AREA FORMULA FOR REGULAR SUBMANIFOLDS OF LOW CODIMENSION IN HEISENBERG GROUPS

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Abstract. We establish an area formula for the spherical measure of intrinsically regular submanifolds of low codimension in Heisenberg groups. The spherical measure is computed with respect to an arbitrary homogeneous distance. Among the arguments of the proof, we point out the differentiability properties of intrinsic graphs and a chain rule for intrinsic differentiable functions.

Contents

1. Introduction

Research in the Analysis and Geometry of simply connected nilpotent Lie groups has spread into several directions, especially in the last decade. Carnot groups, or stratified groups equipped with a homogeneous left invariant distance, are an important class of these nilpotent groups, which are metrically different from Euclidean spaces or Riemannian manifolds, still maintaining a rich algebraic structure. The Heisenberg group $\mathbb{H}^n$ represents one of the prominent models.

More specifically, we are interested in computing the spherical measure of submanifolds in $\mathbb{H}^n$ with respect to a homogeneous distance of the group. For different classes of C^1 smooth submanifolds area formulas are available, [17]. The question has new difficulties, when we consider "intrinsic regular submanifolds" of $\mathbb{H}^n$, that need not be C^1 smooth, nor Lipschitz with respect to the Euclidean distance. These submanifolds in $\mathbb{H}^n$ and their

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characterizations have been studied in several papers, [2], [6], [7], [20]. They are called $\mathbb{H}$-regular surfaces of low codimension in $\mathbb{H}^n$ (Definition 2.10) and can be seen as level sets of continuously differentiable functions from $\mathbb{H}^n$ to $\mathbb{R}^k$, $1 \le k \le n$. We stress that differentiability is always understood with respect to the group operation and dilations, i.e. the so-called Pansu differentiability. In the general setting of intrinsic regular submanifolds in homogeneous groups, $\mathbb{H}$-regular surfaces of low codimension correspond to $(\mathbb{H}^n, \mathbb{R}^k)$-submanifolds of $\mathbb{H}^n$ with $1 \le k \le n$, [14].

An implicit function theorem, proved in [6], states that any of these $\mathbb{H}$-regular surfaces can be locally seen as intrinsic graphs with respect to a special semidirect factorization (Definition 2.5). The parametrizing mapping ϕ associated to the factorization acts between the two factors, which are a vertical and a horizontal subgroup of $\mathbb{H}^n$ (Definition 2.2). In [2] the authors proved that ϕ is uniformly intrinsic differentiable (Definition 2.8), while it is only Hölder continuous with respect to the Euclidean metric. Uniform intrinsic differentiability for maps acting between homogeneous subgroups has been characterized in various ways, [1], [3] [4], [5], [10]. Unexpectedly, this differentiability turns out to be the natural tool to perform the blow-up of intrinsic regular submanifolds and we will apply its characterization stated in Theorem 2.9.

The main result of this work is an area formula for the spherical measure of $\mathbb{H}$-regular surfaces of low codimension. Once a homogeneous distance d on $\mathbb{H}^n$ is fixed, the spherical measure with respect to d can be introduced through the usual Carathéodory construction (Definition 2.15). Precisely, we choose $1 \le k \le n$ and a parametrized $\mathbb{H}$-regular surface Σ of codimension k (Definition 2.12). We associate to Σ a "parametrized measure" μ, using a defining mapping f and a parametrizing map ϕ, according to (1). This measure had already appeared in [6], where the authors introduced it to prove an area formula for the centered Hausdorff measure of Σ, see [20, Theorem 4.5]. The choice of the measure μ is furthermore justified by [4, Theorem 6.1], where it is shown how μ can be rewritten uniquely in terms of intrinsic derivatives of the group valued mapping ϕ.

Our central result is the following theorem.

Theorem 1.1 (Upper blow-up). *Let $\mathbb{H}^n = \mathbb{W} \rtimes \mathbb{V}$ be equipped with a homogeneous distance d and let Σ be a parametrized $\mathbb{H}$-regular surface with respect $(\mathbb{W}, \mathbb{V})$ and having codimension k, with $1 \le k \le n$. Let $\phi : U \to \mathbb{V}$ be its parametrization, where $U \subset \mathbb{W}$ is open and the graph mapping $\Phi : U \to \mathbb{H}^n$ of ϕ defines the intrinsic graph $\Sigma = \Phi(U)$. If $f \in C_h^1(\Omega, \mathbb{R}^k)$ with $\Sigma = f^{-1}(0)$ and $J_H f(x) > 0$ for all $x \in \Sigma$, then*

$$J_{\mathbb{V}} f(y) = \|\nabla_{\mathbb{V}} f_1(y) \wedge \cdots \wedge \nabla_{\mathbb{V}} f_k(y)\|_g > 0$$

for any $y \in \Sigma$. Let $v_1, \dots v_k \in H_1$ be orthonormal vectors such that $\mathbb{V} = span\{v_1, \dots, v_k\}$ and set $V = v_1 \wedge \cdots \wedge v_k$. Consider an orthonormal basis $\{w_{k+1}, \dots, w_{2n}, e_{2n+1}\}$ of $\mathbb{W}$ and define $N = w_{k+1} \wedge \cdots \wedge w_{2n} \wedge e_{2n+1}$. Let us introduce the following measure

$$\mu(B) = \|V \wedge N\|_g \int_{\Phi^{-1}(B)} \frac{J_H f(\Phi(n))}{J_{\mathbb{V}} f(\Phi(n))} \, d\mathcal{H}_E^{2n+1-k}(n) \tag{1}$$

for every Borel set $B \subset \mathbb{H}^n$. Then for every $x \in \Sigma$ we have

$$\theta^{2n+2-k}(\mu, x) = \beta_d(Tan(\Sigma, x)).$$

The geometric functions $\theta^{2n+2-k}(\mu, \cdot)$ and $\beta_d(\cdot)$ are the $(2n+2-k)$-spherical Federer density of μ at x (Definition 2.17) and the spherical factor (Definition 2.18), respectively.

The homogeneous tangent cone $\mathrm{Tan}(\Sigma, x)$ of Σ at x is metrically defined (Definition 2.11) and for $\mathbb{H}$-regular surfaces of low codimension equals ker $Df(x)$, see [6, Proposition 3.29], or [14, Theorem 1.7] in the setting of homogeneous groups. More information on the notions involved in Theorem 1.1 is provided in Section 2.

The terminology "upper blow-up" goes back to [16], where a Federer density was first computed with applications to sets of finite perimeter in stratified groups. Indeed, the Federer density is a suitable limit superior of the ratio between the measure we wish to differentiate and the gauge of the spherical measure with respect to a class of sets, see [15] for more information.

In our higher codimensional framework, the proof of the upper blow-up involves some new features. Three key aspects must be emphasized. First, the intrinsic differentiability of the parametrizing map ϕ (Theorem 2.7) is crucial in establishing the limit of the set (24) in the proof of the upper blow-up. Second, we prove an "intrinsic chain rule" (Theorem 2.2) that permits us to connect the kernel of Df with the intrinsic differential of ϕ, according to (26). However, to make our chain rule work we have slightly modified the well known notion of intrinsic differentiability associated to a factorization (Definition 2.9). We will deserve more attention on this differentiability and the chain rule for next investigations, since they may have an independent interest. Third, we establish a delicate algebraic lemma for computing the Jacobian of projections between vertical subgroups, that are associated to two semidirect factorizations with the same horizontal factor (Lemma 3.1).

By combining Theorem 1.1 with an abstract measure theoretic area formula, see [15, Theorem 11] or [17, Theorem 7.2], we arrive at the area formula for an $\mathbb{H}$-regular surface, using μ and the spherical measure $\mathcal{S}^{2n+2-k}$ with respect to any homogeneous distance d. In the assumptions of Theorem 1.1, for any Borel set $B \subset \Sigma$ we have

$$\mu(B) = \int_B \beta_d(\mathrm{Tan}(\Sigma, x))\ d\mathcal{S}_0^{2k+2-k}(x).$$

If the factors of the semidirect product are orthogonal, then the measure μ can be written in terms of the intrinsic partial derivatives of the parametrization ϕ of Σ as follows.

Theorem 1.2. *In the assumptions of Theorem 1.1, if in addition $\mathbb{W}$ is orthogonal to $\mathbb{V}$, then for every Borel set $B \subset \Sigma$ we have*

$$\int_B \beta_d(\mathit{Tan}(\Sigma, x))\ d\mathcal{S}_0^{2k+2-k}(x) = \int_{\Phi^{-1}(B)} J^\phi\phi(w)\ d\mathcal{H}_E^{2n+1-k}(w) \tag{2}$$

where $J^\phi\phi$ is the natural intrinsic Jacobian of ϕ, introduced in Definition 2.14.

If the distance d is invariant under some classes of symmetries (Definition 2.19) or it is multiradial (Definition 2.21), then the area formula simplifies (Theorem 4.2). Precisely, in these cases the spherical factor only depends on the distance and on the fixed scalar product on $\mathbb{H}^n$, becoming a geometric constant. We may denote it by $\omega_d(2n+1-k)$ and include it in the spherical measure by defining $\mathcal{S}_d^{2n+1-k} = \omega_d(2n+1-k)\ \mathcal{S}_0^{2n+1-k}$. Thus, by the area formula (2), we have

$$\mathcal{S}_d^{2n+2-k}(B) = \int_{\Phi^{-1}(B)} J^\phi\phi(w)\ d\mathcal{H}_E^{2n+1-k}(w)$$

for any Borel set $B \subset \Sigma$. For instance, the previous formula holds for the Cygan-Korányi distance and the sub-Riemannian distance.

Some additional applications follow from our results. By a slight modification of the proof of Theorem 1.1, a standard blow-up theorem computing the centered density of μ at any point of Σ can be established (Theorem 3.2). By combining it with the measure theoretic area formula for the centered density [8, Theorem 3.1], we obtain an area formula for the centered Hausdorff measure of Σ, that extends the one of [6] to any homogeneous distance (Theorem 4.3). We finally provide the cases where spherical measure and centered Hausdorff measure do coincide (Corollary 4.4).

In conclusion, we wish to point out that our scheme to establish the area formula for intrinsic regular submanifolds has actually a more general scope. It may also include more general classes of stratified groups, although some of our tools are still missing. For this reason, these questions are the object of our subsequent investigations.

2. Definitions and preliminary results

The next sections will introduce notions, notations and basic tools that will be used throughout the paper.

2.1. **Coordinates in the Heisenberg group.** The purpose of this section is to introduce the Heisenberg group, along with the special coordinates that allow us to identify $\mathbb{H}^n$ with $\mathbb{R}^{2n+1}$. The Heisenberg group $\mathbb{H}^n$ can be represented as a direct sum of two linear subspaces

$$\mathbb{H}^n = H_1 \oplus H_2$$

with $\dim(H_1)= 2n$ and $\dim(H_2)= 1$, endowed with a symplectic form ω on H_1 and a fixed nonvanishing element e_{2n+1} of H_2. We denote by π_{H_1} and π_{H_2} the canonical projections on H_1 and H_2, which are associated to the direct sum.

We can give to $\mathbb{H}^n$ a structure of Lie algebra by setting

$$[p,q] = \omega(\pi_{H_1}(p), \pi_{H_1}(q))\ e_{2n+1}. \tag{3}$$

Then the Baker-Campbell-Hausdorff formula ensures that

$$pq = p + q + \frac{[p,q]}{2} \tag{4}$$

defines a Lie group operation on $\mathbb{H}^n$. For $t > 0$, the linear mapping $\delta_t : \mathbb{H}^n \to \mathbb{H}^n$ such that

$$\delta_t(w) = t^k w \quad \text{ if } w \in H_k,$$

$k = 1, 2$, defines *intrinsic dilation.*

Given $p \in \mathbb{H}^n$, we denote by l_p the translation by p. Any left invariant vector field on $\mathbb{H}^n$ is of the form $X_v(p) = dl_p(0)(v)$ for any $p \in \mathbb{H}^n$ and some $v \in \mathbb{H}^n$, where we have identified $\mathbb{H}^n$ with $T_0\mathbb{H}^n$. Through the Baker-Campbell-Hausdorff formula, one can check that the Lie algebra of left invariant vector fields $\mathrm{Lie}(\mathbb{H}^n)$ is isomorphic to the given Lie algebra $(\mathbb{H}^n, [\cdot,\cdot])$.

We fix a symplectic basis $(e_1, \dots, e_{2n})$ of (H_1, ω), namely

$$\omega(e_i, e_{n+j}) = \delta_{ij}, \quad \omega(e_i, e_j) = \omega(e_{n+i}, e_{n+j}) = 0$$

for every $i, j = 1, \dots, n$, where δ_{ij} is the Kronecker delta. Thus, we have obtained a *Heisenberg basis*

$$\mathcal{B} = (e_1, \dots, e_{2n+1}),$$

that allows us to identify $\mathbb{H}^n$ with $\mathbb{R}^{2n+1}$. The associated linear isomorphism is defined as

$$\pi_{\mathcal{B}}:\mathbb{H}^n\to\mathbb{R}^{2n+1},\quad \pi_{\mathcal{B}}(p)=(x_1,\dots,x_{2n+1}) \tag{5}$$

for $p=\sum_{j=1}^{2n+1}x_je_j$. We can read the given Lie product on $\mathbb{R}^{2n+1}$ as follows

$$\begin{aligned}[(x_1,\dots,x_{2n+1}),(y_1,\dots,y_{2n+1})]&=\pi_{\mathcal{B}}([\sum_{i=1}^{2n+1}x_ie_i,\sum_{i=1}^{2n+1}y_ie_i])\\&=\left(0,\dots,0,\sum_{i=1}^{n}(x_iy_{i+n}-x_{i+n}y_i)\right)\end{aligned}$$

then the group product takes the following form on $\mathbb{R}^{2n+1}$

$$(x_1,\dots,x_{2n+1})(y_1,\dots,y_{2n+1})=\left(x_1+y_1,\dots,x_{2n+1}+y_{2n+1}+\sum_{i=1}^{n}\frac{x_iy_{i+n}-x_{i+n}y_i}{2}\right). \tag{6}$$

Taking into account (6), in our coordinates we obtain the following basis of left invariant vector fields

$$\begin{aligned}X_j(p)&=\partial_{x_j}-\frac{1}{2}x_{j+n}\partial_{x_{2n+1}}\qquad j=1,\dots,n\\Y_j(p)&=\partial_{x_{n+j}}+\frac{1}{2}x_j\partial_{x_{2n+1}}\qquad j=1,\dots,n\\T(p)&=\partial_{x_{2n+1}}.\end{aligned} \tag{7}$$

They clearly constitute a basis $(X_1,\dots,X_{2n+1})$ of $\mathrm{Lie}(\mathbb{H}^n)$ such that $X_j(0)=e_j$ for every $j=1,\dots,2n{+}1$. Any linear combination of $X_1,\dots,X_{2n}$ is called a *left invariant horizontal vector field of* $\mathbb{H}^n$.

2.2. **Metric structure.** We fix a scalar product $\langle\cdot,\cdot\rangle$ that makes our Heisenberg basis $\mathcal{B}=(e_1,\dots,e_{2n+1})$ orthonormal. In the sequel, any Heisenberg basis will be understood to be orthonormal. We denote by $|\cdot|$ both the Euclidean metric on $\mathbb{R}^{2n+1}$ and the norm induced by $\langle\cdot,\cdot\rangle$ on $\mathbb{H}^n$. The symmetries of the Heisenberg group are detected through the the isometry

$$J:H_1\to H_1,$$

that is defined by the Heisenberg basis

$$J(e_i)=e_{n+i}\quad\text{and}\quad J(e_{n+i})=-e_i$$

for all $i=1,\dots,n$. It is then easy to check that

$$\langle p,q\rangle=\omega(p,Jq)\quad\text{and}\quad J^2=-I$$

for all $p,q\in H_1$.

A *homogeneous distance* d on $\mathbb{H}^n$ is a function $d:\mathbb{H}^n\times\mathbb{H}^n\to[0,+\infty)$ such that

$$d(zx,zy)=d(x,y)\quad\text{and}\quad d(\delta_t(x),\delta_t(y))=td(x,y)$$

for every $x,y,z\in\mathbb{H}^n$ and $t>0$. Any two homogeneous distances are bi-Lipschitz equivalent. We also introduce the *homogeneous norm* $\|x\|=d(x,0)$, $x\in\mathbb{H}^n$, associated to a homogeneous distance d. Notice that this norm satisfies

$$\|xy\|\le\|x\|+\|y\|\quad\text{and}\quad\|\delta_rx\|=r\|x\|$$

for $x, y \in \mathbb{H}^n$ and $r > 0$.

By identifying $T_0\mathbb{H}^n$ with $\mathbb{H}^n$ and by left translating the fixed scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{H}^n$ we obtain a left invariant Riemannian metric g on $\mathbb{H}^n$. Its associated Riemannian norm is denoted by $\| \cdot \|_g$. We may restrict the identification of $T_0\mathbb{H}^n$ with $\mathbb{H}^n$ to the so called horizontal subspace, by identifying H_1 with

$$H_0\mathbb{H}^n \subset T_0\mathbb{H}^n.$$

Then the horizontal fiber at $p \in \mathbb{H}^n$ is $H_p\mathbb{H}^n = dl_p(0)(H_0\mathbb{H}^n)$. The collection of all horizontal fibers constitutes the so-called *horizontal subbundle* $H\mathbb{H}^n$. If we restrict the left invariant metric g to the horizontal subbundle $H\mathbb{H}^n$, we obtain a scalar product on each horizontal fiber, that is the sub-Riemannian metric. This leads in a standard way to the so-called *Carnot-Carathéodory distance*, or *sub-Riemannian distance*, see for instance [9]. This is an example of homogeneous distance.

2.3. **Differentiability and factorizations.** Natural notions of differentiability are well known in $\mathbb{H}^n$ and general Carnot groups, starting from the notion of Pansu differentiability, [18]. Throughout the paper Ω denotes an open subset of $\mathbb{H}^n$ and d is a fixed homogeneous distance. Let $f : \Omega \to \mathbb{R}^k$, $x \in \Omega$ and $v \in H_1$. If there exists

$$\lim_{t \to 0} \frac{f(x(tv)) - f(x)}{t} \in \mathbb{R}^k,$$

then we say that it is the *horizontal partial derivative* at x along X_v, that is the unique left invariant vector field such that $X_v(0) = v$. The above limit is denoted by $X_vf(x)$. Notice that X_v is precisely a left invariant horizontal vector field. We say that $f \in C^1_h(\Omega, \mathbb{R}^k)$ if for every $x \in \Omega$ and every horizontal vector field $X \in \mathrm{Lie}(\mathbb{H}^n)$ the horizontal derivative $Xf(x)$ exists and it is continuous with respect to $x \in \Omega$.

A linear mapping $L : \mathbb{H}^n \to \mathbb{R}^k$ that is homogeneous, i.e. $tL(v) = L(\delta_t v)$ for all $t > 0$ and $v \in \mathbb{H}^n$, is an *h-homomorphism*, that stands for "homogeneous homomorphism". If there exists an h-homomorphism $L : \mathbb{H}^n \to \mathbb{R}^k$ that satisfies

$$|f(xw) - f(x) - Df(x)(w)| = o(d(w, 0)) \qquad \text{as} \quad d(w, 0) \to 0,$$

then it is unique and it is called the *h-differential*, or *Pansu differential*, of f at x. We denote it by $Df(x)$. Notice that $f \in C^1_h(\Omega, \mathbb{R}^k)$ if and only if it is everywhere Pansu differentiable and $x \to Df(x)$ is continuous as a map from Ω to the space of h-homomorphisms, see for instance [13, Section 3].

Definition 2.1. Let $\Omega \subset \mathbb{H}^n$ be an open set and let $f \in C^1_h(\Omega, \mathbb{R})$. We call *horizontal gradient* of f at $x \in \Omega$ the unique vector $\nabla_H f(x)$ of H_1 such that $Df(x)(z) = \langle \nabla_H f(x), z \rangle$ for every $z \in \mathbb{H}^n$.

When differentiability meets the factorizations of the Heisenberg group, the notion of intrinsic differentiability comes up naturally, see [20] for more information. Now we introduce some algebraic properties of factorizations in $\mathbb{H}^n$ in order to define intrinsic differentiability and its basic properties.

Definition 2.2. If a Lie subgroup of $\mathbb{H}^n$ is closed under intrinsic dilations, we call it a *homogeneous subgroup*. Homogeneous subgroups of $\mathbb{H}^n$ containing H_2 are called *vertical subgroups*. Homogeneous subgroups contained in H_1 are called *horizontal subgroups*.

It is easy to realize that any homogeneous subgroup of $\mathbb{H}^n$ is either horizontal or vertical. We also notice that normal homogeneous subgroups of $\mathbb{H}^n$ coincide with vertical subgroups.

Definition 2.3. Let $\mathbb{M}$ and $\mathbb{V}$ be a vertical subgroup and a horizontal subgroup of $\mathbb{H}^n$, respectively. We say that $\mathbb{H}^n$ is the *semidirect product* of $\mathbb{M}$ and $\mathbb{V}$ and write $\mathbb{H}^n = \mathbb{M} \rtimes \mathbb{V}$ if $\mathbb{H}^n = \mathbb{M}\mathbb{V}$ and $\mathbb{M} \cap \mathbb{V} = \{0\}$.

Definition 2.4. Let $\mathbb{M}$, $\mathbb{W}$ and $\mathbb{V}$ be homogeneous subgroups of $\mathbb{H}^n$ such that

$$\mathbb{H}^n = \mathbb{M} \rtimes \mathbb{V} = \mathbb{W} \rtimes \mathbb{V}. \tag{8}$$

The semidirect product $\mathbb{W} \rtimes \mathbb{V}$ automatically yields the unique projections

$$\pi_{\mathbb{W}} : \mathbb{H}^n \to \mathbb{W} \quad \text{and} \quad \pi_{\mathbb{V}} : \mathbb{H}^n \to \mathbb{V}$$

such that $x = \pi_{\mathbb{W}}(x)\pi_{\mathbb{V}}(x)$ for every $x \in \mathbb{H}^n$. If necessary, to emphasize the dependence on the semidirect factorization we will also introduce the notation $\pi_{\mathbb{W}}^{\mathbb{W},\mathbb{V}} = \pi_{\mathbb{W}}$ and $\pi_{\mathbb{V}}^{\mathbb{W},\mathbb{V}} = \pi_{\mathbb{V}}$. The same holds for $\mathbb{M} \rtimes \mathbb{V}$. We define the following restrictions

$$\pi_{\mathbb{W},\mathbb{M}}^{\mathbb{W},\mathbb{V}} = \pi_{\mathbb{W}}^{\mathbb{W},\mathbb{V}}|_{\mathbb{M}} : \mathbb{M} \to \mathbb{W} \quad \text{and} \quad \pi_{\mathbb{M},\mathbb{W}}^{\mathbb{M},\mathbb{V}} = \pi_{\mathbb{M}}^{\mathbb{M},\mathbb{V}}|_{\mathbb{W}} : \mathbb{W} \to \mathbb{M}$$

Remark 2.1. The uniqueness of the factorizations (8) implies that both restrictions $\pi_{\mathbb{W},\mathbb{M}}^{\mathbb{W},\mathbb{V}}$ and $\pi_{\mathbb{M},\mathbb{W}}^{\mathbb{M},\mathbb{V}}$ are invertible and

$$\pi_{\mathbb{W},\mathbb{M}}^{\mathbb{W},\mathbb{V}} = (\pi_{\mathbb{M},\mathbb{W}}^{\mathbb{M},\mathbb{V}})^{-1}. \tag{9}$$

If $\mathbb{H}^n = \mathbb{M} \rtimes \mathbb{V}$, then by the local compactness of $\mathbb{H}^n$, it is immediate to observe that there exists a constant $c_0 \in (0, 1)$, possibly depending on $\mathbb{M}$ and $\mathbb{V}$, such that for all $m \in \mathbb{M}$ and $v \in \mathbb{V}$ the following holds

$$c_0 \ (\|m\| + \|v\|) \le \|mv\| \le \|m\| + \|v\|. \tag{10}$$

Remark 2.2. Whenever two homogeneous subgroups $\mathbb{W}$ and $\mathbb{V}$ of $\mathbb{H}^n$ satisfy

$$\mathbb{H}^n = \mathbb{W}\mathbb{V} \quad \text{and} \quad \mathbb{W} \cap \mathbb{V} = \{0\} \, ,$$

then one of them must be necessarily vertical and the other one must be horizontal.

We recall now definitions and results about intrinsic graphs of functions between two homogeneous subgroups. For more information, see [20].

Definition 2.5. Let $\mathbb{H}^n = \mathbb{W} \rtimes \mathbb{V}$ and let $U \subset \mathbb{W}$ be a set. If $\phi : U \to \mathbb{V}$, then we define its *intrinsic graph* as the set

$$\mathrm{graph}(\phi) = \{m\phi(m) : m \in U\}.$$

We denote by $\Phi : U \to \Sigma$, $\Phi(m) = m\phi(m)$. We call Φ the *graph map* of ϕ.

Remark 2.3. The notion of graph is intrinsic, since both translations and dilations send intrinsic graphs to new intrinsic graphs, with respect to a different function.

Translating intrinsic graphs requires some preliminary notions.

Definition 2.6. Let $\mathbb{H}^n = \mathbb{W} \rtimes \mathbb{V}$ be a semidirect product. Let us consider $x \in \mathbb{H}^n$. We define $\sigma_x : \mathbb{W} \to \mathbb{W}$ as follows

$$\sigma_x(\eta) = \pi_{\mathbb{W}}(l_x(\eta)) = x\eta(\pi_{\mathbb{V}}(x))^{-1}.$$

Given a set $U \subset \mathbb{W}$ and a function $\phi : U \to \mathbb{V}$, the *translation of* ϕ *at* x, $\phi_x : \sigma_x(U) \to \mathbb{V}$ is defined as

$$\phi_x(\eta) = \pi_{\mathbb{V}}(x)\phi(x^{-1}\eta\pi_{\mathbb{V}}(x)) = \pi_{\mathbb{V}}(x)\phi(\sigma_{x^{-1}}(\eta)). \tag{11}$$

Remark 2.4. The map σ_x is invertible on $\mathbb{W}$

$$\sigma_{x^{-1}}(\eta) = x^{-1}\eta\pi_{\mathbb{V}}(x^{-1})^{-1} = x^{-1}\eta\pi_{\mathbb{V}}(x) = \sigma_x^{-1}(\eta).$$

Then for $\eta \in \sigma_x(U)$ we may also write

$$\phi_x(\eta) = \pi_{\mathbb{V}}(x)\phi(\sigma_x^{-1}(\eta)). \tag{12}$$

Next we recall the content of [2, Propositions 3.6].

Proposition 2.1. *Let* $\mathbb{H}^n = \mathbb{W} \rtimes \mathbb{V}$ *be a semidirect product. Let* $U \subset \mathbb{W}$ *be an open set and* $\phi : U \to \mathbb{V}$ *be a function. Then we have*

$$l_x(\mathrm{graph}(\phi)) = \{\eta\phi_x(\eta) : \eta \in \sigma_x(U)\}.$$

Definition 2.7. Let $\mathbb{H}^n = \mathbb{W} \rtimes \mathbb{V}$ be a semidirect product. Let $U \subset \mathbb{W}$ be an open set and let $\phi : U \to \mathbb{V}$ be a function. Let us take $\bar{w} \in U$ and define $x = \bar{w}\phi(\bar{w})$. The function ϕ is intrinsic differentiable at $\bar{w}$ if there exists an h-homomorphism $L : \mathbb{W} \to \mathbb{V}$ such that

$$d(L(w), \phi_{x^{-1}}(w)) = o(\|w\|) \tag{13}$$

as $w \to 0$. The function L is called the *intrinsic differential* of ϕ at $\bar{w}$, it is uniquely defined and we denote it by $d\phi_{\bar{w}}$.

Remark 2.5. By virtue of [2, Proposition 3.23], in our setting any intrinsic linear function is actually an h-homomorphisms. We also observe that the assumption $\bar{w} \in U$ implies that $0 \in \sigma_{x^{-1}}(U)$. In addition $\sigma_{x^{-1}}(U)$ is an open set, hence the limit (13) is entirely justified.

Remark 2.6. By [2, Proposition 3.25], condition (13) is equivalent to ask that for all $w \in U$

$$\|d\phi_{\bar{w}}(\bar{w}^{-1}w)^{-1}\phi(\bar{w})^{-1}\phi(w)\| = o(\|\phi(\bar{w})^{-1}\bar{w}^{-1}w\phi(\bar{w})\|)$$

as $\|\phi(\bar{w})^{-1}\bar{w}^{-1}w\phi(\bar{w})\| \to 0$.

Definition 2.8. Let $\mathbb{H}^n = \mathbb{W} \rtimes \mathbb{V}$ be a semidirect product of $\mathbb{H}^n$. Let $U \subset \mathbb{W}$ be an open set and let $\phi : U \to \mathbb{V}$ be a function. The map ϕ is uniformly intrinsic differentiable on U if for any point $\bar{w} \in U$ there exists an h-homomorphism $d\phi_{\bar{w}} : \mathbb{W} \to \mathbb{V}$ such that

$$\lim_{\delta\to 0} \sup_{0<\|\bar{w}^{-1}w'\|<\delta} \sup_{0<\|w\|<\delta} \frac{d(d\phi_{\bar{w}}(w), \phi_{\Phi(w')^{-1}}(w))}{\|w\|} = 0 \tag{14}$$

where Φ is the graph map of ϕ.

The following definition is a slight modification of the notion of intrinsic differentiability.

Definition 2.9. Let $\mathbb{W}$ be a vertical subgroup of $\mathbb{H}^n$, let $U \subset \mathbb{W}$ be an open set and let $F : U \to \mathbb{R}^k$ with $u \in U$. We fix any horizontal subgroup $\mathbb{V} \subset \mathbb{H}^n$ such that $\mathbb{H}^n = \mathbb{W} \rtimes \mathbb{V}$ and choose $v \in \mathbb{V}$. We define $x = uv$ in $\mathbb{H}^n$ and the corresponding translated function

$$F_{x^{-1}}(w) = F(\sigma_x(w)) - F(u)$$

for $w \in \sigma_{x^{-1}}(U)$. We say that F is *extrinsically differentiable at u with respect to* $(\mathbb{W}, \mathbb{V}, x)$ if there exists an h-homomorphism $L : \mathbb{W} \to \mathbb{R}^k$ such that

$$\frac{|F_{x^{-1}}(w) - L(w)|}{\|w\|} \to 0 \quad \text{as } w \to 0. \tag{15}$$

The uniqueness of L allows us to denote it by $d_x^{\mathbb{W},\mathbb{V}} F$.

The terminology *extrinsic differentiabilty* arises from the fact that the subgroup $\mathbb{V}$ and the point x cannot be detected from the information we have on F. They are actually artifically added from outside. However, when the factor $\mathbb{V}$ as metric spaces replaces $\mathbb{R}^k$ and $v = F(w)$, hence in the numerator of (15) becomes $d(F_{x^{-1}}(w), L(w))$, then extrinsic differentiability yields intrinsic differentiability. We have introduced this notion only to make sense of the following chain rule involving intrinsic differentiability. Somehow extrinsic and intrinsic differentiability compensate each other in the following theorem.

Theorem 2.2 (Chain rule). *Let $\mathbb{H}^n = \mathbb{W} \rtimes \mathbb{V}$ be a semidirect product. Let us consider two open sets $U \subset \mathbb{W}$, $\Omega \subset \mathbb{H}^n$ and two functions $f : \Omega \to \mathbb{R}^k$, $\phi : U \to \mathbb{V}$. Assume $\Phi(U) \subset \Omega$ where Φ is the graph function of ϕ. Let us consider $x_{\mathbb{W}} \in U$ and set $x = \Phi(x_{\mathbb{W}})$. If f and ϕ are h-differentiable at x and intrinsic differentiable at $x_{\mathbb{W}}$, respectively, then the composition $F = f \circ \Phi : U \to \mathbb{R}^k$, given by*

$$F(u) = f(u\phi(u)) \quad \textit{for all } u \in U,$$

is extrinsically differentiable at $x_{\mathbb{W}}$ with respect to $(\mathbb{W}, \mathbb{V}, x)$. For every $w \in \mathbb{W}$ the formula

$$d_x^{\mathbb{W},\mathbb{V}} F(w) = Df(x)(w d\phi_{x_{\mathbb{W}}}(w)) \tag{16}$$

holds. If in addition $f(w\phi(w)) = c$ for every $w \in U$ and some $c \in \mathbb{R}$, then we obtain

$$\ker(Df(x)) = \mathrm{graph}(d\phi_{x_{\mathbb{W}}}). \tag{17}$$

Proof. Let us first show that F is extrinsically differentiable at $x_{\mathbb{W}}$ with respect to $(\mathbb{W}, \mathbb{V}, x)$. We define

$$L(w) = Df(x)(w d\phi_{x_{\mathbb{W}}}(w)) = Df(x)(w) + Df(x)(d\phi_{x_{\mathbb{W}}}(w))$$

for $w \in \mathbb{W}$, that is an h-homomorphism. For w small enough, we have

$$\begin{aligned}
\frac{|F_{x^{-1}}(w) - L(w)|}{\|w\|} &= \frac{|f(xwx_{\mathbb{V}}^{-1}\phi(xwx_{\mathbb{V}}^{-1})) - f(x) - L(w)|}{\|w\|} \\
&= \frac{|f(xw\phi_{x^{-1}}(w)) - f(x) - Df(x)(w d\phi_{x_{\mathbb{W}}}(w))|}{\|w\|} \\
&\leq \frac{|f(xw\phi_{x^{-1}}(w)) - f(x) - Df(x)(w\phi_{x^{-1}}(w))|}{\|w\|} \\
&\quad + \frac{|Df(x)(w\phi_{x^{-1}}(w)) - Df(x)(w d\phi_{x_{\mathbb{W}}}(w))|}{\|w\|}.
\end{aligned}$$

Let us consider the last two addends separately:

$$\begin{aligned}
&\frac{|f(xw\phi_{x^{-1}}(w)) - f(x) - Df(x)(w\phi_{x^{-1}}(w))|}{\|w\|} \\
&= \frac{|f(xw\phi_{x^{-1}}(w)) - f(x) - Df(x)(w\phi_{x^{-1}}(w))|}{\|w\phi_{x^{-1}}(w)\|} \frac{\|w\phi_{x^{-1}}(w)\|}{\|w\|} \to 0
\end{aligned}$$

as $\|w\| \to 0$, by the Pansu differentiability of f at x and by the validity of

$$\frac{\|w\phi_{x^{-1}}(w)\|}{\|w\|} \le 1 + \frac{\|\phi_{x^{-1}}(w)\|}{\|w\|} = 1 + \left\| d\phi_{x_{\mathbb{W}}}\left(\frac{w}{\|w\|}\right)\right\| + \frac{\|d\phi_{x_{\mathbb{W}}}(w)^{-1}\phi_{x^{-1}}(w)\|}{\|w\|} \le C_x$$

for all $w \neq 0$ and sufficiently small. It is indeed a consequence of the intrinsic differentiability of ϕ at $x_{\mathbb{W}}$. For the second addend, the previous intrinsic differentiability yields

$$\frac{|Df(x)(d\phi_{x_{\mathbb{W}}}(w)^{-1}\phi_{x^{-1}}(w))|}{\|w\|} = \left| Df(x)\left(\frac{d\phi_{x_{\mathbb{W}}}(w)^{-1}\phi_{x^{-1}}(w))}{\|w\|}\right)\right| \to 0$$

as $w \to 0$. This complete the proof of the first claim and also establishes formula (16).

Let us now assume the constancy of $w \to f(w\phi(w))$ on U. Since we have proved that F is extrinsically differentiable at $x_{\mathbb{W}}$ with respect to $(\mathbb{W}, \mathbb{V}, x)$, being in this case $F_{x^{-1}}$ identically vanishing, we obtain

$$d_x^{\mathbb{W},\mathbb{V}} F(w) = o(\|w\|)$$

as $w \to 0$. Therefore, for any $u \in \mathbb{W}$, we have

$$\|Df(x)(\delta_t u d\phi_{x_{\mathbb{W}}}(\delta_t u))\| = o(t)$$

as $t \to 0$. Due to the h-linearity, it follows that

$$Df(x)(ud\phi_{x_{\mathbb{W}}}(u)) = 0.$$

We have proved the inclusion $\mathrm{graph}(d\phi_{x_{\mathbb{W}}}) \subset \ker(Df(x))$ of homogeneous subgroups with the same dimension, hence formula (17) is established. □

The notion of $\mathbb{H}$-regular surface in $\mathbb{H}^n$ was first given in [6].

Definition 2.10. Let $\Sigma \subset \mathbb{H}^n$ be a set and let $1 \le k \le n$. We say that Σ is an $\mathbb{H}$-*regular surface of low codimension* if for every $x \in \Sigma$ there exist an open neighbourhood Ω such that $x \in \Omega$ and a function $f = (f_1, \dots, f_k) \in C_h^1(\Omega, \mathbb{R}^k)$ such that

(i) $\Sigma \cap U = \{y \in \Omega : f(y) = 0\}$,
(ii) $\nabla_H f_1(y) \wedge \cdots \wedge \nabla_H f_k(y) \neq 0$ for all $y \in \Omega$.

We can characterize the metric tangent cone of an $\mathbb{H}$-regular surface of codimension k.

Definition 2.11. For any set $A \subset \mathbb{H}^n$, $x \in A$, the *homogeneous tangent cone* is given by the set

$$\mathrm{Tan}(A, x) = \left\{\nu \in \mathbb{H}^n : \nu = \lim_{h\to\infty} \delta_{r_h}(x^{-1}x_h),\ r_h > 0,\ x_h \in A,\ x_h \to x\right\}.$$

From [6, Proposition 3.29], we have the following characterization.

Proposition 2.3. *If Σ is an $\mathbb{H}$-regular surface of low codimension and $f \in C_h^1(\Omega, \mathbb{R}^k)$ is as in Definition 2.10, then*

$$\ker Df(x) = Tan(\Sigma, x)$$

for all $x \in \Sigma \cap \Omega$.

Given an open subset $\Omega \subset \mathbb{H}^n$, a function $f \in C_h^1(\Omega, \mathbb{R}^k)$ and $x \in \Omega$, we define the *horizontal Jacobian*

$$J_H f(x) = \|\nabla_H f_1(x) \wedge \cdots \wedge \nabla_H f_k(x)\|_g,$$

where the norm is given through our fixed left invariant metric g.

If $\mathbb{V} \subset H_1$ is a k dimensional subspace, we set $\nabla_{\mathbb{V}} f(x)$ as the unique vector of $\mathbb{V}$ such that $Df(x)(z) = \langle \nabla_{\mathbb{V}} f(x), z \rangle$ for every $z \in \mathbb{V}$. As a consequence, we can also define the *Jacobian with respect to* $\mathbb{V}$, namely

$$J_{\mathbb{V}} f(x) = \|\nabla_{\mathbb{V}} f_1(x) \wedge \cdots \wedge \nabla_{\mathbb{V}} f_k(x)\|_g.$$

The next implicit function theorem is proved in [6, Theorem 3.27]. Its general version in the framework of homogeneous groups is given in [14, Theorem 1.3].

Theorem 2.4 (Implicit function theorem)**.** *Let $\Omega \subset \mathbb{H}^n$ be an open set, let $f \in C^1_h(\Omega, \mathbb{R}^k)$ be a function and consider a point $x_0 \in \Omega$ such that $J_H f(x_0) > 0$. Then there exists a horizontal subgroup $\mathbb{V}$ such that $J_{\mathbb{V}} f(x_0) > 0$. We set $\Sigma = \{x \in \Omega : f(x) = f(x_0)\}$ and we fix a homogeneous subgroup $\mathbb{W}$ complementary to $\mathbb{V}$. Setting $\pi_{\mathbb{W}}(x_0) = \eta_0$ and $\pi_{\mathbb{V}}(x_0) = \upsilon_0$, there exist an open set $\Omega' \subset \Omega \subset \mathbb{H}^n$, with $x_0 \in \Omega'$, an open set $U \subset \mathbb{W}$ with $\eta_0 \in U$ and a unique continuous function $\phi : U \to \mathbb{V}$ such that $\phi(\eta_0) = \upsilon_0$ and*

$$\Sigma \cap \Omega' = \{w\phi(w) : w \in U\}.$$

Definition 2.12 (Parametrized $\mathbb{H}$-regular surface)**.** Let $\Sigma \subset \Omega$ be an $\mathbb{H}$-regular surface and assume that there exist a semidirect factorization $\mathbb{H}^n = \mathbb{W} \rtimes \mathbb{V}$, an open set $U \subset \mathbb{W}$ and a continuous mapping $\phi : U \to \mathbb{V}$ such that $\Sigma = \{u\phi(u) \in \mathbb{H}^n : u \in U\}$. We say that Σ is a *parametrized $\mathbb{H}$-regular surface with respect to* $(\mathbb{W}, \mathbb{V})$. We say that ϕ is the *parametrization of* Σ.

Proposition 2.5. *Let $f \in C^1_h(\Omega, \mathbb{R}^k)$ be such that $f^{-1}(f(p)) = \Sigma$ for some $p \in \Omega$, and suppose that for some horizontal subgroup $\mathbb{V} \subset \mathbb{H}^n$ it holds $J_{\mathbb{V}} f(x) > 0$ for all $x \in \Sigma$. If $\mathbb{W} \subset \mathbb{H}^n$ is any vertical subgroup such that $\mathbb{H}^n = \mathbb{W} \rtimes \mathbb{V}$, then Σ is a parametrized $\mathbb{H}$-regular surface with respect to $(\mathbb{W}, \mathbb{V})$.*

Proof. The proof follows by exploiting the local factorization given by the implicit function theorem of [6, Proposition 3.13]. □

The proof of the following proposition is a simple application of Theorem 2.2.

Proposition 2.6. *Let $\phi : U \to \mathbb{V}$, where $U \subset \mathbb{W}$ be open, and assume that ϕ is everywhere intrinsic differentiable. Let $\Sigma = \{n\phi(n) : n \in U\}$ and let $f : \Omega \to \mathbb{R}^k$ be everywhere h-differentiable with*

$$\Sigma = f^{-1}(f(p)) \cap (U\mathbb{V})$$

for some $p \in U\mathbb{V}$. If $J_H f(x) > 0$ for all $x \in \Sigma$, then $J_{\mathbb{V}} f(x) > 0$ for all $x \in \Sigma$.

Proof. We consider $x = w\phi(w)$, so by Theorem 2.2 the function $F = f \circ \Phi$ is extrinsically differentiable at w with respect to $(\mathbb{W}, \mathbb{V}, x)$ and

$$0 = d_x^{\mathbb{W},\mathbb{V}} F(v) = Df(x)(v d\phi_{x_{\mathbb{W}}}(v)) = D_{\mathbb{W}} f(x)(v) + D_{\mathbb{V}} f(x)(d\phi_{x_{\mathbb{W}}}(v))$$

where $v \in \mathbb{W}$ and $D_S f(x) = Df(x)|_S$ for any homogeneous subgroup S of $\mathbb{H}^n$. If by contradiction $D_{\mathbb{V}} f(x) : \mathbb{V} \to \mathbb{V}$ would not be a isomorphism, then its image T would have linear dimension less than k. Then the previous equalities would imply that the image of $D_{\mathbb{W}} f(x)$ would be contained in T, hence the same would hold for the image of $Df(x)$. This conflicts with the fact that $Df(x)$ is surjective. □

We conclude this section by pointing out that the intrinsic graph in the implicit function theorem is suitably differentiable.

Theorem 2.7 ([2, Theorem 4.2]). *In the assumption of Theorem 2.4, ϕ is uniformly intrinsic differentiable on U.*

2.4. **Intrinsic derivatives.** In this section we recall some results about uniform intrinsic differentiability in Heisenberg groups. Throughout this section, we assume that $\mathbb{H}^n$ is a semidirect product $\mathbb{W} \rtimes \mathbb{V}$ with $\mathbb{W}$ orthogonal to $\mathbb{V}$. The following proposition ensures that we can always find a Heisenberg basis which is adapted to this factorization.

Proposition 2.8. *Let $\mathbb{H}^n = \mathbb{W} \rtimes \mathbb{V}$ be a semidirect product, where the horizontal subgroup $\mathbb{V}$ is spanned by an orthonormal basis $v_1, \ldots, v_k \in H_1$ and $\mathbb{W}$ is orthogonal to $\mathbb{V}$, hence $k \le n$. Then there exist $v_{k+1}, \ldots, v_n, w_1, \ldots, w_n \in H_1$ such that $(v_{k+1}, \ldots, v_n, w_1, \ldots, w_n, e_{2n+1})$ is an orthonormal basis of $\mathbb{W}$ and $(v_1, \ldots, v_n, w_1, \ldots, w_n, e_{2n+1})$ is a Heisenberg basis of $\mathbb{H}^n$.*

Proof. Since $\mathbb{V}$ is commutative, an element $v = J(w)$ with $v, w \in \mathbb{V}$ satisfies

$$|v|^2 = \langle v, J(w) \rangle = -\omega(v, w) = 0,$$

therefore $\mathbb{V} \cap J(\mathbb{V}) = \{0\}$. We set $J(v_i) = w_i \in \mathbb{W}$ for $i = 1, \ldots, k$ and define the $2k$-dimensional subspace

$$\mathbb{S}_1 = \mathbb{V} \oplus J(\mathbb{V}) \subset H_1.$$

We notice that $\dim(\mathbb{S}_1^\perp \cap H_1) = 2(n-k)$. If $k < n$, we pick a vector $v_{k+1} \in \mathbb{S}_1^\perp \cap H_1$ of unit norm and define $w_{k+1} = Jv_{k+1}$. It is easily observed that both w_{k+1} and v_{k+1} are orthogonal to $\mathbb{S}_1$, so that $(v_1, \ldots, v_{k+1}, w_1, \ldots, w_{k+1}, e_{2n+1})$ is a Heisenberg basis of

$$\mathbb{S}_2 \oplus \operatorname{span}\{e_{2n+1}\},$$

where we have defined $\mathbb{S}_2 = \mathbb{V} \oplus \operatorname{span}\{v_{k+1}\} \oplus J(\mathbb{V} \oplus \operatorname{span}\{v_{k+1}\})$. Indeed, the previous subspace has the structure of a $2k+3$-dimensional Heisenberg group. One can iterate this process until a Heisenberg basis of $\mathbb{H}^n$ is found. □

From now on, we assume that $(v_1, \ldots, v_n, w_1, \ldots, w_n, e_{2n+1})$ is the Heisenberg basis provided by Proposition 2.8. We can identify $\mathbb{V}$ with $\mathbb{R}^k$ and $\mathbb{W}$ with $\mathbb{R}^{2n+1-k}$ through the following diffeomorphisms

$$i_{\mathbb{V}} : \mathbb{V} \to \mathbb{R}^k, \ i_{\mathbb{V}}\left(\sum_{i=1}^k x_i v_i\right) = (x_1, \ldots, x_k),$$

$$i_{\mathbb{W}} : \mathbb{W} \to \mathbb{R}^{2n+1-k},$$

$$i_{\mathbb{W}}\left(z e_{2n+1} + \sum_{i=k+1}^n (x_i v_i + y_i w_i) + \sum_{i=1}^k \eta_i v_i\right) = (x_{k+1}, \ldots, x_n, \eta_1, \ldots, \eta_k, y_{k+1}, \ldots, y_n, z).$$

We identify any function from an open subset $U \subset \mathbb{W}$, $\phi : U \to \mathbb{V}$ with the corresponding function from an open subset $\widetilde{U} \subset \mathbb{R}^{2n+1-k}$, $\widetilde{\phi} : \widetilde{U} \to \mathbb{R}^k$:

$$\widetilde{\phi}(w) = i_{\mathbb{V}}(\phi(i_{\mathbb{W}}^{-1}(w))) \quad \forall w \in \widetilde{U} = i_{\mathbb{W}}(U) \subset \mathbb{R}^{2n+1-k}.$$

Any h-homomorphism $L : \mathbb{W} \to \mathbb{V}$ can be associated as before to a map $\widetilde{L} : \mathbb{R}^{2n+1-k} \to \mathbb{R}^k$, and so it can be identified with a $k \times (2n-k)$ matrix M_L with real coefficients such that for every $w \in \mathbb{R}^{2n+1-k}$

$$\widetilde{L}(w) = M_L \pi(w)^t,$$

where $\pi : \mathbb{R}^{2n+1-k} \to \mathbb{R}^{2n-k}$ is the canonical projection on the first $2n-k$ components.

If $U \subset \mathbb{W}$ is an open set and $\phi : U \to \mathbb{V}$ is intrinsic differentiable at a point $w \in U$, we denote by $D^{\phi}\phi(w)$ the matrix associated to $d\phi_w$ and we call it *intrinsic Jacobian matrix* of ϕ at w. If $\mathcal{U} \subset \mathbb{R}^{2n+1-k}$ is an open set and $\psi = (\psi_1, \dots, \psi_k) : \mathcal{U} \to \mathbb{R}^k$ is a function, we define the family of $2n-k$ vector fields:

$$W_j^{\psi} = \begin{cases} (i_{\mathbb{W}})^*(X_{j+k}) & j = 1, \dots, n-k \\ \nabla^{\psi_{j-n+k}} = \partial_{\eta_{j-n+k}} + \psi_{j-n+k}\partial_z & j = n-k+1, \dots, n \\ (i_{\mathbb{W}})^*(Y_{j+k}) & j = n+1, \dots, 2n-k. \end{cases}$$

Definition 2.13 (Intrinsic derivatives)**.** Let $U \subset \mathbb{W}$ be an open set and let $\bar{w}$ be a point of U. Let $\phi : U \to \mathbb{V}$ be a continuous function. Let $j \in \{1, \dots, 2n-k\}$, we say that ϕ has ∂^{ϕ_j}-derivative at $\bar{w}$ if and only if there exists a vector $(\alpha_{1,j}, \dots, \alpha_{k,j}) \in \mathbb{R}^k$ such that for all $\gamma^j : (-\delta, \delta) \to \widetilde{U}$ integral curves of $W_j^{\widetilde{\phi}}$ with $\gamma^j(0) = i_{\mathbb{W}}(\bar{w})$ the limit

$$\lim_{t \to 0} \frac{\widetilde{\phi}(\gamma^j(t)) - \widetilde{\phi}(\bar{w})}{t}$$

exists and it is equal to $(\alpha_{1,j}, \dots, \alpha_{k,j})$. For all $j = 1, \dots, 2n-k$ we denote it by

$$\partial^{\phi_j}\phi(\bar{w}) = \begin{pmatrix} \partial^{\phi_j}\phi_1 \\ \vdots \\ \partial^{\phi_j}\phi_k \end{pmatrix}(\bar{w}) = \begin{pmatrix} \alpha_{1,j} \\ \vdots \\ \alpha_{k,j}. \end{pmatrix}.$$

Uniform intrinsic differentiability has been characterized in various ways using intrinsic derivatives and the intrinsic Jacobian matrix.

Theorem 2.9 ([4, Theorem 5.7])**.** *Let $U \subset \mathbb{W}$ be an open set. Let $\phi : U \to \mathbb{V}$ be a function. We define $\Sigma = \Phi(U)$ where Φ is the graph mapping of ϕ. Then the following are equivalent:*

(i) *ϕ is uniformly intrinsic differentiable on U.*
(ii) *$\phi \in C^0(U)$ and for every $w \in U$ there exist $\partial^{\phi_j}\phi(w)$ for $j = 1, \dots 2n-k$ and the functions*
$$\partial^{\phi_j}\phi : U \to \mathbb{R}^k,$$
are continuous.
(iii) *ϕ is intrinsic differentiable on U and the map $D^{\phi}\phi : U \to M_{k,2n-k}(\mathbb{R})$ is continuous.*
(iv) *There is an open set $\Omega \subset \mathbb{H}^n$ and $f = (f_1, ..., f_k) \in C_h^1(\Omega, \mathbb{R}^k)$ such that $\Sigma = \{x \in \Omega : f(x) = 0\}$, and $\det([X_i f_j]_{i,j=1,\dots,k}(x)) \neq 0$ for all $x \in \Sigma$.*

Definition 2.14. Let $U \subset \mathbb{W}$ be an open set. Let $\phi : U \to \mathbb{V}$ be an intrinsic differentiable function at $\bar{w} \in U$. We define the *intrinsic Jacobian of ϕ at $\bar{w}$* as

$$J^{\phi}\phi(\bar{w}) = \sqrt{1 + \sum_{\ell=1}^{k} \sum_{I \in \mathcal{I}_\ell} (M_I^{\phi}(\bar{w}))^2},$$

where we have defined $\mathcal{I}_\ell$ as the set of multiindexes

$$\{(i_1, \dots, i_\ell, j_1, \dots, j_\ell)) \in \mathbb{N}^{2l} : 1 \leq i_1 < i_2 < \dots < i_\ell \leq 2n-k,\ 1 \leq j_1 < j_2 \dots < j_\ell \leq k\}.$$

We have also introduced the minors

$$M_I^{\phi}(\bar w)=\det\begin{pmatrix}\partial^{\phi_{i_1}}\phi_{j_1} & \dots & \partial^{\phi_{i_\ell}}\phi_{j_1}\\ \dots & \dots & \dots\\ \partial^{\phi_{i_1}}\phi_{j_\ell} & \dots & \partial^{\phi_{i_\ell}}\phi_{j_\ell}\end{pmatrix}(\bar w).$$

2.5. **Measures and area formulas.** If $\mathbb{H}^n$ is endowed with a homogeneous distance d, we denote by $\mathbb{B}(x,r)=\{y\in\mathbb{H}^n : d(x,y)\le r\}$ and for $S\subset\mathbb{H}^n$,

$$\operatorname{diam}(S)=\sup\{d(x,y): x,y\in S\}.$$

Notice that $\operatorname{diam}(\mathbb{B}(x,r))=2r$ for all $x\in\mathbb{H}^n$ and $r>0$.

Definition 2.15 (Carathéodory's construction)**.** Let $\mathcal{F}\subset\mathcal{P}(\mathbb{H}^n)$ be a non-empty family of closed subsets of $\mathbb{H}^n$, equipped with a homogeneous distance d. Let be $\alpha>0$. If $\delta>0$, and $A\subset\mathbb{H}^n$, we define

$$\phi_\delta^\alpha(A)=\inf\left\{\sum_{j=0}^{\infty}c_\alpha\operatorname{diam}(B_j)^\alpha : A\subset\bigcup_{j=0}^{\infty}B_j,\ \operatorname{diam}(B_j)\le\delta,\ B_j\in\mathcal{F}\right\},\tag{18}$$

If $\mathcal{F}$ coincides with the family of closed balls, $\mathcal{F}_b$, with respect to the distance d, then

$$\mathcal{S}_0^\alpha(A)=\sup_{\delta>0}\phi_\delta^\alpha(A)$$

is the *α-spherical measure* of E. If in (18) we choose $c_\alpha=2^{-\alpha}$, then we use the symbol $\mathcal{S}_0^\alpha$.

In the case $\mathcal{F}$ is the family of all closed sets and $k\in\{1,\dots,2n+1\}$, we define

$$c_k=\frac{\mathcal{L}^k(\{x\in\mathbb{R}^k:|x|\le1\})}{2^k}$$

where $\mathcal{L}^k$ denotes the Lebesgue measure. Then the corresponding k-dimensional Hausdorff measure is given by

$$\mathcal{H}_E^k(A)=\sup_{\delta>0}\phi_\delta^k(A)$$

where $\mathbb{H}^n$ is equipped with the Euclidean distance induced through the identification with $\mathbb{R}^{2n+1}$. These measures are Borel regular on subsets of $\mathbb{H}^n$. For our purposes, it is useful to recall a less known Hausdorff-type measure, first introduced in [19]. Given $\alpha\in[0,\infty)$, $\delta\in(0,\infty)$, we define the *$m-$dimensional centered Hausdorff measure* $\mathcal{C}_0^\alpha$ of a set $A\subset\mathbb{H}^n$ as

$$\mathcal{C}_0^\alpha(A)=\sup_{E\subset A}\mathcal{D}^\alpha(E)$$

where $\mathcal{D}^\alpha(E)=\lim_{\delta\to0+}\mathcal{C}_\delta^\alpha(E)$, and, in turn, $\mathcal{C}_\delta^\alpha(E)=0$ if $E=\emptyset$ and if $E\neq\emptyset$

$$C_\delta^\alpha(E)=\inf\left\{\sum_i r_i^\alpha : E\subset\cup_i\mathbb{B}(x_i,r_i),\ x_i\in E,\ \operatorname{diam}(\mathbb{B}(x_i,r_i))\le\delta\right\}.$$

Definition 2.16. Let $\alpha>0$, $x\in\mathbb{H}^n$ and μ be a Borel regular measure on $\mathbb{H}^n$. If

$$\liminf_{r\to0}\frac{\mu(\mathbb{B}(x,r))}{r^\alpha}=\limsup_{r\to0}\frac{\mu(\mathbb{B}(x,r))}{r^\alpha}$$

we define the *spherical α-centered density of μ at x* as

$$\theta_c^\alpha(\mu,x)=\lim_{r\to0}\frac{\mu(\mathbb{B}(x,r))}{r^\alpha}.$$

Theorem 2.10 ([8, Theorem 3.1]). *Let $\alpha > 0$ and let μ be a Borel regular measure on $\mathbb{H}^n$ such that there exists a countable open covering of $\mathbb{H}^n$, whose elements have μ finite measure. Let $B \subset A \subset \mathbb{H}^n$ be Borel sets. If $\mathcal{C}_0^\alpha(A) < \infty$ and $\mu \llcorner A$ is absolutely continuous with respect to $\mathcal{C}_0^\alpha \llcorner A$, then we have that $\theta_c^\alpha(\mu, \cdot)$ is a Borel function on A and*

$$\mu(B) = \int_B \theta_c^\alpha(\mu, x)\ d\mathcal{C}_0^\alpha(x).$$

We introduce now a crucial definition of density.

Definition 2.17. Let $\mathcal{F}_b$ be the family of closed balls with positive radius in $\mathbb{H}^n$ endowed with an homogeneous distance d. Let $\alpha > 0$, $x \in \mathbb{H}^n$ and μ be a Borel regular measure on $\mathbb{H}^n$. We call *spherical α-Federer density of μ at x* the real number

$$\theta^\alpha(\mu, x) = \inf_{\epsilon>0} \sup \left\{ \frac{2^\alpha \mu(\mathbb{B})}{\operatorname{diam}(\mathbb{B})^\alpha} : x \in \mathbb{B} \in \mathcal{F}_b,\ \operatorname{diam}(\mathbb{B}) < \epsilon \right\}.$$

This density naturally appears in representing a Borel regular measure that is absolutely continuous with respect to the α-dimensional spherical measure.

Theorem 2.11 ([15, Theorem 11]). *Let $\alpha > 0$ and let μ be a Borel regular measure on $\mathbb{H}^n$ such that there exists a countable open covering of $\mathbb{H}^n$ whose elements have μ finite measure. If $B \subset A \subset \mathbb{H}^n$ are Borel sets, then $\theta^\alpha(\mu, \cdot)$ is a Borel function on A. If in addition $\mathcal{S}_0^\alpha(A) < \infty$ and $\mu \llcorner A$ is absolutely continuous with respect to $\mathcal{S}_0^\alpha \llcorner A$, then*

$$\mu(B) = \int_B \theta^\alpha(\mu, x)\ d\mathcal{S}_0^\alpha(x).$$

Definition 2.18 (Spherical factor). Let d be a homogeneous distance in $\mathbb{H}^n$. If $\mathbb{W}$ is a linear subspace of topological dimension p of $\mathbb{H}^n$, then we define the *spherical factor* of $\mathbb{W}$ with respect to d as

$$\beta_d(\mathbb{W}) = \max_{z \in \mathbb{B}(0,1)} \mathcal{H}_E^p(\mathbb{W} \cap \mathbb{B}(z, 1)).$$

When we deal with a homogeneous distance d that preserves some symmetries, then the spherical factor can become a geometric constant. The following definition detects those homogeneous distances giving a constant spherical factor. It extends [16, Definition 6.1] to higher codimension.

Definition 2.19. Let d be a homogeneous distance on $\mathbb{H}^n$ and let $p = 1, \dots, 2n+1$. If $p = 1$ or $p = 2n+1$, then d is called p-vertically symmetric. In the case $1 < p \le 2n$, the p-vertical symmetry requires the following conditions.

We refer to the fixed graded scalar product $\langle \cdot, \cdot \rangle$ and we assume that there exists a family $\mathcal{F} \subset O(H_1)$ of isometries such that for any couple of p-dimensional subspaces $S_1, S_2 \subset H_1$, there exists $L \in \mathcal{F}$ that satisfies the condition $L(S_1) = S_2$. Taking into account that H_1 and H_2 are orthogonal, we introduce the class of isometries

$$\mathcal{O} = \{T \in O(\mathbb{H}^n) : T|_{H_2} = \operatorname{Id}|_{H_2},\ T|_{H_1} \in \mathcal{F}\}.$$

Then we say that d is p-vertically symmetric if the following holds:

- $\pi_{H_1}(\mathbb{B}(0,1)) = \mathbb{B}(0,1) \cap H_1 = \{h \in H_1 : \theta(|\pi_{H_1}(h)|) \le r_0\}$ for some monotone non-decreasing function $\theta : [0, +\infty) \to [0, +\infty)$ and $r_0 > 0$,
- $T(\mathbb{B}(0,1)) = \mathbb{B}(0,1)$ for all $T \in \mathcal{O}$.

More information on p-vertically symmetric distances can be found in [11]. For instance, the sub-Riemannian distance in the Heisenberg group is vertically symmetric. Vertically symmetric distances were already introduced in [16].

The next theorem specializes to the Heisenberg group a general result from [11].

Theorem 2.12. *If $p = 1, \dots, 2n+1$ and d is a homogeneous p-vertically symmetric distance on $\mathbb{H}^n$, then the spherical factor $\beta_d(\mathbb{W})$ is constant on every p-dimensional vertical subgroup $\mathbb{W} \subset \mathbb{H}^n$.*

The previous theorem motivates the following definition.

Definition 2.20. If we have a class of p-dimensional homogeneous subgroups $\mathcal{F}$ for which $\beta_d(S)$ remains constant as $S \in \mathcal{F}$, then we denote the spherical factor by $\omega_d(p)$, without indicating the special class of subgroups.

Definition 2.21 ([17, Definition 8.5])**.** Let d be a homogeneous distance on $\mathbb{H}^n$. We say that d is *multiradial* if there exists a function $\theta : [0, +\infty)^2 \to [0, +\infty)$, which is continuous and monotone non-decreasing on each single variable, with

$$d(x, 0) = \theta(|\pi_{H_1}(x)|, |\pi_{H_2}(x)|).$$

The function θ is also assumed to be coercive in the sense that $\theta(x) \to +\infty$ as $|x| \to +\infty$.

Proposition 2.13. *If $d : \mathbb{H}^n \times \mathbb{H}^n \to [0, \infty)$ is multiradial, then it is also p-vertically symmetric for every $p = 1, \dots, 2n+1$.*

A more general statement can be found in [17]. One may also check that both d_∞ and the Cygan-Korányi distance are multiradial.

It is also possible to find conditions under which the spherical factor has a simpler representation. The next theorem is established in [11].

Theorem 2.14. *If $p = 1, \dots, 2n+1$ and d is a homogeneous distance in $\mathbb{H}^n$ whose unit ball $\mathbb{B}(0,1)$ is convex, then for every p-dimensional vertical subgroup $\mathbb{W}$ we have*

$$\beta_d(\mathbb{W}) = \mathcal{H}_E^p(\mathbb{W} \cap \mathbb{B}(0,1)).$$

3. Low codimensional blow-up in Heisenberg groups

Our main result needs the following algebraic lemma.

Lemma 3.1. *We consider two vertical subgroups $\mathbb{M}$, $\mathbb{W}$ of $\mathbb{H}^n$ and a k-dimensional horizontal subgroup $\mathbb{V} \subset \mathbb{H}^n$ such that*

$$\mathbb{H}^n = \mathbb{M} \rtimes \mathbb{V} = \mathbb{W} \rtimes \mathbb{V}.$$

We introduce the multivectors

$$V = v_1 \wedge \cdots \wedge v_k, \quad N = w_1 \wedge \cdots \wedge w_{2n-k} \wedge e_{2n+1}, \quad M = m_1 \wedge \cdots \wedge m_{2n-k} \wedge e_{2n+1},$$

where $(v_1, \dots, v_k)$, $(w_1, \dots w_{2n-k}, e_{2n+1})$ and $(m_1, \dots, m_{2n-k}, e_{2n+1})$ are orthonormal bases of $\mathbb{V}$, $\mathbb{W}$ and $\mathbb{M}$, respectively. Then for every Borel set $B \subset \mathbb{M}$, we have

$$(\pi_{\mathbb{M},\mathbb{W}}^{\mathbb{M},\mathbb{V}})_\sharp \mathcal{H}_E^{2n+1-k}(B) = \mathcal{H}_E^{2n+1-k}(\pi_{\mathbb{W},\mathbb{M}}^{\mathbb{W},\mathbb{V}}(B)) = \frac{\|V \wedge M\|_g}{\|V \wedge N\|_g} \mathcal{H}_E^{2n+1-k}(B),$$

where the projections $\pi^{\mathbb{M},\mathbb{V}}_{\mathbb{M},\mathbb{W}}$ and $\pi^{\mathbb{W},\mathbb{V}}_{\mathbb{W},\mathbb{M}}$ have been introduced in Definition 2.4. The norms of $V \wedge M$ and $V \wedge N$ are taken with respect to the Hilbert structure of $\Lambda_{2n+1}(\mathbb{H}^n)$ induced by our scalar product on $\mathbb{H}^n$.

Proof. It is clearly not restrictive to relabel the bases of $\mathbb{M}$ and $\mathbb{W}$ as $w_{k+1}, \dots, w_{2n}, e_{2n+1}$ and $m_{k+1}, \dots, m_{2n}, e_{2n+1}$. We define the isomorphisms $i_{\mathbb{W}} : \mathbb{W} \to \mathbb{R}^{2n+1-k}$,

$$i_{\mathbb{W}}\left(x_{2n+1}e_{2n+1} + \sum_{i=k+1}^{2n} x_i w_i\right) = (x_{k+1}, \dots, x_{2n+1})$$

and $i_{\mathbb{M}} : \mathbb{M} \to \mathbb{R}^{2n+1-k}$,

$$i_{\mathbb{M}}\left(x_{2n+1}e_{2n+1} + \sum_{i=k+1}^{2n} x_i m_i\right) = (x_{k+1}, \dots, x_{2n+1})$$

and $i_{\mathbb{V}} : \mathbb{V} \to \mathbb{R}^k$

$$i_{\mathbb{V}}\left(\sum_{i=i}^{k} x_i v_i\right) = (x_1, \dots, x_k).$$

We introduce $\Psi_1 : \mathbb{R}^{2n+1} \to \mathbb{H}^n$,

$$\Psi_1(x_1, \dots, x_{2n+1}) = \Big(x_{2n+1}e_{2n+1} + \sum_{i=k+1}^{2n} x_i w_i\Big)\Big(\sum_{j=1}^{k} x_i v_i\Big). \tag{19}$$

We now notice that $J\Psi_1(x) = \|V \wedge N\|_g$ for every $x = (x_1, \dots, x_{2n+1}) \in \mathbb{R}^{2n+1}$. It suffices to observe that

$$J\Psi_1 = \|\partial_{x_1}\Psi_1 \wedge \cdots \partial_{x_{2n+1}}\Psi_{2n+1}\|_g$$

and use the explicit form of (19). We define another map $\Psi_2 : \mathbb{R}^{2n+1} \to \mathbb{H}^n$,

$$\Psi_2(x_1, \dots, x_{2n+1}) = \Big(x_{2n+1}e_{2n+1} + \sum_{i=k+1}^{2n} x_i m_i\Big)\Big(\sum_{j=1}^{k} x_i v_i\Big),$$

and we observe in the same way that $J\Psi_2(x) = \|V \wedge M\|_g$. We introduce the embedding $q : \mathbb{R}^{2n+1-k} \to \mathbb{R}^{2n+1}$,

$$q(x_1, \dots, x_{2n+1-k}) = (0, \dots, 0, x_1, \dots, x_{2n+1-k})$$

and the projection $p : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1-k}$,

$$p(x_1, \dots, x_{2n+1}) = (x_{k+1}, \dots, x_{2n+1}).$$

For every $z \in \mathbb{H}^n$, we observe that

$$\Psi_1^{-1}(z) = (i_{\mathbb{V}} \circ \pi_{\mathbb{V}}(z), i_{\mathbb{W}} \circ \pi_{\mathbb{W}}(z)).$$

It follows that

$$i_{\mathbb{W}}^{-1} \circ p \circ \Psi_1^{-1} = \pi_{\mathbb{W}}.$$

If we take any $m \in \mathbb{M}$, then

$$\begin{aligned}\pi_{\mathbb{W}}(m) &= i_{\mathbb{W}}^{-1} \circ p \circ \Psi_1^{-1} \circ \Psi_2 \circ \Psi_2^{-1}(m) \\ &= i_{\mathbb{W}}^{-1} \circ p \circ \Psi_1^{-1} \circ \Psi_2 \circ q \circ i_{\mathbb{M}}(m) \\ &= \pi_{\mathbb{M}}^{\mathbb{W},\mathbb{V}}(m).\end{aligned} \tag{20}$$

We notice that $\Psi_1^{-1} \circ \Psi_2$ is a polynomial diffeomorphism, whose Jacobian matrix at x has the following form

$$\begin{pmatrix} I & R_1 & 0 \\ 0 & R_2 & 0 \\ L_1(x) & L_2(x) & 1 \end{pmatrix} \in \mathbb{R}^{(2n+1)\times(2n+1)},$$

where $I \in \mathbb{R}^{k\times k}$, $R_1 \in \mathbb{R}^{k\times(2n-k)}$, $R_2 \in \mathbb{R}^{(2n+1-k)\times(2n+1-k)}$ and $L_1 : \mathbb{R}^{2n+1} \to \mathbb{R}^k$, $L_2 : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n-k}$ are affine functions. From the definition of $q : \mathbb{R}^{2n+1-k} \to \mathbb{R}^{2n+1}$ and $p : \mathbb{R}^{2n+1} \longrightarrow \mathbb{R}^{2n+1-k}$, by explicit computation, it follows that

$$J(\Psi_1^{-1} \circ \Psi_2) = |\det R_2| = J(p \circ \Psi_1^{-1} \circ \Psi_2 \circ q). \tag{21}$$

As a consequence, taking into account that

$$\frac{\|V \wedge M\|_g}{\|V \wedge N\|_g} = J(\Psi_1^{-1} \circ \Psi_2)$$

the following equalities hold

$$\begin{aligned}\mathcal{H}_E^{2n+1-k}(B) &= \mathcal{L}^{2n+1-k}(j(B)) \\ &= \frac{\|V \wedge N\|_g}{\|V \wedge M\|_g} \mathcal{L}^{2n+1-k}(p(L_1((L_2^{-1}(q(j(B))))) \\ &= \frac{\|V \wedge N\|_g}{\|V \wedge M\|_g} \mathcal{H}_E^{2n+1-k}(i^{-1}(p(L_1(L_2^{-1}(q(j(B))))) \\ &= \frac{\|V \wedge N\|_g}{\|V \wedge M\|_g} \mathcal{H}_E^{2n+1-k}(\pi_{\mathbb{W},\mathbb{M}}^{\mathbb{W},\mathbb{V}}(B)).\end{aligned}$$

□

We are now in the position to prove our main result.

Proof of Theorem 1.1. Let us consider $x \in \Sigma$. By formula (1), for any $y \in \Omega$, taking $t > 0$ sufficiently small, we can write

$$\mu(\mathbb{B}(y,t)) = \|V \wedge N\|_g \int_{\Phi^{-1}(\mathbb{B}(y,t))} \frac{J_H f(\Phi(n))}{J_{\mathbb{V}} f(\Phi(n))} \, d\mathcal{H}_E^{2n+1-k}(n). \tag{22}$$

We denote by $\zeta \in U$ the element such that

$$x = \Phi(\zeta) = \zeta\phi(\zeta).$$

We now perform the change of variables

$$n = \sigma_x(\Lambda_t(\eta)) = x(\Lambda_t\eta)(\pi_{\mathbb{V}}(x))^{-1} = x(\Lambda_t\eta)(\phi(\zeta))^{-1},$$

where $\Lambda_t = \delta_t|_{\mathbb{W}}$. The Jacobian of Λ_t is t^{2n+2-k}. It is well known that σ_x has unit Jacobian (see for instance [7, Lemma 2.20]). Setting $\alpha(x) = J_H f(x)/J_{\mathbb{V}} f(x)$, we obtain that

$$\frac{\mu(\mathbb{B}(y,t))}{t^{2n+2-k}} = \|V \wedge N\|_g \int_{\Lambda_{1/t}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(y,t))))} (\alpha \circ \Phi)(\sigma_x(\Lambda_t(\eta)))) \, d\mathcal{H}_E^{2n+1-k}(\eta).$$

By the general definition of Federer density we obtain that

$$\begin{aligned}\theta^{2n+2-k}(\mu, x) &= \inf_{r>0} \sup_{\substack{y \in \mathbb{B}(x,t) \\ 0<t<r}} \frac{\mu(\mathbb{B}(y,t))}{t^{2n+2-k}} \\ &= \inf_{r>0} \sup_{\substack{y \in \mathbb{B}(x,t) \\ 0<t<r}} \|V \wedge N\|_g \int_{\Lambda_{1/t}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(y,t))))} (\alpha \circ \Phi)(\sigma_x(\Lambda_t(\eta))) \, d\mathcal{H}_E^{2n+1-k}(\eta).\end{aligned}$$

There exists $R_0 > 0$ such that for $t > 0$ and $y \in \mathbb{B}(x,t)$ we have the following inclusion

$$\Lambda_{1/t}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(y,t)))) \subset \mathbb{B}_{\mathbb{W}}(0, R_0), \tag{23}$$

where the translated function $\phi_{x^{-1}}$ is defined according to formula (11) and we have set

$$\mathbb{B}_{\mathbb{W}}(0, R_0) = \mathbb{B}(0, R_0) \cap \mathbb{W}.$$

To see (23), we write more explicitly $\Lambda_{1/t}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(y,t))))$, that is

$$\left\{\eta \in \Lambda_{1/t}(\sigma_x^{-1}(U)) : \left\|y^{-1}x(\Lambda_t\eta)\phi(\zeta)^{-1}\phi(x(\Lambda_t\eta)\phi(\zeta)^{-1})\right\| \le t\right\}.$$

It can be written as follows

$$\left\{\eta \in \Lambda_{1/t}(\sigma_x^{-1}(U)) : \left\|(\delta_{1/t}(y^{-1}x))\eta\left(\frac{\phi(\zeta)^{-1}\phi(x(\Lambda_t\eta)\phi(\zeta)^{-1})}{t}\right)\right\| \le 1\right\}.$$

According to (11), the translated function of ϕ at x^{-1} is

$$\phi_{x^{-1}}(\eta) = \pi_{\mathbb{V}}(x^{-1})\phi(x\eta\pi_{\mathbb{V}}(x^{-1})) = \phi(\zeta)^{-1}\phi(x\eta\phi(\zeta)^{-1}).$$

We finally get

$$\Lambda_{1/t}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(y,t)))) = \left\{\eta \in \Lambda_{1/t}(\sigma_x^{-1}(U)) : \left\|(\delta_{1/t}(y^{-1}x))\eta\left(\frac{\phi_{x^{-1}}(\Lambda_t\eta)}{t}\right)\right\| \le 1\right\}, \tag{24}$$

hence for $\eta \in \Lambda_{1/t}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(y,t)))$, taking into account the previous equality, we have established that

$$\eta\left(\frac{\phi_{x^{-1}}(\Lambda_t\eta)}{t}\right) \in \mathbb{B}(0,2).$$

From the estimate (10), we know that

$$c_0\left(\|\eta\| + \left\|\frac{\phi_{x^{-1}}(\Lambda_t\eta)}{t}\right\|\right) \le \left\|\eta\left(\frac{\phi_{x^{-1}}(\Lambda_t\eta)}{t}\right)\right\| \le 2,$$

hence the inclusion (23) holds with $R_0 = 2/c_0$. As a consequence, we have that

$$\theta^{2n+2-k}(\mu, x) < \infty.$$

There exist a positive sequence $\{t_p\}$ converging to zero and $y_p \in \mathbb{B}(x, t_p)$ such that

$$\|V \wedge N\|_g \int_{\Lambda_{1/t_p}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(y_p,t_p))))} \frac{J_H f(\Phi(\sigma_x(\Lambda_{t_p}(\eta))}{J_{\mathbb{V}} f(\Phi(\sigma_x(\Lambda_{t_p}(\eta)))} \, d\mathcal{H}^{2n+1-k_E}(\eta) \to \theta^{2n+2-k}(\mu, x)$$

as $p \to \infty$. Up to extracting a subsequence, since $y_p \in \mathbb{B}(x, t_p)$ for every p, there exists $z \in \mathbb{B}(0,1)$ such that

$$\lim_{p\to\infty} \delta_{1/t_p}(x^{-1}y_p) = z.$$

For the sake of simplicity, we use the notation

$$\mathbb{M}_x = \ker Df(x).$$

Using the projection introduced in Definition 2.4, we set

$$S_z = \pi^{\mathbb{W},\mathbb{V}}_{\mathbb{W},\mathbb{M}_x}(\mathbb{M}_x \cap \mathbb{B}(z,1)) \subset \mathbb{W}.$$

Claim 1: For each $\omega \in \mathbb{W} \setminus S_z$, there exists

$$\lim_{p\to\infty} \mathbf{1}_{\Lambda_{1/t_p}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(y_p,t_p))))}(\omega) = 0.$$

By contradiction, if we had a subsequence of the integers p such that

$$(\delta_{1/t_p}(y_p^{-1}x))\omega\left(\frac{\phi_{x^{-1}}(\Lambda_{t_p}\omega)}{t}\right) \in \mathbb{B}(0,1),$$

then by a slight abuse of notation, we could still call t_p the sequence such that

$$(\delta_{1/t_p}(y_p^{-1}x))\omega d\phi_\zeta(\omega)\left(\frac{(d\phi_\zeta(\Lambda_{t_p}\omega))^{-1}\phi_{x^{-1}}(\Lambda_{t_p}\omega)}{t_p}\right) \in \mathbb{B}(0,1) \tag{25}$$

for all p, where we have used the homogeneity of the intrinsic differential $d\phi_\zeta$ of ϕ, see Definition 2.7 for the notion of intrinsic differential. Indeed, by Theorem 2.7, the function ϕ is in particular intrinsic differentiable at ζ. Due to the intrinsic differentiability, taking into account (25) as $p \to \infty$, it follows that

$$\omega d\phi_\zeta(\omega) \in \mathbb{B}(z,1).$$

It is now interesting to observe that the chain rule of Theorem 2.2 yields

$$\mathrm{graph}(d\phi_\zeta) = \ker(Df(x))) = \mathbb{M}_x. \tag{26}$$

As a consequence, $\omega d\phi_\zeta(\omega) \in \mathbb{B}(z,1) \cap \mathbb{M}_x$ and then

$$\omega = \pi^{\mathbb{W},\mathbb{V}}_{\mathbb{W},\mathbb{M}_x}(\omega d\phi_\zeta(\omega)) \in \pi^{\mathbb{W},\mathbb{V}}_{\mathbb{W},\mathbb{M}_x}(\mathbb{M}_x \cap \mathbb{B}(z,1)) = S_z, \tag{27}$$

that is not possible by our assumption. This concludes the proof of Claim 1.

Now we introduce the density function

$$\alpha(t,\eta) = \frac{J_H f(\Phi(\sigma_x(\Lambda_t(\eta))))}{J_{\mathbb{V}} f(\Phi(\sigma_x(\Lambda_t(\eta))))}$$

to write

$$\|V \wedge N\|_g \int_{\Lambda_{1/t_p}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(y_p,t_p))))} \alpha(t_p,\eta)\ d\mathcal{H}_E^{2n+1-k}(\eta) = I_p + J_p.$$

The sequence I_p, defined in the following equality, satisfies the estimate

$$\begin{aligned} I_p &= \|V \wedge N\|_g \int_{S_z \cap \Lambda_{1/t_p}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(y_p,t_p))))} \alpha(t_p,\eta)\ d\mathcal{H}_E^{2n+1-k}(\eta) \\ &\le \|V \wedge N\|_g \int_{S_z} \alpha(t_p,\eta)\ d\mathcal{H}_E^{2n+1-k}(\eta). \end{aligned}$$

Analogously for J_p, we find

$$J_p = \|V \wedge N\|_g \int_{\Lambda_{1/t_p}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(y_p,t_p))))\setminus S_z} \alpha(t_p,\eta)\ d\mathcal{H}_E^{2n+1-k}(\eta)$$
$$\leq \|V \wedge N\|_g \int_{\mathbb{B}_{\mathbb{W}}(0,R_0)\setminus S_z} \mathbf{1}_{\Lambda_{1/t_p}(\sigma_x^{-1}((\Phi^{-1}(\mathbb{B}(y_p,t_p))))}(\eta)\ \alpha(t_p,\eta)\ d\mathcal{H}_E^{2n+1-k}(\eta).$$

Claim 1 joined with the dominated convergence theorem prove that $J_p \to 0$ as $p \to \infty$, hence $I_p \to \theta^{2n+2-k}(\mu,x)$. To study the asymptotic behavior of I_p, we first observe that

$$\alpha(t_p,\eta) \to \frac{J_H f(x)}{J_{\mathbb{V}} f(x)} = c(x)$$

as $p \to \infty$. It follows that

$$\theta^{2n+2-k}(\mu,x) = \lim_{p\to\infty} I_p \leq \|V \wedge N\|_g\ c(x)\ \mathcal{H}_E^{2n+1-k}(S_z). \tag{28}$$

Claim 2. We set $\mathbb{M}_x = \ker(Df(x))$ and consider $N_x = m_{k+1} \wedge \cdots \wedge m_{2n} \wedge e_{2n+1}$ such that $(m_{k+1},\ldots,m_{2n},e_{2n+1})$ is an orthonormal basis of $\mathbb{M}_x$. We have that

$$c(x) = \frac{J_H f(x)}{J_{\mathbb{V}} f(x)} = \frac{1}{\|V \wedge N_x\|_g}. \tag{29}$$

Since $\mathrm{span}\{\nabla_H f_1(x),\ldots,\nabla_H f_k(x)\}$ is orthogonal to $\mathbb{M}_x$, it is a standard fact that

$$m_{k+1} \wedge \cdots \wedge m_{2n} \wedge e_{2n+1} = *(\nabla_H f_1(x) \wedge \cdots \wedge \nabla_H f_k(x))\lambda \tag{30}$$

for some $\lambda \in \mathbb{R}$, see for instance [12, Lemma 5.1]. Here we have defined the Hodge operator $*$ in $\mathbb{H}^n$ with respect to the fixed orientation

$$\mathbf{e} = e_1 \wedge \ldots e_{2n} \wedge e_{2n+1}$$

and the fixed scalar product $\langle\cdot,\cdot\rangle$. Precisely, we are referring to an orthonormal Heisenberg basis $(e_1,\ldots,e_{2n},e_{2n+1})$, according to Sections 2.1 and 2.2. Therefore $*\eta$ is the unique $(2n+1-k)$-vector such that

$$\xi \wedge *\eta = \langle \xi,\eta\rangle\ \mathbf{e} \tag{31}$$

for all k-vectors ξ. Since the Hodge operator is an isometry, we get

$$|\lambda| = \frac{1}{\|\nabla_H f_1(x) \wedge \ldots \nabla_H f_k(x)\|_g}. \tag{32}$$

Due to (32) and (31), we have

$$\begin{aligned}
\|V \wedge N_x\|_g &= |\lambda| \|v_1 \wedge \cdots \wedge v_k \wedge (*(\nabla_H f_1(x) \wedge \cdots \wedge \nabla_H f_k(x)))\|_g \\
&= \frac{\|\langle v_1 \wedge \cdots \wedge v_k, \nabla_H f_1(x) \wedge \cdots \wedge \nabla_H f_k(x)\rangle \mathbf{e}\|_g}{\|\nabla_H f_1(x) \wedge \cdots \wedge \nabla_H f_k(x)\|_g} \\
&= \frac{|\langle v_1 \wedge \cdots \wedge v_k, \nabla_H f_1(x) \wedge \cdots \wedge \nabla_H f_k(x)\rangle|}{\|\nabla_H f_1(x) \wedge \cdots \wedge \nabla_H f_k(x)\|_g} \\
&= \frac{\|\nabla_{\mathbb{V}} f_1(x) \wedge \cdots \wedge \nabla_{\mathbb{V}} f_k(x)\|_g}{\|\nabla_H f_1(x) \wedge \cdots \wedge \nabla_H f_k(x)\|_g} \\
&= \frac{J_{\mathbb{V}} f(x)}{J_H f(x)},
\end{aligned}$$

hence establishing Claim 2.

As a result, taking into account (28), we have proved that

$$\theta^{2n+2-k}(\mu, x) \leq \frac{\|V \wedge N\|_g}{\|V \wedge N_x\|_g} \, \mathcal{H}_E^{2n+1-k}(S_z). \tag{33}$$

By Lemma 3.1, for $B = \mathbb{M}_x \cap \mathbb{B}(z,1)$, the following formula holds

$$\mathcal{H}_E^{2n+1-k}(\pi_{\mathbb{W},\mathbb{M}_x}^{\mathbb{W},\mathbb{V}}(\mathbb{M}_x \cap \mathbb{B}(z,1))) = \frac{\|V \wedge N_x\|_g}{\|V \wedge N\|_g} \mathcal{H}_E^{2n+1-k}(\mathbb{M}_x \cap \mathbb{B}(z,1)). \tag{34}$$

It follows that

$$\theta^{2n+2-k}(\mu, x) \leq \mathcal{H}_E^{2n+1-k}(\mathbb{M}_x \cap \mathbb{B}(z,1)) \leq \ \mathcal{H}_E^{2n+1-k}(\mathbb{M}_x \cap \mathbb{B}(z_0,1)), \tag{35}$$

where $z_0 \in \mathbb{B}(0,1)$ is chosen such that $\beta_d(\mathbb{M}_x) = \mathcal{H}_E^{2n+1-k}(\mathbb{M}_x \cap \mathbb{B}(z_0,1))$.

For the opposite inequality, we follow the scheme in the proof of [16, Theorem 3.1]. We consider a specific family of points $y_t^0 = x\delta_t z_0 \in \mathbb{B}(x,t)$ and fix $\lambda > 1$. We have that

$$\sup_{0<t<r} \frac{\mu(\mathbb{B}(y_t^0, \lambda t))}{(\lambda t)^{2n+2-k}} \leq \sup_{\substack{y \in \mathbb{B}(x,t), \\ 0<t<\lambda r}} \frac{\mu(\mathbb{B}(y,t))}{t^{2n+2-k}}$$

for every $r > 0$, therefore

$$\limsup_{t \to 0^+} \frac{\mu(\mathbb{B}(y_t^0, \lambda t))}{(\lambda t)^{2n+2-k}} \leq \theta^{2n+2-k}(\mu, x). \tag{36}$$

We introduce the set

$$\begin{aligned} A_t^0 &= \Lambda_{1/\lambda t}(\sigma_x^{-1}(\Phi^{-1}(\mathbb{B}(y_t^0, \lambda t))) \\ &= \left\{ \eta \in \Lambda_{1/\lambda t}(\sigma_x^{-1}(U)) : \eta\left(\frac{\phi_{x^{-1}}(\Lambda_{\lambda t}\eta)}{\lambda t}\right) \in \mathbb{B}(\delta_{1/\lambda} z_0, 1) \right\}. \end{aligned}$$

The second equality can be deduced from (24). Then we can rewrite

$$\begin{aligned} \frac{\mu(\mathbb{B}(y_t^0, \lambda t))}{(\lambda t)^{2n+2-k}} &= \|V \wedge N\|_g \int_{A_t^0} \alpha(\lambda t, \eta) d\mathcal{H}_E^{2n+1-k}(\eta) \\ &= \frac{\|V \wedge N\|_g}{\lambda^{2n+2-k}} \int_{\delta_\lambda A_t^0} \alpha(\lambda t, \delta_{1/\lambda}\eta) d\mathcal{H}_E^{2n+1-k}(\eta) \end{aligned} \tag{37}$$

The domain of integration satisfies

$$\delta_\lambda A_t^0 = \left\{ \eta \in \Lambda_{1/t}(\sigma_x^{-1}(U)) : \eta\left(\frac{\phi_{x^{-1}}(\Lambda_t \eta)}{t}\right) \in \mathbb{B}(z_0, \lambda) \right\}.$$

Due to (23) and the definition of A_t^0, it holds

$$\delta_\lambda A_t^0 \subset \mathbb{B}_{\mathbb{W}}(0, \lambda R_0).$$

Claim 3: For every $\eta \in \pi_{\mathbb{W},\mathbb{M}_x}^{\mathbb{W},\mathbb{V}}(\mathbb{M}_x \cap B(z_0, \lambda))$, we have

$$\lim_{t \to 0^+} \mathbf{1}_{\delta_\lambda A_t^0}(\eta) = 1. \tag{38}$$

The intrinsic differentiability of ϕ at ζ shows that

$$\eta\left(\frac{\phi_{x^{-1}}(\Lambda_t \eta)}{t}\right) \to \eta d\phi_\zeta(\eta) \quad \text{as} \quad t \to 0.$$

Taking into account (9) and (27), we get

$$\pi^{\mathbb{M}_x,\mathbb{V}}_{\mathbb{M}_x,\mathbb{W}}(\eta) = \eta d\phi_\zeta(\eta),$$

hence our assumption on η can be written as follows

$$d\left(\eta d\phi_\zeta(\eta), z_0\right) < \lambda.$$

We conclude that $\eta \in \delta_\lambda A^0_t$ for any $t > 0$ sufficiently small, therefore the limit (38) holds and the proof of Claim 3 is complete.

By Fatou's lemma, taking into account (36) and (37) we get

$$\frac{\|V \wedge N\|_g}{\lambda^{2n+2-k}} \int_{\pi^{\mathbb{W},\mathbb{V}}_{\mathbb{W},\mathbb{M}_x}(\mathbb{M}_x \cap B(z_0,\lambda))} \liminf_{t\to 0} \left(\mathbf{1}_{\delta_\lambda A^0_t}(\eta)\alpha(\lambda t, \delta_{1/\lambda}\eta)\right) d\mathcal{H}^{2n+1-k}_E(\eta) \le \theta^{2n+2-k}(\mu, x).$$

Claim 3 joined with (29) yield

$$\frac{1}{\lambda^{2n+2-k}} \frac{\|V \wedge N\|_g}{\|V \wedge N_x\|_g} \mathcal{H}^{2n+1-k}_E \left(\pi^{\mathbb{W},\mathbb{V}}_{\mathbb{W},\mathbb{M}_x}(\mathbb{M}_x \cap \mathbb{B}(z_0, 1))\right) \le \theta^{2n+2-k}(\mu, x).$$

Applying again (34), we obtain

$$\frac{1}{\lambda^{2n+2-k}} \mathcal{H}^{2n+1-k}_E(\mathbb{M}_x \cap \mathbb{B}(z_0, 1)) \le \theta^{2n+2-k}(\mu, x).$$

Taking the limit as $\lambda \to 1^+$ and considering the opposite inequality (35), the proof is complete. □

Adding some natural modifications in the proof of the previous theorem, we can also obtain the following "centered blow-up".

Theorem 3.2. *In the assumptions of Theorem 1.1, for every $x \in \Sigma$, we have*

$$\theta^{2n+2-k}_c(\mu, x) = \mathcal{H}^{2n+1-k}_E(\mathit{Tan}(\Sigma, x) \cap \mathbb{B}(0, 1)).$$

4. Applications

Combining Theorems 2.11 and Theorem 1.1 we immediately get the following result.

Theorem 4.1 (Area formula)**.** *In the assumptions of Theorem 1.1, for any Borel set $B \subset \Sigma$ we have*

$$\mu(B) = \int_B \beta_d(\mathit{Tan}(\Sigma, x))\; d\mathcal{S}^{2k+2-k}_0(x). \tag{39}$$

We are now in the position to prove Theorem 1.2.

Proof of Theorem 1.2. Since $\mathbb{W}$ and $\mathbb{V}$ are orthogonal, by Proposition 2.8 we can fix a Heisenberg basis $(v_1, \ldots, v_k, v_{k+1}, \ldots, v_n, w_1, \ldots w_{2n}, e_{2n+1})$ such that $\mathbb{V} = \operatorname{span}\{v_1, \ldots, v_k\}$ and $\mathbb{W} = \operatorname{span}\{v_{k+1}, \ldots, v_n, w_i, \ldots, w_n, e_{2n+1}\}$. Our claim follows by representing the measure μ in terms of the intrinsic partial derivatives of the parametrization ϕ of Σ, arguing as in the proof [4, Theorem 6.1]. For the reader's convenience we report the main points of the proof.

Taking into account Theorem 2.7, $\Sigma = \Phi(\Omega)$ is the graph of a uniformly intrinsic differentiable function ϕ. Arguing as in the proof of [5, Theorem 4.1] or [2, Theorem 4.2], there exist an open set $\Omega' \subset \mathbb{H}^n$ and a function $g \in C^1_h(\Omega', \mathbb{R}^k)$ such that $\Sigma \subset g^{-1}(0)$ and for every $m \in U$ the following holds

$$Dg(\Phi(m)) = \begin{bmatrix} \nabla_H g_1(\Phi(m)) \\ \dots \\ \nabla_H g_k(\Phi(m)) \end{bmatrix} = \begin{bmatrix} \mathbb{I}_k & -D^\phi \phi(m) \end{bmatrix}. \tag{40}$$

By Theorem 4.1, for any Borel set $B \subset \Sigma$,

$$\mu(B) = \int_B \beta_d(\mathrm{Tan}(\Sigma, x))\, d\mathcal{S}_0^{2k+2-k}(x) = \int_{\Phi^{-1}(B)} \frac{J_H g(\Phi(n))}{J_{\mathbb{V}} g(\Phi(n))} d\mathcal{H}_E^{2n+1-k}(n).$$

Notice that $J_{\mathbb{V}} g(\Phi(m)) = 1$ for every $m \in U$. By Definition 2.14, taking into account the form of $Dg(\Phi(m))$ in (40), the proof is achieved. □

We now restrict our attention to homogeneous distances with symmetries. Using both Theorem 2.12 and Proposition 2.13 we obtained simpler versions of the area formula.

Theorem 4.2. *Let d be either a $(2n+1-k)$-vertically symmetric distance or a multiradial distance of $\mathbb{H}^n$. Then in the assumptions of Theorem 1.1, we have that*

$$\mu = \omega_d(2n+1-k) \mathcal{S}_0^{2k+2-k} \llcorner \Sigma. \tag{41}$$

Therefore, by defining $\mathcal{S}_d^{2n+2-k} = \omega_d(2n+1-k)\mathcal{S}_0^{2n+1-k}$, we have

$$\mathcal{S}_d^{2n+2-k} \llcorner \Sigma = \|V \wedge N\|_g \; \Phi_\sharp \left(\frac{J_H f}{J_{\mathbb{V}} f} \circ \Phi \right) \mathcal{H}_E^{2n+1-k} \llcorner \mathbb{W}. \tag{42}$$

In the assumptions of the previous theorem, assuming in addition that $\mathbb{W}$ and $\mathbb{V}$ are orthogonal, equation (42) can be rewritten for any Borel set $B \subset \Sigma$ as

$$\mathcal{S}_d^{2n+2-k} \llcorner \Sigma(B) = \int_{\Phi^{-1}(B)} J^\phi \phi(w)\, d\mathcal{H}_E^{2n+1-k}(w), \tag{43}$$

where $J^\phi \phi$ the intrinsic Jacobian of ϕ defined in 2.14.

By Theorem 2.10 and Theorem 3.2 we have the area formula for the centered Hausdorff measure. It is the analogous of Theorem 4.1.

Theorem 4.3. *In the assumptions of Theorem 1.1, for any Borel set $B \subset \Sigma$ we have*

$$\mu(B) = \int_B \mathcal{H}_E^{2n+1-k}(\mathit{Tan}(\Sigma, x) \cap \mathbb{B}(0,1))\, d\mathcal{C}_0^{2k+2-k}(x). \tag{44}$$

When $d = d_\infty$ the previous theorem recovers [6, Theorem 4.1].

Corollary 4.4. *Let d be an homogeneous distance on $\mathbb{H}^n$ such that $\mathbb{B}(0,1)$ is convex. In the assumptions of Theorem 1.1, for every $x \in \Sigma$ we obtain*

$$\theta_c^{2n+2-k}(\mu, x) = \theta^{2n+2-k}(\mu, x) \quad \text{and} \quad \mathcal{C}_0^{2n+2-k} \llcorner \Sigma = \mathcal{S}_0^{2n+2-k} \llcorner \Sigma. \tag{45}$$

Proof. By Theorem 2.14 and Theorem 3.2, for every $x \in \Sigma$ we have

$$\beta_d(\ker(Df(x)) = \mathcal{H}^{2n+1-k}(\ker(Df(x)) \cap \mathbb{B}(0,1)) = \theta_c^{2n+2-k}(\mu, x).$$

Finally, the area formulas (39) and (44) conclude the proof. □

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