RIGIDITY OF SOME FUNCTIONAL INEQUALITIES ON RCD SPACES

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ABSTRACT. We study the cases of equality and prove a rigidity theorem concerning the 1-Bakry-Émery inequality. As an application, we prove the rigidity and identify the extremal functions of the Gaussian isoperimetric inequality, the logarithmic Sobolev inequality and the Poincaré inequality in the setting of $\mathrm{RCD}(K,\infty)$ metric measure spaces. This unifies and extends to the non-smooth setting the results of Carlen-Kerce [20], Morgan [43], Bouyrie [19], Ohta-Takatsu [44], Cheng-Zhou [23].

Examples of non-smooth spaces fitting our setting are measured-Gromov Hausdorff limits of Riemannian manifolds with uniform Ricci curvature lower bound, and Alexandrov spaces with curvature lower bound. Some results including the rigidity of the 1-Bakry-Émery inequality, the rigidity of Φ-entropy inequalities are of particular interest even in the smooth setting.

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1. Introduction

In this paper, we prove some rigidity theorems concerning the 1-Bakry-Émery inequality and some other important functional inequalities on $\mathrm{RCD}(K,\infty)$ metric measure spaces for positive K. Metric measure space satisfying Riemannian curvature-dimension condition $\mathrm{RCD}(K,\infty)$ was introduce by Ambrosio-Gigli-Savaré in [9], as a refinement of the Lott-Sturm-Villani's $\mathrm{CD}(K,\infty)$ condition introduced in [42] and [48]. Important examples of spaces satisfying $\mathrm{RCD}(K,\infty)$ condition include: measured-Gromov Hausdorff limits of Riemannian manifolds with $\mathrm{Ric} \geq K$ (c.f. [32]), Alexandrov spaces with curvature $\geq K$ (c.f. [50]). We refer the readers to the survey [1] for an overview of this fast-growing field and bibliography.

Let us briefly explain the primary motivation of this paper. It is now well-known that the Bakry-Émery theory is an efficient tool in the study of geometric and functional inequalities (c.f. [14] and [15]). Many important inequalities such as the logarithmic-Sobolev inequality and the Gaussian isoperimetric inequality, have proofs using heat flow or the Γ_2-calculus of Bakry-Émery. It was noticed (e.g. by Otto-Villani [45] and Bouyrie [19]) that the cases of equality in the Γ_2-inequality $\Gamma_2 \geq K\Gamma$ is a key to proving rigidity of these inequalities. More precisely, if there is a function attaining the equality in one of these inequalities, there exists a (**possibly different**) function attaining the equality in the Γ_2-inequality. For example, when $K > 0$, any extreme function $f = f_p$ attaining the equality in the sharp Poincaré inequality

$$\int f^2 \,\mathrm{d}\mathfrak{m} \leq \frac{1}{K} \int \Gamma(f) \,\mathrm{d}\mathfrak{m} \tag{1.1}$$

satisfies $\Gamma_2(f_p) = K\Gamma(f_p)$, and any extreme function $f = f_l$ attaining the equality in the sharp logarithmic-Sobolev inequality

$$\int f \ln f \,\mathrm{d}\mathfrak{m} \leq \frac{1}{2K} \int \frac{\Gamma(f)}{f} \,\mathrm{d}\mathfrak{m} \tag{1.2}$$

satisfies $\Gamma_2(\ln f_l) = K\Gamma(\ln f_l)$.

An interesting observation is that both f_p, f_l attain the equality in the **same** 1-Bakry-Émery inequality

$$\sqrt{\Gamma(P_t f)} \leq e^{-Kt} P_t \sqrt{\Gamma(f)} \tag{1.3}$$

where $(P_t)_{t\geq 0}$ is the heat flow associated with the Dirichlet form $\mathbb{E}(\cdot) := \int \Gamma(\cdot) \,\mathrm{d}\mathfrak{m}$ and the 'carré du champ' Γ. Furthermore, both $\mathrm{div}\left(\frac{\nabla P_t f_p}{|\mathrm{P}_t f_p|}\right)$ and $\mathrm{div}\left(\frac{\nabla P_t f_l}{|\mathrm{P}_t f_l|}\right)$ attain the equalities in the Γ_2-inequality and the 2-Bakry-Émery inequality. The main aim of this paper is to understand this observation in general cases and an abstract framework.

1.1. Bakry-Émery's curvature criterion.

Let $(M, \mathrm{g}, e^{-V}\mathrm{Vol}_{\mathrm{g}})$ be a weighted Riemannian manifold equipped with a weighted volume measure $e^{-V}\mathrm{Vol}_{\mathrm{g}}$. The canonical diffusion operator associated with this smooth metric measure space is $\mathrm{L} = \Delta - \nabla V$, where Δ is the Laplace-Beltrami operator. We say that $(M, \mathrm{g}, e^{-V}\mathrm{Vol}_{\mathrm{g}})$ satisfies the $\mathrm{BE}(K,\infty)$ condition for some $K \in \mathbb{R}$, in the sense of Bakry-Émery if

$$\mathrm{Ric}_V := \mathrm{Ric} + \mathrm{Hess}_V \geq K,$$

where Ric denotes the Ricci curvature tensor and Hess_V denotes the Hessian of V.

There are several equivalent characterizations of $\mathrm{BE}(K,\infty)$ condition, which have their own advantages in studying different problems. For example, the following ones are known to be equivalent to the $\mathrm{BE}(K,\infty)$ curvature criterion. Even in the non-smooth $\mathrm{RCD}(K,\infty)$ framework, these characterizations are equivalent (in proper forms), see [9, 10, 35, 47] for more discussions on this topic.

a) Γ_2-inequality: $\Gamma_2(f) \geq K\Gamma(f)$ for all $f \in C_c^\infty(M)$, where Γ_2 and Γ are defined by

$$\Gamma_2(f) := \frac{1}{2}\mathrm{L}\Gamma(f,f) - \Gamma(f, \mathrm{L}f), \qquad \Gamma(f,f) := \frac{1}{2}\mathrm{L}(f^2) - f\mathrm{L}f = \mathrm{g}(\nabla f, \nabla f).$$

b) p-Bakry-Émery inequality for $p > 1$:

$$\sqrt{\Gamma(P_t f)}^p \leq e^{-pKt} P_t\big(\sqrt{\Gamma(f)}^p\big), \qquad \forall\, f \in W^{1,p}(M, e^{-V}\mathrm{Vol}_{\mathrm{g}}) \tag{1.4}$$

where $(P_t)_{t>0}$ is the semigroup generated by the diffusion operator L.

c) 1-Bakry-Émery inequality:

$$\sqrt{\Gamma(P_t f)} \leq e^{-Kt} P_t\big(\sqrt{\Gamma(f)}\big), \qquad \forall\, f \in W^{1,1}(M, e^{-V}\mathrm{Vol}_{\mathrm{g}}). \tag{1.5}$$

Naturally, one would ask the following questions: what if the equalities hold in these different characterizations of $\mathrm{BE}(K,\infty)$? It will not be surprising that the equalities in Γ_2-inequality, 2-Bakry-Émery inequality, and some other 'second-order' inequalities, are all equivalent and any non-constant extreme function is affine and induces a splitting map. For any $p > 1$, by Hölder inequality, the equality in the p-Bakry-Émery inequality yields the equality in the 1-Bakry-Émery inequality. Conversely, from the examples of the Poincaé inequality and the log-Sobolev inequality, the equality in the 1-Bakry-Émery inequality is **strictly weaker** than the equality in the 2-Bakry-Émery inequality. So we would ask: what if the equality in the 1-Bakry-Émery inequality is attained by a non-constant function. Inspired by a recent work of Ambrosio-Brué-Semola [2] concerning $\mathrm{RCD}(0,N)$ spaces, we conjecture that on an $\mathrm{RCD}(K,\infty)$ space with $K > 0$, the existence of a non-constant function attaining the equality in the 1-Bakry-Émery inequality yields the splitting theorem.

In the first theorem, we prove the rigidity of the 1-Bakry-Émery inequality on dimension-free $\mathrm{RCD}(K,\infty)$ spaces with $K > 0$.

Theorem 1.1 (Lemma 2.9, Theorem 3.7, Proposition 3.13, 3.14)**.** *Let $(X, \mathrm{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K,\infty)$ probability space with $K > 0$. Let $u \in \mathrm{D}(\Delta)$ be a non-constant function with $\Delta u \in \mathbb{V}$. Then the following statements are equivalent.*

(1) (Γ_2-inequality) $\Gamma_2(u;\varphi) = K\int \varphi\Gamma(u)\,\mathrm{d}\mathfrak{m}$ for any $\varphi \in L^\infty$ with $\Delta\varphi \in L^\infty$;

(2) $\int(\Delta u)^2\,\mathrm{d}\mathfrak{m} = K\int\Gamma(u)\,\mathrm{d}\mathfrak{m}$;

(3) (Spectral gap) $-\Delta u = Ku$;

(4) (Poincaré inequality) $\int\Gamma(u)\,\mathrm{d}\mathfrak{m} = K\int u^2\,\mathrm{d}\mathfrak{m}$;

(5) (2-Bakry-Émery inequality) $\Gamma(P_t u) = e^{-2Kt}P_t\Gamma(u)$ for some $t > 0$.

If u satisfies one of the properties above, it holds

(6) (1-Bakry-Émery inequality) $\sqrt{\Gamma(P_t u)} = e^{-Kt}P_t\sqrt{\Gamma(u)}$ for all $t > 0$;

(7) $\mathbf{Ric}(u,u) = K\Gamma(u)\,\mathrm{d}\mathfrak{m}$;

(8) u is an affine function, this means $\mathrm{Hess}_u = 0$ and $\Gamma(u)$ is a positive constant;

(9) the gradient flow of u induces a one-parameter semigroup of isometries of (X,d).

If u attains the equality in the 1-Bakry-Émery inequality (6), we have

(10) $\frac{\nabla P_t u}{|\nabla P_t u|} =: b$ does not depend on $t > 0$;

(11) $\Delta\mathrm{div}(b) = -K\mathrm{div}(b)$, thus $\mathrm{div}(b)$ attains the equality in the 2-Barky-Émery inequality;

(12) $\nabla \mathrm{div}(b) = -Kb$;

(13) *there exists an* $\mathrm{RCD}(K, \infty)$ *probability space* $(Y, \mathrm{d}_Y, \mathfrak{m}_Y)$, *such that the metric measure space* $(X, \mathrm{d}, \mathfrak{m})$ *is isometric to the product space*

$$\left(\mathbb{R}, |\cdot|, \sqrt{K/(2\pi)} \exp(-Kt^2/2)\,\mathrm{d}t\right) \times (Y, \mathrm{d}_Y, \mathfrak{m}_Y)$$

equipped with the L^2*-product metric and the product measure;*

(14) u *can be represented in the coordinate of the product space* $\mathbb{R} \times Y$ *by*

$$u(r, y) = \int_0^r g(s)\,\mathrm{d}s \qquad \forall (r, y) \in \mathbb{R} \times Y$$

for some non-negative $g \in L^2(\mathbb{R}, \sqrt{K/(2\pi)} \exp(-Kt^2/2)\,\mathrm{d}t)$. *In particular, if* u *attains equality in the 2-Bakry-Émery inequality, there is a constant* C *such that*

$$P_t u(r, y) = Ce^{Kt} r \qquad \forall (r, y) \in \mathbb{R} \times Y, \quad t > 0.$$

1.2. **Gaussian isoperimetric inequality.** For $K > 0$, let $\phi_K(t) = \sqrt{\frac{K}{2\pi}} \exp(-\frac{Kt^2}{2})$ be a Gaussian-type (probability) density function on $\mathbb{R}$. It is known that $(\mathbb{R}, |\cdot|, \phi_K \mathcal{L}^1)$ is a model space with synthetic Ricci curvature lower bound K.

Let Φ_K denote the error function

$$\Phi_K(t) := \int_{-\infty}^t \phi_K(s)\,\mathrm{d}s.$$

It can be seen that Φ_K is continuous and strictly increasing, so its inverse Φ_K^{-1} is well-defined. We define the Gaussian isoperimetric profile $I_K : (0, 1) \mapsto [0, \sqrt{\frac{K}{2\pi}}]$ by

$$I_K(t) := \phi_K \circ \Phi_K^{-1}(t), \tag{1.6}$$

and we define $I_K(t) = 0$ for $t = 0, 1$. It can be seen that $I_K = \sqrt{K} I_1$ and $I_K'' I_K = -K$. In particular, $I_K(t)$ is strictly concave in t and increasing in K.

Let $\gamma_n = \Pi_{i=1}^n \phi_1(x_i)\mathrm{d}x_i$ be the n-dimensional standard Gaussian measure on $\mathbb{R}^n$. Based on an isoperimetric inequality on the discrete cube and central limit theorem, Bobkov [17] proved the following functional version of the Gaussian isoperimetric inequality

$$I_1\left(\int f\,\mathrm{d}\gamma_n\right) \le \int \sqrt{I_1(f)^2 + |\nabla f|^2}\,\mathrm{d}\gamma_n \tag{1.7}$$

for any Lipschitz function f on $(\mathbb{R}^n, |\cdot|, \gamma_n)$ with values in $[0, 1]$.

In [16], Bakry and Ledoux proved the Bobkov's inequality (1.7) on smooth metric measure spaces using a semigroup method. Recently, by adopting the argument of Bakry-Ledoux, Ambrosio-Mondino [11] obtain the Bobkov's inequality in the non-smooth $\mathrm{RCD}(K, \infty)$ setting.

One interesting problem is: when does the equality hold in the Bobkov's inequality (1.7)? In [20, Section 2], by extending ideas of Ledoux [38], Carlen and Kerce characterized the cases of equality in (1.7) for Gaussian space. Recently, Carlen-Kerce's technique is adopted by Bouyrie [19] to study this problem on weighted Riemannian manifolds satisfying the $\mathrm{BE}(K, \infty)$ condition with $K > 0$.

In this paper, we will study the cases of equality in the Bobkov's inequality on $\mathrm{RCD}(K,\infty)$ spaces. We will identify all the extremal functions, and prove that any non-trivial extreme function induces an isometry map from this space to a product space.

Let us explain how to formulate Bobkov's inequality on an $\mathrm{RCD}(K,\infty)$ metric measure space $(X,\mathrm{d},\mathfrak{m})$. Denote by $\mathbb{V}$ the space of 2-Sobolev functions, defined as the collection of functions $f \in L^2(X,\mathfrak{m})$ such that there exists a sequence $(f_n)_n \subset \mathrm{Lip}(X,\mathrm{d})$ converging to f in L^2 and $\mathrm{lip}(f_n) \to G$ in L^2 for some G, where $\mathrm{lip}(f_n)$ is the local Lipschitz constant of f_n defined by

$$\mathrm{lip}(f_n)(x) := \varlimsup_{y\to x} \frac{|f_n(y)-f_n(x)|}{\mathrm{d}(y,x)}$$

(and we define $\mathrm{lip}(f_n)(x) = 0$ if x is an isolated point). It is known that there exists a minimal function in $\mathfrak{m}$-a.e. sense, denoted by $|\nabla f|$, called minimal weak upper gradient. If (X,d) is a Riemannian manifold and $\mathfrak{m} = \mathrm{Vol}_{\mathrm{g}}$ is its volume measure, we know that $|\nabla f| = \mathrm{lip}(f)$ for any $f \in \mathrm{Lip}$ (c.f. [22, Theorem 6.1]).

On $\mathrm{RCD}(K,\infty)$ spaces, it is known that (c.f. [8, 9]) the functional $\mathbb{V} \ni f \mapsto \mathbb{E}(f) = \int |\nabla f|^2 \,\mathrm{d}\mathfrak{m}$ is lower semi-continuous (w.r.t. weak L^2-convergence), and it is a quasi-regular, strongly local, conservative Dirichlet form admitting a carré du champ $\Gamma(f) := |\nabla f|^2$.

Let $(P_t)_{t\geq 0}$ be the L^2-gradient flow of $\mathbb{E}$ with generator Δ. If $(X,\mathrm{d},\mathfrak{m})$ is a smooth Riemannian manifold with boundary, it is known that (P_t) is the Neumann heat flow and Δ is the (Neumann) Laplace-Beltrami operator. For any $f \in L^1$ with values in $[0,1]$ and $K > 0$, we define $J_K(f) \in [0,+\infty]$ by

$$J_K(f) := \varliminf_{t\to 0} \int \sqrt{I_K(P_t f)^2 + |\nabla P_t f|^2}\,\mathrm{d}\mathfrak{m}. \tag{1.8}$$

Definition 1.2 (Bobkov's inequality on metric measure spaces)**.** We say that a general metric measure space $(X,\mathrm{d},\mathfrak{m})$ supports the K-Bobkov's isoperimetric inequality if for all measurable $f \in L^1(X,\mathfrak{m})$ with values in $[0,1]$,

$$I_K\left(\int f\,\mathrm{d}\mathfrak{m}\right) \leq J_K(f). \tag{1.9}$$

Remark 1.3. It is known that $\mathfrak{m}(X) < \infty$ if $(X,\mathrm{d},\mathfrak{m})$ satisfies $\mathrm{RCD}(K,\infty)$ with $K > 0$ (c.f. [48, Theorem 4.26]). Without loss of generality, we can assume that $\mathfrak{m}$ is a probability measure. Furthermore, the assumption '$f \in L^1(X,\mathfrak{m})$' in Definition 1.2 could be removed.

Applying (1.9) with a characteristic function $f = \chi_E$ for a Borel set $E \subset X$, we get the following *Gaussian isoperimetric inequality*

$$P(E) \geq I_K\big(\mathfrak{m}(E)\big) \tag{1.10}$$

where $P(E)$ is the perimeter function defined by $P(E) := |\mathrm{D}\chi_E|_{\mathrm{TV}}(X)$, and $|\mathrm{D}\chi_E|_{\mathrm{TV}}$ is the total variation of χ_E (c.f. [3, 4] for more details above BV functions and the perimeter function on metric measure spaces).

By lower semi-continuity of weak gradients and the Bakry-Émery's gradient estimate $|\mathrm{lip}(P_t f)|^2 \leq e^{-2Kt}P_t\big(|\nabla f|^2\big)$ (see [9, Theorem 6.2]), we can see that

$$J_K(f) = \int \sqrt{I_K(f)^2 + |\nabla f|^2}\,\mathrm{d}\mathfrak{m}$$

for $f \in \mathrm{Lip}$. In addition, we can see that the Bakry-Émery's gradient estimate yields the irreducible of $\mathbb{E}$, i.e. $|\nabla f| = 0$ implies that f is constant. Since irreducibility implies ergodicity of the heat flow (see for instance [15, Section 3.8]), we know $P_t f \to \int f \,\mathrm{d}\mathfrak{m}$ in L^2 as $t \to \infty$. Notice that by 2-Bakry-Émery inequality, $\lim_{n\to\infty} |\nabla P_t f| = 0$ in L^2. Thus we get

$$\varliminf_{t\to\infty} \int \sqrt{I_K(P_t f)^2 + |\nabla P_t f|^2}\,\mathrm{d}\mathfrak{m} = I_K\left(\int f\,\mathrm{d}\mathfrak{m}\right).$$

In Proposition 4.1 we prove that the function $t \mapsto J_K(P_t f)$ is non-increasing on $\mathrm{RCD}(K,\infty)$ spaces with positive K. From the discussions above we know these spaces support the Bobkov's inequality. In particular, f attains the equality in the Bobkov's inequality if and only if $J_K(P_t f)$ is a constant function in t. Then, in Proposition 4.3 we prove the rigidity of the Bobkov's inequality, which extends [20, Theorem 1] and [19, Theorem 1.4] to the non-smooth setting.

Theorem 1.4 (Proposition 4.1 and 4.3)**.** *Assume that a metric measure space $(X,\mathrm{d},\mathfrak{m})$ satisfies $\mathrm{RCD}(K,\infty)$ for some $K>0$. Then $(X,\mathrm{d},\mathfrak{m})$ supports the K-Bobkov's isoperimetric inequality.*

Furthermore, $I_K\left(\int f\,\mathrm{d}\mathfrak{m}\right) = J_K(f)$ for some non-constant $f \in L^\infty$ if and only if

$$(X,\mathrm{d},\mathfrak{m}) \cong \left(\mathbb{R}, |\cdot|, \sqrt{K/(2\pi)}e^{-Kt^2/2}\mathrm{d}t\right) \times (Y,\mathrm{d}_Y,\mathfrak{m}_Y)$$

for some $\mathrm{RCD}(K,\infty)$ space $(Y,\mathrm{d}_Y,\mathfrak{m}_Y)$, and up to change of variables, f is either the indicator function of a half space

$$f(r,y) = \chi_E, \qquad E = (-\infty, e] \times Y, (r,y) \in \mathbb{R}\times Y$$

where $e \in \mathbb{R}\cup\{+\infty\}$ with $\int_{-\infty}^e \phi_K(s)\,\mathrm{d}s = \int f\,\mathrm{d}\mathfrak{m}$; or else, there are $a = (2\int f)^{-1}$ and $b = \Phi_K^{-1}\big(f(0,y)\big)$ such that

$$f(t,y) = \Phi_K(at+b) = \int_{-\infty}^{at+b} \phi_K(s)\,\mathrm{d}s.$$

1.3. **Φ-entropy inequalities.** Let Φ be a continuous function defined on an interval $I \subset \mathbb{R}$. For any I-valued function f, the Φ-entropy of f is defined by

$$\mathrm{Ent}^\Phi_{\mathfrak{m}}(f) := \int \Phi(f)\,\mathrm{d}\mathfrak{m}.$$

Using a similar method as Chafaï [21] (see also Bolley-Gentil [18]), we can prove the following Φ-entropy inequality on $\mathrm{RCD}(K,\infty)$ spaces. It can be seen that the Poincaré inequality and the log-Sobolev inequality are both Φ-entropy inequalities.

Proposition 1.5 (Proposition 4.5)**.** *Let $(X,\mathrm{d},\mathfrak{m})$ be a metric measure space satisfying $\mathrm{RCD}(K,\infty)$ condition for some $K>0$. Let Φ be a C^2-continuous strictly convex function on an interval $I\subset\mathbb{R}$ such that $\frac{1}{\Phi''}$ is concave. Then $(X,\mathrm{d},\mathfrak{m})$ supports the following Φ-entropy inequality:*

$$\mathrm{Ent}^\Phi_{\mathfrak{m}}(f) - \Phi\Big(\int f\,\mathrm{d}\mathfrak{m}\Big) \le \frac{1}{2K}\int \Phi''(f)\Gamma(f)\,\mathrm{d}\mathfrak{m} \tag{1.11}$$

for any I-valued function f.

Furthermore, we completely characterize the cases of equality in Φ-entropy inequalities. In particular, we prove that the Poincaré inequality and the log-Sobolev inequality are essentially the only Φ-entropy inequalities that the equalities could be attained.

Theorem 1.6. *Let $(X,\mathrm{d},\mathfrak{m})$ be a metric measure space satisfying $\mathrm{RCD}(K,\infty)$ for some $K>0$. Assume there is a function Φ which fulfils the conditions in Proposition 1.5, and a non-constant function f attaining the equality in the corresponding Φ-entropy inequality. Then*

(1) f attains the equality in the 1-Bakry-Émery inequality, so that $(X, \mathrm{d}, \mathfrak{m})$ is isometric to

$$\left(\mathbb{R}, |\cdot|, \sqrt{K/(2\pi)}e^{-Kt^2/2}\mathrm{d}t\right) \times (Y, \mathrm{d}_Y, \mathfrak{m}_Y)$$

for some $\mathrm{RCD}(K, \infty)$ space $(Y, \mathrm{d}_Y, \mathfrak{m}_Y)$;

(2) $\Phi'(f)$ attains the equality in the 2-Bakry-Émery inequality;

(3) up to affine coordinate transforms, additive and multiplicative constants, $\Phi = x^2$ or $x \ln x$. In these cases, $f(r, y)$ can be written as $a_p r$ or $e^{a_l r - a_l^2/2K}$ for some constants $a_p, a_l \in \mathbb{R}$.

Remark 1.7. It is known that the Bobkov's isoperimetric inequality yields some important inequalities (even without any curvature condition). For example, from [16, Theorem 3.2] we know the K-Bobkov's inequality yields the K-logarithmic Sobolev inequality

$$\int f \ln f \,\mathrm{d}\mathfrak{m} \leq \frac{1}{2K} \int \frac{|\nabla f|^2}{f} \,\mathrm{d}\mathfrak{m} \tag{1.12}$$

for any non-negative locally Lipschitz function f with $\int f \,\mathrm{d}\mathfrak{m} = 1$. It is known (c.f. Lott-Villani [41], Gigli-Ledoux [31]) that the K-logarithmic Sobolev inequality implies the K-Talagrand inequality

$$W_2^2(f\mathfrak{m}, \mathfrak{m}) \leq \frac{2}{K} \int f \ln f \,\mathrm{d}\mathfrak{m} \tag{1.13}$$

for any f with $\int f \,\mathrm{d}\mathfrak{m} = 1$. It is known (using Hamilton-Jacobi semigroup, c.f. [40, Theorem 1.8] and [8, Section 3]) that the K-Talagrand inequality implies the K-Poincaré inequality (or K-spectral gap)

$$\int f^2 \,\mathrm{d}\mathfrak{m} \leq \frac{1}{K} \int |\nabla f|^2 \,\mathrm{d}\mathfrak{m} \tag{1.14}$$

for any locally Lipschitz function f with $\int f \,\mathrm{d}\mathfrak{m} = 0$.

Inspired by the implications of the Bobkov's inequality discussed above, one would ask whether we can deduce the rigidity of the Poincaré inequality and the log-Sobolev inequality (Theorem 1.6) from the rigidity of the Bobkov's inequality (Theorem 1.4) or not. For example, assume there is a non-constant function attaining the equality in the Poincaré inequality, then $(X, \mathrm{d}, \mathfrak{m})$ does not support the $(K + \frac{1}{n})$-Bobkov's inequality for any $n \in \mathbb{N}$. So for any $n \in N$ there is $f_n \in \mathrm{Lip}(X, \mathrm{d}) \cap L^\infty$ such that

$$\sqrt{\frac{K}{2\pi}} \geq I_{K+\frac{1}{n}}\left(\int f_n \,\mathrm{d}\mathfrak{m}\right) > J_{K+\frac{1}{n}}(f_n) \geq 0. \tag{1.15}$$

Thus there is a subsequence of (f_n) converging to some f in L^2. Letting $n \to \infty$ in (1.15), by continuity of $(K, t) \mapsto I_K(t)$, Fatou's lemma and lower semi-continuity of $\mathbb{E}$, we obtain

$$I_K\left(\int f \,\mathrm{d}\mathfrak{m}\right) \geq J_K(f).$$

Combining with the K-Bobkov's inequality we get $I_K\left(\int f \,\mathrm{d}\mathfrak{m}\right) = J_K(f)$.

However, we can not assert that f is not constant, because we do not know much about (f_n) except its existence.

Remark 1.8. Concerning an extremal function f of the log-Sobolev inequality, it was conjectured by Otto-Villani [45, Page 391] that $\ln f$ attains the equality in the Γ_2-inequality $\Gamma_2 \geq K\Gamma$. Unfortunately, due to lack of regularity, we can not use second-order differentiation formula as suggested in [45] on curved spaces.

Recently, Ohta-Takatsu [44] give a rigorous proof to the rigidity of the log-Sobolev inequality on smooth metric measure spaces, using a localization argument which benefits from a breakthrough of Klartag [37]. As mentioned in [44, §4], the rigidity of the log-Sobolev inequality on $\mathrm{RCD}(K,\infty)$ spaces was an open problem due to lack of 'needle decomposition' on dimension-free $\mathrm{RCD}(K,\infty)$ spaces.

Thus the novelty of our result is that it gives an affirmative answer to the conjecture of Otto-Villani, and extends the result of Ohta-Takatsu to $\mathrm{RCD}(K,\infty)$ spaces.

1.4. **Structure of the paper.** In the first part of Section 2 we review some basic results about the non-smooth Bakry-Émery theory and calculus on metric measure spaces. Most of these results can be found in the papers of Ambrosio-Gigli-Savaré [10, 8, 9], Gigli [27] and Savaré [47]. In the second part, we study the cases of equality in the 2-Bakry-Émery inequality.

In Section 3 we prove the rigidity of the 1-Bakry-Émery inequality. This extends the result of Ambrosio-Brué-Semola [2] to dimension-free $\mathrm{RCD}(K,\infty)$ spaces with $K>0$. Some important tools used there are the continuity equation theory in the non-smooth framework developed by Ambrosio-Trevisan [13], and the functional analysis tools by Gigli [27]. We remark that the proof in [2] relies on a two-sides heat kernel estimate, and it seems that the proof works only for $K=0$ case.

In Section 4, we apply the results obtained in the previous two sections to study the rigidity of the Bobkov's Gaussian isoperimetric inequality and Φ-inequalities. The arguments in this section are not totally new, similar semigroup arguments were used by Carlen-Kerce [20], Chafaï [21] etc. in the study of related problems on smooth metric measure spaces.

2. Synthetic curvature-dimension conditions

2.1. Γ_2-calculus on metric measure spaces.

Definition 2.1 (Lott-Sturm-Villani's curvature-dimension condition, c.f. [42, 48])**.** We say that a metric measure space $(X,\mathrm{d},\mathfrak{m})$ is $\mathrm{CD}(K,\infty)$ for some $K\in\mathbb{R}$ if the entropy functional $\mathrm{Ent}_{\mathfrak{m}}$ is K-displacement convex on the L^2-Wasserstein space $(\mathcal{P}_2(X),W_2)$. This means, for any two probability measures $\mu_0,\mu_1\in\mathcal{P}_2(X)$ with $\mu_0,\mu_1\ll\mathfrak{m}$, there is a L^2-Wasserstein geodesic $(\mu_t)_{t\in[0,1]}$ such that

$$\frac{K}{2}t(1-t)W_2^2(\mu_0,\mu_1)+\mathrm{Ent}_{\mathfrak{m}}(\mu_t)\leq t\mathrm{Ent}_{\mathfrak{m}}(\mu_1)+(1-t)\mathrm{Ent}_{\mathfrak{m}}(\mu_0) \tag{2.1}$$

where $\mathrm{Ent}_{\mathfrak{m}}(\mu_t)$ is defined as $\int\rho_t\ln\rho_t\,\mathrm{d}\mathfrak{m}$ if $\mu_t=\rho_t\,\mathfrak{m}$, otherwise $\mathrm{Ent}_{\mathfrak{m}}(\mu_t)=+\infty$.

As we introduced in the Introduction section, the energy form $\mathbb{E}(\cdot)$ is defined on $L^2(X,\mathfrak{m})$ by

$$\begin{aligned}\mathbb{E}(f) &:= \inf\Big\{\liminf_{n\to\infty}\int_X \mathrm{lip}(f_n)^2\mathrm{d}\mathfrak{m} : f_n\in\mathrm{Lip}_b(X),\ f_n\to f \text{ in } L^2(X,\mathfrak{m})\Big\}\\ &= \int_X |\nabla f|^2\,\mathrm{d}\mathfrak{m}\end{aligned}$$

where $\mathrm{lip}(f)(x):=\limsup_{y\to x}|f(x)-f(y)|/\mathrm{d}(x,y)$ denotes the local Lipschitz slope at $x\in X$ and $|\nabla f|$ denotes the minimal weak upper gradient. We refer the readers to [8, 22] for details about the theory of Sobolev space on metric measure spaces.

We say that $(X,\mathrm{d},\mathfrak{m})$ is an $\mathrm{RCD}(K,\infty)$ space if it is $\mathrm{CD}(K,\infty)$, and $\mathbb{E}(\cdot)$ is a quadratic form. In this case, it is known that $\mathbb{E}$ defines a quasi-regular, strongly local, conservative Dirichlet form admitting a carré du champ $\Gamma(f) := |\nabla f|^2$ (c.f. [10] and [14]). Denote $\mathbb{V} = \mathrm{D}(\mathbb{E}) = \{f : \mathbb{E}(f) < \infty\}$. For any $f, g \in \mathbb{V}$, by polarization, we define

$$\Gamma(f,g) := \frac{1}{4}\big(\Gamma(f+g) - \Gamma(f-g)\big),$$

and

$$\mathbb{E}(f,g) = \int \Gamma(f,g)\,\mathrm{d}\mathfrak{m}.$$

The heat flow (P_t) is defined as the gradient flow of $\mathbb{E}$ in $L^2(\mathfrak{m})$. It is known that P_t is linear and self-adjoint (c.f. [9]). We recall the following regularization properties of (P_t), ensured by the theory of gradient flows and maximal monotone operators.

Lemma 2.2 (A priori estimates)**.** *For every $f \in L^2(\mathfrak{m})$ and $t > 0$ it holds*

(1) $\|P_t f\|_{L^2} \le \|f\|_{L^2}$*;*
(2) $\mathbb{E}(P_t f) \le \frac{1}{2t}\|f\|_{L^2}^2$*;*
(3) $\|\Delta P_t f\|_{L^2} \le \frac{1}{t}\|f\|_{L^2}$.

Let us recall the notion of non-smooth vector fields introduced by Weaver in [49] (see also [13] and [27]).

Definition 2.3. We say that a linear functional $b : \mathrm{Lip}(X,\mathrm{d}) \mapsto L^0(X,\mathfrak{m})$ is an L^2-derivation, and write $b \in L^2(TX)$ (or $b \in L^2_{\mathrm{loc}}(TX)$ resp.), if it satisfies the following properties.

(1) Leibniz rule: for any $f, g \in \mathrm{Lip}(X,\mathrm{d})$ it holds

$$b(fg) = b(f)g + fb(g).$$

(2) L^2-bound: there exists $g \in L^2(X,\mathfrak{m})$ (or $L^2_{\mathrm{loc}}(X,\mathfrak{m})$ resp.) such that

$$|b(f)| \le g|\mathrm{lip}(f)|, \qquad \mathfrak{m}-\text{a.e. on } X,$$

for any $f \in \mathrm{Lip}$ and we denote by $|b|$ the minimal (in the $\mathfrak{m}$-a.e. sense) g satisfying such property.

In [27] Gigli introduces the so-called tangent and cotangent modules over metric measure spaces, and proves the identification results between L^2-derivations and elements of the tangent module $L^2(TX)$.

Proposition 2.4 (Section 2.2, [27])**.** *Let $\mathbb{E}$ be the Dirichlet form associated with the metric measure space $(X,\mathrm{d},\mathfrak{m})$, and let Γ be the carré du champ defined on $\mathbb{V}$. Then there exists a L^∞-Hilbert module $L^2(TX)$ satisfying the following properties.*

(1) For any $f \in \mathbb{V}$, there is a derivation $\nabla f \in L^2(TX)$ defined by the formula

$$\nabla f(g) = \Gamma(f,g), \qquad \forall g \in \mathrm{Lip}(X,\mathrm{d}).$$

(2) $L^2(TX)$ is a module over the commutative ring $L^\infty(X,\mathfrak{m})$.
(3) $L^2(TX)$ is a Hilbert space equipped with the norm $\|\cdot\|$ which is compatible with the semi-norm $\mathbb{E}$ on $\mathbb{V}$, i.e. it holds the following correspondence

$$\mathbb{V} \ni f \mapsto \nabla f \in L^2(TX), \quad \textit{s.t.} \quad \|\nabla f\|^2 = \mathbb{E}(f).$$

(4) *The norm $\|\cdot\|$ is induced by a pointwise inner product $\langle\cdot,\cdot\rangle$ satisfying*

$$\langle\nabla f,\nabla g\rangle=\Gamma(f,g),\qquad \mathfrak{m}-a.e.$$

and

$$\langle h\nabla f,\nabla g\rangle=h\langle\nabla f,\nabla g\rangle,\qquad \mathfrak{m}-a.e.$$

for any $f,g\in\mathbb{V}$ and $h\in L^\infty_{\rm loc}$.

(5) *$L^2(TX)$ is generated by $\{\nabla g:g\in\mathbb{V}\}$ in the following sense. For any $v\in L^2(TX)$, there exists a sequence $v_n=\sum_{i=1}^{M_n}a_{n,i}\nabla g_{n,i}$ with $a_{n,i}\in L^\infty$ and $g_{n,i}\in\mathbb{V}$, such that $\|v-v_n\|\to 0$ as $n\to\infty$.*

Via integration by parts, we can define the divergence of vector fields.

Definition 2.5. Let $b\in L^2_{\rm loc}(TX)$. We say that $b\in{\rm D}({\rm div})$ if there exists $g\in L^2(X,\mathfrak{m})$ such that

$$\int\langle b,\nabla f\rangle\,{\rm d}\mathfrak{m}=\int b(f)\,{\rm d}\mathfrak{m}=-\int gf\,{\rm d}\mathfrak{m}\qquad\text{for any }\ f\in{\rm Lip}_{\rm bs}(X,{\rm d}).$$

By a density argument it is easy to check that such function g is unique (when it exists) and we will denote it by ${\rm div}(b)$.

In particular, the Dirichlet form $\mathbb{E}$ induces a densely defined selfadjoint operator $\Delta:{\rm D}(\Delta)\subset\mathbb{V}\mapsto L^2$ satisfying $\mathbb{E}(f,g)=-\int g\Delta f\,{\rm d}\mathfrak{m}$ for all $g\in\mathbb{V}$.

Put

$$\Gamma_2(f;\varphi):=\frac12\int\Gamma(f)\Delta\varphi\,{\rm d}\mathfrak{m}-\int\Gamma(f,\Delta f)\varphi\,{\rm d}\mathfrak{m}$$

and ${\rm D}(\Gamma_2):=\Big\{(f,\varphi):f,\varphi\in{\rm D}(\Delta),\ \Delta f\in\mathbb{V},\ \varphi,\Delta\varphi\in L^\infty\Big\}$.

It is proved in [9] (and also [6] for σ-finite case) that ${\rm RCD}(K,\infty)$ implies the following non-smooth Bakry-Émery condition ${\rm BE}(K,\infty)$.

Proposition 2.6 (The Bakry-Émery condition)**.** *Let $(X,{\rm d},\mathfrak{m})$ be an ${\rm RCD}(K,\infty)$ space. Then the corresponding Dirichlet form $\mathbb{E}$ satisfies the following ${\rm BE}(K,\infty)$ condition*

$$\Gamma_2(f;\varphi)\geq K\int\varphi\Gamma(f)\,{\rm d}\mathfrak{m}\tag{2.2}$$

for all $(f,\varphi)\in{\rm D}(\Gamma_2)$ with $\varphi\geq 0$.

Under some natural regularity assumptions on the distance canonically associated with the Dirichlet form, the converse implication is also true, see [10] for more details.

We have the following crucial properties obtained by Savaré [47] and Gigli [27]. Recall that the space of test functions is defined as ${\rm TestF}:=\big\{f\in{\rm D}(\Delta)\cap L^\infty:\Delta f\in\mathbb{V},\,\Gamma(f)\in L^\infty\big\}$. It is known that ${\rm TestF}$ is dense in $\mathbb{V}$ (c.f. [27, (3.1.6)]).

Proposition 2.7. *Let $(X,{\rm d},\mathfrak{m})$ be an ${\rm RCD}(K,\infty)$ space. Then*

(1) *For any $f\in{\rm TestF}$, we have $\Gamma(f)\in\mathbb{V}$ and*

$$\mathbb{E}\Big(\Gamma(f)\Big)\leq-\int\Big(2K\Gamma(f)^2+2\Gamma(f)\Gamma(f,\Delta f)\Big)\,{\rm d}\mathfrak{m}.$$

(2) *For every* $f \in \mathrm{D}(\Delta)$*, we have* $\Gamma(f)^{1/2} \in \mathbb{V}$ *and*

$$\mathbb{E}\Big(\Gamma(f)^{1/2}\Big) \leq \int (\Delta f)^2 \, \mathrm{d}\mathfrak{m} - K \cdot \mathbb{E}(f).$$

(3) *For any* $f \in \mathrm{D}(\Delta)$ *there is a continuous symmetric* L^∞*-bilinear map* $\mathrm{Hess}_f(\cdot,\cdot)$ *defined on* $[L^2(TX)]^2$*, with values in* $L^0(X,\mathfrak{m})$ *(c.f.* [27, Corollary 3.3.9]*). In particular, if* $f, g, h \in \mathrm{TestF}$ *(c.f.* [27, Proposition 3.3.22], [47, Lemma 3.2]*),* $\mathrm{Hess}_f(\cdot,\cdot)$ *is given by the following formula:*

$$2\mathrm{Hess}_f(\nabla g, \nabla h) = \Gamma\big(g, \Gamma(f,h)\big) + \Gamma\big(h, \Gamma(f,g)\big) - \Gamma\big(f, \Gamma(g,h)\big). \tag{2.3}$$

To introduce the measure-valued 'Ricci tensor', we briefly recall the notion of measure-valued Laplacian $\mathbf{\Delta}$ (c.f. [47, 26]). We say that $f \in \mathrm{D}(\mathbf{\Delta}) \subset \mathbb{V}$ if there exists a signed Borel measure $\mu = \mu_+ - \mu_- \in \mathrm{Meas}(X)$ charging no capacity zero sets such that

$$\int \overline{\varphi} \, \mathrm{d}\mu = -\int \Gamma(\varphi.f) \, \mathrm{d}\mathfrak{m}$$

for any $\varphi \in \mathbb{V}$ with quasi-continuous representative $\overline{\varphi} \in L^1(X, |\mu|)$. If μ is unique, we denote it by $\mathbf{\Delta} f$. If $\mathbf{\Delta} f \ll \mathfrak{m}$, we also denote its density by Δf if there is no ambiguity.

Proposition 2.8 (See [27], §3 and [47], Lemma 3.2)**.** *Let* $(X, \mathrm{d}, \mathfrak{m})$ *be a* $\mathrm{RCD}(K,\infty)$ *space. Then for any* $f \in \mathrm{TestF}_{\mathrm{loc}} := \big\{ f \in \mathrm{D}(\mathbf{\Delta}) \cap L^\infty_{\mathrm{loc}} : \Delta f \in \mathbb{V}_{\mathrm{loc}}, \, \Gamma(f) \in L^\infty_{\mathrm{loc}} \big\}$*, it holds* $\Gamma(f) \in \mathrm{D}(\mathbf{\Delta})$ *and the following non-smooth Bochner inequality*

$$\mathbf{\Gamma}_2(f) := \frac{1}{2}\mathbf{\Delta}\Gamma(f) - \Gamma(f, \Delta f)\,\mathfrak{m} \geq \Big(K\Gamma(f) + \|\mathrm{Hess}_f\|^2_{\mathrm{HS}}\Big)\,\mathfrak{m}.$$

Furthermore, define $\mathrm{TestV}_{\mathrm{loc}} := \{\Sigma_{i=1}^n a_i \nabla f_i : n \in \mathbb{N}, a_i, f_i \in \mathrm{TestF}_{\mathrm{loc}}\}$*, there is a measure-valued symmetry bilinear map* $\mathbf{Ric} : [\mathrm{TestV}_{\mathrm{loc}}]^2 \mapsto \mathrm{Meas}(X)$ *satisfying the following properties*

(1) *for any* $f \in \mathrm{TestF}_{\mathrm{loc}}$*,*

$$\mathbf{Ric}(\nabla f, \nabla f) := \underbrace{\frac{1}{2}\mathbf{\Delta}\Gamma(f) - \Gamma(f, \Delta f)\,\mathfrak{m}}_{=\mathbf{\Gamma}_2(f)} - \|\mathrm{Hess}_f\|^2_{\mathrm{HS}}\,\mathfrak{m};$$

(2) *for any* $f \in \mathrm{TestF}_{\mathrm{loc}}$*,*

$$\mathbf{Ric}(\nabla f, \nabla f) \geq K\Gamma(f)\,\mathfrak{m};$$

(3) *for any* $f, g, h \in \mathrm{TestF}_{\mathrm{loc}}$*,*

$$\mathbf{Ric}(h\nabla f, \nabla g) = h\mathbf{Ric}(\nabla f, \nabla g).$$

2.2. **Equality in the 2-Bakry-Émery inequality.** In the next lemma, we study the equality in the 2-Bakry-Émery inequality. The argument for the proof is standard, we just need to pay attention to the regularity issues appearing in the non-smooth framework.

Lemma 2.9 (Equality in the 2-Bakry-Émery inequality)**.** *Let* $(X, \mathrm{d}, \mathfrak{m})$ *be a* $\mathrm{RCD}(K,\infty)$ *probability space for some* $K > 0$ *and let* $u \in \mathbb{V} \cap \mathrm{D}(\Delta)$ *be a non-constant function with* $\Delta u \in \mathbb{V}$ *and* $\int u \, \mathrm{d}\mathfrak{m} = 0$*. Then the following statements are equivalent.*

(1) $u \in \mathrm{TestF}_{\mathrm{loc}}$ *and* $\mathbf{\Gamma}_2(u) = K\Gamma(u)\,\mathfrak{m}$*;*
(2) $\Gamma_2(u; \varphi) = K\int \varphi\Gamma(u)\,\mathrm{d}\mathfrak{m}$ *for all non-negative* $\varphi \in L^\infty$ *with* $\Delta\varphi \in L^\infty$*;*
(3) $\int (\Delta u)^2 \, \mathrm{d}\mathfrak{m} = K\int \Gamma(u)\,\mathrm{d}\mathfrak{m}$*;*
(4) $-\Delta u = Ku$*;*

(5) $\int \Gamma(u)\,\mathrm{d}\mathfrak{m} = K\int u^2\,\mathrm{d}\mathfrak{m}$*;*
(6) $\Gamma(P_t u) = e^{-2Kt}P_t\Gamma(u)$ *for some* $t>0$*.*

In particular, $P_s u$ *satisfies the properties above for all* $s>0$*. Furthermore,* $P_s u$ *satisfies one of these properties for all* $s\in[0,t]$ *if and only if*

$$\int (P_t u)^2\,\mathrm{d}\mathfrak{m} = e^{-2Kt}\int u^2\,\mathrm{d}\mathfrak{m}.$$

If u *attains the equality in the 2-Bakry-Émery inequality (6) above, it holds*

a) $|\nabla P_t u| = e^{-Kt}P_t|\nabla u|$ *for all* $t>0$*;*
b) u *is a non-constant affine function, this means* $\mathrm{Hess}_u = 0$ *and* $\Gamma(u)$ *is a positive constant;*
c) $u \in \mathrm{TestF_{loc}}$ *and* $\mathbf{Ric}(u,u) = K\Gamma(u)\,\mathrm{d}\mathfrak{m}$*;*
d) *the gradient flow of* u *induces a one-parameter semigroup of isometries of* (X,d)*.*

Proof. **Part 1**: We will prove (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5) $\Longrightarrow$ (4) $\Longrightarrow$ (6) $\Longrightarrow$ (2). Statement (1) is a consequence of b) and c) which will be proved in **Part 2**.

(1) $\Longrightarrow$ (2): Integrating φ w.r.t. the measures $\mathbf{\Gamma}_2(u)$, $K\Gamma(u)\,\mathfrak{m}$ we get the answer.

(2) $\Longrightarrow$ (3): Notice that the constant function $\varphi\equiv 1$ is admissible, and $\Gamma_2(u,1) = \int(\Delta u)^2\,\mathrm{d}\mathfrak{m}$.

(3) $\Longrightarrow$ (4): Applying Proposition 2.6 with $\varphi\equiv 1$ (or by Proposition 2.7, (2)), we can see that

$$\int (\Delta f)^2\,\mathrm{d}\mathfrak{m} \geq K\int \Gamma(f)\,\mathrm{d}\mathfrak{m}$$

for $f\in \mathrm{D}(\Delta)$. Let $f = u\pm\epsilon g$ for some $g\in\mathrm{D}(\Delta)$ and $\epsilon\in\mathbb{R}$. We obtain

$$\int \big(\Delta(u\pm\epsilon g)\big)^2\,\mathrm{d}\mathfrak{m} \geq K\int \Gamma(u\pm\epsilon g)\,\mathrm{d}\mathfrak{m}. \tag{2.4}$$

Differentiating (2.4) (w.r.t. the variable ϵ), and combining with the equality in (3) we get

$$\pm\int \Delta u\Delta g\,\mathrm{d}\mathfrak{m} \geq \pm K\int \Gamma(u,g)\,\mathrm{d}\mathfrak{m}.$$

Therefore

$$\int \Delta u\Delta g\,\mathrm{d}\mathfrak{m} = K\int\Gamma(u,g)\,\mathrm{d}\mathfrak{m} = -K\int u\Delta g\,\mathrm{d}\mathfrak{m}. \tag{2.5}$$

Notice that $\mathrm{D}(\Delta)$ is dense in $\mathbb{V}$, and by Poincaré inequality it holds $\overline{\Delta\big(\mathrm{D}(\Delta)\big)}^{L^2} = \mathbb{V}\setminus\{u\equiv c : c\in\mathbb{R}, c\neq 0\}$. Hence (2.5) yields (4).

(4) $\Longrightarrow$ (5) Multiplying u on both sides of $-\Delta u = Ku$ and integrating w.r.t. $\mathfrak{m}$, we obtain the equality in the Poincaré inequality.

(5) $\Longrightarrow$ (4): By Poincaré inequality, we have $\int\Gamma(u+g)\,\mathrm{d}\mathfrak{m} \geq K\int(u+g)^2\,\mathrm{d}\mathfrak{m}$ for all $g\in\mathbb{V}$ with $\int g\,\mathrm{d}\mathfrak{m} = 0$. Then similar to (3) $\Longrightarrow$ (4), we can prove the spectral gap equality by a standard variation argument.

(4) $\Longrightarrow$ (6): Denote $\phi(t) := \int\Big(\Gamma(P_t u) - e^{-2Kt}P_t\Gamma(u)\Big)\,\mathrm{d}\mathfrak{m}$. By (4) we have $-\Delta P_t u = KP_t u$ for any $t\geq 0$, so $\int(\Delta P_t u)^2\,\mathrm{d}\mathfrak{m} = K\int\Gamma(P_t u)\,\mathrm{d}\mathfrak{m}$. It is known that (c.f. [10, Lemma 2.1]) $\phi\in C^1$,

and

$$\begin{aligned}\phi'(t) &= 2\int \Big(-\big(\Delta P_t u\big)^2 + K e^{-2Kt}\Gamma(u)\Big)\,\mathrm{d}\mathfrak{m}\\ &= 2\int \Big(-K\Gamma(P_t u) + K e^{-2Kt}\Gamma(u)\Big)\,\mathrm{d}\mathfrak{m}\\ &\geq 0.\end{aligned}$$

Therefore $\phi(t) \geq \phi(0) = 0$. Note that by 2-Barky-Émery inequality $\Gamma(P_t u) \leq e^{-2Kt}P_t\Gamma(u)$, it holds $\phi \leq 0$. So $\phi \equiv 0$ and $\Gamma(P_t u) = e^{-2Kt}P_t\Gamma(u)$ for all $t > 0$ which is the thesis.

(6) $\Longrightarrow$ (2): It is known that $[0,t] \ni s \mapsto \Phi_{t,\varphi}(s) := \frac{1}{2}\int e^{-2Ks}P_s\varphi\Gamma(P_{t-s}u)\,\mathrm{d}\mathfrak{m}$ is C^1-continuous for any positive $\varphi \in L^\infty$ with $\Delta\varphi \in L^\infty$, and

$$\Phi'_{t,\varphi}(s) = e^{-2Ks}\Big(\Gamma_2(P_{t-s}u; P_s\varphi) - K\int P_s\varphi\Gamma(P_{t-s}u)\,\mathrm{d}\mathfrak{m}\Big) \geq 0.$$

By 2-Bakry-Émery inequality, (6) holds if and only if $\Phi'_{t,\varphi}(s) = 0$ for any $s \in [0,t]$ and any admissible function φ, i.e.

$$\Gamma_2(P_{t-s}u; P_s\varphi) = K\int P_s\varphi\Gamma(P_{t-s}u)\,\mathrm{d}\mathfrak{m},\ \forall s \in [0,t]. \tag{2.6}$$

Notice that u attains the equality in the 2-Bakry-Émery inequality for $t > 0$ if and only if it holds for all $t' \in [0,t]$, thus (2.6) implies

$$\Gamma_2(P_s u; \varphi) = K\int \varphi\Gamma(P_s u)\,\mathrm{d}\mathfrak{m}, \qquad \varphi, \Delta\varphi \in L^\infty,\ 0 \leq s \leq t \tag{2.7}$$

which yields (2).

Part 2: Let $u_s = P_s u$. If u satisfies one of the properties (1)-(6), from the discussion in the first part we know $\Delta u = -Ku$. So $\Delta u_s = P_s\Delta u = -Ku_s$, and u_s also satisfies these properties.

Note that $\frac{\mathrm{d}}{\mathrm{d}s}u_s = \Delta u_s$. By Poincaré inequality, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{2}\int u_s^2\,\mathrm{d}\mathfrak{m} &= \int u_s\frac{\mathrm{d}}{\mathrm{d}s}u_s\,\mathrm{d}\mathfrak{m}\\ &= \int u_s\Delta u_s\,\mathrm{d}\mathfrak{m}\\ &= -\int \Gamma(u_s)\,\mathrm{d}\mathfrak{m}\\ &\leq -K\int u_s^2\,\mathrm{d}\mathfrak{m}.\end{aligned}$$

By Grönwall's lemma, we obtain

$$\int u_s^2\,\mathrm{d}\mathfrak{m} \leq e^{-2Ks}\int u^2\,\mathrm{d}\mathfrak{m}. \tag{2.8}$$

Therefore, (2.8) is an equality for some $t > 0$ if and only if u_s attains the equality in the Poincaré inequality (5).

Part 3: Furthermore, by 1-Bakry-Émery inequality and Cauchy-Schwarz inequality, we have

$$|\nabla P_t u| \leq e^{-Kt}P_t|\nabla u| \leq e^{-Kt}\sqrt{P_t\Gamma(u)}.$$

So if u attains the equality in the 2-Bakry-Émery inequality (6), it holds $|\nabla P_t u| = e^{-Kt} P_t |\nabla u|$.

In addition, integrating the non-smooth Bochner inequality in Proposition 2.8 we obtain

$$\int (\Delta u)^2 \,\mathrm{d}\mathfrak{m} \geq K \int \Gamma(u) \,\mathrm{d}\mathfrak{m} + \int \|\mathrm{Hess}_u\|_{\mathrm{HS}}^2 \,\mathrm{d}\mathfrak{m}.$$

Thus the validity of (3) yields $\mathrm{Hess}_u = 0$. In particular, for any $v \in \mathbb{V}$, it holds

$$\Gamma\big(\Gamma(u), v\big) = 2\mathrm{Hess}_u(\nabla u, \nabla v) = 0,$$

so $\Gamma(\Gamma(u)) = 0$ and $\Gamma(u) = |\nabla u|^2 \equiv c$ for some constant $c \geq 0$. In particular, $u \in \mathrm{TestF}_{\mathrm{loc}}$. If $c = 0$, f is constant. If $c \neq 0$, by [34, Theorem 1.2] we know that the regular Lagrangian flow $(F_r)_{r \in \mathbb{R}^+}$ associated with ∇u induces a family of isometries, i.e. $\mathrm{d}(F_r(x), F_r(y)) = \mathrm{d}(x, y)$ for any $x, y \in X$ and $r > 0$.

Furthermore, by definition of $\mathbf{Ric}$ (c.f. Proposition 2.8) and statement (2) proved in **Part 1**, for any $\varphi \in L^\infty \cap \mathrm{D}(\Delta)$ with $\Delta\varphi \in L^\infty$ we have

$$\Gamma_2(u; \varphi) = \int \varphi \|\mathrm{Hess}_u\|_{\mathrm{HS}}^2 \,\mathrm{d}\mathfrak{m} + \int \varphi \,\mathrm{d}\mathbf{Ric}(u, u) = K \int \varphi \Gamma(u) \,\mathrm{d}\mathfrak{m}.$$

Combining with $\mathrm{Hess}_u = 0$ we obtain

$$\mathbf{\Gamma}_2(u) = \mathbf{Ric}(u, u) = K\Gamma(u)\,\mathfrak{m}$$

and we complete the proof.

$\square$

The following proposition plays a key role in studying Φ-entropy inequalities in §4.2.

Proposition 2.10. *Let $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space satisfying $\mathrm{RCD}(K, \infty)$ condition for some $K > 0$. Let Φ be a C^2-continuous convex function on an interval $I \subset \mathbb{R}$ such that $\frac{1}{\Phi''}$ is concave and strictly positive. Then for all $t > 0$, we have*

$$\Phi''(P_t u)\Gamma(P_t u) \leq e^{-2Kt} P_t\big(\Phi''(u)\Gamma(u)\big) \tag{2.9}$$

for any I-valued function $u \in \mathbb{V}$. In particular, the function $t \mapsto e^{2Kt} \int \Phi''(P_t u)\Gamma(P_t u) \,\mathrm{d}\mathfrak{m}$ is non-increasing.

Furthermore, it holds the equality in (2.9) if and only if the following properties are satisfied.

(1) $(\Phi'')^{-1}$ is affine on the image of u which is defined as $\mathrm{supp}\, u_\sharp \mathfrak{m}$ (by Lemma 2.11 below we know $\mathrm{supp}\, u_\sharp \mathfrak{m}$ is a closed interval or a point).

(2) For any $s \in [0, t]$, there is a constant $c = c(s) > 0$ with $c(s) = e^{-2Ks} c(0)$, such that

$$\sqrt{\Gamma(P_s u)} = e^{-Ks} P_s \sqrt{\Gamma(u)} \;\; \text{and} \;\; \Gamma\big(\Phi'(P_s u)\big) = c.$$

Proof. Denote $P_t u$ by u_t. We have the following 1-Bakry-Émery inequality,

$$\sqrt{\Gamma(u_t)} \leq e^{-Kt} P_t \sqrt{\Gamma(u)}, \qquad \forall\, t \geq 0,\ \forall u \in \mathbb{V}. \tag{2.10}$$

By concavity of $\frac{1}{\Phi''}$ and Jensen's inequality, we have

$$\Phi''(u_t) \leq \Big(P_t\big(1/\Phi''(u)\big)\Big)^{-1}. \tag{2.11}$$

Combining with (2.10) we get the following inequality

$$\Phi''(u_t)\Gamma(u_t) \leq e^{-2Kt}\Big(P_t\sqrt{\Gamma(u)}\Big)^2 \Big(P_t\big(1/\Phi''(u)\big)\Big)^{-1}. \tag{2.12}$$

By Cauchy-Schwarz inequality we know

$$\Big(P_t\sqrt{\Gamma(u)}\Big)^2 \le \Big(P_t\big(\Phi''(u)\Gamma(u)\big)\Big)\Big(P_t\big(1/\Phi''(u)\big)\Big). \tag{2.13}$$

Combining (2.12) and (2.13), we obtain

$$\Phi''(u_t)\Gamma(u_t) \le e^{-2Kt}P_t\big(\Phi''(u)\Gamma(u)\big) \tag{2.14}$$

which is (2.9). Integrating (2.14) w.r.t. $\mathfrak{m}$, we obtain

$$e^{2Kt}\int \Phi''(u_t)\Gamma(u_t)\,\mathrm{d}\mathfrak{m} \le \int \Phi''(u)\Gamma(u)\,\mathrm{d}\mathfrak{m}. \tag{2.15}$$

By semigroup property, we can see that $e^{2Kt}\int \Phi''(u_t)\Gamma(u_t)\,\mathrm{d}\mathfrak{m}$ is non-increasing in t.

Furthermore, since $t \mapsto e^{2Kt}\int \Phi''(u_t)\Gamma(u_t)\,\mathrm{d}\mathfrak{m}$ is non-increasing in t, equality in (2.9) holds for some t_0 implies the equality for any $t \le t_0$. Hence the equality in (2.9) holds for some $t_0 > 0$ if and only if the equalities in (2.10) (2.11) and (2.13) hold for all $0 \le t \le t_0$. The equality in (2.11) holds iff $(\Phi'')^{-1}$ is affine on the image of u, and the validity of the equality in (2.13) if and only if

$$\Phi''(u_t)\Gamma\big(u_t\big) = \frac{c}{\Phi''(u_t)} \tag{2.16}$$

for some constant $c = c(t) > 0$. Moreover, for any $t \le t_0$ we have

$$\sqrt{c(t)} \overset{(2.16)}{=} \Phi''(u_t)\sqrt{\Gamma\big(u_t\big)} \overset{(2.10)(2.11)}{=} e^{-Kt}\frac{P_t\big(\sqrt{\Gamma(u)}\big)}{P_t\big(1/\Phi''(u)\big)} \overset{(2.16)}{=} e^{-Kt}\sqrt{c(0)}$$

which is the thesis.

$\square$

Lemma 2.11. *Let* $(X, \mathrm{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, \infty)$ *metric measure space and* $u \in \mathbb{V}$. *Then the image of* u*, defined as* $\operatorname{supp} u_\sharp\mathfrak{m}$*, is a closed interval in* $\mathbb{R}$ *or a point in which case* u *is constant.*

Proof. Denote $\operatorname{ess\,sup} u = b \in \mathbb{R} \cup \{+\infty\}$ and $\operatorname{ess\,inf} u = a \in \mathbb{R} \cup \{-\infty\}$. We will show that $\operatorname{supp} u_\sharp\mathfrak{m} = [a, b]$.

If $a = b$, u is constant, the assertion is obvious. Otherwise, $a < b$. For any $c \in (a, b)$ and $\epsilon > 0$ small enough such that $(c-\epsilon, c+\epsilon) \subset (a+\epsilon, b-\epsilon)$. Pick bounded measurable sets $A, B \subset X$ with positive $\mathfrak{m}$-volume such that $A \subset u^{-1}\big((a, a+\epsilon)\big)$ and $B \subset u^{-1}\big((b-\epsilon, b)\big)$. By [33] there is a unique L^2-Wasserstein geodesic (μ_t) from $\mu_0 := \frac{\chi_A}{\mathfrak{m}(A)}\mathfrak{m}$ to $\mu_1 := \frac{\chi_B}{\mathfrak{m}(B)}\mathfrak{m}$. There is $\Pi \in \mathcal{P}_2(\mathrm{Geod}(\mathrm{X}, \mathrm{d}))$ such that $(e_t)_\sharp\Pi = \mu_t$ (c.f. [5, Theorem 2.10]). By [46, Lemma 3.1] we know $\frac{\mathrm{d}\mu_t}{\mathrm{d}\mathfrak{m}}$ is uniformly bounded, so Π is a test plan (in the sense of [8, Definition 5.1]). By an equivalent characterization of Sobolev functions using test plans (c.f. [8, §5, Proposition 5.7 and §6]), we know $u \circ \gamma \in W^{1,2}([0, 1])$ for Π-a.e. γ. Hence $t \mapsto u \circ \gamma(t)$ is absolutely continuous for Π-a.e. γ. So for almost every γ, there is an open interval I_γ such that $u \circ \gamma(I_\gamma) \subset (c-\epsilon, c+\epsilon)$. By Fubini's theorem, there is $t_c \in (0, 1)$ and $\Gamma_c \subset \operatorname{supp}\Pi$ with positive measure, such that $u \circ \gamma(t_c) \in (c-\epsilon, c+\epsilon)$ for all $\gamma \in \Gamma_c$. Therefore

$$\mu_{t_c}\Big(\big\{\gamma(t_c) : \gamma \in \Gamma_c\big\}\Big) = (e_{t_c})_\sharp\Pi|_{\Gamma_c}(X) > 0.$$

From the definition of $\mathrm{CD}(K, \infty)$ condition (see (2.1)) we know $\mu_{t_c} \ll \mathfrak{m}$, so

$$u_\sharp\mathfrak{m}\big((c-\epsilon, c+\epsilon)\big) = \mathfrak{m}\Big(u^{-1}\big((c-\epsilon, c+\epsilon)\big)\Big) \ge \mathfrak{m}\Big(\big\{\gamma(t_c) : \gamma \in \Gamma_c\big\}\Big) > 0.$$

Hence $c \in \operatorname{supp} u_\sharp \mathfrak{m}$. Since the choice of c is arbitrary and $\operatorname{supp} u_\sharp \mathfrak{m}$ is closed, we know $\operatorname{supp} u_\sharp \mathfrak{m} = [a, b]$. ☐

Corollary 2.12. *Under the same assumption as Proposition 2.10, if there exists a non-constant $u \in \mathbb{V}$ attaining the equality in (2.9) for all $t > 0$, then up to additive and multiplicative constants, and affine coordinate transforms, $\Phi(x) = x \ln x$ or $\Phi(x) = x^2$. In any of these cases, the function $P_t u$ attains the equality in the 1-Bakry-Émery inequality and the function $\Phi'(P_t u)$ attains the equality in the Poincaré inequality. In particular, $\Phi'(P_t u) - \int \Phi'(P_t u)\,\mathrm{d}\mathfrak{m}$ satisfies the properties (1)-(6) in Lemma 2.9 for all $t > 0$.*

Proof. By Proposition 2.10 and Lemma 2.11 we know $(\Phi'')^{-1}$ is linear on an interval I. So for $x \in I$, $\Phi''(x) = \frac{1}{c_1 x + c_2}$ for some constants c_1, c_2. If $c_1 = 0$, $\Phi = x^2$ up to an additive constant and an affine coordinate transformation. If $c_1 \neq 0$, up to an affine coordinate transform, Φ can be written as $x \ln x + c_3 x + c_4$. In the latter case, we can write Φ as $\Phi(x) = \frac{1}{e^{c_3}}\big((e^{c_3} x) \ln(e^{c_3} x)\big) + c_4$, which is the thesis.

Furthermore, by Proposition 2.10 we know $\Gamma(u_s) = c(s)/\big(\Phi''(u_s)\big)^2$ for all $s > 0$, and $c(s) = e^{-2Ks} c(0)$. Thus for any $t > 0$, we have

$$
\begin{aligned}
& \int \big(\Phi'(u_t)\big)^2 \,\mathrm{d}\mathfrak{m} - \Big(\Phi'\Big(\int u \,\mathrm{d}\mathfrak{m}\Big)\Big)^2 \\
= {} & \int_{+\infty}^{t} \frac{\mathrm{d}}{\mathrm{d}s} \int \big(\Phi'(u_s)\big)^2 \,\mathrm{d}\mathfrak{m}\,\mathrm{d}s \\
\text{By [8, Theorem 4.16]} \quad = {} & \int_t^{+\infty} \int 2\Big(\big(\Phi''(u_s)\big)^2 + \Phi'(u_s)\Phi^{(3)}(u_s)\Big)\Gamma(u_s) \,\mathrm{d}\mathfrak{m}\,\mathrm{d}s \\
= {} & \int_t^{+\infty} c(s) \int 2\Big(1 + \frac{\Phi'\Phi^{(3)}}{(\Phi'')^2}(u_s)\Big) \,\mathrm{d}\mathfrak{m}\,\mathrm{d}s \\
= {} & \int_t^{+\infty} 2e^{-2K(s-t)} c(t) \int \Big(1 + \frac{\Phi'\Phi^{(3)}}{(\Phi'')^2}(u_s)\Big) \,\mathrm{d}\mathfrak{m}\,\mathrm{d}s.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
& \Big(\int \Phi'(u_t) \,\mathrm{d}\mathfrak{m}\Big)^2 - \Big(\Phi'\Big(\int u \,\mathrm{d}\mathfrak{m}\Big)\Big)^2 \\
= {} & \int_{+\infty}^{t} \frac{\mathrm{d}}{\mathrm{d}s} \Big(\int \Phi'(u_s) \,\mathrm{d}\mathfrak{m}\Big)^2 \mathrm{d}s \\
= {} & \int_t^{+\infty} 2\Big(\int \Phi'(u_s) \,\mathrm{d}\mathfrak{m}\Big) \int \Phi^{(3)}(u_s)\Gamma(u_s) \,\mathrm{d}\mathfrak{m}\,\mathrm{d}s \\
= {} & \int_t^{+\infty} 2c(s)\Big(\int \Phi'(u_s) \,\mathrm{d}\mathfrak{m}\Big) \int \frac{\Phi^{(3)}}{(\Phi'')^2}(u_s) \,\mathrm{d}\mathfrak{m}\,\mathrm{d}s \\
= {} & \int_t^{+\infty} 2e^{-2K(s-t)} c(t)\Big(\int \Phi'(u_s) \,\mathrm{d}\mathfrak{m}\Big) \int \frac{\Phi^{(3)}}{(\Phi'')^2}(u_s) \,\mathrm{d}\mathfrak{m}\,\mathrm{d}s.
\end{aligned}
$$

Since $(\Phi'')^{-1}$ is linear, we can see that $\eta := \frac{\Phi^{(3)}}{(\Phi'')^2} = -\Big(\frac{1}{\Phi''}\Big)'$ is constant, so

$$\begin{aligned}
&\int \big(\Phi'(u_t)\big)^2 \,\mathrm{d}\mathfrak{m} - \Big(\int \Phi'(u_t)\,\mathrm{d}\mathfrak{m}\Big)^2 \\
= \;& \int_t^{+\infty} 2e^{-2K(s-t)}c(t)\Big(\int \big(1+\eta\Phi'(u_s)\big)\,\mathrm{d}\mathfrak{m} - \eta\int \Phi'(u_s)\,\mathrm{d}\mathfrak{m}\Big)\,\mathrm{d}s \\
= \;& \int_t^{+\infty} 2e^{-2K(s-t)}c(t)\,\mathrm{d}s \\
= \;& \frac{1}{K}c(t) \\
= \;& \frac{1}{K}\int \Gamma\big(\Phi'(u_t)\big)\,\mathrm{d}\mathfrak{m}.
\end{aligned}$$

This means that $\Phi'(u_t)$ attains the equality in the Poincaré inequality.

□

2.3. **One-dimensional cases.** In this part, we will prove the rigidity of the 2-Bakry-Émery inequality in 1-dimensional cases. This result is a simple application of Lemma 2.9, and it will be used in the study of higher-dimensional spaces.

Proposition 2.13. *Let h be a $\mathrm{CD}(K,\infty)$ probability density supported on a closed set $I \subset \mathbb{R}$, this means, $h\mathcal{L}^1$ is a probability measure such that $(I, |\cdot|, h\mathcal{L}^1)$ is a $\mathrm{CD}(K,\infty)$ space. If there is a non-constant function f satisfying one of the properties (1)-(6) in Lemma 2.9, then $I = \mathbb{R}$ and $h(t) = \phi_K(t) = \sqrt{\frac{K}{2\pi}}\exp(-\frac{Kt^2}{2})$ up to a translation. Furthermore, there is a constant $C = |f'| > 0$ such that*

$$P_t f(x) = Ce^{Kt}x, \qquad \forall\, t \geq 0.$$

Proof. Since h is a $\mathrm{CD}(K,\infty)$ density, by [36] we know $-\ln h$ is K-convex and $\operatorname{supp} h$ is a closed interval $I := [a,b]$ with $a \in \mathbb{R}\cup\{-\infty\}$ and $b \in \mathbb{R}\cup\{+\infty\}$. In particular, h is locally Lipschitz. By Rademacher's theorem, $h'(x)$ exists for $\mathcal{L}^1$-a.e. $x \in I$. Furthermore, $(\ln h)'$ is a BV function and $-(\ln h)'' \geq K$ in weak sense, i.e.

$$\int \varphi'(\ln h)'\,\mathrm{d}\mathcal{L}^1 \geq K\int \varphi\,\mathrm{d}\mathcal{L}^1 \tag{2.17}$$

for all $\varphi \in C^1$ with $\varphi \geq 0$ and $\varphi'(a) = \varphi'(b) = 0$.

Consider the Γ_2-calculus on the metric measure space $(I, |\cdot|, h\mathcal{L}^1)$. For $f \in \mathrm{D}(\Delta_I)$, by Proposition 2.7 we know $f' \in W^{1,2}(I)$. So it is absolutely continuous, and $f''(x)$ exists at almost every $x \in I$. By assumption and Lemma 2.9, we know $\mathrm{Hess}_f = f'' = 0$ and f' is constant. By integration by part formula, we know $f'|_{\{a,b\}\setminus\{\pm\infty\}} = 0$, and

$$\Delta_h f = f'' - (\ln h)'f' = -(\ln h)'f'. \tag{2.18}$$

Since f is not constant, there must be $\{a,b\} = \{\pm\infty\}$ and $I = \mathbb{R}$.

By (2.18) and (2.17), we have

$$\int \big(\Delta_h f\big)^2 h\,\mathrm{d}\mathcal{L}^1 = \int \big((\ln h)'f'\big)^2 h\,\mathrm{d}\mathcal{L}^1 = (f')^2\int (\ln h)'h'\,\mathrm{d}\mathcal{L}^1 \geq K\int (f')^2 h\,\mathrm{d}\mathcal{L}^1. \tag{2.19}$$

By assumption, it holds the equality in (2.19). Hence there must be $(\ln h)'' = K$ in usual sense. Up to a translation, $h(x) = \sqrt{\frac{K}{2\pi}}\exp(-\frac{Kx^2}{2}) = \phi_K(x)$ for $x \in \operatorname{supp} h = \mathbb{R}$.

Furthermore, by Lemma 2.9 we have $(P_t f)'' = 0$, and $(P_t f)'$ is constant for any $t \geq 0$. So there exist smooth functions $a = a(t), b = b(t) \in \mathbb{R}$ such that

$$P_t f(x) = a(t)x + b(t).$$

Notice that $\frac{\mathrm{d}}{\mathrm{d}t} P_t f = (P_t f)'' - (\ln h)'(P_t f)'$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} a(t)x + \frac{\mathrm{d}}{\mathrm{d}t} b(t) = Kxa(t).$$

Hence $a(t) = Ce^{Kt}$ with $C = |f'| > 0$, and $b \equiv 0$.

□

3. Rigidity of the 1-Bakry-Émery inequality

3.1. **Equality in the 1-Barky-Émery inequality.** In this part, we will prove one of the most important results in this paper, concerning the equality in the 1-Bakry-Émery inequality. Several intermediate steps, which corresponds to the results in [2, §2] of Ambrosio-Brué-Semola, will be proved in separate lemmas before the main Theorem 3.7. We remark that some arguments used in [2] concerning $\mathrm{RCD}(0, N)$ spaces are not available now. For example, there is no two-sides heat kernel estimate or uniform volume doubling property for general $\mathrm{RCD}(K, \infty)$ spaces. Fortunately, we can overcome these difficulties by making full use of the heat flow and the functional analysis tools developed by Gigli in [27].

Lemma 3.1. *Let $(X, \mathrm{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, \infty)$ space with $K \in \mathbb{R}$. Assume there exists a non-constant function $f \in \mathbb{V}$ satisfying*

$$|\nabla P_{t_0} f| = e^{-Kt_0} P_{t_0} |\nabla f| \quad \textit{for some } t_0 > 0.$$

For any $s \in (0, t_0)$, denote

$$A_s := \Big\{ |\nabla P_s f| = 0 \Big\}.$$

Then it holds

$$\mathfrak{m}(A_s) = 0.$$

In particular,

$$\mathfrak{m}\Big(\big\{ P_s f = c \big\} \Big) = 0, \qquad \forall\, c \in \mathbb{R}.$$

Proof. Assume by contradiction that $\mathfrak{m}(A_s) > 0$ for some $s > 0$. Since f is non-constant, we know $\mathfrak{m}(A_s) \in (0, 1)$.

Recall that f attains the equality in the 1-Barky-Émery inequality, we have

$$P_s |\nabla f| = e^{Ks} |\nabla P_s f| = 0, \qquad \text{on } A_s.$$

Thus

$$0 = \int_{A_s} P_s |\nabla f| \,\mathrm{d}\mathfrak{m} = \int P_s(\chi_{A_s}) |\nabla f| \,\mathrm{d}\mathfrak{m}.$$

Denote $A_0^c := \Big\{|\nabla f| > 0\Big\}$. We can see that

$$\int_{A_0^c} P_s(\chi_{A_s})\,\mathrm{d}\mathfrak{m} = 0, \tag{3.1}$$

i.e. $P_s(\chi_{A_s}) = 0$ on A_0^c. Note that $P_s(\chi_{A_s})$ is Lipschitz continuous, and by dimension-free Harnack inequality on $\mathrm{RCD}(K,\infty)$ spaces proved by H.-Q. Li in [39, Theorem 3.1], it holds

$$\big((P_s\chi_{A_s})(y)\big)^2 \leq (P_s\chi_{A_s})(x)\exp\Big\{\frac{K\mathrm{d}^2(x,y)}{e^{2Ks}-1}\Big\}.$$

So $P_s(\chi_{A_s})(x) > 0$ at every point $x \in X$. Thus $\mathfrak{m}(A_0^c) = 0$ and $\mathfrak{m}(A_0) = 1$, which contradicts to the assumption that f is non-constant.

Finally, by locality of the weak gradient (c.f. [8, Proposition 5.16]), it holds $|\nabla P_s f| = 0$ $\mathfrak{m}$-a.e. on $\{P_s f = c\}$. So $\mathfrak{m}\big(\{P_s f = c\}\big) \leq \mathfrak{m}(A_s) = 0$.

□

Lemma 3.2. *Under the same assumption as Lemma 3.1. Denote $b_s := \frac{\nabla P_s f}{e^{-Ks}|\nabla P_s f|}$. Then for any $g \in \mathbb{V}$ and $s,t \in \mathbb{R}^+$ with $s+t < t_0$, it holds*

$$\langle b_{t+s}, \nabla P_t g\rangle = P_t\langle b_s, \nabla g\rangle.$$

Proof. By 1-Bakry-Émery inequality and the assumption, for any $s,t,r \in (0,t_0)$ with $s+t+r = t_0$, we can see that

$$\begin{aligned}
0 &\geq e^{-Kr}P_r\Big(|\nabla P_{t+s}f| - e^{-Kt}P_t|\nabla P_s f|\Big)\\
&= \Big(e^{-Kr}P_r|\nabla P_{t+s}f| - \underbrace{e^{-K(t+s+r)}P_{t+s+r}|\nabla f|}_{e^{-Kt_0}P_{t_0}|\nabla f|}\Big)\\
&\quad + \Big(e^{-K(t+s+r)}P_{t+s+r}|\nabla f| - e^{-K(t+r)}P_{t+r}|\nabla P_s f|\Big)\\
&= \Big(e^{-Kr}P_r|\nabla P_{t+s}f| - \underbrace{|\nabla P_{t+s+r}f|}_{|\nabla P_{t_0}f|}\Big) + \Big(e^{-K(t+s+r)}P_{t+s+r}|\nabla f| - e^{-K(t+r)}P_{t+r}|\nabla P_s f|\Big)\\
&\geq 0.
\end{aligned}$$

Thus

$$|\nabla P_{t+s}f| = e^{-Kt}P_t|\nabla P_s f| \tag{3.2}$$

for any $s,t \in \mathbb{R}^+$ with $s+t < t_0$ (c.f. [2, Lemma 2.4, 2.7]).

Fix $t > 0$ and consider the Euler equation associated with the functional

$$\Psi(h) := \int \big(e^{-Kt}P_t|\nabla h| - |\nabla P_t h|\big)\varphi\,\mathrm{d}\mathfrak{m}, \qquad h \in \mathbb{V}, \varphi \in \mathrm{Lip}_{bs}(X,\mathrm{d}).$$

From Lemma 3.1 we know $\frac{\nabla P_s f}{|\nabla P_s f|}$ is well-defined and $\left|\frac{\nabla P_s f}{|\nabla P_s f|}\right| = 1$ $\mathfrak{m}$-a.e.. Using a standard variation argument (c.f. [2, proof of Proposition 2.6]), for any $g \in \mathbb{V}$ and $s > 0$ with $s+t < t_0$, we get

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}|_{\epsilon=0}\Psi(P_s f+\epsilon g) \\ &= \int \Big(e^{-Kt}P_t\big(\frac{\langle \nabla P_s f, \nabla g\rangle}{|\nabla P_s f|}\big)-\frac{\langle \nabla P_{t+s} f, \nabla P_t g\rangle}{|\nabla P_{t+s} f|}\Big)\varphi \,\mathrm{d}\mathfrak{m} \\ &= e^{-K(t+s)}\int \Big(P_t\langle b_s, \nabla g\rangle - \langle b_{t+s}, \nabla P_t g\rangle\Big)\varphi\,\mathrm{d}\mathfrak{m}. \end{aligned}$$

Then the conclusion follows from the arbitrariness of φ.

□

Lemma 3.3. *Let $(X, \mathrm{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K,\infty)$ probability space. Assume there is a non-constant function $f \in \mathbb{V}$ satisfying*

$$|\nabla P_{t_0} f| = e^{-Kt_0}P_{t_0}|\nabla f| \qquad \text{for } t_0>0,$$

and denote $b_s := \frac{\nabla P_s f}{e^{-Ks}|\nabla P_s f|}$.

Then $b_s \in \mathrm{D}(\mathrm{div})$ for any $s\in(0,t_0)$. Furthermore, for any $s,t>0$ with $s+t<t_0$,

$$P_t \mathrm{div}(b_{t+s}) = \mathrm{div}(b_s). \tag{3.3}$$

In particular, $\mathrm{div}(b_s)\in \mathrm{D}(\Delta)$ and $\Delta\mathrm{div}(b_s)\in\mathbb{V}$.

Proof. For any $g\in\mathbb{V}$, we have

$$\begin{aligned} \left|\int \langle b_s, \nabla g\rangle\,\mathrm{d}\mathfrak{m}\right| &= \left|\int P_t\langle b_s,\nabla g\rangle\,\mathrm{d}\mathfrak{m}\right| \\ \text{By Lemma 3.2} \quad &= \left|\int \langle b_{t+s},\nabla P_t g\rangle\,\mathrm{d}\mathfrak{m}\right| \\ &\le \int |b_{t+s}||\nabla P_t g|\,\mathrm{d}\mathfrak{m} \\ \text{By } |b_r|=e^{Kr} \text{ and Cauchy-Schwartz inequality} \quad &\le e^{(t+s)K}\sqrt{\mathbb{E}(P_t g)}. \end{aligned}$$

Note that it holds a standard estimate (c.f. Lemma 2.2) $\mathbb{E}(P_t g) \le \frac{1}{2t}\|g\|^2_{L^2}$. Hence by Riesz representation theorem, $b_s \in \mathrm{D}(\mathrm{div})$.

At last, the identity (3.3) follows immediately from Lemma 3.2. □

Proposition 3.4. *Keep the same assumption and notations as in Lemma 3.3. It holds*

$$\int \big(\mathrm{div}(b_s)\big)^2\,\mathrm{d}\mathfrak{m} = e^{2Ks}\int\big(\mathrm{div}(b_0)\big)^2\,\mathrm{d}\mathfrak{m}$$

for all $s\in[0,t_0]$.

Proof. **Step 1**:

Given $g\in\mathbb{V}$. Consider the following function $t\mapsto\psi(t,g)$ defined on $\mathbb{R}^+$

$$\psi(t,g) := \int e^{Kt}|\nabla P_t g|\,\mathrm{d}\mathfrak{m}.$$

From 1-Bakry-Émery inequality we know ψ is non-increasing in t, thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(t,g) = \int Ke^{Kt}|\nabla P_t g| + \langle b^g_t, \nabla\Delta P_t g\rangle\,\mathrm{d}\mathfrak{m} \le 0$$

where $b_t^g := e^{Kt}\frac{\nabla P_t g}{|\nabla P_t g|} \in L^2(TX)$. Note also that $b_t^f = b_t$.

Fix $s \in (0, t_0)$. By assumption, the function $t \mapsto \psi(t, P_s f)$ is constant on $[0, t_0 - s]$. So $\frac{\mathrm{d}}{\mathrm{d}t}\psi(t, P_s f) = 0$ for $t \in [0, t_0 - s]$, this means

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(t, P_s f) = \int K e^{Kt}|\nabla P_{t+s} f| \,\mathrm{d}\mathfrak{m} + \int \langle b_t^{P_s f}, \nabla \Delta P_{t+s} f\rangle \,\mathrm{d}\mathfrak{m} = 0 \qquad \forall t \in [0, t_0 - s].$$

Fix t and consider the following functional

$$\mathbb{V} \ni g \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\psi(t, g) = \int K e^{Kt}|\nabla P_t g| + \langle b_t^g, \nabla \Delta P_t g\rangle \,\mathrm{d}\mathfrak{m} \leq 0$$

which attains its maximum at $g = P_s f$.

Thus for any $\epsilon \in \mathbb{R}$,

$$\begin{aligned}
& 0 \\
\geq \;& \frac{\mathrm{d}}{\mathrm{d}t}\psi(t, P_s f + \epsilon g) - \frac{\mathrm{d}}{\mathrm{d}t}\psi(t, P_s f) \\
= \;& \underbrace{\int K e^{Kt}\Big(|\nabla P_t(P_s f + \epsilon g)| - |\nabla P_{t+s} f|\Big) \,\mathrm{d}\mathfrak{m}}_{I} \\
& + \underbrace{\int \Big(\langle b_t^{P_s f}, \nabla \Delta P_t(P_s f + \epsilon g)\rangle - \langle b_t^{P_s f}, \nabla \Delta P_{t+s} f\rangle\Big) \,\mathrm{d}\mathfrak{m}}_{II} \\
& + \underbrace{\int \Big(\langle b_t^{P_s f + \epsilon g}, \nabla \Delta P_t(P_s f + \epsilon g)\rangle - \langle b_t^{P_s f}, \nabla \Delta P_t(P_s f + \epsilon g)\rangle\Big) \,\mathrm{d}\mathfrak{m}}_{III}.
\end{aligned}$$

Define $\mathrm{F}_t \subset \mathbb{V}$ by

$$\mathrm{F}_t := \Big\{ g : g \in \mathbb{V} \cap L^\infty(X, \mathfrak{m}), \frac{|\nabla P_t g|}{|\nabla P_{t+s} f|} \in L^\infty(X, \mathfrak{m}) \Big\}. \tag{3.4}$$

By Lemma 3.5,

$$\mathrm{F}_0 \subset \mathrm{F}_r \subset \mathrm{F}_t, \qquad \forall\, 0 \leq r \leq t,$$

and F_0 is an algebra.

For any $g \in \mathrm{F}_t$ and ϵ small enough, we can write I, II, III in the following ways

$$I \;=\; K e^{Kt} \int \int_0^\epsilon \frac{\langle \nabla P_t(P_s f + \tau g), \nabla P_t g\rangle}{|\nabla P_t(P_s f + \tau g)|} \,\mathrm{d}\tau \,\mathrm{d}\mathfrak{m},$$

$$II \;=\; \epsilon \int \langle b_t^{P_s f}, \nabla \Delta P_t g\rangle \,\mathrm{d}\mathfrak{m},$$

and

$$\begin{aligned}
III &= e^{Kt}\int \langle \frac{|\nabla P_{t+s}f|\nabla P_t(P_sf+\epsilon g)-|\nabla P_t(P_sf+\epsilon g)|\nabla P_{t+s}f}{|\nabla P_{t+s}f||\nabla P_t(P_sf+\epsilon g)|}, \nabla\Delta P_t(P_sf+\epsilon g)\rangle\,\mathrm{d}\mathfrak{m}\\
&= e^{Kt}\int \langle \frac{|\nabla P_{t+s}f|\nabla P_t(P_sf+\epsilon g)-|\nabla P_{t+s}f|\nabla P_{t+s}f}{|\nabla P_{t+s}f||\nabla P_t(P_sf+\epsilon g)|}, \nabla\Delta P_t(P_sf+\epsilon g)\rangle\,\mathrm{d}\mathfrak{m}\\
&\quad +e^{Kt}\int \langle \frac{|\nabla P_{t+s}f|\nabla P_{t+s}f-|\nabla P_t(P_sf+\epsilon g)|\nabla P_{t+s}f}{|\nabla P_{t+s}f||\nabla P_t(P_sf+\epsilon g)|}, \nabla\Delta P_t(P_sf+\epsilon g)\rangle\,\mathrm{d}\mathfrak{m}\\
&= \epsilon e^{Kt}\int \langle \frac{\nabla P_t g}{|\nabla P_t(P_sf+\epsilon g)|}, \nabla\Delta P_t(P_sf+\epsilon g)\rangle\,\mathrm{d}\mathfrak{m}\\
&\quad +e^{Kt}\int \Big(\int_\epsilon^0 \frac{\langle \nabla P_t(P_sf+\tau g), \nabla P_t g\rangle}{|\nabla P_t(P_sf+\tau g)||\nabla P_t(P_sf+\epsilon g)|}\,\mathrm{d}\tau\Big)\langle \frac{\nabla P_{t+s}f}{|\nabla P_{t+s}f|}, \nabla\Delta P_t(P_sf+\epsilon g)\rangle\,\mathrm{d}\mathfrak{m}\\
&= O(\epsilon).
\end{aligned}$$

From the discussions above, for any $g\in \mathrm{F}_t$, we can see that $\epsilon \to \frac{\mathrm{d}}{\mathrm{d}t}\psi(t, P_sf+\epsilon g) = I+II+III$ is absolutely continuous and hence differentiable when ϵ is small.

Similar to the proof of Lemma 3.2, by a variational argument we get

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0}\frac{\mathrm{d}}{\mathrm{d}t}\psi(t, P_sf+\epsilon g)\\
&= \underbrace{\int \Big(K\langle b_t^{P_sf}, \nabla P_t g\rangle + \langle b_t^{P_sf}, \nabla\Delta P_t g\rangle\Big)\,\mathrm{d}\mathfrak{m}}_{V_t^1(\nabla P_t g)}\\
&\quad \underbrace{+e^{Kt}\int \Big(\frac{1}{|\nabla P_{t+s}f|}\langle \nabla P_t g, \nabla\Delta P_{t+s}f\rangle - \frac{1}{|\nabla P_{t+s}f|^3}\langle \nabla P_{t+s}f, \nabla P_t g\rangle\langle \nabla P_{t+s}f, \nabla\Delta P_{t+s}f\rangle\Big)\,\mathrm{d}\mathfrak{m}}_{V_t^2(\nabla P_t g)}.
\end{aligned}$$

Step 2:

Define

$$\mathrm{D}_t := \mathrm{Span}\left(\left\{\nabla g : g\in \mathbb{V}, \frac{|\nabla g|}{|\nabla P_{t+s}f|}\in L^\infty(X,\mathfrak{m})\right\}\right).$$

where $\mathrm{Span}(S)$ means the sub-module of $L^2(TX)$ consisting of all finite L^∞-linear combinations of the elements in S. By definition of F_t, we can see that

$$\left\{\nabla P_t g : g\in \mathrm{F}_t\right\}\subset \mathrm{D}_t. \tag{3.5}$$

Furthermore, by linearity V_t^1, V_t^2 can be uniquely defined on D_t by:

$$\begin{aligned}
V_t^1(\nabla g) &:= \int \Big(K\langle b_t^{P_sf}, \nabla g\rangle + \langle b_t^{P_sf}, \nabla\Delta g\rangle\Big)\,\mathrm{d}\mathfrak{m}\\
&= \int \Big(K\langle b_t^{P_sf}, \nabla g\rangle + \langle \nabla \mathrm{div}(b_t^{P_sf}), \nabla g\rangle\Big)\,\mathrm{d}\mathfrak{m}
\end{aligned}$$

and

$$V_t^2(\nabla g) := e^{Kt}\int \Big(\frac{\langle \nabla g, \nabla\Delta P_{t+s}f\rangle}{|\nabla P_{t+s}f|} - \frac{\langle \nabla P_{t+s}f, \nabla g\rangle\langle \nabla P_{t+s}f, \nabla\Delta P_{t+s}f\rangle}{|\nabla P_{t+s}f|^3}\Big)\,\mathrm{d}\mathfrak{m}.$$

From the discussion above we can be seen that

$$V_t^1(\nabla P_t g) + V_t^2(\nabla P_t g) = 0, \qquad \forall g \in \mathrm{F}_t. \tag{3.6}$$

Define

$$\mathrm{Adm} := \Big\{\varphi : \varphi \in C^1(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R}) \cap L^\infty(\mathbb{R})\Big\}$$

and

$$\mathrm{F} := \Big\{g : g = \varphi(P_s f), \varphi \in \mathrm{Adm}\Big\}.$$

From Lemma 3.5, we know $\mathrm{F} \subset \mathrm{F}_0 \subset \mathrm{F}_t$ for any $t \in [0, t_0 - s]$. By (3.5) we get

$$\Big\{\nabla P_t g : g \in \mathrm{F}\Big\} \subset \Big\{\nabla P_t g : g \in \mathrm{F}_0\Big\} \subset \Big\{\nabla P_t g : g \in \mathrm{F}_t\Big\} \subset \mathrm{D}_t.$$

Combining with (3.6) we know

$$V_t^1(\nabla P_t g) + V_t^2(\nabla P_t g) = 0, \qquad \forall g \in \mathrm{F}_0. \tag{3.7}$$

Letting $t \to 0$ in (3.7), by dominated convergence theorem and the fact that F_0 is an algebra, we obtain

$$V_0^1\big(\nabla(gh)\big) + V_0^2\big(\nabla(gh)\big) = 0, \qquad \forall g \in \mathrm{F},\ h \in \mathrm{F}_0. \tag{3.8}$$

Thus for any $h \in \mathrm{F}_0$, $g = \varphi(\epsilon P_s f) \in \mathrm{F}$ with $\varphi = \arctan \in \mathrm{Adm}$ and $\epsilon > 0$, it holds

$$V_0^1\big(g\nabla h\big) + \epsilon V_0^1\big(h\varphi'(\epsilon P_s f)\nabla \mathrm{P}_s f\big) + V_0^2\big(g\nabla h\big) + \epsilon V_0^2\big(h\varphi'(\epsilon P_s f)\nabla P_s f\big) = 0, \tag{3.9}$$

Dividing ϵ on both sides of (3.9) and letting $\epsilon \to \infty$, by dominated convergence theorem, we obtain

$$V_0^1\big(h\nabla \mathrm{P}_s f\big) + V_0^2\big(h\nabla P_s f\big) = 0, \qquad \forall\, h \in \mathrm{F}_0. \tag{3.10}$$

From the structure of V_0^2, we can see that

$$\begin{aligned} & V_0^2(h\nabla P_s f) \\ = \ & e^{Kt}\int h\Big(\frac{1}{|\nabla P_s f|}\langle \nabla P_s f, \nabla\Delta P_s f\rangle - \frac{1}{|\nabla P_s f|^3}\langle \nabla P_s f, \nabla P_s f\rangle\langle \nabla P_s f, \nabla\Delta P_s f\rangle\Big)\,\mathrm{d}\mathfrak{m} \\ = \ & 0. \end{aligned}$$

By (3.10), for any $h \in \mathrm{F}_0$, it holds

$$V_0^1\big(h\nabla \mathrm{P}_s f\big) = \int \Big(K\langle b_s, \nabla P_s f\rangle + \langle \nabla \mathrm{div}(b_s), \nabla P_s f\rangle\Big) h\,\mathrm{d}\mathfrak{m} = 0. \tag{3.11}$$

By Lemma 3.6, (3.11) yields

$$K\langle b_s, \nabla P_s f\rangle + \langle \nabla \mathrm{div}(b_s), \nabla P_s f\rangle = 0.$$

Hence we can pick $h = \frac{1}{|\nabla P_s f|}$ in (3.11), so that

$$\int K|b_s|^2 - \big(\mathrm{div}(b_s)\big)^2\,\mathrm{d}\mathfrak{m} = 0.$$

Note that $|b_s| = e^{Ks}$, it holds

$$\int \big(\mathrm{div}(b_s)\big)^2\,\mathrm{d}\mathfrak{m} = \int K|b_s|^2 = e^{2Ks}\int K|b_0|^2 = e^{2Ks}\int \big(\mathrm{div}(b_0)\big)^2\,\mathrm{d}\mathfrak{m}$$

which is the thesis.

□

In the following two lemmas, we keep the same notions as in the proof of Proposition 3.4.

Lemma 3.5. *For any $r \le t \le t_0 - s$, we have $F \subset \mathrm{F}_r \subset \mathrm{F}_t$. In particular, F_0 is an algebra.*

Proof. For any $g = \varphi(P_s f) \in \mathrm{F}$, by chain rule (c.f. [27, Theorem 2.2.6]) we know

$$\nabla g = \nabla \varphi(P_s f) = \varphi'(P_s f)\nabla P_s f, \qquad \forall \varphi \in \mathrm{Adm}. \tag{3.12}$$

Since $\varphi \in \mathrm{Adm}$, $\varphi' \in L^\infty(\mathbb{R})$, we know $\varphi'(P_s f) \in L^\infty(X, \mathfrak{m})$. So by 1-Bakry-Émery inequality and the assumption that f attains the equality in the 1-Bakry-Émery inequality, we have

$$\begin{array}{rcl} |\nabla P_t g| & \le & e^{-Kt} P_t |\nabla g| \\ \text{By (3.12)} \qquad & \le & e^{-Kt} P_t\big(|\varphi'(P_s f)||\nabla P_s f|\big) \\ & \le & \|\varphi'(P_s f)\|_{L^\infty} e^{-Kt} P_t |\nabla P_s f| \\ \text{By (3.2)} \qquad & = & \|\varphi'(P_s f)\|_{L^\infty} |\nabla P_{t+s} f|. \end{array}$$

Thus by definition of F_t (see (3.4)), $\mathrm{F} \subset \mathrm{F}_t$ for any $t \le t_0 - s$. Furthermore, for any $r \le t \le t_0 - s$ and $g \in \mathrm{F}_r$, there is $C_2 = \big\| \frac{|\nabla P_r g|}{|\nabla P_{r+s} f|} \big\|_{L^\infty} > 0$ such that

$$\begin{array}{rcl} |\nabla P_t g| & \le & e^{-K(t-r)} P_{t-r} |\nabla P_r g| \\ & \le & C_2 e^{-K(t-r)} P_{t-r}\big(|\nabla P_{r+s} f|\big) \\ & = & C_2 |\nabla P_{t+s} f|. \end{array}$$

Hence $\mathrm{F}_r \subset \mathrm{F}_t$.

In particular, for any $g, h \in \mathrm{F}_0$, there is $C_3 > 0$ such that

$$|\nabla(gh)| \le \|g\|_{L^\infty} |\nabla h| + \|h\|_{L^\infty} |\nabla g| \le C_3 |\nabla P_s f|,$$

so by definition $gh \in \mathrm{F}_0$ and F_0 is an algebra.

$\square$

Next we will show that the set F_0 includes all Lipschitz functions with bounded support.

Lemma 3.6. *The set $\mathrm{Lip}_{bs}(X, \mathrm{d})$ of Lipschitz functions with bounded support is a subset of F_0. In particular, if there is $H \in L^1(X, \mathfrak{m})$ such that*

$$\int H h \, \mathrm{d}\mathfrak{m}, \qquad \forall\, h \in \mathrm{F}_0.$$

Then $H = 0$.

Proof. Given $g \in \mathrm{Lip}_{bs}$ with $\operatorname{supp} g \subset B_R(x)$ for some $R > 0$ and $x \in X$. By definition, $|\nabla g| \le \mathrm{Lip}(g)$ where $\mathrm{Lip}(g)$ is a non-negative real constant.

By assumption $|\nabla P_s f| = e^{-Ks} P_s |\nabla f|$ and $|\nabla f| \ne 0$. Pick a non-zero non-negative function $G \in L^\infty$ satisfying $G^2 \le \min\{|\nabla f|, 1\}$. So by Lipschitz regularization of the heat flow, $P_s G^2$ is Lipschitz and

$$P_s G^2 \le P_s |\nabla f| = e^{-Ks} |\nabla P_s f|.$$

By dimension-free Harnack inequality [39, Theorem 3.1], for any $y_1, y_2 \in X$,

$$\big((P_s G^2)(y_1)\big)^2 \le \big((P_s G)(y_1)\big)^2 \le \big(P_s G^2\big)(y_2) \exp\Big\{ \frac{K \mathrm{d}^2(y_1, y_2)}{e^{2Ks} - 1} \Big\}. \tag{3.13}$$

Let $y_2 = x$ in (3.13), since G is non-zero, we know $(P_s G^2)(x) > 0$. Let $y_1 = x$ and $y_2 \in B_R(x)$ (3.13), we know $\inf_{y\in B_R(x)} P_s G^2 > 0$. Thus there is $C > 0$ such that

$$|\nabla g| \le \mathrm{Lip}(g) < C \inf_{y\in B_R(x)} P_s G^2 \le C e^{-Ks} |\nabla P_s f| \quad \text{on} \quad B_R(x)$$

which is the thesis.

Furthermore, if

$$\int H h \,\mathrm{d}\mathfrak{m}, \qquad \forall \quad h \in \mathrm{F}_0.$$

Via approximation by Lipschitz function with bounded support, we can prove that $\int_E H \,\mathrm{d}\mathfrak{m} = 0$ for all measurable set $E \subset X$. So $H \equiv 0$. □

Theorem 3.7 (Equality in the 1-Bakry-Émery inequality)**.** *Let $(X, \mathrm{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, \infty)$ probability space with $K \in \mathbb{R}$. Assume there exists a non-constant $f \in \mathbb{V}$ attaining the equality in the 1-Bakry-Émery inequality*

$$|\nabla P_{t_0} f| = e^{-Kt_0} P_{t_0} |\nabla f| \qquad \textit{for some } t_0 > 0.$$

Denote $b_s := e^{Ks} \frac{\nabla P_s f}{|\nabla P_s f|}$. Then it holds the following properties:

a) $\frac{\nabla P_s f}{|\nabla P_s f|} = e^{-Ks} b_s =: b$ *is independent of $s \in (0, t_0)$;*
b) $\nabla \mathrm{div}(b) = -Kb$*;*
c) $\Delta \mathrm{div}(b) = -K \mathrm{div}(b)$*, thus $f = \mathrm{div}(b)$ attains the equality in the 2-Barky-Émery inequality.*

Furthermore, denote by $(F_t)_{t\in\mathbb{R}^+}$ the regular Lagrangian flow associated with b, we have

$$(F_t)_\sharp \mathfrak{m} = e^{-\frac{K}{2}\left(t^2 + \frac{2}{K} t \mathrm{div}(b)\right)} \mathfrak{m} \qquad \textit{if } K \neq 0, \tag{3.14}$$

and

$$(F_t)_\sharp \mathfrak{m} = \mathfrak{m} \qquad \textit{if } K = 0. \tag{3.15}$$

Proof. **Part 1**:

By Lemma 3.3 we know $b_s \in \mathrm{D(div)}$ for any $s \in (0, t_0)$. For any $\varphi \in \mathrm{D}(\Delta)$ and $s, t, h > 0$ with $h < \frac{1}{2} t$ and $s + t + h < t_0$, we have

$$\begin{aligned} & \int \Big(P_{t+h}\varphi - P_t \varphi\Big) \mathrm{div}(b_{t+s}) \,\mathrm{d}\mathfrak{m} \\ = & \int \big(P_{t+h}\varphi\big) \mathrm{div}(b_{t+s+h}) \,\mathrm{d}\mathfrak{m} - \int \big(P_t \varphi\big) \mathrm{div}(b_{t+s}) \,\mathrm{d}\mathfrak{m} \\ & - \int \big(P_{t+h}\varphi\big) \Big(\mathrm{div}(b_{t+h+s}) - \mathrm{div}(b_{t+s})\Big) \,\mathrm{d}\mathfrak{m} \\ \text{By Lemma 3.2} \quad = & \int \varphi \mathrm{div}(b_s) \,\mathrm{d}\mathfrak{m} - \int \varphi \mathrm{div}(b_s) \,\mathrm{d}\mathfrak{m} - \int \big(P_h \varphi\big) \Big(\mathrm{div}(b_{h+s}) - \mathrm{div}(b_s)\Big) \,\mathrm{d}\mathfrak{m} \\ = & -\int \big(P_h \varphi\big) \Big(\mathrm{div}(b_{h+s}) - \mathrm{div}(b_s)\Big) \,\mathrm{d}\mathfrak{m}. \end{aligned}$$

Therefore,

$$\int \Big(\frac{P_{t+h}\varphi - P_t\varphi}{h}\Big) \mathrm{div}(b_{t+s}) \,\mathrm{d}\mathfrak{m} = -\int \big(P_h\varphi\big) \Big(\frac{\mathrm{div}(b_{h+s}) - \mathrm{div}(b_s)}{h}\Big) \,\mathrm{d}\mathfrak{m}. \tag{3.16}$$

By Cauchy-Schwarz inequality and the estimate $\|\Delta P_t\varphi\|_{L^2} \leq \frac{1}{t}\|\varphi\|_{L^2}$ (c.f. Lemma 2.2), we get the following estimate from (3.16)

$$\begin{aligned}
\left|\int P_h\varphi\Big(\mathrm{div}(b_{h+s}) - \mathrm{div}(b_s)\Big)\,\mathrm{d}\mathfrak{m}\right| &\leq \big\|P_{t+h}\varphi - P_t\varphi\big\|_{L^2}\big\|\mathrm{div}(b_{t+s})\big\|_{L^2} \\
&= \left\|\int_t^{t+h}\Delta P_s\varphi\,\mathrm{d}s\right\|_{L^2}\big\|\mathrm{div}(b_{t+s})\big\|_{L^2} \\
&\leq \left(h\int_t^{t+h}\|\Delta P_{s-h}(P_h\varphi)\|_{L^2}^2\,\mathrm{d}s\right)^{\frac{1}{2}}\big\|\mathrm{div}(b_{t+s})\big\|_{L^2} \\
&\leq h\frac{2}{t}\|P_h\varphi\|_{L^2}\big\|\mathrm{div}(b_{t+s})\big\|_{L^2}.
\end{aligned}$$

Thus by arbitrariness of φ and the density of $P_h\big(L^2(X,\mathfrak{m})\big)$ in $L^2(X,\mathfrak{m})$, we obtain

$$\big\|\mathrm{div}(b_{h+s}) - \mathrm{div}(b_s)\big\|_{L^2} \lesssim h.$$

Therefore $s \mapsto \mathrm{div}(b_s)$ is absolutely continuous and differentiable in L^2 for a.e. $s \in [0,t_0]$. Furthermore, for $s \in [0,t_0]$ where $\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{div}(b_s)$ exists, it holds

$$\begin{aligned}
& \int(\Delta\varphi)\mathrm{div}(b_s)\,\mathrm{d}\mathfrak{m} \\
\text{By Lemma 3.2} \quad &= \int(\Delta P_t\varphi)\mathrm{div}(b_{t+s})\,\mathrm{d}\mathfrak{m} \\
&= \int(\frac{\mathrm{d}}{\mathrm{d}t}P_t\varphi)\mathrm{div}(b_{t+s})\,\mathrm{d}\mathfrak{m} \\
\text{Letting } h \to 0 \text{ in (3.16)} \quad &= -\int\varphi\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{div}(b_s)\,\mathrm{d}\mathfrak{m}.
\end{aligned}$$

Therefore, for a.e. $s \in [0,t_0]$,

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{div}(b_s) = -\Delta\mathrm{div}(b_s). \tag{3.17}$$

So by Poincaré inequality, we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{2}\int\big(\mathrm{div}(b_s)\big)^2\,\mathrm{d}\mathfrak{m} &= \int\mathrm{div}(b_s)\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{div}(b_s)\,\mathrm{d}\mathfrak{m} \\
\text{By (3.17)} \quad &= -\int\mathrm{div}(b_s)\Delta\mathrm{div}(b_s)\,\mathrm{d}\mathfrak{m} \\
&= \int|\nabla\mathrm{div}(b_s)|^2\,\mathrm{d}\mathfrak{m} \\
\text{By Poincaré inequality} \quad &\geq K\int\big(\mathrm{div}(b_s)\big)^2\,\mathrm{d}\mathfrak{m}.
\end{aligned}$$

By Grönwall's lemma, we obtain

$$\int\big(\mathrm{div}(b_s)\big)^2\,\mathrm{d}\mathfrak{m} \geq e^{2Ks}\int\big(\mathrm{div}(b_0)\big)^2\,\mathrm{d}\mathfrak{m}. \tag{3.18}$$

By Proposition 3.4, the inequality in (3.18) is actually an equality. So for any $s \in (0, t_0)$, $\mathrm{div}(b_s)$ attains the equality in the Poincaré inequality. By Lemma 2.9 we know

$$\Delta \mathrm{div}(b_s) = -K \mathrm{div}(b_s). \tag{3.19}$$

For any $\varphi \in \mathbb{V}$, we have

$$\int \langle \nabla \varphi, \nabla \mathrm{div}(b_s) \rangle = -\int \varphi \Delta \mathrm{div}(b_s) = \int \varphi K \mathrm{div}(b_s) = \int -K \langle b_s, \nabla \varphi \rangle.$$

Thus

$$\nabla \mathrm{div}(b_s) = -K b_s. \tag{3.20}$$

In addition, by (3.17) and (3.19), it holds $\frac{\mathrm{d}}{\mathrm{d}s} \mathrm{div}(b_s) = K \mathrm{div}(b_s)$ and

$$\frac{\mathrm{d}}{\mathrm{d}s} e^{-Ks} \mathrm{div}(b_s) = -K e^{-Ks} \mathrm{div}(b_s) + e^{-Ks} \frac{\mathrm{d}}{\mathrm{d}s} \mathrm{div}(b_s) = 0.$$

Combining with (3.20) we know $b := e^{-Ks} b_s$ is independent of s.

Finally, by (3.19) and (3.20) we get

$$\Delta \mathrm{div}(b) = -K \mathrm{div}(b) \tag{3.21}$$

and

$$\nabla \mathrm{div}(b) = -K b. \tag{3.22}$$

Part 2:

The identities (3.14) and (3.15) can be proved using similar argument as [30, §4] (and [2, §2]). For reader's convenience, we offer more details here.

Firstly, by c) and Lemma 2.9, we know $\mathrm{div}(b) \in \mathrm{TestF}_{\mathrm{loc}}$ and $\mathrm{Hess}_{\mathrm{div}(b)} = 0$. Secondly, by b) and c) we know $-K \nabla_{sym} b = \mathrm{Hess}_{\mathrm{div}(b)} = 0$ (c.f. [13, §5] or [27, §3.4] for details about the covariant derivative). If $K \neq 0$, $\nabla_{sym} b = 0$. If $K = 0$, by b) it holds $\nabla \mathrm{div}(b) = 0$ so $\mathrm{div}(b)$ is constant. Note that $\int \mathrm{div}(b) \, \mathrm{d}\mathfrak{m} = 0$, so $\mathrm{div}(b) = 0$. Then following the argument in [2, proof of Proposition 2.8] we can still prove $\nabla_{sym} b = 0$.

Combining [13, Theorems 9.7] of Ambrosio-Trevisan and a truncation argument (c.f. [30, Theorem 4.2]), we can prove that the regular Lagrangian flow $F_t(x)$ associated with b exists for all $(t, x) \in \mathbb{R}^+ \times X$. Thus the curve $(F_t)_\sharp \mathfrak{m}$ is well-defined for all $t \in \mathbb{R}^+$.

By definition of regular Lagrangian flow (F_t) (c.f. [13, §8]), for any $g \in \mathbb{V}$, $\mu_t = (F_t)_\sharp \mathfrak{m}$ solves the following continuity equation

$$\mu_0 = \mathfrak{m}, \qquad \frac{\mathrm{d}}{\mathrm{d}t} \int g \, \mathrm{d}\mu_t = \int b(g) \, \mathrm{d}\mu_t = \int \langle b, \nabla g \rangle \, \mathrm{d}\mu_t \tag{3.23}$$

for a.e. $t \in \mathbb{R}^+$. It has been proved in [13, §5] that the continuity equation (3.23) has a unique solution. If $K = 0$, it can be seen from $\mathrm{div}(b) = 0$ that $\mu_t \equiv \mathfrak{m}$ solves (3.23). For $K \neq 0$, we just need to check that $\mu_t := e^{-\frac{K}{2}\left(t^2 + \frac{2}{K} t \mathrm{div}(b)\right)} \mathfrak{m}$ verifies (3.23).

Given $g \in \mathbb{V}$, by computation,

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}t} \int g\, e^{-\frac{K}{2}\left(t^2 + \frac{2}{K} t \mathrm{div}(b)\right)} \,\mathrm{d}\mathfrak{m} \\
= & \int g\big(-Kt - \mathrm{div}(b)\big)\, e^{-\frac{K}{2}\left(t^2 + \frac{2}{K} t \mathrm{div}(b)\right)} \,\mathrm{d}\mathfrak{m} \\
\text{By c)} \quad = & \int g\Big(-Kt + \frac{1}{K}\Delta\big(\mathrm{div}(b)\big)\Big)\, e^{-\frac{K}{2}\left(t + \frac{2}{K} t \mathrm{div}(b)\right)} \,\mathrm{d}\mathfrak{m} \\
\text{By b)} \quad = & \int -Ktg\, e^{-\frac{K}{2}\left(t + \frac{2}{K} t \mathrm{div}(b)\right)} \,\mathrm{d}\mathfrak{m} + \int \langle b, \nabla g\rangle\, e^{-\frac{K}{2}\left(t + \frac{2}{K} t \mathrm{div}(b)\right)} \,\mathrm{d}\mathfrak{m} \\
& + \int Ktg|b|^2\, e^{-\frac{K}{2}\left(t + \frac{2}{K} t \mathrm{div}(b)\right)} \,\mathrm{d}\mathfrak{m} \\
= & \int \langle b, \nabla g\rangle\, e^{-\frac{K}{2}\left(t + \frac{2}{K} t \mathrm{div}(b)\right)} \,\mathrm{d}\mathfrak{m}
\end{aligned}
$$

which is the thesis.

□

Corollary 3.8. *Let $(X, \mathrm{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, \infty)$ probability space with $K \leq 0$. Then there is no non-constant function attaining the equality in the 1-Bakry-Émery inequality.*

Proof. By c) of Theorem 3.7, $\Delta \mathrm{div}(b) = -K \mathrm{div}(b)$. Thus

$$0 \leq \int |\nabla \mathrm{div}(b)|^2 \,\mathrm{d}\mathfrak{m} = -\int \mathrm{div}(b) \Delta \mathrm{div}(b) \,\mathrm{d}\mathfrak{m} = K \int \mathrm{div}(b)^2 \,\mathrm{d}\mathfrak{m} \leq 0.$$

So $\mathrm{div}(b) = 0$ and $b = 0$. □

Let u be a non-constant affine function (c.f. b) of Lemma 2.9). We know that $|\nabla u|$ is a positive constant and u is Lipschitz. By [30, Theorem 4.4] (or [34, Theorem 3.16]), we know that the gradient flow $(F_t)_{t \geq 0}$ of u, which is also the regular Lagrangian flow associated with $-\nabla u$ in the sense of Ambrosio-Trevisan [13, §8], satisfies the following equality (see also [28])

$$\int \Big(u(x) - u\big(F_t(x)\big)\Big) \,\mathrm{d}\mathfrak{m} = \frac{1}{2}\int_0^t \int |\nabla u|^2 \circ F_s \,\mathrm{d}\mathfrak{m} \,\mathrm{d}s + \frac{1}{2}\int_0^t \int |\dot{F}_s|^2 \circ F_s \,\mathrm{d}\mathfrak{m} \,\mathrm{d}s \tag{3.24}$$

and it induces a family of isometries

$$\mathrm{d}\big(F_t(x), F_t(y)\big) = \mathrm{d}(x, y) \tag{3.25}$$

for any $x, y \in X, t > 0$. More generally, if there is a vector field $b \in L^2(TX)$ with $\mathrm{div}(b) \in L^\infty_{\mathrm{loc}}$ and $\nabla_{sym} b = 0$, by By [2, Theorem 2.1] (or [34, Theorem 3.18]), the regular Lagrangian flow associated with b induces a family of isometries.

In particular, there is a decomposition of X in the form $\{X_q\}_{q \in Q}$, where Q is the set of indices, such that $x_0, x_1 \in X_q$ for some q if and only if there is $t \geq 0$ such that $F_t(x_0) = x_1$ or $F_t(x_1) = x_0$. In this case, X_q is an interval which can be parametrized by $(F_t)_t$ (or u). Define the quotient map $\mathfrak{Q} : X \mapsto Q$ by

$$q = \mathfrak{Q}(x) \Longleftrightarrow x \in X_q.$$

There is a disintegration of $\mathfrak{m}$ consistent with $\mathfrak{Q}$ in the following sense.

Definition 3.9 (Disintegation on sets, c.f. [7], Theorem 5.3.1)**.** Let $(X, \mathscr{X}, \mathfrak{m})$ denote a measure space. Given any family $\{X_q\}_{q\in Q}$ of subsets of X, a *disintegration of* $\mathfrak{m}$ *on* $\{X_q\}_{q\in Q}$ is a measure-space structure $(Q, \mathscr{Q}, \mathfrak{q})$ and a map

$$Q \ni q \longmapsto \mathfrak{m}_q \in \mathcal{M}(X, \mathscr{X})$$

so that:

(1) For $\mathfrak{q}$-a.e. $q \in Q$, $\mathfrak{m}_q$ is concentrated on X_q.
(2) For all $B \in \mathscr{X}$, the map $q \mapsto \mathfrak{m}_q(B)$ is $\mathfrak{q}$-measurable.
(3) For all $B \in \mathscr{X}$, $\mathfrak{m}(B) = \int_Q \mathfrak{m}_q(B)\,\mathfrak{q}(\mathrm{d}q)$; this is abbreviated by $\mathfrak{m} = \int_Q \mathfrak{m}_q \mathfrak{q}(\mathrm{d}q)$.

From Theorem 3.7 and Lemma 2.9, we know there is a decomposition $\{X_q\}_{q\in Q}$ induced by b (or $-\frac{1}{K}\nabla \mathrm{div}(b)$ when $K > 0$) satisfying the following properties.

Corollary 3.10. *Keep the same assumptions and notations as in Theorem 3.7, assume further that $K > 0$. Then there exists a decomposition $\{X_q\}_{q\in Q}$ of X induced by the regular Lagrangian flow (F_t) associated with b, such that:*

(1) for any $q \in \mathfrak{V}$, X_q is a geodesic line in (X, d);
(2) for any $q \in \mathfrak{V}$, $x_1, x_2 \in X_q$, there is a unique t such that

$$t = t|b| = \mathrm{d}(x_1, x_2).$$

and $F_t(x_0) = x_1$ or $F_t(x_1) = x_0$;
(3) there exists a disintegration of $\mathfrak{m}$ on $\{X_q\}_{q\in Q}$

$$\mathfrak{m} = \int_Q \mathfrak{m}_q\,\mathfrak{q}(\mathrm{d}q), \qquad \mathfrak{q}(Q) = 1;$$

(4) for $\mathfrak{q}$-a.e. $q \in Q$ and any $t > 0$, it holds

$$(F_t)_\sharp \mathfrak{m}_q = e^{-\frac{K}{2}\left(t^2 + \frac{2}{K} t\mathrm{div}(b)\right)} \mathfrak{m}_q,$$

and the 1-dimensional metric measure space $(X_q, \mathrm{d}, \mathfrak{m}_q)$ satisfies $\mathrm{CD}(K, \infty)$;
(5) for $\mathfrak{q}$-a.e. $q \in Q$, $\mathrm{div}(b)|_{X_q}$ can be represented by

$$\mathrm{div}(b)(x) = \mathrm{sign}\big(\mathrm{div}(b)(x)\big) K \mathrm{d}(x, x_q),\ x \in X_q,$$

where x_q is the unique point in X_q such that $\mathrm{div}(b)(x_q) = 0$. In particular,

$$\int \mathrm{div}(b)\,\mathrm{d}\mathfrak{m}_q = 0, \qquad \mathfrak{q} - a.e.\ q \in Q. \tag{3.26}$$

Proof. From the construction of the decomposition discussed before, it is not hard to see the validity of assertions (1)- (3). Assertion (4) is a consequence of (3.14) in Theorem 3.7. We will just prove (5). For $u := \frac{1}{K}\mathrm{div}(b)$, by (3.24) and Lemma 3.11 below we have

$$\begin{aligned}
&\int_Q \int_{X_q} \Big(u(x) - u\big(F_t(x)\big)\Big)\,\mathrm{d}\mathfrak{m}_q\,\mathrm{d}\mathfrak{q}(q)\\
= &\int \Big(u(x) - u\big(F_t(x)\big)\Big)\,\mathrm{d}\mathfrak{m}\\
= &\frac{1}{2}\int_0^t \int |\nabla u|^2 \circ F_s\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s + \frac{1}{2}\int_0^t \int |\dot{F}_s|^2 \circ F_s\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s\\
\geq &\frac{1}{2}\int_Q \Big(\int_0^t \int_{X_q} |\mathrm{lip}(u|_{X_q})|^2 \circ F_s\,\mathrm{d}\mathfrak{m}_q\,\mathrm{d}s + \frac{1}{2}\int_0^t \int |\dot{F}_s|^2 \circ F_s\,\mathrm{d}\mathfrak{m}_q\,\mathrm{d}s\Big)\,\mathrm{d}\mathfrak{q}(q).
\end{aligned}$$

Thus for a.e. $q \in Q$, X_q is the trajectories of the gradient flow of $u = \frac{1}{K}\mathrm{div}(b)$:

$$\left|\frac{1}{K}\mathrm{div}(b)(x_1) - \frac{1}{K}\mathrm{div}(b)(x_2)\right| = \frac{1}{K}|\nabla \mathrm{div}(b)|\mathrm{d}(x_1,x_2) = \mathrm{d}(x_1,x_2),\ \forall x_1,x_2 \in X_q.$$

As u is non-constant, there is a unique point $x_q \in X_q$ such that $\mathrm{div}(b)(x_q) = 0$. So $\mathrm{div}(b)$ can be represented by

$$\mathrm{div}(b)(x) = \mathrm{sign}\big(\mathrm{div}(b)(x)\big)K\mathrm{d}(x,x_q), \qquad \forall x \in X_q,$$

□

Lemma 3.11. *For any $g \in \mathbb{V} \cap \mathrm{Lip}(X,\mathrm{d})$ and $s \in [0,t]$, it holds the following inequality*

$$\int |\nabla g|^2 \circ F_s\,\mathrm{d}\mathfrak{m} \geq \int_Q \int_{X_q} |\mathrm{lip}(g|_{X_q})|^2 \circ F_s\,\mathrm{d}\mathfrak{m}_q \mathrm{d}\mathfrak{q}(q). \tag{3.27}$$

Proof. Let $(g_n)_n \subset L^2$ be a sequence of Lipschitz functions such that $g_n \to g$ and $|\mathrm{lip}(g_n)| \to |\nabla g|$ in $L^2(X,(F_s)_\sharp \mathfrak{m})$. Note that $(F_s)_\sharp\mathfrak{m} = \int_Q \big((F_s)_\sharp \mathfrak{m}_q\big)\,\mathrm{d}\mathfrak{q}(q)$, there is a subsequence of (g_n), still denoted by (g_n), such that $g_n|_{X_q} \to g|_{X_q}$ in $L^2(X_q,(F_s)_\sharp\mathfrak{m}_q)$ for $\mathfrak{q}$-a.e. $q \in Q$.

Notice that $|\mathrm{lip}(g_n)||_{X_q} \geq |\mathrm{lip}(g_n|_{X_q})|$, and it is known that $|\mathrm{lip}(g|_{X_q})| = |\nabla g|_{X_q}|$ $\mathfrak{m}_q$-a.e. on X_q (c.f. [22, Theorem 6.1]). Then we have

$$\begin{aligned}
\int |\nabla g|^2 \circ F_s \,\mathrm{d}\mathfrak{m} &= \int |\nabla g|^2 \,\mathrm{d}(F_s)_\sharp \mathfrak{m} \\
&= \lim_{n\to\infty} \int |\mathrm{lip}(g_n)|^2 \,\mathrm{d}(F_s)_\sharp \mathfrak{m} \\
&= \lim_{n\to\infty} \int_Q \Big(\int_{X_q} |\mathrm{lip}(g_n)|^2 \,\mathrm{d}(F_s)_\sharp \mathfrak{m}_q \Big) \mathrm{d}\mathfrak{q}(q) \\
\text{By Fatou's lemma} \quad &\geq \int_Q \varliminf_{n\to\infty} \Big(\int_{X_q} |\mathrm{lip}(g_n)|^2 \,\mathrm{d}(F_s)_\sharp \mathfrak{m}_q \Big) \mathrm{d}\mathfrak{q}(q) \\
&\geq \int_Q \varliminf_{n\to\infty} \Big(\int_{X_q} |\mathrm{lip}(g_n|_{X_q})|^2 \,\mathrm{d}(F_s)_\sharp \mathfrak{m}_q \Big) \mathrm{d}\mathfrak{q}(q) \\
\text{By definition of the energy form } \mathbb{E} \quad &\geq \int_Q \Big(\int_{X_q} |\nabla g|_{X_q}|^2 \,\mathrm{d}(F_s)_\sharp \mathfrak{m}_q \Big) \mathrm{d}\mathfrak{q}(q) \\
&= \int_Q \int_{X_q} |\mathrm{lip}(g|_{X_q})|^2 \,\mathrm{d}(F_s)_\sharp \mathfrak{m}_q \mathrm{d}\mathfrak{q}(q) \\
&= \int_Q \int_{X_q} |\mathrm{lip}(g|_{X_q})|^2 \circ F_s \,\mathrm{d}\mathfrak{m}_q \mathrm{d}\mathfrak{q}(q)
\end{aligned}$$

which is the thesis. □

Remark 3.12. Unlike the well-known result of Cheeger [22, Theorem 6.1] which tells us that $|\nabla g| = |\mathrm{lip}(g)|$ $\mathfrak{m}$-a.e. if $(X, \mathrm{d}, \mathfrak{m})$ satisfies volume doubling property and supports a local Poincaré inequality, it is still unknown whether this result is still true on $\mathrm{RCD}(K, \infty)$ spaces or not. In [29], the author and Gigli prove that $|\nabla g|_p = |\nabla g|$ for all $p > 1$ on $\mathrm{RCD}(K, \infty)$ spaces. But it is still possible that $|\nabla g| < |\mathrm{lip}(g)|$.

3.2. **Proof of the rigidity.** In this part, we will complete the proof of Theorem 1.1 by proving the following Proposition 3.13, 3.14.

In Proposition 2.13, we proved the rigidity of the 2-Bakry-Émery inequality for 1-dimensional spaces. Generally, it is proved by Gigli-Ketterer-Kuwada-Ohta [30] that $(X, \mathrm{d}, \mathfrak{m})$ is isometric to the product space of the 1-dimensional Gaussian space and an $\mathrm{RCD}(K, \infty)$ space, if there is a non-constant function attaining the equality in the Poincaré inequality. As a consequence of Theorem 3.7, Lemma 2.9 and the result of Gigli-Ketterer-Kuwada-Ohta, we get the following proposition.

Proposition 3.13 (c.f. [30], Theorem 1.1)**.** *Let $(X, \mathrm{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, \infty)$ space with $K > 0$. Assume there is a non-constant $f \in \mathbb{V}$ attaining the equality in the 1-Bakry-Émery inequality. Then there exists an $\mathrm{RCD}(K, \infty)$-space $(Y, \mathrm{d}_Y, \mathfrak{m}_Y)$, such that the metric space $(X, \mathrm{d}, \mathfrak{m})$ is isometric to the product space*

$$\Big(\mathbb{R}, |\cdot|, \sqrt{K/(2\pi)}\exp(-Kt^2/2)\mathrm{d}t\Big) \times (Y, \mathrm{d}_Y, \mathfrak{m}_Y)$$

equipped with the L^2-product metric and product measure.

Sketch of the proof. By (c) of Theorem 3.7 and Lemma 2.9, $u = \frac{1}{K}\mathrm{div}\Big(\frac{\nabla P_t f}{|\nabla P_t f|}\Big)$ attains the equality in the Poincaré inequality. Then the assertion follows from [30, Theorem 1.1].

For reader's convenience, we offer more details here. By Theorem 3.7 and Lemma 2.9, $\mathrm{Hess}_u = 0$ and $|\nabla u| = 1$, so that $-\nabla u$ induces a family of isometries (F_t). By Corollary 3.10, there is a disintegration $\mathfrak{m} = \mathfrak{m}_q \mathfrak{q}(\mathrm{d}q)$ associated with the one-to-one map $\Psi : \mathbb{R} \times u^{-1}(0) \ni (r, x) \mapsto F_r(x) \in X$.

In addition, assume (in the coordinate of Ψ) that $u\big((0, y)\big) = 0$. By (4) and (5) of Corollary 3.10, up to a reflection, we may write

$$u\big((r, y)\big) = r,$$

and

$$(F_r)_\sharp \mathfrak{m}_q = e^{-\frac{K}{2}\left(r^2+2ur\right)} \mathfrak{m}_q.$$

Hence $\mathfrak{m}_q \ll \mathcal{H}^1|_{X_q}$ with continuous density h_q, and

$$h_q\big((r, y)\big) = e^{-\frac{K}{2}\left(r^2+2u((0,y))r\right)} h_q\big((0, y)\big) = e^{-\frac{Kr^2}{2}} h_q\big((0, y)\big).$$

So $\mathfrak{m}$ is isomorphic to a product measure $\Phi_K \times \mathfrak{m}_Y$.

Following Gigli's strategy of the splitting theorem [24], one can prove that the map Ψ induces an isometry between the Sobolev spaces $W^{1,2}\big(\Psi^{-1}(X)\big)$ and $W^{1,2}\big(\mathbb{R} \times u^{-1}(0)\big)$. Then from Sobolev-to-Lipschitz property we know that Ψ is an isometry between metric measure spaces (see [24, §6], [25], and [30, §5] for details). $\square$

Finally, we have the following characterization of extreme functions.

Proposition 3.14. *Under the same assumption and keep the same notations as Proposition 3.13, f can be represented in the coordinate of the product space $\mathbb{R} \times Y$, by*

$$f(r, y) = \int_0^r g(s)\,\mathrm{d}s, \qquad (r, y) \in \mathbb{R} \times Y$$

for some non-negative $g \in L^2(\mathbb{R}, \phi_K \mathcal{L}^1)$. In particular, if f attains the equality in the 2-Bakry-Émery inequality, it holds $P_t f(r, y) = Ce^{Kt} r$ for some constant C.

Proof. By Theorem 3.7 and the proof of Proposition 3.13, we know

$$\frac{\nabla f}{|\nabla f|} = \nabla \frac{1}{K} \mathrm{div}\Big(\frac{\nabla f}{|\nabla f|}\Big) = \nabla r. \tag{3.28}$$

So for $\mathfrak{m}_Y$-a.e. $y \in Y$,

$$f(r, y) - f(0, y) = \int_0^r |\nabla f|(s, y)\,\mathrm{d}s.$$

Given $r \in \mathbb{R}$, from (3.28) we can see that $f(r, y)$ in independent of $y \in Y$, so we can assume $f(0, y) = 0$ and denote $g(s) := |\nabla f|(s, y)$ which is the thesis.

If f also attains the equality in the 2-Bakry-Émery inequality, by Lemma 3.11, (3.26), and a standard localization argument we can see that $f(\cdot, y)$ attains the quality in the 1-dimensional Poincaré inequality for $\mathfrak{m}_Y$-a.e. $y \in Y$. Then the second assertion follows from Proposition 2.13. $\square$

4.1. **Equality in the Bobkov's inequality.** In this part, we will study the cases of equality in the Bobkov's inequality, as well as the Gaussian isoperimetric inequality, and prove the corresponding rigidity theorems,

Using an argument of Carlen-Kerce [20, Section 2] (which was firstly used by Ledoux in [38], see also a recent work of Bouyrie [19]), we can prove the following monotonicity formula concerning $\mathrm{RCD}(K,\infty)$ spaces for $K>0$.

Proposition 4.1. *Let $(X,\mathrm{d},\mathfrak{m})$ be a $\mathrm{RCD}(K,\infty)$ space with $K>0$. For any $f:X\mapsto[0,1]$, $t>0$, denote $f_t=P_tf$ and define*

$$J_K(f_t):=\int\sqrt{I_K(f_t)^2+\Gamma(f_t)}\,\mathrm{d}\mathfrak{m} \tag{4.1}$$

where I_K is the Gaussian isoperimetric profile defined in (1.6).

Then for $\mathcal{L}^1$-a.e. t, it holds

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}J_K(f_t)\\ =\;&-\int G_K^{-\frac{3}{2}}\Big(\big\|I_K\mathrm{Hess}_{f_t}-I_K'\nabla f_t\otimes\nabla f_t\big\|_{\mathrm{HS}}^2+\|\mathrm{Hess}_{f_t}\|_{\mathrm{HS}}^2\Gamma(f_t)-\frac{1}{4}\Gamma\big(\Gamma(f_t)\big)\Big)\,\mathrm{d}\mathfrak{m}\\ &-\int G_K^{-\frac{1}{2}}\Big(\mathrm{d}\mathbf{Ric}(f_t,f_t)-K\Gamma(f_t)\,\mathrm{d}\mathfrak{m}\Big)\end{aligned}$$

where $G_K=I_K(f_t)^2+\Gamma(f_t)$.

In particular, $J_K(f_t)$ is non-increasing in t.

Proof. If f is constant, $J_K(f_t)$ is also a constant function of t, there is nothing to prove. So we assume that f is not constant. In addition, similar to [11, Proof of Theorem 3.1, Step 1], it suffices to prove the assertion for every $f\in\mathrm{Lip}(X,\mathrm{d})$ taking values in $[\epsilon,1-\epsilon]$, for some $\epsilon\in(0,\frac{1}{2})$. In fact, for general f, we can replace f by $f^\epsilon:=\frac{1}{1+2\epsilon}(f+\epsilon)$, then letting $\epsilon\downarrow 0$ we will get the answer.

It is known that $f_t\in L^\infty(X,\mathfrak{m})\cap\mathrm{D}(\Delta)$, and $\Delta f_t\in\mathbb{V}$. By Lipschitz regularization of P_t (c.f. [9, Theorem 6.5]), we also have $f_t\in\mathrm{Lip}(X,\mathrm{d})$ for any $t\in(0,\infty)$, so $f_t\in\mathrm{TestF}$. From [11, Lemma 3.2] we know $t\mapsto J_K(f_t)$ is Lipschitz, and for $\mathcal{L}^1$-a.e. t we have

$$\begin{aligned}\frac{\mathrm{d}J_K}{\mathrm{d}t}&=\frac{\mathrm{d}}{\mathrm{d}t}\int\sqrt{G_K(f_t)}\,\mathrm{d}\mathfrak{m}\\ &=\int G_K(f_t)^{-\frac{1}{2}}\Big(I_K(f_t)I_K'(f_t)\Delta f_t+\Gamma(f_t,\Delta f_t)\Big)\,\mathrm{d}\mathfrak{m},\end{aligned}$$

where $G_K(f)$ denotes the function $I_K(f)^2+\Gamma(f)$. Notice that by minimal (maximal) principle, $G_K(f_t)>\delta$ for some $\delta>0$. Thus the formula above is well-posed.

From the definition of $\mathbf{Ric}$ in Proposition 2.8, we can see that

$$\begin{aligned}\frac{\mathrm{d}J_K}{\mathrm{d}t}=\;&\underbrace{\int G_K^{-\frac{1}{2}}I_K(f_t)I_K'(f_t)\Delta f_t\,\mathrm{d}\mathfrak{m}}_{J_1}-\underbrace{\int\frac{1}{2}\Gamma\big(G_K^{-\frac{1}{2}},\Gamma(f_t)\big)+G_K^{-\frac{1}{2}}\Big(\|\mathrm{Hess}_{f_t}\|_{\mathrm{HS}}^2+K\Gamma(f_t)\Big)\,\mathrm{d}\mathfrak{m}}_{J_2}\\ &-\int G_K^{-\frac{1}{2}}\Big(\mathrm{d}\mathbf{Ric}(f_t,f_t)-K\Gamma(f_t)\,\mathrm{d}\mathfrak{m}\Big).\end{aligned}$$

Thus the non-smooth Bochner inequality in Proposition 2.8 yields

$$\frac{\mathrm{d}J_K}{\mathrm{d}t} \le J_1 + J_2. \tag{4.2}$$

By computation,

$$\begin{aligned}
J_1 &= -\int \Gamma\big(G_K^{-\frac{1}{2}} I_K I_K', f_t\big)\,\mathrm{d}\mathfrak{m}\\
&= -\int G_K^{-\frac{1}{2}} (I_K I_K')' \Gamma(f_t)\,\mathrm{d}\mathfrak{m} + \frac{1}{2}\int G_K^{-\frac{3}{2}} I_K I_K' \Gamma(G_K, f_t)\,\mathrm{d}\mathfrak{m}\\
&= -\int G_K^{-\frac{1}{2}} (I_K I_K')' \Gamma(f_t)\,\mathrm{d}\mathfrak{m}\\
&\quad +\frac{1}{2}\int G^{-\frac{3}{2}} I_K I_K' \Big(2 I_K I_K' \Gamma(f_t, f_t) + \Gamma\big(\Gamma(f_t), f_t\big)\Big)\,\mathrm{d}\mathfrak{m}\\
&= -\int G_K^{-\frac{1}{2}} \big((I_K')^2 - K\big)\Gamma(f_t)\,\mathrm{d}\mathfrak{m} + \int G_K^{-\frac{3}{2}} (I_K I_K')^2 \Gamma(f_t)\,\mathrm{d}\mathfrak{m}\\
&\quad + \int G_K^{-\frac{3}{2}} I_K I_K' \mathrm{Hess}_{f_t}(f_t, f_t)\,\mathrm{d}\mathfrak{m}\\
&= -\int G_K^{-\frac{3}{2}} \Big((I_K')^2 \Gamma(f_t)^2 \underbrace{-K I_K^2 \Gamma(f_t) - K\Gamma(f_t)^2}_{=-K\Gamma(f_t) G_K(f_t)} - I_K I_K' \mathrm{Hess}_{f_t}(f_t, f_t)\Big)\,\mathrm{d}\mathfrak{m}
\end{aligned}$$

where in the fourth equality we use the identity $(I_K I_K')' = (I_K')^2 - K$ which follows from $I_K I_K'' = -K$.

Similarly,

$$\begin{aligned}
&-\frac{1}{2}\int \Gamma\big(G_K^{-\frac{1}{2}}, \Gamma(f_t)\big)\,\mathrm{d}\mathfrak{m}\\
=& \int \frac{1}{4} G_K^{-\frac{3}{2}} \Big(2 I_K I_K' \Gamma\big(f_t, \Gamma(f_t)\big) + \Gamma\big(\Gamma(f_t)\big)\Big)\,\mathrm{d}\mathfrak{m}\\
=& \int G_K^{-\frac{3}{2}} \Big(I_K I_K' \mathrm{Hess}_{f_t}(f_t, f_t) + \frac{1}{4}\Gamma\big(\Gamma(f_t)\big)\Big)\,\mathrm{d}\mathfrak{m}.
\end{aligned}$$

In summary, we get

$$\begin{aligned}
&J_1 + J_2\\
=& -\int G_K^{-\frac{3}{2}} \Big((I_K')^2 \Gamma(f_t)^2 - \frac{1}{4}\Gamma\big(\Gamma(f_t)\big) - 2 I_K I_K' \mathrm{Hess}_{f_t}(f_t, f_t) + \|\mathrm{Hess}_{f_t}\|_{\mathrm{HS}}^2 \big(I_K^2 + \Gamma(f_t)\big)\Big)\,\mathrm{d}\mathfrak{m}\\
=& -\int G_K^{-\frac{3}{2}} \Big(\big\|I_K \mathrm{Hess}_{f_t} - I_K' \nabla f_t \otimes \nabla f_t\big\|_{\mathrm{HS}}^2 + \|\mathrm{Hess}_{f_t}\|_{\mathrm{HS}}^2 \Gamma(f_t) - \frac{1}{4}\Gamma\big(\Gamma(f_t)\big)\Big)\,\mathrm{d}\mathfrak{m}.
\end{aligned}$$

Recall that by definition

$$\begin{aligned}
\Gamma\big(\Gamma(f_t)\big) &= 2\mathrm{Hess}_{f_t}\big(\nabla f_t, \nabla \Gamma(f_t)\big)\\
&\le 2\|\mathrm{Hess}_{f_t}\|_{\mathrm{HS}} \sqrt{\Gamma(f_t)} \sqrt{\Gamma\big(\Gamma(f_t)\big)},
\end{aligned}$$

thus

$$\|\mathrm{Hess}_{f_t}\|_{\mathrm{HS}}^2 \Gamma(f_t) \ge \frac{1}{4}\Gamma\big(\Gamma(f_t)\big).$$

Combining with (4.2) we have

$$\frac{\mathrm{d}J_K}{\mathrm{d}t} \le J_1 + J_2 \le 0,$$

so $t \mapsto J_K(f_t)$ is non-increasing. $\square$

Appying Proposition 4.1, we obtain the functional version of Gaussian isoperimetric inequality of Bobkov on $\mathrm{RCD}(K,\infty)$ spaces, which had been proved by Ambrosio-Mondino in [11] using a different proof (see also [15, Chapter 8.5.2] for more discussions).

Proposition 4.2. *Let $(X,\mathrm{d},\mathfrak{m})$ be a metric measure space satisfying $\mathrm{RCD}(K,\infty)$ condition for some $K > 0$. Then $(X,\mathrm{d},\mathfrak{m})$ supports the K-Bobkov's isoperimetric inequality in the sense of Definition 1.2,*

$$I_K\left(\int f\,\mathrm{d}\mathfrak{m}\right) \le J_K(f)$$

for all measurable function f with values in $[0,1]$.

Proof. Let f be a measurable function with values in $[0,1]$. By Proposition 4.1 and definition of $J_K(f)$ we know

$$\varlimsup_{t\to+\infty} J_K(f_t) \le \varliminf_{t\to 0} J_K(f_t) = J_K(f).$$

Combining with the ergodicity of heat flow and the 2-Bakry-Émery inequality

$$\lim_{t\to+\infty} J_K(f_t) = I_K\left(\int f\,\mathrm{d}\mathfrak{m}\right),$$

we get the Bobkov's isoperimetric inequality. $\square$

In the next proposition, we discover the cases of equality in the Bobkov's inequality. By Proposition 4.1, we simultaneously obtain the rigidity of the Gaussian isoperimetric inequality. We refer the readers to [20, Section 2] for related discussions on $\mathbb{R}^n$.

Proposition 4.3 (Equality in the Bobkov's inequality)**.** *Let $(X,\mathrm{d},\mathfrak{m})$ be a $\mathrm{RCD}(K,\infty)$ metric measure space with $K > 0$. Then there exists a non-constant f attaining the equality $I_K\left(\int f\,\mathrm{d}\mathfrak{m}\right) = J_K(f)$ if and only if*

$$(X,\mathrm{d},\mathfrak{m}) \cong \left(\mathbb{R}, |\cdot|, \sqrt{K/(2\pi)}e^{-Kt^2/2}\mathrm{d}t\right) \times (Y,\mathrm{d}_Y,\mathfrak{m}_Y)$$

for some $\mathrm{RCD}(K,\infty)$ space $(Y,\mathrm{d}_Y,\mathfrak{m}_Y)$, and up to change of variables, f is either the indicator function of a half space

$$f(r,y) = \chi_E, \qquad E = (-\infty, e] \times Y,$$

where $e \in \mathbb{R} \cup \{+\infty\}$ with $\int_{-\infty}^{e} \phi_K(s)\,\mathrm{d}s = \int f\,\mathrm{d}\mathfrak{m}$; or else, there are $a = (2\int f)^{-1}$ and $b = \Phi_K^{-1}\big(f(0,y)\big)$ such that

$$f(y,t) = \Phi_K(at+b) = \int_{-\infty}^{at+b} \phi_K(s)\,\mathrm{d}s.$$

Proof. **Part 1**: Denote $f_t = P_t f$ and $h_t = \Phi_K^{-1}(f_t)$. We will show that h_t satisfies $\mathbf{\Gamma}_2(h_t) = K\Gamma(h_t)\,\mathfrak{m}$ (c.f. Proposition 2.8), and thus satisfies (1) in Lemma 2.9.

By Proposition 4.1 we know $I_K\left(\int f\,\mathrm{d}\mathfrak{m}\right) = J_K(f)$ if and only if

$$I_K\left(\int f\,\mathrm{d}\mathfrak{m}\right) = I_K\left(\int f_t\,\mathrm{d}\mathfrak{m}\right) = J_K(f_t) \qquad \text{for all } t \ge 0,$$

which is equivalent to $\frac{\mathrm{d}J_K}{\mathrm{d}t} = 0$ for all $t > 0$. From Proposition 4.1, we know that $\frac{\mathrm{d}J_K}{\mathrm{d}t} = 0$ if and only if the following equalities (4.3) (4.4) (4.5) are satisfied

$$\mathbf{Ric}(f_t, f_t) = K\Gamma(f_t)\,\mathfrak{m}, \tag{4.3}$$

$$I_K \mathrm{Hess}_{f_t} - I_K' \nabla f_t \otimes \nabla f_t = 0 \tag{4.4}$$

and

$$\|\mathrm{Hess}_{f_t}\|_{\mathrm{HS}}^2 \Gamma(f_t) - \frac{1}{4}\Gamma\big(\Gamma(f_t)\big) = 0. \tag{4.5}$$

By definitions,

$$I_K(f_t) = \phi_K(h_t), \qquad f_t = \Phi_K(h_t). \tag{4.6}$$

By (4.6) and chain rule (c.f. [27, Theorem 2.2.6])

$$I_K'(f_t)\nabla f_t = -h_t \phi_K(h_t)\nabla h_t, \qquad \nabla f_t = \phi_K(h_t)\nabla h_t.$$

Then we have

$$I_K'(f_t) = -h_t, \qquad \nabla f_t = I_K(f_t)\nabla h_t,$$

and

$$\mathrm{Hess}_{f_t} = -h_t\phi_K(h_t)\nabla h_t \otimes \nabla h_t + \phi_K(h_t)\mathrm{Hess}_{h_t}.$$

In conclusion, we obtain

$$\nabla h_t = I_K^{-1}(f_t)\nabla f_t \tag{4.7}$$

and

$$\mathrm{Hess}_{f_t} = I_K'(f_t) I_K^{-1}(f_t)\nabla f_t \otimes \nabla f_t + I_K(f_t)\mathrm{Hess}_{h_t}. \tag{4.8}$$

By (4.7) and the bi-linearity of $\mathbf{Ric}(\cdot,\cdot)$, (4.3) is equivalent to

$$\mathbf{Ric}(h_t, h_t) = K\Gamma(h_t)\,\mathfrak{m}. \tag{4.9}$$

Compare (4.8) and (4.4), we can see that f_t satisfies (4.4) if and only if $\mathrm{Hess}_{h_t} = 0$, which is equivalent to

$$\|\mathrm{Hess}_{h_t}\|_{\mathrm{HS}} = 0. \tag{4.10}$$

By (4.8) and (4.10), we have

$$\|\mathrm{Hess}_{f_t}\|_{\mathrm{HS}} = \|I_K' I_K^{-1} \nabla f_t \otimes \nabla f_t\|_{\mathrm{HS}} = I_K' I_K^{-1}\Gamma(f_t)$$

and

$$\begin{aligned}
\Gamma\big(\Gamma(f_t)\big) &= 2\mathrm{Hess}_{f_t}\big(\nabla f_t, \nabla\Gamma(f_t)\big)\\
\text{By (4.8)} \qquad &= 2I_K' I_K^{-1}\Gamma(f_t)\Gamma\big(f_t, \Gamma(f_t)\big)\\
&= 4I_K' I_K^{-1}\Gamma(f_t)\mathrm{Hess}_{f_t}(\nabla f_t, \nabla f_t)\\
\text{By (4.8)} \qquad &= 4I_K' I_K^{-1}\Gamma(f_t)\Big(I_K' I_K^{-1}\big(\Gamma(f_t)\big)^2\Big).
\end{aligned}$$

Therefore,

$$\|\mathrm{Hess}_{f_t}\|_{\mathrm{HS}}^2\Gamma(f_t) - \frac{1}{4}\Gamma\big(\Gamma(f_t)\big) = (I_K' I_K^{-1})^2\big(\Gamma(f_t)\big)^3 - (I_K' I_K^{-1})^2\big(\Gamma(f_t)\big)^3 = 0 \tag{4.11}$$

which is exactly (4.5).

In conclusion, (4.3) (4.4) (4.5) $\Longleftrightarrow$ (4.9) (4.10), and the latter ones are equivalent to

$$\mathbf{\Gamma}_2(h_t) = K\Gamma(h_t)\,\mathfrak{m} \tag{4.12}$$

which is the thesis.

Part 2: By Proposition 3.14 we just need to study the 1-dimensional cases. By Proposition 2.13 we know $h_t = \Phi_K^{-1}(f_t)$ is an affine function on $\mathbb{R}$ for any $t > 0$, there exist $a = a(t), b = b(t) \in \mathbb{R}$ such that

$$f_t(x) = \Phi_K(ax + b) = \int_{-\infty}^{ax+b} \phi_K(s)\,\mathrm{d}s.$$

By [20, Theorem 1] there is $s \geq 0$ such that

$$f_t = P_{t+s}(\chi_E),\ \forall\, t \geq 0$$

where E is the half-line such that $\int_E \phi_K \,\mathrm{d}\mathcal{L}^1 = \int f \,\mathrm{d}\mathfrak{m}$.

Therefore, if $s = 0$, $f = \chi_E$. Otherwise, $a(t), b(t)$ are continuous on $[0, +\infty)$, so

$$f = \Phi_K(a_0 x + b_0) = \int_{-\infty}^{a_0 x + b_0} \phi_K(s)\,\mathrm{d}s$$

where $a_0 = (2\int f)^{-1}$, $b_0 = \Phi_K^{-1}(f(0))$. □

Applying Proposition 4.3, we obtain the rigidity of the Gaussian isoperimetric inequality.

Corollary 4.4 (Rigidity of the Gaussian isoperimetric inequality)**.** *Let* $(X, \mathrm{d}, \mathfrak{m})$ *be a* $\mathrm{RCD}(K, \infty)$ *metric measure space with* $K > 0$. *If there is a Borel set* $E \subset X$ *with positive* $\mathfrak{m}$*-measure such that*

$$P(E) = J_K(\chi_E) = I_K\big(\mathfrak{m}(E)\big).$$

Then

$$(X, \mathrm{d}, \mathfrak{m}) \cong \Big(\mathbb{R}, |\cdot|, \sqrt{K/(2\pi)}e^{-Kt^2/2}\mathrm{d}t\Big) \times (Y, \mathrm{d}_Y, \mathfrak{m}_Y)$$

for some $\mathrm{RCD}(K, \infty)$ *space* $(Y, \mathrm{d}_Y, \mathfrak{m}_Y)$*, and* $E \cong (-\infty, e] \times Y$ *with* $e = \Phi_K^{-1}\big(\mathfrak{m}(E)\big)$.

4.2. **Equalities in Φ-entropy inequalities.** In this part we will characterize the cases of equalities in the logarithmic Sobolev inequality, the Poincaré inequality, and more generally, Φ-entropy inequalities of Chafaï [21] and Bolley-Gentil [18] on $\mathrm{RCD}(K, \infty)$ metric measure spaces.

First of all, we prove a general Φ-entropy inequality. For more discussions about admissible Φ's, see [21, Page 330], [18, Section 1.3] and the references therein.

Proposition 4.5. *Let* $(X, \mathrm{d}, \mathfrak{m})$ *be a metric measure space satisfying* $\mathrm{RCD}(K, \infty)$ *condition for some* $K > 0$. *Let* Φ *be a* C^2*-continuous strictly convex function on an interval* $I \subset \mathbb{R}$ *such that* $\frac{1}{\Phi''}$ *is concave. Then* $(X, \mathrm{d}, \mathfrak{m})$ *satisfies the following* Φ*-entropy inequality:*

$$\underbrace{\mathrm{Ent}^{\Phi}_{\mathfrak{m}}(f)}_{:=\int \Phi(f)\,\mathrm{d}\mathfrak{m}} - \Phi\Big(\int f \,\mathrm{d}\mathfrak{m}\Big) \leq \frac{1}{2K}\int \Phi''(f)\Gamma(f)\,\mathrm{d}\mathfrak{m} \tag{4.13}$$

for all I*-valued functions* f.

Proof. Let f be an I-valued function and denote $f_t := P_t f$. By the ergodicity of the heat flow, we have

$$\begin{aligned}
\mathrm{Ent}^{\Phi}_{\mathfrak{m}}(f) - \Phi\Big(\int f \,\mathrm{d}\mathfrak{m}\Big) &= -\int_0^{+\infty} \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Ent}^{\Phi}_{\mathfrak{m}}(f_t)\,\mathrm{d}t \\
\text{By [8, Theorem 4.16]} &= \int_0^{+\infty}\int \Phi''(f_t)\Gamma(f_t)\,\mathrm{d}\mathfrak{m}\,\mathrm{d}t \\
\text{By (2.9), Proposition 2.10} &\leq \int_0^{+\infty} e^{-2Kt}\int P_t\big(\Phi''(f)\Gamma(f)\big)\,\mathrm{d}\mathfrak{m}\,\mathrm{d}t \\
&= \frac{1}{2K}\int \Phi''(f)\Gamma(f)\,\mathrm{d}\mathfrak{m}
\end{aligned}$$

which is the thesis. □

Finally, we complete the proof of Theorem 1.6.

Proof of Theorem 1.6. We keep the same notations as in §1.3. If there is a function f attaining the equality in (4.13), from the proof of Proposition 4.5, we can see that

$$\Phi''(P_t f)\Gamma(P_t f) = e^{-2Kt}P_t\big(\Phi''(f)\Gamma(f)\big)$$

for almost every $t > 0$. If f is not constant, by Proposition 2.10 (or Corollary 2.12) and Proposition 3.13 we know $(X, \mathrm{d}, \mathfrak{m})$ is isometric to the product $\big(\mathbb{R}, |\cdot|, \phi_K\mathcal{L}^1\big) \times (Y, \mathrm{d}_Y, \mathfrak{m}_Y)$ of two $\mathrm{RCD}(K,\infty)$ metric measure spaces. Concerning the extreme functions, by Corollary 2.12 and Proposition 3.14 we just need to consider the following two cases

a) Poincaré inequality: $\Phi = x^2$ for $x \in \mathbb{R}$. If there is a non-constant function $f \in \mathbb{V}$ with $\int f \,\mathrm{d}\mathfrak{m} = 0$ such that
$$\int f^2 \,\mathrm{d}\mathfrak{m} = \frac{1}{K}\int |\nabla f|^2 \,\mathrm{d}\mathfrak{m}.$$
Then f itself satisfies the properties in Lemma 2.9. In this case $f(r,y) = a_p r$ for a constant $a_p \in \mathbb{R}$.

b) Logarithmic Sobolev inequality: $\Phi(x) = x\ln x$ for $x \in \mathbb{R}^+$. If there is a non-negative function $f \in \mathbb{V}$ with $\int f \,\mathrm{d}\mathfrak{m} = 1$ such that
$$\int f\ln f \,\mathrm{d}\mathfrak{m} = \frac{1}{2K}\int \frac{|\nabla f|^2}{f}\,\mathrm{d}\mathfrak{m}.$$
Then by Corollary 2.12, $\ln f$ attains the equality in the 2-Bakry-Émery inequality. In this case $f(r,y) = e^{a_l r - a_l^2/2K}$ for a constant $a_l \in \mathbb{R}$.

□

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