STOCHASTIC HOMOGENISATION OF FREE-DISCONTINUITY FUNCTIONALS IN RANDOMLY PERFORATED DOMAINS

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Abstract. In this paper we study the asymptotic behaviour of a family of random free-discontinuity energies E_ε defined in a randomly perforated domain, as ε goes to zero. The functionals E_ε model the energy associated to displacements of porous random materials that can develop cracks. To gain compactness for sequences of displacements with bounded energies, we need to overcome the lack of equi-coerciveness of the functionals. We do so by means of an extension result, under the assumption that the random perforations cannot come too close to one another. The limit energy is then obtained in two steps. As a first step we apply a general result of stochastic convergence of free-discontinuity functionals to a modified, coercive version of E_ε. Then the effective volume and surface energy densities are identified by means of a careful limit procedure.

Keywords: Homogenisation, Γ-convergence, free-discontinuity problems, randomly perforated domains, Neumann boundary conditions, porous materials, brittle fracture.

MSC 2010: 49J45, 49Q20, 74Q05.

1. Introduction

In this paper we prove a stochastic homogenisation result for free-discontinuity functionals defined in randomly perforated domains. More precisely we consider the functionals E_ε given by

$$E_\varepsilon(\omega)(u,A) = \int_{A\setminus\varepsilon K(\omega)} f\left(\omega, \frac{x}{\varepsilon}, \nabla u\right) dx + \int_{S_u\cap(A\setminus\varepsilon K(\omega))} g\left(\omega, \frac{x}{\varepsilon}, \nu_u\right) d\mathcal{H}^{n-1}, \tag{1.1}$$

for $u \in SBV(A)$; here $A \subset \mathbb{R}^n$ is a bounded, Lipschitz domain, and $SBV(A)$ denotes the set of special functions of bounded variation in A. In (1.1) the parameter ω belongs to the sample space Ω of a given probability space $(\Omega, \mathcal{T}, P)$, whereas $\varepsilon > 0$ sets the geometric scale of the problem. The integrands f and g are stationary random variables, thus they are to be interpreted as an ensemble of coefficients; f satisfies standard p-growth assumptions, for $p > 1$, and g is bounded (see Section 2.2). Note that the Mumford-Shah functional is a special case of our class of energies. The integration in (1.1) is performed only on the set $A \setminus \varepsilon K(\omega)$, where $K(\omega)$ denotes a collection of randomly distributed n-dimensional balls with random radii (see (2.4)), and models random perforations inside the material occupying the reference configuration A. Energies of this type can be used to describe the elastic energy of a *porous brittle random* material.

In the deterministic *periodic* setting, the limit behaviour of energies of type (1.1) has been studied both in the case of Dirichlet conditions on the perforations [20] and in the case of natural boundary conditions [3, 9, 21]. Only very recently, in [11], the stochastic homogenisation of free-discontinuity functionals was considered, under quite general assumptions on the volume and surface integrands, and in the vector-valued case (see [10], and [5, 22] for the deterministic counterpart). In [11], however, the volume and surface integrands must satisfy non-degenerate lower bounds, which is not the case for E_ε, due to the presence of the perforations.

The study of the asymptotic behaviour of elliptic problems in *randomly* perforated domains has a long history starting with the seminal work of Jikov [24]. We refer the reader to the book [25] and the references therein for the classical results on this subject. More recently the random counterpart of the work by Cioranescu and Murat [14] has been also considered [12, 13, 23]. In this case, sequences u_ε of equi-bounded energy can be trivially extended to zero inside $\varepsilon K(\omega)$, due to the homogeneous *Dirichlet boundary conditions*, and hence can be assumed from the onset to satisfy a priori bounds on the whole domain. In the Dirichlet setting the main difficulty in the analysis lies then in the characterisation of the limiting "capacitary" term. Since in this case no extension result for the u_ε is needed, the assumptions on the geometry of the perforations can be rather mild [23].

In this paper we assume instead that sequences u_ε of equi-bounded energy satisfy *natural boundary conditions* on the perforations, which makes the compactness of minimising sequences subtle. In this setting the classical way to obtain compactness is to *extend* the functions u_ε inside the perforations in a way that keeps the functionals on the extended functions comparable with the functionals on u_ε. In the periodic case, and for Sobolev functions, the use of extension theorems as a powerful technique to treat degenerate problems is due to Khruslov [26], Cioranescu and Paulin [15], and to Tartar [27]. In that setting, the most general extension result is due to Acerbi, Chiadò Piat, Dal Maso, and Percivale [1], and has been proved under minimal assumptions on the geometry of the periodic perforations, which in particular can be connected in dimension $n > 2$.

In the random case a common approach to the homogenisation of perforated (or porous) materials is to assume the existence of an extension operator as a property of the domain (see, *e.g.*, [25, Chapter 8]). More precisely, it is often assumed that the perforated domain $A \setminus \varepsilon K(\omega)$ is a random set, that it is open and connected, that its density (namely the expectation of its characteristic function) is strictly positive, and that there exists an extension operator from the perforated to the full domain. These assumptions guarantee compactness of sequences with equi-bounded energies, and allow to prove existence of the Γ-limit, and non-degeneracy of the limit energy. Alternatively, simplified random geometries are considered, for which one can prove directly that the random domain satisfies the assumptions above. This is the case for a class of disperse media, the so-called *random spherical structure*; *i.e.,* a system of many hard sphere particles. In the simplest case of such structure the domain has an underlying ε-periodic grid, and in each ε-cell the random perforation is a ball - with random radius and centre - which is $\varepsilon\delta$-separated from the boundary of the cell where it is contained, for a given $\delta > 0$. A more general geometry is given by the case where the spherical holes are $2\varepsilon\delta$-separated from one another, but no underlying periodic "safety" grid is postulated. For random spherical structures it is shown, *e.g.*, in [25, Section 8.4] that if the spherical holes are not too close to one another, then the density of the domain is strictly positive, and some extension operator exists in the Sobolev setting.

Our approach is in the same spirit, and we now explain it in some detail.

1.1. **Overview of the main results.** In what follows we give a brief overview of the main results contained in this paper: An extension result for special functions of bounded variation in a randomly perforated domain, and the Γ-convergence of the functionals E_ε in (1.1).

The extension property in SBV**.** The geometry we consider for the randomly perforated domain is the following: We assume that the perforations $K(\omega)$ are disjoint balls of random centres and radii, and that the radii are bounded from above by a deterministic constant $r_* > 0$. Moreover, we require that the minimal distance between any two of them is 2δ, where $\delta > 0$ is independent of the realisation ω. In other words, not only the perforations are separated, but also their δ-neighbourhoods are so. Our first main result is an extension property for this class of domains in SBV (Lemma 4.1 and Theorem 4.2). We recall that the existence of an extension operator in SBV, for the Mumford-Shah functional, has been proved by Cagnetti and Scardia [9] in the periodic case. This result, however, cannot be applied directly to our case since the domain $A \setminus \varepsilon K(\omega)$ is in general not periodic. Intuitively, we would like to apply the deterministic result in a δ-neighbourhood of each component of $K(\omega)$, since by assumption such neighbourhoods are pairwise disjoint. If we did it naively, however, then we could have for each component of $K(\omega)$ a different extension operator norm bound, since the components of $K(\omega)$ are balls with possibly different centres and radii from one another. Consequently, we would not be able to obtain uniform bounds for the extended function, which are crucial for equi-coerciveness.

To illustrate how we obtain uniform bounds, we now focus on a generic perforation $B(\theta(\omega), r(\omega))$, where $\theta(\omega)$ and $r(\omega)$ are the (random) centre and radius, with $r(\omega) < r_*$. We need to construct an extension operator from the annulus $A_\delta(\omega) := B(\theta(\omega), r(\omega)+\delta) \setminus B(\theta(\omega), r(\omega))$ (which is contained in $A \setminus \varepsilon K(\omega)$, thanks to the δ-separation of the perforations) to $B(\theta(\omega), r(\omega) + \delta)$. Essentially, there are two different cases to be considered separately: the case $r(\omega) < \delta$ and the opposite case

$r(\omega) \geq \delta$. If $r(\omega) < \delta$ we follow [9] to extend from $B(\theta(\omega), 2r(\omega)) \setminus B(\theta(\omega), r(\omega)) \subset A_\delta(\omega)$; if instead $r(\omega) \geq \delta$ we follow [9] to extend from $B(\theta(\omega), r(\omega)(1+\delta/r_*)) \setminus B(\theta(\omega), r(\omega)) \subset A_\delta(\omega)$. Since the deterministic extension constructed in [9] is invariant under translations and homotheties of the domains, in both cases the extension constant is independent of $r(\omega)$ and $\theta(\omega)$ (see Lemma 4.1). We then repeat this procedure for every inclusion, and obtain an extension operator from $A \setminus \varepsilon K(\omega)$ to A, with an extension constant independent of ε and of ω (Theorem 4.2). This is a key ingredient in the proof of the compactness (strongly in L^1) for sequences with bounded energies $E_\varepsilon(\omega)$ (Proposition 4.7).

The Γ-convergence result. Once the compactness result is established, we prove the stochastic Γ-convergence of $E_\varepsilon(\omega)$ for $\varepsilon \to 0$ (Theorems 5.1 and 5.3). Our strategy is to resort to a perturbation argument. Namely, we first introduce a perturbed functional $E^k_\varepsilon(\omega)$, with volume and surface densities given by $f^k := a^k f$ and $g^k := a^k g$, where

$$a^k(\omega, x) := \begin{cases} 1 & \text{if } x \in \mathbb{R}^n \setminus K(\omega), \\ \frac{1}{k} & \text{if } x \in K(\omega). \end{cases}$$

In other words, $E^k_\varepsilon(\omega)$ is obtained from $E_\varepsilon(\omega)$ by filling the holes with a coefficient $\frac{1}{k}$, with $k \in \mathbb{N}$. The perturbed functionals are non-degenerate and coercive, hence for fixed k the Γ-limit of E^k_ε for $\varepsilon \to 0$ exists almost surely by [11, Theorem 3.12]. Moreover, we can identify the limit volume and surface energy densities, which are given by

$$f^k_{\rm hom}(\omega, \xi) = \lim_{t \to +\infty} \frac{1}{t^n} \inf \left\{ \int_{tQ} f^k(\omega, x, \nabla u)\, dx : u \in W^{1,p}(tQ),\ u = \xi \cdot x \ \text{ near } \partial(tQ) \right\}, \quad (1.2)$$

and

$$g^k_{\rm hom}(\omega, \nu) = \lim_{t \to +\infty} \frac{1}{t^{n-1}} \inf \left\{ \int_{S_u \cap tQ^\nu} g^k(\omega, x, \nu_u)\, d\mathcal{H}^{n-1} : u \in \mathcal{P}(tQ^\nu),\ u = u_{0,1,\nu} \ \text{ near } \partial(tQ^\nu) \right\}, \quad (1.3)$$

where $\xi \in \mathbb{R}^n$, $\nu \in \mathbb{S}^{n-1}$, Q^ν is the rotated unit cube centred at the origin with one face perpendicular to ν, $u_{0,1,\nu}$ is the piecewise constant function equal to 1 in the half-space in the positive direction of ν and 0 in the complement, and $\mathcal{P}$ denotes the set of partitions with values in $\{0,1\}$.

The volume and the surface densities $f_{\rm hom}$ and $g_{\rm hom}$ of the Γ-limit of $E_\varepsilon(\omega)$ are then obtained as the limits for $k \to +\infty$ of $f^k_{\rm hom}$ and $g^k_{\rm hom}$, respectively. The most delicate part in the proof is to show that these limits coincide with

$$\lim_{t \to +\infty} \frac{1}{t^n} \inf \left\{ \int_{tQ \setminus K(\omega)} f(\omega, x, \nabla u)\, dx : u \in W^{1,p}(tQ),\ u = \xi \cdot x \ \text{ near } \partial(tQ) \right\} \quad (1.4)$$

and

$$\lim_{t \to +\infty} \frac{1}{t^{n-1}} \inf \left\{ \int_{S_u \cap (tQ^\nu \setminus K(\omega))} g(\omega, x, \nu_u)\, d\mathcal{H}^{n-1} : u \in \mathcal{P}(tQ^\nu),\ u = u_{0,1,\nu} \ \text{ near } \partial(tQ^\nu) \right\}, \quad (1.5)$$

respectively. This step requires a careful use of extension techniques for Sobolev functions (Lemma 4.5) and for Caccioppoli partitions (Lemma 4.6) separately, in order to construct, starting from a competitor for the minimisation problem in (1.4) (resp. (1.5)) a competitor for the minimisation problem in (1.2) (resp. (1.3)). Lemma 4.6, in particular, requires the use of a technical lemma proved by Congedo and Tamanini in [16] (see also [17]), which establishes some regularity properties for minimisers of the perimeter functional. These regularity properties, in turn, ensure that minimising partitions are constant on a sphere around each perforation, from which we can then perform a trivial extension at no additional energetic cost.

Finally, our assumptions on the geometry of $K(\omega)$ allow us to prove that the limit densities $f_{\rm hom}$ and $g_{\rm hom}$ are non-degenerate.

1.2. **Conclusions and outlook.** In this paper we prove a stochastic homogenisation result for free-discontinuity functionals on randomly perforated domains, without imposing any boundary conditions on the perforations. Our approach relies on the construction of an extension operator guaranteeing that, given a function in the perforated domain, the extended function in the whole domain is bounded, in energy, in terms of the original function. The construction of the extension operator, in turn, is guaranteed by our assumptions on the geometry of the randomly perforated domain. In particular, the assumption of δ-separation of the holes is crucial in our analysis. This assumption, moreover, also ensures that the *density* of the random domain is strictly positive, and hence the non-degeneracy of the limit energy.

It would be interesting to investigate whether our result could work in the more general case where the existence of a fixed safety distance δ is replaced by a more global condition of "average" separation, *e.g.* in the spirit of [25, Section 8.4].

2. Setting of the problem and statement of the main result

2.1. **Notation.** We introduce here all the notation that we need.

- $\mathbb{N}^* := \{z \in \mathbb{Z} : z \geq 1\}$;
- For $\rho > 0$ and $\theta \in \mathbb{R}^n$ we define $Q_\rho(\theta) := \{x \in \mathbb{R}^n : |x_i - \theta_i| < \frac{\rho}{2},\ i = 1,..,n\}$; we use the shorthands $Q_\rho = Q_\rho(0)$ and $Q = Q_1$;
- For $\rho > 0$ and $\theta \in \mathbb{R}^n$ we define $B(\theta, \rho) := \{x \in \mathbb{R}^n : |x - \theta| < \rho\}$;
- For $0 < r < s$ and $\theta \in \mathbb{R}^n$ we define the open annulus $B_{r,s}(\theta) := B(\theta, s) \setminus \overline{B}(\theta, r)$ and denote $B_{r,s} = B_{r,s}(0)$;
- $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$;
- $\mathcal{L}^n$ denotes the Lebesgue measure on $\mathbb{R}^n$ and $\mathcal{H}^{n-1}$ the $(n-1)$-dimensional Hausdorff measure on $\mathbb{R}^n$;
- $\mathcal{A}$ denotes the family of bounded domains of $\mathbb{R}^n$ with Lipschitz boundary;
- We denote with $\mathscr{B}^n$ the Borel σ-algebra on $\mathbb{R}^n$ and with $\mathscr{B}(\mathbb{S}^{n-1})$ the Borel σ-algebra on $\mathbb{S}^{n-1}$;
- For $\xi \in \mathbb{R}^n$, we denote with ℓ_ξ the linear function $\ell_\xi(x) = \xi \cdot x$ for $x \in \mathbb{R}^n$;
- For $x \in \mathbb{R}^n$, $t > 0$ and $\nu \in \mathbb{S}^{n-1}$, we denote with $Q_t^\nu(x)$ the cube of side-length $t > 0$, centred at x with one face orthogonal to ν;
- For $x \in \mathbb{R}^n$ and $\nu \in \mathbb{S}^{n-1}$, we set

$$u_{x,1,\nu}(y) := \begin{cases} 1 & \text{if } (y - x) \cdot \nu \geq 0 \\ 0 & \text{if } (y - x) \cdot \nu < 0. \end{cases}$$

The functional setting for our analysis is that of *generalised special functions of bounded variation.* We recall some basic definitions and refer to [2] for a more comprehensive introduction to the topic.

For $A \in \mathcal{A}$, the space of special functions of bounded variation in A is defined as

$$SBV(A) = \{u \in BV(A) : \ Du = \nabla u\, \mathcal{L}^n + (u^+ - u^-)\nu_u \mathcal{H}^{n-1} \llcorner S_u\}.$$

Here S_u denotes the approximate discontinuity set of u, ν_u is the generalised normal to S_u, u^+ and u^- are the traces of u on both sides of S_u. We also consider the space

$$\mathcal{P}(A) = \{u \in SBV(A) : \nabla u = 0,\ u \in \{0, 1\}\ \mathcal{L}^n\text{-a.e.}, \mathcal{H}^{n-1}(S_u) < +\infty\};$$

hence every u in $\mathcal{P}(A)$ is a partition in the sense of [2, Definition 4.21].

For $p > 1$, we define the following vector subspace of $SBV(A)$:

$$SBV^p(A) = \{u \in SBV(A) : \ \nabla u \in L^p(A) \text{ and } \mathcal{H}^{n-1}(S_u) < +\infty\}.$$

We consider also the larger space of *generalised special functions of bounded variation* in A,

$$GSBV(A) = \{u \in L^1(A) : \ (u \wedge m) \vee (-m) \in SBV(A) \text{ for all } m \in \mathbb{N}\}.$$

By analogy with the case of SBV functions, we write

$$GSBV^p(A) = \{u \in GSBV(A) : \ \nabla u \in L^p(A) \text{ and } \mathcal{H}^{n-1}(S_u) < +\infty\}.$$

2.2. **Volume and surface integrands.** Let $p > 1$, $0 < c_1 \leq c_2 < +\infty$, $L > 0$, and let $f\colon \mathbb{R}^n \times \mathbb{R}^n \longrightarrow [0, +\infty)$ be a Borel function on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying the following conditions:

($f1$) (lower bound) for every $x \in \mathbb{R}^n$ and every $\xi \in \mathbb{R}^n$

$$c_1|\xi|^p \leq f(x, \xi);$$

($f2$) (upper bound) for every $x \in \mathbb{R}^n$ and every $\xi \in \mathbb{R}^n$

$$f(x, \xi) \leq c_2(1 + |\xi|^p);$$

($f3$) (continuity in ξ) for every $x \in \mathbb{R}^n$ we have

$$|f(x, \xi_1) - f(x, \xi_2)| \leq L\big(1 + |\xi_1|^{p-1} + |\xi_2|^{p-1}\big)|\xi_1 - \xi_2|$$

for every ξ_1, $\xi_2 \in \mathbb{R}^n$.

Let $0 < c_3 \leq c_4 < +\infty$ and let $g\colon \mathbb{R}^n \times \mathbb{S}^{n-1} \longrightarrow [0, +\infty)$ be a Borel function on $\mathbb{R}^n \times \mathbb{S}^{n-1}$ satisfying the following conditions:

($g1$) (lower bound) for every $x \in \mathbb{R}^n$ and every $\nu \in \mathbb{S}^{n-1}$

$$c_3 \leq g(x, \nu);$$

($g2$) (upper bound) for every $x \in \mathbb{R}^n$ and every $\nu \in \mathbb{S}^{n-1}$

$$g(x, \nu) \leq c_4;$$

($g3$) (symmetry) for every $x \in \mathbb{R}^n$ and every $\nu \in \mathbb{S}^{n-1}$

$$g(x, \nu) = g(x, -\nu).$$

2.3. **Stochastic framework.** Let $(\Omega, \mathcal{T}, P)$ be a complete probability space. We start by recalling some definitions.

Definition 2.1 (Group of P-preserving transformations)**.** A group of P-preserving transformations on $(\Omega, \mathcal{T}, P)$ is a family $(\tau_y)_{y\in\mathbb{R}^n}$ of $\mathcal{T}$-measurable mappings $\tau_y\colon \Omega \to \Omega$ satisfying the following properties:

- (measurability) the map $(\omega, y) \mapsto \tau_y(\omega)$ is $(\mathcal{T} \otimes \mathscr{B}^n, \mathcal{T})$-measurable;
- (bijectivity) τ_y is bijective for every $y \in \mathbb{R}^n$;
- (invariance) $P(\tau_y(E)) = P(E)$, for every $E \in \mathcal{T}$ and every $y \in \mathbb{R}^n$;
- (group property) $\tau_0 = \mathrm{id}_\Omega$ (the identity map on Ω) and $\tau_{y+y'} = \tau_y \circ \tau_{y'}$ for every $y, y' \in \mathbb{R}^n$.

If, in addition, every set $E \in \mathcal{T}$ which satisfies $\tau_y(E) = E$ for every $y \in \mathbb{R}^n$ has probability 0 or 1, then $(\tau_y)_{y\in\mathbb{R}^n}$ is called *ergodic*.

We are now in a position to define the notion of stationary random integrand.

Definition 2.2 (Stationary random integrand)**.** Let $(\tau_y)_{y\in\mathbb{R}^n}$ be a group of P-preserving transformations on $(\Omega, \mathcal{T}, P)$. We say that $f\colon \Omega \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow [0, +\infty)$ is a *stationary random volume integrand* if

(a) f is $(\mathcal{T} \otimes \mathscr{B}^n \otimes \mathscr{B}^n)$-measurable;

(b) $f(\omega, \cdot, \cdot)$ satisfies ($f1$)–($f3$) for every $\omega \in \Omega$, with c_1, c_2 independent of ω;

(c) $f(\omega, x + y, \xi) = f(\tau_y(\omega), x, \xi)$, for every $\omega \in \Omega$, $x, y \in \mathbb{R}^n$, and $\xi \in \mathbb{R}^n$.

Similarly, we say that $g\colon \Omega \times \mathbb{R}^n \times \mathbb{S}^{n-1} \longrightarrow [0, +\infty)$ is a *stationary random surface integrand* if

(d) g is $(\mathcal{T} \otimes \mathscr{B}^n \otimes \mathscr{B}(\mathbb{S}^{n-1}))$-measurable;

(e) $g(\omega, \cdot, \cdot)$ satisfies ($g1$)–($g3$) for every $\omega \in \Omega$, with c_3, c_4 independent of ω;

(f) $g(\omega, x + y, \nu) = g(\tau_y(\omega), x, \nu)$, for every $\omega \in \Omega$, $x, y \in \mathbb{R}^n$, and $\nu \in \mathbb{S}^{n-1}$.

If in addition $(\tau_y)_{y\in\mathbb{R}^n}$ is an ergodic group of P-preserving transformations, then we say that f and g are ergodic.

We also recall the definition of random domain. The main difference with the classical definition given in, *e.g.*, [25, Chapter 8] is that we do not assume any ergodicity for the group $(\tau_y)_{y\in\mathbb{R}^n}$.

Definition 2.3 (Random domain)**.** Let $(\tau_y)_{y\in\mathbb{R}^n}$ be a group of P-preserving transformations on $(\Omega,\mathcal{T},P)$. A *random domain* is a map $\omega\mapsto D(\omega)$ from Ω to the subsets of $\mathbb{R}^n$ such that:

- the map $(\omega,x)\mapsto\chi_{D(\omega)}(x)$ is $(\mathcal{T}\otimes\mathscr{B}^n)$-measurable;
- for every $\omega\in\Omega$, $x,y\in\mathbb{R}^n$ it holds

$$\chi_{D(\omega)}(x+y)=\chi_{D(\tau_y\omega)}(x). \tag{2.1}$$

In short, we say that D is a random domain. We refer to (2.1) as the stationarity condition for the domain. If in addition $(\tau_y)_{y\in\mathbb{R}^n}$ is ergodic, then we say that the random domain is ergodic.

Remark 2.4. We note that D is a random domain if and only if for every $\omega\in\Omega$

$$D(\omega)=\big\{x\in\mathbb{R}^n\colon\tau_x\omega\in\widetilde{D}\big\} \tag{2.2}$$

for some $\widetilde{D}\in\mathcal{T}$. Indeed if $D(\omega)$ is as in (2.2) for some $\widetilde{D}\in\mathcal{T}$, then it is immediate to check that $\chi_{D(\omega)}$ satisfies (2.1). If on the other hand $\chi_{D(\omega)}$ satisfies (2.1), we have that $\chi_{D(\omega)}(x)=\chi_{D(\tau_x\omega)}(0)$ and therefore

$$\widetilde{D}=\{\omega\in\Omega\colon 0\in D(\omega)\}.$$

Definition 2.5 (Density of a random domain)**.** Let D be a random domain, let $\widetilde{D}\in\mathcal{T}$ be as in (2.2), and let $\mathscr{I}\subset\mathcal{T}$ denote the σ-algebra of $(\tau_y)_{y\in\mathbb{R}^n}$-invariant sets; that is, $\mathscr{I}:=\{E\in\mathcal{T}:\tau_y(E)=E\quad\forall\,y\in\mathbb{R}^n\}$. The function $\kappa\colon\Omega\to[0,+\infty)$ defined for every $\omega\in\Omega$ as $\kappa(\omega):=\mathbb{E}[\chi_{\widetilde{D}}|\mathscr{I}](\omega)$ is called the pointwise density of D.

Remark 2.6. By the definition of conditional expectation we have that

$$\int_\Omega\mathbb{E}[\chi_{\widetilde{D}}|\mathscr{I}](\omega)\,dP(\omega)=\int_\Omega\chi_{\widetilde{D}}(\omega)\,dP(\omega)=P(\widetilde{D}),$$

since $\Omega\in\mathscr{I}$. The nonnegative number $\bar\kappa:=P(\widetilde{D})$ is usually referred to as the (average) density of D (see e.g. [25, Chapter 8]).

Remark 2.7 (Birkhoff's Ergodic Theorem). Let D be a random domain and let $\varepsilon>0$. For every $\omega\in\Omega$ and $x\in\mathbb{R}^n$, we set

$$\kappa_\varepsilon(\omega,x):=\chi_{\varepsilon D(\omega)}(x). \tag{2.3}$$

Then, the Birkhoff Ergodic Theorem ensures that for P-a.e. $\omega\in\Omega$

$$\kappa_\varepsilon(\omega,\cdot)\overset{*}{\rightharpoonup}\kappa(\omega)\quad\text{in}\quad L^\infty_{\rm loc}(\mathbb{R}^n)$$

as $\varepsilon\to0$. If moreover D is ergodic then

$$\kappa_\varepsilon(\omega,\cdot)\overset{*}{\rightharpoonup}\bar\kappa\quad\text{in}\quad L^\infty_{\rm loc}(\mathbb{R}^n).$$

We require the following additional assumptions on the geometry of the random domain D.

Definition 2.8 (Random perforated domain)**.** Let $\delta>0$ and $r_*>\delta$ be fixed and independent of ω, let K be a random domain, and set, for $\omega\in\Omega$, $D(\omega):=\mathbb{R}^n\setminus K(\omega)$. We say that D is a random perforated domain if:

(K1) for every $\omega\in\Omega$ the set $K(\omega)$ is the union of closed balls with radius smaller than r_*;

(K2) for every $\omega\in\Omega$ the distance between any two distinct balls in $K(\omega)$ is larger that 2δ.

Properties (K1) and (K2) can be rephrased as follows:

- $K(\omega)$ is a countable union of balls of the form

$$K(\omega):=\bigcup_{i\in\mathcal{I}}\overline{B}(\theta_i(\omega),r_i(\omega)), \tag{2.4}$$

  with $r_i(\omega)\in(0,r_*)$, $\theta_i(\omega)\in\mathbb{R}^n$, and with $\theta_i(\omega)\neq\theta_j(\omega)$ for $i\neq j$, for every $i,j\in\mathcal{I}$ and for every $\omega\in\Omega$;
- for every $i,j\in\mathcal{I}$ with $i\neq j$

$$B(\theta_i(\omega),r_i(\omega)+\delta)\cap B(\theta_j(\omega),r_j(\omega)+\delta)=\emptyset. \tag{2.5}$$

The set $K(\omega)$ is a special type of *random spherical structure*, as defined in [25, Definition 8.19]. It is special because of the strong 2δ-separation of the spherical perforations, which is crucial in our analysis.

Remark 2.9 (Example of a random perforated domain). The simplest example of a random perforated domain can be obtained as follows. Let $\mathcal{L} \subset \mathbb{R}^n$ be a regular Bravais lattice (*e.g.*, the cubic lattice or the triangular lattice for $n = 2$). Let $Q(\mathcal{L})$ denote the periodicity cell of the lattice, and let $B \subset\subset Q(\mathcal{L})$ be a ball well contained in the cell. Then an admissible set of perforations is given by

$$K_{\mathcal{L}}(\omega) = \bigcup_{y \in \mathcal{Y}(\omega)} (B + y),$$

where $\mathcal{Y}(\omega) \subset \mathcal{L}$ is a random set obtained, for instance, by running i.i.d. Bernoulli trials at each $y \in \mathcal{L}$. Then $D(\omega) := \mathbb{R}^n \setminus K_{\mathcal{L}}(\omega)$ is a random perforated domain.

We now show that a random domain as in Definition 2.8 has a positive pointwise density $\kappa(\omega)$ for P-a.e. $\omega \in \Omega$.

Property 2.10. *Let D be a random perforated domain as in Definition 2.8 and let κ be its pointwise density as in Definition 2.5. Then $\kappa(\omega) > 0$ for P-a.e. $\omega \in \Omega$.*

Proof. Let $\varepsilon > 0$ be small and let κ_ε be as in (2.3); then for $\omega \in \Omega$

$$\int_Q \kappa_\varepsilon(\omega, x)\, dx = \int_Q \chi_{\varepsilon D(\omega)}(x)\, dx = \mathcal{L}^n(Q \cap \varepsilon D(\omega)) = \mathcal{L}^n(Q \setminus \varepsilon K(\omega)),$$

where Q denotes the unit cube. By the Birkhoff Ergodic Theorem we deduce that, in particular,

$$\lim_{\varepsilon \to 0} \mathcal{L}^n(Q \setminus \varepsilon K(\omega)) = \kappa(\omega), \tag{2.6}$$

for P-a.e. $\omega \in \Omega$. We now show that, by Definition 2.8, the left hand-side of (2.6) can be estimated from below by a positive constant independent of ω.

Let N_ε denote the number of components $\varepsilon B(\theta_i(\omega), r_i(\omega))$ of $\varepsilon K(\omega)$ such that $\varepsilon B(\theta_i(\omega), r_i(\omega)+\delta)$ is contained in Q, namely the components of $\varepsilon K(\omega)$ that do not intersect $Q \setminus Q_{1-2\varepsilon\delta}$. Note that the total number of perforations intersecting Q is $N_\varepsilon + N_\varepsilon^b$, where N_ε^b denotes the number of "boundary" perforations. We can neglect the boundary perforations in the estimate of $\mathcal{L}^n(Q \cap \varepsilon K(\omega))$ (and hence of $\mathcal{L}^n(Q \setminus \varepsilon K(\omega))$) since they provide an infinitesimal volume contribution. The assumption of $2\delta\varepsilon$-separation of the components of $\varepsilon K(\omega)$ ensures that $N_\varepsilon \le (2\varepsilon\delta)^{-n}$.

If $N_\varepsilon \ll \varepsilon^{-n}$ we immediately get

$$\mathcal{L}^n(Q \cap \varepsilon K(\omega)) \le c_n N_\varepsilon \varepsilon^n r_*^n \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

where $c_n := \mathcal{L}^n(B(0,1))$ and therefore

$$\mathcal{L}^n(Q \setminus \varepsilon K(\omega)) = \mathcal{L}^n(Q) - \mathcal{L}^n(Q \cap \varepsilon K(\omega)) \ge \frac{1}{2}$$

for small enough $\varepsilon > 0$ and every $\omega \in \Omega$.

We now assume that $N_\varepsilon \sim \varepsilon^{-n}$. First of all, by the definition of N_ε, and by the $2\delta\varepsilon$-separation of the components of $\varepsilon K(\omega)$, we have that

$$\varepsilon B(\theta_j(\omega), r_j(\omega) + \delta) \setminus \varepsilon B(\theta_j(\omega), r_j(\omega)) \subset Q \setminus \varepsilon K(\omega).$$

Consequently we have

$$\mathcal{L}^n(Q \setminus \varepsilon K(\omega)) \ge N_\varepsilon c_n (\varepsilon\delta)^n,$$

where to establish the last inequality we have used that

$$\mathcal{L}^n(\varepsilon B(\theta_j(\omega), r_j(\omega) + \delta) \setminus \varepsilon B(\theta_j(\omega), r_j(\omega))) = \varepsilon^n c_n \big((r_j(\omega) + \delta)^n - r_j(\omega)^n\big) \ge c_n \varepsilon^n \delta^n.$$

Therefore also in this case we have that

$$\lim_{\varepsilon \to 0} \mathcal{L}^n(Q \setminus \varepsilon K(\omega)) \ge \lim_{\varepsilon \to 0} N_\varepsilon c_n (\varepsilon\delta)^n = c > 0,$$

and this concludes the proof. □

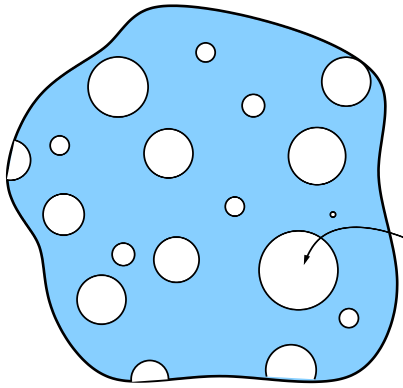

FIGURE 1. The randomly perforated domain $A \setminus \varepsilon K(\omega)$.

2.4. **Energy functionals and statement of the main result.** We now introduce the sequence of functionals we are going to study.

For $\omega \in \Omega$ and $\varepsilon > 0$ we consider the random functionals $E_\varepsilon(\omega)\colon L^1_{\rm loc}(\mathbb{R}^n) \times \mathcal{A} \longrightarrow [0, +\infty]$ defined as

$$E_\varepsilon(\omega)(u, A) := \begin{cases} \displaystyle\int_{A\setminus\varepsilon K(\omega)} f\left(\omega, \frac{x}{\varepsilon}, \nabla u\right) dx + \int_{S_u\cap(A\setminus\varepsilon K(\omega))} g\left(\omega, \frac{x}{\varepsilon}, \nu\right) d\mathcal{H}^{n-1} & \text{if } u_{|A} \in GSBV^p(A), \\ +\infty & \text{otherwise,} \end{cases} \tag{2.7}$$

where f and g are stationary random integrands as in Definition 2.2, and $K(\omega)$ is as in Definition 2.8 (see Figure 1).

Let moreover $F(\omega), G(\omega) : L^1_{\rm loc}(\mathbb{R}^n) \times \mathcal{A} \longrightarrow [0, +\infty]$ be defined as

$$F(\omega)(u, A) := \begin{cases} \displaystyle\int_{A\setminus K(\omega)} f(\omega, x, \nabla u)\, dx & \text{if } u_{|A} \in W^{1,p}(A), \\ +\infty & \text{otherwise,} \end{cases} \tag{2.8}$$

and

$$G(\omega)(u, A) := \begin{cases} \displaystyle\int_{S_u\cap(A\setminus K(\omega))} g(\omega, x, \nu_u)\, d\mathcal{H}^{n-1} & \text{if } u_{|A} \in GSBV^p(A), \\ +\infty & \text{otherwise.} \end{cases} \tag{2.9}$$

Let $A \in \mathcal{A}$ be fixed; for $v \in L^1_{\rm loc}(\mathbb{R}^n)$, with $v_{|A} \in W^{1,p}(A)$, we define

$$m^{1,p}_{F(\omega)}(v, A) := \inf\big\{F(\omega)(u, A) :\ u \in L^1_{\rm loc}(\mathbb{R}^n), u_{|A} \in W^{1,p}(A),\ u = v \text{ near } \partial A\big\}. \tag{2.10}$$

Similarly, for $v \in L^1_{\rm loc}(\mathbb{R}^n)$, with $v_{|A} \in \mathcal{P}(A)$, we define

$$m^{\rm pc}_{G(\omega)}(v, A) := \inf\big\{G(\omega)(u, A) :\ u \in L^1_{\rm loc}(\mathbb{R}^n), u_{|A} \in \mathcal{P}(A),\ u = v \text{ near } \partial A\big\}. \tag{2.11}$$

In the formulas above, by "$u = v$ near ∂A" we mean that there exists a neighbourhood U of ∂A in $\mathbb{R}^n$ such that $u = v$ $\mathcal{L}^n$-a.e. in $U \cap A$.

The following theorem is the main result of this paper.

Theorem 2.11 (Homogenisation theorem)**.** *Let f and g be stationary random volume and surface integrands, and let $D \subset \mathbb{R}^n$ be a random perforated domain as in Definition 2.8. Assume that the stationarity of f, g and D is satisfied with respect to the same group $(\tau_y)_{y\in\mathbb{R}^n}$ of P-preserving transformations on $(\Omega, \mathcal{T}, P)$. Let $\varepsilon > 0$, and let E_ε be the functionals defined as in* (2.7)*.*

I) (Compactness) *Let $\omega \in \Omega$ and $A \in \mathcal{A}$ be fixed; let $(u_\varepsilon) \subset L^1_{\rm loc}(\mathbb{R}^n)$ be such that*

$$\sup_{\varepsilon>0}\Big(E_\varepsilon(\omega)(u_\varepsilon, A) + \|u_\varepsilon\|_{L^\infty(A\setminus\varepsilon K(\omega))}\Big) < +\infty.$$

Then there exist a sequence $(\tilde{u}_\varepsilon) \subset SBV^p(A) \cap L^1_{\rm loc}(\mathbb{R}^n)$ *and a function* $u \in SBV^p(A) \cap L^1_{\rm loc}(\mathbb{R}^n)$ *such that* $\tilde{u}_\varepsilon = u_\varepsilon$ $\mathcal{L}^n$*-a.e. in* $A \setminus \varepsilon K(\omega)$ *and (up to a subsequence not relabelled)* $\tilde{u}_\varepsilon \to u$ *strongly in* $L^1(A)$*.*

II) (Almost sure Γ-convergence) *There exists* $\Omega' \in \mathcal{T}$*, with* $P(\Omega') = 1$*, such that for every* $\omega \in \Omega'$ *the functionals* $E_\varepsilon(\omega)$ Γ*-converge with respect to the* $L^1_{\rm loc}(\mathbb{R}^n)$*-convergence, as* $\varepsilon \to 0$*, to the functional* $E_{\rm hom}(\omega)\colon L^1_{\rm loc}(\mathbb{R}^n) \times \mathcal{A} \longrightarrow [0, +\infty]$ *given by*

$$E_{\rm hom}(u, A) = \begin{cases} \displaystyle\int_A f_{\rm hom}(\omega, \nabla u)\, dx + \int_{A\cap S_u} g_{\rm hom}(\omega, \nu_u)\, d\mathcal{H}^{n-1} & \text{if } u_{|_A} \in GSBV^p(A), \\ +\infty & \text{otherwise.} \end{cases} \tag{2.12}$$

In (2.12)*, for every* $\omega \in \Omega'$*,* $\xi \in \mathbb{R}^n$*, and* $\nu \in \mathbb{S}^{n-1}$*,*

$$f_{\rm hom}(\omega, \xi) := \lim_{t\to+\infty} \frac{1}{t^n} m^{1,p}_{F(\omega)}(\ell_\xi, Q_t(0)),$$

and

$$g_{\rm hom}(\omega, \nu) := \lim_{t\to+\infty} \frac{1}{t^{n-1}} m^{\rm pc}_{G(\omega)}(u_{0,1,\nu}, Q^\nu_t(0)),$$

with $m^{1,p}_{F(\omega)}$ *and* $m^{\rm pc}_{G(\omega)}$ *defined as in* (2.10) *and* (2.11)*, respectively.*

III) (Properties of $f_{\rm hom}$ and $g_{\rm hom}$) *The homogenised volume integrand* $f_{\rm hom}$ *satisfies the following properties:*

i. (measurability) $f_{\rm hom}$ *is* $(\mathcal{T} \otimes \mathscr{B}^n)$*-measurable;*

ii. (bounds) *there exists* $\tilde{c}_0 > 0$ *such that*

$$\tilde{c}_0 |\xi|^p \le f_{\rm hom}(\omega, \xi) \le c_2(1 + |\xi|^p),$$

for every $\omega \in \Omega'$ *and every* $\xi \in \mathbb{R}^n$*, with* c_2 *as in* $(f2)$*;*

iii. (continuity) *there exists* $L' > 0$ *such that*

$$|f_{\rm hom}(\omega, \xi_1) - f_{\rm hom}(\omega, \xi_2)| \le L'\big(1 + |\xi_1|^{p-1} + |\xi_2|^{p-1}\big)|\xi_1 - \xi_2|,$$

for every $\omega \in \Omega'$ *and every* ξ_1*,* $\xi_2 \in \mathbb{R}^n$*.*

Additionally, the homogenised surface integrand $g_{\rm hom}$ *satisfies:*

iv. (measurability) $g_{\rm hom}$ *is* $(\mathcal{T} \otimes \mathscr{B}(\mathbb{S}^{n-1}))$*-measurable;*

v. (bounds) *there exists* $\tilde{c}_0 > 0$ *such that*

$$\tilde{c}_0 \le g_{\rm hom}(\omega, \nu) \le c_4,$$

for every $\omega \in \Omega'$ *and every* $\nu \in \mathbb{S}^{n-1}$*, with* c_4 *as in* $(g2)$*;*

vi. (symmetry) $g_{\rm hom}(\omega, \nu) = g_{\rm hom}(\omega, -\nu)$*, for every* $\omega \in \Omega'$ *and every* $\nu \in \mathbb{S}^{n-1}$*.*

If, in addition, f*,* g *and* D *are ergodic, then* $f_{\rm hom}$ *and* $g_{\rm hom}$ *are independent of* ω*.*

The proof of Theorem 2.11 will be broken up into three main steps which will be, respectively, the object of Proposition 4.7, Theorem 5.1, and Theorem 5.3 below.

3. Preliminaries

In this short section we collect two known results which will be used in what follows. The first one, Theorem 3.1 is an extension result for $GSBV$-functions. The second result, Lemma 3.3, is a regularity result for minimal partitions.

For $p > 1$ and $n \ge 2$ we introduce the shorthand MS^p for the p-Mumford-Shah functional, namely we write

$$MS^p(u, A) := \int_A |\nabla u|^p dx + \mathcal{H}^{n-1}(S_u \cap A),$$

where $A \in \mathcal{A}$ and $u \in GSBV^p(A)$. Moreover, if $u \in GSBV^p(B)$, with $B \in \mathcal{A}$ and $\bar{A} \subset B$, we use the notation

$$MS^p(u, \bar{A}) := MS^p(u, A) + \mathcal{H}^{n-1}(S_u \cap \partial A).$$

We now recall [9, Theorem 1.1].

Theorem 3.1. *Let $p > 1$, let $A, A' \subset \mathbb{R}^n$ be bounded open sets with Lipschitz boundary and assume that A' is connected, $A' \subset A$ and $A \setminus A' \subset\subset A$. Then there exists a linear extension operator $T : GSBV^p(A') \longrightarrow GSBV^p(A)$ and a constant $c = c(n, p, A, A') > 0$ such that*

- $Tu = u \quad \mathcal{L}^n$*-a.e. in* A',
- $\mathcal{H}^{n-1}(S_{Tu} \cap (\partial A' \cap A)) = 0$,
- $MS^p(Tu, A) \leq c MS^p(u, A')$,

for every $u \in GSBV^p(A')$. The constant c is invariant under translations and homotheties.

If in addition $u \in L^\infty(A')$, then $Tu \in SBV^p(A) \cap L^\infty(A)$, and $\|Tu\|_{L^\infty(A)} = \|u\|_{L^\infty(A')}$.

Remark 3.2. The result in [9] is stated and proven in $GSBV^p$ for the most classical case $p = 2$, but the general case of $GSBV^p$ for $p > 1$ follows immediately. In fact, a key tool of the proof in [9] is the density lower bound proved in [19] (see also [18]), which is actually valid for any $p > 1$ (see for instance [2, Theorem 7.21]).

We now state a technical lemma (see [16, Lemma 4.5], and see also [17, Lemma 2.5] for a more general version of the result) for (locally) minimal partitions.

Lemma 3.3. *Let $n \geqslant 2$ and $\tau \in (0, 1]$ be fixed. There exists a constant $\gamma = \gamma(n, \tau) > 0$ such that if $0 < s \leq r$, and $u \in \mathcal{P}(B_{r,r+s})$ verifies the following hypotheses:*

(*H*1) $\mathcal{H}^{n-1}(S_u \cap B_{r,r+s}) \leq \mathcal{H}^{n-1}(S_v \cap B_{r,r+s})$ *for every competitor* $v \in \mathcal{P}(B_{r,r+s})$ *satisfying* $\operatorname{supp}(u - v) \subset B_{r,r+s}$;

(*H*2) $\mathcal{H}^{n-1}(S_u \cap B_{r,r+s}) \leq \gamma s^{n-1}$;

then for every r_0 and s_0 such that $r \leq r_0 < r_0 + s_0 \leq r + s$ and $s_0 \geq \tau s$, there exists a radius $\bar{r} \in (r + s_0/3, r_0 + 2s_0/3)$ with the property that

$$S_u \cap \partial B_{\bar{r}} = \emptyset.$$

4. Extension results and compactness

In this section we prove a compactness result for sequences $(u_\varepsilon) \subset L^1_{\rm loc}(\mathbb{R}^n)$ satisfying the bound

$$E_\varepsilon(\omega)(u_\varepsilon, A) + \|u_\varepsilon\|_{L^\infty(A \setminus \varepsilon K(\omega))} \leq C \quad \text{for every } \varepsilon > 0, \tag{4.1}$$

for a constant $C > 0$ independent of $\varepsilon > 0$, where $A \in \mathcal{A}$ and E_ε is defined as in (2.7), and $\omega \in \Omega$.

By definition of the functionals $E_\varepsilon(\omega)$, the bound in (4.1) does not provide any information on the BV-norm of u_ε in $A \cap \varepsilon K(\omega)$. To gain the desired bound, we show that (u_ε) can be actually replaced by a sequence $(\tilde{u}_\varepsilon) \subset SBV^p(A)$ satisfying the two following properties:

$$\tilde{u}_\varepsilon = u_\varepsilon \quad \mathcal{L}^n\text{-a.e. in } A \setminus \varepsilon K(\omega) \quad \text{and} \quad \sup_\varepsilon \|\tilde{u}_\varepsilon\|_{BV(A)} < +\infty. \tag{4.2}$$

In particular, $\tilde{u}_\varepsilon$ is energetically equivalent to u_ε. To prove the existence of such a sequence, we resort to a new extension result for functions defined on a perforated domain without assuming any periodicity on the distribution of the perforations (cf. Cagnetti and Scardia [9] for the case of periodically distributed perforations).

4.1. Extension.

The main result of this subsection is a $GSBV$-extension result from $A \setminus \varepsilon K(\omega)$ to A (cf. Theorem 4.2). Since this result is proven for $\omega \in \Omega$ fixed, in what follows we omit the dependence of the set $K(\omega)$ on the random parameter ω. Hence below K denotes any subset of $\mathbb{R}^n$ satisfying the two properties (2.4) and (2.5) (cf. Definition 2.8).

Loosely speaking, to prove the desired $GSBV$-extension result we would like to apply Theorem 3.1 in a δ-neighbourhood of each component $B(\theta_i, r_i)$ of K (which are pairwise disjoint by assumption (2.5)). If we did it naively, however, we could have for each $B(\theta_i, r_i)$ a different extension constant. Lemma 4.1 below ensures that the extension constant can be actually taken to be independent of θ_i and r_i.

Lemma 4.1 ($GSBV$-extension in an annulus)**.** *Let $n \geq 2$ and $p > 1$; let $\delta, r_* > 0$ be fixed, with $r_* > \delta$. Let $\theta \in \mathbb{R}^n$ and $0 < r < r_*$; then there exist a linear extension operator $T_{\theta,r} : GSBV^p(B_{r,r+\delta}(\theta)) \to GSBV^p(B(\theta, r+\delta))$ and a constant $c = c(n,p,\delta,r_*) > 0$ such that*

$$T_{\theta,r}u = u \ \mathcal{L}^n\text{-a.e. in } \ B_{r,r+\delta}(\theta),$$
$$\mathcal{H}^{n-1}(S_{T_{\theta,r}u} \cap \partial B(\theta,r)) = 0,$$
$$MS^p(T_{\theta,r}u, B(\theta, r+\delta)) \leq c\, MS^p\big(u, B_{r,r+\delta}(\theta)\big)$$

for every $u \in GSBV^p(B_{r,r+\delta}(\theta))$. The constant c is invariant under translations and homotheties. If in addition $u \in L^\infty(B_{r,r+\delta}(\theta))$, then $T_{\theta,r}u \in SBV^p(B(\theta,r+\delta)) \cap L^\infty(B(\theta, r+\delta))$, and $\|T_{\theta,r}u\|_{L^\infty(B(\theta,r+\delta))} = \|u\|_{L^\infty(B_{r,r+\delta}(\theta))}$.

Proof. Let $u \in GSBV^p(B_{r,r+\delta}(\theta))$. We treat the cases $r < \delta$ and $r \geq \delta$ separately.

Case 1: $r < \delta$**.** Note that in this case $B_{r,2r}(\theta) \subset B_{r,r+\delta}(\theta)$.

Let $v := u_{|B_{r,2r}(\theta)}$. By applying Theorem 3.1 with $A' = B_{r,2r}(\theta)$ and $A = B(\theta, 2r)$, we deduce the existence of a constant $c = c(n,p) > 0$ (independent of θ and r) and a function $w \in GSBV^p(B(\theta,2r))$ satisfying $w = v = u$ $\mathcal{L}^n$-a.e. in $B_{r,2r}(\theta)$ and

$$MS^p(w, B(\theta,2r)) \leq c\, MS^p(v, B_{r,2r}(\theta)) = c\, MS^p(u, B_{r,2r}(\theta)) \leq c\, MS^p(u, B_{r,r+\delta}(\theta)). \tag{4.3}$$

We now define the function $\tilde{u}$ in $B(\theta, r+\delta)$ as follows:

$$\tilde{u} := \begin{cases} u & \text{in } B_{r,r+\delta}(\theta), \\ w_{|\overline{B}(\theta,r)} & \text{in } \overline{B}(\theta,r). \end{cases} \tag{4.4}$$

Clearly, $\tilde{u} \in GSBV^p(B(\theta, r+\delta))$, $\tilde{u} = u$ in $B_{r,r+\delta}(\theta)$, and

$$\begin{aligned} MS^p(\tilde{u}, B(\theta, r+\delta)) &= MS^p(u, B_{r,r+\delta}(\theta)) + MS^p(w, \overline{B}(\theta,r)) \\ &= MS^p(u, B_{r,r+\delta}(\theta)) + MS^p(w, B(\theta,r)) \\ &\leq (1+c) MS^p(u, B_{r,r+\delta}(\theta)), \end{aligned}$$

where $c > 0$ is the same constant as in (4.3).
The desired extension operator $T_{\theta,r} \colon GSBV^p(B_{r,r+\delta}(\theta)) \longrightarrow GSBV^p(B(\theta, r+\delta))$ is then the one associating to any $u \in GSBV^p(B_{r,r+\delta}(\theta))$ the function $\tilde{u}$ defined by (4.4).

Case 2: $r \geq \delta$**.** Since $r < r_*$, we have that $B_{r,r(1+\delta/r_*)}(\theta) \subset B_{r,r+\delta}(\theta)$.

Let $v := u_{|B_{r,r(1+\delta/r_*)}(\theta)}$. Proceeding as in Case 1, we apply Theorem 3.1 with $A' = B_{r,r(1+\delta/r_*)}(\theta)$ and $A = B(\theta, r(1+\delta/r_*))$, and deduce the existence of a constant $c = c(n,p,\delta,r^*) > 0$ (independent of θ and r) and a function $w \in GSBV^p(B(\theta, r(1+\delta/r_*)))$ satisfying $w = v = u$ $\mathcal{L}^n$-a.e. in $B_{r,r(1+\delta/r_*)}(\theta)$ and

$$\begin{aligned} MS^p(w, B(\theta, r(1+\delta/r_*)) &\leq c\, MS^p(v, B_{r,r(1+\delta/r_*)}(\theta)) \\ &= c\, MS^p(u, B_{r,r(1+\delta/r_*)}(\theta)) \leq c\, MS^p(u, B_{r,r+\delta}(\theta)). \end{aligned} \tag{4.5}$$

The desired extension operator $T_{\theta,r} \colon GSBV^p(B_{r,r+\delta}(\theta)) \longrightarrow GSBV^p(B(\theta, r+\delta))$ is then the one associating to any $u \in GSBV^p(B_{r,r+\delta}(\theta))$ the function $T_{\theta,r}u$ defined as

$$T_{\theta,r}u := \begin{cases} u & \text{in } B_{r,r+\delta}(\theta), \\ w_{|\overline{B}(\theta,r)} & \text{in } \overline{B}(\theta,r). \end{cases}$$

Indeed, clearly $T_{\theta,r}u \in GSBV^p(B(\theta, r+\delta))$, $T_{\theta,r}u = u$ in $B_{r,r+\delta}(\theta)$, and

$$\begin{aligned} MS^p(T_{\theta,r}u, B(\theta, r+\delta)) &= MS^p(u, B_{r,r+\delta}(\theta)) + MS^p(w, \overline{B}(\theta,r)) \\ &\leq MS^p(u, B_{r,r+\delta}(\theta)) + MS^p(w, B(\theta,2r)) \\ &\leq (1+c) MS^p(u, B_{r,r+\delta}(\theta)), \end{aligned}$$

where $c > 0$ is the same constant as in (4.5).

□

We now make use of Lemma 4.1 to prove the desired $GSBV$-extension result from $A \setminus \varepsilon K$ to A.

Theorem 4.2 ($GSBV$-extension in $A \setminus \varepsilon K$)**.** *Let $A \in \mathcal{A}$, let $K \subset \mathbb{R}^n$ satisfy* (2.4) *and* (2.5)*, and let $\varepsilon > 0$. Let $p > 1$. Then there exists a linear extension operator $T_\varepsilon : GSBV^p(A \setminus \varepsilon K) \longrightarrow GSBV^p(A)$ and a constant $c = c(n, p, \delta, r_*) > 0$ such that*

(E1) $T_\varepsilon u = u$ $\mathcal{L}^n$*-a.e. in* $A \setminus \varepsilon K$,

(E2) $\mathcal{H}^{n-1}(S_{T_\varepsilon u} \cap (\partial(\varepsilon K) \cap A)) = 0$,

(E3) $MS^p(T_\varepsilon u, A) \le c\big(MS^p(u, A \setminus \varepsilon K) + \mathcal{H}^{n-1}(\partial A)\big)$

for every $u \in GSBV^p(A \setminus \varepsilon K)$. Moreover, the constant c is invariant under homotheties and translations.

If in addition $u \in L^\infty(A \setminus \varepsilon K)$, then

(E4) $T_\varepsilon u \in L^\infty(A)$, *and* $\|T_\varepsilon u\|_{L^\infty(A)} = \|u\|_{L^\infty(A \setminus \varepsilon K)}$.

Proof. Let $\bar{u} : \mathbb{R}^n \setminus \varepsilon K \to \mathbb{R}$ denote the trivial extension of u to $\mathbb{R}^n \setminus \varepsilon K$; *i.e.*,

$$\bar{u} := \begin{cases} u & \text{in } A \setminus \varepsilon K \\ 0 & \text{in } (\mathbb{R}^n \setminus A) \setminus \varepsilon K. \end{cases}$$

Then $\bar{u} = u$ in $A \setminus \varepsilon K$, and

$$MS^p(\bar{u}, \mathbb{R}^n \setminus \varepsilon K) \le MS^p(u, A \setminus \varepsilon K) + \mathcal{H}^{n-1}(\partial A). \tag{4.6}$$

Let $\mathcal{I}_\varepsilon$ be the set of indices $j \in \mathcal{I}$ such that $\varepsilon \overline{B}(\theta_j, r_j)$ intersects A. For $j \in \mathcal{I}_\varepsilon$ we use the shorthand A_j for the open annulus $B_{r_j, r_j+\delta}(\theta_j)$, and we denote with $T_{j,\varepsilon} : GSBV^p\big(\varepsilon A_j\big) \longrightarrow GSBV^p(\varepsilon B(\theta_j, r_j + \delta))$ the extension operator provided by Lemma 4.1. Finally, we define the function $\tilde{u}_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ as

$$\tilde{u}_\varepsilon := \begin{cases} T_{j,\varepsilon}\big(\bar{u}_{|\varepsilon A_j}\big) & \text{in } \varepsilon B(\theta_j, r_j + \delta),\ j \in \mathcal{I}_\varepsilon, \\ \bar{u} & \text{otherwise}. \end{cases}$$

Clearly $\tilde{u}_\varepsilon \in GSBV^p(A \cup \bigcup_{j \in \mathcal{I}_\varepsilon} \varepsilon B(\theta_j, r_j + \delta))$. Moreover,

$$\begin{aligned} MS^p\Big(\tilde{u}_\varepsilon, A \cup \bigcup_{j \in \mathcal{I}_\varepsilon} \varepsilon B(\theta_j, r_j + \delta)\Big) &\le \sum_{j \in \mathcal{I}_\varepsilon} MS^p\left(T_{j,\varepsilon}\big(\bar{u}_{|\varepsilon A_j}\big), \varepsilon B(\theta_j, r_j + \delta)\right) + MS^p(\bar{u}, \mathbb{R}^n \setminus \varepsilon K) \\ &\le c(n, p, \delta, r_*) \sum_{j \in \mathcal{I}_\varepsilon} MS^p\left(\bar{u}, \varepsilon A_j\right) + MS^p(\bar{u}, \mathbb{R}^n \setminus \varepsilon K) \\ &\le \left(c(n, p, \delta, r_*) + 1\right)\left(MS^p(u, A \setminus \varepsilon K) + \mathcal{H}^{n-1}(\partial A)\right), \end{aligned}$$

where we have used (4.6), and the fact that, since for each of the operators $T_{j,\varepsilon}$ the constant provided by Lemma 4.1 is invariant under translations and homotheties, it is in particular independent of j and ε. Finally, the claim follows by defining $T_\varepsilon u := \tilde{u}^\varepsilon_{|A}$. □

Remark 4.3. A careful inspection of the proof of Theorem 4.2 shows that, as in [1], one can obtain the following estimate, alternative to $(E3)$:

$$MS^p(T_\varepsilon u, A') \le c MS^p(u, A \setminus \varepsilon K), \quad \forall A' \in \mathcal{A}, A' \subset\subset A,$$

upon choosing $\varepsilon > 0$ small enough, so that perforations that are possibly cut by ∂A do not intersect A'. Indeed, the additional boundary contribution in $(E3)$ is due to the possible presence of perforations that are cut by ∂A, and for which the extension result Lemma 4.1 does not apply. This boundary term is clearly no longer necessary if we accept to control the Mumford-Shah of the extended function only far from the boundary.

Remark 4.4. In Theorem 4.2 it is not necessary to assume that the connected components of K are balls. For instance, the case where each component of K is a smooth strictly convex domain does not essentially differ from the case of spherical inclusions.

For later use we also state the analogue of Lemma 4.1 for Sobolev functions (Lemma 4.5) and for partitions (Lemma 4.6).

Lemma 4.5 (Sobolev-extension in an annulus). *Let $n \geq 2$ and $p > 1$; let $\delta, r_* > 0$ be fixed, with $r_* > \delta$. Let $\theta \in \mathbb{R}^n$ and $0 < r < r_*$; then there exist an extension operator $T_{\theta,r} : W^{1,p}(B_{r,r+\delta}(\theta)) \to W^{1,p}(B(\theta, r+\delta))$ and a constant $c = c(n, p, \delta, r_*) > 0$ such that*

$$T_{\theta,r}u = u \quad \mathcal{L}^n\text{-a.e. in} \quad B_{r,r+\delta}(\theta),$$
$$\|T_{\theta,r}u\|_{L^p(B(\theta,r+\delta))} \leq c\,\|u\|_{L^p(B_{r,r+\delta}(\theta))},$$
$$\|D(T_{\theta,r}u)\|_{L^p(B(\theta,r+\delta))} \leq c\,\|Du\|_{L^p(B_{r,r+\delta}(\theta))},$$

for every $u \in W^{1,p}(B_{r,r+\delta}(\theta))$. The constant c is invariant under translations and homotheties.

Proof. The proof can be obtained by repeating every step of the proof of Lemma 4.1, up to invoking the extension result [1, Lemma 2.6] instead of Theorem 3.1. □

Lemma 4.6 (Extension of a partition in an annulus). *Let $n \geq 2$, and let $\delta, r_* > 0$ be fixed, with $r_* > \delta$. Let $\theta \in \mathbb{R}^n$ and $0 < r < r_*$; then there exist an extension operator $T_{\theta,r} : \mathcal{P}(B_{r,r+\delta}(\theta)) \to \mathcal{P}(B(\theta, r+\delta))$ and a constant $c = c(n, \delta, r_*) > 0$ such that*

$$T_{\theta,r}u = u \quad \mathcal{L}^n\text{-a.e. in} \quad B_{r,r+\delta}(\theta),$$
$$\mathcal{H}^{n-1}(S_{T_{\theta,r}u} \cap B(\theta, r+\delta)) \leq c\mathcal{H}^{n-1}(S_u \cap B_{r,r+\delta}(\theta)),$$

for every $u \in \mathcal{P}(B_{r,r+\delta}(\theta))$. The constant c is invariant under translations and homotheties.

Proof. The proof is obtained by combining an adaptation of the proof of Theorem 3.1 ([9, Theorem 1.1]) with the proof of Lemma 4.1.

Case 1: $r < \delta$. In this case we extend from $B_{r,2r}(\theta) \subset B_{r,r+\delta}(\theta)$ to $B(\theta, 2r)$. Up to a translation and a rescaling we reduce to extending a partition v from $B_{1,2}(0)$ to $B(0,2)$. Let $\Phi : B_{\frac{1}{2},1}(0) \to B_{1,\frac{3}{2}}(0)$ denote the reflection map with $\Phi = \mathrm{Id}$ on $\partial B(0,1)$, which associates to a point $z \in B_{\frac{1}{2},1}(0)$ the point $\tilde{z} \in B_{1,\frac{3}{2}}(0)$ on the line joining z with 0, with $(z + \tilde{z})/2 \in \partial B(0,1)$. Then the function

$$\tilde{v} := \begin{cases} v & \text{in } B_{1,2}(0) \\ v \circ \Phi & \text{in } B_{\frac{1}{2},1}(0) \end{cases}$$

satisfies $\tilde{v} \in \mathcal{P}(B_{\frac{1}{2},2}(0))$ and

$$\mathcal{H}^{n-1}(S_{\tilde{v}} \cap B_{\frac{1}{2},2}(0)) \leq c\mathcal{H}^{n-1}(S_v \cap B_{1,2}(0)), \tag{4.7}$$

where $c > 0$ is a constant depending only on the dimension n. Finally, we modify $\tilde{v}$ in the annulus $B_{\frac{1}{2},1}(0)$, and substitute it with a minimiser of the perimeter. More precisely, we let $\hat{v} \in \mathcal{P}(B_{\frac{1}{2},2}(0))$ be a solution of the following minimisation problem

$$\inf\left\{\mathcal{H}^{n-1}(S_w \cap B_{\frac{1}{2},2}(0) : \ w \in L^1_{\mathrm{loc}}(\mathbb{R}^n), w_{|B_{\frac{1}{2},2}(0)} \in \mathcal{P}(B_{\frac{1}{2},2}(0)), \ w = v \ \text{ in } B_{1,2}(0)\right\}.$$

Then, (4.7) gives

$$\mathcal{H}^{n-1}(S_{\hat{v}} \cap B_{\frac{1}{2},2}(0)) \leq \mathcal{H}^{n-1}(S_{\tilde{v}} \cap B_{\frac{1}{2},2}(0)) \leq c\mathcal{H}^{n-1}(S_v \cap B_{1,2}(0)). \tag{4.8}$$

We now distinguish the cases of a "small" or "large" jump set of $\hat{v}$ in the annulus $B_{\frac{1}{2},1}(0)$. We say that $\hat{v}$ has a small jump set if

$$\mathcal{H}^{n-1}(S_{\hat{v}} \cap B_{\frac{1}{2},1}(0)) \leq \frac{\gamma}{2^{n-1}}, \tag{4.9}$$

where $\gamma = \gamma(n) > 0$ is the universal constant as in Lemma 3.3 (applied with $\tau = 1$). We note that if (4.9) holds true, then the function $\hat{v}$ satisfies the assumptions of Lemma 3.3 in $B_{\frac{1}{2},1}(0)$. Indeed, $(H1)$ follows by the local minimality of $\hat{v}$ in the annulus, and $(H2)$ is exactly (4.9). Therefore Lemma 3.3 (with $r = s = r_0 = s_0 = \frac{1}{2}$ and $\tau = 1$) yields the existence of $\bar{r} \in (\frac{4}{6}, \frac{5}{6})$ such that

$$S_{\hat{v}} \cap \partial B(0, \bar{r}) = \emptyset,$$

namely the trace of $\hat{v}$ is constant on $\partial B(0,\bar{r})$. We denote this constant value by m, and we define the function $\bar{v}$ in $B(0,2)$ as

$$\bar{v} := \begin{cases} \hat{v} & \text{in } B_{\bar{r},2}(0), \\ m & \text{in } \overline{B}(0,\bar{r}). \end{cases}$$

Then $\bar{v} \in \mathcal{P}(B(0,2))$ and, by (4.8),

$$\mathcal{H}^{n-1}(S_{\bar{v}} \cap B(0,2)) = \mathcal{H}^{n-1}(S_{\hat{v}} \cap B_{\bar{r},2}(0)) \leq \mathcal{H}^{n-1}(S_{\hat{v}} \cap B_{\frac{1}{2},2}(0)) \leq c\mathcal{H}^{n-1}(S_v \cap B_{1,2}(0)).$$

Hence the function $\bar{v}$ is the required extension.

If instead (4.9) is not satisfied, then the extension is obtained by simply filling the perforation with, *e.g.*, the constant value 0. In doing so the additional perimeter created by the discontinuity on $\partial B(0,1)$ is comparable to $\frac{\gamma}{2^{n-1}}$, up to a multiplicative constant. More precisely, we set

$$\bar{v} := \begin{cases} v & \text{in } B_{1,2}(0), \\ 0 & \text{in } \overline{B}(0,1). \end{cases}$$

Clearly $\bar{v} \in \mathcal{P}(B(0,2))$, and by (4.8)

$$\begin{aligned} \mathcal{H}^{n-1}(S_{\bar{v}} \cap B(0,2)) &\leq \mathcal{H}^{n-1}(S_v \cap B_{1,2}(0)) + s_n \\ &< \mathcal{H}^{n-1}(S_v \cap B_{1,2}(0)) + \frac{s_n 2^{n-1}}{\gamma}\mathcal{H}^{n-1}(S_{\hat{v}} \cap B_{\frac{1}{2},1}(0)) \\ &\leq c\,\mathcal{H}^{n-1}(S_v \cap B_{1,2}(0)), \end{aligned} \tag{4.10}$$

where $s_n := \mathcal{H}^{n-1}(\partial B(0,1))$ and $c = c(n) > 0$. Hence also in this case the function $\bar{v}$ is the required extension.

Case 2: $r \geq \delta$**.** Since $r < r_*$, we have that

$$B(\theta, r(1+\delta/r_*)) \subset B(\theta, r+\delta).$$

We now extend from $B(\theta, r(1+\delta/r_*)) \setminus \overline{B}(\theta,r)$ to $B(\theta,r) \setminus \overline{B}(\theta, r(1+\delta/r_*)^{-1})$. Up to a translation and a rescaling, we can restrict our attention to the case $\theta = 0$ and $r = 1$; *i.e.,* we extend from the set $A_1 := B(0,(1+\delta/r_*)) \setminus \overline{B}(0,1)$ to $A_2 := B(0,1) \setminus \overline{B}(0,(1+\delta/r_*)^{-1})$. Let $v \in \mathcal{P}(A_1)$; then by denoting with $\Phi : A_2 \to A_1$ the reflection map with $\Phi = \mathrm{Id}$ on $\partial B(0,1)$, we have that the function

$$\tilde{v} := \begin{cases} v & \text{in } A_1 \\ v \circ \Phi & \text{in } A_2 \end{cases}$$

satisfies $\tilde{v} \in \mathcal{P}(A_1 \cup A_2')$, where $A_2' := A_2 \cup \partial B(0,1)$, and

$$\mathcal{H}^{n-1}(S_{\tilde{v}} \cap (A_1 \cup A_2')) \leq c\mathcal{H}^{n-1}(S_v \cap A_1), \tag{4.11}$$

with $c = c(n,\delta,r_*) > 0$. Again, as in Case 1, we denote with $\hat{v} \in \mathcal{P}(A_1 \cup A_2')$ a minimiser of the perimeter in $A_1 \cup A_2'$ such that $\hat{v} = v$ in A_1. We then apply Lemma 3.3 to obtain the desired extension. Since $A_2 = B(0,1) \setminus B(0, \frac{r_*}{r_*+\delta})$, we have that $r = \frac{r_*}{r_*+\delta}$ and $s = \frac{\delta}{r_*+\delta}$ (and note that $s \leq r$ since $\delta < r_*$).

In this case we say that $\hat{v}$ has a small jump set in A_2 if

$$\mathcal{H}^{n-1}(S_{\hat{v}} \cap A_2) \leq \gamma \left(\frac{\delta}{r_*+\delta}\right)^{n-1}, \tag{4.12}$$

where $\gamma = \gamma(n) > 0$ is the universal constant as in Lemma 3.3 (applied with $\tau = 1$). We note that the function $\hat{v}$ satisfies the assumptions of Lemma 3.3 in A_2. Therefore Lemma 3.3 (with $r_0 = r$ and $s_0 = s$) yields the existence of $\bar{r} \in \frac{1}{3}(\frac{3r_*+\delta}{r_*+\delta}, \frac{3r_*+2\delta}{r_*+\delta})$ such that

$$S_{\hat{v}} \cap \partial B(0,\bar{r}) = \emptyset,$$

namely the trace of $\hat{v}$ is constant on $\partial B(0,\bar{r})$, with value, say, $m \in \{0,1\}$. Proceeding as in the previous case yields the conclusion.

□

4.2. **Compactness.** In this subsection we use Theorem 4.2 to prove that a sequence (u_ε) with equibounded energy $E_\varepsilon(\omega)$ can be replaced, without changing the energy, with a sequence which is precompact with respect to the strong L^1-convergence.

Proposition 4.7 (Compactness)**.** *Let $\omega \in \Omega$ and $A \in \mathcal{A}$ be fixed. Let $(u_\varepsilon) \subset L^1(A)$ be a sequence satisfying*

$$\sup_{\varepsilon>0} \Big(E_\varepsilon(\omega)(u_\varepsilon, A) + \|u_\varepsilon\|_{L^\infty(A\setminus\varepsilon K(\omega))} \Big) < +\infty. \tag{4.13}$$

Then there exist a sequence $(\tilde{u}_\varepsilon) \subset SBV^p(A)$ and a function $u \in SBV^p(A)$ such that $\tilde{u}_\varepsilon = u_\varepsilon$ $\mathcal{L}^n$-a.e. in $A \setminus \varepsilon K(\omega)$ and (up to a subsequence) $\tilde{u}_\varepsilon \to u$ strongly in $L^1(A)$.

Proof. We start observing that (4.13) yields $(u_\varepsilon) \subset SBV^p(A \setminus \varepsilon K(\omega)) \cap L^\infty(A \setminus \varepsilon K(\omega))$. Let T^ω_ε be the extension operator from $A \setminus \varepsilon K(\omega)$ to A as in Theorem 4.2 and set

$$\tilde{u}_\varepsilon := T^\omega_\varepsilon\big(u_{\varepsilon|A\setminus\varepsilon K(\omega)}\big).$$

Then $\tilde{u}_\varepsilon \in SBV^p(A) \cap L^\infty(A)$, $\tilde{u}_\varepsilon = u_\varepsilon$ $\mathcal{L}^n$- a.e. in $A \setminus \varepsilon K(\omega)$, and $(E3)$ gives

$$\begin{aligned} MS^p(\tilde{u}_\varepsilon, A) &\le c(n,p,\delta,r_*) \big(MS^p(u_\varepsilon, A\setminus\varepsilon K(\omega)) + \mathcal{H}^{n-1}(\partial A) \big) \\ &\le c(n,p,\delta,r_*) \left(\frac{1}{c_1 \wedge c_3} + 1 \right) \big(E_\varepsilon(\omega)(u_\varepsilon, A) + \mathcal{H}^{n-1}(\partial A) \big). \end{aligned} \tag{4.14}$$

Since moreover by $(E4)$ the extension operator T^ω_ε preserves the L^∞-norm, by combining (4.13) and (4.14) we immediately deduce that

$$\sup_{\varepsilon>0} \big(MS^p(\tilde{u}_\varepsilon, A) + \|\tilde{u}_\varepsilon\|_{L^\infty(A)} \big) < +\infty.$$

Therefore by Ambrosio's Compactness Theorem [2, Theorem 4.8], up to subsequences not relabelled, $\tilde{u}_\varepsilon \to u$ strongly in $L^1(A)$, for some $u \in SBV^p(A)$.

□

Remark 4.8 (Weak coerciveness)*.* Let $\omega \in \Omega$ be fixed and let $(u_\varepsilon) \subset L^1(A)$ be such that

$$\sup_{\varepsilon>0} \Big(E_\varepsilon(\omega)(u_\varepsilon, A) + \|u_\varepsilon\|_{L^\infty(A\setminus\varepsilon K(\omega))} \Big) < +\infty.$$

Then, for P-a.e. $\omega \in \Omega$, up to a subsequence not relabelled, we have

$$u_\varepsilon \chi_{(\mathbb{R}^n\setminus\varepsilon K(\omega))} = u_\varepsilon \kappa_\varepsilon(\omega,\cdot) \rightharpoonup u\,\kappa(\omega) \quad \text{weakly in } L^1(A), \tag{4.15}$$

for some $u \in SBV^p(A)$, with $\kappa(\omega)$ as in Definition 2.5.

Indeed, Proposition 4.7 yields the existence of a sequence $(\tilde{u}_\varepsilon) \subset SBV^p(A)$ and a function $u \in SBV^p(A)$ such that $\tilde{u}_\varepsilon = u_\varepsilon$ in $A \setminus \varepsilon K(\omega)$ and (up to a subsequence not relabelled)

$$\tilde{u}_\varepsilon \to u \quad \text{strongly in } L^1(A). \tag{4.16}$$

On the other hand, by the Birkhoff's Ergodic Theorem (see Remark 2.7) for P-a.e. $\omega \in \Omega$ we have

$$\chi_{(\mathbb{R}^n\setminus\varepsilon K(\omega))} = \kappa_\varepsilon(\omega,\cdot) \rightharpoonup \kappa(\omega) \quad \text{weakly}^* \text{ in } L^\infty(A). \tag{4.17}$$

Then the conclusion follows from the equality $u_\varepsilon \chi_{(\mathbb{R}^n\setminus\varepsilon K(\omega))} = \tilde{u}_\varepsilon \chi_{(\mathbb{R}^n\setminus\varepsilon K(\omega))}$, by combining (4.16) and (4.17).

5. Homogenisation result

In this section we prove both the existence of the homogenisation formulas defining f_{hom} and g_{hom} and the almost sure Γ-convergence of $E_\varepsilon(\omega)$ towards $E_{\text{hom}}(\omega)$ stated in Theorem 2.11.

The existence of the homogenisation formulas is achieved in two steps. The first step consists in applying [11, Theorem 3.12] to a coercive perturbation of E_ε. Then in the second step we pass to the limit in the perturbation parameter and show that this procedure leads to f_{hom} and g_{hom}. This last step requires the separate extension results for Sobolev functions (Lemma 4.5) and for partitions (Lemma 4.6).

Theorem 5.1 (Homogenisation formulas)**.** *Let f and g be stationary random volume and surface integrands, and let $D \subset \mathbb{R}^n$ be a random perforated domain as in Definition 2.8. Assume that the stationarity of f, g, and D is satisfied with respect to the same group $(\tau_y)_{y\in\mathbb{R}^n}$ of P-preserving transformations on $(\Omega,\mathcal{T},P)$. For $\omega\in\Omega$, let $F(\omega)$ and $G(\omega)$ be as in (2.8) and (2.9), respectively. Let moreover $m^{1,p}_{F(\omega)}$ and $m^{\rm pc}_{G(\omega)}$ be defined by (2.10) and (2.11), respectively. Then there exists $\Omega'\in\mathcal{T}$, with $P(\Omega')=1$, such that for every $\omega\in\Omega'$, for every $x,\xi\in\mathbb{R}^n$, and every $\nu\in\mathbb{S}^{n-1}$ the limits*

$$\lim_{t\to+\infty}\frac{m^{1,p}_{F(\omega)}(\ell_\xi,Q_t(tx))}{t^n}\quad\text{and}\quad\lim_{t\to+\infty}\frac{m^{\rm pc}_{G(\omega)}(u_{tx,1,\nu},Q^\nu_t(tx))}{t^{n-1}}$$

exist and are independent of x. More precisely, there exist a $(\mathcal{T}\otimes\mathscr{B}^n)$-measurable function $f_{\rm hom}\colon\Omega\times\mathbb{R}^n\to[0,+\infty)$ and a $(\mathcal{T}\otimes\mathscr{B}(\mathbb{S}^{n-1}))$-measurable function $g_{\rm hom}\colon\Omega\times\mathbb{S}^{n-1}\to[0,+\infty)$ such that, for every $x\in\mathbb{R}^n$, $\xi\in\mathbb{R}^n$, and $\nu\in\mathbb{S}^{n-1}$

$$f_{\rm hom}(\omega,\xi)=\lim_{t\to+\infty}\frac{1}{t^n}m^{1,p}_{F(\omega)}(\ell_\xi,Q_t(tx))=\lim_{t\to+\infty}\frac{1}{t^n}m^{1,p}_{F(\omega)}(\ell_\xi,Q_t(0)),\tag{5.1}$$

$$g_{\rm hom}(\omega,\nu)=\lim_{t\to+\infty}\frac{1}{t^{n-1}}m^{\rm pc}_{G(\omega)}(u_{tx,1,\nu},Q^\nu_t(tx))=\lim_{t\to+\infty}\frac{1}{t^{n-1}}m^{\rm pc}_{G(\omega)}(u_{0,1,\nu},Q^\nu_t(0)).\tag{5.2}$$

If, in addition, f, g, and D are ergodic, then $f_{\rm hom}$ and $g_{\rm hom}$ are independent of ω, and

$$f_{\rm hom}(\xi)=\lim_{t\to+\infty}\frac{1}{t^n}\int_\Omega m^{1,p}_{F(\omega)}(\ell_\xi,Q_t(0))\,dP(\omega),$$

$$g_{\rm hom}(\nu)=\lim_{t\to+\infty}\frac{1}{t^{n-1}}\int_\Omega m^{\rm pc}_{G(\omega)}(u_{0,1,\nu},Q^\nu_t(0))\,dP(\omega).$$

Proof. For $k\in\mathbb{N}^*$ we set $f^k(\omega,x,\xi):=a^k(\omega,x)f(\omega,x,\xi)$ and $g^k(\omega,x,\nu):=a^k(\omega,x)g(\omega,x,\nu)$, where

$$a^k(\omega,x):=\begin{cases}1 & \text{if } x\in\mathbb{R}^n\setminus K(\omega),\\ \frac{1}{k} & \text{if } x\in K(\omega),\end{cases}\tag{5.3}$$

and consider the coercive functionals $F^k(\omega),G^k(\omega):L^1_{\rm loc}(\mathbb{R}^n)\times\mathcal{A}\longrightarrow[0,+\infty]$ defined as

$$F^k(\omega)(u,A):=\begin{cases}\displaystyle\int_A f^k(\omega,x,\nabla u)\,dx & \text{if } u_{|A}\in W^{1,p}(A),\\ +\infty & \text{otherwise,}\end{cases}$$

and

$$G^k(\omega)(u,A):=\begin{cases}\displaystyle\int_{S_u\cap A} g^k(\omega,x,\nu_u)\,d\mathcal{H}^{n-1} & \text{if } u_{|A}\in GSBV^p(A),\\ +\infty & \text{otherwise.}\end{cases}$$

Moreover, we denote with $m^{1,p}_{F^k(\omega)}$ and $m^{\rm pc}_{G^k(\omega)}$ the corresponding minimisation problems as in (2.10) and (2.11), respectively.

For every fixed $k\in\mathbb{N}^*$ the functions f^k and g^k satisfy the assumptions of [11, Theorem 3.12]. Hence we can deduce the existence of a set $\Omega^k\subset\Omega$, with $\Omega^k\in\mathcal{T}$ and $P(\Omega^k)=1$, such that for every $\omega\in\Omega^k$ and for every $x,\xi\in\mathbb{R}^n,\nu\in\mathbb{S}^{n-1}$ it holds

$$\lim_{t\to+\infty}\frac{m^{1,p}_{F^k(\omega)}(\ell_\xi,Q_t(tx))}{t^n}=\lim_{t\to+\infty}\frac{m^{1,p}_{F^k(\omega)}(\ell_\xi,Q_t(0))}{t^n}=:f^k_{\rm hom}(\omega,\xi)\tag{5.4}$$

and

$$\lim_{t\to+\infty}\frac{m^{\rm pc}_{G^k(\omega)}(u_{tx,1,\nu},Q^\nu_t(tx))}{t^{n-1}}=\lim_{t\to+\infty}\frac{m^{\rm pc}_{G^k(\omega)}(u_{0,1,\nu},Q^\nu_t(0))}{t^{n-1}}=:g^k_{\rm hom}(\omega,\nu).\tag{5.5}$$

Furthermore, $f^k_{\rm hom}$ is $(\mathcal{T}\otimes\mathscr{B}^n)$-measurable while $g^k_{\rm hom}$ is $(\mathcal{T}\otimes\mathscr{B}(\mathbb{S}^{n-1}))$-measurable. Now we set

$$\Omega':=\bigcap_{k\in\mathbb{N}^*}\Omega^k;\tag{5.6}$$

clearly $\Omega'\in\mathcal{T}$, $P(\Omega')=1$, and for every $\omega\in\Omega'$ and every $k\in\mathbb{N}^*$, the limits in (5.4) and (5.5) exist. We note moreover that for every $\omega\in\Omega'$, $\xi\in\mathbb{R}^n$, and $\nu\in\mathbb{S}^{n-1}$ the sequences $f^k_{\rm hom}(\omega,\xi)$

and $g^k_{\rm hom}(\omega,\nu)$ are decreasing in k. Therefore, for every $\omega\in\Omega'$, $\xi\in\mathbb{R}^n$, and $\nu\in\mathbb{S}^{n-1}$ we define the functions $f_{\rm hom}$ and $g_{\rm hom}$ as follows:

$$\lim_{k\to+\infty} f^k_{\rm hom}(\omega,\xi)=\inf_{k\in\mathbb{N}^*} f^k_{\rm hom}(\omega,\xi)=: f_{\rm hom}(\omega,\xi) \tag{5.7}$$

and

$$\lim_{k\to+\infty} g^k_{\rm hom}(\omega,\nu)=\inf_{k\in\mathbb{N}^*} g^k_{\rm hom}(\omega,\nu)=: g_{\rm hom}(\omega,\nu). \tag{5.8}$$

By definition, we clearly have that $f_{\rm hom}$ is $(\mathcal{T}\otimes\mathscr{B}^n)$-measurable and $g_{\rm hom}$ is $(\mathcal{T}\otimes\mathscr{B}(\mathbb{S}^{n-1}))$-measurable. We now show that the functions $f_{\rm hom}$ and $g_{\rm hom}$ satisfy (5.1) and (5.2), respectively.

For every $\omega\in\Omega'$, $x,\xi\in\mathbb{R}^n$, and $\nu\in\mathbb{S}^{n-1}$ set

$$\overline{f}(\omega,x,\xi):=\limsup_{t\to+\infty}\frac{m^{1,p}_{F(\omega)}(\ell_\xi,Q_t(tx))}{t^n},$$

$$\underline{f}(\omega,x,\xi):=\liminf_{t\to+\infty}\frac{m^{1,p}_{F(\omega)}(\ell_\xi,Q_t(tx))}{t^n},$$

and

$$\overline{g}(\omega,x,\nu):=\limsup_{t\to+\infty}\frac{m^{\rm pc}_{G(\omega)}(u_{tx,1,\nu},Q^\nu_t(tx))}{t^{n-1}},$$

$$\underline{g}(\omega,x,\nu):=\liminf_{t\to+\infty}\frac{m^{\rm pc}_{G(\omega)}(u_{tx,1,\nu},Q^\nu_t(tx))}{t^{n-1}}.$$

Then, to conclude it is enough to show that

$$\overline{f}=\underline{f}=f_{\rm hom} \tag{5.9}$$

and

$$\overline{g}=\underline{g}=g_{\rm hom}, \tag{5.10}$$

with $f_{\rm hom}$ and $g_{\rm hom}$ as in (5.7) and (5.8), respectively. We prove the two claims above in two separate steps.

Step 1: Proof of (5.9). By definition $0\le f\chi_{A\setminus K(\omega)}\le f^k$ for every $k\in\mathbb{N}^*$, hence by the monotonicity of the integral we immediately deduce that $\overline{f}\le f^k_{\rm hom}$ for every $k\in\mathbb{N}^*$. Therefore

$$\overline{f}(\omega,x,\xi)\le\inf_{k\in\mathbb{N}^*} f^k_{\rm hom}(\omega,\xi)=f_{\rm hom}(\omega,\xi), \tag{5.11}$$

for every $\omega\in\Omega'$, $x,\xi\in\mathbb{R}^n$.

We now show that $f_{\rm hom}\le\underline{f}$. To this end let $t\gg1$, $\omega\in\Omega'$, $x\in\mathbb{R}^n$ and $\xi\in\mathbb{R}^n$ be fixed. For $\eta>0$ let $\hat u\in W^{1,p}(Q_t(tx))$ be such that $\hat u=\ell_\xi$ near $\partial Q_t(tx)$ and

$$F(\omega)(\hat u,Q_t(tx))\le m^{1,p}_{F(\omega)}(\ell_\xi,Q_t(tx))+\eta t^n. \tag{5.12}$$

Since ℓ_ξ is a competitor for $m^{1,p}_{F(\omega)}(\ell_\xi,Q_t(tx))$ we immediately get

$$\begin{aligned}\int_{Q_t(tx)\setminus K(\omega)}|D\hat u|^p\,dy&\le\frac{1}{c_1}F(\omega)(\hat u,Q_t(tx))\\&\le\frac{1}{c_1}\Big(m^{1,p}_{F(\omega)}(\ell_\xi,Q_t(tx))+\eta t^n\Big)\le\frac{1}{c_1}\Big(c_2(1+|\xi|^p)t^n+\eta t^n\Big).\end{aligned} \tag{5.13}$$

Starting from $\hat u$ we now construct a competitor for $m^{1,p}_{F^k(\omega)}(\ell_\xi,Q_t(tx))$. First of all, we extend $\hat u$ by setting $\hat u=\ell_\xi$ in $\mathbb{R}^n\setminus Q_t(tx)$. Now, let $\mathcal{J}\subset\mathcal{I}$ denote the set of indices j such that $B(\theta_j(\omega),r_j(\omega))\cap Q_t(tx)\neq\emptyset$. We clearly have

$$Q_t(tx)\subset Q_t(tx)\cup\bigcup_{j\in\mathcal{J}}B(\theta_j(\omega),r_j(\omega)+\delta).$$

For every $j\in\mathcal{J}$ we set $\hat u_j:=\hat u_{|A_j(\omega)}$, where $A_j(\omega)$ denotes the open annulus $B_{r_j(\omega),r_j(\omega)+\delta}(\theta_j(\omega))$. By applying Lemma 4.5 in every $A_j(\omega)$ we deduce the existence of an extension operator T^ω_j :

$W^{1,p}(A_j(\omega)) \longrightarrow W^{1,p}(B(\theta_j(\omega), r_j(\omega)+\delta)$ and a constant $c>0$ independent of j and ω, such that

$$\|D\big(T_j^\omega \hat{u}_j\big)\|_{L^p(B(\theta_j(\omega),r_j(\omega)+\delta)} \leq c\|D\hat{u}_j\|_{L^p(A_j(\omega))}.$$

We then define the function $\tilde{u}:\mathbb{R}^n\to\mathbb{R}$ as follows

$$\tilde{u} = \sum_{j\in\mathcal{J}} (T_j^\omega \hat{u}_j)\chi_{B(\theta_j(\omega),r_j(\omega)+\delta)} + \hat{u}\,\chi_{Q_t(tx)\setminus K^\delta(\omega)},$$

where

$$K^\delta(\omega) := \bigcup_{j\in\mathcal{J}} B(\theta_j(\omega), r_j(\omega)+\delta).$$

By construction $\tilde{u}_{|Q_t(tx)} \in W^{1,p}(Q_t(tx))$. Moreover,

$$\|D\tilde{u}\|^p_{L^p(Q_t(tx))} \leq c\bigg(\|D\hat{u}\|^p_{L^p(Q_t(tx)\setminus K(\omega))} + |\xi|^p\mathcal{L}^n\bigg(\bigcup_{j\in\mathcal{J}} B_{r_j(\omega),r_j(\omega)+\delta}(\theta_j(\omega))\setminus Q_t(tx)\bigg)\bigg),$$

therefore from (5.13) we deduce that

$$\int_{Q_t(tx)} |D\tilde{u}|^p\,dy \leq c\int_{Q_t(tx)\setminus K(\omega)} |D\hat{u}|^p\,dy + \tilde{c}\,|\xi|^p t^{n-1} \leq \frac{c}{c_1}\Big(c_2(1+|\xi|^p)t^n + \eta t^n\Big) + \tilde{c}\,|\xi|^p t^{n-1}, \tag{5.14}$$

where $\tilde{c}$ depends on r_*. We note that in general the function $\tilde{u}$ does not coincide with ℓ_ξ in a neighbourhood of $\partial Q_t(tx)$, since we might have altered the boundary value of $\hat{u}$ in the perforations intersecting $\partial Q_t(tx)$. We then need to further modify $\tilde{u}$ in a way such that it attains the boundary datum. To this aim, let $\varphi\in C_0^\infty(Q_t(tx))$ be a cut-off function between $Q_{t-4(r_*+\delta)}(tx)$ and $Q_t(tx)$; *i.e.*, $0\leq\varphi\leq 1$, $\varphi\equiv 1$ in $Q_{t-4(r_*+\delta)}(tx)$, $\varphi\equiv 0$ in $\mathbb{R}^n\setminus Q_t(tx)$, and $\|D\varphi\|_\infty \leq c$, with $c = c(n,r_*,\delta)>0$. Set

$$w := \varphi\,\tilde{u} + (1-\varphi)\ell_\xi;$$

clearly $w\in W^{1,p}(Q_t(tx))$, and $w=\ell_\xi$ in a neighbourhood of $\partial Q_t(tx)$. We now claim that

$$\lim_{t\to+\infty} \frac{1}{t^n}\int_{Q_t(tx)\setminus Q_{t-4(r_*+\delta)}(tx)} |Dw|^p dy = 0. \tag{5.15}$$

To ease the notation, in what follows we set $t' := t-4(r_*+\delta)$. We clearly have

$$\int_{Q_t(tx)\setminus Q_{t'}(tx)} |Dw|^p dy \leq c\int_{Q_t(tx)\setminus Q_{t'}(tx)} |D\tilde{u}|^p dy + c\int_{Q_t(tx)\setminus Q_{t'}(tx)} |\tilde{u}-\ell_\xi|^p dy + c|\xi|^p t^{n-1}. \tag{5.16}$$

We now cover $Q_t(tx)\setminus Q_{t'}(tx)$ with a finite number of possibly overlapping cubes with side-length $2(r_*+\delta)$, having one face on the boundary $\partial Q_t(tx)$. Thus we write

$$Q_t(tx)\setminus Q_{t'}(tx) = \bigcup_{\sigma\in\mathcal{S}} Q^\sigma_{2(r_*+\delta)},$$

where $Q^\sigma_{2(r_*+\delta)} := \sigma + Q_{2(r_*+\delta)}$ and $\mathcal{S}\subset\mathbb{R}^n$ is a finite set of translation vectors such that the volume of this covering is asymptotically equal to the volume of $Q_t(tx)\setminus Q_{t'}(tx)$, for $t\to+\infty$.

We now apply the Poincaré inequality to the function $\tilde{u}-\ell_\xi$ in $Q_t(tx)\setminus Q_{t'}(tx)$. To do so we preliminarily observe that for every $\sigma\in\mathcal{S}$ it holds

$$\mathcal{H}^{n-1}\left(\partial Q^\sigma_{2(r_*+\delta)} \cap \{\tilde{u}=\ell_\xi\}\right) \geq \delta^{n-1}. \tag{5.17}$$

This is clearly true if $\partial Q^\sigma_{2(r_*+\delta)}\cap K(\omega)=\emptyset$, since in that case $\tilde{u}=\ell_\xi$ on the whole face $\partial Q^\sigma_{2(r_*+\delta)}\cap \partial Q_t(tx)$, whose $\mathcal{H}^{n-1}$-measure is larger than δ^{n-1}. If instead $\partial Q^\sigma_{2(r_*+\delta)}\cap K(\omega)\neq\emptyset$, since each ball in $K(\omega)$ has diameter smaller than $2r_*$ and is separated from any other ball by a distance which is at least 2δ, inequality (5.17) holds in this case as well. Therefore the Poincaré inequality applied in every cube $Q^\sigma_{2(r_*+\delta)}$ gives

$$\int_{Q^\sigma_{2(r_*+\delta)}} |\tilde u - \ell_\xi|^p dy \le C \int_{Q^\sigma_{2(r_*+\delta)}} |D\tilde u - \xi|^p dy,$$

where $C = C(n, p, \delta, r_*) > 0$ is independent of σ. Hence by adding up all the cubes $Q^\sigma_{2(r_*+\delta)}$, with $\sigma \in \mathcal{S}$, we get

$$\int_{Q_t(tx)\setminus Q_{t'}(tx)} |\tilde u - \ell_\xi|^p dy \le C \int_{Q_t(tx)\setminus Q_{t'}(tx)} |D\tilde u - \xi|^p dy. \tag{5.18}$$

Then, gathering (5.16) and (5.18) yields

$$\int_{Q_t(tx)\setminus Q_{t'}(tx)} |Dw|^p dy \le c \int_{Q_t(tx)\setminus Q_{t'}(tx)} |D\tilde u|^p dy + c|\xi|^p t^{n-1}.$$

Hence to prove (5.15) it is enough to show that

$$\lim_{t\to+\infty} \frac{1}{t^n} \int_{Q_t(tx)\setminus Q_{t'}(tx)} |D\tilde u|^p dy = 0.$$

The latter is a consequence of the equality

$$\frac{1}{t^n} \int_{Q_t(tx)\setminus Q_{t'}(tx)} |D\tilde u|^p dy = \int_{Q_1(x)\setminus Q_{1-\frac{4(r_*+\delta)}{t}}(x)} |Dv|^p dz,$$

where $v(z) := \frac{1}{t}\tilde u(tz)$ for every $z \in Q_1(x)$. In fact, by (5.14) we have

$$\int_{Q_1(x)} |Dv|^p dz = \frac{1}{t^n} \int_{Q_t(tx)} |D\tilde u|^p \, dy \le \frac{c}{c_1}\Big(c_2(1 + |\xi|^p) + \eta\Big) + \frac{\tilde c}{t} |\xi|^p,$$

thus

$$\lim_{t\to+\infty} \int_{Q_1(x)\setminus Q_{1-\frac{4(r_*+\delta)}{t}}(x)} |Dv|^p dz = 0,$$

by the absolute continuity of the Lebesgue integral.

Since w is a competitor for $m^{1,p}_{F^k(\omega)}(\ell_\xi, Q_t(tx))$, we clearly have

$$m^{1,p}_{F^k(\omega)}(\ell_\xi, Q_t(tx)) \le F^k(\omega)(w, Q_t(tx)). \tag{5.19}$$

We now estimate the right-hand side of the inequality above in terms of $F(\omega)(\hat u, Q_t(tx))$, and hence in terms of $m^{1,p}_{F(\omega)}(\ell_\xi, Q_t(tx))$, thanks to (5.12). By the definition of F^k and by $(f2)$ we bound

$$\begin{aligned}
F^k(\omega)(w, Q_t(tx)) &\le F(\omega)(w, Q_t(tx)) + \frac{c_2}{k}\int_{Q_t(tx)} (1 + |Dw|^p)\, dy \\
&\le F(\omega)(\tilde u, Q_{t'}(tx)) + F(\omega)(w, Q_t(tx) \setminus Q_{t'}(tx)) + \frac{c_2}{k}\int_{Q_t(tx)} (1 + |Dw|^p)\, dy \\
&\le F(\omega)(\hat u, Q_{t'}(tx)) + c_2 \int_{Q_t(tx)\setminus Q_{t'}(tx)} (1 + |Dw|^p)\, dy + \frac{c_2}{k}\int_{Q_t(tx)} (1 + |Dw|^p)\, dy,
\end{aligned}$$

where we have used the definition of w and the fact that $\tilde u = \hat u$ in $Q_t(tx) \setminus K(\omega)$. Again by the definition of w we estimate

$$\int_{Q_t(tx)} (1 + |Dw|^p)\, dy \le \int_{Q_{t'}(tx)} (1 + |D\tilde u|^p)\, dy + \int_{Q_t(tx)\setminus Q_{t'}(tx)} (1 + |Dw|^p)\, dy,$$

so that, by (5.19) and (5.12), and by invoking (5.14) we find

$$\begin{aligned}
m^{1,p}_{F^k(\omega)}(\ell_\xi, Q_t(tx)) &\le m^{1,p}_{F(\omega)}(\ell_\xi, Q_t(tx) + \eta t^n + 2c_2 \int_{Q_t(tx)\setminus Q_{t'}(tx)} (1 + |Dw|^p)\, dy \\
&\quad + \frac{c_2}{k}\Big(t^n + C\big((1 + |\xi|^p) + \eta\big) + c\,|\xi|^p t^{n-1}\Big).
\end{aligned}$$

Therefore, by (5.4) and (5.15), passing to the liminf as $t \to +\infty$ we get

$$f^k_{\rm hom}(\omega,\xi) \le \underline{f}(\omega,x,\xi) + \eta + \frac{c}{k}(1+|\xi|^p),$$

for every $\omega \in \Omega'$, $x,\xi \in \mathbb{R}^n$, and $k \in \mathbb{N}^*$. Thus letting $k \to +\infty$ yields

$$f_{\rm hom}(\omega,\xi) = \inf_{k\in\mathbb{N}^*} f^k_{\rm hom}(\omega,\xi) \le \underline{f}(\omega,x,\xi) + \eta \tag{5.20}$$

for every $\omega \in \Omega'$ and $x,\xi \in \mathbb{R}^n$. Hence, by the arbitrariness of $\eta > 0$, gathering (5.11) and (5.20) eventually gives (5.9) and thus (5.1).

Step 2: Proof of (5.10). By definition $0 \le g\chi_{A\setminus K(\omega)} \le g^k$ for every $k \in \mathbb{N}^*$, hence by the monotonicity of the integral we immediately deduce that $\overline{g} \le g^k_{\rm hom}$ for every $k \in \mathbb{N}^*$. Therefore

$$\overline{g}(\omega,x,\nu) \le \inf_{k\in\mathbb{N}^*} g^k_{\rm hom}(\omega,\nu) = g_{\rm hom}(\omega,\nu), \tag{5.21}$$

for every $\omega \in \Omega'$, $x \in \mathbb{R}^n$ and $\nu \in \mathbb{S}^{n-1}$.

We now show that $g_{\rm hom} \le \underline{g}$. To this end let $t \gg 1$, $\omega \in \Omega'$, $x \in \mathbb{R}^n$ and $\nu \in \mathbb{S}^{n-1}$ be fixed. For $\eta > 0$ let $\hat{u} \in \mathcal{P}(Q^\nu_t(tx))$ be such that $\hat{u} = u_{tx,1,\nu}$ near $\partial Q^\nu_t(tx)$ and

$$G(\omega)(\hat{u}, Q^\nu_t(tx)) \le m^{\rm pc}_{G(\omega)}(u_{tx,1,\nu}, Q^\nu_t(tx)) + \eta t^{n-1}. \tag{5.22}$$

Since $u_{tx,1,\nu}$ is a competitor for $m^{\rm pc}_{G(\omega)}(u_{tx,1,\nu}, Q^\nu_t(tx))$, by ($g1$) and ($g2$) we immediately get

$$\begin{aligned}\frac{\mathcal{H}^{n-1}(S_{\hat{u}} \cap (Q^\nu_t(tx)\setminus K(\omega)))}{t^{n-1}} &\le \frac{G(\omega)(\hat{u}, Q^\nu_t(tx))}{c_3 t^{n-1}} \\ &\le \frac{m^{\rm pc}_{G(\omega)}(u_{tx,1,\nu}, Q^\nu_t(tx))}{c_3 t^{n-1}} + \frac{\eta}{c_3} \le \frac{c_4+\eta}{c_3}.\end{aligned} \tag{5.23}$$

We now modify $\hat{u}$ in order to obtain a competitor for $m^{\rm pc}_{G^k(\omega)}(u_{tx,1,\nu}, Q^\nu_t(tx))$. We preliminarily extend $\hat{u}$ to the whole $\mathbb{R}^n$ by setting $\hat{u} = u_{tx,1,\nu}$ in $\mathbb{R}^n \setminus Q^\nu_t(tx))$. Now, we denote with $\mathcal{J} \subset \mathcal{I}$ the set of indices j such that $B(\theta_j(\omega), r_j(\omega)) \cap Q^\nu_t(tx) \ne \emptyset$. For each $j \in \mathcal{J}$ we set $\hat{u}_j := \hat{u}_{|A_j(\omega)}$, with $A_j(\omega) := B_{r_j(\omega), r_j(\omega)+\delta}(\theta_j(\omega))$.

We divide the proof into three substeps.

Substep 2.1: Extension of $\hat{u}$ in the inner perforations. Let $\mathcal{J}_I \subset \mathcal{J}$ denote the set of indices j such that $B(\theta_j(\omega), r_j(\omega)+\delta) \subset Q^\nu_t(tx)$. By Lemma 4.6 there exists an extension $v_j := T_j\hat{u}_j \in \mathcal{P}(B(\theta_j(\omega), r_j(\omega)+\delta))$ of $\hat{u}_j$ whose jump set in $B(\theta_j(\omega), r_j(\omega)+\delta)$ is controlled, in measure, by the jump set of $\hat{u}_j$ (and hence by the jump of $\hat{u}$ in $A_j(\omega)$).

Substep 2.2: Modification of $\hat{u}$ in the boundary perforations. Let $\mathcal{J}_B := \mathcal{J} \setminus \mathcal{J}_I$, and let $j \in \mathcal{J}_B$. We set

$$w_j := \begin{cases} \hat{u}_j & \text{in } A_j(\omega), \\ u_{tx,1,\nu} & \text{in } \overline{B}(\theta_j(\omega), r_j(\omega)). \end{cases}$$

Clearly, for every $j \in \mathcal{J}_B$, the additional jump created by w_j is controlled by the perimeter of the boundary perforations $B(\theta_j(\omega), r_j(\omega))$. Since the perforations in $K(\omega)$ are pairwise disjoint (and in particular this is true for the boundary perforations), the total additional jump due to the boundary perforations is controlled by the perimeter of $Q^\nu_t(tx)$; *i.e.,* it is equal to $c\,t^{n-1}$ for some $c > 0$ independent of t.

Substep 2.3: Adding up all the contributions. We now denote with $\tilde{u} \in \mathcal{P}(Q^\nu_t(tx))$ the function defined as

$$\tilde{u} := \begin{cases} \hat{u} & \text{in } Q^\nu_t(tx) \setminus K^\delta(\omega), \\ v_j & \text{in } B(\theta_j(\omega), r_j(\omega)+\delta), j \in \mathcal{J}_I \\ w_j & \text{in } B(\theta_j(\omega), r_j(\omega)+\delta) \cap Q^\nu_t(tx), j \in \mathcal{J}_B. \end{cases}$$

By construction the function $\tilde{u}$ satisfies the following properties:

a. $\tilde{u} = u_{tx,1,\nu}$ in a neighbourhood of $\partial Q^\nu_t(tx)$;
b. $\mathcal{H}^{n-1}(S_{\tilde{u}} \cap (Q^\nu_t(tx)\setminus K(\omega))) \le \mathcal{H}^{n-1}(S_{\hat{u}} \cap (Q^\nu_t(tx)\setminus K(\omega)))$;

c. $\mathcal{H}^{n-1}(S_{\tilde{u}} \cap Q_t^\nu(tx)) \le c(\mathcal{H}^{n-1}(S_{\hat{u}} \cap (Q_t^\nu(tx) \setminus K(\omega))) + t^{n-1})$, for some $c > 0$ independent of t.

Since $\tilde{u}$ is a competitor for $m^{\rm pc}_{G^k(\omega)}(u_{tx,1,\nu}, Q_t^\nu(tx))$, by combining b., c., and (5.23) we get

$$\begin{aligned}\frac{m^{\rm pc}_{G^k(\omega)}(u_{tx,1,\nu}, Q_t^\nu(tx))}{t^{n-1}} &\le \frac{1}{t^{n-1}} G^k(\omega)(\tilde{u}, Q_t^\nu(tx)) \\ &\le \frac{1}{t^{n-1}} G(\omega)(\tilde{u}, Q_t^\nu(tx)) + \frac{c_4}{k\, t^{n-1}} \mathcal{H}^{n-1}(Q_t^\nu(tx) \cap S_{\tilde{u}}) \\ &\le \frac{1}{t^{n-1}} G(\omega)(\hat{u}, Q_t^\nu(tx)) + \frac{c}{k} \\ &\le \frac{m^{\rm pc}_{G(\omega)}(u_{tx,1,\nu}, Q_t^\nu(tx))}{t^{n-1}} + \eta + \frac{c}{k},\end{aligned}$$

where we have also used (5.22). Therefore passing to the liminf as $t \to +\infty$ we get

$$g^k_{\rm hom}(\omega, \nu) \le \underline{g}(\omega, x, \nu) + \eta + \frac{c}{k},$$

for every $\omega \in \Omega'$, $x \in \mathbb{R}^n$, $\nu \in \mathbb{S}^{n-1}$, and $k \in \mathbb{N}^*$. Thus finally letting $k \to +\infty$ and then $\eta \to 0$ yields

$$g_{\rm hom}(\omega, \nu) := \inf_{k \in \mathbb{N}^*} g^k_{\rm hom}(\omega, \nu) \le \underline{g}(\omega, x, \nu), \tag{5.24}$$

for every $\omega \in \Omega'$, $x \in \mathbb{R}^n$ and $\nu \in \mathbb{S}^{n-1}$. Hence gathering (5.21) and (5.24) eventually gives (5.10) and thus (5.2).

If f, g, and K are stationary with respect to an ergodic group of P-preserving transformations, then [11, Theorem 3.12] ensures that $f^k_{\rm hom}$ and $g^k_{\rm hom}$ (and hence $f_{\rm hom}$ and $g_{\rm hom}$) are independent of ω. Then, the thesis follows by integrating (5.9) and (5.10) over Ω, by the Dominated Convergence Theorem. □

Remark 5.2 (Γ-convergence of the perturbed functionals). Let f, g and D be as in Theorem 5.1. For $k \in \mathbb{N}^*$ we set $f^k(\omega, x, \xi) := a^k(\omega, x) f(\omega, x, \xi)$ and $g^k(\omega, x, \nu) := a^k(\omega, x) g(\omega, x, \nu)$, where a^k is defined as in (5.3). For $\varepsilon > 0$ and $k \in \mathbb{N}^*$, let $E^k_\varepsilon(\omega)\colon L^1_{\rm loc}(\mathbb{R}^n) \times \mathcal{A} \longrightarrow (0, +\infty]$ be the functionals defined as

$$E^k_\varepsilon(\omega)(u, A) := \begin{cases} \displaystyle\int_A f^k\left(\omega, \frac{x}{\varepsilon}, \nabla u\right) dx + \int_{S_u \cap A} g^k\left(\omega, \frac{x}{\varepsilon}, \nu_u\right) d\mathcal{H}^{n-1} & \text{if } u_{|A} \in GSBV^p(A), \\ +\infty & \text{otherwise.}\end{cases}$$

If Ω' is the set in the statement of Theorem 5.1 (defined as in (5.6)), we deduce from [11, Theorem 3.13] that for every $\omega \in \Omega'$ and $k \in \mathbb{N}^*$ the functionals $E^k_\varepsilon(\omega)$ Γ-converge to the homogeneous free-discontinuity functional $E^k_{\rm hom}(\omega)\colon L^1_{\rm loc}(\mathbb{R}^n) \times \mathcal{A} \longrightarrow (0, +\infty]$ given by

$$E^k_{\rm hom}(u, A) := \begin{cases} \displaystyle\int_A f^k_{\rm hom}(\omega, \nabla u)\, dx + \int_{S_u \cap A} g^k_{\rm hom}(\omega, \nu_u)\, d\mathcal{H}^{n-1} & \text{if } u_{|A} \in GSBV^p(A), \\ +\infty & \text{otherwise,}\end{cases} \tag{5.25}$$

where $f^k_{\rm hom}$ and $g^k_{\rm hom}$ are as in (5.4) and (5.5), respectively.

Theorem 5.3 (Γ-convergence). *Let f and g be stationary random volume and surface integrands, and let $D \subset \mathbb{R}^n$ be a random perforated domain as in Definition 2.8. Assume that the stationarity of f, g and D is satisfied with respect to the same group $(\tau_y)_{y \in \mathbb{R}^n}$ of P-preserving transformations on $(\Omega, \mathcal{T}, P)$. Let E_ε be as in (2.7), let $\Omega' \in \mathcal{T}$ (with $P(\Omega') = 1$), $f_{\rm hom}$, and $g_{\rm hom}$ be as in Theorem 5.1. Then, for every $\omega \in \Omega'$ and every $A \in \mathcal{A}$, the functionals $E_\varepsilon(\omega)(\cdot, A)$ Γ-converge in $L^1_{\rm loc}(\mathbb{R}^n)$ to the homogeneous functional $E_{\rm hom}(\omega)\colon L^1_{\rm loc}(\mathbb{R}^n) \times \mathcal{A} \longrightarrow [0, +\infty]$ defined as*

$$E_{\rm hom}(u, A) := \begin{cases} \displaystyle\int_A f_{\rm hom}(\omega, \nabla u)\ dx + \int_{A \cap S_u} g_{\rm hom}(\omega, \nu_u)\ d\mathcal{H}^{n-1} & \text{if } u_{|_A} \in GSBV^p(A), \\ +\infty & \text{otherwise.}\end{cases} \tag{5.26}$$

Moreover, for every $\omega \in \Omega'$, $\xi, \xi_1, \xi_2 \in \mathbb{R}^n$ *and* $\nu \in \mathbb{S}^{n-1}$ *we have that*

$$\tilde{c}_0|\xi|^p \leq f_{\rm hom}(\omega, \xi) \leq c_2(1 + |\xi|^p), \tag{5.27}$$

and

$$\tilde{c}_0 \leq g_{\rm hom}(\omega, \nu) \leq c_4, \tag{5.28}$$

where $\tilde{c}_0 = \tilde{c}_0(n, \delta) > 0$, *and* c_2 *and* c_4 *are as in* (*f*2) *and* (*g*2). *Furthermore, there exists* $L' > 0$ *such that*

$$|f_{\rm hom}(\omega, \xi_1) - f_{\rm hom}(\omega, \xi_2)| \leq L'\big(1 + |\xi_1|^{p-1} + |\xi_2|^{p-1}\big)|\xi_1 - \xi_2|. \tag{5.29}$$

Proof. In view of (5.7) and (5.8), the Monotone Convergence Theorem yields

$$E_{\rm hom}(\omega)(u, A) = \inf_{k \in \mathbb{N}^*} E^k_{\rm hom}(\omega)(u, A) = \lim_{k \to +\infty} E^k_{\rm hom}(\omega)(u, A) \tag{5.30}$$

for every $\omega \in \Omega'$, $A \in \mathcal{A}$ and $u \in GSBV^p(A)$, where $E^k_{\rm hom}$ is as in (5.25).

We prove the Γ-convergence of E_ε to $E_{\rm hom}$ in two steps.

Step 1: liminf-inequality. Let $\omega \in \Omega'$ and $A \in \mathcal{A}$ be fixed. Let $u \in GSBV^p(A)$ and let $(u_\varepsilon) \subset L^1_{\rm loc}(\mathbb{R}^n)$ be a sequence satisfying $u_\varepsilon \to u$ strongly in $L^1(A)$ and $\sup_\varepsilon E_\varepsilon(\omega)(u_\varepsilon, A) < +\infty$. Note in particular that $(u_\varepsilon) \subset GSBV^p(A)$. For $M > 0$ we consider the truncated function $u^M := (u \wedge M) \vee (-M) \in GSBV^p(A) \cap L^\infty(A)$ and the truncated sequence $(u^M_\varepsilon) \subset GSBV^p(A) \cap L^\infty(A)$; clearly (u^M_ε) converges to u^M in $L^1(A)$ as $\varepsilon \to 0$.

Let $(\tilde{u}_\varepsilon) \subset SBV^p(A) \cap L^\infty(A)$ be the extension provided by Proposition 4.7, such that $\tilde{u}_\varepsilon = u^M_\varepsilon$ a.e. in $A \setminus \varepsilon K(\omega)$, and let $\tilde{u} \in SBV^p(A) \cap L^\infty(A)$ be such that (up to a subsequence) $\tilde{u}_\varepsilon \to \tilde{u}$ strongly in $L^1(A)$. Since the sequences (u_ε) and $(\tilde{u}_\varepsilon)$ coincide in $A \setminus \varepsilon K(\omega)$, we deduce by Property 2.10 that $\tilde{u} = u^M$ a.e. in A. Furthermore, (4.14) gives

$$MS^p(\tilde{u}_\varepsilon, A) \leq \frac{c(n, p, \delta, r_*)}{c_1 \wedge c_3} \left(E_\varepsilon(\omega)(u^M_\varepsilon, A) + \mathcal{H}^{n-1}(\partial A)\right),$$

and therefore we have

$$\begin{aligned} E^k_\varepsilon(\omega)(\tilde{u}_\varepsilon, A) &\leq E_\varepsilon(\omega)(u^M_\varepsilon, A) + \frac{c_2 \vee c_4}{k} MS^p(\tilde{u}_\varepsilon, A \cap \varepsilon K(\omega)) + \frac{c_2}{k}\mathcal{L}^n(A \cap \varepsilon K(\omega)) \\ &\leq \Big(1 + \frac{c}{k}\Big) E_\varepsilon(\omega)(u^M_\varepsilon, A) + \frac{c}{k}\mathcal{H}^{n-1}(\partial A) + \frac{c_2}{k}\mathcal{L}^n(A), \end{aligned}$$

where $c = c(n, p, \delta, r_*)$. Then, by Remark 5.2 we deduce that for every $\omega \in \Omega'$, $A \in \mathcal{A}$ and $k \in \mathbb{N}^*$

$$\begin{aligned} E^k_{\rm hom}(\omega)(u^M, A) &\leq \liminf_{\varepsilon \to 0} E^k_\varepsilon(\omega)(\tilde{u}_\varepsilon, A) \\ &\leq \Big(1 + \frac{c}{k}\Big) \liminf_{\varepsilon \to 0} E_\varepsilon(\omega)(u^M_\varepsilon, A) + \frac{c}{k}\mathcal{H}^{n-1}(\partial A) + \frac{c_2}{k}\mathcal{L}^n(A). \end{aligned}$$

By letting $k \to +\infty$ and using (5.30), we then get

$$E_{\rm hom}(\omega)(u^M, A) = \lim_{k \to +\infty} E^k_{\rm hom}(\omega)(u^M, A) \leq \liminf_{\varepsilon \to 0} E_\varepsilon(\omega)(u^M_\varepsilon, A), \tag{5.31}$$

and hence the liminf-inequality is proved for the truncations, for every $M > 0$. Now we observe that E_ε decreases by truncations up to a quantifiable error, namely

$$E_\varepsilon(\omega)(u^M_\varepsilon, A) \leq E_\varepsilon(\omega)(u_\varepsilon, A) + \int_{A \cap \{|u_\varepsilon| > M\}} f(\omega, x, 0)dx \leq E_\varepsilon(\omega)(u_\varepsilon, A) + c_2\mathcal{L}^n(A \cap \{|u_\varepsilon| > M\}).$$

Therefore, from (5.31) we obtain the improved estimate

$$\begin{aligned} E_{\rm hom}(\omega)(u^M, A) &\leq \liminf_{\varepsilon \to 0} \left(E_\varepsilon(\omega)(u_\varepsilon, A) + c_2\mathcal{L}^n(A \cap \{|u_\varepsilon| > M\})\right) \\ &\leq \liminf_{\varepsilon \to 0} E_\varepsilon(\omega)(u_\varepsilon, A) + c_2 \limsup_{\varepsilon \to 0} \mathcal{L}^n(A \cap \{|u_\varepsilon| > M\}). \end{aligned}$$

Since $u_\varepsilon \to u$ in $L^1(A)$ we have that $\limsup_{\varepsilon \to 0} \mathcal{L}^n(A \cap \{|u_\varepsilon| > M\}) \leq \mathcal{L}^n(A \cap \{|u| > M\})$, and hence

$$E_{\rm hom}(\omega)(u^M, A) \leq \liminf_{\varepsilon \to 0} E_\varepsilon(\omega)(u_\varepsilon, A) + c_2\mathcal{L}^n(A \cap \{|u| > M\}).$$

Finally, since $u^M \to u$ in $L^1(A)$ as $M \to +\infty$, the liminf-inequality follows by the lower semicontinuity of $E_{\rm hom}(\omega)(\cdot, A)$.

Step 2: limsup-inequality. Let $\omega \in \Omega'$ and $A \in \mathcal{A}$ be fixed. Let $u \in GSBV^p(A)$; in view of Remark 5.2 there exists $(u_\varepsilon) \subset GSBV^p(A)$ such that $u_\varepsilon \to u$ in $L^1(A)$ and $\lim_{\varepsilon\to 0} E^k_\varepsilon(\omega)(u_\varepsilon, A) = E^k_{\rm hom}(u, A)$. Then by the definition of E^k_ε we have

$$E^k_{\rm hom}(u, A) = \lim_{\varepsilon\to 0} E^k_\varepsilon(\omega)(u_\varepsilon, A) \geq \limsup_{\varepsilon\to 0} E_\varepsilon(\omega)(u_\varepsilon, A),$$

for every $k \in \mathbb{N}^*$. Then, letting $k \to +\infty$, from (5.30) we finally deduce

$$E_{\rm hom}(u, A) = \lim_{k\to+\infty} E^k_{\rm hom}(u, A) \geq \limsup_{\varepsilon\to 0} E_\varepsilon(\omega)(u_\varepsilon, A)$$

and hence the limsup-inequality is proved.

Step 3: Lower bounds on the limit integrands. We start by proving the lower bound in (5.27). To do so, let $\omega \in \Omega'$ and $\xi \in \mathbb{R}^n$, and let u_ε be a recovery sequence for $E_\varepsilon(\omega)$ at ℓ_ξ in Q. With no loss of generality we can assume that the sequence is bounded in L^∞. If not, we replace u_ε with $u^M_\varepsilon := (u_\varepsilon \wedge M) \vee (-M)$, with $M := 2|\xi|$. Then we still have that $u^M_\varepsilon \to \ell_\xi$ in L^1, and by the estimate of the energy of the truncation in Step 1 we have

$$E_\varepsilon(\omega)(u^M_\varepsilon, A) \leq E_\varepsilon(\omega)(u_\varepsilon, A) + c_2\mathcal{L}^n(A \cap \{|u_\varepsilon| > M\}).$$

By letting $\varepsilon \to 0$ in the previous inequality, since $\mathcal{L}^n(A \cap \{|u_\varepsilon| > M\}) \to 0$ as $\varepsilon \to 0$, we conclude that

$$\limsup_{\varepsilon\to 0} E_\varepsilon(\omega)(u^M_\varepsilon, A) \leq \limsup_{\varepsilon\to 0} E_\varepsilon(\omega)(u_\varepsilon, A) = E_{\rm hom}(\omega)(\ell_\xi, Q),$$

hence also u^M_ε is a recovery sequence for $E_\varepsilon(\omega)$ at ℓ_ξ in Q.

Let $T^\omega_\varepsilon u_\varepsilon$ denote the extension of u_ε in Q provided by Theorem 4.2; note that by Ambrosio's Compactness Theorem $T^\omega_\varepsilon u_\varepsilon$ converges in L^1, and since $T^\omega_\varepsilon u_\varepsilon = u_\varepsilon$ in $A \setminus \varepsilon K(\omega)$, by Property 2.10 we have that $T^\omega_\varepsilon u_\varepsilon \to \ell_\xi$ in L^1, up to a subsequence. By [2, Theorem 4.7], for every $Q' \subset\subset Q$, we have

$$\begin{aligned} MS^p(\ell_\xi, Q') &\leq \liminf_{\varepsilon\to 0} MS^p(T^\omega_\varepsilon u_\varepsilon, Q') \leq c \liminf_{\varepsilon\to 0} MS^p(u_\varepsilon, Q \setminus \varepsilon K(\omega)) \\ &\leq \frac{c}{c_1 \wedge c_3} \liminf_{\varepsilon\to 0} E_\varepsilon(\omega)(u_\varepsilon, Q) = \frac{c}{c_1 \wedge c_3} E_{\rm hom}(\omega)(\ell_\xi, Q), \end{aligned}$$

where we have also used Remark 4.3. In conclusion,

$$\mathcal{L}^n(Q')|\xi|^p \leq \frac{c}{c_1 \wedge c_3} f_{\rm hom}(\omega, \xi),$$

which gives the lower bound in (5.27) for $\tilde{c}_0 := \frac{c_1 \wedge c_3}{c}$, by letting $Q' \nearrow Q$.

For the proof of the lower bound in (5.28) we proceed similarly. Let $\omega \in \Omega'$, $0 < \sigma < 1$, and $\nu \in \mathbb{S}^{n-1}$, and let R^ν_σ denote the rectangle obtained by shrinking the square Q^ν by the factor σ in the direction ν. Let u_ε be a recovery sequence for $E_\varepsilon(\omega)$ at $u_{0,1,\nu}$ in R^ν_σ, and let $T^\omega_\varepsilon u_\varepsilon$ denote the extension of u_ε in R^ν_σ provided by Theorem 4.2. Again by [2, Theorem 4.7] and by Remark 4.3, for every $R' \subset\subset R^\nu_\sigma$, we have

$$\begin{aligned} MS^p(u_{0,1,\nu}, R') &\leq \liminf_{\varepsilon\to 0} MS^p(T^\omega_\varepsilon u_\varepsilon, R') \leq c \liminf_{\varepsilon\to 0} MS^p(u_\varepsilon, R^\nu_\sigma \setminus \varepsilon K(\omega)) \\ &\leq \frac{c}{c_1 \wedge c_3} \liminf_{\varepsilon\to 0} E_\varepsilon(\omega)(u_\varepsilon, R^\nu_\sigma) = \frac{c}{c_1 \wedge c_3} E_{\rm hom}(\omega)(u_{0,1,\nu}, R^\nu_\sigma), \end{aligned}$$

where the constant c is independent of σ. In conclusion,

$$\mathcal{H}^{n-1}(S_{u_{0,1,\nu}} \cap R') \leq \frac{c}{c_1 \wedge c_3} \left(\sigma f_{\rm hom}(\omega, 0) + g_{\rm hom}(\omega, \nu)\right),$$

which, gives the lower bound in (5.28) for $\tilde{c}_0$ defined above, by letting $R' \nearrow R^\nu_\sigma$ and $\sigma \to 0$.

Step 4: Upper bounds on the limit integrands. The upper bound in (5.27) follows immediately by taking, for $\omega \in \Omega'$ and $\xi \in \mathbb{R}^n$, the sequence $u_\varepsilon = \ell_\xi$ and by using the liminf inequality for E_ε in Q and the bound $(f2)$, since

$$c_2(1 + |\xi|^p) \geq \liminf_{\varepsilon\to 0} E_\varepsilon(\omega)(u_\varepsilon, Q) \geq E_{\rm hom}(\omega)(\ell_\xi, Q) = f_{\rm hom}(\omega, \xi).$$

The proof of the upper bound in (5.28) is completely analogous.

Step 5: Lipschitz continuity of f_{hom}. Property (5.29) follows from the bounds in (5.27) and from the convexity of $f_{\mathrm{hom}}(\omega,\cdot)$, see, e.g., [4, Remark 4.13 (iii)]. □

Remark 5.4. In Theorem 5.3 the L^1-topology can be replaced by the weak convergence in (4.15).

In view of Remark 5.4, as a corollary of Theorem 5.3 we obtain a Γ-convergence result for the following (asymptotically degenerate coercive) functionals.

Let $\varepsilon>0$, and let (α_ε) and (β_ε) be two positive sequences, infinitesimal as $\varepsilon\to 0$. For $\omega\in\Omega$, $x,\xi\in\mathbb{R}^n$ and $\nu\in\mathbb{S}^{n-1}$ we define

$$a_\varepsilon(\omega,x) := \begin{cases} 1 & \text{if } x\in\mathbb{R}^n\setminus K(\omega),\\ \alpha_\varepsilon & \text{if } x\in K(\omega),\end{cases} \qquad b_\varepsilon(\omega,x) := \begin{cases} 1 & \text{if } x\in\mathbb{R}^n\setminus K(\omega),\\ \beta_\varepsilon & \text{if } x\in K(\omega),\end{cases}$$

$f_\varepsilon(\omega,x,\xi) := a_\varepsilon(\omega,x)f(\omega,x,\xi)$ and $g_\varepsilon(\omega,x,\nu) := b_\varepsilon(\omega,x)g(\omega,x,\nu)$.

We now consider the functionals $E_\varepsilon^{\alpha_\varepsilon,\beta_\varepsilon}(\omega)\colon L^1_{\mathrm{loc}}(\mathbb{R}^n)\times\mathcal{A}\longrightarrow(0,+\infty]$ defined as

$$E_\varepsilon^{\alpha_\varepsilon,\beta_\varepsilon}(\omega)(u,A) := \begin{cases} \displaystyle\int_A f_\varepsilon\left(\omega,\frac{x}{\varepsilon},\nabla u\right)dx + \int_{S_u\cap A} g_\varepsilon\left(\omega,\frac{x}{\varepsilon},\nu_u\right)d\mathcal{H}^{n-1} & \text{if } u_{|A}\in GSBV^p(A),\\ +\infty & \text{otherwise.}\end{cases} \tag{5.32}$$

For an overview on the behaviour of the functionals in (5.32) in the deterministic case see [8, 28].

Corollary 5.5. *Let $\Omega'\in\mathcal{T}$ (with $P(\Omega')=1$), f_{hom}, and g_{hom} be as in Theorem 5.1. Then, for every $\omega\in\Omega'$ and every $A\in\mathcal{A}$, the functionals $E_\varepsilon^{\alpha_\varepsilon,\beta_\varepsilon}(\omega)(\cdot,A)$ in* (5.32) *Γ-converge with respect to the weak convergence in* (4.15) *to the homogeneous functional $E_{\mathrm{hom}}(\omega)(\cdot,A)$ defined in* (5.26).

Proof. The liminf inequality follows immediately from Theorem 5.26 and Remark 5.4, due to the lower bound $E_\varepsilon^{\alpha_\varepsilon,\beta_\varepsilon}\geq E_\varepsilon$. Let now $A\in\mathcal{A}$. For the limsup inequality, by a standard truncation argument we can reduce to the case of $u\in SBV^p(A)\cap L^\infty(A)$. Let $(u_\varepsilon)\subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$ be a sequence such that $u_\varepsilon\to u$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\lim_{\varepsilon\to0}E_\varepsilon(\omega)(u_\varepsilon,A)=E_{\mathrm{hom}}(\omega)(u,A)$. With no loss of generality we can assume that $\|u_\varepsilon\|_{L^\infty(A)}\leq\|u\|_{L^\infty(A)}$. Let $(\tilde u_\varepsilon)\subset SBV^p(A)\cap L^\infty(A)$ be the extension provided by Theorem 4.2. By Property 2.10, $\tilde u_\varepsilon\to u$ strongly in $L^1(A)$. Furthermore,

$$\begin{aligned} E_\varepsilon^{\alpha_\varepsilon,\beta_\varepsilon}(\omega)(\tilde u_\varepsilon,A) = E_\varepsilon(\omega)(u_\varepsilon,A) &+ \alpha_\varepsilon\int_{\varepsilon K(\omega)\cap A} f\left(\omega,\frac{x}{\varepsilon},\nabla\tilde u_\varepsilon\right)dx\\ &+ \beta_\varepsilon\int_{S_{\tilde u_\varepsilon}\cap(\varepsilon K(\omega)\cap A)} g\left(\omega,\frac{x}{\varepsilon},\nu_{\tilde u_\varepsilon}\right)d\mathcal{H}^{n-1}.\end{aligned}$$

Since

$$\int_{\varepsilon K(\omega)\cap A} f\left(\omega,\frac{x}{\varepsilon},\nabla\tilde u_\varepsilon\right)dx \leq c_2\int_{\varepsilon K(\omega)\cap A}(1+|\nabla\tilde u_\varepsilon|^p)dx \leq c_2\mathcal{L}^n(A)+c_2MS^p(\tilde u_\varepsilon,A),$$

and

$$\int_{S_{\tilde u_\varepsilon}\cap(\varepsilon K(\omega)\cap A)} g\left(\omega,\frac{x}{\varepsilon},\nu_{\tilde u_\varepsilon}\right)d\mathcal{H}^{n-1}\leq c_4MS^p(\tilde u_\varepsilon,A),$$

by Theorem 4.2 we deduce that

$$\lim_{\varepsilon\to0}E_\varepsilon^{\alpha_\varepsilon,\beta_\varepsilon}(\omega)(\tilde u_\varepsilon,A)=\lim_{\varepsilon\to0}E_\varepsilon(\omega)(u_\varepsilon,A)=E_{\mathrm{hom}}(\omega)(u,A).$$

□

Acknowledgments

The work of C. I. Zeppieri was supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) project number ZE 1186/1-1 and under the Germany Excellence Strategy EXC 2044-390685587, Mathematics Münster: Dynamics–Geometry–Structure. L. Scardia acknowledges support by the EPSRC under the grants EP/V00204X/1 and EP/V008897/1.

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