.3.6), we get the following equality

$$\widetilde{\chi_E}(1-\widetilde{\chi_E})\mathrm{div}F = \frac{\langle \mathcal{F}_i, \nu_E\rangle - \langle \mathcal{F}_e, \nu_E\rangle}{4}|D_H\chi_E|. \tag{4.3.17}$$

We now prove sharp estimates on the L^∞-norm of the normal traces. Let us point out that such estimates could not be obtained directly from (4.3.2), employing (4.3.5) and (4.3.6). A more refined argument is necessary, involving the differentiation with respect to the h-perimeter measure.

Proposition 4.3.4. *If $F \in \mathcal{DM}^\infty(H\Omega)$ and $E \subset \Omega$ is a set of finite h-perimeter, then*

$$\| \langle \mathcal{F}_i, \nu_E\rangle \|_{L^\infty(\mathscr{F}_H E; |D_H\chi_E|)} \le \|F\|_{L^\infty(E)}, \tag{4.3.18}$$

$$\| \langle \mathcal{F}_e, \nu_E\rangle \|_{L^\infty(\mathscr{F}_H E; |D_H\chi_E|)} \le \|F\|_{L^\infty(\Omega\setminus E)}, \tag{4.3.19}$$

where the interior and exterior normal traces of F are defined in (4.3.5) *and* (4.3.6).

Proof. By Theorem 1.3.16 the perimeter measure $|D_H\chi_E|(\cdot)$ is a.e. asymptotically doubling. Therefore the following differentiation property holds (see [72, Sections 2.8.17 and 2.9.6]): for $D_H\chi_E$-a.e. $x \in \mathscr{F}_H E$ one has

$$\langle \mathcal{F}_i, \nu_E\rangle (x) = \lim_{r\to 0} \frac{2(\chi_E F, D_H\chi_E)(B(x,r))}{|D_H\chi_E|(B(x,r)}.$$

Let ε_k be the sequence defining (4.3.1) and (4.3.7). By (1.3.10), we obtain that the sequence $| \langle \chi_E F, \rho_{\varepsilon_k} * D_H\chi_E\rangle |\mu$ is uniformly bounded in $\mathcal{M}(\Omega)$. Thus, there exists a weak* converging subsequence, which we do not relabel. Let the positive measure $\lambda_i \in \mathcal{M}(\Omega)$ be its limit.

In an analogous way, one can prove that the measures $| \left\langle \chi_{\Omega\setminus E} F, \rho_{\varepsilon_k} * D_H\chi_E\right\rangle |\mu$ are uniformly bounded in $\mathcal{M}(\Omega)$. So there exists a weak* converging subsequence, which we do not relabel again, and whose limit is the positive Radon measure $\lambda_e \in \mathcal{M}(\Omega)$. We also observe that the sequences $\chi_E|\rho_{\varepsilon_k} * D_H\chi_E|\mu$ and $\chi_{\Omega\setminus E}|\rho_{\varepsilon_k} * D_H\chi_E|\mu$ are bounded in $\mathcal{M}(\Omega)$ and that, if $\gamma \in \mathcal{M}(\Omega)$ is any of their weak* limit points, then $\gamma \le |D_H\chi_E|$, due to (1.3.18).

We can choose a sequence of balls $B(x, r_j)$ with $r_j \to 0$ in such a way that

$$|D_H\chi_E|(\partial B(x,r_j)) = \lambda_i(\partial B(x,r_j)) = \lambda_e(\partial B(x,r_j)) = 0$$

for all j. As a result, taking into account [11, Proposition 1.62] and (4.3.1), we have

$$\begin{aligned}
\left|\frac{2(\chi_E F, D_H\chi_E)(B(x,r_j))}{|D_H\chi_E|(B(x,r_j))}\right| &= \left|\frac{\lim\limits_{\varepsilon_k\to 0} 2\displaystyle\int_{B(x,r_j)} \langle \chi_E F, \rho_{\varepsilon_k} * D_H\chi_E\rangle\, dy}{\lim\limits_{\varepsilon_k\to 0}\displaystyle\int_{B(x,r_j)} |\rho_{\varepsilon_k} * D_H\chi_E|\, dy}\right| \\
&\le 2\|F\|_{L^\infty(E)} \frac{\lim\limits_{\varepsilon_k\to 0}\displaystyle\int_{B(x,r_j)} \chi_E|\rho_{\varepsilon_k} * D_H\chi_E|\, dy}{\lim\limits_{\varepsilon_k\to 0}\displaystyle\int_{B(x,r_j)} |\rho_{\varepsilon_k} * D_H\chi_E|\, dy}.
\end{aligned}$$

The last term can be also written as

$$2\|F\|_{L^\infty(E)}\left(1 - \frac{\lim\limits_{\varepsilon_k\to 0}\displaystyle\int_{B(x,r_j)} \chi_{\Omega\setminus E}|\rho_{\varepsilon_k} * D_H\chi_E|\, dy}{\lim\limits_{\varepsilon_k\to 0}\displaystyle\int_{B(x,r_j)} |\rho_{\varepsilon_k} * D_H\chi_E|\, dy}\right).$$

It follows that

$$\left|\frac{2(\chi_E F, D_H\chi_E)(B(x,r_j))}{|D_H\chi_E|(B(x,r_j))}\right| \le 2\|F\|_{L^\infty(E)}\left(1-\frac{\lim\limits_{\varepsilon_k\to 0}\left|\int_{B(x,r_j)}\chi_{\Omega\setminus E}(\rho_{\varepsilon_k}*D_H\chi_E)\,dy\right|}{\lim\limits_{\varepsilon_k\to 0}\int_{B(x,r_j)}|\rho_{\varepsilon_k}*D_H\chi_E)|\,dy}\right)$$

$$= 2\|F\|_{L^\infty(E)}\left(1-\frac{1}{2}\frac{|D_H\chi_E(B(x,r_j))|}{|D_H\chi_E|(B(x,r_j))}\right).$$

by (2.4.8) and the second limit of (1.3.18). Taking the limit as $j\to\infty$, the definition of reduced boundary immediately yields

$$|\langle\mathcal{F}_i,\nu_E\rangle(x)| = \lim_{k\to\infty}\left|\frac{2(\chi_E F, D_H\chi_E)(B(x,r_j))}{|D_H\chi_E|(B(x,r_j))}\right| \le \|F\|_{L^\infty(E)}.$$

The estimate for the exterior normal trace $\langle\mathcal{F}_e,\nu_E\rangle$ can be obtained in a similar way, hence the proof is complete. $\square$

4.3.1 Locality of normal traces

In this section we show the locality of normal traces, along with their relation with the orientation of the reduced boundary. First, we need to recall some known facts on the locality properties of perimeter in stratified groups. By Theorem 1.3.16 (see also [1, Theorem 4.2]) and Lemma 1.3.17, for any set E of finite h-perimeter in Ω, there exists a Borel function θ_E, such that $\theta_E\ge\alpha>0$ and

$$|D_H\chi_E|(B) = \int_{B\cap\mathscr{F}_H E}\theta_E\,d\mathscr{S}^{Q-1}, \tag{4.3.20}$$

which implies $\theta\in L^1(\Omega;\mathscr{S}^{Q-1}\llcorner\mathscr{F}_H E)$. By this representation, a property holds $|D_H\chi_E|$-a.e. if and only if it holds $\mathscr{S}^{Q-1}$-a.e. on $\mathscr{F}_H E$, see also Remark 1.3.18.

Given two sets E_1,E_2 of finite h-perimeter such that $\mathscr{S}^{Q-1}(\mathscr{F}_H E_1\cap\mathscr{F}_H E_2)>0$, by [20, Theorem 2.9], for any Borel set $B\subset\mathscr{F}_H E_1\cap\mathscr{F}_H E_2$ we have

$$|D_H\chi_{E_1}|(B) = |D_H\chi_{E_2}|(B).$$

Hence, (4.3.20) implies that

$$\theta_{E_1}(x)=\theta_{E_2}(x)\quad\text{for}\quad\mathscr{S}^{Q-1}\text{-a.e. }\ x\in\mathscr{F}_H E_1\cap\mathscr{F}_H E_2. \tag{4.3.21}$$

Moreover, [20, Corollary 2.6] implies that, for $\mathscr{S}^{Q-1}$-a.e. $x\in\mathscr{F}_H E_1\cap\mathscr{F}_H E_2$, we have $\nu_{E_1}(x)=\pm\nu_{E_2}(x)$.

Lemma 4.3.5. *If E_1 and E_2 have finite h-perimeter in Ω with $\mathscr{S}^{Q-1}(\mathscr{F}_H E_1\cap\mathscr{F}_H E_2)>0$, then we have*

$$|D_H(\chi_{E_1}-\chi_{E_2})|(B(x,r)) = o(|D_H\chi_{E_j}|(B(x,r))) \tag{4.3.22}$$

for $\mathscr{S}^{Q-1}$-a.e. $x\in\mathscr{F}_H E_1\cap\mathscr{F}_H E_2$ such that $\nu_{E_1}(x)=\nu_{E_2}(x)$, and for $j=1,2$. Analogously, we have

$$|D_H(\chi_{E_1}+\chi_{E_2})|(B(x,r)) = o(|D_H\chi_{E_j}|(B(x,r))) \tag{4.3.23}$$

for $\mathscr{S}^{Q-1}$-a.e. $x\in\mathscr{F}_H E_1\cap\mathscr{F}_H E_2$ such that $\nu_{E_1}(x)=-\nu_{E_2}(x)$, and for $j=1,2$. In addition, we have

$$|D_H\chi_{E_1}|(B(x,r)) \sim |D_H\chi_{E_2}|(B(x,r)), \tag{4.3.24}$$

for $\mathscr{S}^{Q-1}$-a.e. $x\in\mathscr{F}_H E_1\cap\mathscr{F}_H E_2$ and $j=1,2$.

Proof. We first define the following sets

$$L := \mathscr{F}_H E_1 \cap \mathscr{F}_H E_2 \quad \text{and} \quad G := \mathscr{F}_H E_1 \Delta \mathscr{F}_H E_2.$$

Then, by (4.3.20) and (4.3.21), we obtain

$$|D_H \chi_{E_1}| \llcorner L = |D_H \chi_{E_2}| \llcorner L, \tag{4.3.25}$$

$$||D_H \chi_{E_1}| - |D_H \chi_{E_2}|| = \theta \mathscr{S}^{Q-1} \llcorner G, \tag{4.3.26}$$

where $\theta = \theta_{E_j}(x)$ for $\mathscr{S}^{Q-1}$-a.e. $x \in \mathscr{F}_H E_j$ and for $j = 1, 2$. Hence, since $L \cap G = \emptyset$, for $\mathscr{S}^{Q-1}$-a.e. $x \in L$, we have

$$||D_H \chi_{E_1}| - |D_H \chi_{E_2}|| \, (B(x,r)) = \int_{G \cap B(x,r)} \theta \, d\mathscr{S}^{Q-1} = o(r^{Q-1}), \tag{4.3.27}$$

by (4.3.26) and standard differentiation of Borel measures. In addition,

$$|D_H \chi_{E_j}|(B(x,r)) \geq c r^{Q-1} \tag{4.3.28}$$

for $\mathscr{S}^{Q-1}$-a.e. $x \in \mathscr{F}_H E_j$, $r > 0$ sufficiently small and $j = 1, 2$, by [1, Theorem 4.3]. Then, (4.3.27) and the triangle inequality imply that, for $j = 1, 2$,

$$||D_H \chi_{E_1}|(B(x,r)) - |D_H \chi_{E_2}|(B(x,r))| = o(|D_H \chi_{E_j}|(B(x,r))),$$

from which we get (4.3.24). Then, we notice that, for $\mathscr{S}^{Q-1}$-a.e. $x \in L$ such that $\nu_{E_1}(x) = \nu_{E_2}(x)$, and $j = 1, 2$, we have

$$\begin{aligned} |D_H(\chi_{E_1} - \chi_{E_2})|(B(x,r)) &= \Big| (\nu_{E_1} - \nu_{E_2}) |D_H \chi_{E_1}| \llcorner L + \nu_{E_1} |D_H \chi_{E_1}| \llcorner G + \\ &\quad - \nu_{E_2} |D_H \chi_{E_2}| \llcorner G \Big| (B(x,r)) \\ &\leq \int_{B(x,r)} |\nu_{E_1} - \nu_{E_2}| \, d|D_H \chi_{E_1}| \llcorner L + \\ &\quad + |D_H \chi_{E_1}|(G \cap B(x,r)) + |D_H \chi_{E_2}|(G \cap B(x,r)) \\ &\leq \int_{B(x,r)} |\nu_{E_1} - \nu_{E_1}(x)| \, d|D_H \chi_{E_1}| + \\ &\quad + \int_{B(x,r)} |\nu_{E_2} - \nu_{E_2}(x)| \, d|D_H \chi_{E_2}| + o(r^{Q-1}) \\ &= o(|D_H \chi_{E_j}|(B(x,r))), \end{aligned}$$

by (4.3.25), (4.3.24), (4.3.28), the triangle inequality and standard differentiation of Borel measures. Thus, we can conclude that (4.3.22) holds. Analogously, (4.3.23) follows for $\mathscr{S}^{Q-1}$-a.e. $x \in \mathscr{F}_H E_1 \cap \mathscr{F}_H E_2$ such that $\nu_{E_1}(x) = -\nu_{E_2}(x)$. □

Theorem 4.3.6. *Let $F \in \mathcal{DM}^\infty(H\Omega)$, and $E_1, E_2 \subset \Omega$ be sets of finite h-perimeter such that $\mathscr{S}^{Q-1}(\mathscr{F}_H E_1 \cap \mathscr{F}_H E_2) > 0$. Then, we have*

$$\langle \mathcal{F}_i, \nu_{E_1} \rangle (x) = \langle \mathcal{F}_i, \nu_{E_2} \rangle (x) \quad \textit{and} \quad \langle \mathcal{F}_e, \nu_{E_1} \rangle (x) = \langle \mathcal{F}_e, \nu_{E_2} \rangle (x), \tag{4.3.29}$$

for $\mathscr{S}^{Q-1}$-a.e. $x \in \{y \in \mathscr{F}_H E_1 \cap \mathscr{F}_H E_2 : \nu_{E_1}(y) = \nu_{E_2}(y)\}$, and

$$\langle \mathcal{F}_i, \nu_{E_1} \rangle (x) = - \langle \mathcal{F}_e, \nu_{E_2} \rangle (x) \quad \textit{and} \quad \langle \mathcal{F}_e, \nu_{E_1} \rangle (x) = - \langle \mathcal{F}_i, \nu_{E_2} \rangle (x), \tag{4.3.30}$$

for $\mathscr{S}^{Q-1}$-a.e. $x \in \{y \in \mathscr{F}_H E_1 \cap \mathscr{F}_H E_2 : \nu_{E_1}(y) = -\nu_{E_2}(y)\}$.

Proof. We recall that, by Theorem 1.3.16, the perimeter measure $|D_H\chi_{E_j}|(\cdot)$ is a.e. asymptotically doubling, for $j = 1, 2$. Therefore, by the definitions (4.3.5) and (4.3.6), and the differentiation of perimeters (see [72, Sections 2.8.17 and 2.9.6]), we have

$$\left\langle \mathcal{F}_i, \nu_{E_j} \right\rangle (x) = 2 \lim_{r\to 0} \frac{(\chi_{E_j} F, D_H\chi_{E_j})(B(x,r))}{|D_H\chi_{E_j}|(B(x,r))},$$
$$\left\langle \mathcal{F}_e, \nu_{E_j} \right\rangle (x) = 2 \lim_{r\to 0} \frac{(\chi_{\Omega\setminus E_j} F, D_H\chi_{E_j})(B(x,r))}{|D_H\chi_{E_j}|(B(x,r))},$$

for $j = 1, 2$, and for $\mathscr{S}^{Q-1}$-a.e. $x \in \mathscr{F}_H E_j$. Let $x \in \mathscr{F}_H E_1 \cap \mathscr{F}_H E_2$ be such that $\nu_{E_1}(x) = \nu_{E_2}(x)$ and (4.3.22) and (4.3.24) hold true. Taking into account that $|\langle \mathcal{F}_i, \nu_{E_1}\rangle (x) - \langle \mathcal{F}_i, \nu_{E_2}\rangle (x)|$ can be written as the limit of the difference

$$2 \lim_{r\to 0} \left| \frac{(\chi_{E_1} F, D_H\chi_{E_1})(B(x,r))}{|D_H\chi_{E_1}|(B(x,r))} - \frac{(\chi_{E_2} F, D_H\chi_{E_2})(B(x,r))}{|D_H\chi_{E_2}|(B(x,r))} \right|$$

using the linearity and the triangle inequality, we get

$$\begin{aligned} |\langle \mathcal{F}_i, \nu_{E_1}\rangle (x) - \langle \mathcal{F}_i, \nu_{E_2}\rangle (x)| \le{}& 2 \limsup_{r\to 0} \left| \frac{(\chi_{E_1} F, D_H(\chi_{E_1} - \chi_{E_2}))(B(x,r))}{|D_H\chi_{E_1}|(B(x,r))} \right| \\ &+ \limsup_{r\to 0} \left| \frac{(\chi_{E_2} F, D_H(\chi_{E_1} - \chi_{E_2}))(B(x,r))}{|D_H\chi_{E_2}|(B(x,r))} \right| \\ &+ \limsup_{r\to 0} \left| \frac{(\chi_{E_1} F, D_H\chi_{E_2})(B(x,r))}{|D_H\chi_{E_1}|(B(x,r))} - \frac{(\chi_{E_2} F, D_H\chi_{E_1})(B(x,r))}{|D_H\chi_{E_2}|(B(x,r))} \right|. \end{aligned}$$

By (4.1.2), we have $|(\chi_{E_j} F, D_H(\chi_{E_1} - \chi_{E_2}))| \le \|F\|_{L^\infty(E_j)} |D_H(\chi_{E_1} - \chi_{E_2})|$, for $j = 1, 2$, and so, by (4.3.22), we conclude that

$$\left| \frac{(\chi_{E_j} F, D_H(\chi_{E_1} - \chi_{E_2}))(B(x,r))}{|D_H\chi_{E_j}|(B(x,r))} \right| \to 0,$$

for $j = 1, 2$. Now we have to deal with the last term, which, by (4.3.24), is infinitesimal as $r \to 0$ if and only if so is

$$\left| \frac{(\chi_{E_1} F, D_H\chi_{E_2})(B(x,r)) - (\chi_{E_2} F, D_H\chi_{E_1})(B(x,r))}{|D_H\chi_{E_2}|(B(x,r))} \right|. \tag{4.3.31}$$

By Theorem 4.1.1, we know that $\chi_{E_j} F \in \mathcal{DM}^\infty(H\Omega)$, for $j = 1, 2$. Let ε_k be the defining sequence for

$$\left\langle \mathcal{F}_e, \nu_{E_j} \right\rangle, \quad \left\langle \mathcal{F}_i, \nu_{E_j} \right\rangle, \quad (\chi_{E_j} F, D_H\chi_{E_j}) \quad \text{and} \quad (\chi_{\Omega\setminus E_j} F, D_H\chi_{E_j})$$

through limits analogous to those of (4.3.1) and (4.3.7) for E_j, $j = 1, 2$, in place of E. It follows that

$$\rho_{\varepsilon_k} * \chi_{E_1} \overset{*}{\rightharpoonup} \widetilde{\chi_{E_1}} \quad \text{in } L^\infty(\Omega; |\mathrm{div} F|)$$

and, by (4.2.2), (4.3.10) and (1.3.14), in $\Omega_{2\varepsilon}^{\mathcal{R}}$ we have

$$\begin{aligned} \mathrm{div}((\rho_{\varepsilon_k} * \chi_{E_1})\chi_{E_2} F) &= \rho_{\varepsilon_k} * \chi_{E_1} \mathrm{div}(\chi_{E_2} F) + \langle \chi_{E_2} F, \nabla_H(\rho_{\varepsilon_k} * \chi_{E_1})\rangle \mu \\ &= (\rho_{\varepsilon_k} * \chi_{E_1})\widetilde{\chi_{E_2}} \mathrm{div}(F) + (\rho_{\varepsilon_k} * \chi_{E_1})(F, D_H\chi_{E_2}) \\ &+ \langle \chi_{E_2} F, \rho_{\varepsilon_k} * D_H\chi_{E_1}\rangle \mu. \end{aligned} \tag{4.3.32}$$

Since $|\rho_{\varepsilon_k} * \chi_{E_1}|(x) \le 1$ for any $x \in \Omega$, up to extracting a further subsequence, we may find $\overline{\chi_{E_1}} \in L^\infty(\Omega; |(F, D_H\chi_{E_2})|)$ such that

$$\rho_{\varepsilon_k} * \chi_{E_1} \overset{*}{\rightharpoonup} \overline{\chi_{E_1}} \quad \text{in } L^\infty(\Omega; |(F, D_H\chi_{E_2})|).$$

Thus, by Lemma 2.4.3, we get the weak* convergence

$$(\rho_{\varepsilon_k} * \chi_{E_1})(F, D_H\chi_{E_2}) \rightharpoonup \overline{\chi_{E_1}}(F, D_H\chi_{E_2}).$$

Moreover, by Lemma 1.3.6, the sequence $\langle \chi_{E_2}F, \rho_{\varepsilon_k} * D_H\chi_{E_1}\rangle\rangle\, \mu$ is uniformly bounded on Ω, hence, up to extracting further subsequences, there exists the weak* limit

$$\langle \chi_{E_2}F, \rho_{\varepsilon_k} * D_H\chi_{E_1}\rangle\, \mu \rightharpoonup (\chi_{E_2}F, D_H\chi_{E_1}).$$

By Remark 2.4.1 we know that $|\mathrm{div}F|$-a.e. there holds $0 \le \widetilde{\chi_{E_2}} \le 1$ and by Lemma 2.4.3, we conclude that

$$(\rho_{\varepsilon_k} * \chi_{E_1})\widetilde{\chi_{E_2}}\mathrm{div}(F) \rightharpoonup \widetilde{\chi_{E_1}}\widetilde{\chi_{E_2}}\mathrm{div}F.$$

Passing now to the limit in (4.3.32) as $\varepsilon_k \to 0$, there holds

$$\mathrm{div}(\chi_{E_1}\chi_{E_2}F) = \widetilde{\chi_{E_1}}\widetilde{\chi_{E_2}}\mathrm{div}F + \overline{\chi_{E_1}}(F, D_H\chi_{E_2}) + (\chi_{E_2}F, D_H\chi_{E_1}). \tag{4.3.33}$$

Arguing in an analogous way, exchanging the role of χ_{E_1} and χ_{E_2}, we get

$$\mathrm{div}(\chi_{E_1}\chi_{E_2}F) = \widetilde{\chi_{E_1}}\widetilde{\chi_{E_2}}\mathrm{div}F + \overline{\chi_{E_2}}(F, D_H\chi_{E_1}) + (\chi_{E_1}F, D_H\chi_{E_2}). \tag{4.3.34}$$

Then (4.3.33) and (4.3.34) yield

$$(\chi_{E_2}F, D_H\chi_{E_1}) - (\chi_{E_1}F, D_H\chi_{E_2}) = \overline{\chi_{E_2}}(F, D_H\chi_{E_1}) - \overline{\chi_{E_1}}(F, D_H\chi_{E_2}). \tag{4.3.35}$$

Joining (2.4.2), Lemma 2.4.3 and (4.3.25), we can conclude that

$$\overline{\chi_{E_1}}(x) = \overline{\chi_{E_2}}(x) = 1/2 \quad \text{for } \mathscr{S}^{Q-1} - a.e.\ x \in \mathscr{F}_H E_1 \cap \mathscr{F}_H E_2 =: L.$$

By (4.1.2), we notice that

$$|(F, D_H\chi_{E_j})| \llcorner L \le \|F\|_{L^\infty(\Omega)} |D_H\chi_{E_j}| \llcorner L, \text{ for } \ j = 1, 2; \tag{4.3.36}$$

and so, by Remark 1.3.18, we obtain

$$\overline{\chi_{E_2}}(F, D_H\chi_{E_1}) \llcorner L = \frac{1}{2}(F, D_H\chi_{E_1}) \llcorner L, \quad \overline{\chi_{E_1}}(F, D_H\chi_{E_2}) \llcorner L = \frac{1}{2}(F, D_H\chi_{E_2}) \llcorner L.$$

Now, if we set $G := \mathscr{F}_H E_1 \Delta \mathscr{F}_H E_2$, we observe that we can rewrite (4.3.35) as

$$\begin{aligned}(\chi_{E_2}F, D_H\chi_{E_1}) - (\chi_{E_1}F, D_H\chi_{E_2}) &= \overline{\chi_{E_2}}(F, D_H\chi_{E_1}) \llcorner G - \overline{\chi_{E_1}}(F, D_H\chi_{E_2}) \llcorner G \\ &\quad + \frac{1}{2}(F, D_H(\chi_{E_1} - \chi_{E_2})) \llcorner L.\end{aligned} \tag{4.3.37}$$

By (4.1.2) and by standard differentiation of Borel measures, we have

$$|(F, D_H\chi_{E_j}) \llcorner G|(B(x,r)) \le \|F\|_{L^\infty(\Omega)} |D_H\chi_{E_j}|(G \cap B(x,r)) = o(r^{Q-1})$$

for $\mathscr{S}^{Q-1}$-a.e. $x \in L$, since $G \cap L = \emptyset$, and $j = 1, 2$. In addition, $|D_H\chi_{E_j}|(B(x,r)) \ge cr^{Q-1}$ for $\mathscr{S}^{Q-1}$-a.e. $x \in \mathscr{F}_H E_j$, $r > 0$ sufficiently small and $j = 1, 2$, by [1, Theorem 4.3]; and so we obtain

$$|(F, D_H\chi_{E_j}) \llcorner G|(B(x,r)) = o(|D_H\chi_{E_2}|(B(x,r))).$$

As for the second term, by (4.1.2) and (4.3.22) we get

$$\begin{aligned}|(F, D_H(\chi_{E_1} - \chi_{E_2}))|(L \cap B(x,r)) &\le \|F\|_{L^\infty(\Omega)} |D_H(\chi_{E_1} - \chi_{E_2})|(B(x,r)) \\ &= o(|D_H\chi_{E_2}|(B(x,r)))\end{aligned}$$

for $\mathscr{S}^{Q-1}$-a.e. $x \in \mathscr{F}_H E_1 \cap \mathscr{F}_H E_2$ such that $\nu_{E_1}(x) = \nu_{E_2}(x)$. This implies that the expression in (4.3.31) goes to zero as $r \to 0$, and so it proves the first part of (4.3.29). Concerning the exterior normal traces, one can argue in a similar way for the sets $\Omega \setminus E_1$ and $\Omega \setminus E_2$. Finally, taking into account (4.3.5), (4.3.6), Remark 4.3.1 and (4.3.29) applied to E_1 and $\Omega \setminus E_2$, and conversely, we arrive at (4.3.30). □

4.3.2 Tripartition by weak* limit of mollified functions

In this section we study the properties of the limit $\widetilde{\chi_E}$ defined in (4.3.7) and the related Leibniz rule. From Remark 2.4.1 we have that $0 \le \widetilde{\chi_E}(x) \le 1$ for $|\mathrm{div}F|$-a.e. $x \in \Omega$.

Proposition 4.3.7. *Let $F \in \mathcal{DM}^\infty(H\Omega)$ and $E \subset \Omega$ be a set of finite h-perimeter. Let $\widetilde{\chi_E}$, $\widetilde{E}^1$, $\widetilde{E}^0$ and $\widetilde{\mathscr{F}_H E}$ be as in (4.3.7), (4.3.8) and (4.3.9), respectively. It holds*

$$|\mathrm{div}F|(\Omega \setminus (\mathscr{F}_H E \cup \widetilde{E}^1 \cup \widetilde{E}^0)) = |\mathrm{div}F|(\Omega \setminus (\widetilde{\mathscr{F}_H E} \cup \widetilde{E}^1 \cup \widetilde{E}^0)) = 0. \tag{4.3.38}$$

In particular, $\widetilde{\chi_E}$ is uniquely determined on $\Omega \setminus \mathscr{F}_H E$, up to $|\mathrm{div}F|$-negligible sets, and we have

$$\widetilde{\chi_E} = \chi_{\widetilde{E}^1} \quad |\mathrm{div}F|\text{-a.e. in } \Omega \setminus \mathscr{F}_H E, \tag{4.3.39}$$

so that

$$\mathrm{div}(\chi_E F) \llcorner (\Omega \setminus \mathscr{F}_H E) = \mathrm{div}F \llcorner (\widetilde{E}^1 \setminus \mathscr{F}_H E). \tag{4.3.40}$$

Proof. From (4.3.17) we immediately conclude that

$$\int_{\Omega \setminus \mathscr{F}_H E} \widetilde{\chi_E}(1 - \widetilde{\chi_E}) \, d|\mathrm{div}F| = 0.$$

Therefore definitions (4.3.8) and (4.3.9) give

$$\Omega = \widetilde{E}^1 \cup \widetilde{E}^0 \cup \widetilde{\mathscr{F}_H E} \cup Z_E^F,$$

where $Z_E^F = \Omega \setminus (\mathscr{F}_H E \cup \widetilde{E}^1 \cup \widetilde{E}^0)$ is $|\mathrm{div}F|$-negligible. This proves (4.3.38). Then, if we restrict (4.3.10) to $\Omega \setminus \mathscr{F}_H E$, we have

$$\mathrm{div}(\chi_E F) \llcorner (\Omega \setminus \mathscr{F}_H E) = \widetilde{\chi_E} \mathrm{div}F \llcorner (\Omega \setminus \mathscr{F}_H E), \tag{4.3.41}$$

which immediately shows that $\widetilde{\chi_E}$ is uniquely determined on $\Omega \setminus \mathscr{F}_H E$, as a function in $L^\infty(\Omega; |\mathrm{div}F|)$. Moreover, by (4.3.38), we have that $\widetilde{\chi_E}(x) \in \{0, 1\}$ for $|\mathrm{div}F|$-a.e. $x \in \Omega \setminus \mathscr{F}_H E$, and this implies (4.3.39). Finally, this immediately shows that (4.3.41) is equivalent to (4.3.40). □

Formula (4.3.39) will be important to show that indeed the set $\widetilde{E}^1$ is uniquely defined up to $|\mathrm{div}F|$-negligible sets.

Remark 4.3.8. We notice that, by Proposition 4.2.9, $\widetilde{\chi_E}(x) = \chi_E^{*,\mathcal{R}}(x)$ for $|\mathrm{div}F|$-a.e. $x \in C_E^{\mathcal{R}}$. In particular, we obtain

$$|\mathrm{div}F| \left((\widetilde{E}^1 \Delta E^{1,\mathcal{R}}) \cap C_E^{\mathcal{R}} \right) = 0 \text{ and } |\mathrm{div}F| \left((\widetilde{E}^0 \Delta E^{0,\mathcal{R}}) \cap C_E^{\mathcal{R}} \right) = 0.$$

If we now assume that

$$|\mathrm{div}F|(\Omega \setminus C_E^{\mathcal{R}}) = 0, \tag{4.3.42}$$

then it follows that $\widetilde{\chi_E}(x) = \chi_E^{*,\mathcal{R}}(x)$ for $|\mathrm{div}F|$-a.e. $x \in \Omega$. In particular, this yields

$$|\mathrm{div}F|(\widetilde{E}^1 \Delta E^{1,\mathcal{R}}) = 0 \quad \text{and} \quad |\mathrm{div}F|(\widetilde{E}^0 \Delta E^{0,\mathcal{R}}) = 0.$$

Thus, we have shown that

$$\partial_H^{*,\mathcal{R}} E \setminus \mathscr{F}_H E = \Omega \setminus (E^{1,\mathcal{R}} \cup E^{0,\mathcal{R}} \cup \mathscr{F}_H E) = (\Omega \setminus (\widetilde{E}^1 \cup \widetilde{E}^0 \cup \mathscr{F}_H E)) \cup \widetilde{Z}_E^F,$$

for some $|\mathrm{div}F|$-negligible set $\widetilde{Z}_E^F$. By (4.3.38), we obtain $|\mathrm{div}F|(\partial_H^{*,\mathcal{R}} E \setminus \mathscr{F}_H E) = 0$. Hence, under the assumption (4.3.42), we can identify $\widetilde{E}^1$ and $\widetilde{E}^0$ with $E^{1,\mathcal{R}}$ and $E^{0,\mathcal{R}}$, up to $|\mathrm{div}F|$-negligible sets, thus obtaining their uniqueness in this special case. In fact, as we shall see below, the uniqueness holds in general, even if (4.3.42) fails to be true.

The following proposition proves that any given set of finite h-perimeter E in Ω yields a tripartition of Ω. More precisely, for $F \in \mathcal{DM}^\infty(H\Omega)$ there exists a representative $\widetilde{\chi_E}$ of χ_E such that $\widetilde{\chi_E}(x) \in \{1, 0, 1/2\}$ for $|\mathrm{div}F|$-a.e. $x \in \Omega$.

Proposition 4.3.9. *Let $F \in \mathcal{DM}^\infty(H\Omega)$ and let $E \subset \Omega$ be a set of finite h-perimeter. Let $\rho \in C_c(B(0,1))$ be a mollifier satisfying $\rho(x) = \rho(x^{-1})$ and $\int_{B(0,1)} \rho(y)\, dy = 1$. If $\widetilde{\chi_E} \in L^\infty(\Omega; |\mathrm{div}F|)$ is defined by (4.3.7), then*

$$\widetilde{\chi_E} = \frac{1}{2} \quad |\mathrm{div}F|\text{-}a.e.\ on\ \ \widetilde{\mathscr{F}_H E}. \tag{4.3.43}$$

In addition, the normal traces of F on the boundary of E satisfy

$$\langle \mathcal{F}_i, \nu_E \rangle = \langle \mathcal{F}_e, \nu_E \rangle \quad |D_H \chi_E|\text{-}a.e.\ on\ \ \widetilde{E}^1 \cup \widetilde{E}^0 \tag{4.3.44}$$

and we have

$$\chi_{\widetilde{\mathscr{F}_H E}} \mathrm{div}F = (\langle \mathcal{F}_i, \nu_E \rangle - \langle \mathcal{F}_e, \nu_E \rangle)|D_H \chi_E|. \tag{4.3.45}$$

Proof. From (4.3.17) it follows immediately that

$$(\langle \mathcal{F}_i, \nu_E \rangle - \langle \mathcal{F}_e, \nu_E \rangle)|D_H \chi_E| = 0 \quad \text{on} \quad \mathscr{F}_H E \cap (\widetilde{E}^1 \cup \widetilde{E}^0),$$

proving (4.3.44). Let ε_k be the defining sequence such that (4.3.1) and (4.3.7) hold. We have

$$(\rho_{\varepsilon_k} * \chi_E)\widetilde{\chi_E}(1 - \widetilde{\chi_E})\mathrm{div}F \rightharpoonup (\widetilde{\chi_E})^2(1 - \widetilde{\chi_E})\mathrm{div}F,$$

by Lemma 2.4.3. Since the traces $\langle \mathcal{F}_i, \nu_E \rangle, \langle \mathcal{F}_e, \nu_E \rangle$ defined in (4.3.5) and (4.3.6) belong to $L^\infty(\Omega; |D_H \chi_E|)$, Remark 2.4.4 and (4.3.17) imply that

$$\begin{aligned}(\rho_{\varepsilon_k} * \chi_E)\frac{\langle \mathcal{F}_i, \nu_E \rangle - \langle \mathcal{F}_e, \nu_E \rangle}{4}|D_H \chi_E| &\rightharpoonup \frac{1}{2}\frac{\langle \mathcal{F}_i, \nu_E \rangle - \langle \mathcal{F}_e, \nu_E \rangle}{4}|D_H \chi_E| \\ &= \frac{1}{2}\widetilde{\chi_E}(1 - \widetilde{\chi_E})\mathrm{div}F.\end{aligned}$$

Again (4.3.17) shows that the previous sequences of measures are equal, hence so are their limits. Taking their difference, we get

$$\left(\widetilde{\chi_E} - \frac{1}{2}\right)\widetilde{\chi_E}(1 - \widetilde{\chi_E})\mathrm{div}F \llcorner \mathscr{F}_H E = 0.$$

This implies $\widetilde{\chi_E} = \dfrac{1}{2}$ $|\mathrm{div}F|$-a.e. on $\widetilde{\mathscr{F}_H E}$. From (4.3.43) and (4.3.17), we obtain (4.3.45). ☐

Remark 4.3.10. Proposition 4.3.9 and (4.3.43) imply that

$$|\mathrm{div}F|\left(\widetilde{\mathscr{F}_H E} \setminus \{x \in \Omega : \widetilde{\chi_E} = 1/2\}\right) = 0.$$

Since Proposition 4.3.7 states that $\Omega = \widetilde{E}^1 \cup \widetilde{E}^0 \cup \widetilde{\mathscr{F}_H E} \cup Z_E^F$, for some $|\mathrm{div}F|$-negligible set Z_E^F, then we get the tripartition

$$\widetilde{\chi_E}(x) \in \{0, 1, 1/2\} \quad \text{for} \quad |\mathrm{div}F|\text{-a.e. } x \in \Omega.$$

As a result, we have shown that for every $F \in \mathcal{DM}^\infty(H\Omega)$, taking any weak* limit of $\rho_\varepsilon * \chi_E$ in $L^\infty(\Omega; |\mathrm{div}F|)$, this limit attains only the three possible values $1, 0, 1/2$ for $|\mathrm{div}F|$-a.e. $x \in \Omega$. This motivates our definitions (4.3.8) and (4.3.9).

Remark 4.3.11. If $F \in \mathcal{DM}^\infty(H\Omega)$ and $E \subset \Omega$ is a set of finite h-perimeter, then

$$(F, D_H\chi_E) = \frac{\langle \mathcal{F}_i, \nu_E\rangle + \langle \mathcal{F}_e, \nu_E\rangle}{2}|D_H\chi_E|. \tag{4.3.46}$$

This follows from (4.3.15) and from the definitions of the normal traces, (4.3.5), (4.3.6).

We are now arrived at our first general result on the Leibniz rule for divergence-measure horizontal fields and characteristic functions of sets of finite h-perimeter in stratified groups.

Theorem 4.3.12. *If $F \in \mathcal{DM}^\infty(H\Omega)$ and $E \subset \Omega$ is a set of finite h-perimeter, then we have*

$$\mathrm{div}(\chi_E F) = \chi_{\widetilde{E}^1}\mathrm{div}F + \langle \mathcal{F}_i, \nu_E\rangle\, |D_H\chi_E|, \tag{4.3.47}$$

$$\mathrm{div}(\chi_E F) = \chi_{\widetilde{E}^1 \cup \widetilde{\mathscr{F}_H E}}\mathrm{div}F + \langle \mathcal{F}_e, \nu_E\rangle\, |D_H\chi_E|, \tag{4.3.48}$$

where $\widetilde{\chi_E} \in L^\infty(\Omega; |\mathrm{div}F|)$ is the weak limit defined in* (4.3.7).

Proof. By Remark 4.3.10, we have

$$\widetilde{\chi_E}(x) = \chi_{\widetilde{E}^1}(x) + \frac{1}{2}\chi_{\widetilde{\mathscr{F}_H E}}(x) \quad \text{for } |\mathrm{div}F| - a.e.\ x \in \Omega.$$

Due to (4.3.46), we can rewrite (4.3.10) as follows

$$\mathrm{div}(\chi_E F) = \chi_{\widetilde{E}^1}\mathrm{div}F + \frac{1}{2}\chi_{\widetilde{\mathscr{F}_H E}}\mathrm{div}F + \frac{\langle \mathcal{F}_i, \nu_E\rangle + \langle \mathcal{F}_e, \nu_E\rangle}{2}|D_H\chi_E|.$$

We can now employ (4.3.45) to substitute the term $\chi_{\widetilde{\mathscr{F}_H E}}\mathrm{div}F$, obtaining

$$\begin{aligned}\mathrm{div}(\chi_E F) = \chi_{\widetilde{E}^1}\mathrm{div}F &+ \frac{\langle \mathcal{F}_i, \nu_E\rangle - \langle \mathcal{F}_e, \nu_E\rangle}{2}|D_H\chi_E| \llcorner \widetilde{\mathscr{F}_H E} + \\ &+ \frac{\langle \mathcal{F}_i, \nu_E\rangle + \langle \mathcal{F}_e, \nu_E\rangle}{2}|D_H\chi_E|.\end{aligned}$$

The previous equality immediately gives (4.3.47). To derive (4.3.48), we simply join (4.3.47) with (4.3.45) and (4.3.44). □

4.3.3 Uniqueness results

The previous results, together with the auxiliary definitions of $\widetilde{E}^1$, $\widetilde{E}^0$ and $\widetilde{\mathscr{F}_H E}$, allow us to obtain the following uniqueness theorem, along with a number of relevant consequences.

Theorem 4.3.13 (Uniqueness)**.** *If $F \in \mathcal{DM}^\infty(H\Omega)$ and $E \subset \Omega$ is a set of finite h-perimeter, then there exists a unique $|\mathrm{div}F|$-measurable subset*

$$E^{1,F} \subset \Omega \setminus \mathscr{F}_H E,$$

up to $|\mathrm{div}F|$-negligible sets, such that

$$\widetilde{\chi_E}(x) = \chi_{E^{1,F}}(x) + \frac{1}{2}\chi_{\mathscr{F}_H E}(x) \quad \textit{for } |\mathrm{div}F|\text{-}a.e.\ x \in \Omega. \tag{4.3.49}$$

In addition, we have

$$|\mathrm{div}F|\left(\mathscr{F}_H E \setminus \widetilde{\mathscr{F}_H E}\right) = 0 \tag{4.3.50}$$

and there exist unique normal traces

$$\langle \mathcal{F}_i, \nu_E\rangle, \langle \mathcal{F}_e, \nu_E\rangle \in L^\infty(\mathscr{F}_H E; |D_H\chi_E|)$$

satisfying

$$\mathrm{div}(\chi_E F) = \chi_{E^{1,F}}\mathrm{div}F + \langle \mathcal{F}_i, \nu_E\rangle\, |D_H\chi_E|, \tag{4.3.51}$$

$$\mathrm{div}(\chi_E F) = \chi_{E^{1,F} \cup \mathscr{F}_H E}\mathrm{div}F + \langle \mathcal{F}_e, \nu_E\rangle\, |D_H\chi_E|. \tag{4.3.52}$$

Proof. Thanks to (4.3.39) the set

$$E^{1,F} = \widetilde{E}^1 \setminus \mathscr{F}_H E \subset \Omega \setminus \mathscr{F}_H E$$

satisfies the equality $\widetilde{\chi_E} = \chi_{E^{1,F}}$ $|\mathrm{div} F|$-a.e. in $\Omega \setminus \mathscr{F}_H E$ and it is uniquely determined up to $|\mathrm{div} F|$-negligible sets. By the tripartition stated in Proposition 4.3.7, we get

$$\widetilde{\chi_E}(x) = \chi_{E^{1,F}}(x) + \chi_{\widetilde{E}^1 \cap \mathscr{F}_H E}(x) + \frac{1}{2}\chi_{\widetilde{\mathscr{F}_H E}}(x) \quad \text{for } |\mathrm{div} F|\text{-a.e. } x \in \Omega, \tag{4.3.53}$$

from which we get

$$\widetilde{E}^1 = E^{1,F} \cup (\widetilde{E}^1 \cap \mathscr{F}_H E) \text{ and } \widetilde{E}^0 = E^{0,F} \cup (\widetilde{E}^0 \cap \mathscr{F}_H E),$$

where $E^{0,F} = \Omega \setminus (E^{1,F} \cup \mathscr{F}_H E)$. Let E be a set of finite h-perimeter in Ω. We notice that $\mathscr{F}_H E$ is a Borel set, by definition, so that, if $F \in \mathcal{DM}^\infty(H\Omega)$, the measure $|\mathrm{div} F| \llcorner \mathscr{F}_H E$ is well defined. Let ε_k be the fixed nonnegative vanishing sequence such that $\rho_{\varepsilon_k} * \chi_E \overset{*}{\rightharpoonup} \widetilde{\chi_E}$ in $L^\infty(\Omega; |\mathrm{div} F|)$. Then, we also have

$$\rho_{\varepsilon_k} * \chi_E \overset{*}{\rightharpoonup} \widetilde{\chi_E} \text{ in } L^\infty(\Omega; |\mathrm{div} F| \llcorner \mathscr{F}_H E). \tag{4.3.54}$$

To see this, it is enough to multiply any test functions $\psi \in L^1(\Omega; |\mathrm{div} F| \llcorner \mathscr{F}_H E)$ with $\chi_{\mathscr{F}_H E}$, getting a function in $L^\infty(\Omega; |\mathrm{div} F|)$.

Now, Theorem 4.2.7 shows that $|\mathrm{div} F| \ll \mathscr{S}^{Q-1}$, so that

$$|\mathrm{div} F| \llcorner \mathscr{F}_H E \ll \mathscr{S}^{Q-1} \llcorner \mathscr{F}_H E,$$

which, by Theorem 1.3.16, gives

$$|\mathrm{div} F| \llcorner \mathscr{F}_H E \ll |D_H \chi_E|. \tag{4.3.55}$$

Then, it is easy to see that (4.3.55) implies $|\mathrm{div} F| \llcorner \mathscr{F}_H E = \theta_{F,E} |D_H \chi_E|$, for some $\theta_{F,E} \in L^1(\Omega; |D_H \chi_E|)$, by Radon-Nikodým theorem. We recall that, by (2.4.2), we have

$$\rho_\varepsilon * \chi_E \overset{*}{\rightharpoonup} 1/2$$

in $L^\infty(\Omega; |D_H \chi_E|)$. Due to Lemma 2.4.3, it follows that

$$(\rho_\varepsilon * \chi_E) |\mathrm{div} F| \llcorner \mathscr{F}_H E \rightharpoonup \frac{1}{2} |\mathrm{div} F| \llcorner \mathscr{F}_H E.$$

On the other hand, (4.3.54) implies

$$(\rho_{\varepsilon_k} * \chi_E) |\mathrm{div} F| \llcorner \mathscr{F}_H E \rightharpoonup \widetilde{\chi_E} |\mathrm{div} F| \llcorner \mathscr{F}_H E,$$

and so we conclude that any weak* limit point of $\{\rho_\varepsilon * \chi_E\}_{\varepsilon > 0}$ in $L^\infty(\Omega; |\mathrm{div} F|)$ must satisfy $\widetilde{\chi_E}(x) = \frac{1}{2}$ for $|\mathrm{div} F| \llcorner \mathscr{F}_H E$-a.e. $x \in \Omega$. Clearly, this means that

$$\widetilde{\chi_E}(x) = \frac{1}{2} \text{ for } |\mathrm{div} F|\text{-a.e. } x \in \mathscr{F}_H E. \tag{4.3.56}$$

As an immediate consequence, we obtain

$$|\mathrm{div} F|(\widetilde{E}^1 \cap \mathscr{F}_H E) = 0 \text{ and } |\mathrm{div} F|(\widetilde{E}^0 \cap \mathscr{F}_H E) = 0,$$

which implies (4.3.50). Thus, combining these results with (4.3.53), we deduce that there exists a unique $|\mathrm{div} F|$-measurable set $E^{1,F} \subset \Omega \setminus \mathscr{F}_H E$ such that (4.3.49) holds. Hence, there exists a unique weak* limit $\widetilde{\chi_E}$ of $\{\rho_\varepsilon * \chi_E\}_{\varepsilon > 0}$ in $L^\infty(\Omega; |\mathrm{div} F|)$. Thanks to (4.3.47) and (4.3.48), we obtain the uniqueness of the normal traces. Indeed, we have

$$\mathrm{div}(\chi_E F) - \chi_{E^{1,F}} \mathrm{div} F = \langle \mathcal{F}_i, \nu_E \rangle \, |D_H \chi_E|,$$
$$\mathrm{div}(\chi_E F) - \chi_{E^{1,F} \cup \mathscr{F}_H E} \mathrm{div} F = \langle \mathcal{F}_e, \nu_E \rangle \, |D_H \chi_E|,$$

and the uniqueness of the terms on the left hand sides implies the uniqueness of those on the right hand sides. $\square$

In view of the previous uniqueness result, we are in the position to introduce the following definition.

Definition 4.3.14. Let $E \subset \Omega$ be a set of finite h-perimeter and let $F \in \mathcal{DM}^\infty(H\Omega)$. We define the *measure theoretic interior of E with respect to F* as $E^{1,F} \subset \Omega \setminus \mathscr{F}_H E$, such that

$$\operatorname{div}(\chi_E F) \llcorner (\Omega \setminus \mathscr{F}_H E) = \chi_{E^{1,F}} \operatorname{div} F. \tag{4.3.57}$$

Analogously, we define the *measure theoretic exterior of E with respect to F* as a set $E^{0,F} \subset \Omega \setminus \mathscr{F}_H E$ such that

$$\operatorname{div}(\chi_{\Omega \setminus E} F) \llcorner (\Omega \setminus \mathscr{F}_H E) = \chi_{E^{0,F}} \operatorname{div} F. \tag{4.3.58}$$

Remark 4.3.15. The existence of $E^{1,F}$, along with its uniqueness up to $|\operatorname{div} F|$-negligible sets, is a direct consequence of restricting (4.3.51) to $\Omega \setminus \mathscr{F}_H E$. Analogously, the existence and uniqueness up to $|\operatorname{div} F|$-negligible sets of $E^{0,F}$ follows from applying (4.3.51) to $\Omega \setminus E$ and restricting it to $\Omega \setminus \mathscr{F}_H E$. In addition, we have

$$|\operatorname{div} F| \left(\Omega \setminus (E^{1,F} \cup E^{0,F} \cup \mathscr{F}_H E) \right) = 0,$$

since

$$\begin{aligned} \chi_{E^{0,F}} \operatorname{div} F &= \operatorname{div}(\chi_{\Omega \setminus E} F) \llcorner (\Omega \setminus \mathscr{F}_H E) = \chi_{\Omega \setminus \mathscr{F}_H E} \operatorname{div} F - \operatorname{div}(\chi_E F) \llcorner (\Omega \setminus \mathscr{F}_H E) \\ &= \left(\chi_{\Omega \setminus \mathscr{F}_H E} - \chi_{E^{1,F}} \right) \operatorname{div} F = \chi_{\Omega \setminus (E^{1,F} \cup \mathscr{F}_H E)} \operatorname{div} F, \end{aligned}$$

thanks to (4.3.57).

Remark 4.3.16. Theorem 4.3.13 shows that the interior and exterior normal traces $\langle \mathcal{F}_i, \nu_E \rangle$ and $\langle \mathcal{F}_e, \nu_E \rangle$ are unique up to $|D_H \chi_E|$-negligible sets. As an immediate consequence of this fact, joined with (4.3.5) and (4.3.6), we see that also the pairings $(\chi_E F, D_H \chi_E)$ and $(\chi_{\Omega \setminus E} F, D_H \chi_E)$ are uniquely determined and do not depend on the approximating sequences $\langle \chi_E F, \nabla_H (\rho_{\varepsilon_k} * \chi_E) \rangle \mu$ and $\left\langle \chi_{\Omega \setminus E} F, \nabla_H (\rho_{\varepsilon_k} * \chi_E) \right\rangle \mu$. In addition, (4.3.46) shows that also the pairing $(F, D_H \chi_E)$ is unique and independent from the choice of the approximating sequence.

We conclude this section with the following easy refinement of (4.3.45).

Corollary 4.3.17. *Let $F \in \mathcal{DM}^\infty(H\Omega)$ and E be a set of finite h-perimeter. Then, we have*

$$\chi_{\mathscr{F}_H E} \operatorname{div} F = (\langle \mathcal{F}_i, \nu_E \rangle - \langle \mathcal{F}_e, \nu_E \rangle) |D_H \chi_E|. \tag{4.3.59}$$

Proof. It is enough to subtract (4.3.51) from (4.3.52). $\square$

4.4 Gauss–Green and integration by parts formulas

This section is devoted to establish different Gauss–Green formulas and integration by parts formula in stratified groups. Throughout we shall use the measure theoretic interior $E^{1,F} \subset \mathbb{G}$ introduced in Definition 4.3.14. We start with a general version of the Gauss–Green formulas, which is a direct consequence of Theorem 4.3.13.

Theorem 4.4.1. *Let $F \in \mathcal{DM}^\infty(H\Omega)$ and $E \Subset \Omega$ be a set of finite h-perimeter. Then, we have*

$$\operatorname{div} F(E^{1,F}) = - \int_{\mathscr{F}_H E} \langle \mathcal{F}_i, \nu_E \rangle \, d|D_H \chi_E|, \tag{4.4.1}$$

$$\operatorname{div} F(E^{1,F} \cup \mathscr{F}_H E) = - \int_{\mathscr{F}_H E} \langle \mathcal{F}_e, \nu_E \rangle \, d|D_H \chi_E|. \tag{4.4.2}$$

Proof. If we evaluate (4.3.51) and (4.3.52) on Ω, we obtain

$$\mathrm{div}(\chi_E F)(\Omega) = \mathrm{div}F(E^{1,F}) + \int_{\mathscr{F}_H E} \langle \mathcal{F}_i, \nu_E \rangle \, d|D_H \chi_E|,$$
$$\mathrm{div}(\chi_E F)(\Omega) = \mathrm{div}F(E^{1,F} \cup \mathscr{F}_H E) + \int_{\mathscr{F}_H E} \langle \mathcal{F}_e, \nu_E \rangle \, d|D_H \chi_E|.$$

Then, we exploit the fact that $\chi_E F \in \mathcal{DM}^\infty(H\Omega)$, thanks to Theorem 4.1.1, and Lemma 4.2.6 in order to conclude that $\mathrm{div}(\chi_E F)(\Omega) = 0$. □

We consider now some special cases in which the normal traces coincides; that is, in which there are no jumps along the reduced boundary of the integration domain. To this purpose, we give the following definition.

Definition 4.4.2. Let $F \in \mathcal{DM}^\infty(H\Omega)$ and let $E \subset \Omega$ be a set of finite h-perimeter. We define the *average normal trace* $\langle \mathcal{F}, \nu_E \rangle$ as the function in $L^\infty(\Omega; |D_H \chi_E|)$ satisfying

$$(F, D_H \chi_E) = \langle \mathcal{F}, \nu_E \rangle \, |D_H \chi_E|. \tag{4.4.3}$$

Remark 4.4.3. Thanks to (4.1.2), we have

$$|(F, D_H \chi_E)| \leq \|F\|_{L^\infty(\Omega)} |D_H \chi_E|,$$

which implies the existence of $\langle \mathcal{F}, \nu_E \rangle \in L^\infty(\Omega; |D_H \chi_E|)$ satisfying (4.4.3), by Radon-Nikodým theorem. In addition, (4.3.46) shows that

$$\langle \mathcal{F}, \nu_E \rangle = \frac{\langle \mathcal{F}_i, \nu_E \rangle + \langle \mathcal{F}_e, \nu_E \rangle}{2} \quad |D_H \chi_E|\text{-a.e. in } \Omega. \tag{4.4.4}$$

Proposition 4.4.4. *Let $F \in \mathcal{DM}^\infty(H\Omega)$ and let $E \subset \Omega$ be a set of finite h-perimeter such that $|\mathrm{div}F|(\mathscr{F}_H E) = 0$. Then we have*

$$\langle \mathcal{F}_i, \nu_E \rangle = \langle \mathcal{F}_e, \nu_E \rangle = \langle \mathcal{F}, \nu_E \rangle \quad |D_H \chi_E|\text{-a.e. in } \Omega. \tag{4.4.5}$$

As a consequence, we obtain

$$\mathrm{div}(\chi_E F) = \chi_{E^{1,F}} \mathrm{div}F + (F, D_H \chi_E) = \chi_{E^{1,F}} \mathrm{div}F + \langle \mathcal{F}, \nu_E \rangle \, |D_H \chi_E|. \tag{4.4.6}$$

Proof. Equality (4.4.5) is an immediate consequence of $|\mathrm{div}F|(\mathscr{F}_H E) = 0$, (4.3.59) and (4.4.4). Then, by combining (4.3.51), (4.4.3) and (4.4.5) we obtain (4.4.6). □

The previous result immediately gives a new version of the Gauss–Green formula without jumps on the reduced boundary of the domain.

Theorem 4.4.5. *Let $F \in \mathcal{DM}^\infty(H\Omega)$ and let $E \Subset \Omega$ be a set of finite h-perimeter with $|\mathrm{div}F|(\mathscr{F}_H E) = 0$. Then there exists a unique normal trace $\langle \mathcal{F}, \nu_E \rangle \in L^\infty(\Omega; |D_H \chi_E|)$ such that*

$$\mathrm{div}F(E^{1,F}) = - \int_{\mathscr{F}_H E} \langle \mathcal{F}, \nu_E \rangle \, d|D_H \chi_E|. \tag{4.4.7}$$

Proof. The existence of a unique normal trace $\langle \mathcal{F}, \nu_E \rangle$ follows from Proposition 4.4.4. Then, we evaluate (4.4.6) on Ω and apply Lemma 4.2.6, and thus we obtain (4.4.7). □

We now prove that, in the case F is continuous, the normal traces are equal and coincide with the scalar product in the horizontal section.

Proposition 4.4.6. *Let $F \in \mathcal{DM}^\infty(H\Omega) \cap C(H\Omega)$ and $E \subset \Omega$ be a set of finite h-perimeter. Then we have*

$$\langle \mathcal{F}_i, \nu_E \rangle (x) = \langle \mathcal{F}_e, \nu_E \rangle (x) = \langle F(x), \nu_E(x) \rangle \quad \textit{for} \;\; |D_H\chi_E|\text{-a.e. } x \in \mathscr{F}_H E \tag{4.4.8}$$

and in particular $|\mathrm{div}F|(\mathscr{F}_H E) = 0$.

Proof. Let $\phi \in C_c(\Omega)$ and let $(\chi_E F, D_H\chi_E)$ be as defined in (4.3.1), hence

$$\int_\Omega \phi \, d(\chi_E F, D_H\chi_E) = \lim_{\varepsilon \to 0} \int_\Omega \langle \phi F, \chi_E(\rho_\varepsilon * D_H\chi_E) \rangle \, dx.$$

We observe that $\phi F \in C_c(\Omega, H\Omega)$ and taking into account that ν_E is the measure theoretic unit interior h-normal, by (2.4.7) we obtain

$$\int_\Omega \phi \, d(\chi_E F, D_H\chi_E) = \int_\Omega \frac{1}{2} \phi \langle F, \nu_E \rangle \, d|D_H\chi_E|.$$

By definition of interior normal trace (4.3.5), we obtain that

$$\langle \mathcal{F}_i, \nu_E \rangle (x) = \langle F(x), \nu_E(x) \rangle \quad \text{for } |D_H\chi_E|\text{-a.e. } x \in \Omega,$$

which implies (4.4.8) for the interior normal trace. The identity for the exterior normal trace in (4.4.8) can be proved in an analogous way, employing (2.4.8) and definition (4.3.6). Finally, in view of (4.3.59), we get $|\mathrm{div}F|(\mathscr{F}_H E) = 0$. □

Theorem 4.4.7. *Let $F \in \mathcal{DM}^\infty(H\Omega) \cap C(H\Omega)$ and let $E \Subset \Omega$ be a set of finite h-perimeter. Then the following formula holds*

$$\mathrm{div}F(E^{1,F}) = -\int_{\mathscr{F}_H E} \langle F, \nu_E \rangle \, d|D_H\chi_E|. \tag{4.4.9}$$

Proof. By Proposition 4.4.6, we have a unique normal trace, that $|D_H\chi_E|$-a.e. in $\mathscr{F}_H E$ equals the scalar product $\langle F, \nu_E \rangle$. In addition, $|\mathrm{div}F|(\mathscr{F}_H E) = 0$. Since $E \Subset \Omega$, we may apply (4.4.7) to conclude the proof. □

Next, we apply the Leibniz rule (Theorem 4.1.1) to derive integration by parts formulas.

Theorem 4.4.8. *Let $F \in \mathcal{DM}^\infty_{\mathrm{loc}}(H\Omega)$, E be a set of locally finite h-perimeter in Ω and $\varphi \in C(\Omega)$ with $\nabla_H\varphi \in L^1_{\mathrm{loc}}(H\Omega)$ such that $\mathrm{supp}(\varphi\chi_E) \Subset \Omega$. Then, there exist interior and exterior normal traces*

$$\langle \mathcal{F}_i, \nu_E \rangle, \langle \mathcal{F}_e, \nu_E \rangle \in L^\infty_{\mathrm{loc}}(\Omega; |D_H\chi_E|)$$

such that the following formulas hold

$$\int_{E^{1,F}} \varphi \, d\mathrm{div}F + \int_E \langle F, \nabla_H\varphi \rangle \, dx = -\int_{\mathscr{F}_H E} \varphi \langle \mathcal{F}_i, \nu_E \rangle \, d|D_H\chi_E|, \tag{4.4.10}$$

$$\int_{E^{1,F} \cup \mathscr{F}_H E} \varphi \, d\mathrm{div}F + \int_E \langle F, \nabla_H\varphi \rangle \, dx = -\int_{\mathscr{F}_H E} \varphi \langle \mathcal{F}_e, \nu_E \rangle \, d|D_H\chi_E|. \tag{4.4.11}$$

In addition, for any open set $U \Subset \Omega$, we have the following estimates

$$\| \langle \mathcal{F}_i, \nu_E \rangle \|_{L^\infty(\mathscr{F}_H E \cap U; |D_H\chi_E|)} \le \|F\|_{L^\infty(E \cap U)}, \tag{4.4.12}$$

$$\| \langle \mathcal{F}_e, \nu_E \rangle \|_{L^\infty(\mathscr{F}_H E \cap U; |D_H\chi_E|)} \le \|F\|_{L^\infty(U \setminus E)}. \tag{4.4.13}$$

Proof. Let $U \Subset \Omega$ be an open set such that $\operatorname{supp}(\varphi\chi_E) \subset U$. Then, we clearly have $F \in \mathcal{DM}^\infty(HU)$, $\chi_E \in BV(U)$ and $\varphi \in C(U) \cap L^\infty(U)$ with $\nabla_H \varphi \in L^1(HU)$. Hence, Theorem 4.1.1 implies that $\chi_E F \in \mathcal{DM}^\infty(HU)$ and so we can apply (4.2.8) to φ and $\chi_E F$, thus obtaining

$$\operatorname{div}(\varphi\chi_E F) = \varphi \operatorname{div}(\chi_E F) + \langle \chi_E F, \nabla_H \varphi \rangle \, \mu \tag{4.4.14}$$

in the sense of Radon measures on U. Now, Theorem 4.3.13 implies the existence of interior and exterior normal traces $\langle \mathcal{F}_i, \nu_E \rangle, \langle \mathcal{F}_e, \nu_E \rangle$ in $L^\infty(U; |D_H \chi_E|)$, and, by (4.3.51) and (4.3.52), we get

$$\operatorname{div}(\varphi\chi_E F) = \chi_{E^{1,F}} \varphi \operatorname{div} F + \varphi \langle \mathcal{F}_i, \nu_E \rangle |D_H \chi_E| + \chi_E \langle F, \nabla_H \varphi \rangle \, \mu, \tag{4.4.15}$$
$$\operatorname{div}(\varphi\chi_E F) = \chi_{E^{1,F} \cup \mathscr{F}_H E} \varphi \operatorname{div} F + \varphi \langle \mathcal{F}_e, \nu_E \rangle |D_H \chi_E| + \chi_E \langle F, \nabla_H \varphi \rangle \, \mu. \tag{4.4.16}$$

Finally, we evaluate (4.4.15) and (4.4.16) on U, and we employ Lemma 4.2.6 and the assumption that $\operatorname{supp}(\varphi\chi_E) \subset U$, so that (4.4.10) and (4.4.11) immediately follow. The estimates (4.4.12) and (4.4.13) follow from the restriction of F and χ_E to U and from (4.3.18) and (4.3.19). $\square$

Remark 4.4.9. We notice that the local statement of Theorem 4.4.8 shows that the field F needs not be essentially bounded on the whole set Ω, but only on an arbitrarily small neighborhood of $\mathscr{F}_H E$. In particular, let $\varepsilon > 0$ and $E \Subset \Omega$. We define

$$E_\varepsilon := \{x \in E : \operatorname{dist}(x, \mathscr{F}_H E) < \varepsilon\},$$
$$E^\varepsilon := \{x \in \Omega \setminus E : \operatorname{dist}(x, \mathscr{F}_H E) < \varepsilon\}.$$

Then, from (4.4.12) and (4.4.13) one can deduce that we have

$$\| \langle \mathcal{F}_i, \nu_E \rangle \|_{L^\infty(\mathscr{F}_H E; |D_H \chi_E|)} \le \inf_{\varepsilon > 0} \|F\|_{L^\infty(E_\varepsilon)},$$
$$\| \langle \mathcal{F}_e, \nu_E \rangle \|_{L^\infty(\mathscr{F}_H E; |D_H \chi_E|)} \le \inf_{\varepsilon > 0} \|F\|_{L^\infty(E^\varepsilon)}.$$

Indeed, it is enough to take the open set $U = E_\varepsilon \cup E^\varepsilon = \{x \in \Omega : \operatorname{dist}(x, \mathscr{F}_H E) < \varepsilon\}$ for some $\varepsilon > 0$ such that $U \Subset \Omega$, and then to pass to the infimum in ε.

As an application of the integration by parts formulas, one can generalize the classical Euclidean Green's identities to $C^1_H(\Omega)$ functions whose horizontal gradients are in $\mathcal{DM}^\infty_{\mathrm{loc}}(H\Omega)$.

In the spirit of Definition 1.3.8, we can define the distributional sub-Laplacian of a locally summable function $u : \Omega \to \mathbb{R}$ with horizontal gradient satisfying $\nabla_H u \in L^1_{\mathrm{loc}}(H\Omega)$ as the distribution

$$C^\infty_c(\Omega) \ni \phi \mapsto -\int_\Omega \langle \nabla_H u, \nabla_H \phi \rangle \, dx. \tag{4.4.17}$$

We shall denote the distributional sub-Laplacian of u by $\Delta_H u$ and, with a little abuse of notation, we shall use the same symbol to denote also the measurable function defining the distribution, whenever it exists. Arguing as in the paragraph after Remark 1.3.9, one can show that, if $u \in C^2_H(\Omega)$, then its distributional sub-Laplacian coincides with the pointwise sub-Laplacian, and so we can write

$$\Delta_H u = \sum_{j=1}^{\mathrm{m}} X_j^2 u.$$

Theorem 4.4.10. *Let $u \in C^1_H(\Omega)$ satisfy $\Delta_H u \in \mathcal{M}_{\mathrm{loc}}(\Omega)$ and let $E \subset \Omega$ be a set of locally finite h-perimeter in Ω. Then for each $v \in C_c(\Omega)$ with $\nabla_H v \in L^1(H\Omega)$, one has*

$$\int_{E^1_u} v \, d\Delta_H u = -\int_{\mathscr{F}_H E} v \, \langle \nabla_H u, \nu_E \rangle \, d|D_H \chi_E| - \int_E \langle \nabla_H v, \nabla_H u \rangle \, dx, \tag{4.4.18}$$

where $E^1_u := E^{1,\nabla_H u}$ is the measure theoretic interior of E with respect to $\nabla_H u$. If $u, v \in C^1_{H,c}(\Omega)$ also satisfy $\Delta_H u, \Delta_H v \in \mathcal{M}(\Omega)$, then we have

$$\int_{E^1_u} v\, d\Delta_H u - \int_{E^1_v} u\, d\Delta_H v = \int_{\mathscr{F}_H E} \langle u\nabla_H v - v\nabla_H u, \nu_E\rangle \, d|D_H\chi_E|, \tag{4.4.19}$$

where $E^1_v := E^{1,\nabla_H v}$, analogously as with u. If $E \Subset \Omega$, one can drop the assumption that u and v have compact support in Ω.

Proof. Arguing as in the proof of Theorem 4.4.8, we can localize to an open set $U \Subset \Omega$ such that $\mathrm{supp}(v\chi_E) \subset U$. Then, we notice that, since $u \in C^1_H(U)$ and $\Delta_H u \in \mathcal{M}(U)$, then $\nabla_H u \in \mathcal{DM}^\infty(HU) \cap C(HU)$. Thus, since E is a set of finite h-perimeter in U, the normal traces of $\nabla_H u$ on $\mathscr{F}_H E \cap U$ coincide with $\langle \nabla_H u(x), \nu_E(x)\rangle$ for $|D_H\chi_E|$-a.e. $x \in U$, by Proposition 4.4.6. In addition, Proposition 4.4.6 gives $|\Delta_H u|(\mathscr{F}_H E) = 0$, and so (4.4.10) implies (4.4.18), if we set $E^{1,\nabla_H u} =: E^1_u$.

If now $u, v \in C^1_{H,c}(\Omega)$ and satisfy $\Delta_H u, \Delta_H v \in \mathcal{M}(\Omega)$, one also has (4.4.18) with the roles of u and v interchanged, and thus with a set E^1_v uniquely determined by $\nabla_H v$, instead. Subtracting these two expressions leads to (4.4.19). If $E \Subset \Omega$, then the assumption on the compact support of u and v are not anymore needed. □

4.5 Absolutely continuous divergence-measure horizontal fields

This section is devoted to some first applications of our previous results, which cover the case of $F \in L^\infty(H\Omega)$ with $\mathrm{div}F \in L^1(\Omega)$, and consequently the cases $F \in W^{1,1}(H\Omega)$ and $F \in \mathrm{Lip}_{H,c}(H\Omega)$.

Theorem 4.5.1. *If $F \in \mathcal{DM}^\infty(H\Omega)$ and $|\mathrm{div}F| \ll \mu$, then for any set of finite h-perimeter $E \subset \Omega$, we have*

$$\mathrm{div}(\chi_E F) = \chi_E \mathrm{div}F + (F, D_H\chi_E) \tag{4.5.1}$$

in the sense of Radon measures on Ω. Therefore, we also obtain (4.4.5),

$$|\mathrm{div}F|(E^{1,F}\Delta E) = 0, \tag{4.5.2}$$

and

$$\mathrm{div}(\chi_E F) = \chi_E \mathrm{div}F + \langle \mathcal{F}, \nu_E\rangle\, |D_H\chi_E|. \tag{4.5.3}$$

Proof. Formula (4.5.1) is a simple application of (4.2.10) to $g = \chi_E$, taking into account (1.3.8). Since the absolute continuity assumption and (1.3.26) give $|\mathrm{div}F|(\mathscr{F}_H E) = 0$, we can apply Proposition 4.4.4 and obtain (4.4.5). In addition, by comparing (4.4.6) and (4.5.1), we get (4.5.2). Thus, (4.5.3) is an immediate consequence of (4.4.6) and (4.5.2). □

Thanks to Theorem 4.5.1, the proofs of Theorem 4.1.3 and Theorem 4.1.4 can be immediately achieved.

Proof of Theorem 4.1.3. We evaluate (4.5.3) on Ω and apply Lemma 4.2.6, thanks to the fact that $E \Subset \Omega$. □

Proof of Theorem 4.1.4. It suffices to combine (4.4.5), (4.5.2) and $|\mathrm{div}F|(\mathscr{F}_H E) = 0$ with Theorem 4.4.8. □

We notice now that Theorem 4.1.4 may be applied to a set of locally finite h-perimeter whose reduced boundary is not rectifiable in the Euclidean sense.

Example 4.5.2. We recall that a set $S \subset \mathbb{G}$ is called a C^1_H-regular surface if, for any $p \in S$, there exists an open set $U \ni p$ and a map $f \in C^1_H(U)$ such that

$$S \cap U = \{q \in U : f(q) = 0 \text{ and } \nabla_H f(q) \neq 0\}.$$

In [140, Theorem 3.1], the authors proved the existence of a C^1_H-regular surface S in the Heisenberg group $\mathbb{H}^1$ such that $\mathscr{H}^{\frac{5-\varepsilon}{2}}_{|\cdot|}(S) > 0$ for any $\varepsilon \in (0,1)$; which means that S is not 2-Euclidean rectifiable. In particular, they showed that there exists a function $f \in C^1_H(\mathbb{H}^1)$ related to S as above, with $U = \mathbb{H}^1$. From [81, Theorem 2.1], it is known that the open set $E = \{p \in \mathbb{H}^1 : f(p) < 0\}$ is of locally finite h-perimeter and $\mathscr{F}_H E = S$. Thus, given any $F \in \mathcal{DM}^\infty_{\rm loc}(H\mathbb{H}^1)$ such that $|{\rm div} F| \ll \mu = \mathscr{L}^3$, we can apply Theorem 4.1.4 to F and E to show that there exists a unique normal trace $\langle \mathcal{F}, \nu_E \rangle \in L^\infty_{\rm loc}(\mathbb{H}^1; |D_H \chi_E|)$. In addition, for any $\varphi \in C_c(\mathbb{H}^1)$ with $\nabla_H \varphi \in L^1(H\mathbb{H}^1)$ we obtain

$$\int_E \varphi \, d{\rm div} F + \int_E \langle F, \nabla_H \varphi \rangle \, dx = - \int_S \varphi \, \langle \mathcal{F}, \nu_E \rangle \, d|D_H \chi_E|.$$

We stress the fact that, on the right hand side, we are integrating on a fractal object, which is an Euclidean unrectifiable set.

Theorem 4.5.3. *Let $u \in C^1_H(\Omega)$ be such that $\Delta_H u \in \mathcal{M}_{\rm loc}(\Omega)$ with $|\Delta_H u| \ll \mu$ and let $E \subset \Omega$ be a set of locally finite h-perimeter in Ω. Then for each $v \in C_c(\Omega)$ with $\nabla_H v \in L^1(H\Omega)$ one has*

$$\int_E v \, d\Delta_H u = - \int_{\mathscr{F}_H E} v \, \langle \nabla_H u, \nu_E \rangle \, d|D_H \chi_E| - \int_E \langle \nabla_H v, \nabla_H u \rangle \, dx. \tag{4.5.4}$$

If $u, v \in C^1_{H,c}(\Omega)$ also satisfy $\Delta_H u, \Delta_H v \in \mathcal{M}(\Omega)$, $|\Delta_H u| \ll \mu, |\Delta_H v| \ll \mu$, one has

$$\int_E v \, d\Delta_H u - u \, d\Delta_H v = \int_{\mathscr{F}_H E} \langle u \nabla_H v - v \nabla_H u, \nu_E \rangle \, d|D_H \chi_E|. \tag{4.5.5}$$

If $E \Subset \Omega$, one can drop the assumption that u and v have compact support in Ω.

Proof. It suffices to combine the results of Theorem 4.4.10 with the fact that, in this case, $E^1_u = E^1_v = E$ up to μ-negligible sets, which follows from Theorem 4.5.1. □

It is worth noticing that one could weaken the absolute continuity assumption on ${\rm div} F$, by asking only $|{\rm div} F|(\partial^{*,\mathcal{R}}_H E) = 0$. We also notice that, in the Euclidean context, this resembles part of the hypotheses assumed by Degiovanni, Marzocchi and Musesti in [66, Theorem 5.2] and Schuricht in [138, Proposition 5.11]. However, we do not require the existence of any suitable smooth approximation of F, as they do: thus, our results are more general, even though we cannot represent the normal traces as the classical scalar product. Before stating our results, we need a preliminary lemma.

Lemma 4.5.4. *Let $F \in \mathcal{DM}^\infty(H\Omega)$ and let $E \subset \Omega$ be a set of finite h-perimeter such that* (4.3.42) *holds. Then, we have*

$$|{\rm div} F|(E^{1,F} \Delta E^{1,\mathcal{R}}) = 0 \quad \text{and} \quad |{\rm div} F|(E^{0,F} \Delta E^{0,\mathcal{R}}) = 0. \tag{4.5.6}$$

Proof. Thanks to Remark 4.3.8, we see that

$$|{\rm div} F| \left((E^{1,F} \Delta E^{1,\mathcal{R}}) \cap C^{\mathcal{R}}_E \right) = 0 \text{ and } |{\rm div} F| \left((E^{0,F} \Delta E^{0,\mathcal{R}}) \cap C^{\mathcal{R}}_E \right) = 0.$$

If (4.3.42) holds, then it is clear that we obtain (4.5.6). □

Proposition 4.5.5. *Let $F \in \mathcal{DM}^\infty(H\Omega)$ and let $E \subset \Omega$ be a set of finite h-perimeter such that $|\mathrm{div}F|(\partial_H^{*,\mathcal{R}} E) = 0$. Then we have* (4.4.5), (4.5.6) *and*

$$\mathrm{div}(\chi_E F) = \chi_{E^{1,\mathcal{R}}} \mathrm{div}F + \langle \mathcal{F}, \nu_E \rangle \, |D_H \chi_E|. \tag{4.5.7}$$

Proof. We notice that $\Omega \setminus C_E^{\mathcal{R}} \subset \partial_H^{*,\mathcal{R}} E$, therefore we can apply Lemma 4.5.4 to get (4.5.6). Thanks to (4.3.51) and (4.5.6), we easily get

$$\mathrm{div}(\chi_E F) = \chi_{E^{1,\mathcal{R}}} \mathrm{div}F + \langle \mathcal{F}_i, \nu_E \rangle \, |D_H \chi_E|. \tag{4.5.8}$$

In addition, (4.5.6) and $|\mathrm{div}F|(\partial_H^{*,\mathcal{R}} E) = 0$ imply that

$$\begin{aligned} |\mathrm{div}F|(\mathscr{F}_H E) &\le |\mathrm{div}F|(E^{1,\mathcal{R}} \cap \mathscr{F}_H E) + |\mathrm{div}F|(E^{0,\mathcal{R}} \cap \mathscr{F}_H E) \\ &= |\mathrm{div}F|(E^{1,F} \cap \mathscr{F}_H E) + |\mathrm{div}F|(E^{0,F} \cap \mathscr{F}_H E) = 0, \end{aligned}$$

since $E^{1,F}, E^{0,F} \subset \Omega \setminus \mathscr{F}_H E$ by Definition 4.3.14. As an immediate consequence of this fact and of (4.5.6), we may rewrite (4.3.52) as

$$\mathrm{div}(\chi_E F) = \chi_{E^{1,\mathcal{R}}} \mathrm{div}F + \langle \mathcal{F}_e, \nu_E \rangle \, |D_H \chi_E|. \tag{4.5.9}$$

Finally, by combining (4.5.8) and (4.5.9), the equalities (4.4.5) and (4.5.7) follow. $\square$

Theorem 4.5.6. *Let $F \in \mathcal{DM}^\infty(H\Omega)$ and let $E \Subset \Omega$ be a set of finite h-perimeter such that $|\mathrm{div}F|(\partial_H^{*,\mathcal{R}} E) = 0$. Then, we have*

$$\mathrm{div}F(E^{1,\mathcal{R}}) = - \int_{\mathscr{F}_H E} \langle \mathcal{F}, \nu_E \rangle \; d|D_H \chi_E|. \tag{4.5.10}$$

Proof. We just need to evaluate (4.5.7) on Ω and we apply Lemma 4.2.6 to get (4.5.10). $\square$

In analogy to the case $|\mathrm{div}F| \ll \mu$, we can obtain similar integration by parts formula and Green's identities in the case $|\mathrm{div}F|(\partial_H^{*,\mathcal{R}} E) = 0$, with $E^{1,\mathcal{R}}$ instead of E in the integration with respect to the divergence and the Laplacian measure, respectively.

Theorem 4.5.7. *Let $F \in \mathcal{DM}^\infty_{\mathrm{loc}}(H\Omega)$, E be a set of locally finite h-perimeter in Ω such that $|\mathrm{div}F|(\partial_H^{*,\mathcal{R}} E) = 0$, and let $\varphi \in C(\Omega)$ with $\nabla_H \varphi \in L^1_{\mathrm{loc}}(H\Omega)$ such that $\mathrm{supp}(\varphi \chi_E) \Subset \Omega$. Then there exists a unique normal trace $\langle \mathcal{F}, \nu_E \rangle \in L^\infty_{\mathrm{loc}}(\Omega; |D_H \chi_E|)$ of F, such that the following formula holds*

$$\int_{E^{1,\mathcal{R}}} \varphi \, d\mathrm{div}F + \int_E \langle F, \nabla_H \varphi \rangle \; dx = - \int_{\mathscr{F}_H E} \varphi \, \langle \mathcal{F}, \nu_E \rangle \; d|D_H \chi_E|.$$

Proof. Proposition 4.5.5 implies that, if $|\mathrm{div}F|(\partial_H^{*,\mathcal{R}} E) = 0$, then we have (4.4.5), (4.4.3) and (4.5.6). One needs just to combine these results with Theorem 4.4.8. $\square$

Theorem 4.5.8. *Let $u \in C^1_H(\Omega)$ satisfy $\Delta_H u \in \mathcal{M}_{\mathrm{loc}}(\Omega)$ and let $E \subset \Omega$ be a set of locally finite h-perimeter in Ω such that $|\Delta_H u|(\partial_H^{*,\mathcal{R}} E) = 0$. Then for each $v \in C_c(\Omega)$ with $\nabla_H v \in L^1(H\Omega)$ one has*

$$\int_{E^{1,\mathcal{R}}} v \, d\Delta_H u = - \int_{\mathscr{F}_H E} v \; \langle \nabla_H u, \nu_E \rangle \; d|D_H \chi_E| - \int_E \langle \nabla_H v, \nabla_H u \rangle \; dx.$$

If $u, v \in C^1_{H,c}(\Omega)$ also satisfy $\Delta_H u, \Delta_H v \in \mathcal{M}(\Omega)$, $|\Delta_H u|(\partial_H^{*,\mathcal{R}} E) = |\Delta_H v|(\partial_H^{*,\mathcal{R}} E) = 0$, one has

$$\int_{E^{1,\mathcal{R}}} v \, d\Delta_H u - u \, d\Delta_H v = \int_{\mathscr{F}_H E} \langle u \nabla_H v - v \nabla_H u, \nu_E \rangle \; d|D_H \chi_E|.$$

If $E \Subset \Omega$, one can drop the assumption that u and v have compact support in Ω.

Proof. It suffices to combine the results of Theorem 4.4.10 with the fact that, by Proposition 4.5.5, $E^1_u = E^1_v = E^{1,\mathcal{R}}$ up to $|\Delta_H u|, |\Delta_H v|$-negligible sets. $\square$

As an easy consequence of Theorem 4.3.6, we obtain the same locality property for the normal trace in the case $|\mathrm{div} F| \ll \mu$ and $|\mathrm{div} F|(\partial^{*,\mathcal{R}}_H E) = 0$.

Proposition 4.5.9. *Let $F \in \mathcal{DM}^\infty(H\Omega)$, and $E_1, E_2 \subset \Omega$ be sets of finite h-perimeter such that $\mathscr{S}^{Q-1}(\mathscr{F}_H E_1 \cap \mathscr{F}_H E_2) > 0$ and $|\mathrm{div} F|(\partial^{*,\mathcal{R}}_H E_j) = 0$, for $j = 1, 2$. Then, we have*

$$\langle \mathcal{F}, \nu_{E_1} \rangle (x) = \langle \mathcal{F}, \nu_{E_2} \rangle (x), \tag{4.5.11}$$

for $\mathscr{S}^{Q-1}$-a.e. $x \in \{y \in \mathscr{F}_H E_1 \cap \mathscr{F}_H E_2 : \nu_{E_1}(y) = \nu_{E_2}(y)\}$, and

$$\langle \mathcal{F}, \nu_{E_1} \rangle (x) = - \langle \mathcal{F}, \nu_{E_2} \rangle (x), \tag{4.5.12}$$

for $\mathscr{S}^{Q-1}$-a.e. $x \in \{y \in \mathscr{F}_H E_1 \cap \mathscr{F}_H E_2 : \nu_{E_1}(y) = -\nu_{E_2}(y)\}$.

Proof. The result follows immediately from Theorem 4.3.6 and (4.4.5), which holds by Proposition 4.5.5. $\square$

4.6 Applications to sets of Euclidean finite perimeter

The underlying linear structure of $\mathbb{G}$ allows for introducing an Euclidean scalar product, for instance using a fixed system of graded coordinates, see Section 1.2.2. With respect to this metric structure the classical sets of finite perimeter can be considered. We will call them sets of *Euclidean finite perimeter* to make a precise distinction with sets of finite h-perimeter.

If $E \subset \mathbb{G}$ is a set of locally finite Euclidean perimeter in $\Omega \subset \mathbb{G}$ and $F \in \mathcal{DM}^\infty_{\mathrm{loc}}(H\Omega)$, then we can refine (4.4.1) and (4.4.2) employing the theory of divergence-measure fields in Euclidean space. From the Euclidean Leibniz rule for essentially bounded divergence-measure fields ([82, Theorem 2.1] of Frid), the uniqueness of the representative $\widetilde{g}$ in Theorem 4.1.1 and of the pairing measure follows.

Recall that we can identify $\mathbb{G}$ with $\mathbb{R}^{\mathrm{q}}$, where q is the topological dimension of $\mathbb{G}$. In this section, we shall denote the Euclidean norm by $|\cdot|$, and the Riemannian one by $|\cdot|_g$. The L^∞-norm $\|\cdot\|_{\infty,\Omega}$ for horizontal fields is the same defined in (1.3.2) using $|\cdot|_g$.

We denote the Euclidean Hausdorff measure by $\mathscr{H}^\alpha_{|\cdot|}$ and the Euclidean ball by

$$B_{|\cdot|}(x, r) := \{y \in \mathbb{R}^{\mathrm{q}} : |x - y| < r\}.$$

Consequently, given $u \in L^1_{\mathrm{loc}}(\mathbb{G})$, we denote by

$$u^*_{|\cdot|}(x) := \begin{cases} \lim\limits_{r \to 0} \fint_{B_{|\cdot|}(x,r)} u(y)\, dy & \text{if the limit exists,} \\ 0 & \text{otherwise,} \end{cases} \tag{4.6.1}$$

the *Euclidean precise representative* of u.

The following useful lemma is a consequence of the rectifiability of the reduced boundary and of the negligibility of characteristic points [112]. Its proof can be found in [139]. For the ease of the reader, we add a short proof.

Lemma 4.6.1. *If E is a set of Euclidean locally finite perimeter in Ω and if we denote by $\mathscr{F}E$ the Euclidean reduced boundary, we have $\mathscr{S}^{Q-1}(\mathscr{F}_H E \Delta \mathscr{F} E) = 0$.*

Proof. By Theorem 1.3.16, Lemma 1.3.17 and [139, Proposition 5.11], we know that

$$|D_H \chi_E| = |\pi_H N_E| \mathscr{H}_{|\cdot|}^{\mathrm{q}-1} \llcorner \mathscr{F} E = \theta_E \mathscr{S}^{Q-1} \llcorner \mathscr{F}_H E, \tag{4.6.2}$$

where $\pi_H N_E$ is the projection of the Euclidean measure theoretic unit interior normal N_E on the horizontal bundle of $\mathbb{G}$. Hence, since $\theta_E \geq \alpha > 0$ by Theorem 1.3.16, we get

$$\mathscr{S}^{Q-1}(\mathscr{F}_H E \setminus \mathscr{F} E) = 0,$$

In addition, [139, Proposition 5.11] implies also that the set

$$\mathrm{Char}(E) := \{x \in \mathscr{F} E : \pi_H N_E(x) = 0\}$$

is $\mathscr{S}^{Q-1}$-negligible. Therefore, by (4.6.2) we have

$$\mathscr{H}_{|\cdot|}^{\mathrm{q}-1}(\mathscr{F} E \setminus (\mathscr{F}_H E \cup \mathrm{Char}(E))) = 0,$$

which implies $\mathscr{S}^{Q-1}(\mathscr{F} E \setminus (\mathscr{F}_H E \cup \mathrm{Char}(E))) = 0$ by [80, Proposition 4.4]. Since $\mathrm{Char}(E)$ is $\mathscr{S}^{Q-1}$-negligible, the proof is complete. □

We now recall the Euclidean Leibniz rule for essentially bounded divergence-measure fields in a stratified group.

Theorem 4.6.2. *Let $F \in \mathcal{DM}^\infty(H\Omega)$ and $g \in L^\infty(\Omega)$ be such that for every $j = 1, \dots, \mathrm{q}$ we have $\partial_{x_j} g \in \mathcal{M}(\Omega)$. It follows that $gF \in \mathcal{DM}^\infty(H\Omega)$ and*

$$\mathrm{div}(gF) = g^*_{|\cdot|} \mathrm{div} F + (F, Dg), \tag{4.6.3}$$

where (F, Dg) is the weak limit of $\langle F, D(\rho_\varepsilon \tilde{*} g)\rangle_{\mathbb{R}^{\mathrm{q}}}\, \mu$ as $\varepsilon \to 0$, denoting by $\tilde{*}$ the Euclidean convolution product, by $\rho \in C_c^\infty(B_{|\cdot|}(0,1))$ a radially symmetric mollifier with $\int \rho \, dx = 1$ and $\rho_\varepsilon(x) = \varepsilon^{-\mathrm{q}} \rho(x/\varepsilon)$.*

Proof. Since $F \in \mathcal{DM}^\infty(H\Omega) \subset \mathcal{DM}^\infty(\Omega)$ and g is an essentially bounded function of Euclidean bounded variation, [82, Theorem 2.1] shows that $gF \in \mathcal{DM}^\infty(\Omega)$ and that we have (4.6.3). Then we clearly have $gF \in \mathcal{DM}^\infty(H\Omega)$, since F is a measurable horizontal section. □

We stress the fact that $g \in BV(\Omega)$ in general does not imply $g \in BV_H(\Omega)$, unless the set Ω is bounded. Since a function of Euclidean bounded variation on Ω belongs only to $BV_{H,\mathrm{loc}}(\Omega)$, we shall need to localize all the following statements.

Theorem 4.6.3. *Let $F \in \mathcal{DM}^\infty(H\Omega)$ and $g \in L^\infty(\Omega)$ be such that $\partial_{x_j} g \in \mathcal{M}(\Omega)$ for $j = 1, \dots, \mathrm{q}$. Then, the measure (F, Dg) satisfies*

$$|(F, Dg)| \llcorner U \leq \|F\|_{L^\infty(U)} |D_H g| \llcorner U, \tag{4.6.4}$$

for any open bounded set $U \subset \Omega$.

Proof. Without loss of generality, we may assume Ω to be bounded, which means that we have $g \in L^\infty(\Omega) \cap BV_H(\Omega)$. By Theorem 4.6.2, we know that

$$\langle F, \nabla(\rho_\varepsilon \tilde{*} g)\rangle_{\mathbb{R}^{\mathrm{q}}}\, \mu \rightharpoonup (F, Dg)$$

as $\varepsilon \to 0$. By (1.3.12) one easily observes that $\langle F, \nabla(\rho_\varepsilon \tilde{*} g)\rangle_{\mathbb{R}^{\mathrm{q}}} = \langle F, \nabla_H(\rho_\varepsilon \tilde{*} g)\rangle$ and this means that

$$\langle F, \nabla_H(\rho_\varepsilon \tilde{*} g)\rangle\, \mu \rightharpoonup (F, Dg). \tag{4.6.5}$$

We notice that, for any $\phi \in C_c(\Omega)$, we have

$$\begin{aligned}\limsup_{\varepsilon\to 0}\left|\int_\Omega \phi \langle F, (\rho_\varepsilon \tilde{*} D_H g)\rangle \, dx\right| &\le \limsup_{\varepsilon\to 0} \|F\|_{L^\infty(\Omega)} \int_\Omega |\phi| \rho_\varepsilon \tilde{*} |D_H g| \, dx \\ &= \|F\|_{L^\infty(\Omega)} \int_\Omega |\phi| \, d|D_H g|,\end{aligned}$$

by well-known properties of Euclidean convolution of measures (see [11, Theorem 2.2]). Now we show that

$$\langle F, \nabla_H(\rho_\varepsilon \tilde{*} g)\rangle \, \mu - \langle F, (\rho_\varepsilon \tilde{*} D_H g)\rangle \, \mu \rightharpoonup 0. \tag{4.6.6}$$

Indeed, if (4.6.6) holds, then, for any $\phi \in C_c(\Omega)$, by (4.6.5) we have

$$\begin{aligned}\left|\int_\Omega \phi \, d(F, Dg)\right| &= \lim_{\varepsilon\to 0}\left|\int \phi \langle F, \nabla_H(\rho_\varepsilon \tilde{*} g)\rangle \, dx\right| \\ &\le \limsup_{\varepsilon\to 0}\left|\int \phi \langle F, \nabla_H(\rho_\varepsilon \tilde{*} g) - (\rho_\varepsilon \tilde{*} D_H g)\rangle \, dx\right| \\ &\quad + \limsup_{\varepsilon\to 0}\left|\int \phi \langle F, \rho_\varepsilon \tilde{*} D_H g\rangle \, dx\right| \\ &\le \|F\|_{L^\infty(\Omega)} \int_\Omega |\phi| \, d|D_H g|,\end{aligned}$$

which implies (4.6.4). Therefore, we need to show a commutation estimate. We recall the fact that $|a^i_j(x) - a^i_j(y)| \le C|x-y|$ on compact sets, for any $j = 1, \dots, \mathrm{m}$ and $i = \mathrm{m}+1, \dots, \mathrm{q}$. Hence, for any $x \in \Omega$ and $\varepsilon > 0$ such that $B_{|\cdot|}(x, \varepsilon) \subset \Omega$, the equality between the modulus of the sum

$$\left|\sum_{i=\mathrm{m}+1}^{\mathrm{q}} a^i_j(x)(\rho_\varepsilon \tilde{*} \partial_{y_i} g)(x) - \rho_\varepsilon \tilde{*} (a^i_j \partial_{y_i} g)(x)\right|$$

and its more explicit version

$$\left|\sum_{i=\mathrm{m}+1}^{\mathrm{q}} \int_{B_{|\cdot|}(x,\varepsilon)} (a^i_j(x) - a^i_j(y)) \rho_\varepsilon(x-y) \, d\partial_{y_i} g(y)\right|$$

leads us to the inequality

$$\left|\sum_{i=\mathrm{m}+1}^{\mathrm{q}} a^i_j(x)(\rho_\varepsilon \tilde{*} \partial_{y_i} g)(x) - \rho_\varepsilon \tilde{*} (a^i_j \partial_{y_i} g)(x)\right| \le C \|\rho\|_{L^\infty(B_{|\cdot|}(0,1))} \frac{|Dg|(B_{|\cdot|}(x,\varepsilon))}{\varepsilon^{\mathrm{q}-1}}. \tag{4.6.7}$$

We now take $\phi \in C_c(\Omega)$ and we employ the fact that $\partial_{x_j}(\rho_\varepsilon \tilde{*} g) = (\rho_\varepsilon \tilde{*} \partial_{x_j} g)$, for any $j = 1, \dots, \mathrm{q}$, to obtain

$$\begin{aligned}\left|\int_\Omega \langle \phi F, \nabla_H(\rho_\varepsilon \tilde{*} g) - (\rho_\varepsilon \tilde{*} D_H g)\rangle \, dx\right| &= \left|\int_\Omega \phi \sum_{j=1}^{\mathrm{m}} F_j \left(\sum_{i=\mathrm{m}+1}^{\mathrm{q}} a^i_j (\rho_\varepsilon \tilde{*} \partial_{x_i} g) - \rho_\varepsilon \tilde{*} (a^i_j \partial_{x_i} g)\right) dx\right| \\ &\le C \|F\|_{L^\infty(\Omega)} \|\rho\|_{L^\infty(B_{|\cdot|}(0,1))} \int_\Omega |\phi(x)| \frac{|Dg|(B_{|\cdot|}(x,\varepsilon))}{\varepsilon^{\mathrm{q}-1}} \, dx\end{aligned}$$

by (4.6.7). Let now $\varepsilon > 0$ be smal enough so that

$$\mathrm{supp}(\phi) \subset \Omega^\varepsilon := \{x \in \Omega : \mathrm{dist}_{|\cdot|}(x, \partial\Omega) > \varepsilon\}.$$

It follows that

$$\begin{aligned}\int_\Omega |\phi(x)| \frac{|Dg|(B_{|\cdot|}(x,\varepsilon))}{\varepsilon^{\mathrm{q}-1}} \, dx &= \int_{\Omega^\varepsilon} \int_{B_{|\cdot|}(x,\varepsilon)} |\phi(x)| \varepsilon^{1-q} \, d|Dg|(y) \, dx \\ &= \int_\Omega \int_{B_{|\cdot|}(y,\varepsilon)} |\phi(x)| \varepsilon^{1-q} \, dx \, d|Dg|(y) \\ &\le \mu\Big(B_{|\cdot|}(0,1)\Big) \, \varepsilon \, \|\phi\|_{L^\infty(\Omega)} |Dg|(\Omega).\end{aligned}$$

We finally conclude that

$$\int_\Omega \phi(x)\frac{|Dg|(B_{|\cdot|}(x,\varepsilon))}{\varepsilon^{\mathrm{q}-1}}\,dx \to 0.$$

All in all, (4.6.6) follows, and this ends the proof of (4.6.4).

□

Thanks to the Leibniz rule of Theorem 4.6.2 and to its refinement given in Theorem 4.6.3, we are able to obtain the Gauss–Green formulas for Euclidean sets of finite perimeter in stratified groups. Even though such results could be proved directly, using (4.6.4) and employing techniques very similar to those presented in [52, Theorems 3.2, 4.1, 4.2], we shall instead first show that, in the case of a set of Euclidean finite perimeter E, the pairing $(F, D_H\chi_E)$ defined in Theorem 4.1.1 actually coincides with $(F, D\chi_E)$ introduced in Theorem 4.6.2. Then, the Gauss–Green formulas will be just a consequence of Theorem 4.3.12.

Let us denote by $E^1_{|\cdot|}$ and $E^0_{|\cdot|}$ the Euclidean measure theoretic interior and exterior of a measurable set $E \subset \Omega$; that is,

$$E^1_{|\cdot|} = \left\{x \in \Omega : \lim_{r\to 0}\frac{\mu(B_{|\cdot|}(x,r)\cap E)}{\mu(B_{|\cdot|}(x,r))} = 1\right\},$$

$$E^0_{|\cdot|} = \left\{x \in \Omega : \lim_{r\to 0}\frac{\mu(B_{|\cdot|}(x,r)\cap E)}{\mu(B_{|\cdot|}(x,r))} = 0\right\}.$$

We recall now that, if E is a set of Euclidean finite perimeter and we denote by $\mathscr{F}E$ the Euclidean reduced boundary, then

$$\chi^*_{E,|\cdot|} = \chi_{E^1_{|\cdot|}} + \frac{1}{2}\chi_{\mathscr{F}E}, \tag{4.6.8}$$

see for instance [52, Lemma 2.13] and the references therein.

We proceed now to show that, in the case $g = \chi_E$ for an Euclidean set of finite perimeter E, the Euclidean Leibniz rule is equivalent to the group one.

Theorem 4.6.4. *Let $F \in \mathcal{DM}^\infty(H\Omega)$ and E be a set of Euclidean finite perimeter in Ω. Then we have $\chi_E F \in \mathcal{DM}^\infty(H\Omega)$,*

$$(F, D\chi_E) = (F, D_H\chi_E), \tag{4.6.9}$$

and

$$\mathrm{div}(\chi_E F) = \chi^*_{E,|\cdot|}\mathrm{div}F + (F, D_H\chi_E). \tag{4.6.10}$$

In addition, for any $\rho \in C_c(B(0,1))$ satisfying $\rho \geq 0$, $\rho(x) = \rho(-x)$, $\int_{B(0,1)}\rho(x)\,dx = 1$, we have $\rho_\varepsilon * \chi_E \overset{*}{\rightharpoonup} \chi^*_{E,|\cdot|}$ in $L^\infty(\Omega; |\mathrm{div}F|)$ and $\langle F, \nabla_H(\rho_\varepsilon * \chi_E)\rangle\,\mu \rightharpoonup (F, D\chi_E)$ in $\mathcal{M}(\Omega)$. In particular, $E^{1,F} = E^1_{|\cdot|}$ and $E^{0,F} = E^0_{|\cdot|}$, up to $|\mathrm{div}F|$-negligible sets.

Proof. It is easy to see that $\chi_E F \in L^\infty(H\Omega)$ and that (4.6.3) with $g = \chi_E$ yields us

$$\mathrm{div}(\chi_E F) = \chi^*_{E,|\cdot|}\mathrm{div}F + (F, D\chi_E), \tag{4.6.11}$$

which means $\chi_E F \in \mathcal{DM}^\infty(H\Omega)$. Notice that this fact would not follow directly from Theorem 4.1.1, since, in our assumptions, the h-perimeter of E is only locally finite. Let us assume now Ω to be bounded, so that E is a set of finite h-perimeter in Ω. By (4.3.10), we immediately obtain

$$\chi^*_{E,|\cdot|}\mathrm{div}F + (F, D\chi_E) = \widetilde{\chi_E}\mathrm{div}F + (F, D_H\chi_E), \tag{4.6.12}$$

where $\widetilde{\chi_E}$ and $(F, D_H\chi_E)$ are unique thanks to Theorem 4.3.13 and Remark 4.3.16. This means that

$$(F, D_H\chi_E) = \chi^*_{E,|\cdot|}\mathrm{div}F + (F, D\chi_E) - \widetilde{\chi_E}\mathrm{div}F. \tag{4.6.13}$$

Hence, if Ω is unbounded, we get (4.6.12) and (4.6.13) in the sense of Radon measures on any bounded open set $U \subset \Omega$. However, since $\mathrm{div}F \in \mathcal{M}(\Omega)$, $\widetilde{\chi_E}, \chi^*_{E,|\cdot|} \in L^\infty(\Omega; |\mathrm{div}F|)$ and $(F, D\chi_E) \in \mathcal{M}(\Omega)$, by Theorem 4.6.2, the right hand side of (4.6.13) is a finite Radon measure on Ω. Thus, it follows that $(F, D_H\chi_E) \in \mathcal{M}(\Omega)$, even if E is only a set of locally finite h-perimeter on Ω. Hence, (4.6.12) holds on the whole Ω. We recall now that, by Lemma 4.6.1, $\mathscr{S}^{Q-1}(\mathscr{F}_H E \Delta \mathscr{F} E) = 0$, which implies

$$|\mathrm{div}F|(\mathscr{F}_H E \Delta \mathscr{F} E) = 0, \tag{4.6.14}$$

by Theorem 4.2.7. Now, we employ (4.3.49) in order to rewrite (4.6.12) as

$$\left(\chi_{E^1_{|\cdot|}} - \chi_{E^{1,F}}\right) \mathrm{div}F = \frac{1}{2}\left(\chi_{\mathscr{F}_H E} - \chi_{\mathscr{F} E}\right) \mathrm{div}F + (F, D_H\chi_E) - (F, D\chi_E). \tag{4.6.15}$$

Thanks to (4.6.14), we have

$$\left(\chi_{\mathscr{F}_H E} - \chi_{\mathscr{F} E}\right) \mathrm{div}F = 0,$$

so that (4.6.15) reduces to

$$\left(\chi_{E^1_{|\cdot|}} - \chi_{E^{1,F}}\right) \mathrm{div}F = (F, D_H\chi_E) - (F, D\chi_E). \tag{4.6.16}$$

If we restrict (4.6.16) to $\mathscr{F}_H E$, we obtain

$$\chi_{E^1_{|\cdot|} \cap \mathscr{F}_H E} \mathrm{div}F = (F, D_H\chi_E) - (F, D\chi_E), \tag{4.6.17}$$

since $E^{1,F} \subset \Omega \setminus \mathscr{F}_H E$, by Definition 4.3.14, and $|(F, D_H\chi_E)|, |(F, D\chi_E)| \ll |D_H\chi_E|$, by (4.1.2) and (4.6.4). We notice that

$$\begin{aligned} |\mathrm{div}F|(E^1_{|\cdot|} \cap \mathscr{F}_H E) &= |\mathrm{div}F|(E^1_{|\cdot|} \cap (\mathscr{F}_H E \cap \mathscr{F} E)) + |\mathrm{div}F|(E^1_{|\cdot|} \cap (\mathscr{F}_H E \setminus \mathscr{F} E)) \\ &\le |\mathrm{div}F|(\mathscr{F}_H E \setminus \mathscr{F} E) = 0, \end{aligned}$$

by (4.6.14) and the fact that $E^1_{|\cdot|} \cap \mathscr{F} E = \emptyset$. Therefore, we obtain

$$\chi_{E^1_{|\cdot|} \cap \mathscr{F}_H E} \mathrm{div}F = 0,$$

so that (4.6.17) implies (4.6.9). Then, combining (4.6.12) and (4.6.9), we obtain

$$\left(\chi^*_{E,|\cdot|} - \widetilde{\chi_E}\right) \mathrm{div}F = 0,$$

which immediately yields $\widetilde{\chi_E}(x) = \chi^*_{E,|\cdot|}(x)$ for $|\mathrm{div}F|$-a.e. $x \in \Omega$. As a consequence, we get $|\mathrm{div}F|(E^{1,F} \Delta E^1_{|\cdot|}) = 0$ and $|\mathrm{div}F|(E^{0,F} \Delta E^0_{|\cdot|}) = 0$. $\square$

Remark 4.6.5. By Theorem 4.6.4, $\chi_E F, \chi_{\Omega\setminus E} F \in \mathcal{DM}^\infty(H\Omega)$ for any $F \in \mathcal{DM}^\infty(H\Omega)$ and any set E of Euclidean finite perimeter in Ω. This means that, by (4.6.9), we have

$$(\chi_E F, D\chi_E) = (\chi_E F, D_H\chi_E) \text{ and } (\chi_{\Omega\setminus E} F, D\chi_E) = (\chi_{\Omega\setminus E} F, D_H\chi_E).$$

Thus, we can define the normal traces of F on the reduced boundary of an Euclidean set of finite perimeter as in (4.3.5) and (4.3.6). We stress the fact that, thanks to Remark 4.3.16, the measures $(\chi_E F, D_H\chi_E)$ and $(\chi_{\Omega\setminus E} F, D_H\chi_E)$ do not depend on the vanishing sequence ε_k.

These results enable us to prove Gauss–Green formulas for sets of Euclidean finite perimeter, Theorem 4.1.5, extending [52, Theorem 4.2] to all geometries of stratified groups.

Proof of Theorem 4.1.5. If we choose $U \Subset \Omega$, it is clear that E is a set of finite Euclidean perimeter in U, and so of finite h-perimeter in U. By Theorem 4.6.4, we know that, up to $|\mathrm{div} F|$-negligible sets, $E^{1,F} = E^1_{|\cdot|}$. Then, (4.1.12), (4.1.13) and (4.1.14) follow immediately from (4.3.51), (4.3.52) and (4.3.59). The estimates on the normal traces are a consequence of (4.4.12) and (4.4.13) in Theorem 4.4.8, since assumptions imply that E is also a set of finite h-perimeter on any bounded open set of Ω. The same theorem shows that (4.1.15) and (4.1.16) are a consequence of (4.4.10) and (4.4.11), taking into account that $|\mathrm{div} F|(E^{1,F} \Delta E^1_{|\cdot|}) = 0$. Thus, we conclude the proof. □

Remark 4.6.6. The normal traces of F on the reduced boundary of an Euclidean set of finite perimeter E satisfy the same locality property stated in Theorem 4.3.6. As a byproduct, we have also provided an alternate proof of the locality of normal traces on reduced boundaries of Euclidean sets of finite perimeter. Such proof does not employ De Giorgi's blow-up theorem, which was essential in [52, Proposition 4.10].

Arguing as for Theorem 4.1.5, we can provide a generalization of Green's identities to stratified groups for sets of Euclidean locally finite perimeter, Theorem 4.1.6, which extends the result of [52, Proposition 4.5] to stratified groups.

Proof of Theorem 4.1.6. It suffices to combine the results of Theorem 4.4.10 with the case of a set of Euclidean finite perimeter. By Theorem 4.6.4, we know that, up to $|\Delta_H u|$-negligible sets, $E^1_u = E^1_{|\cdot|}$, and so we get (4.1.21). The same is clearly true up to $|\Delta_H v|$-negligible sets, and this concludes the proof. □

Chapter 5

Evolution problems for Radon measures

5.1 Introduction

In this chapter we describe a current research project with Luigi Ambrosio, Mark A. Peletier and Oliver Tse, concerning the modelling of dislocations in crystals (for which we refer for instance to [88, 89, 93, 94, 153]) and related systems of evolution equations for couples of nonnegative measures (μ_1, μ_2) of the following form

$$\begin{cases} \frac{d}{dt}\mu_1 &= \operatorname{div}(\mu_1 \nabla(V * \mu)) - \sigma \\ \frac{d}{dt}\mu_2 &= -\operatorname{div}(\mu_2 \nabla(V * \mu)) - \sigma \end{cases} \tag{5.1.1}$$

for $\mu = \mu_1 - \mu_2$, some interaction potential V, and some (possibly nonlinear) dissipation term σ depending on μ_1 and μ_2. Section 5.2 is devoted to a quick overview of the models for the dynamics of dislocations (borrowing especially from [89]) and of the previous works [18, 21, 115] by Ambrosio, Mainini and Serfaty on systems of evolution equations such as (5.1.1). Then, employing techniques from the theory of abstract gradient systems, we try to represent these evolution equations as the gradient flows of a given energy with respect to a suitable distance among couples of nonnegative measures. To this purpose, in Section 5.3 we outline the definition of a family of Hellinger-Kantorovich distances introduced by Liero, Mielke and Savaré in [108, 109]. After focusing on a particular case of such distances between couples of nonnegative Radon measures, $\mathrm{D}_{\mathbb{K}}$, we study its properties, focusing on its Benamou-Brenier formulation in Section 5.4. In particular, we give an alternative representation of $\mathrm{D}_{\mathbb{K}}$ in terms of the minimization of an action functional $\mathcal{A}$, we prove the existence of (weakly) continuous minimizing curves of measures which realize this minimum, we show that $\mathrm{D}_{\mathbb{K}}$ is indeed an (extended) distance on $\mathcal{M}_+(\mathbb{R}^n) \times \mathcal{M}_+(\mathbb{R}^n)$ and we prove that the convergence with respect to $\mathrm{D}_{\mathbb{K}}$ implies the narrow convergence between couples of nonnegative Radon measures. Then, Section 5.5 contains a state of the art description of our research concerning the first variation of $\mathrm{D}_{\mathbb{K}}$ under different types of perturbations. The final goal of this investigation would be to derive Euler-Lagrange equations for the distance $\mathrm{D}_{\mathbb{K}}$. Future research shall go in the direction of analyzing further properties of this Hellinger-Kantorovich distance, such as its dual representation, with the final aim to apply the classical methods of minimizing movements to prove the existence of solutions satisfying some type of energy dissipation equality. However, our investigations met an unexpected obstacle which we describe in Section 5.6: the local (or descending) slope of the self energy

$$\Phi_{\text{self}}(\mu_1, \mu_2) := \mu_1(\mathbb{R}^n) + \mu_2(\mathbb{R}^n)$$

is not lower semicontinuous with respect to the distance $\mathrm{D}_{\mathbb{K}}$, so that classical results from the theory of gradient flows do not seem to apply.

5.2 Models for the dynamics of the densities of dislocation

We start by giving a short summary of the models of evolution of dislocations in crystals, since a full account of it would go beyond the scope of this exposition (for a more detailed description of this framework, we refer to [89, 153]).

Dislocations can be defined as defects in an atomic lattice, and they play a central role in the theory of plastic deformation. In particular, in a three-dimensional lattice a dislocation is a line defect. Thanks to the periodicity of the atomic lattice structure, it is possible to represent straight parallel edge dislocations as points in the two-dimensional plane perpendicular to the dislocation line. The key idea is that the macroscopic plastic deformation is the result of the combined motion of a large quantity of dislocations: at the continuum-level, the effect of the dislocation is described by certain types of measures, the *densities of dislocation*. For this reason, historically, models of plastic deformation have been different according to the chosen scales.

A widely accepted model for the evolution of dislocations at the continuum level was introduced by Groma and Balogh in [93, 94]. The authors considered the evolution of positively and negatively oriented straight parallel edge dislocations in a three-dimensional periodic lattice, and denoted by ρ^+ and ρ^- their respective densities[1]. Considering the interaction between dislocations as controlled by some coupling interaction potential V and the presence of some external forces represented by a smooth external potential U, the Groma-Balogh equations are the following system:

$$\begin{cases} \frac{d}{dt}\rho^+ &= \operatorname{div}\left(\rho^+(\nabla V * (\rho^+ - \rho^-) + \nabla U)\right) \\ \frac{d}{dt}\rho^- &= -\operatorname{div}\left(\rho^-(\nabla V * (\rho^+ - \rho^-) + \nabla U)\right). \end{cases} \tag{5.2.1}$$

The purpose of [89,153] is to show rigorously that indeed is possible to pass from the discrete model of dislocations to the continuum one given by (5.2.1). We give here a short description of the formal argument.

Because of the periodicity, it is possible to consider the dislocations as points in the torus $\mathbb{T}^2$, on which we define the discrete elastic energy

$$\widetilde{E}_m(x;b) = \frac{1}{m^2}\sum_{i=1}^m \sum_{\substack{j=1 \\ j\neq i}}^m b_i b_j V(x_i - x_j) + \frac{1}{m}\sum_{i=1}^m b_i U(x_i),$$

where $x = (x_1, \dots, x_m) \in (\mathbb{T}^2)^m$ and $b = (b_1, \dots, b_m) \in \{\pm 1\}^m$ denote the position and the sign (orientation) of each dislocation. We assume now that the evolution of the dislocations is given by a gradient flow of the energy, namely,

$$\frac{dx(t)}{dt} = -m\nabla \widetilde{E}_m(x(t);b), \tag{5.2.2}$$

which is Orowan's relation. If now we pass to the framework of measures, we can define the empirical measures associated to the discrete densities of the positively and negatively oriented dislocations:

$$\mu_m^+ := \frac{1}{m}\sum_{\substack{i=1 \\ b_i=1}}^m \delta_{x_i}, \quad \mu_m^- := \frac{1}{m}\sum_{\substack{i=1 \\ b_i=-1}}^m \delta_{x_i}.$$

[1]We stress the fact that, in this context, $\rho^\pm$ do not denote the positive and negative part of some given Radon measure ρ.

It is not difficult to check that, at least formally, μ_m^+ and μ_m^- satisfy (5.2.1). Hence, if we assume that $\mu_m^\pm$ converge to some $\rho^\pm$ in a suitable weak sense as $m \to +\infty$, then these limit measures should satisfy (5.2.1) as well. The difficulties related to obtaining a rigorous proof of this convergence are caused by the singularity near the origin of the interaction potential V, which is a Green's type function. This issue is central to the paper [89], and we refer the interested reader to it.

We observe that the discrete evolution system (5.2.2) is a gradient flow, and, at least formally, $\rho^+(t), \rho^-(t)$ being curves of measures limits of discrete dislocations densities, it would seem possible to give an interpretation of (5.2.1) as the gradient flow of some free energy functional with respect to a Wasserstein-like type of distance. In order to avoid some complications, we shall consider the case of a constant external potential U, adding instead a dissipation term σ as in (5.1.1). Along this line of thought, in [18,21] a system of evolution equations very similar to (5.1.1), the Chapman-Rubinstein-Schatzman-E evolution model for superconductivity, has been studied by viewing it as a gradient flow on a particular space of measures in a suitable sense. More precisely, in [18] the authors investigated the case of real-valued signed measures, which is closer to the physical model, and thus they were forced to introduce new concepts of Wasserstein pseudo-distances for signed measures.

The physical mean field model for the evolution of the density of vortices in a type-II superconductor under the effect of an external magnetic field, derived formally by Chapman, Rubinstein and Schatzman in [39] (see also the work of E [68]), is the initial value problem[2]:

$$\begin{cases} \frac{d}{dt}\mu(t) - \operatorname{div}(\nabla h_{\mu(t)}|\mu(t)|) = 0 & \text{in } \ (0,+\infty)\times\Omega \\ \mu(0) = \mu_0 & \text{at } \ t = 0, \end{cases} \tag{5.2.3}$$

where Ω is a bounded open set with smooth boundary in $\mathbb{R}^2$ (or $\mathbb{R}^2$ itself under some additional assumptions), $\mu_0 \in \mathcal{M}(\overline{\Omega}) \cap H^{-1}(\Omega)$ is a signed Radon measure, and h_μ is given by the elliptic boundary value problem

$$\begin{cases} -\Delta h_\mu + h_\mu = \mu & \text{in } \ \Omega \\ h_\mu = 1 & \text{on } \ \partial\Omega. \end{cases} \tag{5.2.4}$$

In [21], the authors dealt with probability measures, and they showed that (5.2.3) can be seen as the gradient flow of the energy functional Φ_λ for the quadratic Wasserstein distance W_2, where

$$\Phi_\lambda(\mu) := \frac{\lambda}{2}|\mu|(\Omega) + \frac{1}{2}\int_\Omega |\nabla h_\mu|^2 + |h_\mu - 1|^2\, dx, \tag{5.2.5}$$

for some suitable $\lambda \geq 0$. Then, it is possible to use the *minimizing movement scheme* ([13,103]) in order to build a solution to this gradient flow in the Wasserstein space; that is, for any $\tau > 0$ we find by recursion a sequence of minimizers (μ_τ^k) for the functional

$$\nu \to \frac{1}{2\tau}W_2^2(\mu_\tau^{k-1}, \nu) + \Phi_\lambda(\nu),$$

we construct a piecewise constant interpolation in time, and then we pass to the limit as $\tau \to 0$.

The key idea of [18] is instead to consider signed measures, and to apply the minimizing movement scheme using a pseudo-distance, which is required to be lower semicontinuous and bounded from below by an actual distance. More precisely, if we let

$$\mu, \nu \in \mathcal{M}_{\kappa,M}(\overline{\Omega}) := \{\mu \in \mathcal{M}(\overline{\Omega}) : \mu(\overline{\Omega}) = \kappa, |\mu|(\overline{\Omega}) \leq M\},$$

such that $|\nu|(\overline{\Omega}) \leq |\mu|(\overline{\Omega})$, then it is possible to define the functional $\mathcal{W}_2^2(\nu, \mu)$ as

$$\inf\{W_2^2(\sigma_1, \mu^+) + W_2^2(\sigma_2, \mu^-) : \sigma_1, \sigma_2 \in \mathcal{M}^+(\overline{\Omega}), \sigma_1 - \sigma_2 = \nu, \sigma_1(\overline{\Omega}) = \mu^+(\overline{\Omega}), \sigma_2(\overline{\Omega}) = \mu^-(\overline{\Omega})\}, \tag{5.2.6}$$

[2]Unless otherwise stated, in this chapter the operators $\nabla, \operatorname{div}, \Delta$ are intended to be acting on the x variable.

where μ^+ and μ^- are the positive and negative part of the measure μ, so that $\mu = \mu^+ - \mu^-$ and $|\mu| = \mu^+ + \mu^-$. It is clear that this functional is not symmetric in general, unless $|\nu|(\overline{\Omega}) = |\mu|(\overline{\Omega})$, since in this latter case $\nu^+(\overline{\Omega}) = \mu^+(\overline{\Omega})$ and $\nu^-(\overline{\Omega}) = \mu^-(\overline{\Omega})$. Thus, in such a case the only couple (σ_1, σ_2) which we can choose is (ν^+, ν^-), and the pseudo-distance $\mathcal{W}_2$ reduces to

$$\mathcal{W}_2(\nu, \mu) = \sqrt{W_2^2(\nu^+, \mu^+) + W_2^2(\nu^-, \mu^-)},$$

which is the 2-Wasserstein distance on the product space $\mathcal{M}_\alpha^+(\overline{\Omega}) \times \mathcal{M}_\beta^+(\overline{\Omega})$, where $\alpha = \mu^+(\overline{\Omega})$, $\beta = \mu^-(\overline{\Omega})$ and

$$\mathcal{M}_\alpha^+(\overline{\Omega}) := \{\theta \in \mathcal{M}^+(\overline{\Omega}) : \theta(\overline{\Omega}) = \alpha\}. \tag{5.2.7}$$

One can also show that $\mathcal{W}_2$ does not satisfy the triangle inequality in general. However, these two properties are not really essential for the minimizing movement scheme, while it is relevant that $\mathcal{W}_2$ is lower semicontinuous with respect to weak convergence of measures. We refer to [18, 114] for investigations on other notions of pseudo-distances for signed Radon measures.

Then, following [18] we can consider the following discrete minimization problem: for a given $\mu \in \mathcal{M}_{\kappa,M}(\overline{\Omega})$ and $\tau > 0$, solve for

$$\min_{\nu \in \mathcal{M}_{\kappa,M}(\overline{\Omega}), |\nu|(\overline{\Omega}) \leq |\mu|(\overline{\Omega})} \Phi_\lambda(\nu) + \frac{1}{2\tau}\mathcal{W}_2^2(\nu, \mu). \tag{5.2.8}$$

The strategy is to consider a perturbed functional where the energy is given by

$$\Phi_\lambda^\delta(\nu) = \Phi_\lambda(\nu) + \delta \int_\Omega (\hat{\nu})^4 \, dx, \tag{5.2.9}$$

for $\delta > 0$, where $\hat{\nu} := \chi_\Omega \nu$ and we identify ν with its density when $\nu \ll \mathscr{L}^2$, with a little abuse of notation. We also let $\tilde{\nu} := \chi_{\partial\Omega}\nu$. It is then proved that the perturbed minimization problem admits a solution μ_τ^δ and that any limit point as $\delta \to 0$ is a solution to (5.2.8). The Euler-Lagrange equations for the perturbed problem allow us to show, by an entropy argument, that, if the initial datum is L^p regular enough ($p \geq 4$), then the solution to the perturbed and unperturbed problem preserves that L^p regularity.

Now, in order to deal with the boundary, the key idea of the discrete time minimization is to start by considering the interior part of the previous minimum and then adding its boundary part to the new step. Given a time step $\tau > 0$ and an initial datum $\mu_\tau^0 = \mu^0 \in \mathcal{M}_{\kappa,M}(\overline{\Omega}) \cap H^{-1}(\Omega)$, assuming that the k-step μ_τ^k has already been defined, we set ν_τ^{k+1} to be the minimizer of

$$\min_{\nu \in \mathcal{M}_{\kappa',M}(\overline{\Omega}), |\nu|(\overline{\Omega}) \leq |\hat{\mu}_\tau^k|(\overline{\Omega})} \Phi_\lambda(\nu) + \frac{1}{2\tau}\mathcal{W}_2^2(\nu, \hat{\mu}_\tau^k),$$

where $\kappa' = \hat{\mu}_\tau^k(\Omega)$. Then we let $\mu_\tau^{k+1} := \nu_\tau^{k+1} + \tilde{\mu}_\tau^k \in \mathcal{M}_{\kappa,M}(\overline{\Omega})$ and we define the piecewise constant interpolation $\bar{\mu}_\tau(t) := \mu_\tau^{\lceil t/\tau \rceil}$ for any $t \geq 0$. After having found a discrete $C^{0,1/2}$ estimate, we see that there exists a sequence $\tau_n \to 0$ and a measure $\mu(t) \in \mathcal{M}_{\kappa,M}(\overline{\Omega})$ such that $\bar{\mu}_{\tau_n}(t)$ converges weakly to $\mu(t)$, preserving the uniform bound in $L^4(\Omega)$ if $\mu^0 \in L^4(\Omega)$.

Finally, calculation made 'by hand', employing again the Euler-Lagrange equations for the discrete minimization problem, allowed the authors to prove their main result, [18, Theorem 1.1].

Theorem 5.2.1. *Let $\mu^0 \in L^4(\Omega)$. The minimizing movement scheme produces a signed measure $\mu(t) \in L^4(\Omega)$ which satisfies $\mu(0) = \mu^0$ and*

$$\frac{d}{dt}\mu(t) - \operatorname{div}(\chi_\Omega \nabla h_{\mu(t)} \rho(t)) = 0 \quad \textit{in the duality with } C_c^\infty((0, +\infty) \times \mathbb{R}^2), \tag{5.2.10}$$

where $\rho(t)$ is a suitable positive measure satisfying $\rho(t) \geq |\mu(t)|$ in Ω.

More precisely, there exist two positive measures on $\overline{\Omega}$, $\rho_1(t), \rho_2(t)$, and a positive measure $\sigma = \sigma_\mu$ on $(0, +\infty) \times \overline{\Omega}$ such that $\mu(t) = \rho_1(t) - \rho_2(t)$, $\rho(t) = \rho_1(t) + \rho_2(t)$ and

$$\begin{cases} \frac{d}{dt}\rho_1(t) - \operatorname{div}(\chi_\Omega \nabla h_{\mu(t)}\rho_1(t)) & = -\sigma(t) \\ \frac{d}{dt}\rho_2(t) + \operatorname{div}(\chi_\Omega \nabla h_{\mu(t)}\rho_2(t)) & = -\sigma(t) \end{cases} \quad \textit{in the duality with } C_c^\infty((0,+\infty) \times \mathbb{R}^2). \tag{5.2.11}$$

It is possible to see that, if μ_0 is positive, then $\mu(t)$ is positive and $\rho(t) = \mu(t)$ (and so, $\rho_1(t) = \mu(t)$, $\rho_2(t) = 0$), coherently with the results stated in [21]. It is conjectured that this scheme could be improved to obtain $\rho(t) = |\mu(t)|$, so that $\rho_1(t) = \mu^+(t)$ and $\rho_2(t) = \mu^-(t)$. However, up to now it is not known whether ρ_1 and ρ_2 are orthogonal in the general case. On the other hand, the measure σ takes into account the mass cancellation, but it still lacks unfortunately an explicit expression.

A first attempt to characterize σ could be to ask for no mass cancellation (or vortex annihilation, in the Ginzburg-Landau model framework). This idea has been carried out in [115], where couples of positive measures with fixed mass and finite second moment are considered instead of signed measures.

We summarize this approach as follows. Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary (or $\Omega = \mathbb{R}^2$) and let us define

$$\mathcal{M}^2_\alpha(\overline{\Omega}) := \left\{ \mu \in \mathcal{M}^+(\overline{\Omega}) : \mu(\overline{\Omega}) = \alpha, \int_{\overline{\Omega}} |x|^2 \, d\mu(x) < \infty \right\}.$$

Given an initial datum

$$(\mu_1^0, \mu_2^0) \in \mathcal{M}^2_\alpha(\overline{\Omega}) \times \mathcal{M}^2_\beta(\overline{\Omega}),$$

for some $\alpha, \beta \geq 0$, such that $\chi_\Omega(\mu_1^0 - \mu_2^0) \in H^{-1}(\Omega)$, we want to find a couple of measures $(\mu_1(t), \mu_2(t))$ which is a solution, in the duality with $C_c^\infty((0, +\infty) \times \mathbb{R}^2)$, to

$$\begin{cases} \frac{d}{dt}\mu_1(t) - \operatorname{div}(\chi_\Omega \nabla h_{\mu(t)}\mu_1(t)) & = 0 \\ \frac{d}{dt}\mu_2(t) + \operatorname{div}(\chi_\Omega \nabla h_{\mu(t)}\mu_2(t)) & = 0 \\ \mu(t) = \mu_1(t) - \mu_2(t), \end{cases} \tag{5.2.12}$$

where $h_{\mu(t)}$ is the solution, for any $t > 0$, to

$$\begin{cases} -\Delta h_{\mu(t)} = \mu(t) & \text{in } \Omega \\ h_{\mu(t)} = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.2.13}$$

which is a variant of (5.2.4). In this setting, the energy functional is defined has

$$(\mu_1, \mu_2) \to \Phi(\mu) := \frac{1}{2} \int_\Omega h_\mu \, d\mu, \tag{5.2.14}$$

where $\mu = \mu_1 - \mu_2 \in \mathcal{M}(\overline{\Omega})$. It is clear that Φ does not see possible overlappings of μ_1 and μ_2, since it depends only on the difference. We can also show that, if $\chi_\Omega \mu \in H^{-1}(\Omega)$, an integration by parts yields

$$\Phi(\mu) = \frac{1}{2} \int_\Omega |\nabla h_\mu|^2 \, dx,$$

obtaining in this way strict convexity, nonnegativity and lower semicontinuity for Φ. In the case $\Omega = \mathbb{R}^2$, it is possible to obtain analogous properties with a slightly more complex argument.

In order to proceed, we apply the same minimizing movement scheme used in [18], with the simplification that now $\mathcal{W}_2$ is just the 2-Wasserstein distance on the product space $\mathcal{M}^2_\alpha(\overline{\Omega}) \times$

$\mathcal{M}^2_\beta(\overline{\Omega})$. Therefore, given initial data $(\mu_1^0,\mu_2^0)\in\mathcal{M}^2_\alpha(\Omega)\times\mathcal{M}^2_\beta(\Omega)$ with finite energy and second moment (and no boundary part), and a time step $\tau>0$, we need to find $(\mu_1)_\tau$ and $(\mu_2)_\tau$ which are solution to

$$\min_{(\nu_1,\nu_2)\in\mathcal{M}^2_\alpha(\overline{\Omega})\times\mathcal{M}^2_\beta(\overline{\Omega})}\Phi(\nu_1-\nu_2)+\frac{1}{2\tau}(W_2^2(\nu_1,\mu_1^0)+W_2^2(\nu_2,\mu_2^0)). \tag{5.2.15}$$

Arguing again very similarly to [18], in [115] it is proved that L^p regularity, for $p\geq 4$, is preserved passing from the perturbed problem to the unperturbed one, where the perturbed energy is given by

$$\Phi^\delta(\mu)=\Phi(\mu)+\delta\int_\Omega(\hat{\mu}_1)^4\,dx+\delta\int_\Omega(\hat{\mu}_2)^4\,dx.$$

Thanks to the Euler-Lagrange equations for the perturbed minimization problem, it is possible to obtain entropy estimates as in [18] and then to show that the weak limit of the piecewise constant interpolation is indeed a solution of (5.2.12). In order to build such interpolation, we consider initial data $(\mu_1^0,\mu_2^0)\in\mathcal{M}^+_\alpha(\overline{\Omega})\times\mathcal{M}^+_\beta(\overline{\Omega})$ with finite energy and second moment, and a time step $\tau>0$, and we find recursively $(\nu_1)^k_\tau$ and $(\nu_2)^k_\tau$ which are solution to

$$\min_{(\nu_1,\nu_2)\in\mathcal{M}^+_{\alpha_k}(\overline{\Omega})\times\mathcal{M}^-_{\beta_k}(\overline{\Omega})}\Phi(\nu_1-\nu_2)+\frac{1}{2\tau}(W_2^2(\nu_1,(\hat{\mu}_1)^{k-1}_\tau)+W_2^2(\nu_2,(\hat{\mu}_2)^{k-1}_\tau)), \tag{5.2.16}$$

where $\alpha_k=(\hat{\mu}_1)^{k-1}_\tau(\Omega)\leq\alpha$ and $\beta_k=(\hat{\mu}_2)^{k-1}_\tau(\Omega)\leq\beta$. We then let

$$(\mu_i)^k_\tau=(\nu_i)^k_\tau+(\tilde{\mu}_i)^{k-1}_\tau,\quad\text{for } i=1,2.$$

After having constructed a couple of piecewise constant interpolations as customary, it is possible to pass to the limit for a suitable subsequence τ_n, thus finding a couple $(\mu_1(t),\mu_2(t))$ which is the weak limit of $((\bar{\mu}_1)_{\tau_n}(t),(\bar{\mu}_2)_{\tau_n}(t))$. In this way, the following theorem, the main result [115, Theorem 1.1], is proved.

Theorem 5.2.2. *Let $(\mu_1^0,\mu_2^0)\in\mathcal{M}^+_\alpha(\overline{\Omega})\times\mathcal{M}^+_\beta(\overline{\Omega})$ be such that $\chi_\Omega\mu_i^0\in L^p(\Omega)$, $p\geq 4$ and $i=1,2$. Then there exists a weakly continuous map $(\mu_1(t),\mu_2(t))$ on $[0,+\infty)$, uniformly bounded in $L^p(\Omega)$, satisfying $\mu_i(0)=\mu_i^0$, $i=1,2$ and (5.2.12), where $\mu(t)=\mu_1(t)-\mu_2(t)$. In addition, the following energy dissipation equality holds:*

$$\Phi(\mu(t))+\int_s^t\int_\Omega|\nabla h_{\mu(r)}|^2\,d(\mu_1(r)+\mu_2(r))\,dr=\Phi(\mu(s)) \tag{5.2.17}$$

for any $t\geq s\geq 0$.

It is plain to see that Theorem 5.2.2 is not affected by any of the uncertainties of Theorem 5.2.1, as a consequence of the absence of the dissipation term in (5.2.12). However, we want to deal with the general case (5.1.1) for $\sigma\neq 0$: to this purpose, in the following section we consider new types of distances between couples of nonnegative finite Radon measures.

5.3 The Hellinger-Kantorovich distance

Following the works [108, 109], it is possible to redefine the framework of the minimizing movement scheme itself, by passing from the product Wasserstein distance (or one of the possible pseudodistances considered in [114]) to a version of the Hellinger-Kantorovich distance, defined through an Onsager operator $\mathbb{K}$.

The Onsager operators appear in the context of abstract gradient systems. A triple $(X,\mathcal{F},\Psi)$ is called a gradient system if X is a Banach space, if the functional $\mathcal{F}:X\to\mathbb{R}\cup\{\infty\}$ has a Fréchet subdifferential $D\mathcal{F}(u)\in X^*$ well defined on a suitable subset of X, and if the dissipation potential Ψ satisfies the following properties:

- for any $u \in X$, $\Psi(u, \cdot) : X \to [0, +\infty]$ is lower semicontinuous and convex;
- $\Psi(u, 0) = 0$.

In classical gradient systems, $\Psi(u, \cdot)$ is a quadratic form; that is, there exists a symmetric and positive (semi-)definite operator $\mathbb{G}(u) : X \to X^*$ (called the Riemannian operator) such that

$$\Psi(u, v) = \frac{1}{2}\langle \mathbb{G}(u)v, v\rangle$$

for any $u, v \in X$.

If we denote by $\Psi^*(u, \cdot)$ the Legendre-Fenchel transform of the dissipation potential, then we can define the Onsager operator $\mathbb{K}$ as the symmetric and positive (semi-)definite operator $\mathbb{K}(u) : X \to X^*$ such that

$$\Psi^*(u, v) = \frac{1}{2}\langle \mathbb{K}(u)v, v\rangle$$

for any $u, v \in X$. It is possible to see that $\mathbb{K}(u) = \mathbb{G}(u)^{-1}$ and $\mathbb{K}(u)^{-1} = \mathbb{G}(u)$.

In this setting, the gradient evolution is given by the equation

$$\frac{d}{dt}u = -\mathbb{K}(u)D\mathcal{F}(u).$$

In our context[3],

$$X = \left\{(\mu_1, \mu_2) \in \mathcal{M}_+(\mathbb{R}^n) \times \mathcal{M}_+(\mathbb{R}^n) : \int_{\mathbb{R}^n} |x|^2 \, d\mu_i(x) < \infty, i = 1, 2\right\},$$

and, letting $\mu := \mu_1 - \mu_2$, we consider as the free energy

$$\Phi(\mu_1, \mu_2) := \frac{1}{2}\int_{\mathbb{R}^n} (V * \mu)\, d\mu + \mu_1(\mathbb{R}^n) + \mu_2(\mathbb{R}^n), \tag{5.3.1}$$

which is the sum of an interaction energy, for some suitably regular potential V, and the so-called self energy. If $n = 2$, the interaction part can be for instance chosen to be the standard Dirichlet energy, as in [115],

$$\int_{\mathbb{R}^2} h_\mu(x)\, d\mu(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log|x - y|\, d\mu(y)\, d\mu(x)$$

for h_μ defined as in (5.2.13) with $\Omega = \mathbb{R}^2$ and no boundary conditions, and $\mu \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ with finite second moment. As for the second term in the energy, it has been conjectured to be the Γ-limit of a sequence of discrete energies arising from dislocation models, under some suitable assumptions (see [153, Conjecture 5.5.1]).

Then, we consider a type of Onsager operator which arises in reaction-diffusion systems (for a detailed study, we refer to [122]):

$$\mathbb{K}(\mu_1, \mu_2)\xi := -\mathrm{div}(\mathbb{M}(\mu_1, \mu_2)\nabla\xi) + \mathbb{H}(\mu_1, \mu_2)\xi, \tag{5.3.2}$$

where $\mathbb{M}$ is a symmetric and positive definite matrix and $\mathbb{H}$ is the reaction matrix.

We choose

$$\mathbb{M}(\mu_1, \mu_2) := \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} \tag{5.3.3}$$

[3]For the purpose of modelling the dynamics of the densities of dislocations it would be enough to consider the dimension $n = 2$; however, the abstract gradient flow structure allows us to deal directly with $\mathbb{R}^n$.

and

$$\mathbb{H}(\mu_1,\mu_2) := \frac{1}{2} f(\mu_1,\mu_2) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \tag{5.3.4}$$

where $f : [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is a continuous function such that $f(x,y) = f(y,x)$ and $f(x,0) = f(0,y) = 0$.

Now, we see that, at least formally, the Fréchet differential of Φ is given by

$$D\Phi(\mu_1,\mu_2) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} (V * \mu) + \begin{pmatrix} \chi_{\{\mu_1>0\}} \\ \chi_{\{\mu_2>0\}} \end{pmatrix}. \tag{5.3.5}$$

Hence, we have that the (formal) gradient system evolution is given by

$$\begin{cases} \frac{d}{dt}\mu_1 & = \operatorname{div}(\mu_1 \nabla((V*\mu) + \chi_{\{\mu_1>0\}})) - \frac{1}{2} f(\mu_1,\mu_2)(\chi_{\{\mu_1>0\}} + \chi_{\{\mu_2>0\}}) \\ \frac{d}{dt}\mu_2 & = \operatorname{div}(\mu_2 \nabla(-(V*\mu) + \chi_{\{\mu_2>0\}})) - \frac{1}{2} f(\mu_1,\mu_2)(\chi_{\{\mu_1>0\}} + \chi_{\{\mu_2>0\}}), \end{cases}$$

which reduces to

$$\begin{cases} \frac{d}{dt}\mu_1 & = \operatorname{div}(\mu_1 \nabla(V*\mu)) - f(\mu_1,\mu_2) \\ \frac{d}{dt}\mu_2 & = -\operatorname{div}(\mu_2 \nabla(V*\mu)) - f(\mu_1,\mu_2), \end{cases} \tag{5.3.6}$$

since clearly $f(\mu_1,\mu_2) \neq 0$ only if $\mu_1 > 0$ and $\mu_2 > 0$, and, at least formally, the distribution $D\chi_{\{\mu_i>0\}}$ is supported on the set $\{\mu_i = 0\}$; thus $\mu_i D\chi_{\{\mu_i>0\}} = 0$. In this way, we see that the diffusion part of $\mathbb{K}$ deals only with the interaction energy, while the reaction part only with the self energy: therefore, there is a decoupling, as if the two parts of the Onsager operator and the energy have orthogonal roles.

Since $\mathbb{K}$ is the inverse of the metric tensor $\mathbb{G}$, it can be used to define a Hellinger-Kantorovich distance (for a detailed exposition, we refer to [108]) as

$$\begin{aligned} \mathrm{D}^2_{\mathbb{K}}((\nu_1,\nu_2),(\mu_1,\mu_2)) := \inf \Big\{ & \int_0^1 \int_{\mathbb{R}^n} \nabla\xi : \mathbb{M}(\rho_1,\rho_2)\nabla\xi + \xi \cdot \mathbb{H}(\rho_1,\rho_2)\xi \, ds, \\ & \frac{d}{ds}\begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = -\operatorname{div}\left(\mathbb{M}(\rho_1,\rho_2)\nabla\xi\right) + \mathbb{H}(\rho_1,\rho_2)\xi, \\ & \rho_i(0) = \mu_i, \rho_i(1) = \nu_i, i = 1,2 \Big\}. \end{aligned}$$

The action minimization which defines this distance can be rewritten slightly more explicitly in the following way:

$$\begin{aligned} \mathrm{D}^2_{\mathbb{K}}((\nu_1,\nu_2),(\mu_1,\mu_2)) := \inf \Big\{ & \int_0^1 \int_{\mathbb{R}^n} \rho_1|\nabla\xi_1|^2 + \rho_2|\nabla\xi_2|^2 + \frac{1}{2} f(\rho_1,\rho_2)(\xi_1+\xi_2)^2 \, ds, \\ & \frac{d}{ds}\begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = -\operatorname{div}\begin{pmatrix} \rho_1\nabla\xi_1 \\ \rho_2\nabla\xi_2 \end{pmatrix} + \frac{1}{2} f(\rho_1,\rho_2)\begin{pmatrix} \xi_1+\xi_2 \\ \xi_1+\xi_2 \end{pmatrix}, \\ & \rho_i(0) = \mu_i, \rho_i(1) = \nu_i, i = 1,2 \Big\}. \end{aligned} \tag{5.3.7}$$

In order to derive rigorously the existence of a solution to (5.3.6), we can use this distance and the energy Φ, and then apply the machinery of the minimizing movements scheme. To this purpose, we first study more in detail the properties of a slightly modified version of the distance $\mathrm{D}_{\mathbb{K}}$.

5.4 The Benamou-Brenier formulation of $\mathrm{D}_{\mathbb{K}}$

We define a distance of Hellinger-Kantorovich type for couples of measures

$$(\mu_1, \mu_2), (\nu_1, \nu_2) \in \mathcal{M}_+(\mathbb{R}^n) \times \mathcal{M}_+(\mathbb{R}^n).$$

To this purpose, we consider a continuous concave function $f : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ satisfying

$$f(x, y) = f(y, x) > 0 \text{ for } x, y > 0, f(x, 0) = f(0, y) = 0 \quad \text{and} \quad f(x, y) \le C(x + y), \tag{5.4.1}$$

for some constant $C > 0$; and we denote by f_∞ its recession function, given by

$$f_\infty(x, y) := \lim_{z \to +\infty} \frac{f(zx, zy)}{z}.$$

Clearly, $f_\infty(0, y) = f_\infty(x, 0) = 0$ and $0 \le f_\infty(x, y) \le C(x + y)$.

Then, we set

$$\mathrm{D}^2_{\mathbb{K}}((\nu_1, \nu_2), (\mu_1, \mu_2)) := \inf \left\{ \int_0^1 \left(\int_{\mathbb{R}^n} |v_1|^2 \, d\rho_{1,t} + |v_2|^2 \, d\rho_{2,t} + \frac{|\xi|^2}{2} \, d\mathbf{f}(\rho_{1,t}, \rho_{2,t}) \right) dt, \right. \tag{5.4.2}$$

$$\frac{d}{dt} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = -\mathrm{div} \begin{pmatrix} v_1 \rho_1 \\ v_2 \rho_2 \end{pmatrix} + \frac{\xi}{2} \mathbf{f}(\rho_1, \rho_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{5.4.3}$$

$$(\rho_1, \rho_2) \in C([0, 1]; \mathcal{M}_+(\mathbb{R}^n) \times \mathcal{M}_+(\mathbb{R}^n)), \tag{5.4.4}$$

$$\left. \rho_{i,0} = \nu_i, \rho_{i,1} = \mu_i, i = 1, 2 \right\}, \tag{5.4.5}$$

where $\mathbf{f} : \mathcal{M}_+(\mathbb{R}^n) \times \mathcal{M}_+(\mathbb{R}^n) \to \mathcal{M}_+(\mathbb{R}^n)$ is the map given by

$$\mathbf{f}(\rho_1, \rho_2) = f\left(\frac{d\rho_1^{\mathrm{ac}}}{dx}, \frac{d\rho_2^{ac}}{dx} \right) dx + f_\infty \left(\frac{d\rho_1^{\mathrm{s}}}{d|(\rho_1^{\mathrm{s}}, \rho_2^{\mathrm{s}})|}, \frac{d\rho_2^{\mathrm{s}}}{d|(\rho_1^{\mathrm{s}}, \rho_2^{\mathrm{s}})|} \right) d|(\rho_1^{\mathrm{s}}, \rho_2^{\mathrm{s}})|.$$

We stress the fact that (5.4.3) and (5.4.5) have to be intended in a distributional sense; that is,

$$\begin{aligned} -\int_0^1 \int_{\mathbb{R}^n} \frac{\partial \varphi(t, x)}{\partial t} \, d\rho_{i,t}(x) \, dt = \int_0^1 \int_{\mathbb{R}^n} \nabla \varphi(t, x) \cdot v_i(t, x) \, d\rho_{i,t}(x) \, dt + \\ + \int_{\mathbb{R}^n} \varphi(t, x) \frac{\xi(t, x)}{2} \, d\mathbf{f}(\rho_{1,t}(x), \rho_{2,t}(x)) \, dt + \\ + \int_{\mathbb{R}^n} \varphi(0, x) \, d\nu_i(x) - \int_{\mathbb{R}^n} \varphi(1, x) \, d\mu_i(x) \end{aligned} \tag{5.4.6}$$

for $i = 1, 2$ and any $\varphi \in C_c^\infty([0, 1] \times \mathbb{R}^n)$. Actually, thanks to a simple regularization argument via convolution, it is possible to show that the test functions φ may be taken in $C_c^1([0, 1] \times \mathbb{R}^n)$. In addition, if we select test functions of the form $\varphi(t, x) = \eta(t)\zeta(x)$, for some $\eta \in C_c^\infty([0, 1])$ and $\zeta \in C_c^\infty(\mathbb{R}^n)$, then (5.4.6) reduces to

$$\frac{d}{dt} \int_{\mathbb{R}^n} \zeta(x) \, d\rho_{i,t}(x) = \int_{\mathbb{R}^n} \nabla \zeta(x) \cdot v_i(t, x) \, d\rho_{i,t}(x) + \frac{1}{2} \int_{\mathbb{R}^n} \zeta(x) \xi(t, x) \, d\mathbf{f}(\rho_{1,t}(x), \rho_{2,t}(x)) \tag{5.4.7}$$

for $i = 1, 2$, all $t \in (0, 1)$ and any $\zeta \in C_c^\infty(\mathbb{R}^n)$. Actually, this formulation is equivalent to (5.4.6), by the density of the space Span $\langle \{ \eta\zeta : \eta \in C_c^\infty([0, 1]), \zeta \in C_c^\infty(\mathbb{R}^n) \} \rangle$ in $C_c^1([0, 1] \times \mathbb{R}^n)$.

Remark 5.4.1. We observe that, if $\mu_i(\mathbb{R}^n) = \nu_i(\mathbb{R}^n)$ for $i = 1, 2$, then

$$\mathrm{D}^2_{\mathbb{K}}((\nu_1, \nu_2), (\mu_1, \mu_2)) \leq W_2^2(\nu_1, \mu_1) + W_2^2(\nu_2, \mu_2).$$

Indeed, we can choose $\xi = 0$ and ρ_1, ρ_2, v_1, v_2 to be the solutions of the continuity equations

$$\frac{d}{dt}\rho_i = -\mathrm{div}(v_i \rho_i), i = 1, 2,$$

which are well known to exist, thanks to the Benamou-Brenier representation of the Wasserstein distance (for which we refer for instance to [13, Chapter 8]).

For the scope of this section, it is useful to recall the notion of *narrow convergence* of nonnegative finite Radon measures.

Definition 5.4.2. Let $\mu, \mu_k \in \mathcal{M}^+(\mathbb{R}^n), k \in \mathbb{N}$. Then we say that μ_k narrowly convergence to μ as $k \to +\infty$ if

$$\lim_{k \to +\infty} \int_{\mathbb{R}^n} \varphi \, d\mu_k = \int_{\mathbb{R}^n} \varphi \, d\mu$$

for any $\varphi \in C_{\mathrm{b}}(\mathbb{R}^n)$.

Remark 5.4.3. Given a sequence of nonnegative finite Radon measures (σ_1^k, σ_2^k) which is narrowly converging to some (σ_1, σ_2), then, for any converging subsequence of $(\mathbf{f}(\sigma_1^k, \sigma_2^k))_k$ (which we do not relabel), we have

$$\lim_{k \to +\infty} \int_{\mathbb{R}^n} \psi \, d\mathbf{f}(\sigma_1^k, \sigma_2^k) \leq \int_{\mathbb{R}^n} \psi \, d\mathbf{f}(\sigma_1, \sigma_2), \tag{5.4.8}$$

for any $\psi \in C_{\mathrm{b}}(\mathbb{R}^n)$, $\psi \geq 0$. This means the $\mathbf{f}$ is a narrowly upper semicontinuous nonnegative measure valued map. In particular, (5.4.8) holds for $\psi \equiv 1$, so that we have

$$\lim_{k \to +\infty} \int_{\mathbb{R}^n} d\mathbf{f}(\sigma_1^k, \sigma_2^k) \leq \int_{\mathbb{R}^n} d\mathbf{f}(\sigma_1, \sigma_2). \tag{5.4.9}$$

Indeed, $\mathbf{f}(\sigma_1^k, \sigma_2^k)(\mathbb{R}^n) \leq C(\sigma_1^k + \sigma_2^k)(\mathbb{R}^n) \leq \tilde{C}$, since the sequence (σ_1^k, σ_2^k) is narrowly converging. Hence, there exists a narrowly converging subsequence whose limit we denote by $\mathbf{f}_\sigma$. By concavity, we have

$$\begin{aligned} f(x, y) &= \inf_j \{a_j x + b_j y + c_j\}, \\ f_\infty(x, y) &= \inf_j \{a_j x + b_j y\}, \end{aligned}$$

for some suitable sequences of real numbers $(a_j), (b_j), (c_j)$ (see [11, Proposition 2.31 and Lemma 2.33]).

Then, for any $\psi \in C_{\mathrm{b}}(\mathbb{R}^n)$, $\psi \geq 0$, we have

$$\int_{\mathbb{R}^n} \psi \, d\mathbf{f}_\sigma = \lim_{k \to +\infty} \int_{\mathbb{R}^n} \psi \, \mathbf{f}(\sigma_1^k, \sigma_2^k) \leq \lim_{k \to +\infty} \int_{\mathbb{R}^n} \psi \, d(a_j \sigma_1^k + b_j \sigma_2^k + c_j) = \int_{\mathbb{R}^n} \psi \, d(a_j \sigma_1 + b_j \sigma_2 + c_j).$$

Thus, by splitting the σ_i's in absolutely continuous and singular parts and passing to the infimum in $j \in \mathbb{N}$, we conclude our argument.

In order to prove that the functional $\mathrm{D}_{\mathbb{K}}$ is indeed a distance, we exploit an alternative representation. To this purpose, we recall the notion of generalised product of Radon measures (for which we refer to [11, Section 2.5]).

Definition 5.4.4. Let $U \subset \mathbb{R}^k, V \subset \mathbb{R}^m$ be open sets, $\mu \in \mathcal{M}_+(U)$, and $z \to \nu_z$ a function which associates to each $z \in U$ a finite Radon measure ν_z on V. We say that this map is μ-measurable if $z \to \nu_z(B)$ is μ-measurable for any $B \in \mathcal{B}(V)$.

If we assume in addition that

$$\int_U |\nu_z|(V)\, d\mu(z) < \infty,$$

we define the generalized product $\mu \otimes \nu_z$ as the finite Radon measure on $U \times V$ which satisfies

$$\mu \otimes \nu_z(B) = \int_U \left(\int_V \chi_B(z,y)\, d\nu_z(y) \right) d\mu(z) \text{ for any } B \in \mathcal{B}(U \times V).$$

Thanks to [11, Proposition 2.26], we know that the measure $\sigma := \mu \otimes \nu_z$ is well defined, belongs to $\mathcal{M}(U \times V)$ and satisfies

$$\int_{U \times V} \psi(z,y)\, d\sigma(z,y) = \int_U \left(\int_V \psi(z,y)\, d\nu_z(y) \right) d\mu(z)$$

for any bounded Borel function $\psi : U \times V \to \mathbb{R}$. In addition, if $\pi : \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^n$ is the projection on the first coordinate, $\pi(z,y) = z$, then $\pi_{\#}\sigma = \nu_z(\mathbb{R}^n)\,\mu$.

In view of this definition, we set

$$\sigma_i := \mathscr{L}^1 \llcorner (0,1) \otimes \rho_{i,t},\ i = 1,2 \text{ and } \Sigma := \mathscr{L}^1 \llcorner (0,1) \otimes \mathbf{f}(\rho_{1,t}, \rho_{2,t}). \tag{5.4.10}$$

Then, we define the following Radon measures on $(0,1) \times \mathbb{R}^n$:

$$w_i := v_i \sigma_i, \text{ for } i = 1,2, \text{ and } \eta := \xi \Sigma. \tag{5.4.11}$$

Thanks to (5.4.10) and (5.4.11), we see that

$$\int_0^1 \left(\int_{\mathbb{R}^n} |v_1|^2 d\rho_1 + |v_2|^2 d\rho_1 + \frac{|\xi|^2}{2} d\mathbf{f}(\rho_1,\rho_2) \right) dt = \int_0^1 \int_{\mathbb{R}^n} \left| \frac{dw_1}{d\sigma_1} \right|^2 d\sigma_1 + \left| \frac{dw_2}{d\sigma_2} \right|^2 d\sigma_2 + \frac{1}{2} \left| \frac{d\eta}{d\Sigma} \right|^2 d\Sigma.$$

Arguing in this way, we can give an alternative definition of $\mathrm{D}_{\mathbb{K}}$:

$$\widetilde{\mathrm{D}_{\mathbb{K}}}^2((\nu_1,\nu_2),(\mu_1,\mu_2)) := \inf \left\{ \int_0^1 \int_{\mathbb{R}^n} \left| \frac{dw_1}{d\sigma_1} \right|^2 d\sigma_1 + \left| \frac{dw_2}{d\sigma_2} \right|^2 d\sigma_2 + \frac{1}{2} \left| \frac{d\eta}{d\Sigma} \right|^2 d\Sigma \right\}, \tag{5.4.12}$$

where the infimum is taken over all the $\sigma_1, \sigma_2, \Sigma \in \mathcal{M}_+((0,1)\times\mathbb{R}^n)$, $w_1, w_2 \in \mathcal{M}((0,1)\times\mathbb{R}^n;\mathbb{R}^n)$ and $\eta \in \mathcal{M}((0,1)\times\mathbb{R}^n)$ such that

$$\frac{d}{dt}\sigma_i = -\mathrm{div} w_i + \frac{\eta}{2} + (\delta_0 \otimes \nu_i - \delta_1 \otimes \mu_i),\ i = 1,2. \tag{5.4.13}$$

In addition, if $\widetilde{\pi}(t,x) = t$, we require that

$$\widetilde{\pi}_{\#}\sigma_i \ll \mathscr{L}^1 \llcorner (0,1), \quad \frac{d\widetilde{\pi}_{\#}\sigma_i}{d\mathscr{L}^1 \llcorner (0,1)} \in L^\infty((0,1)),\ i = 1,2, \tag{5.4.14}$$

and

$$\widetilde{\pi}_{\#}\Sigma \ll \mathscr{L}^1 \llcorner (0,1), \quad \left(\frac{d\widetilde{\pi}_{\#}\Sigma}{d\mathscr{L}^1 \llcorner (0,1)} \right)_t \Sigma_t = \mathbf{f}\left(\frac{d\pi_{\#}\sigma_1}{d\mathscr{L}^1 \llcorner (0,1)} \sigma_{1,t}, \frac{d\pi_{\#}\sigma_2}{d\mathscr{L}^1 \llcorner (0,1)} \sigma_{2,t} \right), \tag{5.4.15}$$

where $\sigma_{i,t}, i = 1,2$ and Σ_t for $t \in (0,1)$ are positive finite Radon measures on $\mathbb{R}^n$ coming from the disintegration of σ_i and Σ_t, respectively, so that $\sigma_i = \widetilde{\pi}_{\#}\sigma_i \otimes \sigma_{i,t}$ and $\Sigma = \widetilde{\pi}_{\#}\Sigma \otimes \Sigma_t$ (see [11, Theorem 2.28]).

We remark that (5.4.13) has to be intended in a distributional sense; that is, for any $\psi \in C_c^1([0,1]\times\mathbb{R}^n)$ and $i=1,2$, we have

$$-\int_0^1\int_{\mathbb{R}^n}\frac{\partial\psi(t,x)}{\partial t}\,d\sigma_i(t,x)=\int_0^1\int_{\mathbb{R}^n}\nabla\psi(t,x)\cdot\,dw_i(t,x)+\int_0^1\int_{\mathbb{R}^n}\frac{\psi(t,x)}{2}\,d\eta(t,x)+ \qquad (5.4.16)$$
$$+\int_{\mathbb{R}^n}\psi(0,x)\,d\nu_i-\int_{\mathbb{R}^n}\psi(1,x)\,d\mu_i.$$

It is plain to see that (5.4.16) is equivalent to (5.4.6), thanks to (5.4.10) and (5.4.11). Therefore, it seems natural to wonder whether $\mathrm{D}_{\mathbb{K}}$ and $\widetilde{\mathrm{D}_{\mathbb{K}}}$ are indeed different representations of the same distance. To this purpose, we need to show that, if there exists weak solutions $\rho_1,\rho_2\in L^\infty((0,1);\mathcal{M}_+(\mathbb{R}^n))$ of (5.4.3), then they admit a continuous representative, under some additional assumptions on the velocity fields v_1,v_2 and the reaction factor ξ. This technical result and its proof are very similar to their analogues for the continuity equation as stated in [13, Lemma 8.1.2].

Lemma 5.4.5. *Let $(\rho_1,\rho_2,v_1,v_2,\xi)$ be a Borel family of measures satisfying* (5.4.7) *and*

$$\rho_i\in L^\infty((0,1);\mathcal{M}_+(\mathbb{R}^n)),\ |v_i|\in L^1((0,1);L^1(\mathbb{R}^n;\rho_i)),\ \xi\in L^1((0,1);L^1(\mathbb{R}^n;\mathbf{f}(\rho_1,\rho_2))), \qquad (5.4.17)$$

for $i=1,2$.

Then there exist continuous curves $\tilde\rho_i\in C([0,1];\mathcal{M}_+(\mathbb{R}^n))$ such that $\tilde\rho_{i,t}=\rho_{i,t}$ for $\mathscr{L}^1$-a.e. $t\in(0,1)$. In addition, if $\zeta\in C_c^1([0,1]\times\mathbb{R}^n)$ and $0\le t_1\le t_2\le 1$, for $i=1,2$ we have

$$\int_{\mathbb{R}^n}\zeta(t_2,x)\,d\tilde\rho_{i,t_2}(x)-\int_{\mathbb{R}^n}\zeta(t_1,x)\,d\tilde\rho_{i,t_1}(x)=\int_{t_1}^{t_2}\int_{\mathbb{R}^n}\frac{\partial\zeta}{\partial t}(t,x)\,d\rho_{i,t}(x)\,dt \qquad (5.4.18)$$
$$+\int_{t_1}^{t_2}\int_{\mathbb{R}^n}\nabla\zeta(t,x)\cdot v_i(t,x)\,d\rho_{i,t}(x)\,dt$$
$$+\int_{t_1}^{t_2}\int_{\mathbb{R}^n}\frac{1}{2}\zeta(t,x)\,\xi(t,x)\,d\mathbf{f}(\rho_{1,t}(t),\rho_{2,t}(t))\,dt.$$

Proof. For any fixed $\zeta\in C_c^\infty(\mathbb{R}^n)$ and $i=1,2$, we define the maps

$$P_i(\zeta,t):=\int_{\mathbb{R}^n}\zeta(x)\,d\rho_{i,t}(x),\ \ t\in(0,1),$$

which are just the representation of the action of the finite Radon measure $\rho_{i,t}(\cdot)$ on $C_c^\infty(\mathbb{R}^n)\subset C_c(\mathbb{R}^n)$. Thanks to (5.4.7) and (5.4.17), it follows that $P_i(\zeta,\cdot)$ is well defined and belongs to $W^{1,1}(0,1)$, with weak derivative given by

$$\dot P_i(\zeta,t)=\int_{\mathbb{R}^n}\nabla\zeta(x)\cdot v_i(t,x)\,d\rho_{i,t}(x)+\frac{1}{2}\int_{\mathbb{R}^n}\zeta(x)\xi(t,x)\,d\mathbf{f}(\rho_{1,t}(x),\rho_{2,t}(x)). \qquad (5.4.19)$$

In addition, we get

$$|\dot P_i(\zeta,t)|\le\|\zeta\|_{C^1(\mathbb{R}^n)}(V_i(t)+\Xi(t)), \qquad (5.4.20)$$

where

$$\|\zeta\|_{C^1(\mathbb{R}^n)}:=\max\{\|\zeta\|_{L^\infty(\mathbb{R}^n)};\|\nabla\zeta\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)}\},$$
$$V_i(t):=\int_{\mathbb{R}^n}|v_i(t,x)|\,d\rho_{i,t}(x),$$
$$\Xi(t):=\int_{\mathbb{R}^n}|\xi(t,x)|\,d\mathbf{f}(\rho_{1,t}(x),\rho_{2,t}(x)).$$

We notice that $V_i,\Xi\in L^1((0,1))$, because of (5.4.17), so that $V_i(t),\Xi(t)<\infty$ for any $t\in F_{V_i,\Xi}$, with $\mathscr{L}^1((0,1)\setminus F_{V_i,\Xi})=0$. Hence, (5.4.20) implies that, for any $t\in F_{V_i,\Xi}$, $P_i(\cdot,t)$ can be extended to a continuous linear functional on $C^1_c(\mathbb{R}^n)$. If we denote by $L_{P_i(\zeta,\cdot)}$ the set of Lebesgue points of $P_i(\zeta,\cdot)$, we know that $\mathscr{L}^1((0,1)\setminus L_{P_i(\zeta,\cdot)})=0$. Hence, thanks to the separability of $C^1_c(\mathbb{R}^n)$ with respect to the norm $\|\cdot\|_{C^1(\mathbb{R}^n)}$, we can find a countable dense set Γ and define $L_{\Gamma,i} := \bigcap_{\zeta\in\Gamma} L_{P_i(\zeta,\cdot)}$. If we set $L_i := F_{V_i,\Xi}\cap L_{\Gamma,i}$ is clear that $\mathscr{L}^1((0,1)\setminus L_i)=0$, and we notice that the restriction of P_i to L_i is a uniformly continuous family of linear bounded functionals on $C^1_c(\mathbb{R}^n)$, since (5.4.20) yields

$$|P_i(\zeta,t)-P_i(\zeta,s)| \le \|\zeta\|_{C^1(\mathbb{R}^n)} \int_s^t V_i(u)+\Xi(u)\,du \tag{5.4.21}$$

for any $\zeta\in C^1_c(\mathbb{R}^n)$ and any $s,t\in L_i$, $s<t$. Thus, $\{P_i(\cdot,t)\}_{t\in L_i}$ can be extended to a continuous curve $\{\tilde{P}_i(\cdot,t)\}_{t\in(0,1)}$ in the dual of $C^1_c(\mathbb{R}^n)$: it is enough now to prove that $\{\tilde{P}_i(\cdot,t)\}_{t\in(0,1)}$ belongs indeed to the dual of $C_c(\mathbb{R}^n)$ in order to conclude the existence of a continuous representative $\tilde{\rho}_{i,t}$ of $\rho_{i,t}$. To this purpose, we claim that the uniformly bounded family $\{P_i(\cdot,t)\}_{t\in L_i}$ is also uniformly tight: then, by applying Prokhorov's theorem [27, Theorem 8.6.2], we conclude that any accumulation point of that family has to be a finite nonnegative Radon measure, so that $\{\tilde{P}_i(\cdot,t)\}_{t\in(0,1)}$ can be actually represented by a continuous curve $\{\tilde{\rho}_{i,t}\}_{t\in(0,1)}\subset\mathcal{M}_+(\mathbb{R}^n)$.

In order to prove the claim, we start by noticing that $P_i(\cdot,t)$ can be extended to a functional on $C^1_b(\mathbb{R}^n)$, thanks to (5.4.17). Then, let us consider a family of cutoff functions $\zeta_k\in C^\infty_c(\mathbb{R}^n)$ satisfying

$$0\le\zeta_k\le 1,\ \zeta_k(x)\equiv 1 \text{ if } |x|\le k,\ \zeta_k(x)\equiv 0 \text{ if } |x|\ge k+1,\ |\nabla\zeta_k|\le 2.$$

Without loss of generality, we can assume that $\zeta_k\in\Gamma$ for any $k\in\mathbb{N}$. By (5.4.19), for any $k\in N$ and $s,t\in L_i$, $s<t$, we have

$$\begin{aligned}|P_i(1-\zeta_k,t)-P_i(1-\zeta_k,s)| &\le 2\int_0^1\int_{B(0,k+1)\setminus B(0,k)} |v_i(u,x)|\,d\rho_{i,u}(x)\,du\\ &\quad+\frac12\int_0^1\int_{\mathbb{R}^n\setminus B(0,k)}|\xi(u,x)|\,d\mathbf{f}(\rho_{1,u}(x),\rho_{2,u}(x))\,du.\end{aligned}$$

By (5.4.17), it is clear that the right hand side of this inequality goes to zero as $k\to+\infty$, so that, for any $\varepsilon>0$ there exists $k_0\in\mathbb{N}$ such that

$$|P_i(1-\zeta_k,t)-P_i(1-\zeta_k,s)|<\varepsilon$$

for any $k\ge k_0$. Analogously, $P_i(1-\zeta_k,s)<\varepsilon$ for any $k\ge k_1$, for some $k_1\in\mathbb{N}$. Hence, by the triangle inequality, we have

$$\int_{\mathbb{R}^n\setminus\overline{B(0,k+1)}} d\rho_{i,t}(x)\le P_i(1-\zeta_k,t)<2\varepsilon$$

for any $t\in L_i$ and $k\ge\max\{k_0,k_1\}$. This proves the uniform tightness of the family $\{\rho_{i,t}\}_{t\in L_i}$, and enables us to apply Prokhorov's theorem.

Finally, we pass to the proof of (5.4.18). Let at first $0<t_1\le t_2<1$. We select $\varphi\in C^1_c([0,1]\times\mathbb{R}^n)$ and a sequence $\eta_k\in C^\infty_c(t_1,t_2)$ satisfying

$$0\le\eta_k(t)\le 1,\ \eta_k(t)\to\chi_{(t_1,t_2)}(t) \text{ for any } t\in(t_1,t_2),\ \eta_k'\mathscr{L}^1\llcorner(0,1)\rightharpoonup\delta_{t_1}-\delta_{t_2}.$$

Then, using (5.4.7) we get

$$\begin{aligned}0&=\int_0^1\int_{\mathbb{R}^n}\left(\partial_t(\eta_k\varphi)+\nabla(\eta_k\varphi)\cdot v_i\right)\,d\rho_{i,t}\,dt+\frac12\int_0^1\int_{\mathbb{R}^n}\eta_k\varphi\xi\,d\mathbf{f}(\rho_{1,t},\rho_{2,t})\,dt\\ &=\int_0^1\eta_k\left(\int_{\mathbb{R}^n}\left(\partial_t\varphi+\nabla\varphi\cdot v_i\right)\,d\rho_{i,t}+\frac12\int_{\mathbb{R}^n}\varphi\xi\,d\mathbf{f}(\rho_{1,t},\rho_{2,t})\right)dt+\int_0^1\eta_k'\int_{\mathbb{R}^n}\varphi\,d\tilde{\rho}_{i,t}\,dt.\end{aligned}$$

Thus, we conclude by passing to the limit in $k\to+\infty$ and employing the continuity of $\tilde{\rho}_{i,t}$. Then, if we have for $t_1=0$ (or $t_2=1$), we employ (5.4.18) with $t_1=1/k$ (or $t_2=1-1/k$, respectively), and pass to the limit in $k\to+\infty$ using again the continuity of $\tilde{\rho}_{i,t}$. $\square$

Corollary 5.4.6. $\mathrm{D}_{\mathbb{K}} = \widetilde{\mathrm{D}_{\mathbb{K}}}$.

Proof. Given $\rho_1, \rho_2, v_1, v_2, \xi$ satisfying (5.4.3), (5.4.4) and (5.4.5), it is plain to see that, if we define $\sigma_1, \sigma_2, \Sigma, w_1, w_2, \eta$ as in (5.4.10) and (5.4.11), then (5.4.13) is satisfied, in view of (5.4.6) and (5.4.16). In addition, for $i = 1, 2$,

$$\tilde{\pi}_{\#}\sigma_i = \rho_{i,t}(\mathbb{R}^n)\,\mathscr{L}^1 \llcorner (0,1),$$

which gives us (5.4.14), thanks to (5.4.4). This shows that $\mathrm{D}_{\mathbb{K}} \geq \widetilde{\mathrm{D}_{\mathbb{K}}}$, which in particular means that, if $\widetilde{\mathrm{D}_{\mathbb{K}}} = +\infty$, then $\mathrm{D}_{\mathbb{K}} = +\infty$. Without loss of generality, let $\nu_1, \nu_2, \mu_1, \mu_2 \in \mathcal{M}_+(\mathbb{R}^n)$ be such that $\widetilde{\mathrm{D}_{\mathbb{K}}}((\nu_1, \nu_2), (\mu_1, \mu_2)) < \infty$. Given $\sigma_1, \sigma_2, \Sigma, w_1, w_2, \eta$ satisfying (5.4.13) and (5.4.14), let us assume that

$$\int_0^1 \int_{\mathbb{R}^n} \left|\frac{dw_1}{d\sigma_1}\right|^2 d\sigma_1 + \left|\frac{dw_2}{d\sigma_2}\right|^2 d\sigma_2 + \frac{1}{2}\left|\frac{d\eta}{d\Sigma}\right|^2 d\Sigma < \infty.$$

In particular, since $\sigma_i((0,1) \times \mathbb{R}^n) < \infty$, $i = 1, 2$, this means that

$$\left|\frac{dw_i}{d\sigma_i}\right| \in L^1((0,1) \times \mathbb{R}^n; \sigma_i),\ \frac{d\eta}{d\Sigma} \in L^1((0,1) \times \mathbb{R}^n; \Sigma),$$

for $i = 1, 2$. Then, we consider the disintegration of the measures $\sigma_i, i = 1, 2$: by [11, Theorem 2.28], we know that there exist positive finite Radon measures $\sigma_{i,t}$ for $t \in (0,1)$ such that $t \to \sigma_{i,t}$ is $\tilde{\pi}_{\#}\sigma_i$-measurable and $\sigma_{i,t}(\mathbb{R}^n) = 1$ for $\tilde{\pi}_{\#}\sigma_i$-a.e. $t \in (0,1)$. However, thanks to (5.4.14), we see that $\tilde{\pi}_{\#}\sigma_i \ll \mathscr{L}^1 \llcorner (0,1)$, with density in $L^\infty((0,1))$, so that

$$\frac{d\pi_{\#}\sigma_i}{d\mathscr{L}^1 \llcorner (0,1)}\sigma_{i,\cdot} \in L^\infty((0,1); \mathcal{M}_+(\mathbb{R}^n)).$$

Now, we set

$$\rho_{i,t} := \left(\frac{d\pi_{\#}\sigma_i}{d\mathscr{L}^1 \llcorner (0,1)}\right)_t \sigma_{i,t},\ v_i := \frac{dw_i}{d\sigma_i}, i = 1, 2, \text{ and } \xi := \frac{d\eta}{d\Sigma},$$

and we employ (5.4.15) to show that

$$\Sigma = \tilde{\pi}_{\#}\Sigma \otimes \Sigma_t = \mathscr{L}^1 \llcorner (0,1) \otimes \left(\frac{d\tilde{\pi}_{\#}\Sigma}{d\mathscr{L}^1 \llcorner (0,1)}\right)_t \Sigma_t = \mathscr{L}^1 \llcorner (0,1) \otimes \mathbf{f}(\rho_{1,t}, \rho_{2,t}),$$

where Σ_t comes from the disintegration of Σ, in analogous way as we did above with $\sigma_{i,t}$. Thus, in this way we obtain a weak solution to (5.4.3) and (5.4.5) satisfying the assumptions of Lemma 5.4.5, which implies the existence of a representative continuous in t for $\rho_{1,t}$ and $\rho_{2,t}$. This ends the proof. □

Since we showed that to any measure $\sigma_i, i = 1, 2$, as in the definition of $\widetilde{\mathrm{D}_{\mathbb{K}}}$ can be associated a continuous curve of measures $\rho_{i,t}$, from this point on we set $\sigma_i := \mathscr{L}^1 \llcorner (0,1) \otimes \rho_{i,t}, i = 1, 2$. Analogously, we set $\Sigma(\rho_1, \rho_2) := \mathscr{L}^1 \llcorner (0,1) \otimes \mathbf{f}(\rho_{1,t}, \rho_{2,t})$.

The main advantage of the different representation given by (5.4.12) lies in the possibility to achieve a form of weak lower semicontinuity for the the action functional

$$\mathcal{A}(\rho_1, \rho_2, w_1, w_2, \eta) := \int_0^1 \left(\int_{\mathbb{R}^n} \left|\frac{dw_1}{d\sigma_1}\right|^2 d\rho_1 + \left|\frac{dw_2}{d\sigma_2}\right|^2 d\rho_2 + \frac{1}{2}\left|\frac{d\eta}{d\Sigma(\rho_1, \rho_2)}\right|^2 d\mathbf{f}(\rho_1, \rho_2)\right) dt \tag{5.4.22}$$

for $\rho_i \in C([0,1]; \mathcal{M}_+(\mathbb{R}^n))$, $w_i \in \mathcal{M}((0,1) \times \mathbb{R}^n; \mathbb{R}^n)$, with $w_i \ll \sigma_i$, and $\eta \in \mathcal{M}((0,1) \times \mathbb{R}^n)$, with $\eta \ll \Sigma(\rho_1, \rho_2)$.

To this purpose, we recall a technical lemma on a joint lower semicontinuity properties of sequence of measures and functions.

Lemma 5.4.7. *Let (X,d) be a metric space and $\mu_k, \mu \in \mathcal{M}_+(X)$. If $\mu_k \rightharpoonup \mu$ and $f_k \in L^2(X;\mu_k)$ with $\|f_k\|_{L^2(X;\mu_k)} \le C < \infty$ for any k, then there exists a measure σ such that, up to a subsequence,*

1. $f_k\mu_k \rightharpoonup \sigma$,
2. $\sigma = f\mu$ *for some* $f \in L^2(X,\mu)$, *and so* $\sigma \ll \mu$,
3. $\|f\|_{L^2(X;\mu)} \le \liminf_{k\to+\infty} \|f_k\|_{L^2(X;\mu_k)}$.

It is possible to show (see [11, Theorem 2.34 and Example 2.36]) that this result is equivalent to the joint lower semicontinuity with respect to the weak convergence of the functional

$$F(\sigma,\mu) := \begin{cases} \displaystyle\int_X \left|\frac{\sigma}{\mu}\right|^2 d\mu & \text{if } \sigma \ll \mu, \\ +\infty & \text{otherwise.} \end{cases}$$

Lemma 5.4.8. *Let*

$$\begin{gathered} \rho^k_{1,t}, \rho^k_{2,t} \in L^\infty((0,1);\mathcal{M}_+(\mathbb{R}^n)),\ \sigma^k_i = \mathscr{L}^1 \llcorner (0,1) \otimes \rho^k_{i,t}, \\ w^k_1, w^k_2 \in \mathcal{M}((0,1)\times\mathbb{R}^n;\mathbb{R}^n),\ w^k_i \ll \sigma^k_i, i = 1,2, \\ \eta^k \in \mathcal{M}((0,1)\times\mathbb{R}^n),\ \eta^k \ll \Sigma(\rho^k_1,\rho^k_2) \end{gathered}$$

for any k, with

$$\begin{aligned} \liminf_{k\to+\infty} \|\rho^k_{i,\cdot}(\mathbb{R}^n)\|_{L^\infty((0,1))} &< \infty, \\ \liminf_{k\to+\infty} \int_0^1\int_{\mathbb{R}^n} \left|\frac{dw^k_i}{d\sigma^k_i}\right|^2 d\rho^k_i\,dt &< \infty, \\ \liminf_{k\to+\infty} \int_0^1\int_{\mathbb{R}^n} \left|\frac{d\eta^k}{d\Sigma(\rho^k_1,\rho^k_2)}\right|^2 d\mathbf{f}(\rho^k_1,\rho^k_2)\,ds &< \infty. \end{aligned}$$

If $(\sigma^k_1,\sigma^k_2,w^k_1,w^k_2,\eta^k) \rightharpoonup (\sigma_1,\sigma_2,w_1,w_2,\eta)$, then we have

1. $\sigma_i = \mathscr{L}^1 \llcorner (0,1) \otimes \rho_{i,t}$, *for some* $\rho_i \in L^\infty((0,1);\mathcal{M}_+(\mathbb{R}^n))$ *with*
$$\|\rho_{i,\cdot}(\mathbb{R}^n)\|_{L^\infty((0,1))} \le \liminf_{k\to+\infty} \|\rho^k_{i,\cdot}(\mathbb{R}^n)\|_{L^\infty((0,1))},\ i = 1,2; \tag{5.4.23}$$
2. $w_i \ll \sigma_i$, *with*
$$\int_0^1\int_{\mathbb{R}^n} \left|\frac{dw_i}{d\sigma_i}\right|^2 d\rho_i\,dt \le \liminf_{k\to+\infty} \int_0^1\int_{\mathbb{R}^n} \left|\frac{dw^k_i}{d\sigma^k_i}\right|^2 d\rho^k_i\,dt,\ i = 1,2; \tag{5.4.24}$$
3. $\eta \ll \Sigma(\rho_1,\rho_2)$, *with*
$$\int_0^1\int_{\mathbb{R}^n} \left|\frac{d\eta}{d\Sigma(\rho_1,\rho_2)}\right|^2 d\mathbf{f}(\rho_1,\rho_2)\,dt \le \liminf_{k\to+\infty} \int_0^1\int_{\mathbb{R}^n} \left|\frac{d\eta^k}{d\mathbf{f}(\rho^k_1,\rho^k_2)}\right|^2 d\Sigma(\rho^k_1,\rho^k_2)\,ds. \tag{5.4.25}$$

Proof. Since $\sigma^k_i \rightharpoonup \sigma_i$, then $\widetilde{\pi}_\#\sigma^k_i \rightharpoonup \widetilde{\pi}_\#\sigma_i$. In addition, $\widetilde{\pi}_\#\sigma^k_i = \rho^k_{i,\cdot}(\mathbb{R}^n)\mathscr{L}^1 \llcorner (0,1)$, and, up to a subsequence, $\rho^k_{i,\cdot}(\mathbb{R}^n)$ is uniformly bounded in $L^\infty((0,1))$, and so in $L^2((0,1))$. Hence, we may apply Lemma 5.4.7 to the sequence of functions $\rho^k_{i,\cdot}(\mathbb{R}^n)$ and of measures $\mu_k = \mathscr{L}^1 \llcorner (0,1)$, in order to get $\widetilde{\pi}_{\#}\sigma_i \ll \mathscr{L}^1 \llcorner (0,1)$. Exploiting the disintegration of the measure σ_i (see [11, Theorem 2.28]) as in the proof of Corollary 5.4.6, we obtain

$$\sigma_i = \widetilde{\pi}_{\#}\sigma_i \otimes \sigma_{i,t},$$

for some finite nonnegative Radon measures $\sigma_{i,t}$ satisfying $\sigma_{i,t}(\mathbb{R}^n) = 1$ for $\mathscr{L}^1$-a.e. $t \in (0,1)$, thanks to the absolute continuity $\widetilde{\pi}_{\#}\sigma_i \ll \mathscr{L}^1 \llcorner (0,1)$. This allows us to define

$$\rho_{i,t} := \left(\frac{d\pi_{\#}\sigma_i}{d\mathscr{L}^1 \llcorner (0,1)} \right)_t \sigma_{i,t}.$$

Then, $\sigma_i^k \rightharpoonup \sigma_i$ implies that, for any $\psi \in C_c((0,1) \times \mathbb{R}^n)$,

$$\int_0^1 \int_{\mathbb{R}^n} \psi \, d\rho_{i,t}^k \, dt = \int_0^1 \int_{\mathbb{R}^n} \psi \, d\sigma_i^k \to \int_0^1 \int_{\mathbb{R}^n} \psi \, d\sigma_i = \int_0^1 \int_{\mathbb{R}^n} \psi \, d\rho_{i,t} \, dt$$

so that we obtain

$$\left| \int_0^1 \int_{\mathbb{R}^n} \psi \, d\rho_{i,t} \, dt \right| \leq \|\psi\|_{L^1((0,1);L^\infty(\mathbb{R}^n))} \liminf_{k \to +\infty} \|\rho_{i,\cdot}^k(\mathbb{R}^n)\|_{L^\infty(0,1)},$$

from which we get (5.4.23), by passing to the supremum in

$$\psi \in C_c((0,1) \times \mathbb{R}^n), \|\psi\|_{L^1((0,1);L^\infty(\mathbb{R}^n))} \leq 1.$$

Let us consider now w_i^k. Up to a subsequence, we notice that, by the assumptions, $w_i^k = v_i^k \sigma_i^k$ for some v_i^k in $L^2((0,1) \times \mathbb{R}^n; \sigma_i^k)$ with $\|v_i^k\|_{L^2((0,1)\times\mathbb{R}^n;\sigma_i^k)} \leq C < \infty$ for any k and $i = 1,2$. We apply Lemma 5.4.7 to the sequences $(\sigma_i^k)_k$ and $(v_i^k)_k$ and we immediately deduce that the weak limit w_i of w_i^k satisfies $w_i \ll \sigma_i$ and (5.4.24).

As for the term involving $\mathbf{f}$, we notice that $\mathbf{f}(\rho_1^k, \rho_2^k) \leq C(\rho_1^k + \rho_2^k)$, by (5.4.1). This fact and the assumptions on ρ_i^k clearly imply the uniform boundedness of $\Sigma(\rho_1^k, \rho_2^k) = \mathscr{L}^1 \llcorner (0,1) \otimes \mathbf{f}(\rho_1^k, \rho_2^k)$. Hence, there exists a converging subsequence (which we do not relabel) $\Sigma(\rho_1^k, \rho_2^k) \rightharpoonup \Sigma_\rho$, for some $\Sigma_\rho \in \mathcal{M}_+((0,1) \times \mathbb{R}^n)$.

Therefore, employing again Lemma 5.4.7 we obtain that $\eta \ll \Sigma_\rho$ and

$$\int_0^1 \int_{\mathbb{R}^n} \left| \frac{d\eta}{d\Sigma_\rho} \right|^2 d\Sigma_\rho \leq \liminf_{k \to +\infty} \int_0^1 \int_{\mathbb{R}^n} \left| \frac{d\eta^k}{d\Sigma(\rho_1^k, \rho_2^k)} \right|^2 d\mathbf{f}(\rho_1^k, \rho_2^k) \, dt < \infty. \tag{5.4.26}$$

It is not difficult to see that Remark 5.4.3 implies $\Sigma_\rho \leq \Sigma(\rho_1, \rho_2) = \mathscr{L}^1 \llcorner (0,1) \otimes \mathbf{f}(\rho_1, \rho_2)$, which means that

$$\Sigma_\rho = h\Sigma(\rho_1, \rho_2),$$

for some $h \in L^1((0,1) \times \mathbb{R}^n; \Sigma(\rho_1, \rho_2)), 0 \leq h \leq 1$. We deduce that $\eta \ll \Sigma(\rho_1, \rho_2)$, and, since $\eta = g\Sigma_\rho$ for some $g \in L^1((0,1) \times \mathbb{R}^n; \Sigma_\rho)$, we conclude that

$$\eta = gh\Sigma(\rho_1, \rho_2).$$

Therefore, it follows that

$$\left| \frac{d\eta}{d\Sigma_\rho} \right|^2 \Sigma_\rho = |g|^2 \Sigma_\rho = |g| \, |\eta| \geq |g| h \, |\eta| = |g|^2 h^2 \, \Sigma(\rho_1, \rho_2) = \left| \frac{d\eta}{d\Sigma(\rho_1, \rho_2)} \right|^2 \mathscr{L}^1 \llcorner (0,1) \otimes \mathbf{f}(\rho_1, \rho_2).$$

Hence, (5.4.26) implies (5.4.25). □

It should be pointed out that the narrow upper semicontinuity of $\mathbf{f}$ is essential in the last part of the proof of Lemma 5.4.8.

Thanks to this type of weak lower semicontinuity of $\mathcal{A}$, we can prove the existence of a narrowly continuous minimizing curve $(\rho_{1,t}, \rho_{2,t})$ for the distance $\mathrm{D}_{\mathbb{K}}$.

Proposition 5.4.9. *Let $(\mu_1, \mu_2), (\nu_1, \nu_2) \in \mathcal{M}_+(\mathbb{R}^n) \times \mathcal{M}_+(\mathbb{R}^n)$ be such that*

$$\mathrm{D}_{\mathbb{K}}((\nu_1, \nu_2), (\mu_1, \mu_2)) < \infty.$$

Then there exist a weakly continuous curve $(\rho_1, \rho_2) \in C([0,1]; \mathcal{M}_+(\mathbb{R}^n) \times \mathcal{M}_+(\mathbb{R}^n))$, vector fields $v_1, v_2 \in L^2((0,1); L^2(\mathbb{R}^n; \rho_i))$ and a scalar reaction term $\xi \in L^2((0,1); L^2(\mathbb{R}^n; \mathbf{f}(\rho_1, \rho_2)))$ satisfying

$$\mathrm{D}^2_{\mathbb{K}}((\nu_1, \nu_2), (\mu_1, \mu_2)) = \mathcal{A}(\rho_1, \rho_2, w_1, w_2, \eta), \tag{5.4.27}$$

where $w_i = v_i \sigma_i$ for $i = 1, 2$ and $\eta = \xi \Sigma(\rho_1, \rho_2)$. In addition, for any $0 \le s < t \le 1$ and $i = 1, 2$, we have

$$\rho_{i,t}(\mathbb{R}^n) - \rho_{i,s}(\mathbb{R}^n) = \int_s^t \int_{\mathbb{R}^n} \frac{\xi}{2} \, d\mathbf{f}(\rho_{1,u}, \rho_{2,u}) \, du. \tag{5.4.28}$$

Proof. Since $\mathrm{D}_{\mathbb{K}}((\nu_1, \nu_2), (\mu_1, \mu_2)) < \infty$, then we can find a sequence of admissible curves, vector fields and reaction terms $(\rho_1^k, \rho_2^k, v_1^k, v_2^k, \xi^k)$ satisfying (5.4.3) and such that, if we define w_i^k and η^k as above, then

$$\mathrm{D}^2_{\mathbb{K}}((\nu_1, \nu_2), (\mu_1, \mu_2)) = \lim_{k \to \infty} \mathcal{A}(\rho_1^k, \rho_2^k, w_1^k, w_2^k, \eta^k).$$

We proceed now to show that the sequences ρ_1^k, ρ_2^k are uniformly bounded in $L^\infty((0,1); \mathcal{M}_+(\mathbb{R}^n))$.

We recall the definition of $\sigma_i^k = \mathscr{L}^1 \llcorner (0,1) \otimes \rho_{i,t}^k$, $i = 1, 2$. Since $(\sigma_1^k, \sigma_2^k, w_1^k, w_2^k, \eta^k)$ is a weak solution to (5.4.13), by (5.4.16) we have

$$\begin{aligned} -\int_0^1 \int_{\mathbb{R}^n} \frac{\partial \psi(t,x)}{\partial t} \, d\sigma_i^k(t,x) = \int_0^1 \int_{\mathbb{R}^n} \nabla \psi(t,x) \cdot \, dw_i^k(t,x) + \int_0^1 \int_{\mathbb{R}^n} \frac{\psi(t,x)}{2} \, d\eta^k(t,x) + \\ + \int_{\mathbb{R}^n} \psi(0,x) \, d\nu_i - \int_{\mathbb{R}^n} \psi(1,x) \, d\mu_i \end{aligned} \tag{5.4.29}$$

for any $\psi \in C_c^1([0,1] \times \mathbb{R}^n)$ and $i = 1, 2$. Hence, if we choose $\psi(t,x) = \phi(x)$ in (5.4.29) for some $\phi \in C_c^1(\mathbb{R}^n)$, we deduce that, for any $t \in [0,1]$,

$$\int_{\mathbb{R}^n} \phi \, d\rho_{i,t}^k = \int_0^t \int_{\mathbb{R}^n} \nabla \phi \cdot \, dw_i^k + \frac{1}{2} \int_0^t \int_{\mathbb{R}^n} \phi \, d\eta^k + \int_{\mathbb{R}^n} \phi \, d\nu_i. \tag{5.4.30}$$

Let $\varphi \in C_c^1(B(0,2))$ such that $\varphi \equiv 1$ on $B(0,1)$. For some $R > 0$, we define $\varphi_R(x) = \varphi(x/R)$: this function satisfies $\varphi_R \equiv 1$ on $B(0,R)$, $\varphi_R \equiv 0$ in $\mathbb{R}^n \setminus B(0,2R)$ and $|\nabla \varphi_R| \le C/R$. In particular, we notice that

$$\left| \int_{\mathbb{R}^n \setminus B(0,R)} \varphi_R \, d\rho_{i,t}^k \right| \le \|\varphi_R\|_{L^\infty(\mathbb{R}^n)} \rho_{i,t}^k(\mathbb{R}^n \setminus B(0,R)),$$

$$\begin{aligned} \left| \int_0^t \int_{\mathbb{R}^n} \nabla \varphi_R \cdot dw_i^k \right| &\le \left(\int_0^1 \int_{\mathbb{R}^n} \left| \frac{dw_i^k}{d\sigma_i^k} \right|^2 d\rho_{i,s}^k \, ds \right)^{\frac{1}{2}} \left(\int_0^1 \int_{\mathbb{R}^n} |\nabla \varphi_R|^2 \, d\rho_{i,s}^k \, ds \right)^{\frac{1}{2}} \\ &\le \frac{C}{R} \left(\int_0^1 \rho_{i,s}^k(\mathbb{R}^n) \, ds \right)^{\frac{1}{2}}, \end{aligned} \tag{5.4.31}$$

$$\left|\int_0^t\int_{\mathbb{R}^n\setminus B(0,R)}\varphi_R\,d\eta^k\right|\le\left(\int_0^1\int_{\mathbb{R}^n}\left|\frac{d\eta^k}{d\Sigma(\rho_1^k,\rho_2^k)}\right|^2 d\mathbf{f}(\rho_1^k,\rho_2^k)\,ds\right)^{\frac{1}{2}}\left(\int_0^1\int_{\mathbb{R}^n\setminus B(0,R)} d\mathbf{f}(\rho_1^k,\rho_2^k)\,ds\right)^{\frac{1}{2}}$$
$$\le C\left(\int_0^1\int_{\mathbb{R}^n\setminus B(0,R)} d\mathbf{f}(\rho_{1,s}^k,\rho_{2,s}^k)\,ds\right)^{\frac{1}{2}}$$
$$\le C\left(\int_0^1 \rho_{1,s}^k(\mathbb{R}^n\setminus B(0,R))+\rho_{2,s}^k(\mathbb{R}^n\setminus B(0,R))\,ds\right)^{\frac{1}{2}}$$

since $(\rho_1^k,\rho_2^k,w_1^k,w_2^k,\eta^k)$ is a minimizing sequence, and by (5.4.1). This means that, for any fixed k, all these terms goes to zero as $R\to+\infty$, since $\rho_i^k\in C([0,1];\mathcal{M}_+(\mathbb{R}^n))$ and (5.4.17) holds. Therefore, if we set $\phi=\varphi_R$ in (5.4.30) and we pass to the limit as $R\to+\infty$, we get

$$\rho_{i,t}^k(\mathbb{R}^n)-\nu_i(\mathbb{R}^n)=\int_0^t\int_{\mathbb{R}^n}\frac{1}{2}\,d\eta^k\,ds$$
$$\le\frac{1}{2}\left(\int_0^1\int_{\mathbb{R}^n}\left|\frac{d\eta^k}{d\mathbf{f}(\rho_1^k,\rho_2^k)}\right|^2 d\mathbf{f}(\rho_1^k,\rho_2^k)\,ds\right)^{\frac{1}{2}}\left(\int_0^t\int_{\mathbb{R}^n} d\mathbf{f}(\rho_1^k,\rho_2^k)\,ds\right)^{\frac{1}{2}}$$
$$\le C_0+\frac{C_2}{2}\int_0^t\rho_{1,s}^k(\mathbb{R}^n)+\rho_{2,s}^k(\mathbb{R}^n)\,ds,$$

hence, by summing these inequalities for $i=1,2$, we obtain

$$\rho_{1,t}^k(\mathbb{R}^n)+\rho_{2,t}^k(\mathbb{R}^n)\le C_1+C_2\int_0^t\rho_{1,s}^k(\mathbb{R}^n)+\rho_{2,s}^k(\mathbb{R}^n)\,ds. \tag{5.4.32}$$

Therefore, by Gronwall's lemma and (5.4.32), it follows that

$$\rho_{1,t}^k(\mathbb{R}^n)+\rho_{2,t}^k(\mathbb{R}^n)\le C_1e^{C_2t}. \tag{5.4.33}$$

Thus, (5.4.33) implies that ρ_i^k is uniformly bounded in $L^\infty((0,1);\mathcal{M}_+(\mathbb{R}^n))$ for $i=1,2$. It is then obvious to see that the sequence $\sigma_i^k:=\mathscr{L}^1\llcorner(0,1)\otimes\rho_{i,t}^k$ is uniformly bounded in $\mathcal{M}_+((0,1)\times\mathbb{R}^n)$, so that there exists a subsequence converging to some measure σ_i. Hence, by Lemma 5.4.8 we conclude the existence of a curve of measures $\rho_i\in L^\infty((0,1);\mathcal{M}_+(\mathbb{R}^n))$ such that $\sigma_i=\mathscr{L}^1\llcorner(0,1)\otimes\rho_{i,t}$, for $i=1,2$.

Let us then show that (w_1^k,w_2^k,η^k) is bounded in $\mathcal{M}((0,1)\times\mathbb{R}^n;\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R})$. We have

$$\int_0^1\int_{\mathbb{R}^n}d|w_i^k|\le\left(\int_0^1\int_{\mathbb{R}^n}\left|\frac{dw_i^k}{d\sigma_i^k}\right|^2 d\rho_i^k\,ds\right)^{\frac{1}{2}}\left(\int_0^1\int_{\mathbb{R}^n}\rho_{i,s}^k\,ds\right)^{\frac{1}{2}}\le C\left(\int_0^1 C_1e^{C_2s}\,ds\right)^{\frac{1}{2}}\le\tilde{C},$$

and

$$\int_0^1\int_{\mathbb{R}^n}d|\eta^k|\le\left(\int_0^1\int_{\mathbb{R}^n}\left|\frac{d\eta^k}{d\Sigma(\rho_1^k,\rho_2^k)}\right|^2 d\mathbf{f}(\rho_1^k,\rho_2^k)\,ds\right)^{\frac{1}{2}}\left(\int_0^1\int_{\mathbb{R}^n}d\mathbf{f}(\rho_1^k,\rho_2^k)\,ds\right)^{\frac{1}{2}}$$
$$\le C\left(\int_0^1\rho_{1,s}^k(\mathbb{R}^n)+\rho_{2,s}^k(\mathbb{R}^n)\,ds\right)^{\frac{1}{2}}\le C\left(\int_0^1 C_1e^{C_2s}\,ds\right)^{\frac{1}{2}}\le\tilde{C},$$

by (5.4.33).

Thus, we find that there exist narrowly convergent subsequences (w_i^k) in $\mathcal{M}((0,1)\times\mathbb{R}^n;\mathbb{R}^n)$ and (η^k) in $\mathcal{M}((0,1)\times\mathbb{R}^n)$, which converge to some w_i and η, respectively. In addition, since $\mathcal{A}(\rho_1^k,\rho_2^k,w_1^k,w_2^k,\eta^k)<\infty$, we see that the conditions of Lemma 5.4.8 are satisfied, and so we can conclude that we have $w_i\ll\sigma_i$, $\eta\ll\Sigma(\rho_1,\rho_2)$, together with (5.4.24) and (5.4.25). This shows that there exist velocity fields $v_i \in L^2((0,1); L^2(\mathbb{R}^n; \rho_i))$ such that $w_i = v_i \sigma_i$, and a scalar reaction term $\xi \in L^2((0,1); L^2(\mathbb{R}^n; \mathbf{f}(\rho_1, \rho_2)))$ such that $\eta = \xi \Sigma(\rho_1, \rho_2)$.

Now, if we pass to the limit as $k \to +\infty$ in (5.4.29), we easily obtain

$$- \int_0^1 \int_{\mathbb{R}^n} \frac{\partial \psi(t,x)}{\partial t} \, d\sigma_i(t,x) = \int_0^1 \int_{\mathbb{R}^n} \nabla \psi(t,x) \cdot \, dw_i(t,x) + \int_0^1 \int_{\mathbb{R}^n} \frac{\psi(t,x)}{2} \, d\eta(t,x) + \\ + \int_{\mathbb{R}^n} \psi(0,x) \, d\nu_i - \int_{\mathbb{R}^n} \psi(1,x) \, d\mu_i$$

for any $i = 1,2$ and $\psi \in C_c^1([0,1] \times \mathbb{R}^n)$, which implies (5.4.7) by substituting

$$\sigma_i = \mathscr{L}^1 \llcorner (0,1) \otimes \rho_{i,t}, \;\; w_i = v_i \sigma_i, \;\; \eta = \xi \Sigma(\rho_1, \rho_2), \;\; \Sigma(\rho_1, \rho_2) = \mathscr{L}^1 \llcorner (0,1) \otimes \mathbf{f}(\rho_{1,t}, \rho_{2,t}).$$

Thus, $(\rho_1, \rho_2, v_1, v_2, \xi)$ is a weak solution of (5.4.3) and we can employ Lemma 5.4.5 to ensure the existence of a continuous representative for the curves $\rho_{1,t}$ and $\rho_{2,t}$.

Finally, we notice that, if we choose $\zeta(t,x) = \varphi_R(x)$ in (5.4.18), and we use (5.4.31) in order to pass to the limit as $R \to +\infty$, we deduce (5.4.28). □

From this point on, we shall refer to couples of continuous curves of measures (ρ_1, ρ_2) satisfying (5.4.27) as to *minimizing curves* with respect to $\mathrm{D}_{\mathbb{K}}$. Before proceeding with the description of the properties of $\mathrm{D}_{\mathbb{K}}$, we show a simple estimate on the total masses of any minimizing curve.

Lemma 5.4.10. *Let $(\mu_1, \mu_2), (\nu_1, \nu_2) \in \mathcal{M}_+(\mathbb{R}^n) \times \mathcal{M}_+(\mathbb{R}^n)$ be such that*

$$\mathrm{D}_{\mathbb{K}}((\nu_1, \nu_2), (\mu_1, \mu_2)) < \infty.$$

Then any minimizing curve (ρ_1, ρ_2) with respect to $\mathrm{D}_{\mathbb{K}}$ from (ν_1, ν_2) to (μ_1, μ_2) satisfies

$$\rho_{1,t}(\mathbb{R}^n) + \rho_{2,t}(\mathbb{R}^n) \leq \left(\nu_1(\mathbb{R}^n) + \nu_2(\mathbb{R}^n) + \frac{1}{2} \mathrm{D}_{\mathbb{K}}((\nu_1, \nu_2), (\mu_1, \mu_2))^2 \right) e^{\frac{C}{2} t} \tag{5.4.34}$$

for any $t \in [0,1]$, where $C := \sup_{x,y>0} f(x,y)/(x+y)$.

Proof. By (5.4.28) with $s = 0$, for any $t \in [0,1]$ we have

$$\rho_{1,t}(\mathbb{R}^n) + \rho_{2,t}(\mathbb{R}^n) \leq \nu_1(\mathbb{R}^n) + \nu_2(\mathbb{R}^n) + \int_0^t \int_{\mathbb{R}^n} |\xi| \, d\mathbf{f}(\rho_{1,u}, \rho_{2,u}) \, du,$$

where $\xi \in L^2((0,1); L^2(\mathbb{R}^n; \mathbf{f}(\rho_1, \rho_2)))$ is the scalar reaction term associated to the minimizing curve (ρ_1, ρ_2). Then, by Cauchy-Schwarz inequality, (5.4.27) and the fact that $f(x,y) \leq C(x+y)$ for some $C > 0$, we have

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^n} |\xi| \, d\mathbf{f}(\rho_{1,u}, \rho_{2,u}) \, du &\leq \left(\int_0^t \int_{\mathbb{R}^n} |\xi|^2 \, d\mathbf{f}(\rho_{1,u}, \rho_{2,u}) \, du \right)^{\frac{1}{2}} \left(\int_0^t \int_{\mathbb{R}^n} d\mathbf{f}(\rho_{1,u}, \rho_{2,u}) \, du \right)^{\frac{1}{2}} \\ &\leq \frac{1}{2} \left(\int_0^1 \int_{\mathbb{R}^n} |\xi|^2 \, d\mathbf{f}(\rho_{1,u}, \rho_{2,u}) \, du + \int_0^t \int_{\mathbb{R}^n} d\mathbf{f}(\rho_{1,u}, \rho_{2,u}) \, du \right) \\ &\leq \frac{1}{2} \left(\mathrm{D}_{\mathbb{K}}((\nu_1, \nu_2), (\mu_1, \mu_2))^2 + C \int_0^t \rho_{1,u}(\mathbb{R}^n) + \rho_{2,u}(\mathbb{R}^n) \, du \right). \end{aligned}$$

All in all, we obtain

$$\rho_{1,t}(\mathbb{R}^n) + \rho_{2,t}(\mathbb{R}^n) \leq C_1 + C_2 \int_0^t \rho_{1,u}(\mathbb{R}^n) + \rho_{2,u}(\mathbb{R}^n) \, du,$$

where $C_1 = \nu_1(\mathbb{R}^n) + \nu_2(\mathbb{R}^n) + \frac{1}{2} \mathrm{D}_{\mathbb{K}}((\nu_1, \nu_2), (\mu_1, \mu_2))^2$ and $C_2 = \frac{C}{2}$. Thus, a straightforward application of Gronwall's inequality allows us to obtain (5.4.34). □

We shall now prove that the functional $\mathrm{D}_{\mathbb{K}}$ is indeed an extended distance, borrowing in particular a classical scaling argument from the Wasserstein distance's framework.

Proposition 5.4.11. *$\mathrm{D}_{\mathbb{K}}$ is an extended distance on $\mathcal{M}_+(\mathbb{R}^n) \times \mathcal{M}_+(\mathbb{R}^n)$.*

Proof. It is clear that $\mathrm{D}_{\mathbb{K}} \geq 0$ and that it is symmetric.

Let now $\mathrm{D}_{\mathbb{K}}((\nu_1,\nu_2),(\mu_1,\mu_2)) = 0$, and let (ρ_1,ρ_2) be a minimizing curve of couples of nonnegative measures from (ν_1,ν_2) to (μ_1,μ_2), with vector fields v_1, v_2 and scalar reaction term ξ, whose existence is ensured by Proposition 5.4.9. Then, for $\mathscr{L}^1$-a.e. $t \in (0,1)$, we have clearly $v_i = 0$ for $\rho_{i,t}$-a.e. x, and $\xi = 0$ for $\mathbf{f}(\rho_{1,t},\rho_{2,t})$-a.e. x. This means that $\dot\rho_i = 0$ in the duality with $C_c^\infty([0,1]\times\mathbb{R}^n)$, and so ρ_i must be constant in time, but this is possible if and only if $\nu_i = \mu_i$ for $i = 1,2$. This shows the nondegeneracy.

As for the triangular inequality, let us consider three couples of nonnegative Radon measures $(\mu_1,\mu_2),(\nu_1,\nu_2),(\sigma_1,\sigma_2)$ such that $\mathrm{D}_{\mathbb{K}}((\nu_1,\nu_2),(\sigma_1,\sigma_2)), \mathrm{D}_{\mathbb{K}}((\sigma_1,\sigma_2),(\mu_1,\mu_2)) < \infty$, otherwise there is nothing to prove.

Let (r_1,r_2) be the minimizing continuous curve from (σ_1,σ_2) to (μ_1,μ_2), with velocity fields (Ξ_1,Ξ_2) and scalar reaction term z; while (s_1,s_2) is the minimizing continuous curve from (ν_1,ν_2) to (σ_1,σ_2), with velocity fields (u_1,u_2) and scalar reaction term y. Then, we define an admissible curve from (ν_1,ν_2) to (μ_1,μ_2) together with its velocity fields and scalar reaction term by setting

$$\rho_i(t,x) := \begin{cases} s_i(t/T,x) & 0 \leq t < T, \\ r_i((t-T)/(1-T),x) & T \leq t \leq 1, \end{cases}$$

$$v_i(t,x) := \begin{cases} \frac{1}{T}u_i(t/T,x) & 0 \leq t < T, \\ \frac{1}{1-T}\Xi_i((t-T)/(1-T),x) & T \leq t \leq 1, \end{cases}$$

$$\xi(t,x) := \begin{cases} \frac{1}{T}y(t/T,x) & 0 \leq t < T, \\ \frac{1}{1-T}z((t-T)/(1-T),x) & T \leq t \leq 1, \end{cases}$$

where we choose $T \in (0,1)$ such that

$$\frac{T}{1-T} = \frac{\mathrm{D}_{\mathbb{K}}((\nu_1,\nu_2),(\sigma_1,\sigma_2))}{\mathrm{D}_{\mathbb{K}}((\sigma_1,\sigma_2),(\mu_1,\mu_2))}.$$

It is easy to check that $(\rho_1,\rho_2,v_1,v_2,\xi)$ is admissible. To simplify notation, we set

$$\mathscr{A}(\rho_1,\rho_2,v_1,v_2,\xi)(t) := \int_{\mathbb{R}^n} |v_1(t,x)|^2\,d\rho_{1,t}(x) + |v_2(t,x)|^2\,d\rho_{2,t}(x) + \frac{|\xi(t,x)|^2}{2}\mathbf{f}(\rho_{1,t},\rho_{2,t}).$$

Hence, it follows that

$$\begin{aligned} \mathrm{D}^2_{\mathbb{K}}((\nu_1,\nu_2),(\mu_1,\mu_2)) &\leq \int_0^1 \mathscr{A}(\rho_1,\rho_2,v_1,v_2,\xi)(t)\,dt \\ &= \frac{1}{T^2}\int_0^T \mathscr{A}(s_1,s_2,u_1,u_2,y)(t/T)\,dt + \\ &+ \frac{1}{(1-T)^2}\int_T^1 \mathscr{A}(r_1,r_2,\Xi_1,\Xi_2,z)((t-T)/(1-T))\,dt \\ &= \frac{1}{T}\int_0^1 \mathscr{A}(s_1,s_2,u_1,u_2,y)(t)\,dt + \frac{1}{(1-T)}\int_0^1 \mathscr{A}(r_1,r_2,\Xi_1,\Xi_2,z)(t)\,dt \\ &= (\mathrm{D}_{\mathbb{K}}((\nu_1,\nu_2),(\sigma_1,\sigma_2)) + \mathrm{D}_{\mathbb{K}}((\sigma_1,\sigma_2),(\mu_1,\mu_2)))^2. \end{aligned}$$

This ends the proof. $\square$

As a byproduct of this proof, we can deduce the following technical result on the minimizing curves ρ_i.

Lemma 5.4.12. *Let $\mu_1, \mu_2, \nu_1, \nu_2 \in \mathcal{M}_+(\mathbb{R}^n)$ such that $\mathrm{D}_{\mathbb{K}}((\nu_1,\nu_2),(\mu_1,\mu_2)) < \infty$, and let $(\rho_1,\rho_2) \in C([0,1];\mathcal{M}_+(\mathbb{R}^n)\times\mathcal{M}_+(\mathbb{R}^n))$ be a minimizing curve from (ν_1,ν_2) to (μ_1,μ_2) with respect to $\mathrm{D}_{\mathbb{K}}$. Then, for any $s \in [0,1]$ we have*

$$\mathrm{D}_{\mathbb{K}}((\nu_1,\nu_2),(\rho_{1,s},\rho_{2,s})) \le \sqrt{s}\,\mathrm{D}_{\mathbb{K}}((\nu_1,\nu_2),(\mu_1,\mu_2)), \tag{5.4.35}$$

$$\mathrm{D}_{\mathbb{K}}((\rho_{1,s},\rho_{2,s}),(\mu_1,\mu_2)) \le \sqrt{1-s}\,\mathrm{D}_{\mathbb{K}}((\nu_1,\nu_2),(\mu_1,\mu_2)). \tag{5.4.36}$$

Proof. By Proposition 5.4.9, there exist continuous curves of nonnegative Radon measures $\rho_{1,t}, \rho_{2,t}$, vector fields

$$v_1, v_2 \in L^2((0,1);L^2(\mathbb{R}^n;\rho_i))$$

and a scalar reaction term

$$\xi \in L^2((0,1);L^2(\mathbb{R}^n;\mathbf{f}(\rho_1,\rho_2)))$$

such that (5.4.3) holds,

$$\mathrm{D}_{\mathbb{K}}((\nu_1,\nu_2),(\mu_1,\mu_2))^2 = \int_0^1 \left(\int_{\mathbb{R}^n} |v_1(t)|^2\,d\rho_{1,t} + |v_2(t)|^2\,d\rho_{2,t} + \frac{|\xi|^2(t)}{2}\,d\mathbf{f}(\rho_{1,t},\rho_{2,t})\right) dt,$$

and $\rho_{i,0} = \nu_i$, $\rho_{i,1} = \mu_i$, for $i = 1,2$. Let us now fix $s \in [0,1]$ and let $\widetilde{\rho}_{i,t} := \rho_{i,ts}$ for $i = 1,2$. By the continuity, it is clear that $\widetilde{\rho}_{i,0} = \nu_i$ and $\widetilde{\rho}_{i,1} = \rho_{i,s}$. In addition, it is not difficult to see that (5.4.3) implies

$$\frac{d}{dt}\begin{pmatrix}\widetilde{\rho}_1\\ \widetilde{\rho}_2\end{pmatrix} = -\mathrm{div}\begin{pmatrix} sv_1\widetilde{\rho}_1\\ sv_2\widetilde{\rho}_2\end{pmatrix} + \frac{s\xi}{2}\mathbf{f}(\widetilde{\rho}_1,\widetilde{\rho}_2)\begin{pmatrix}1\\1\end{pmatrix},$$

so that, if we set $\widetilde{v}_i(t) := sv_i(st)$, $i = 1,2$, and $\widetilde{\xi}(t) := s\xi(st)$, then $(\widetilde{\rho}_{1,t},\widetilde{\rho}_{2,t},\widetilde{v}_1(t),\widetilde{v}_2(t),\widetilde{\xi}(t))$ is a distributional solution to (5.4.3). Hence, by the definition of the distance $\mathrm{D}_{\mathbb{K}}$, we obtain

$$\begin{aligned}
\mathrm{D}_{\mathbb{K}}((\nu_1,\nu_2),(\rho_{1,s},\rho_{2,s}))^2 &\le \int_0^1 \left(\int_{\mathbb{R}^n} |\widetilde{v}_1(t)|^2\,d\widetilde{\rho}_{1,t} + |\widetilde{v}_2(t)|^2\,d\widetilde{\rho}_{2,t} + \frac{|\widetilde{\xi}(t)|^2}{2}\,d\mathbf{f}(\widetilde{\rho}_{1,t},\widetilde{\rho}_{2,t})\right) dt\\
&= s^2\int_0^1 \left(\int_{\mathbb{R}^n} |v_1(st)|^2\,d\rho_{1,ts} + |v_2(st)|^2\,d\rho_{2,ts} + \frac{|\xi|^2(st)}{2}\,d\mathbf{f}(\rho_{1,ts},\rho_{2,ts})\right) dt\\
&= [ts = u]\\
&= s\int_0^s \left(\int_{\mathbb{R}^n} |v_1(u)|^2\,d\rho_{1,u} + |v_2(u)|^2\,d\rho_{2,u} + \frac{|\xi(u)|^2}{2}\,d\mathbf{f}(\rho_{1,u},\rho_{2,u})\right) du\\
&\le s\int_0^1 \left(\int_{\mathbb{R}^n} |v_1|^2\,d\rho_1 + |v_2|^2\,d\rho_2 + \frac{1}{2}|\xi|^2\,d\mathbf{f}(\rho_1,\rho_2)\right) du\\
&\le s\mathrm{D}_{\mathbb{K}}((\nu_1,\nu_2),(\mu_1,\mu_2))^2.
\end{aligned}$$

Hence, (5.4.35) immediately follows. If instead we set $\widetilde{\rho}_{i,t} := \rho_{i,s+t(1-s)}$, $i = 1,2$, we get (5.4.36) with a similar argument. ☐

As an consequence of (5.4.28), we can derive necessary and sufficient conditions on the measures μ_i and ν_i to ensure the finiteness of $\mathrm{D}_{\mathbb{K}}((\nu_1,\nu_2),(\mu_1,\mu_2))$.

To this purpose, we also notice that for any $m, q \ge 0$, not both zero, there exists $\alpha > 0$ such that $f(x, mx+q) \ge \alpha x$ as $x \to 0$. This is a consequence of the concavity of f and the fact that $f(0,y) = 0$, for any $y \ge 0$, and $f(x,y) > 0$ for any $x, y > 0$.

Corollary 5.4.13. $\mathrm{D}_{\mathbb{K}}((\nu_1,\nu_2),(\mu_1,\mu_2)) < \infty$ *if and only if*

$$\nu_1(\mathbb{R}^n) - \nu_2(\mathbb{R}^n) = \mu_1(\mathbb{R}^n) - \mu_2(\mathbb{R}^n). \tag{5.4.37}$$

Proof. Let us at first assume that $\mathrm{D}_{\mathbb{K}}((\nu_1,\nu_2),(\mu_1,\mu_2)) < \infty$. Then there exist continuous curves ρ_1,ρ_2 and a scalar reaction term ξ which realize the minimum and such that (5.4.28) holds for any $t \in [0,1]$. In particular, we have

$$\mu_1(\mathbb{R}^n) - \nu_1(\mathbb{R}^n) = \int_0^1 \int_{\mathbb{R}^n} \frac{1}{2}\xi \, d\mathbf{f}(\rho_1,\rho_2)\, ds = \mu_2(\mathbb{R}^n) - \nu_2(\mathbb{R}^n),$$

which immediately implies (5.4.37).

Let us now suppose that (5.4.37) holds and that none of the measures μ_i,ν_i is the null measure. Then, we notice that we can transport the measures μ_i,ν_i into the absolutely continuous measures $\bar\mu_i := \mu_i(\mathbb{R}^n)\mathscr{L}^n \llcorner (0,1)^n$ and $\bar\nu_i := \nu_i(\mathbb{R}^n)\mathscr{L}^n \llcorner (0,1)^n$. By Remark 5.4.1 and the triangle inequality, we need just to show that $(\bar\mu_1,\bar\mu_2)$ and $(\bar\nu_1,\bar\nu_2)$ have finite distance.

We may now look for solutions to (5.4.3) such that $v_i \equiv 0$. To this purpose, we observe that any admissible curve (ρ_1,ρ_2) must satisfy $\dot\rho_1 = \dot\rho_2$ in the sense of distributions. Then, for any test function $\phi \in C_{\mathrm{b}}(\mathbb{R}^n)$ and for any $t \in (0,1)$ we have

$$\int_{\mathbb{R}^n} \phi \, d(\rho_{1,t} - \bar\nu_1) = \int_{\mathbb{R}^n} \phi \, d(\rho_{2,t} - \bar\nu_2).$$

This means that $\rho_{2,t} = \rho_{1,t} + \bar\nu_2 - \bar\nu_1$. We can clearly choose the following linear interpolation, due to the identity (5.4.37):

$$\rho_{i,t} = ((1-t)\nu_i(\mathbb{R}^n) + t\mu_i(\mathbb{R}^n))\, \mathscr{L}^n \llcorner (0,1)^n.$$

By definition of $\bar\mu_i$ and $\bar\nu_i$, it is easy to check that $\rho_{i,0} = \bar\nu_i$ and $\rho_{i,1} = \bar\mu_i$. We denote by $\rho_i(t,x)$ the density of $\rho_{i,t}$ with respect to the Lebesgue measure.

The scalar term ξ is then a measurable function which satisfies the equation

$$(\mu_1(\mathbb{R}^n) - \nu_1(\mathbb{R}^n))\chi_{(0,1)^n}(x) = \frac{1}{2} f(\rho_1(t,x),\rho_2(t,x))\xi(t,x).$$

Since $f(0,0) = 0$, we can choose $\xi(t,x) \equiv 0$ for $x \notin (0,1)^n$ and any $t \in (0,1)$. On the other hand, for $x \in (0,1)^n$, $\xi(t,x)$ is a bounded function, since the concavity of f implies

$$\begin{aligned} f(\rho_1(t,x),\rho_2(t,x)) &\geq (1-t) f(\nu_1(\mathbb{R}^n),\nu_2(\mathbb{R}^n)) + t f(\mu_1(\mathbb{R}^n),\mu_2(\mathbb{R}^n)) \\ &\geq \min\{f(\nu_1(\mathbb{R}^n),\nu_2(\mathbb{R}^n)), f(\mu_1(\mathbb{R}^n),\mu_2(\mathbb{R}^n))\} > 0 \end{aligned}$$

for any $x \in (0,1)^n$. This clearly shows that

$$\int_0^1 \int_{\mathbb{R}^n} \frac{1}{2} f(\rho_1,\rho_2)|\xi|^2 \, dx\, dt < \infty,$$

for this choice of $(\rho_1,\rho_2,v_1,v_2,\xi)$, thus proving the finiteness of the distance.

If instead $\mu_1 = \nu_1 = 0$, then (5.4.37) implies $\nu_2(\mathbb{R}^n) = \mu_2(\mathbb{R}^n)$ and Remark 5.4.1 implies that

$$\mathrm{D}_{\mathbb{K}}((0,\nu_2),(0,\mu_2)) \leq W_2(\nu_2,\mu_2) < \infty.$$

If $\nu_1 = \nu_2 = 0$, then (5.4.37) implies $\mu_1(\mathbb{R}^n) = \mu_2(\mathbb{R}^n)$, which we can assume to be non zero. We can take then $\rho_1(t,x) = \rho_2(t,x) = g(t)\chi_{(0,1)^n}(x)$, for some continuous increasing function g such that $g(0) = 0$ and $g(1) = \mu_1(\mathbb{R}^n)$. Therefore, it follows that we can select

$$\xi(t,x) = \frac{2g'(t)}{f(g(t),g(t))}\chi_{(0,1)^n}(x),$$

and we have to prove that there exists an admissible g such that

$$\int_0^1 2\frac{(g'(t))^2}{f(g(t),g(t))}\, dt < \infty.$$

Now we select $g(t) = \mu_1(\mathbb{R}^n)t^\gamma$, for $\gamma > 0$, and then we have that, as $t \to 0$,

$$\frac{(g'(t))^2}{f(g(t), g(t))} \leq Ct^{\gamma-2}.$$

Therefore, it is enough to choose $\gamma > 1$ to obtain integrabiliy.

Finally, if $\nu_1 = 0$ and the other measures are non trivial, we have $\rho_2(t,x) = \rho_1(t,x) + \nu_2(\mathbb{R}^n)\chi_{(0,1)^n}(x)$. Hence, we can argue as in the previous case, this time having $\rho_2(t,x) = (g(t) + \nu_2(\mathbb{R}^n))\chi_{(0,1)^n}(x)$. Therefore, we have

$$\xi(t,x) = \frac{2g'(t)}{f(g(t), g(t)+c)}\chi_{(0,1)^n}(x),$$

where $c = \nu_2(\mathbb{R}^n) > 0$, and we have to prove that there exists an admissible g such that

$$\int_0^1 2\frac{(g'(t))^2}{f(g(t), g(t)+c)}\,dt < \infty.$$

Arguing analogously as before, we choose $g(t) = \mu_1(\mathbb{R}^n)t^\gamma$, for $\gamma > 1$, and we conclude. □

We show now that the convergence with respect to the $\mathrm{D}_{\mathbb{K}}$ distance implies the narrow convergence of each measure in the couple.

Proposition 5.4.14. *Let $(\mu_1, \mu_2), (\nu_1, \nu_2) \in \mathcal{M}^+(\mathbb{R}^n) \times \mathcal{M}^+(\mathbb{R}^n)$, then, for any $\varphi \in \mathrm{Lip_b}(\mathbb{R}^n)$ and $i = 1, 2$, we have*

$$\left|\int_{\mathbb{R}^n} \varphi\, d\mu_i - \int_{\mathbb{R}^n} \varphi\, d\nu_i\right| \leq C(\|\varphi\|_{L^\infty(\mathbb{R}^n)} + \mathrm{Lip}(\varphi))\mathrm{D}_{\mathbb{K}}((\mu_1, \mu_2), (\nu_1, \nu_2)). \tag{5.4.38}$$

Proof. Clearly, we can assume that $\mathrm{D}_{\mathbb{K}}((\mu_1, \mu_2), (\nu_1, \nu_2)) < \infty$, otherwise there is nothing to prove.

Let $\rho_1, \rho_2, v_1, v_2, \xi$ be the minimizing curves, velocity fields and scalar reaction term which realize the distance $\mathrm{D}_{\mathbb{K}}((\mu_1, \mu_2), (\nu_1, \nu_2))$. Then, for any $\varphi \in \mathrm{Lip_b}(\mathbb{R}^n)$ and $i = 1, 2$, we obtain

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} \varphi\, d\mu_i - \int_{\mathbb{R}^n} \varphi\, d\nu_i\right| &= \left|\int_0^1 \frac{d}{dt}\int_{\mathbb{R}^n} \varphi\, d\rho_i\, dt\right| = \left|\int_0^1 \left(\int_{\mathbb{R}^n} v_i \cdot \nabla\varphi\, d\rho_i + \int_{\mathbb{R}^n} \varphi\frac{\xi}{2}\, d\mathbf{f}(\rho_1, \rho_2)\right)\, dt\right| \\
&\leq \mathrm{Lip}(\varphi)\left(\int_0^1 \int_{\mathbb{R}^n} |v_i|^2\, d\rho_i\, dt\right)^{\frac{1}{2}}\left(\int_0^1 \rho_i(\mathbb{R}^n)\, dt\right)^{\frac{1}{2}} \\
&+ \|\varphi\|_{L^\infty(\mathbb{R}^n)}\left(\int_0^1 \int_{\mathbb{R}^n} \frac{|\xi|^2}{2}\, d\mathbf{f}(\rho_1, \rho_2)\, dt\right)^{\frac{1}{2}}\left(\int_0^1 \int_{\mathbb{R}^n} d\mathbf{f}(\rho_1, \rho_2)\, dt\right)^{\frac{1}{2}}.
\end{aligned}$$

Now, by the sublinearity of $\mathbf{f}$ and (5.4.33), we can conclude that there exists $C > 0$ such that

$$\left(\int_0^1 \rho_i(\mathbb{R}^n)\, dt\right)^{\frac{1}{2}} \leq C,$$
$$\left(\int_0^1 \int_{\mathbb{R}^n} d\mathbf{f}(\rho_1, \rho_2)\, dt\right)^{\frac{1}{2}} \leq C.$$

This allows us to conclude. □

Corollary 5.4.15. *Let $(\mu_1, \mu_2), (\mu_1^k, \mu_2^k) \in \mathcal{M}^+(\mathbb{R}^n) \times \mathcal{M}^+(\mathbb{R}^n)$ be such that*

$$\mathrm{D}_{\mathbb{K}}((\mu_1, \mu_2), (\mu_1^k, \mu_2^k)) \to 0$$

as $k \to +\infty$. Then μ_i^k narrowly converges to μ_i, and, in particular, $\mu_i^k(\mathbb{R}^n) \to \mu_i(\mathbb{R}^n)$ for $i = 1, 2$.

Proof. By (5.4.38), we have
$$\int_{\mathbb{R}^n} \psi \, d\mu_i^k \to \int_{\mathbb{R}^n} \psi \, d\mu_i$$
as $k \to +\infty$ for any $\psi \in \mathrm{Lip_b}(\mathbb{R}^n)$. Given then $\varphi \in C_\mathrm{b}(\mathbb{R}^n)$, there exist two sequences $\varphi_j, \phi_j \in \mathrm{Lip_b}(\mathbb{R}^n)$ such that $\varphi_j \uparrow \varphi$ and $\phi_j \downarrow \varphi$. Hence, we obtain

$$\limsup_{k\to+\infty} \int_{\mathbb{R}^n} \varphi \, d\mu_i^k \le \limsup_{k\to+\infty} \int_{\mathbb{R}^n} \phi_j \, d\mu_i^k = \int_{\mathbb{R}^n} \phi_j \, d\mu_i$$

and
$$\liminf_{k\to+\infty} \int_{\mathbb{R}^n} \varphi \, d\mu_i^k \ge \liminf_{k\to+\infty} \int_{\mathbb{R}^n} \varphi_j \, d\mu_i^k = \int_{\mathbb{R}^n} \varphi_j \, d\mu_i.$$
Thus, we conclude by passing to the limit in j. Finally, if we take $\varphi \equiv 1$ in (5.4.38), we immediately obtain the convergence $\mu_i^k(\mathbb{R}^n) \to \mu_i(\mathbb{R}^n)$ for $i = 1, 2$. □

Thanks to Lemma 5.4.12 and Corollary 5.4.15, we can show a narrow convergence result for the minimizing curves.

Lemma 5.4.16. *Let* $(\mu_1, \mu_2), (\mu_1^k, \mu_2^k) \in \mathcal{M}^+(\mathbb{R}^n) \times \mathcal{M}^+(\mathbb{R}^n)$ *be such that*

$$\mathrm{D}_\mathbb{K}((\mu_1, \mu_2), (\mu_1^k, \mu_2^k)) \to 0$$

as $k \to +\infty$. *Let* $(\rho_1^k, \rho_2^k) \in C([0,1]; \mathcal{M}_+(\mathbb{R}^n) \times \mathcal{M}_+(\mathbb{R}^n))$ *be a geodetic curve of couples of nonnegative measures from* (μ_1, μ_2) *to* (μ_1^k, μ_2^k) *with respect to* $\mathrm{D}_\mathbb{K}$. *Then, for any* $s \in [0, 1]$, *we have*
$$\mathrm{D}_\mathbb{K}((\mu_1, \mu_2), (\rho_{1,s}^k, \rho_{2,s}^k)) \to 0$$
and $\rho_{i,s}^k$ *narrowly converges to* μ_i, $i = 1, 2$ *as* $k \to +\infty$.

Proof. Thanks to (5.4.35), we immediately get the convergence of $(\rho_{1,s}^k, \rho_{2,s}^k)$ to (μ_1, μ_2) with respect to $\mathrm{D}_\mathbb{K}$ for any $s \in [0, 1]$. Then, Corollary 5.4.15 implies the narrow convergence. □

However, we point out that the convergence with respect to the $\mathrm{D}_\mathbb{K}$ distance does not imply the convergence of the total mass of the vector valued measures, as we show in the following example.

Example 5.4.17. Let us consider the following sequences of measures:

$$\mu_1^k := g_k(x_1) \, \mathscr{L}^n \llcorner (0,1)^n,$$
$$\mu_2^k := (1 - g_k(x_1)) \, \mathscr{L}^n \llcorner (0,1)^n,$$

where $g_k : [0,1] \to \mathbb{R}$ is given by

$$g_k(t) := \begin{cases} 1 & x \in \displaystyle\bigcup_{m=0}^{2^{k-1}-1} \left[\frac{2m}{2^k}, \frac{2m+1}{2^k} \right], \\ 0 & \text{otherwise,} \end{cases}$$

for $k \ge 1$. It is not difficult to show that

$$\mu_1^k \rightharpoonup \frac{1}{2} \mathscr{L}^n \llcorner (0,1)^n =: \mu_1,$$
$$\mu_2^k \rightharpoonup \frac{1}{2} \mathscr{L}^n \llcorner (0,1)^n =: \mu_2,$$

and that
$$\mu_1^k(\mathbb{R}^n) = \mu_2^k(\mathbb{R}^n) = \mu_1(\mathbb{R}^n) = \mu_2(\mathbb{R}^n) = \frac{1}{2}.$$

In addition, we notice that μ_i^k have compact support inside $[0,1]^n$ for any $k \geq 1$ and $i = 1, 2$, which implies that the second moments are uniformly integrable. Hence, [13, Proposition 7.1.5] implies that

$$W_2(\mu_i^k, \mu_i) \to 0 \quad \text{for } i = 1, 2.$$

The equality of masses and Remark 5.4.1 allow us to obtain

$$\mathrm{D}_{\mathbb{K}}^2((\mu_1^k, \mu_2^k), (\mu_1, \mu_2)) \leq W_2^2(\mu_1^k, \mu_1) + W_2^2(\mu_2^k, \mu_2),$$

and this yields that

$$(\mu_1^k, \mu_2^k) \overset{\mathrm{D}_{\mathbb{K}}}{\to} (\mu_1, \mu_2).$$

However, since μ_1^k and μ_2^k are concentrated on disjoint sets, we have

$$|(\mu_1^k, \mu_2^k)|(\mathbb{R}^n) = \frac{1}{2} + \frac{1}{2} = 1.$$

Instead, $\mu_1 = \mu_2$, and so

$$|(\mu_1, \mu_2)|(\mathbb{R}^n) = \frac{\sqrt{2}}{2}.$$

This shows that

$$\mathrm{D}_{\mathbb{K}}((\mu_1^k, \mu_2^k), (\mu_1, \mu_2)) \to 0$$

does not imply $|(\mu_1^k, \mu_2^k)|(\mathbb{R}^n) \to |(\mu_1, \mu_2)|(\mathbb{R}^n)$.

5.5 First variation of $\mathrm{D}_{\mathbb{K}}$

In analogy with the theory of Wasserstein distance (see for instance [13, Chapter 10] and [19]), it is of interest to study the behaviour of the distance $\mathrm{D}_{\mathbb{K}}$ under smooth perturbations of the endpoint measures. This is indeed a fundamental step in order to derive the Euler-Lagrange equations for the minimizing movement scheme related to the gradient flow of the energy with respect to $\mathrm{D}_{\mathbb{K}}$, in analogy with the approach of [18, 21].

Let $\mu_i, \nu_i \in \mathcal{M}_+(\mathbb{R}^n)$, $\Phi_i \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$ and $\psi_i \in C_c^\infty(\mathbb{R}^n)$ be smooth perturbations, and set $\mu_i^\varepsilon := \phi_{i,\#}^\varepsilon(e^{\varepsilon\psi_i}\mu_i)$, where $\phi_i^\varepsilon := \mathrm{Id} + \varepsilon\Phi_i$. Let ρ_i be optimal curves satisfying (5.4.3).

We perturb the optimal curves ρ_i by taking vector fields and scalar functions

$$\Phi_i^s \in C_c^\infty([0,1] \times \mathbb{R}^n; \mathbb{R}^n), \psi_i^s \in C_c^\infty([0,1] \times \mathbb{R}^n) \text{ such that } \Phi_i^0 \equiv 0, \psi_i^0 \equiv 0, \Phi_i^1 = \Phi_i, \psi_i^1 = \psi_i.$$

We set

$$\rho_{i,s}^\varepsilon := \phi_{i,\#}^{\varepsilon,s}(e^{\varepsilon\psi_i^s}\rho_{i,s}),$$

where $\phi_i^{\varepsilon,s} := \mathrm{Id} + \varepsilon\Phi_i^s$.

If $(\rho_1, \rho_2, v_1, v_2, \xi)$ is a weak solution to the system (5.4.3), then we can obtain a definition of perturbed vector fields v_i^ε and scalar function ξ^ε with a standard procedure (similar to the one exploited in [19]). For any test function $g \in C_c^1(\mathbb{R}^n)$, we have

$$\begin{aligned}
\frac{d}{ds}\int_{\mathbb{R}^n} g\, d\rho_{i,s}^\varepsilon &= \frac{d}{ds}\int_{\mathbb{R}^n} g(\phi_i^{\varepsilon,s}) e^{\varepsilon\psi_i^s}\, d\rho_{i,s} \\
&= \int_{\mathbb{R}^n} \varepsilon e^{\varepsilon\psi_i^s}\left((\nabla g)(\phi_i^{\varepsilon,s})\frac{\partial\Phi_i^s}{\partial s} + g(\phi_i^{\varepsilon,s})\frac{\partial\psi_i^s}{\partial s}\right) d\rho_{i,s} + \\
&\quad + \int_{\mathbb{R}^n} g(\phi_i^{\varepsilon,s}) e^{\varepsilon\psi_i^s} d\left(-\mathrm{div}(v_i\rho_i) + \frac{\xi}{2}\mathbf{f}(\rho_1, \rho_2)\right) \\
&= \int_{\mathbb{R}^n} (\nabla g)(\phi_i^{\varepsilon,s}) e^{\varepsilon\psi_i^s}\left(v_i + \varepsilon\nabla\Phi_i^s \cdot v_i + \varepsilon\frac{\partial\Phi_i^s}{\partial s}\right) d\rho_i + \\
&\quad + g(\phi_i^{\varepsilon,s}) e^{\varepsilon\psi_i^s} d\left(\frac{\xi}{2}\mathbf{f}(\rho_1, \rho_2) + \varepsilon\nabla\psi_i^s \cdot v_i\rho_i + \varepsilon\frac{\partial\psi_i^s}{\partial s}\rho_i\right).
\end{aligned}$$

Hence, we may take

$$v_i^\varepsilon = \left(v_i + \varepsilon \nabla \Phi_i^s \cdot v_i + \varepsilon \frac{\partial \Phi_i^s}{\partial s} \right) \circ (\phi_i^{\varepsilon,s})^{-1}. \tag{5.5.1}$$

As for the scalar reaction term, we get

$$\xi^\varepsilon = \frac{\phi_{i,\#}^{\varepsilon,s}\left(e^{\varepsilon \psi_i^s} \left(\mathbf{f}(\rho_1,\rho_2)\xi + 2\varepsilon \frac{\partial \psi_i^s}{\partial s}\rho_i + 2\varepsilon \nabla \psi_i^s \cdot v_i \rho_i \right) \right)}{\mathbf{f}(\rho_1^\varepsilon, \rho_2^\varepsilon)}. \tag{5.5.2}$$

Hence, we must find conditions under which these two expressions for $i = 1, 2$ are equal. In other words, we need to choose only perturbations ϕ_i^ε and ψ_i^ε satisfying

$$\begin{aligned} \phi_{1,\#}^{\varepsilon,s} \left(e^{\varepsilon \psi_1^s} \left(\mathbf{f}(\rho_1,\rho_2)\xi + 2\varepsilon \frac{\partial \psi_1^s}{\partial s}\rho_1 + 2\varepsilon \nabla \psi_1^s \cdot v_1 \rho_1 \right) \right) = \\ = \phi_{2,\#}^{\varepsilon,s} \left(e^{\varepsilon \psi_2^s} \left(\mathbf{f}(\rho_1,\rho_2)\xi + 2\varepsilon \frac{\partial \psi_2^s}{\partial s}\rho_2 + 2\varepsilon \nabla \psi_2^s \cdot v_2 \rho_2 \right) \right). \end{aligned} \tag{5.5.3}$$

This is indeed a quite serious issue, since it does not seem straightforward to derive such conditions, except for a few simple cases which we list here.

- **(The case $\Phi_1^s = \Phi_2^s$)** If we assume $\Phi_1^s = \Phi_2^s = \Phi^s$, we look for ψ_1^s, ψ_2^s satisfying

$$\begin{aligned} \xi(e^{\varepsilon\psi_1} - e^{\varepsilon\psi_2})\mathbf{f}(\rho_1,\rho_2) &= 2\varepsilon(e^{\varepsilon\psi_2}\frac{\partial \psi_2}{\partial s}\rho_2 + e^{\varepsilon\psi_2}\nabla\psi_2 \cdot v_2\rho_2 - e^{\varepsilon\psi_1}\frac{\partial \psi_1}{\partial s}\rho_1 - e^{\varepsilon\psi_1}\nabla\psi_1 \cdot v_1\rho_1) \\ &= 2(D_{s,x}e^{\varepsilon\psi_2}\cdot(1,v_2)\rho_2 - D_{s,x}e^{\varepsilon\psi_1}\cdot(1,v_1)\rho_1). \end{aligned} \tag{5.5.4}$$

  If we set $\gamma_i^\varepsilon := e^{\varepsilon\psi_i}$, then we obtain some type of nonlinear transport equation:

$$\xi(\gamma_1^\varepsilon - \gamma_2^\varepsilon)\mathbf{f}(\rho_1,\rho_2) = 2(D_{s,x}\gamma_2^\varepsilon \cdot (1,v_2)\rho_2 - D_{s,x}\gamma_1^\varepsilon \cdot (1,v_1)\rho_1).$$

  In the particular case $f(x,y) = \sqrt{xy}$ and $\rho_i \ll \mathscr{L}^n$, this reduces to

$$\frac{1}{2}\xi(\gamma_1^\varepsilon - \gamma_2^\varepsilon) = D_{s,x}\gamma_2^\varepsilon \cdot (1,v_2)\sqrt{\frac{\rho_2}{\rho_1}} - D_{s,x}\gamma_1^\varepsilon \cdot (1,v_1)\sqrt{\frac{\rho_1}{\rho_2}},$$

  on $\mathrm{supp}(\rho_1) \cap \mathrm{supp}(\rho_2)$.

- **(The case $\psi_1^s = \psi_2^s = 0$)** If we assume $\psi_1^s = \psi_2^s \equiv 0$, then (5.5.3) reduces to

$$\phi_{1,\#}^{\varepsilon,s}(\xi\mathbf{f}(\rho_1,\rho_2)) = \phi_{2,\#}^{\varepsilon,s}(\xi\mathbf{f}(\rho_1,\rho_2)).$$

  It follows immediately that this implies $\phi_1^{\varepsilon,s} = \phi_2^{\varepsilon,s}$ on $\mathrm{supp}(\rho_1)\cap\mathrm{supp}(\rho_2)\cap\mathrm{supp}(\xi)$, and so $\Phi_1^s = \Phi_2^s = \Phi^s$. Hence, if we do not change the masses of μ_1 and μ_2, the only perturbation which is allowed is the one with the same push-forward in both components. This can be also seen as the trivial solution to (5.5.3), since $\Phi_1^s = \Phi_2^s = \Phi^s$ and $\psi_1^s = \psi_2^s \equiv 0$.

  One could argue that this implies a strong rigidity, which seems to depend only on the reaction term. Indeed, in the limiting case of $f \equiv 0$, we would have the Wasserstein distance, for which no such condition is required. However, it might be that, since the model describes cancellations, then there could be some sort of balance between ρ_1 and ρ_2: by moving them independently in the space, we might make this balance condition fail.

We focus now on this latter particular case; that is, $\Phi_1^s = \Phi_2^s = \Phi^s$ and $\psi_1^s = \psi_2^s \equiv 0$. By (5.5.1) and (5.5.2), we obtain

$$
\begin{aligned}
v_i^\varepsilon &= \left(v_i + \varepsilon \nabla \Phi^s \cdot v_i + \varepsilon \frac{\partial \Phi^s}{\partial s}\right) \circ (\phi^{\varepsilon,s})^{-1}, \\
\xi^\varepsilon &= \frac{\phi_\#^{\varepsilon,s}(\mathbf{f}(\rho_1,\rho_2)\xi)}{\mathbf{f}(\rho_1^\varepsilon,\rho_2^\varepsilon)}.
\end{aligned}
$$

Therefore, standard calculations yield

$$
\begin{aligned}
\mathrm{D}_{\mathbb{K}}((\nu_1,\nu_2),(\mu_1^\varepsilon,\mu_2^\varepsilon)) - \mathrm{D}_{\mathbb{K}}((\nu_1,\nu_2),(\mu_1,\mu_2)) \le 2\varepsilon \int_0^1 \int_{\mathbb{R}^n} &(v_1 \cdot \nabla \Phi^s \cdot v_1 + \frac{\partial \Phi^s}{\partial s} \cdot v_1)\rho_1 + \\
&+ (v_2 \cdot \nabla \Phi^s \cdot v_2 + \frac{\partial \Phi^s}{\partial s} \cdot v_2)\rho_2 \, ds + \\
&+ \frac{1}{2}\int_0^1 \int_{\mathbb{R}^n} \left| \frac{\phi_\#^{\varepsilon,s}(\mathbf{f}(\rho_1,\rho_2)\xi)}{\mathbf{f}(\rho_1^\varepsilon,\rho_2^\varepsilon)} \right|^2 \mathbf{f}(\rho_1^\varepsilon,\rho_2^\varepsilon) - |\xi|^2 \mathbf{f}(\rho_1,\rho_2)\, ds.
\end{aligned}
$$

If we assume now that $\rho_i \ll \mathscr{L}^n$, then

$$
\xi^\varepsilon = \frac{(f(\rho_1,\rho_2)\xi)}{f\left(\frac{\rho_1}{\det(\nabla\phi^{\varepsilon,s})}, \frac{\rho_2}{\det(\nabla\phi^{\varepsilon,s})}\right)\det(\nabla \phi^{\varepsilon,s})} \circ (\phi^{\varepsilon,s})^{-1},
$$

and so we get

$$
\begin{aligned}
\int_{\mathbb{R}^n} |\xi^\varepsilon|^2 \, d\mathbf{f}(\rho_1^\varepsilon,\rho_2^\varepsilon) &= \int_{\mathbb{R}^n} |\xi|^2 f(\rho_1,\rho_2) \frac{f(\rho_1,\rho_2)}{f\left(\frac{\rho_1}{\det(\nabla\phi^{\varepsilon,s})}, \frac{\rho_2}{\det(\nabla\phi^{\varepsilon,s})}\right)\det(\nabla \phi^{\varepsilon,s})} \, dx \\
&= \int_{\mathbb{R}^n} |\xi|^2 f(\rho_1,\rho_2) - \varepsilon|\xi|^2 f(\rho_1,\rho_2)\mathrm{Tr}\nabla\Phi^s + \\
&\quad + \varepsilon \mathrm{Tr}\nabla\Phi^s |\xi|^2 \left(\frac{\partial f}{\partial x}(\rho_1,\rho_2)\rho_1 + \frac{\partial f}{\partial y}(\rho_1,\rho_2)\rho_2\right) + o(\varepsilon)\, dx.
\end{aligned}
$$

Hence, the contribution to the first variation of the reaction part is

$$
\frac{1}{2}\int_0^1 \int_{\mathbb{R}^n} \mathrm{Tr}\nabla\Phi^s |\xi|^2 \left(\frac{\partial f}{\partial x}(\rho_1,\rho_2)\rho_1 + \frac{\partial f}{\partial y}(\rho_1,\rho_2)\rho_2 - f(\rho_1,\rho_2)\right) dx\, ds. \tag{5.5.5}
$$

In particular, if f is 1-homogeneous (for instance, $f(x,y) = \sqrt{xy}$), then the reaction term does not contribute to the first variation, since, by Euler theorem on homogeneous functions,

$$
f(x,y) = x\frac{\partial f}{\partial x}(x,y) + y\frac{\partial f}{\partial y}(x,y).
$$

We can now conclude that, at least under the absolute continuity assumption on ρ_1, ρ_2, the first variation of $\mathrm{D}_{\mathbb{K}}$ with respect to the push-forward perturbation ϕ is given by

$$
\begin{aligned}
(\Phi,\Phi) \to 2\int_0^1 \int_{\mathbb{R}^n} &(v_1 \cdot D\Phi^s \cdot v_1 + \frac{\partial \Phi^s}{\partial s} \cdot v_1)\rho_1 + (v_2 \cdot D\Phi^s \cdot v_2 + \frac{\partial \Phi^s}{\partial s} \cdot v_2)\rho_2 \, ds + \\
+ \frac{1}{2}\int_0^1 \int_{\mathbb{R}^n} &\mathrm{Tr} D\Phi^s |\xi|^2 \left(\frac{\partial f}{\partial x}(\rho_1,\rho_2)\rho_1 + \frac{\partial f}{\partial y}(\rho_1,\rho_2)\rho_2 - f(\rho_1,\rho_2)\right) ds.
\end{aligned} \tag{5.5.6}
$$

It is worth to notice that in the case f is a 1-homogeneous function, then the second term is identically zero, so that only the transport part contributes to this first variation.

Clearly, the absolute continuity assumption on the curves ρ_i is quite strong, hence, it would be interesting to consider the case of $\mu_i, \nu_i \ll \mathscr{L}^n$ and to investigate whether under this condition we would indeed have $\rho_i \ll \mathscr{L}^n$, in analogy with the classical case of the Wasserstein distance.

5.6 The descending slope of the energy

Following the idea behind the minimizing movements scheme (for a detailed exposition, we refer to [13, Chapter 2]), it seems natural to look for the existence of solutions to the system (5.3.6), by seeing it as a gradient flow of the energy

$$\Phi(\mu_1, \mu_2) = \frac{1}{2}\int_{\mathbb{R}^n} (V * \mu)\, d\mu + \mu_1(\mathbb{R}^n) + \mu_2(\mathbb{R}^n)$$

with respect to the distance $\mathrm{D}_{\mathbb{K}}$. By [13, Theorem 2.3.3], if Φ satisfies some lower semicontinuity and coercivity assumptions and the relaxed slope $\partial^-\Phi$ of Φ (the sequentially lower semicontinuous envelope of the local slope of Φ) is a strong upper gradient for Φ itself, then any curve obtained as limits of the (generalized) minimizing movements scheme is a curve of maximal slope for $|\partial^-\Phi|$ which satisfies the following *energy dissipation equality*:

$$\frac{1}{2}\int_0^T |(\mu_1,\mu_2)'|(t)^2\, dt + \frac{1}{2}\int_0^T |\partial^-\Phi(\mu_1(t),\mu_2(t))|^2 + \Phi(\mu_1(T),\mu_2(T)) = \Phi(\mu_1(0),\mu_2(0))$$

for any $T > 0$, where $|(\mu_1,\mu_2)'|(t)$ is the metric derivative of the curve $(\mu_1(t),\mu_2(t))$ with respect to the distance $D_{\mathbb{K}}$; that is,

$$|(\mu_1,\mu_2)'|(t) := \lim_{h\to 0} \frac{\mathrm{D}_{\mathbb{K}}((\mu_1(t+h),\mu_2(t+h)),(\mu_1(t),\mu_2(t)))}{|h|}.$$

However, it turns out that, in general, the self-energy

$$\Phi_{\mathrm{self}}(\mu_1,\mu_2) := \mu_1(\mathbb{R}^n) + \mu_2(\mathbb{R}^n)$$

admits a local slope $|\partial\Phi_{\mathrm{self}}|$ which is upper semicontinuous and not lower semicontinuous with respect to $\mathrm{D}_{\mathbb{K}}$.

We devote this section to proving this claim.

By the definition of the *local (or descending) slope*, by setting $\mu := (\mu_1,\mu_2)$, we have

$$|\partial\Phi_{\mathrm{self}}|(\mu) := \limsup_{\mu^k \overset{\mathrm{D}_{\mathbb{K}}}{\to} \mu} \frac{(\Phi_{\mathrm{self}}(\mu) - \Phi_{\mathrm{self}}(\mu^k))^+}{\mathrm{D}_{\mathbb{K}}(\mu^k,\mu)}. \tag{5.6.1}$$

Proposition 5.6.1. *We have*

$$|\partial\Phi_{\mathrm{self}}|(\mu_1,\mu_2) \le \sqrt{2\int_{\mathbb{R}^n} d\mathbf{f}(\mu_1,\mu_2)}. \tag{5.6.2}$$

Proof. Let $((\mu_1^k,\mu_2^k))_{k\ge 1}$ be a sequence of couples of nonnegative measures on $\mathbb{R}^n$ converging to (μ_1,μ_2) with respect to $\mathrm{D}_{\mathbb{K}}$. In particular, we can assume that $\mathrm{D}_{\mathbb{K}}((\mu_1^k,\mu_2^k),(\mu_1,\mu_2)) < \infty$ for any $k \ge 1$. Hence, by Proposition 5.4.9, there exist $(\rho_1^k,\rho_2^k) \in C([0,1];\mathcal{M}_+(\mathbb{R}^n)\times\mathcal{M}_+(\mathbb{R}^n))$, $v_1^k, v_2^k \in L^2((0,1);L^2(\mathbb{R}^n;\rho_i^k))$ and $\xi^k \in L^2((0,1);L^2(\mathbb{R}^n;\mathbf{f}(\rho_1^k,\rho_2^k)))$ such that

$$\mathrm{D}_{\mathbb{K}}((\mu_1^k,\mu_2^k),(\mu_1,\mu_2))^2 = \int_0^1 \left(\int_{\mathbb{R}^n} |v_1^k|^2\, d\rho_1^k + |v_2^k|^2\, d\rho_2^k + \frac{1}{2}|\xi^k|^2\, d\mathbf{f}(\rho_1^k,\rho_2^k)\right) ds.$$

In particular, if we set $\mu := (\mu_1,\mu_2)$ and $\mu^k := (\mu_1^k,\mu_2^k)$, we get

$$\mathrm{D}_{\mathbb{K}}(\mu,\mu^k) \ge \sqrt{\int_0^1\int_{\mathbb{R}^n} \frac{1}{2}|\xi^k|^2\, d\mathbf{f}(\rho_1^k,\rho_2^k)\, ds}.$$

Employing now (5.4.28) with $t=1$ and $s=0$ and Cauchy-Schwarz inequality, we get

$$\begin{aligned}\Big(\Phi_{\text{self}}(\mu)-\Phi_{\text{self}}(\mu^k)\Big)^+ &= \Big(\mu_1(\mathbb{R}^n)+\mu_2(\mathbb{R}^n)-\mu_1^k(\mathbb{R}^n)-\mu_2^k(\mathbb{R}^n)\Big)^+\\ &= \Big(\int_0^1\int_{\mathbb{R}^n}\xi^k\, d\mathbf{f}(\rho_1^k,\rho_2^k)\,ds\Big)^+\\ &\le \sqrt{\int_0^1\int_{\mathbb{R}^n}|\xi^k|^2\, d\mathbf{f}(\rho_1^k,\rho_2^k)\,ds}\sqrt{\int_0^1\int_{\mathbb{R}^n} d\mathbf{f}(\rho_1^k,\rho_2^k)\,ds}.\end{aligned}$$

All in all, we obtain

$$|\partial\Phi_{\text{self}}|(\mu_1,\mu_2)\le \limsup_{\mu^k\xrightarrow{\mathrm{D}_{\mathbb{K}}}\mu}\sqrt{2\int_0^1\int_{\mathbb{R}^n} d\mathbf{f}(\rho_1^k,\rho_2^k)\,ds}.$$

We notice now that, thanks to (5.4.34), we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} d\mathbf{f}(\rho_{1,s}^k,\rho_{2,s}^k) &\le C\left(\rho_{1,s}^k(\mathbb{R}^n)+\rho_{2,s}^k(\mathbb{R}^n)\right)\\ &\le C\left(\mu_1(\mathbb{R}^n)+\mu_2(\mathbb{R}^n)+\frac{1}{2}\mathrm{D}_{\mathbb{K}}((\mu_1,\mu_2),(\mu_1^k,\mu_2^k))\right)e^{\frac{C}{2}s},\end{aligned}$$

which yields the existence of a majorant in $L^1((0,1))$ for the functions $s\to\int_{\mathbb{R}^n} d\mathbf{f}(\rho_{1,s}^k,\rho_{2,s}^k)$. Thus, we can employ the continuity of the square root, Reverse Fatou's Lemma, Lemma 5.4.16 and (5.4.9) to conclude that

$$|\partial\Phi_{\text{self}}|(\mu_1,\mu_2)\le\sqrt{2\int_0^1\limsup_{\mu^k\xrightarrow{\mathrm{D}_{\mathbb{K}}}\mu}\int_{\mathbb{R}^n} d\mathbf{f}(\rho_1^k,\rho_2^k)\,ds}\le\sqrt{2\int_0^1\int_{\mathbb{R}^n} d\mathbf{f}(\mu_1,\mu_2)\,ds}=\sqrt{2\int_{\mathbb{R}^n} d\mathbf{f}(\mu_1,\mu_2)}.$$

□

Under some additional assumptions on f, it is actually possible to obtain an equality in (5.6.4). As a consequence of this and Remark (5.4.3), we deduce that the local slope of the self-energy is upper semicontinuous with respect to the distance $\mathrm{D}_{\mathbb{K}}$, but not lower semicontinuous.

Proposition 5.6.2. *Let $f:[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ be a continuous concave function such that* (5.4.1) *holds and satisfying also*

$$f(x,y)\le C\min\{x,y\}, \tag{5.6.3}$$

for some constant $C>0$. Then, we have

$$|\partial\Phi_{\text{self}}|(\mu_1,\mu_2)=\sqrt{2\int_{\mathbb{R}^n} d\mathbf{f}(\mu_1,\mu_2)}. \tag{5.6.4}$$

In particular, $|\partial\Phi_{\text{self}}|$ is upper and not lower semicontinuous with respect to the distance $\mathrm{D}_{\mathbb{K}}$.

Proof. Thanks to (5.6.4), we need only to show that there exists a suitable sequence of μ^k such that

$$\lim_{k\to+\infty}\frac{(\Phi_{\text{self}}(\mu)-\Phi_{\text{self}}(\mu^k))^+}{\mathrm{D}_{\mathbb{K}}(\mu^k,\mu)}\ge\sqrt{2\int_{\mathbb{R}^n} d\mathbf{f}(\mu_1,\mu_2)}. \tag{5.6.5}$$

Let $\mu_i^k=\mu_i-\varepsilon_k\mathbf{f}(\mu_1,\mu_2)$, for some nonnegative sequence $\varepsilon_k\to 0$. Thanks to (5.6.3), it is clear that $\mu_i^k\ge 0$ if we assume $\varepsilon_k C\le 1$. It is easy to notice that

$$\Big(\Phi_{\text{self}}(\mu_1,\mu_2)-\Phi_{\text{self}}(\mu_1^k,\mu_2^k)\Big)^+=2\varepsilon_k\int_{\mathbb{R}^n} d\mathbf{f}(\mu_1,\mu_2).$$

We observe now that a solution to (5.4.3) is given by $\rho^k_{i,s} = (1-s)\mu_i + s\mu_i^k$, $v_i = 0$, for $i = 1, 2$, and $\xi^k \in L^2((0,1); L^2(\mathbb{R}^n; \mathbf{f}(\rho_1^k, \rho_2^k)))$ satisfying

$$-\varepsilon_k \mathbf{f}(\mu_1, \mu_2) = \frac{\xi^k}{2} \mathbf{f}(\rho_i^k, \rho_2^k). \tag{5.6.6}$$

We check now the integrability property of such ξ^k. Since $0 \le f(x, y) \le C(x + y)$, then there exists a function $g \in L^1(\mathbb{R}^n; \mu_1 + \mu_2)$ such that

$$\mathbf{f}(\mu_1, \mu_2) = g\,(\mu_1 + \mu_2) \quad \text{and} \quad 0 \le g \le C. \tag{5.6.7}$$

In addition,

$$0 \le \mathbf{f}(\rho^k_{1,s}, \rho^k_{2,s}) \le C\,(\mu_1 + \mu_2 - 2\varepsilon_k s \mathbf{f}(\mu_1, \mu_2)) \le C\,(\mu_1 + \mu_2),$$

so that there exists $g_{k,s} \in L^1(\mathbb{R}^n; \mu_1 + \mu_2)$ such that

$$\mathbf{f}(\rho^k_{1,s}, \rho^k_{2,s}) = g_{k,s}\,(\mu_1 + \mu_2) \quad \text{and} \quad 0 \le g_{k,s} \le C. \tag{5.6.8}$$

Combining (5.6.7) and (5.6.8), we obtain the following expression for ξ^k:

$$\xi^k = -2\varepsilon_k \frac{g}{g_{k,s}}. \tag{5.6.9}$$

We now let $\gamma := \mathbf{f}(\mu_1, \mu_2)$ and we use the concavity of f to get

$$\begin{aligned}
\mathbf{f}(\rho^k_{1,s}, \rho^k_{2,s}) &= \mathbf{f}\left(\mu_1(1 - \sqrt{\varepsilon_k}) + (\mu_1 - \sqrt{\varepsilon_k} s\gamma)\sqrt{\varepsilon_k}, \mu_2(1 - \sqrt{\varepsilon_k}) + (\mu_2 - \sqrt{\varepsilon_k} s\gamma)\sqrt{\varepsilon_k}\right) \\
&\ge (1 - \sqrt{\varepsilon_k})\mathbf{f}(\mu_1, \mu_2) + \sqrt{\varepsilon_k}\mathbf{f}(\mu_1 - \sqrt{\varepsilon_k} s\gamma, \mu_2 - \sqrt{\varepsilon_k} s\gamma) \\
&\ge (1 - \sqrt{\varepsilon_k})\mathbf{f}(\mu_1, \mu_2),
\end{aligned}$$

where we used (5.6.3) to ensure that $\mu_i - \sqrt{\varepsilon_k} s \mathbf{f}(\mu_1, \mu_2) \ge 0$, for any $i \in \{1, 2\}$ and $s \in [0, 1]$, provided that $\sqrt{\varepsilon_k} C \le 1$, which can be assumed without loss of generality. As an immediate consequence, we obtain the following relation between g and $g_{k,s}$:

$$g_{k,s} \ge (1 - \sqrt{\varepsilon_k}) g.$$

It is easy to see that this and (5.6.9) imply

$$|\xi^k| \le 2 \frac{\varepsilon_k}{1 - \sqrt{\varepsilon_k}}, \tag{5.6.10}$$

from which we conclude that $\xi^k \in L^2((0,1); L^2(\mathbb{R}^n; \mathbf{f}(\rho_1^k, \rho_2^k)))$. In addition, it is easy to notice that $\xi^k \le 0$. Hence, thanks to (5.4.2), (5.6.6) and (5.6.10), we get

$$\begin{aligned}
\mathrm{D}_{\mathbb{K}}((\mu_1^k, \mu_2^k), (\mu_1, \mu_2))^2 &\le \int_0^1 \int_{\mathbb{R}^n} \frac{|\xi|^2}{2}\, d\mathbf{f}(\rho^k_{1,s}, \rho^k_{2,s})\, ds \\
&\le \frac{\varepsilon_k}{1 - \sqrt{\varepsilon_k}} \int_0^1 \int_{\mathbb{R}^n} (-\xi_k)\, d\mathbf{f}(\rho^k_{1,s}, \rho^k_{2,s})\, ds \\
&= \frac{\varepsilon_k}{1 - \sqrt{\varepsilon_k}} \int_0^1 \int_{\mathbb{R}^n} 2\varepsilon_k\, d\mathbf{f}(\mu_1, \mu_2)\, ds = 2\frac{\varepsilon_k^2}{1 - \sqrt{\varepsilon_k}} \int_{\mathbb{R}^n} d\mathbf{f}(\mu_1, \mu_2).
\end{aligned}$$

All in all, we obtain

$$\frac{\left(\Phi_{\mathrm{self}}(\mu_1, \mu_2) - \Phi_{\mathrm{self}}(\mu_1^k, \mu_2^k)\right)^+}{\mathrm{D}_{\mathbb{K}}((\mu_1^k, \mu_2^k), (\mu_1, \mu_2))} \ge \sqrt{2(1 - \sqrt{\varepsilon_k}) \int_{\mathbb{R}^n} d\mathbf{f}(\mu_1, \mu_2)},$$

which easily implies (5.6.5), so that (5.6.4) is now proved.

Finally, Remark (5.4.3) implies immediately the upper semicontinuity of $|\partial\Phi_{\mathrm{self}}|$, while, in order to prove that $|\partial\Phi_{\mathrm{self}}|$ is not lower semicontinuous with respect to the distance $\mathrm{D}_{\mathbb{K}}$, it is enough to construct an example which shows that $\int_{\mathbb{R}^n} d\mathbf{f}(\cdot,\cdot)$ is not lower semicontinuous with respect to $\mathrm{D}_{\mathbb{K}}$.

We recall that $f(0,x) = 0 = f(x,0)$ for any $x > 0$. Without loss of generality, we assume that $f\left(\frac{1}{2},\frac{1}{2}\right) > 0$. Then, we take the sequences of measures (μ_1^k,μ_2^k) of Example 5.4.17, which satisfy

$$(\mu_1^k,\mu_2^k) \overset{\mathrm{D}_{\mathbb{K}}}{\to} (\mu_1,\mu_2)$$

for $\mu_1 = \mu_2 = \frac{1}{2}\mathscr{L}^n \llcorner (0,1)^n$.

Since $g_k(x_1)(1-g_k(x_1)) = 0$ for any $x_1 \in [0,1]$, it is clear that

$$\int_{\mathbb{R}^n} d\mathbf{f}(\mu_1^k,\mu_2^k) = \int_0^1 f(g_k(x_1), 1-g_k(x_1))\, dx_1 = 0$$

for any $k \geq 1$, while

$$\int_{\mathbb{R}^n} d\mathbf{f}(\mu_1,\mu_2) = f\left(\frac{1}{2},\frac{1}{2}\right) > 0.$$

Thus, this means that $\int_{\mathbb{R}^n} d\mathbf{f}(\cdot,\cdot)$ is not lower semicontinuous with respect to the $\mathrm{D}_{\mathbb{K}}$ distance, and so is $|\partial\Phi_{\mathrm{self}}|$. $\square$

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