BMO-TYPE SEMINORMS GENERATING SOBOLEV FUNCTIONS

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Abstract. In the recent literature certain BMO-type seminorms provide characterizations of Sobolev functions. In the same order of ideas, we obtain the norm of the gradient of a function in $L^p(\Omega)$, where $\Omega \subset \mathbb{R}^n$, $n > 1$ and $p > 1$, as limit of BMO-type seminorms involving families of pairwise disjoint sets with arbitrary orientation, the sets being not necessarily cubes or tessellation cells. An analogous result is obtained when rotations are not allowed.

1. Introduction

In [4], the Authors introduced a new function space $\mathcal{B}$, defined through a generalization of the well known space of functions with bounded mean oscillation (BMO) introduced by John and Nirenberg.

For $n \geq 1$, given $Q = (0,1)^n$, $f \in L^1(Q)$ the space B is defined as

$$\mathcal{B} = \left\{ f \in L^1(Q) : \|f\|_B < \infty \right\}$$

where

$$\|f\|_{\mathcal{B}} = \sup_{0<\varepsilon\leq 1} \varepsilon^{n-1} \sup_{\mathcal{F}_\varepsilon} \sum_{Q_\varepsilon \in \mathcal{F}_\varepsilon} \fint_{Q_\varepsilon} \left| f(x) - \fint_{Q_\varepsilon} f \right| dx\,,$$

and the supremum is taken over all collections $\mathcal{F}_\varepsilon$ of disjoint ε-cubes $Q_\varepsilon \subset Q$ with faces parallel to the coordinate axes such that $\sharp\mathcal{F}_\varepsilon \leq \varepsilon^{1-n}$. Clearly, when $n = 1$, $\mathcal{B} = BMO$; otherwise, for $n \geq 2$ the space BMO and BV (the space of bounded variation functions) are strictly embedded in $\mathcal{B}$.

Later, some of the ideas contained in [4] have been extended in [1] in order to give a new characterization of sets of finite perimeter.

Recently, in [10], the Authors introduced a new BMO seminorm and they gave a representation formula of the norm of the gradient for a Sobolev function which does not make use of the distributional derivatives. In particular, given an open set $\Omega \subset \mathbb{R}^n$, a function $f \in L^p_{\text{loc}}(\Omega)$, $p \geq 1$, for any $\varepsilon > 0$, they consider

$$k_\varepsilon(f, p, \Omega) := \varepsilon^{n-p} \sup_{\mathcal{G}_\varepsilon} \sum_{Q' \in \mathcal{G}_\varepsilon} \fint_{Q'} \left| f(x) - \fint_{Q'} f \right|^p dx\,,$$

where the supremum is taken over all families $\mathcal{G}_\varepsilon$ of disjoint ε-cubes Q' of side length ε and arbitrary orientation contained in Ω. They proved that, if $1 < p < \infty$, given $f \in L^p_{\text{loc}}(\Omega)$

$$f \in W^{1,p}(\Omega) \iff \liminf_{\varepsilon\to 0^+} k_\varepsilon(f, p, \Omega) < \infty. \tag{1.1}$$

Moreover, if $f \in W^{1,p}(\Omega)$ and $p \geq 1$, then

$$\lim_{\varepsilon\to 0} k_\varepsilon(f, p, \Omega) = \gamma(n, p) \int_\Omega |\nabla f|^p\, dx,$$

where $\gamma(n, p) := \max_{\nu\in\mathbb{S}^{n-1}} \int_Q |x \cdot \nu|^p\, dx$.

Very recently, a similar representation formula for the gradient norm of a Sobolev function is studied in [6], by considering tessellations of Ω inspired by M.C. Escher, not necessarily generated by cubic cells.

In this paper, we give a representation formula for the gradient norm of a Sobolev function, by considering the more general case of an isotropic family formed by copies of pairwise ε dilation of a bounded connected

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open set with locally Lipschitz boundary D, with arbitrary orientation. We extend the previous representation formulae in [10] and [6] since the sets involved are not necessarily cubes or tessellation cells.

Let $\Omega \subset \mathbb{R}^n$, $n > 1$, be an open set, and let $D \subset \mathbb{R}^n$ be a bounded connected open set with locally Lipschitz boundary. Given a function $f \in L^p(\Omega)$, $1 \leq p < \infty$, for $\varepsilon > 0$ we consider the following seminorm:

$$K_\varepsilon^D(f, p, \Omega) := \varepsilon^{n-p} \sup_{\mathcal{K}_\varepsilon} \sum_{D' \in \mathcal{K}_\varepsilon} \fint_{D'} \left| f(x) - \fint_{D'} f \right|^p dx \,, \tag{1.2}$$

where the supremum on the right hand side is computed over all the families $\mathcal{K}_\varepsilon$ constituted by pairwise disjoint images D' of εD by isometries of $\mathbb{R}^n$ contained in Ω. In this case, we will say that $\mathcal{K}_\varepsilon$ is a collection of disjoint translations D' of εD with *arbitrary orientation*.

Our main result reads as follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ and $D \subset \mathbb{R}^n$ as above. If $p > 1$ and $f \in L^p_{loc}(\Omega)$, then*

$$|\nabla f| \in L^p(\Omega) \iff \liminf_{\varepsilon \to 0} K_\varepsilon^D(f, p, \Omega) < \infty. \tag{1.3}$$

Moreover, if $f \in W^{1,p}(\Omega)$ and $p \geq 1$, we have also

$$\lim_{\varepsilon \to 0} K_\varepsilon^D(f, p, \Omega) = \gamma_p \int_\Omega |\nabla f|^p \, dx \tag{1.4}$$

where $\gamma_p = \gamma_p(n, p, D)$ is a constant such that

$$\gamma_p \leq \frac{1}{|D|^2} \max_{\nu \in \mathbb{S}^{n-1}} \int_D |x \cdot \nu|^p \, dx. \tag{1.5}$$

When $D = Q = (-1/2, 1/2)^n$ is the unit cube, the exact value of γ_p is known and

$$\gamma_p = \max_{\nu \in \mathbb{S}^{n-1}} \int_D |x \cdot \nu|^p \, dx := \gamma(n, p)$$

as stated in Theorem 2.2 of [10]. We recall that the exact value of $\gamma(n, p)$ is known for few values of n and p: it easy to see that $\gamma(n, 1) = 1/4$, $\gamma(n, 2) = 1/12$ and $\gamma(2, p) \geq 2^{1-p/2}/(p + 1)(p + 2)$.

It is interesting to analyze the equality case in (1.5). In order to obtain $\gamma_p = \gamma(n, p)$ it seems to be necessary to "cover" the whole Ω without gaps or overlaps with copies of D. In [6], considering the smaller class of tessellation objects D, the Authors proved a version of Theorem 1.1 with the equality in (1.5). We extend the main result of [6] by considering other techniques.

A crucial fact for the validity of the representation formula (1.4) is that the sets D' can be chosen with arbitary orientation. Things goes very differently when rotations are not allowed. This theme has been further investigated in [2] where the Authors considered the more general case of anisotropic coverings formed by translations of ε-dilation of the set D, to give a new characterization of the perimeter of a measurable set in $\mathbb{R}^n$, see also [8] for an extension of the construction to SBV functions (the space of special BV functions whose gradient measure has no Cantor part). Here, we present a representation formula of the L^p- gradient norm of a Sobolev function considering an anisotropic variant of the BMO–type seminorm, by using families made by translations of a given bounded connected open set with Lipschitz boundary. More precisely, given a function $f \in L^p(\Omega)$, for any $\varepsilon > 0$ we consider the following quantity

$$H_\varepsilon^D(f, p, \Omega) := \varepsilon^{n-p} \sup_{\mathcal{H}_\varepsilon} \sum_{D' \in \mathcal{H}_\varepsilon} \fint_{D'} \left| f(x) - \fint_{D'} f \right|^p dx \,, \tag{1.6}$$

where $\mathcal{H}_\varepsilon$ is any pairwise disjoint family of translations D' of εD contained in Ω.

Note that since D is bounded and Ω is an open set, for ε sufficiently small the family $\mathcal{H}_\varepsilon$ is nonempty and $\mathcal{H}_\varepsilon \subset \mathcal{K}_\varepsilon$. Hence,

$$H_\varepsilon^D(f, p, \Omega) \leq K_\varepsilon^D(f, p, \Omega)$$

We are able to prove the following result.

Theorem 1.2. *Let $p \geq 1$ and $f \in W^{1,p}(\Omega)$. If $D \subset \mathbb{R}^n$ is a bounded connected open set with locally Lipschitz boundary, then there exists a Lipschitz continuous p-homogeneous function $\psi_p^D : \mathbb{R}^n \to [0, +\infty]$, strictly positive on $\mathbb{R}^n \setminus \{0\}$, such that if Ω is an open subset of $\mathbb{R}^n$ then*

$$\lim_{\varepsilon\to 0} H_\varepsilon^D(f, p, \Omega) = \int_\Omega \psi_p^D(\nabla f(x))\, dx. \tag{1.7}$$

In the special case $p = 1$, formula (1.7) is proved in [8].

We observe that when $D = B_1$ is the unitary ball, $H_\varepsilon^{B_1}$ coincides with $K_\varepsilon^{B_1}$ since B_1 is rotational invariant and

$$\lim_{\varepsilon\to 0} K_\varepsilon^{B_1}(f, p, \Omega) = \lim_{\varepsilon\to 0} H_\varepsilon^{B_1}(f, p, \Omega) = \gamma_p \int_\Omega |\nabla f|^p\, dx.$$

In this case, the constant γ_p is less or equal than $\gamma(n, p)$ since the ball is not a "covering"object.

We conclude by observing that for a general BV function f no such representation formula for the total variation of the gradient measure Df may hold (see [9]). However, in Corollary 4.2 we characterize BV functions in terms of the anisotropic variant of the BMO- type seminorm.

2. Preliminaries and Notation

In this section we list some notations and preliminary results useful in the paper.

Given a measurable set $A \subset \mathbb{R}^n$ we denote by $|A|$ its Lebesgue measure. We denoted by $\mathcal{L}^n$ the Lebesgue measure and by $\mathcal{H}^{n-1}$ the Hausdorff $(n-1)$-dimensional measure. By $\sharp K$, we indicate the cardinality of a set K.For a given set $A \subset \mathbb{R}^n$, with $0 < |A| < \infty$, and for a given measurable function $g : A \to \mathbb{R}$, we shall denote by

$$g_A := \fint_A g(x)\, dx := \frac{1}{|A|} \int_A g(x)\, dx$$

the average of g on A.

For $\eta > 0$ and $A \subset \mathbb{R}^n$, we denote by $I_\delta(A)$ the η–neighborhood of A which is defined as follows

$$I_\eta(A) := \{x \in \mathbb{R}^n : \ \mathrm{dist}\,(x, A) < \eta\}.$$

For any $z \in \mathbb{R}^n$, $\nu \in \mathbb{S}^{n-1}$ and $\rho > 0$, we denote by $B_\rho(z)$ the open ball of radius ρ centered in z and by $Q_\nu(z, \rho)$ a generic open cube, centered in z, having sidelenght ρ, and with two faces orthogonal to ν. If the center is at the origin and $\rho = 1$ we shall simply write Q_ν instead of $Q_\nu(0, 1)$. Throughout the paper C will denote a positive constant whose value may change from line to line .

We recall here the following inequalities that will be used in the paper.

Given $\delta \in (0, 1)$, from the convexity of the function $t \to |t|^p$ we get for every $a, b \in \mathbb{R}$

$$|a + b|^p = \left|\frac{1}{(1+\delta)}(1+\delta)a + \frac{\delta}{1+\delta}\frac{1+\delta}{\delta}b\right|^p \leq (1+\delta)^p|a|^p + \frac{(1+\delta)^p}{\delta^p}|b|^p \tag{2.1}$$

Given $\xi, \eta \in \mathbb{R}^n$ it holds

$$\left||\xi|^p - |\eta|^p\right| \leq p\,(|\xi| + |\eta|)^{p-1}\,|\xi - \eta| \tag{2.2}$$

and, given $\xi, \eta \in \mathbb{R}^n \setminus \{0\}$ it holds

$$\left|\frac{\xi}{|\xi|} - \frac{\eta}{|\eta|}\right| \leq 2\frac{|\xi - \eta|}{|\xi|}. \tag{2.3}$$

If $D \subset \Omega$ is a bounded open set with Lipschitz boundary, for any $h \in W^{1,q}(\Omega)$ with $q < n$, the following Sobolev–Poincaré inequality holds (see for example, Problem 7.12 in [11]),

$$\|h - h_D\|_{L^{q^*}(D)} \leq C\|Dh\|_{L^q(D)}\,, \tag{2.4}$$

where the constant C depends on q and D.

The quantities $K_\varepsilon^D(h,p,\Omega)$ and $H_\varepsilon^D(h,p,\Omega)$, defined respectively in (1.2) and (1.6), are strictly related to the L^p norm of the gradient of h. Indeed, one can choose $q<n$ such that $1\le q\le p<q^*$. By Hölder's inequality, we get:

$$\|h\|_{L^p(D)} \le \|h\|_{L^{q^*}(D)}|D|^{\frac{1}{p}-\frac{1}{q^*}} \tag{2.5}$$

and

$$\|\nabla h\|_{L^q(D)} \le \|\nabla h\|_{L^p(D)}|D|^{\frac{1}{q}-\frac{1}{p}}. \tag{2.6}$$

Thus, from (2.4), using (2.5) and (2.6), we get that there exists a constant $C>0$ depending only on D and p such that if $D'=\varepsilon D+x_0\subset\Omega$, then

$$\varepsilon^{n-p}\fint_{D'}\left|h(x)-\fint_{D'}h\right|^p\,dx\le C\int_{D'}|\nabla h|^p$$

with $C=C(p,D)$.

Hence,

$$\varepsilon^{n-p}\sum_{D'\in\mathcal{H}_\varepsilon}\fint_{D'}\left|h(x)-\fint_{D'}h\right|^p\,dx\le C\,\|\nabla h\|^p_{L^p(\Omega)}$$

and thus,

$$H_\varepsilon^D(h,p,\Omega)\le C\,\|\nabla h\|^p_{L^p(\Omega)}\,. \tag{2.7}$$

Similarly,

$$K_\varepsilon^D(h,p,\Omega)\le C\,\|\nabla h\|^p_{L^p(\Omega)}\,. \tag{2.8}$$

3. The functionals $H_\pm$

Given a function $f\in L^p(\Omega)$, $1\le p<\infty$, we define the following quantities

$$H_+^D(f,p,\Omega)=\limsup_{\varepsilon\to0}H_\varepsilon^D(f,p,\Omega)\,,$$

$$H_-^D(f,p,\Omega)=\liminf_{\varepsilon\to0}H_\varepsilon^D(f,p,\Omega)\,.$$

Clearly, we have $H_-^D(f,p,\Omega)\le H_+^D(f,p,\Omega)$.

$$H_\varepsilon^{\lambda D}(f,p,\Omega)=\lambda^{p-n}H^D_{\varepsilon\lambda}(f,p,\Omega)\qquad\qquad H_\pm^{\lambda D}(f,p,\Omega)=\lambda^{p-n}H^D_\pm(f,p,\Omega)\,. \tag{3.1}$$

Throughout the whole paper we shall assume without loss of generality that diam $(D)=1$. Indeed, setting $\tilde D:=D/\mathrm{diam}(D)$, (3.1) gives that

$$H_\varepsilon^D(f,p,\Omega)=(\mathrm{diam}(D))^{p-n}\,H^{\tilde D}_{\varepsilon\,\mathrm{diam}\,D}(f,p,\Omega)\,,\qquad\qquad H_\pm^D(f,p,\Omega)=(\mathrm{diam}(D))^{p-n}\,H^{\tilde D}_\pm(f,p,\Omega)\,.$$

In the following, since the set D is fixed, we drop the superscript D and we only write H_ε and $H_\pm$.

3.1. Properties of the functionals H_ε and $H_\pm$.

We list some properties of $H_\varepsilon(f,p,\Omega)$ and $H_\pm(f,p,\Omega)$, omitting the elementary proofs. To this end we denote by $\mathcal{A}_\Omega$ the family of all open subsets of Ω.

- *Translation invariance*: for any $\tau\in\mathbb{R}^n$, we have
$$H_\varepsilon(f(\cdot-\tau),p,\Omega+\tau)=H_\varepsilon(f,p,\Omega)\quad\text{and}\quad H_\pm(f(\cdot-\tau),p,\Omega+\tau)=H_\pm(f,p,\Omega)\,;$$
- *Monotonicity*: $H_\varepsilon(f,p,\cdot)$ and $H_\pm(f,p,\cdot)$ are increasing with respect to set inclusion;
- *Superadditivity of H_ε*: if $A_1,\,A_2\in\mathcal{A}_\Omega$ and $A_1\cap A_2=\emptyset$, we have
$$H_\varepsilon(f,p,A_1\cup A_2)\ge H_\varepsilon(f,p,A_1)+H_\varepsilon(f,p,A_2)\,;$$
- *Homogeneity*: for any $t>0$, we have
$$H_{t\varepsilon}(f(\cdot/t),p,t\Omega)=t^{n-p}H_\varepsilon(f,p,\Omega)\,;$$

- *Superadditivity of H_-*: if $A_1,\, A_2 \in \mathcal{A}_\Omega$ and $A_1 \cap A_2 = \emptyset$, we have
$$H_-(f,p,A_1\cup A_2) \geq H_-(f,p,A_1) + H_-(f,p,A_2)\,.$$

Using the same argument as in Proposition 3.1 in [8] we obtain the subadditivity of $H_+(f,p,\cdot)$.

Given $A_1, A_2 \in \mathcal{A}_\Omega$ and a function $f \in L^p(\Omega)$, from the definition of H_+ it is plain to see that for any $\delta > 0$
$$H_+(f,p,A_1\cup A_2) \leq H_+(f,p,W_1) + H_+(f,p,W_2)\,. \tag{3.2}$$
where $W_i = I_\delta(A_i)\cap(A_1\cup A_2)$, for $i = 1, 2$.

Proposition 3.1. *Let Ω be an open set. Then for all $A \in \mathcal{A}_\Omega$*
$$H_+(f,p,A) = \sup\{H_+(f,p,A') \,:\, A' \subset\subset A,\, A' \in \mathcal{A}_\Omega\}\,. \tag{3.3}$$
Moreover $H_+(f,p,\cdot)$ is σ-subadditive on $\mathcal{A}_\Omega$.

In the particular case that f is the linear function $f_\nu(x) := x\cdot\nu$ with $x \in \Omega$ and $\nu \in \mathbb{S}^{n-1}$, some of the elementary properties of the functionals H_ε and $H_\pm$ read as

- *Translation invariance*: for any $\tau \in \mathbb{R}^n$, we have
$$H_\varepsilon(f_\nu,p,\Omega+\tau) = H_\varepsilon(f_\nu,p,\Omega) \quad \text{and} \quad H_\pm(f_\nu,p,\Omega+\tau) = H_\pm(f_\nu,p,\Omega)\,; \tag{3.4}$$
- *Homogeneity*: for any $t > 0$
$$H_{t\varepsilon}(f_\nu(\cdot),p,t\Omega) = t^n H_\varepsilon(f_\nu,p,\Omega) \qquad \text{and} \qquad H_\pm(f_\nu,p,t\Omega) = t^n H_\pm(f_\nu,p,\Omega)\,. \tag{3.5}$$

3.2. **Definition of ψ_p^D.** We begin by proving a proposition, where we show that the functionals H_- and H_+ coincide if they act on a linear function $f_\nu(x) = x\cdot\nu$, $\nu\in\mathbb{S}^{n-1}$, and on any unitary cube centered in the origin.

Proposition 3.2. *Let $1 \leq p < +\infty$, $\nu \in \mathbb{S}^{n-1}$ and $f_\nu(x) = x\cdot\nu$. For any unitary cube $\tilde{Q}$ centered in the origin, we have*
$$H_+(f_\nu,p,\tilde{Q}) = H_-(f_\nu,p,\tilde{Q}) = \sup_{0<s\leq 1} H_s(f_\nu,p,\tilde{Q}) < +\infty$$
Moreover $H_+(f_\nu,p,\tilde{Q})$ is bounded from above.

Proof. By (3.5), we have
$$H_-(f_\nu,p,\tilde{Q}) = \liminf_{\varepsilon\to 0} H_\varepsilon(f_\nu,p,\tilde{Q}) = \liminf_{\varepsilon\to 0} \varepsilon^n H_1(f_\nu,p,(1/\varepsilon)\tilde{Q})\,.$$
Fixed $\varepsilon < s \leq 1$, the cube $(1/\varepsilon)\tilde{Q}$ contains the union of at least $\lfloor (s/\varepsilon)\rfloor^n$ open disjoint cubes of side $1/s$ and $H_1(f_\nu,p,(1/s)\tilde{Q}) = H_1(f_\nu,p,z+(1/s)\tilde{Q})$ for any $z \in \mathbb{R}^n$. By the monotonicity in the second argument, the superadditivity of H_1 and the homogeneity, it holds
$$H_-(f_\nu,p,\tilde{Q}) \geq \liminf_{\varepsilon\to 0} \varepsilon^n \lfloor (s/\varepsilon)\rfloor^n H_1(f_\nu,p,(1/s)\tilde{Q}) = t^n H_1(f_\nu,p,(1/s)\tilde{Q}) = H_s(f_\nu,p,\tilde{Q})\,.$$
Hence,
$$H_-(f_\nu,p,\tilde{Q}) \geq \sup_{0<s\leq 1} H_s(f_\nu,p,\tilde{Q})\,.$$
Moreover,
$$H_-(f_\nu,p,\tilde{Q}) \leq H_+(f_\nu,p,\tilde{Q}) = \limsup_{\varepsilon\to 0} H_\varepsilon(f_\nu,p,\tilde{Q}) = \lim_{\varepsilon\to 0}\sup_{0<s<\varepsilon} H_s(f_\nu,p,\tilde{Q}) \leq \sup_{0<s\leq 1} H_s(f_\nu,p,\tilde{Q})\,.$$
So we have $H(f_\nu,p,\tilde{Q}) := H_+(f_\nu,p,\tilde{Q}) = H_-(f_\nu,p,\tilde{Q}) = \sup_{0<s\leq 1} H_s(f_\nu,p,\tilde{Q})$. The upper bound on H_+ is an immediate consequence of (2.7). $\square$

We set now

$$\psi_p : \ \mathbb{S}^{n-1} \ni \nu \ \to \ \psi_p(\nu) := H(f_\nu, p, Q) \in (0, +\infty) , \tag{3.6}$$

where $Q = \left(-\frac{1}{2}, \frac{1}{2}\right)^n$ is the canonical unit cube with edges parallel to the coordinate axes. Our next result shows that the values of the function ψ_p do not depend on the choice of the unitary cube centered in the origin, at the right hand side of (3.6).

Proposition 3.3. *Let $1 \le p < +\infty$, $\nu \in \mathbb{S}^{n-1}$ and $f_\nu(x) = x \cdot \nu$. For any unitary cube $\tilde{Q}$ centered in the origin, we have*

$$\psi_p(\nu) = H(f_\nu, p, \tilde{Q}) .$$

Proof. We can cover the cube Q with m open cubes $x_i + r\tilde{Q}$ up to a set A_r of Lebesgue measure going to 0 as $r \to 0$. By the subadditivity of $H_+(f_\nu, p, \cdot)$, the translation invariance and the homogeneity,

$$H_+(f_\nu, p, Q) \le \sum_{i=1}^{m} H_+(f_\nu, p, x_i + r\tilde{Q}) + H_+(f_\nu, p, A_r) \le H_+(f_\nu, p, \tilde{Q}) m r^n + C|A_r| .$$

Since $mr^n \le 1$, $H_+(f_\nu, p, Q) \le H_+(f_\nu, p, \tilde{Q})$. Interchanging the role of Q and $\tilde{Q}$, we get $H_+(f_\nu, p, Q) = H_+(f_\nu, p, \tilde{Q})$. By Proposition 3.2, we have $H(f_\nu, p, Q) = H(f_\nu, p, \tilde{Q})$. □

We claim now that ψ_p is Lipschitz continuous.

Proposition 3.4. *The function ψ_p is Lipschitz continuous and bounded away from zero.*

Proof. Fix $\nu, \tau \in \mathbb{S}^{n-1}$ and $\delta > 0$. There exists ε_0 such that for $0 < \varepsilon < \varepsilon_0$

$$\psi_p(\nu) - \psi_p(\tau) \le H_\varepsilon(f_\nu, p, Q) - H_\varepsilon(f_\tau, p, Q) + \delta .$$

There exists a family $\mathcal{H}_\varepsilon$ of translated copies D' of εD in Q such that

$$\psi_p(\nu) - \psi_p(\tau) \le \varepsilon^{n-p} \sum_{D' \in \mathcal{H}_\varepsilon} \left[⨍_{D'} \left| f_\nu - ⨍_{D'} f_\nu \right|^p dx - ⨍_{D'} \left| f_\tau - ⨍_{D'} f_\tau \right|^p dx \right] + 2\delta \le C|\nu - \tau| + 2\delta ,$$

where the last estimate follows by the triangular and Poincaré inequalities. The Lipschitz continuity of ψ_p then follows by letting $\delta \to 0$ and then interchanging the role of ν and τ.

To conclude the proof observe that $\frac{1}{2}D \subset Q$ since $\mathrm{diam}(D) = 1$. Therefore, from Proposition 3.2 we have

$$\min_{\nu \in \mathbb{S}^{n-1}} \psi_p(\nu) = \psi_p(\nu_0) \ge H_{\frac{1}{2}}(f_{\nu_0}, p, Q) \ge C ⨍_{\frac{1}{2}D} \left| f_{\nu_0} - ⨍_{\frac{1}{2}D} f_{\nu_0} \right|^p dx > 0 .$$

Indeed, if the latter integral would be zero, then

$$x - ⨍_{\frac{1}{2}D} y \, dy \in \{x \in \mathbb{R}^n : \ x \cdot \nu_0 = 0\}$$

for $\mathcal{L}^n$–a.e. $x \in \frac{1}{2}D$ and this is not the case. □

In the following we remove also the subscript p from ψ_p, denoting it simply by ψ.

Let us consider now the p-homogeneous extension $\tilde{\psi}$ of ψ to $\mathbb{R}^n$, which is defined by setting $\tilde{\psi}(0) = 0$ and

$$\tilde{\psi}(\tau) = |\tau|^p \psi\left(\frac{\tau}{|\tau|}\right), \qquad \text{for all } \tau \in \mathbb{R}^n \setminus \{0\} .$$

4. $W^{1,p}$ functions: the anisotropic case

We start by a simple covering lemma whose elementary proof is omitted.

Lemma 4.1. *Let Ω be a bounded open set, $f \in C^1(\overline{\Omega})$ and $t > 0$. For every $\sigma > 0$ there exist $r > 0$ and a finite family of pairwise disjoint open cubes $Q(x_i; r)$ with edges parallel to coordinate axes contained in $U_t := \{x \in \Omega : |\nabla f(x)| > t\}$, $i = 1, \dots, m$, such that*

$$\Big| U_t \setminus \bigcup_{i=1}^{m} Q(x_i; r) \Big| < \sigma, \tag{4.1}$$

$$|\nabla f(x) - \nabla f(y)| < \sigma \qquad \text{for all } x, y \in Q(x_i; r), \tag{4.2}$$

For a generalized version of this Lemma, see [6].

We can now prove the main result of this section.

Proof of Theorem 1.2. We have to prove that if $p \geq 1$ and $f \in W^{1,p}(\Omega)$, then

$$H_+(f, p, \Omega) = H_-(f, p, \Omega) = \int_\Omega \psi(\nabla f)\, dx\,. \tag{4.3}$$

We divide the proof in two steps.

Step 1. *For $f \in W^{1,p}(\Omega)$, we have*

$$H_-(f, p, \Omega) \geq \int_\Omega \psi(\nabla f)\, dx\,. \tag{4.4}$$

To prove this inequality assume first that Ω is a bounded open set and that $f \in C^1(\overline{\Omega})$. Fix $t > 0$ and $\sigma > 0$ and take the cubes $Q(x_i; r)$, $i = 1, \dots m$, as in Lemma 4.1. Fix i and $\varepsilon > 0$ and consider a family $\mathcal{H}_\varepsilon$ of pairwise disjoint sets D_j of the form $z_j + \varepsilon D \subset Q(x_i; r)$, for $j = 1, \dots, k$. For every $x \in z_j + \varepsilon D$ we may write

$$f(x) = f(z_j) + \nabla f(z_j) \cdot (x - z_j) + R_j(x)\,,$$

where $R_j(x) = (\nabla f(\overline{x}) - \nabla f(z_j)) \cdot (x - z_j)$ for some $\overline{x} \in Q(x_i; r)$. Thus, using the estimate (4.2), we have that $|R_j(x)| \leq \sigma\varepsilon$. Thus, using again (4.2),

$$\begin{aligned}
\varepsilon^{n-p} \sum_{j=1}^{k} \fint_{D_j} \Big| f(x) - \fint_{D_j} f \Big|^p dx &\geq \varepsilon^{n-p} \sum_{j=1}^{k} \fint_{D_j} \Big| \nabla f(z_j) \cdot (x - z_j) - \fint_{D_j} \nabla f(z_j) \cdot (y - z_j)\, dy \Big|^p dx - 2k\sigma^p \varepsilon^n \\
&\geq \varepsilon^{n-p} \sum_{j=1}^{k} \fint_{D_j} \Big| \nabla f(x_i) \cdot (x - z_j) - \fint_{D_j} \nabla f(x_i) \cdot (y - z_j)\, dy \Big|^p dx - Ck\sigma^p \varepsilon^n \\
&\geq \varepsilon^{n-p} |\nabla f(x_i)|^p \sum_{j=1}^{k} \fint_{D_j} \Big| \frac{\nabla f(x_i)}{|\nabla f(x_i)|} \cdot x - \fint_{D_j} \frac{\nabla f(x_i)}{|\nabla f(x_i)|} \cdot y\, dy \Big|^p dx - C\sigma^p r^n\,,
\end{aligned}$$

where in the last inequality we used the fact that $k\varepsilon^n = |\cup_{j=1}^k D_j|/|D| \leq r^n/|D|$. Thus, from the inequality above, taking the supremum with respect to all families $\mathcal{H}_\varepsilon$ and the lim inf with respect to ε, we have

$$H_-(f, p, Q(x_i; r)) \geq r^n |\nabla f(x_i)|^p \psi\left(\frac{\nabla f(x_i)}{|\nabla f(x_i)|} \right) - C\sigma^p r^n \tag{4.5}$$

Now, we observe that

$$\begin{aligned}
&r^n |\nabla f(x_i)|^p \psi\left(\frac{\nabla f(x_i)}{|\nabla f(x_i)|} \right) \\
&\geq \int_{Q(x_i, r)} |\nabla f(x)|^p \psi\left(\frac{\nabla f(x)}{|\nabla f(x)|} \right) dx - \int_{Q(x_i, r)} \left| |\nabla f(x)|^p \psi\left(\frac{\nabla f(x)}{|\nabla f(x)|} \right) - |\nabla f(x_i)|^p \psi\left(\frac{\nabla f(x_i)}{|\nabla f(x_i)|} \right) \right| dx
\end{aligned} \tag{4.6}$$

On the other hand we have

(4.7)
$$\begin{aligned}
&\int_{Q(x_i,r)}\left||\nabla f(x)|^p\psi\left(\frac{\nabla f(x)}{|\nabla f(x)|}\right)-|\nabla f(x_i)|^p\psi\left(\frac{\nabla f(x_i)}{|\nabla f(x_i)|}\right)\right|\,dx\\
&\quad\le\int_{Q(x_i,r)}\psi\left(\frac{\nabla f(x)}{|\nabla f(x)|}\right)\left||\nabla f(x)|^p-|\nabla f(x_i)|^p\right|\,dx+\int_{Q(x_i,r)}|\nabla f(x_i)|^p\left|\psi\left(\frac{\nabla f(x)}{|\nabla f(x)|}\right)-\psi\left(\frac{\nabla f(x_i)}{|\nabla f(x_i)|}\right)\right|\,dx\\
&\quad\le\|\psi\|_{L^\infty(\mathbb{S}^{n-1})}\int_{Q(x_i,r)}\left||\nabla f(x)|^p-|\nabla f(x_i)|^p\right|\,dx+\mathrm{Lip}(\psi)\int_{Q(x_i,r)}|\nabla f(x_i)|^p\left|\frac{\nabla f(x)}{|\nabla f(x)|}-\frac{\nabla f(x_i)}{|\nabla f(x_i)|}\right|\,dx.
\end{aligned}$$

Then, using (2.2) and (2.3), we get

$$\begin{aligned}
&\int_{Q(x_i,r)}\left||\nabla f(x)|^p\psi\left(\frac{\nabla f(x)}{|\nabla f(x)|}\right)-|\nabla f(x_i)|^p\psi\left(\frac{\nabla f(x_i)}{|\nabla f(x_i)|}\right)\right|\,dx\\
&\quad\le 2^{p-1}p\|\psi\|_{L^\infty(\mathbb{S}^{n-1})}\|\nabla f\|^{p-1}_{L^\infty(\Omega)}\sigma r^n+\|\nabla f\|^{p-1}_{L^\infty(\Omega)}\mathrm{Lip}(\psi)\int_{Q(x_i,r)}|\nabla f(x_i)|\left|\frac{\nabla f(x)}{|\nabla f(x)|}-\frac{\nabla f(x_i)}{|\nabla f(x_i)|}\right|\,dx\\
&\quad\le 2^{p-1}p\|\psi\|_{L^\infty(\mathbb{S}^{n-1})}\|\nabla f\|^{p-1}_{L^\infty(\Omega)}\sigma r^n+2\|\nabla f\|^{p-1}_{L^\infty(\Omega)}\mathrm{Lip}(\psi)\sigma r^n\le C\sigma r^n
\end{aligned}\tag{4.8}$$

for some constant C depending on p, the Lipschitz constant of ψ, the L^∞–norms of ∇f and ψ. Finally, taking into account (4.6), by the p–homogeneity of ψ, we obtain

$$H_-(f,p,Q(x_i,r))\ge\int_{Q(x_i,r)}\psi(\nabla f(x))\,dx-C\sigma r^n\,,\tag{4.9}$$

Thus, summing up with respect to i, by the superadditivity of $H_-(f,p,\cdot)$, we get

$$H_-(f,p,\Omega)\ge\sum_{i=1}^m H_-(f,p,Q(x_i,r))\ge\sum_{i=1}^m\int_{Q(x_i,r)}\psi(\nabla f(x))\,dx-C\sigma|\Omega|\ge\int_{U_t}\psi(\nabla f(x))\,dx-C\sigma\,,$$

where the last inequality follows from (4.1) and the constant C depends on $|\Omega|,|D|$, p and on $\|\nabla f\|_{L^\infty(\Omega)}$. Then (4.4) follows at once letting first $\sigma\to 0$ and then $t\to 0$ in the previous inequality.

Without loss of generality, we take now a smooth open set Ω (an exhaustion of Ω by compact and smooth domains is possible thanks to Lemma 1 in [12]) and $f\in W^{1,p}(\Omega)$. Given $\sigma>0$, take $f_\sigma\in C^1(\Omega)$ such that $\|f-f_\sigma\|_{W^{1,p}(\Omega)}<\sigma$. Given an open bounded set $A\subset\subset\Omega$, for any family $\mathcal{H}_\varepsilon$ of pairwise disjoint sets $D'\subset A$ of the type $z+\varepsilon D$ we have

$$\begin{aligned}
\varepsilon^{n-p}\sum_{D'\in\mathcal{H}_\varepsilon}\fint_{D'}\left|f(x)-\fint_{D'}f\right|^p\,dx&\ge\varepsilon^{n-p}\frac{1}{(1+\delta)^p}\sum_{D'\in\mathcal{H}_\varepsilon}\fint_{D'}\left|f_\sigma(x)-\fint_{D'}f_\sigma\right|^p\,dx+\\
&\qquad-\varepsilon^{n-p}\frac{1}{\delta^p}\sum_{D'\in\mathcal{H}_\varepsilon}\fint_{D'}\left|(f-f_\sigma)(x)-\fint_{D'}(f-f_\sigma)\right|^p\,dx\\
&\ge\frac{\varepsilon^{n-p}}{(1+\delta)^p}\sum_{D'\in\mathcal{H}_\varepsilon}\fint_{D'}\left|f_\sigma(x)-\fint_{D'}f_\sigma\right|^p\,dx+\\
&\qquad-\frac{C(D)\varepsilon^n}{\delta^p}\sum_{D'\in\mathcal{H}_\varepsilon}\fint_{D'}|\nabla f(x)-\nabla f_\sigma(x)|^p\,dx\\
&\ge\frac{\varepsilon^{n-p}}{(1+\delta)^p}\sum_{D'\in\mathcal{H}_\varepsilon}\fint_{D'}\left|f_\sigma(x)-\fint_{D'}f_\sigma\right|^p\,dx-\frac{C(D)}{|D|\delta^p}\int_A|\nabla f(x)-\nabla f_\sigma(x)|^p\,dx\,,
\end{aligned}$$

where $C(D)$ is the Poincaré constant of D. Thus, passing to the supremum with respect to the family $\mathcal{H}_\varepsilon$ and letting ε tend to 0 we have, from (4.4) applied to f_σ we have

$$H_-(f,p,\Omega) \geq \frac{1}{(1+\delta)^p} H_-(f_\sigma,p,A) - \frac{C}{\delta^p}\|f-f_\sigma\|_{W^{1,p}(A)} \geq \frac{1}{(1+\delta)^p}\int_A \psi(\nabla f_\sigma(x))\,dx - \frac{C\sigma}{\delta^p}\,,$$

where the constant C depends only on D. Recalling Proposition 3.4, (4.4) then follows letting first $\sigma \to 0$, $\delta \to 0$ and then $A \uparrow \Omega$.

Step 2. *If $f \in W^{1,p}(\Omega)$, then*

$$H_+(f,p,\Omega) \leq \int_\Omega \psi(\nabla f)\,dx\,. \tag{4.10}$$

Using (3.3) and the same argument of the previous step, we may always assume that Ω is a bounded open set with locally Lipschitz boundary. Moreover, by an approximation argument similar to the one used in the final part of the previous step, we may assume without loss of generality that $f \in C^1(\Omega)$.

Recall that $|\partial U_t| = 0$ for all but countably many $t > 0$. Then fix t so that $|\partial U_t| = 0$ and $\sigma > 0$ and consider the same cubes $Q(x_i;r)$, $i = 1,\dots m$, as before. Using the subadditivity of $H_+(f,p,\cdot)$ we have

$$H_+(f,p,\Omega) \leq \sum_{i=1}^m H_+(f,p,Q(x_i,r)) + H_+(f,p,\Omega\setminus\overline{U}_t) + H_+(f,p,W_t)\,,$$

where $W_t \subset \Omega$ is an open set such that $\overline{U}_t \setminus \cup_{i=1}^m Q(x_i,r) \subset W_t$ and $|W_t| < \sigma$. Note that this choice of W_t is possible thanks to (4.1) and to the fact that $|\partial U_t| = 0$. Then, arguing as in the proof of (4.5) we get that for every $i = 1,\dots,m$

$$H_+(f,p,Q(x_i;r)) \leq \int_{Q(x_i;r)} \psi(\nabla f(x))\,dx + C\sigma r^n\,,$$

for some positive constant C depending only on $|D|$, p and $\|\nabla f\|_{L^\infty(\Omega)}$. Thus, from the two previous inequalities, recalling (2.7), we have

$$H_+(f,p,\Omega) \leq \int_\Omega \psi(\nabla f(x))\,dx + C\sigma|\Omega| + C\int_{\{|\nabla f|\leq t\}} |\nabla f(x)|^p\,dx + C\sigma^p\|\nabla f\|_{L^\infty(\Omega)}\,.$$

Then (4.10) follows letting first $\sigma \to 0$ and then $t \to 0$. □

As a consequence of Theorem 1.2 we obtain the following Corollary.

Corollary 4.2. *Let $p > 1$ and $f \in L^p(\Omega)$. If $H_-(f,p,\Omega)$ is finite then $f \in W^{1,p}(\Omega)$. Conversely, if $f \in W^{1,p}((\Omega)$ then $H_+(f,\Omega)$ is finite.*

Proof. We prove only that if $H_-(f,p,\Omega) < \infty$ then f is in $W^{1,p}(\Omega)$, since the other implication follows at once from (2.7). Fix an open set $A \subset\subset \Omega$ and $0 < \sigma < \mathrm{dist}(A,\partial\Omega)$. For all $x \in A$ set $f_\sigma(x) = (\varrho_\sigma * f)(x)$, where ϱ is a standard mollifier with compact support in the unit ball B and $\varrho_\sigma(x) = \sigma^{-n}\varrho(x/\sigma)$. Then, given any family $\mathcal{H}_\varepsilon$ of pairwise disjoint sets D' of the form $z + \varepsilon D' \subset A$, using the definition of f_σ, Jensen inequality and Fubini's theorem, we get, recalling that $\int_B \varrho\,dx = 1$,

$$\begin{aligned}
\varepsilon^{n-p}\sum_{D'\in\mathcal{H}_\varepsilon} \fint_{D'}\left|f_\sigma(x) - \fint_{D'} f_\sigma\right|^p dx &= \varepsilon^{n-p}\sum_{D'\in\mathcal{H}_\varepsilon}\fint_{D'}\left|\int_B \varrho(y)f(x-\sigma y)\,dy - \fint_{D'}\int_B \varrho(y)f(z-\sigma y)\,dydz\right|^p dx \\
&\leq \varepsilon^{n-p}\int_B \varrho(y)\left(\sum_{D'\in\mathcal{H}_\varepsilon}\fint_{D'}\left|f(x-\sigma y) - \fint_{D'} f(z-\sigma y)\,dz\right|^p dx\right)dy \\
&= \varepsilon^{n-p}\int_B \varrho(y)\left(\sum_{D'\in\mathcal{H}_\varepsilon}\fint_{D'-\sigma y}\left|f(x) - \fint_{D'-\sigma y} f\right|^p dx\right)dy \leq H_\varepsilon(f,p,\Omega)\,.
\end{aligned}$$

Therefore, taking the supremum over all families $\mathcal{H}_\varepsilon$, recalling that ψ is bounded away from zero and letting $\varepsilon \to 0$, we get for all $\sigma > 0$ sufficiently small

$$\int_A |\nabla f_\sigma|^p \, dx \le C \int_A \psi(\nabla f_\sigma) \, dx = CH_-(f_\sigma, p, A) \le CH_-(f, p, \Omega) .$$

Hence the conclusion follows by letting first $\sigma \to 0$ and then $A \uparrow \Omega$. □

We conclude by observing that Corollary 4.2 does not hold in general if $f \in W^{1,1}$. In fact, [8, Corollary 4.2], the Authors characterize the functions in $BV(\Omega)$ as the function $f \in L^1(\Omega)$ such that $H_+(f, 1, \Omega)$ is finite (see also [7]). As a consequence, it is possible to show the following characterization of functions of bounded variation.

Corollary 4.3. *The following are equivalent*

i) $f \in BV(\Omega)$*;*

ii)

$$\sup_{\mathcal{H}_\varepsilon} \sum_{D' \in \mathcal{H}_\varepsilon} \varepsilon^{n-1} \fint_{D'} |f - f_{D'}| < +\infty;$$

iii)

$$\sup_{\mathcal{G}_\varepsilon} \sum_{D' \in \mathcal{G}_\varepsilon} \|f - f_{D'}\|_{L^{\frac{n}{n-1}}(D')} < \infty$$

where $f_{D'} := \fint_{D'} f$ *and the supremum is taken over all families* $\mathcal{G}_\varepsilon$ *of disjoint images* D' *of* εD *by isometries of* $\mathbb{R}^n$ *contained in* Ω*.*

Proof. The equivalence *i)* $\Leftrightarrow$ *ii)* is proved by Corollary 4.2 in [8].

We prove now that *i)* $\Rightarrow$ *iii)*. By Poincaré–Wirtinger inequality, we have that for any $f \in BV(D)$,

$$\|f - f_D\|_{L^{\frac{n}{n-1}}(D)} \le C|Df|(D) \tag{4.11}$$

where C is a constant depending only on D.

Then, by (4.11), we obtain

$$\|f - f_{D'}\|_{L^{\frac{n}{n-1}}(D')} \le C|Df|(D')$$

where C is a constant depending only on D. Then, summing over all sets D' in $\mathcal{G}_\varepsilon$, we obtain *iii)*.

It remains to prove that *iii)* $\Rightarrow$ *ii)*. By Hölder's inequality

$$\varepsilon^{n-1} \fint_{D'} |f - f_{D'}| \, dx \le C\|f - f_{D'}\|_{L^{\frac{n}{n-1}}(D')}, \tag{4.12}$$

where C is a constant depending only on D. The conclusion follows again by summing over all sets D' in $\mathcal{G}_\varepsilon$. □

5. $W^{1,p}$ FUNCTIONS: THE ISOTROPIC CASE

Proof of Theorem 1.1. Let $p \ge 1$ and $f \in W^{1,p}$. We observe that, since ψ_p^D is a p-homogeneous function, with a slight abuse of notation we shall still denote by $\psi_p^D : \mathbb{S}^{n-1} \to [0, +\infty)$ the restriction of ψ_p^D to the unitary sphere. Hence (1.7) is equivalent to

$$\lim_{\varepsilon \to 0} H_\varepsilon^D(f, p, \Omega) = \int_\Omega |\nabla f|^p \psi_p^D \left(\frac{\nabla f(x)}{|\nabla f(x)|} \right) dx. \tag{5.1}$$

Following along the line the proof of Proposition 3.2, we can redefine

$$\tilde{\psi}_p^D := K(f_\nu, p, \tilde{Q}) = \sup_{0<s\le 1} K_s(f_\nu, p, \tilde{Q}).$$

Moreover, retrace the proof of Theorem 1.2, we have

$$\lim_{\varepsilon\to 0} K_\varepsilon^D(f,p,\Omega) = \int_\Omega |\nabla f|^p \tilde{\psi}_p^D\left(\frac{\nabla f(x)}{|\nabla f(x)|}\right) dx.$$

We observe now that, since in the family $\mathcal{K}_\varepsilon$ considered in the functional K_ε^D rotations are allowed, the function $\tilde{\psi}_p(\nu)$ is constant. Indeed, for any $\nu,\tau \in \mathbb{S}^{n-1}$, we denote by O the rotation that takes ν into τ and we have

$$\fint_{x_0+\varepsilon R(D)} \left| x\cdot\nu - \fint_{x_0+\varepsilon R(D)} t\cdot\nu \right|^p dx = \fint_{\varepsilon O^{-1}(R(D))} \left| y\cdot\tau - \fint_{\varepsilon O^{-1}(R(D))} z\cdot\tau \right|^p dy\,.$$

Therefore multiplying by ε^{n-p}, summing up above all possible $R(D)$ and passing to the supremum on $\mathcal{K}_\varepsilon$ and ε we have proved that $\tilde{\psi}_p^D(\nu) \le \tilde{\psi}_p^D(\tau)$. Interchanging the role of ν and τ we obtain that $\tilde{\psi}_p^D$ is constant.

It remains to prove that $\tilde{\psi}_p^D \le \frac{1}{|D|^2}\max_{\nu\in\mathbb{S}^{n-1}} \int_D |x\cdot\nu|^p\,dx$. Without loss of generality we can assume that the barycenter of D is zero: it is sufficient to observe that $K_\varepsilon^D(f_\nu,p,Q) = K_\varepsilon^{x_0+D}(f_\nu,p,Q)$. By a change of variable and observing that $\varepsilon^n|D|\sharp\mathcal{K}_\varepsilon \le 1$, it easy to obtain that

$$\sum_{D'\in\mathcal{K}_\varepsilon} \varepsilon^{n-p} \fint_{\varepsilon D} \left| x\cdot\nu - \fint_{\varepsilon D} t\cdot\nu\right|^p dx = \sum_{D'\in\mathcal{K}_\varepsilon} \frac{\varepsilon^n}{|D|}\int_D |x\cdot\nu|^p\,dx \le \frac{1}{|D|^2}\int_D |x\cdot\nu|^p\,dx, \tag{5.2}$$

Taking the supremum on $\mathcal{K}_\varepsilon$ and ε, we obtain

$$\tilde{\psi}^D \le \frac{1}{|D|^2}\max_{\nu\in\mathbb{S}^{n-1}} \int_D |x\cdot\nu|^p\,dx.$$

The other implication in (1.3) follows repeating the same arguments as in Corollary 4.2. □

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