AN ASYMPTOTIC EXPANSION FOR THE FRACTIONAL p-LAPLACIAN AND GRADIENT DEPENDENT NONLOCAL OPERATORS

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ABSTRACT. We obtain asymptotic representation formulas for harmonic functions in the viscosity sense with respect to the fractional p-Laplacian and to gradient dependent nonlocal operators.

CONTENTS

1. INTRODUCTION

One of the most famous basic fact of partial differential equations is that a smooth function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is harmonic (i.e. $\Delta u = 0$) if and only if it satisfies the mean value property, that is

$$u(x) = \fint_{B_r(x)} u(y)dy, \qquad \text{whenever } B_r(x) \subset\subset \Omega. \tag{1.1}$$

Such a characterization holds in some sense for harmonic functions with respect to more general differential operators. In fact, similar properties can be obtained for quasi-linear operators such as the p-Laplace operator $\Delta_p u$, in an asymptotic form. More precisely, in 2010 Manfredi, Parviainen and Rossi proved in [14] that, if $p \in (1, \infty]$, a continuous function $u : \Omega \to \mathbb{R}$ is p-harmonic in Ω if and only if (in the viscosity sense)

$$u(x) = \frac{2+n}{p+n} \fint_{B_r(x)} u(y)dy + \frac{p-2}{2p+2n}\Big(\max_{B_r(x)} u + \min_{B_r(x)} u\Big) + o(r^2), \tag{1.2}$$

as the radius r of the ball vanishes. Notice that formula (1.2) boils down to (1.1) for $p = 2$, up to a rest of order $o(r^2)$ and that it holds true in the classical sense at those points $x \in \Omega$ for which u is C^2 around x and the gradient of u does not vanish. In the case $p = \infty$ the formula fails in the classical sense, since $|x|^{4/3} - |y|^{4/3}$ is ∞-harmonic in $\mathbb{R}^2$ in the viscosity sense but (1.2) fails to hold

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point-wisely. If $p \in (1,\infty)$ and $n = 2$ the characterization holds in the classical sense (see [2, 13]). Finally, the limiting case $p = 1$ was investigated in 2012 in [9].

Once the local (linear and nonlinear) case is rather well understood, it is natural to investigate the validity of some kind of asymptotic mean value property in the *nonlocal* case, for instance, letting $s \in (0,1)$, for s-harmonic functions (i.e. such that $(-\Delta)^s u = 0$), where formally

$$(-\Delta)^s u(x) := C(n,s) \lim_{r\to 0} \int_{\mathbb{R}^n\setminus B_r} \frac{u(x)-u(x-y)}{|y|^{n+2s}}\, dy, \qquad C(n,s) = \frac{2^{2s} s\Gamma\left(\frac{n}{2}+s\right)}{\pi^{\frac{n}{2}}\Gamma(1-s)}.$$

The equivalence between s-harmonic functions and the fractional mean value property is proved in [1] (see also [11], [5]), with the fractional mean kernel given by

$$\mathscr{M}_r^s u(x) = c(n,s) r^{2s} \int_{\mathbb{R}^n\setminus B_r} \frac{u(x-y)}{(|y|^2-r^2)^s |y|^n}\, dy, \tag{1.3}$$

where $c(n,s) = \Gamma(n/2)\sin \pi s/\pi^{n/2+1}$. Furthermore, in [6] the authors obtain an asymptotic expansion for harmonic functions with respect to a fractional anisotropic operator (that includes the case of the fractional Laplacin). Precisely, a continuous function u is harmonic in the viscosity sense if and only if (1.3) holds in a viscosity sense up to a rest of order two, namely

$$u(x) = c(n,s) r^{2s} \int_{\mathbb{R}^n\setminus B_r} \frac{u(x-y)}{(|y|^2-r^2)^s |y|^n}\, dy + o(r^2), \tag{1.4}$$

The goal of this paper is to continue the analysis of the nonlocal case and to provide a nonlocal counterpart (in some sense) of the result by Manfredi, Parviainen and Rossi [14] for the (s,p)-Laplacian $(-\Delta)^s_p$. Up to the authors' knowledge, this is the first attempt to obtain similar properties in the nonlocal, nonlinear, case.

Namely, the fractional p-Laplacian is the differential (in a suitable Banach space) of the convex functional

$$u \mapsto \frac{1}{p}[u]^p_{s,p} := \frac{1}{p}\iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dxdy.$$

More precisely, the (s,p)-Laplacian is formally defined as

$$(-\Delta)^s_p u(x) = \lim_{r\to 0} \mathcal{L}_r^{s,p} u(x), \qquad \mathcal{L}_r^{s,p} u(x) := \int_{|y|>r} \frac{|u(x)-u(x-y)|^{p-2}(u(x)-u(x-y))}{|y|^{n+sp}}\, dy.$$

This definition is consistent, up to a normalization, with the linear operator $(-\Delta)^s$.
We suppose here and all through Section 2 that u is not a constant function and that $p \geq 2$. Then we define

$$\mathcal{D}_r^{s,p} u(x) := \int_{|y|>r} \left(\frac{|u(x)-u(x-y)|}{|y|^s}\right)^{p-2} \frac{dy}{|y|^n(|y|^2-r^2)^s},$$

which, to make an analogy with the local case, plays the "nonlocal" role of $\nabla u(x)$ (see also and Proposition 2.9, for the limit as $s \nearrow 1$), and

$$\mathscr{M}_r^{s,p} u(x) := (\mathcal{D}_r^{s,p} u(x))^{-1} \int_{|y|>r} \left(\frac{|u(x)-u(x-y)|}{|y|^s}\right)^{p-2} \frac{u(x-y)}{|y|^n(|y|^2-r^2)^s}\, dy, \tag{1.5}$$

playing the role of a (s,p)-mean kernel. Both $\mathcal{D}_r^{s,p}$ and $\mathscr{M}_r^{s,p}$ naturally appear when we make an asymptotic expansion for smooth functions (see Theorem 2.3). Notice also that for $p = 2$, $\mathscr{M}_r^{s,2} u$ is given by (1.4) (and $\mathcal{D}_r^{s,2} u(x) = c(n,s)^{-1} r^{-2s}$).

The main result relative to the fractional p-Laplacian, that we prove in Section 2 (Theorem 2.7), is the following.

Main result 1. *Let $p \geq 2$, $\Omega \subset \mathbb{R}^n$ be an open set and let $u \in C(\Omega) \cap L^\infty(\mathbb{R}^n)$ be a non constant function. Then*

$$(-\Delta)_p^s u(x) = 0$$

in the viscosity sense if and only if

$$\lim_{r \searrow 0} \mathcal{D}_r^{s,p} u(x)\big(u(x) - \mathscr{M}_r^{s,p} u(x)\big) = 0 \tag{1.6}$$

holds in the viscosity sense for all $x \in \Omega$.

Notice that for $p = 2$ the result in [6] is recovered. In the case we consider here, however, the dependence of $\mathcal{D}_r^{s,p}$ of the function u does not allow a simplification of the formula obtained.

In the second part of the paper (Section 3) we investigate a different nonlocal version of the p-Laplace operator $(-\Delta)_{p,\pm}^s u$ and of the infinity Laplace operator $(-\Delta)_\infty^s u$, that arise in tug-of war games, introduced in [3, 4]. For these operators we obtain an asymptotic representation formula in the viscosity sense, see Theorems 3.6 and 3.10. We summarize the results on these two nonlocal operators in the following theorem, denoting by $(-\Delta)_s^\#$ the nonlocal p-Laplace and infinity Laplacian respectively, and $\mathscr{M}_r^\#$ playing the role of the nonlocal mean kernel and of the infinity mean kernel, respectively.

Main result 2. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u \in C(\Omega) \cap L^\infty(\mathbb{R}^n)$. Then*

$$(-\Delta)_s^\# u(x) = 0$$

in the viscosity sense if and only if

$$u(x) - \mathscr{M}_r^\# u(x) = o(r^{2s})$$

holds for all $x \in \Omega$ in the viscosity sense.

Furthermore, both in Sections 2 and 3 we study the asymptotic properties of the Laplace operators and mean kernels as $s \nearrow 1$. In Appendix A we insert some basic integral asymptotics.

2. The fractional p-Laplacian

2.1. An asymptotic expansion.

Let $p \geq 2$. Throughout Section 2, we consider u to be a non constant function. The next proposition motivates this choice, and justifies (1.5) as a good definition.

Proposition 2.1. *Unless u is a constant function, for any $x \in \mathbb{R}^n$ there exist some $r_x > 0$ and $c_x > 0$ such that, for all $r < r_x$, it holds that $\mathcal{D}_r^{s,p} u(x) \geq c_x$.*

Proof. We have that

$$|y|^2 - r^2 \leq |y|^2,$$

hence

$$\mathcal{D}_r^{s,p} u(x) \geq \int_{|y|>r} \frac{|u(x) - u(x-y)|^{p-2}}{|y|^{n+sp}}\, dy,$$

and by changing variables

$$\mathcal{D}_r^{s,p} u(x) \geq \int_{\mathcal{C}B_r(x)} \frac{|u(x) - u(y)|^{p-2}}{|x-y|^{n+sp}}\, dy.$$

If u is not constant, for any $x \in \mathbb{R}^n$ there exists z_x such that $u(x) \neq u(z_x)$, hence there exists $r_x < |x - z_x|/2$ such that

$$u(x) \neq u(z), \qquad \text{for all } z \in B_{r_x}(z_x).$$

Let $\tilde{r} = |z_x - x|/2$, then for any $r < \tilde{r}$

$$\mathcal{D}_r^{s,p} u(x) \geq \int_{\mathcal{C}B_{\tilde{r}}(x)} \frac{|u(x)-u(y)|^{p-2}}{|x-y|^{n+sp}}\, dy \geq \int_{B_{r_x}(z_x)} \frac{|u(x)-u(y)|^{p-2}}{|x-y|^{n+sp}}\, dy := c_x,$$

with c_x positive, independent of r. □

Remark 2.2. Notice that it is quite natural to assume that u is not constant and it is similar to what is required in the local case, namely $\nabla u(x) \neq 0$ (see the proof of [14, Theorem 2]).

We obtain an asymptotic property for smooth functions.

Theorem 2.3. *Let $u \in C^2_{\rm loc}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then*

$$\mathcal{D}_r^{s,p} u(x)\big(u(x) - \mathscr{M}_r^{s,p} u(x)\big) = (-\Delta)_p^s u(x) + \mathcal{O}(r^{2-2s})$$

as $r \to 0$.

Proof. We note that the constants may change value from line to line. We fix an arbitrary $\bar{\varepsilon}$, the corresponding $r := r(\bar{\varepsilon})$ as in (2.8), and some number $0 < \varepsilon < \min\{\bar{\varepsilon}, r\}$, to be taken arbitrarily small.

Starting from the definition, we have that

$$\begin{aligned}
\mathcal{L}_\varepsilon^{s,p} u(x) = &\int_{\varepsilon<|y|<r} \frac{|u(x)-u(x-y)|^{p-2}(u(x)-u(x-y))}{|y|^{n+sp}}\, dy \\
&+ \int_{|y|>r} \frac{|u(x)-u(x-y)|^{p-2}(u(x)-u(x-y))}{|y|^{n+s(p-2)}} \left(\frac{1}{|y|^{2s}} - \frac{1}{(|y|^2-r^2)^s} \right) dy \\
&+ \int_{|y|>r} \frac{|u(x)-u(x-y)|^{p-2}(u(x)-u(x-y))}{|y|^{n+s(p-2)}(|y|^2-r^2)^s}\, dy.
\end{aligned}$$

Thus we obtain that

$$\begin{aligned}
&\mathcal{L}_\varepsilon^{s,p} u(x) + \int_{|y|>r} \frac{|u(x)-u(x-y)|^{p-2}u(x-y)}{|y|^{n+s(p-2)}(|y|^2-r^2)^s} dy \\
&= u(x) \int_{|y|>r} \frac{|u(x)-u(x-y)|^{p-2}}{|y|^{n+s(p-2)}(|y|^2-r^2)^s} dy \\
&\quad + \int_{\varepsilon<|y|<r} \frac{|u(x)-u(x-y)|^{p-2}(u(x)-u(x-y))}{|y|^{n+sp}} dy \\
&\quad + \int_{|y|>r} \frac{|u(x)-u(x-y)|^{p-2}(u(x)-u(x-y))}{|y|^{n+s(p-2)}} \left(\frac{1}{|y|^{2s}} - \frac{1}{(|y|^2-r^2)^s} \right) dy \\
&:= u(x) \int_{|y|>r} \frac{|u(x)-u(x-y)|^{p-2}}{|y|^{n+s(p-2)}(|y|^2-r^2)^s} dy + I_\varepsilon^s(r) + J(r).
\end{aligned} \tag{2.1}$$

Since $u \in C^2_{loc}(\mathbb{R}^n)$, using (2.10), (2.14) and (2.16), we get that

$$\lim_{\varepsilon \searrow 0} I_\varepsilon^s(r) = \mathcal{O}(r^{p(1-s)}), \tag{2.2}$$

(see also [10, Lemma 3.6]). Looking for an estimate on $J(r)$, we split it into two parts

$$\begin{aligned}
J(r) &= \int_{|y|>r} \frac{|u(x)-u(x-y)|^{p-2}(u(x)-u(x-y))}{|y|^{n+s(p-2)}} \left(\frac{1}{|y|^{2s}} - \frac{1}{(|y|^2-r^2)^s} \right) dy \\
&= r^{-sp} \int_{|y|>1} \frac{|u(x)-u(x-ry)|^{p-2}(u(x)-u(x-ry))}{|y|^{n+s(p-2)}} \left(\frac{1}{|y|^{2s}} - \frac{1}{(|y|^2-1)^s} \right) dy \\
&= r^{-sp} \Bigg[\int_{|y|>\frac{1}{r}} \frac{|u(x)-u(x-ry)|^{p-2}(u(x)-u(x-ry))}{|y|^{n+s(p-2)}} \left(\frac{1}{|y|^{2s}} - \frac{1}{(|y|^2-1)^s} \right) dy \\
&+ \int_{1<|y|<\frac{1}{r}} \frac{|u(x)-u(x-ry)|^{p-2}(u(x)-u(x-ry))}{|y|^{n+s(p-2)}} \left(\frac{1}{|y|^{2s}} - \frac{1}{(|y|^2-1)^s} \right) \Bigg] dy \\
&=: r^{-sp} \left(J_1(r) + J_2(r) \right).
\end{aligned}$$

We have that

$$|u(x)-u(x-y)|^{p-1} \leq c(|u(x)|^{p-1} + |u(x-y)|^{p-1}),$$

thus we obtain the bound

$$|J_1(r)| \leq C \|u\|^{p-1}_{L^\infty(\mathbb{R}^n)} \int_{\frac{1}{r}}^{\infty} \frac{dt}{t^{sp+1}} \left| 1 - \frac{1}{(1-\frac{1}{t^2})^s} \right|.$$

The fact that

$$J_1(r) = \mathcal{O}(r^{2+sp})$$

follows from Proposition A.1. For J_2, by symmetry we write

$$\begin{aligned}
J_2(r) &= \frac{1}{2} \int_{1<|y|<\frac{1}{r}} \frac{|u(x)-u(x-ry)|^{p-2}(u(x)-u(x-ry))}{|y|^{n+s(p-2)}} \left(\frac{1}{|y|^{2s}} - \frac{1}{(|y|^2-1)^s} \right) dy \\
&+ \frac{1}{2} \int_{1<|y|<\frac{1}{r}} \frac{|u(x)-u(x+ry)|^{p-2}(u(x)-u(x+ry))}{|y|^{n+s(p-2)}} \left(\frac{1}{|y|^{2s}} - \frac{1}{(|y|^2-1)^s} \right) dy \\
&= \frac{1}{2} \int_{1<|y|<\frac{1}{r}} \frac{|u(x)-u(x-ry)|^{p-2}(2u(x)-u(x-ry)-u(x+ry))}{|y|^{n+s(p-2)}} \left(\frac{1}{|y|^{2s}} - \frac{1}{(|y|^2-1)^s} \right) dy \\
&+ \frac{1}{2} \int_{1<|y|<\frac{1}{r}} \frac{\left(|u(x)-u(x+ry)|^{p-2} - |u(x)-u(x-ry)|^{p-2}\right)(u(x)-u(x+ry))}{|y|^{n+s(p-2)}} \\
&\qquad \left(\frac{1}{|y|^{2s}} - \frac{1}{(|y|^2-1)^s} \right) dy.
\end{aligned}$$

We proceed using (2.13) and (2.15) (and passing $\bar{\varepsilon}$ to 0). We have that

$$\begin{aligned}
|J_2(R)| &\leq C r^p \int_1^{1/r} \rho^{p-1-s(p-2)} \left(\frac{1}{(\rho^2-1)^s} - \frac{1}{\rho^{2s}} \right) d\rho \\
&\leq C r^{p-(p-2)(1-s)} \int_1^{1/r} \rho \left(\frac{1}{(\rho^2-1)^s} - \frac{1}{\rho^{2s}} \right) d\rho. \\
&\leq C r^{p-(p-2)(1-s)}.
\end{aligned} \tag{2.3}$$

This yields that $J_2(r) = \mathcal{O}(r^{2+sp-2s})$.

It follows that

$$J(r) = \mathcal{O}(r^{2-2s}).$$

Looking back at (2.1), using this and recalling (2.2), by sending $\varepsilon \to 0^+$, we obtain that

$$(-\Delta)_p^s u(x) + \int_{|y|>r} \frac{|u(x)-u(x-y)|^{p-2}u(x-y)}{|y|^{n+s(p-2)}(|y|^2-r^2)^s} dy = u(x)\mathcal{D}_r^{s,p}u(x) + \mathcal{O}(r^{2-2s}).$$

This concludes the proof of the Theorem. □

It is a property of mean value kernels that $\mathscr{M}_r u(x)$ converges to $u(x)$ as $r \searrow 0$ both in the local (linear and nonlinear) and in the nonlocal linear setting. In our case, due to the presence of $\mathcal{D}_r^{s,p}u$, we have this property when $\nabla u(x) \neq 0$ only for a limited range of values of p depending on s and becoming larger as $s \nearrow 1$. For other ranges of p, we were not able to obtain such a result. More precisely, we have the following proposition.

Proposition 2.4. *If $u \in C^2_{\rm loc}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and s,p are such that*

$$p \in \left[2, \frac{2}{1-s}\right)$$

then for any $x \in \mathbb{R}^n$ such that $\nabla u(x) \neq 0$, it holds that

$$\lim_{r\searrow 0} \mathscr{M}_r^{s,p} u(x) = u(x).$$

Proof. There is some $r>0$ such that $\nabla u(y) \neq 0$ for all $y \in B_{2r}(x)$. Then

$$\begin{aligned}\mathcal{D}_r^{s,p}u(x) &\geq \int_{B_{2r}\setminus B_r} \frac{|u(x)-u(x-y)|^{p-2}}{|y|^{n+s(p-2)}(|y|^2-r^2)^s} dy \\ &= \int_{B_{2r}\setminus B_r} \left|\nabla u(\xi)\cdot \frac{y}{|y|}\right|^{p-2} |y|^{(p-2)(1-s)-n}(|y|^2-r^2)^{-s}\, dy,\end{aligned}$$

where $\xi \in B_{2r}(x)$. Therefore, using (2.22)

$$\mathcal{D}_r^{s,p}u(x) \geq C_{p,n}|\nabla u(\xi)|^{p-2} r^{(p-2)(1-s)-2s},$$

which for p in the given range, allows to say that

$$\lim_{r\searrow 0} \mathcal{D}_r^{s,p}u(x) = \infty.$$

From Proposition 2.1 and Theorem 2.3, we obtain

$$\lim_{r\searrow 0} \left(u(x) - \mathscr{M}_r^{s,p}u(x)\right) = \lim_{r\searrow 0} \left(\mathcal{D}_r^{s,p}u(x)\right)^{-1} \left((-\Delta)_p^s u(x) + \mathcal{O}(r^{2-2s})\right),$$

and the conclusion is settled. □

2.2. **Viscosity setting.** For the viscosity setting of the (s,p)-Laplacian, see the paper [12] (and also [7, 10, 15]). As a first thing, we recall the definition of viscosity solutions.

Definition 2.5. *A function $u \in L^\infty(\mathbb{R}^n)$, upper (lower) semi-continuous in $\overline{\Omega}$ is a viscosity subsolution (supersolution) in Ω of*

$$(-\Delta)_p^s u = 0, \qquad \text{and we write} \quad (-\Delta)_p^s u \leq\ (\geq)\, 0$$

if for every $x \in \Omega$, any neighborhood $U = U(x) \subset \Omega$ and any $\varphi \in C^2(\overline{U})$ such that

$$\begin{aligned}&\varphi(x) = u(x)\\ &\varphi(y) >\ (<)\, u(y), \qquad \text{for any } y \in U \setminus \{x\},\end{aligned} \tag{2.4}$$

if we let

$$v = \begin{cases} \varphi, & \text{in } U \\ u, & \text{in } \mathbb{R}^n \setminus U\end{cases} \tag{2.5}$$

then

$$(-\Delta)^s_p v(x) \leq \ (\geq)\, 0.$$

A viscosity solution of $(-\Delta)^s_p u = 0$ *is a (continuous) function that is both a subsolution and a supersolution.*

We define here what we mean for an asymptotic expansion to hold in the viscosity sense.

Definition 2.6. *Let* $\varepsilon > 0$ *be fixed and let* $u \in L^\infty(\mathbb{R}^n)$ *be upper (lower) semi-continuous in* Ω*. We say that*

$$\lim_{r\to 0} \mathcal{D}^{s,p}_r u(x)\big(u(x) - \mathscr{M}^{s,p}_r u(x)\big) = 0$$

holds in the viscosity sense if for any neighborhood $U = U(x) \subset \Omega$ *and any* $\varphi \in C^2(\overline{U})$ *such that* (2.4) *holds, and if we let* v *be defined as in* (2.5)*, then both*

$$\liminf_{r\searrow 0} \mathcal{D}^{s,p}_r u(x)\big(u(x) - \mathscr{M}^{s,p}_r u(x)\big) \geq 0$$

and

$$\limsup_{r\searrow 0} \mathcal{D}^{s,p}_r u(x)\big(u(x) - \mathscr{M}^{s,p}_r u(x)\big) \leq 0$$

hold point wisely.

The result for viscosity solutions is a consequence of the asymptotic expansion for smooth functions, and goes as follows.

Theorem 2.7. *Let* $\Omega \subset \mathbb{R}^n$ *be an open set and let* $u \in C(\Omega) \cap L^\infty(\mathbb{R}^n)$*. Then*

$$(-\Delta)^s_p u(x) = 0$$

in the viscosity sense if and only if

$$\lim_{r\to 0} \mathcal{D}^{s,p}_r u(x)\big(u(x) - \mathscr{M}^{s,p}_r u(x)\big) = 0 \tag{2.6}$$

holds for all $x \in \Omega$ *in the viscosity sense.*

Proof. For $x \in \Omega$ and any $U(x)$ neighborhood of x, defining v as in (2.5), we have that $v \in C^2(U(x)) \cap L^\infty(\mathbb{R}^n)$. By Theorem 2.3 we have that

$$\mathcal{D}^{s,p}_r v(x)\big(v(x) - \mathscr{M}^{s,p}_r v(x)\big) = (-\Delta)^s_p v(x) + \mathcal{O}(r^{2-2s}), \tag{2.7}$$

which allows to obtain the conclusion. □

2.3. **Asymptotics as** $s \nearrow 1$**.** We prove here that sending $s \nearrow 1$, for a smooth enough function the fractional p-Laplace operator approaches the p-Laplacian. The result is known in the mathematical community, see [8]. We give here a complete proof, on the one hand for the reader convenience and on the other hand since some estimates here introduced are heavily used throughout Section 2.

Theorem 2.8. *Let* $u \in C^2_{\rm loc}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ *be such that* $\nabla u(x) \neq 0$*. Then*

$$\lim_{s\nearrow 1} (1-s)(-\Delta)^s_p u(x) = -C_{p,n}\, \Delta_p u(x),$$

where $C_{p,n} > 0$*.*

Proof. Since $u \in C^2_{\rm loc}(\mathbb{R}^n)$ we have that for any $\bar\varepsilon > 0$ there exists $r = r(\bar\varepsilon) > 0$ such that

$$\text{for any } |y| < r, \qquad |D^2 u(x) - D^2 u(x+y)| < \bar\varepsilon. \tag{2.8}$$

We fix an arbitrary $\bar\varepsilon$, the corresponding r and some number $0<\varepsilon<\min\{\bar\varepsilon, r\}$, to be taken arbitrarily small.
We notice that

$$(-\Delta)_p^s u(x) = \lim_{\varepsilon\to 0} \mathcal{L}_\varepsilon^{s,p} u(x) = \mathcal{L}_r^{s,p} u(x) + \lim_{\varepsilon\to 0}\left(\mathcal{L}_\varepsilon^{s,p} u(x) - \mathcal{L}_r^{s,p} u(x)\right).$$

As for the first term in this sum, we have that

$$\begin{aligned}\mathcal{L}_r^{s,p} u(x) &= \int_{|y|>r} \frac{|u(x)-u(x-y)|^{p-2}(u(x)-u(x-y))}{|y|^{n+sp}}\, dy \le 2^{p-1}\|u\|_{L^\infty(\mathbb{R}^n)}^{p-1}\omega_n \int_r^\infty \rho^{-1-sp}\, d\rho \\ &= C(n,p,\|u\|_{L^\infty(\mathbb{R}^n)})\frac{r^{-sp}}{sp}.\end{aligned}$$

Notice that

$$\lim_{s\nearrow 1}(1-s)\mathcal{L}_r^{s,p} u(x) = 0. \tag{2.9}$$

Now by symmetry

$$\begin{aligned} 2\Big(\mathcal{L}_\varepsilon^{s,p} u(x) - \mathcal{L}_r^{s,p} u(x)\Big) &= 2\int_{B_r\setminus B_\varepsilon} \frac{|u(x)-u(x-y)|^{p-2}(u(x)-u(x-y))}{|y|^{n+sp}}\, dy \\ = &\int_{B_r\setminus B_\varepsilon} \frac{|u(x)-u(x-y)|^{p-2}(u(x)-u(x-y))}{|y|^{n+sp}}\, dy \\ &+ \int_{B_r\setminus B_\varepsilon} \frac{|u(x)-u(x+y)|^{p-2}(u(x)-u(x+y))}{|y|^{n+sp}}\, dy \\ = &\int_{B_r\setminus B_\varepsilon} \frac{|u(x)-u(x-y)|^{p-2}(2u(x)-u(x-y)-u(x+y))}{|y|^{n+sp}}\, dy \\ &+ \int_{B_r\setminus B_\varepsilon} \frac{\Big(|u(x)-u(x+y)|^{p-2}-|u(x)-u(x-y)|^{p-2}\Big)(u(x)-u(x+y))}{|y|^{n+sp}}\, dy \\ = &\, I_{r,\varepsilon}(x) + J_{r,\varepsilon}(x). \end{aligned} \tag{2.10}$$

Using a Taylor expansion, there exist $\underline{\delta}, \overline{\delta} \in (0,1)$ such that

$$u(x)-u(x-y) = \nabla u(x)\cdot y - \frac12 \langle D^2 u(x-\underline{\delta} y)y, y\rangle, \qquad u(x)-u(x+y) = -\nabla u(x)\cdot y - \frac12 \langle D^2 u(x+\overline{\delta} y)y, y\rangle.$$

Having that $|\underline{\delta} y|, |\overline{\delta} y| \le |y| < r$, recalling (2.8), we get that $\big|\langle (D^2u(x) - D^2 u(x-\underline{\delta} y))y, y\rangle\big| \le \bar\varepsilon |y|^2$, hence

$$\begin{aligned} 2u\,(x) - u(x-y) - u(x+y) &= -\langle D^2 u(x)y,y\rangle \\ &+ \frac12(\langle D^2u(x)y,y\rangle - \langle D^2u(x-\underline{\delta} y)y,y\rangle) + \frac12(\langle D^2u(x)y,y\rangle - \langle D^2u(x+\overline{\delta} y)y,y\rangle) \\ =\; & -\langle D^2u(x)y,y\rangle + T_1, \qquad\qquad \text{with} \quad |T_1| \le \bar\varepsilon|y|^2, \end{aligned} \tag{2.11}$$

according to (2.8). Also denoting $\omega = y/|y| \in \mathbb{S}^{n-1}$ and taking the Taylor expansion for the function $f(x) = |a - xb|^{p-2}$, we obtain

(2.12)

$$\begin{aligned} |u(x)-u(x-y)|^{p-2} &= |y|^{p-2}\Big|\nabla u(x)\cdot\omega - \frac{|y|}{2}\langle D^2u(x-\underline{\delta}y)\omega,\omega\rangle\Big|^{p-2} \\ &= |y|^{p-2}\Big|\nabla u(x)\cdot\omega - \frac{|y|}{2}\left(\langle D^2u(x)\omega,\omega\rangle + \big\langle\big(D^2u(x-\underline{\delta}y)-D^2u(x)\big)\,\omega,\omega\big\rangle\right)\Big|^{p-2} \\ &= |y|^{p-2}\Big|\nabla u(x)\cdot\omega - \frac{|y|}{2}\left(\langle D^2u(x)\omega,\omega\rangle + \mathcal{O}(\bar\varepsilon)\right)\Big|^{p-2} \\ &= |y|^{p-2}|\nabla u(x)\cdot\omega|^{p-2} + T_2, \qquad\qquad \text{with}\quad |T_2| \le C(1+\bar\varepsilon)|y|^{p-1}. \end{aligned}$$

Thus we have that

$$\begin{aligned} |u(x)-u(x-y)|^{p-2}(2u(x)-u(x-y)-u(x+y)) = &-|y|^p|\nabla u(x)\cdot\omega|^{p-2}\langle D^2u(x)\omega,\omega\rangle \\ &+ T_1|y|^{p-2}|\nabla u(x)\cdot\omega|^{p-2} + T_3, \\ &\text{with}\quad |T_3| \le C(1+\bar\varepsilon)|y|^{p+1}. \end{aligned} \tag{2.13}$$

Passing to hyper-spherical coordinates, we have that

$$\begin{aligned} I_{r,\varepsilon}(x) = &-\int_\varepsilon^r \rho^{p-1-sp}\,d\rho\int_{\mathbb{S}^{n-1}}|\nabla u(x)\cdot\omega|^{p-2}\langle D^2u(x)\omega,\omega\rangle\,d\omega + I^1_{r,\varepsilon}(x) + I^2_{r,\varepsilon}(x) \\ = &-\frac{r^{p(1-s)}-\varepsilon^{p(1-s)}}{p(1-s)}\int_{\mathbb{S}^{n-1}}|\nabla u(x)\cdot\omega|^{p-2}\langle D^2u(x)\omega,\omega\rangle\,d\omega + I^1_{r,\varepsilon}(x) + I^2_{r,\varepsilon}(x). \end{aligned} \tag{2.14}$$

Here

$$I^1_{r,\varepsilon} \le \bar\varepsilon\, C\frac{r^{p(1-s)}-\varepsilon^{p(1-s)}}{p(1-s)}$$

and

$$|I^2_{r,\varepsilon}| \le C(1+\bar\varepsilon)\int_{B_r\setminus B_\varepsilon}|y|^{p-sp}\,dy = C(1+\bar\varepsilon)\,\frac{r^{p-sp+1}-\varepsilon^{p-sp+1}}{p(1-s)+1}.$$

This means that

$$\lim_{s\nearrow 1}\lim_{\varepsilon\searrow 0}(1-s)\left(I^1_{r,\varepsilon}(x)+I^2_{r,\varepsilon}(x)\right) = \mathcal{O}(\bar\varepsilon).$$

Thus we get

$$\lim_{s\nearrow 1}\lim_{\varepsilon\searrow 0}(1-s)I_{r,\varepsilon}(x) = -\frac{1}{p}\int_{\mathbb{S}^{n-1}}|\nabla u(x)\cdot\omega|^{p-2}\langle D^2u(x)\omega,\omega\rangle\,d\omega + \mathcal{O}(\bar\varepsilon).$$

Using again that $\left|\langle(D^2u(x)-D^2u(x-\underline{\delta}y))y,y\rangle\right| \le \bar\varepsilon|y|^2$, we also have that

$$\begin{aligned} u(x)-u(x-y) &= \nabla u(x)\cdot y - \frac12\langle D^2u(x)y,y\rangle + \frac12\left(\langle D^2u(x)y,y\rangle - \langle D^2u(x-\underline{\delta}y)y,y\rangle\right) \\ &= \nabla u(x)\cdot y - \frac12\langle D^2u(x)y,y\rangle + \mathcal{O}(\bar\varepsilon)|y|^2 \\ u(x)-u(x+y) &= -\nabla u(x)\cdot y - \frac12\langle D^2u(x)y,y\rangle + \frac12\left(\langle D^2u(x)y,y\rangle - \langle D^2u(x+\overline{\delta}y)y,y\rangle\right) \\ &= -\nabla u(x)\cdot y - \frac12\langle D^2u(x)y,y\rangle + \mathcal{O}(\bar\varepsilon)|y|^2. \end{aligned}$$

Taking the second order expansion (i.e, taking the following order of the expansion in (2.12), with third order reminder) we obtain

$$\begin{aligned}&|u(x)-u(x+y)|^{p-2}-|u(x)-u(x-y)|^{p-2}\\ =&|y|^{p-1}(p-2)(\nabla u(x)\cdot\omega)|\nabla u(x)\cdot\omega|^{p-4}\left(\langle D^2u(x)\omega,\omega\rangle+\mathcal{O}(\bar\varepsilon)\right)+T_4, \qquad |T_4|\le C|y|^p.\end{aligned}$$

Thus

$$\begin{aligned}&(|u(x)-u(x+y)|^{p-2}-|u(x)-u(x-y)|^{p-2})(u(x)-u(x+y))\\ &=-|y|^p(p-2)|\nabla u(x)\cdot\omega|^{p-2}\left(\langle D^2u(x)\omega,\omega\rangle+\mathcal{O}(\bar\varepsilon)\right)+T_5, \qquad |T_5|\le C|y|^{p+1}.\end{aligned} \tag{2.15}$$

Therefore we get that

$$J_{r,\varepsilon}(x)=-(p-2)\frac{r^{p-sp}-\varepsilon^{p-sp}}{p(1-s)}\left(\int_{\mathbb{S}^{n-1}}|\nabla u(x)\cdot\omega)|^{p-2}\langle D^2u(x)\omega,\omega\rangle+\mathcal{O}(\bar\varepsilon)\right)+\tilde J_{r,\varepsilon}(x), \tag{2.16}$$

and

$$|\tilde J_{r,\varepsilon}(x)|\le C\frac{r^{p+1-sp}-\varepsilon^{p+1-sp}}{p(1-s)+1}.$$

We obtain that

$$\lim_{s\nearrow1}\lim_{\varepsilon\searrow0}J_{r,\varepsilon}(x)=-\frac{p-2}{p}\int_{\mathbb{S}^{n-1}}|\nabla u(x)\cdot\omega)|^{p-2}\langle D^2u(x)\omega,\omega\rangle\,d\omega+\mathcal{O}(\bar\varepsilon).$$

It follows that

$$\begin{aligned}\lim_{s\nearrow1}(1-s)(-\Delta)^s_pu(x)=&\lim_{s\nearrow1}(1-s)\left(\mathcal{L}^{s,p}_ru(x)+\lim_{\varepsilon\searrow0}\left(\mathcal{L}^{s,p}_\varepsilon u(x)-\mathcal{L}^{s,p}_ru(x)\right)\right)\\ =&-\frac{p-1}{2p}\int_{\mathbb{S}^{n-1}}|\nabla u(x)\cdot\omega)|^{p-2}\langle D^2u(x)\omega,\omega\rangle\,d\omega+\mathcal{O}(\bar\varepsilon).\end{aligned}$$

Sending $\bar\varepsilon$ to zero, we get that

$$\lim_{s\nearrow1}(1-s)(-\Delta)^s_pu(x)=-\frac{p-1}{2p}|\nabla u(x)|^{p-2}\int_{\mathbb{S}^{n-1}}|z(x)\cdot\omega|^{p-2}\langle D^2u(x)\omega,\omega\rangle\,d\omega,$$

with $z(x)=\nabla u(x)/|\nabla u(x)|$. We follow here the ideas in [8]. Let $U(x)\in\mathbb{M}^{n\times n}(\mathbb{R})$ be an orthogonal matrix, such that $z(x)=U(x)e_n$, where e_k denotes the k^{th} vector of the canonical basis of $\mathbb{R}^n$. Changing coordinates $\omega'=U(x)\omega$ we obtain

$$\begin{aligned}\mathcal{I}=&\int_{\mathbb{S}^{n-1}}|z(x)\cdot\omega'|^{p-2}\langle D^2u(x)\omega',\omega'\rangle\,d\omega=\int_{\mathbb{S}^{n-1}}|e_n\cdot\omega|^{p-2}\langle U(x)^{-1}D^2u(x)U(x)\omega,\omega\rangle\,d\omega\\ =&\int_{\mathbb{S}^{n-1}}|\omega_n|^{p-2}\langle B(x)\omega,\omega\rangle\,d\omega\end{aligned}$$

where $B(x)=U(x)^{-1}D^2u(x)U(x)\in\mathbb{M}^{n\times n}(\mathbb{R})$. Then we get that

$$\mathcal{I}=\sum_{i,j=1}^n b_{ij}(x)\int_{\mathbb{S}^{n-1}}|\omega_n|^{p-2}\omega_i\omega_j d\omega=\sum_{j=1}^n b_{jj}(x)\int_{\mathbb{S}^{n-1}}|\omega_n|^{p-2}\omega_j^2\,d\omega$$

by symmetry. Now

$$\int_{\mathbb{S}^{n-1}}|\omega_n|^{p-2}\omega_j^2\,d\omega=\begin{cases}\gamma_p, & \text{if } j\ne n\\ \gamma_p', & \text{if } j=n\end{cases}$$

with γ_p, γ'_p two constants[1] for which $\gamma'_p/\gamma_p = p-1$, so

$$\mathcal{I} = \; = \gamma_p \sum_{j=1}^n b_{jj}(x) + (\gamma'_p - \gamma_p) b_{nn}(x) = \gamma_p \left(\sum_{j=1}^n b_{jj}(x) + (p-2) b_{nn}(x) \right).$$

We notice that, since $U(x)$ is orthogonal and A is symmetric,

$$\sum_{j=1}^n b_{jj}(x) = \mathrm{Tr} B(x) = \mathrm{Tr}(U(x)^{-1} D^2 u(x) U(x)) = \mathrm{Tr}(D^2 u(x)) = \Delta u(x)$$

and

$$\begin{aligned} b_{nn}(x) &= \langle U(x)^{-1} D^2 u(x) U(x) e_n, e_n \rangle = \langle D^2 u(x) U(x) e_n, U(x) e_n \rangle = \langle D^2 u(x) z(x), z(x) \rangle \\ &= |\nabla u|^{-2} \langle D^2 u(x) \nabla u(x), \nabla u(x) \rangle = \Delta_\infty u(x). \end{aligned}$$

Therefore

$$\mathcal{I} = \frac{\gamma_p (p-1)}{p} (\Delta u(x) + (p-2) \Delta_\infty u(x)), \tag{2.17}$$

and this leads to

$$\lim_{s \nearrow 1} (1-s)(-\Delta)^s_p u(x) = -\frac{\gamma_p (p-1)}{2p} |\nabla u(x)|^{p-2} (\Delta u(x) + (p-2) \Delta_\infty u(x)).$$

Recalling that

$$\Delta_p u(x) = |\nabla u(x)|^{p-2} (\Delta u(x) + (p-2) \Delta_\infty u(x))$$

we conclude the proof of the Lemma. $\square$

Next we study the asymptotic behaviour of $\mathscr{M}^p_r$ as $s \nearrow 1$ and we obtain an asymptotic expansion for the p-Laplace operator.

Proposition 2.9. *For any $u \in C^2_{\mathrm{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, denoting*

$$\mathscr{M}^p_r u(x) := \int_{\mathbb{S}^{n-1}} |u(x) - u(x - r\omega)|^{p-2} u(x - r\omega) \, d\omega \left(\int_{\mathbb{S}^{n-1}} |u(x) - u(x - r\omega)|^{p-2} \, d\omega \right)^{-1}$$

it holds that

$$\lim_{s \nearrow 1} \mathscr{M}^{s,p}_r u(x) = \mathscr{M}^p_r u(x) \tag{2.18}$$

and that

$$\lim_{s \nearrow 1} (1-s) \mathcal{D}^{s,p}_r u(x) = \int_{\mathbb{S}^{n-1}} |u(x) - u(x - r\omega)|^{p-2} \, d\omega. \tag{2.19}$$

In addition,

$$\Big(|\nabla u|^{p-2} + \mathcal{O}(r) \Big) \Big(u(x) - \mathscr{M}^p_r u(x) \Big) = -c_{n,p} r^2 \Delta_p u(x) + \mathcal{O}(r^3). \tag{2.20}$$

---

[1] Precisely (see Lemma 2.1 in [8])

$$\gamma_p = \frac{\Gamma\left(\frac{1}{2}\right)^{n-2} \Gamma\left(\frac{3}{2}\right) \Gamma\left(\frac{p-1}{2}\right)}{\Gamma\left(\frac{p+n}{2}\right)}, \qquad \gamma'_p = \frac{\Gamma\left(\frac{1}{2}\right)^{n-1} \Gamma\left(\frac{p-1}{2}\right)}{\Gamma\left(\frac{p+n}{2}\right)}.$$

Proof. Let $\varepsilon \in (0,1/2)$, to be take arbitrarily small in the sequel. We have that

$$\begin{aligned}\mathcal{D}^{s,p}_r u(x) = &\int_{|y|>(1+\varepsilon)r} \frac{|u(x)-u(x-y)|^{p-2}}{|y|^{n+sp-2s}(|y|^2-r^2)^s}\,dy + \int_{r<|y|<(1+\varepsilon)r} \frac{|u(x)-u(x-y)|^{p-2}}{|y|^{n+sp-2s}(|y|^2-r^2)^s}\,dy\\ =& I_1^{s,\varepsilon} + I_2^{s,\varepsilon}.\end{aligned}$$

Given that for $|y|>r(1+\varepsilon)$ one has that $|y|^2-r^2 \geq \varepsilon(\varepsilon+2)(1+\varepsilon)^{-2}|y|^2$, we get

$$\begin{aligned}|I_1^{s,\varepsilon}| \leq& \frac{(1+\varepsilon^2)^s}{\varepsilon^s(\varepsilon+2)^s} c_{n,p,\|u\|_{L^\infty(\mathbb{R}^n)}} \int_{(1+\varepsilon)r}^\infty \rho^{-1-sp}\,d\rho\\ =& \frac{(1+\varepsilon^2)^s}{\varepsilon^s(\varepsilon+2)^s} c_{n,p,\|u\|_{L^\infty(\mathbb{R}^n)}} \frac{[(1+\varepsilon)r]^{-sp}}{s}\end{aligned} \tag{2.21}$$

and it follows that

$$\lim_{s\nearrow 1}(1-s)I_1^{s,\varepsilon} = 0.$$

On the other hand, integrating by parts we get that

$$\begin{aligned}I_2^{s,\varepsilon} =& \int_{\mathbb{S}^{n-1}} d\omega\left(\int_r^{(1+\varepsilon)r} |u(x)-u(x-\rho\omega)|^{p-2}\rho^{-1-sp+2s}(\rho^2-r^2)^{-s}\,d\rho\right)\\ =& \int_{\mathbb{S}^{n-1}} d\omega\Bigg[\frac{(\rho-r)^{1-s}}{1-s}|u(x)-u(x-\rho\omega)|^{p-2}\rho^{-1-sp+2s}(\rho+r)^{-s}\Bigg|_r^{(1+\varepsilon)r}\\ &- \int_r^{(1+\varepsilon)r} \frac{(\rho-r)^{1-s}}{1-s}\frac{d}{d\rho}\left(|u(x)-u(x-\rho\omega)|^{p-2}\rho^{-1-sp+2s}(\rho+r)^{-s}\right)\,d\rho\Bigg]\\ =& \int_{\mathbb{S}^{n-1}} d\omega\Bigg[\frac{(\varepsilon r)^{1-s}}{1-s}|u(x)-u(x-(1+\varepsilon)r\omega)|^{p-2}[(1+\varepsilon)r]^{-1-sp+2s}[(2+\varepsilon)r]^{-s}\\ &- \int_r^{(1+\varepsilon)r} \frac{(\rho-r)^{1-s}}{1-s}\frac{d}{d\rho}\left(|u(x)-u(x-\rho\omega)|^{p-2}\rho^{-1-sp+2s}(\rho+r)^{-s}\right)\,d\rho\Bigg]\end{aligned}$$

Notice that

$$\begin{aligned}&\left|\int_r^{(1+\varepsilon)r} \frac{(\rho-r)^{1-s}}{1-s}\frac{d}{d\rho}\left(|u(x)-u(x-\rho\omega)|^{p-2}\rho^{-1-sp+2s}(\rho+r)^{-s}\right)\,d\rho\right|\\ &\qquad\leq C\max\{r^{-sp}, r^{1-sp}\}\frac{\varepsilon^{2-s}}{1-s},\end{aligned}$$

hence

$$\begin{aligned}&\lim_{s\nearrow 1}(1-s)\int_{\mathbb{S}^{n-1}} d\omega\left[\int_r^{(1+\varepsilon)r} \frac{(\rho-r)^{1-s}}{1-s}\frac{d}{d\rho}\left(|u(x)-u(x-\rho\omega)|^{p-2}\rho^{-1-sp+2s}(\rho+r)^{-s}\right)\,d\rho\right]\\ &\qquad= \mathcal{O}(\varepsilon).\end{aligned}$$

On the other hand

$$\begin{aligned}&\lim_{s\nearrow 1}(1-s)\int_{\mathbb{S}^{n-1}} d\omega\frac{(\varepsilon r)^{1-s}}{1-s}|u(x)-u(x-(1+\varepsilon)r\omega)|^{p-2}[(1+\varepsilon)r]^{-1-sp+2s}[(2+\varepsilon)r]^{-s}\\ =& \frac{1}{(1+\varepsilon)^{p-1}(2+\varepsilon)^{-1}r^p}\int_{\mathbb{S}^{n-1}} d\omega|u(x)-u(x-(1+\varepsilon)r\omega)|^{p-2}.\end{aligned}$$

Finally,

$$\lim_{\varepsilon\searrow 0}\lim_{s\nearrow 1}(1-s)I_2^{s,\varepsilon}=\frac{2}{r^p}\int_{\mathbb{S}^{n-1}}|u(x)-u(x-r\omega)|^{p-2}\,d\omega$$

and one gets (2.19). In exactly the same fashion, one proves that

$$\begin{aligned}\lim_{s\nearrow 1}(1-s)\int_{|y|>r}&\left(\frac{|u(x)-u(x-y)|}{|y|^s}\right)^{p-2}\frac{u(x-y)}{|y|^n(|y|^2-r^2)^s}\,dy\\&=\frac{2}{r^p}\int_{\mathbb{S}^{n-1}}|u(x)-u(x-r\omega)|^{p-2}u(x-r\omega)\,d\omega\end{aligned}$$

and (2.18) can be concluded.

In order to prove (2.20), one uses (2.13) and (2.15), the computations and notations in Theorem 2.8 (in particular (2.13), (2.15) and (2.17)) to obtain that

$$\begin{aligned}&\left(\int_{\mathbb{S}^{n-1}}|u(x)-u(x-r\omega)|^{p-2}\,d\omega\right)(u(x)-\mathscr{M}_r^p u(x))\\&=-\frac{p-1}{2}r^p\int_{\mathbb{S}^{n-1}}|\nabla u(x)\cdot\omega|^{p-2}\langle D^2u(x)\omega,\omega\rangle+\mathcal{O}(r^{p+1})\\&=-r^p\frac{\gamma_p(p-1)}{2p}|\nabla u(x)|^{p-2}(\Delta u(x)+(p-2)\Delta_\infty u(x))+\mathcal{O}(r^{p+1})\\&=-r^p\frac{\gamma_p(p-1)}{2p}\Delta_p u(x)+\mathcal{O}(r^{p+1}).\end{aligned}$$

Proving in the same way by (2.12), that

$$\int_{\mathbb{S}^{n-1}}|u(x)-u(x-r\omega)|^{p-2}\,d\omega=r^{p-2}\int_{\mathbb{S}^{n-1}}|\nabla u(x)\cdot\omega|^{p-2}\,d\omega+\mathcal{O}(r^{p-1}),$$

and recalling that

$$\int_{\mathbb{S}^{n-1}}|\nabla u(x)\cdot\omega|^{p-2}\,d\omega=C_{n,p}|\nabla u(x)|^{p-2},\tag{2.22}$$

we get that

$$\Big(|\nabla u(x)|^{p-2}+\mathcal{O}(r)\Big)\Big(u(x)-\mathscr{M}_r^p u(x)\Big)=-\tilde{c}_{n,p}r^2\Delta_p u(x)+\mathcal{O}(r^3),$$

with

$$\tilde{c}_{n,p}=\frac{\gamma_p(p-1)}{pC_{n,p}}=\frac{(p-1)(p-3)}{2p(p+n-2)}.$$

This concludes the proof of the Theorem. □

Remark 2.10. Let us point out that (2.20) gives an asymptotic expansion for the p-Laplace which differs from the one given in [14]. The very nice formula in [14] says that

$$|\nabla u|^{p-2}\left(u(x)-\tilde{\mathcal{M}}_p u(x)\right)=-\bar{c}_{p,n}r^2\Delta_p u(x)+o(r^2),$$

with

$$\tilde{\mathcal{M}}_p u(x)=\frac{2+n}{p+n}\fint_{B_r(x)}u(y)\,dy+\frac{p-2}{2(p+n)}\left(\max_{\overline{B_r(x)}}u(y)+\min_{\overline{B_r(x)}}u(y)\right)$$

and

$$\bar{c}_{p,n}=\frac{1}{2(p+n)}.$$

The statement (2.20), even though it appears weaker, still allows us to conclude that in the viscosity sense, at points $x \in \mathbb{R}^n$ for which the test functions $v(x)$ satisfy $\nabla v(x) \neq 0$, if u satisfies the mean value property, then $\Delta_p u(x) = 0$.

3. Gradient dependent operators

3.1. **The "nonlocal" p-Laplacian.** In this section, we are interested in a nonlocal version of the p-Laplace operator, that arises in tug-of war game, introduced in [3].

This operator is the nonlocal version of the p-Laplacian given in a non-divergence form, and deprived of the $|\nabla u|^{p-2}$ factor. Let $p \in (1, +\infty)$ and denote for any $A \in \mathscr{M}^{n\times n}(\mathbb{R})$ and $\xi \in \mathbb{R}^n$,

$$\langle A\xi, \xi\rangle = \sum_{i,j=1}^n a_{ij}\xi_i\xi_j.$$

Precisely, in the classical setting, the p-Laplace operator for $\nabla u \neq 0$ is

$$\Delta_p u := \Delta_{p,\pm} u = \Delta u + (p-2)|\nabla u|^{-2}\langle D^2 u \nabla u, \nabla u\rangle.$$

By convention, when $\nabla u = 0$, as in [3],

$$\Delta_{p,+} u := \Delta u + (p-2) \sup_{\xi \in \mathbb{S}^{n-1}} \langle D^2 u\, \xi, \xi\rangle$$

and

$$\Delta_{p,-} u := \Delta u + (p-2) \inf_{\xi \in \mathbb{S}^{n-1}} \langle D^2 u\, \xi, \xi\rangle.$$

Let $s \in (1/2, 1)$ and $p \in [2, +\infty)$. In the nonlocal setting we have the following definition given in [3, Section 4].

Definition 3.1. *When $\nabla u(x) = 0$ we define*

$$(-\Delta)^s_{p,+} u(x) := \frac{1}{\alpha_p} \sup_{\xi \in \mathbb{S}^{n-1}} \int_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}} \chi_{[c_p,1]}\left(\frac{y}{|y|}\cdot \xi\right) dy$$

and

$$(-\Delta)^s_{p,-} u(x) := \frac{1}{\alpha_p} \inf_{\xi \in \mathbb{S}^{n-1}} \int_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}} \chi_{[c_p,1]}\left(\frac{y}{|y|}\cdot \xi\right) dy.$$

When $\nabla u(x) \neq 0$ then

$$(-\Delta)^s_p u(x) = (-\Delta)^s_{p,\pm} = \frac{1}{\alpha_p} \int_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}} \chi_{[c_p,1]}\left(\frac{y}{|y|}\cdot z(x)\right) dy$$

with

$$z(x) = \frac{\nabla u(x)}{|\nabla u(x)|}.$$

Here, c_p, α_p are positive constants.

We remark that the case $p \in (1,2)$ is defined with the kernel $\chi_{[0,c_p]}\left(\frac{y}{|y|}\cdot z(x)\right)$ for some $c_p > 0$, and can be treated in the same way.

In particular, for $p \in [2, +\infty)$ we consider

$$\begin{aligned} \alpha_p &:= \frac{1}{2}\int_{\mathbb{S}^{n-1}} (\omega\cdot e_2)^2 \chi_{[c_p,1]}(\omega\cdot e_1)\, d\omega, \\ \beta_p &:= \frac{1}{2}\int_{\mathbb{S}^{n-1}} (\omega\cdot e_1)^2 \chi_{[c_p,1]}(\omega\cdot e_1)\, d\omega - \alpha_p, \end{aligned} \tag{3.1}$$

and

$$c_p \quad \text{such that} \quad \frac{\beta_p}{\alpha_p} = p-2. \tag{3.2}$$

With these constants, one gets [3, Subsections 4.2.1, 4.2.2], which affirms that if $u \in C^2(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$, then

$$\lim_{s\nearrow 1}(1-s)\Delta_p^s u(x) = \Delta_p u(x).$$

We define now a (s,p)-mean kernel for the nonlocal p-Laplacian.

Definition 3.2. *For any $r>0$ and $u\in L^\infty(\mathbb{R}^n)$, when $\nabla u(x)\neq 0$, let*

$$M_r^{s,p}u(x) := \frac{C_{s,p}r^{2s}}{2}\int_{\mathcal{C}B_r}\frac{u(x+y)+u(x-y)}{|y|^n(|y|^2-r^2)^s}\chi_{[c_p,1]}\left(\frac{y}{|y|}\cdot z(x)\right)dy, \qquad z(x)=\frac{\nabla u(x)}{|\nabla u(x)|},$$

where[2]*.*

$$C_{s,p}=c_s\gamma_p, \quad \textit{with} \quad c_s := \left(\int_1^\infty \frac{d\rho}{\rho(\rho^2-1)^s}\right)^{-1}, \quad \gamma_p := \left(\int_{\mathbb{S}^{n-1}}\chi_{[c_p,1]}(\omega\cdot e_1)d\omega\right)^{-1}.$$

When $\nabla u(x)=0$, we define

$$M_r^{s,p,+}u(x) := \frac{C_{s,p,+}r^{2s}}{2}\sup_{\xi\in\mathbb{S}^{n-1}}\int_{\mathcal{C}B_r}\frac{u(x+y)+u(x-y)}{|y|^n(|y|^2-r^2)^s}\chi_{[c_p,1]}\left(\frac{y}{|y|}\cdot\xi\right)dy,$$

and

$$M_r^{s,p,-}u(x) := \frac{C_{s,p,-}r^{2s}}{2}\inf_{\xi\in\mathbb{S}^{n-1}}\int_{\mathcal{C}B_r}\frac{u(x+y)+u(x-y)}{|y|^n(|y|^2-r^2)^s}\chi_{[c_p,1]}\left(\frac{y}{|y|}\cdot\xi\right)dy,$$

with

$$C_{s,p,+}=c_s\gamma_{p,+} \quad \textit{with} \quad \gamma_{p,+} := \left(\sup_{\xi\in\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}}\chi_{[c_p,1]}(\omega\cdot\xi)d\omega\right)^{-1}$$

respectively

$$C_{s,p,-}=c_s\gamma_{p,-} \quad \textit{with} \quad \gamma_{p,-} := \left(\inf_{\xi\in\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}}\chi_{[c_p,1]}(\omega\cdot\xi)d\omega\right)^{-1}.$$

We have the next asymptotic expansion for smooth functions.

Theorem 3.3. *Let $u\in C^2_{\mathrm{loc}}(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$. Then*

$$u(x) = M_r^{s,p,\pm}u(x) + c(n,s,p)r^{2s}(-\Delta)^s_{p,\pm}u(x) + \mathcal{O}(r^{2s+2}),$$

as $r\searrow 0$.

Proof. We prove the result for $\nabla u(x)\neq 0$ (the proof goes the same for $\nabla u(x)=0$).
Since $u\in C^2_{\mathrm{loc}}(\mathbb{R}^n)$ we have that for any $\bar\varepsilon>0$ there exists $r=r(\bar\varepsilon)>0$ such that (2.8) is satisfied. Passing to spherical coordinates we have that

$$C_{s,p}r^{2s}\int_{\mathcal{C}B_r}\frac{dy}{|y|^n(|y|^2-r^2)^s}\chi_{[c_p,1]}\left(\frac{y}{|y|}\cdot z(x)\right) = C_{s,p}\int_1^\infty\frac{d\rho}{\rho(\rho^2-1)^s}\int_{\mathbb{S}^{n-1}}\chi_{[c_p,1]}\left(\omega\cdot z(x)\right)d\omega$$
$$= C_{s,p}\int_1^\infty\frac{dy}{\rho(\rho^2-1)^s}\int_{\mathbb{S}^{n-1}}\chi_{[c_p,1]}\left(\omega\cdot e_1\right)d\omega = 1,$$

where the last line follows after a rotation (one takes $U\in\mathscr{M}^{n\times n}(\mathbb{R})$ an orthogonal matrix such that $U^{-1}(x)z(x)=e_1$ and changes variables).

[2]It holds that $c(s)=\frac{2\sin\pi s}{\pi}$

It follows that for any $r > 0$,

$$u(x) - M_r^{s,p} u(x) = \frac{C_{s,p} r^{2s}}{2} \int_{\mathcal{C}B_r} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^n (|y|^2 - r^2)^s} \chi_{[c_p,1]} \left(\frac{y}{|y|} \cdot z(x) \right) dy.$$

Therefore we have that

$$\begin{aligned}
u(x) - M_r^{s,p} u(x) = &\ \frac{C_{s,p} \alpha_p r^{2s}}{2} (-\Delta)_p^s u(x) \\
&- \frac{C_{s,p} r^{2s}}{2} \int_{B_r} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}} \chi_{[c_p,1]} \left(\frac{y}{|y|} \cdot z(x) \right) dy \\
&+ \frac{C_{s,p} r^{2s}}{2} \int_{\mathcal{C}B_r} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}} \left(\frac{|y|^{2s}}{(|y|^2 - r^2)^s} - 1 \right) \chi_{[c_p,1]} \left(\frac{y}{|y|} \cdot z(x) \right) dy \\
=: &\ \frac{C_{s,p} \alpha_p r^{2s}}{2} (-\Delta)_p^s u(x) - I_r + J_r
\end{aligned}$$

and

$$\begin{aligned}
J_r = &\ \frac{C_{s,p}}{2} \int_{\mathcal{C}B_1} \frac{2u(x) - u(x+ry) - u(x-ry)}{|y|^{n+2s}} \left(\frac{|y|^{2s}}{(|y|^2 - 1)^s} - 1 \right) \chi_{[c_p,1]} \left(\frac{y}{|y|} \cdot z(x) \right) dy \\
= &\ \frac{C_{s,p}}{2} \int_{B_{\frac{1}{r}} \setminus B_1} \frac{2u(x) - u(x+ry) - u(x-ry)}{|y|^{n+2s}} \left(\frac{|y|^{2s}}{(|y|^2 - 1)^s} - 1 \right) \chi_{[c_p,1]} \left(\frac{y}{|y|} \cdot z(x) \right) dy \\
&+ \frac{C_{s,p}}{2} \int_{\mathcal{C}B_{\frac{1}{r}}} \frac{2u(x) - u(x+ry) - u(x-ry)}{|y|^{n+2s}} \left(\frac{|y|^{2s}}{(|y|^2 - 1)^s} - 1 \right) \chi_{[c_p,1]} \left(\frac{y}{|y|} \cdot z(x) \right) dy \\
= &\ J_r^1 + J_r^2.
\end{aligned}$$

We obtain

$$\begin{aligned}
|J_r^2| &\le 4 \|u\|_{L^\infty(\mathbb{R}^n)} \frac{C_{n,s,p}}{2} \int_{\frac{1}{r}}^\infty \frac{d\rho}{\rho^{1+2s}} \left(\frac{\rho^{2s}}{(\rho^2 - 1)^s} - 1 \right) \int_{\mathbb{S}^{n-1}} \chi_{[c_p,1]} (\omega \cdot z(x)) \, d\omega \\
&\le C_{s,p} \int_{\frac{1}{r}}^\infty \frac{d\rho}{\rho^{1+2s}} \left(\frac{\rho^{2s}}{(\rho^2 - 1)^s} - 1 \right),
\end{aligned}$$

and using Proposition A.1

$$J_r^2 = \mathcal{O}(r^{2+2s}).$$

We have that

$$\begin{aligned}
J_r^1 - I_r = \frac{C_{s,p}}{2} \Bigg[&\int_{B_{\frac{1}{r}} \setminus B_1} \frac{2u(x) - u(x+ry) - u(x-ry)}{|y|^n (|y|^2 - 1)^s} \chi_{[c_p,1]} \left(\frac{y}{|y|} \cdot z(x) \right) dy \\
&- \int_{B_{\frac{1}{r}}} \frac{2u(x) - u(x+ry) - u(x-ry)}{|y|^{n+2s}} \chi_{[c_p,1]} \left(\frac{y}{|y|} \cdot z(x) \right) dy \Bigg]
\end{aligned}$$

which, by (2.11) and Proposition A.1, gives that

$$J_r^1 - I_r = \mathcal{O}(r^{2s+2}).$$

It follows that

$$u(x) - M_r^{s,p} u(x) = C(s,p) r^{2s} (-\Delta)_p^s u(x) + \mathcal{O}(r^{2s+2})$$

for $r \searrow 0$, hence the conclusion. □

We recall the viscosity setting introduced in [3].

Definition 3.4. *A function $u \in L^\infty(\mathbb{R}^n)$, upper (lower) semi-continuous in $\overline{\Omega}$ is a viscosity subsolution (supersolution) in Ω of*

$$(-\Delta)^s_{p,\pm} u = 0, \qquad \text{and we write} \quad (-\Delta)^s_{p,\pm} u \leq \, (\geq)\, 0$$

if for every $x \in \Omega$, any neighborhood $U = U(x) \subset \Omega$ and any $\varphi \in C^2(\overline{U})$ such that (2.4) holds if we let v as in (2.5)

$$(-\Delta)^s_{p,\pm} v(x) \leq \, (\geq)\, 0.$$

A viscosity solution of $(-\Delta)^s_{p,\pm} u = 0$ is a (continuous) function that is both a subsolution and a supersolution.

Furthermore, we define an asymptotic expansion in the viscosity sense.

Definition 3.5. *Let $u \in L^\infty(\mathbb{R}^n)$ upper (lower) semi-continuous in Ω. We say that*

$$\lim_{r \searrow 0} \left(u(x) - M_r^{s,p} u(x)\right) = o(r^{2s})$$

holds in the viscosity sense if for any neighborhood $U = U(x) \subset \Omega$ and any $\varphi \in C^2(\overline{U})$ such that (2.4) holds, and if we let v be defined as in (2.5), then both

$$\liminf_{r \searrow 0} \frac{u(x) - M_r^{s,p} u(x)}{r^{2s}} \geq 0$$

and

$$\limsup_{r \searrow 0} \frac{u(x) - M_r^{s,p} u(x)}{r^{2s}} \leq 0$$

hold point wisely.

The result for viscosity solutions, which is a direct consequence of Theorem 3.3 applied to the test function v, goes as follows.

Theorem 3.6. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u \in C(\Omega) \cap L^\infty(\mathbb{R}^n)$. Then*

$$(-\Delta)^s_{p,\pm} u(x) = 0$$

in the viscosity sense if and only if

$$\lim_{r \searrow 0} \left(u(x) - M_r^{s,p} u(x)\right) = o(r^{2s})$$

holds for all $x \in \Omega$ in the viscosity sense.

We study also the limit case as $s \nearrow 1$ of this version of the (s,p)-mean kernel, and obtain an asymptotic expansion in the local case.

Proposition 3.7. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $u \in C^1(\Omega) \cap L^\infty(\mathbb{R}^n)$. For any $r > 0$ small denoting*

$$M_r^p u(x) := \frac{1}{2} \int_{\partial B_r} \left(u(x+y) - u(x-y)\right) \chi_{[c_p,1]} \left(\frac{y}{|y|} \cdot z(x)\right) dy$$

it holds that

$$\lim_{s \nearrow 1} M_r^{s,p} u(x) = M_r^p u(x), \tag{3.3}$$

for every $x \in \Omega, r > 0$ such that $B_{2r}(x) \subset \Omega$. In addition,

$$u(x) - M_r^p u(x) = -c_{n,p} r^2 (-\Delta)_{p,\pm} u(x) + \mathcal{O}(r^3). \tag{3.4}$$

Proof. It holds that

$$M_r^{s,p}u(x) = \frac{C_{s,p}}{2}\int_{\mathcal{C}B_1} \frac{u(x+ry)+u(x-ry)}{|y|^n(|y|^2-1)^s}\chi_{[c_p,1]}\left(\frac{y}{|y|}\cdot z(x)\right)dy.$$

Let $\varepsilon > 0$ be fixed (to be taken arbitrarily small). Then

$$|J_\varepsilon(x)| := \left|\int_{B_{1+\varepsilon}} \frac{u(x+ry)+u(x-ry)}{|y|^n(|y|^2-1)^s}\chi_{[c_p,1]}\left(\frac{y}{|y|}\cdot z(x)\right)dy\right| \le \frac{2\|u\|_{L^\infty(\mathbb{R}^n)}}{\gamma_p}\int_{1+\varepsilon}^\infty \frac{dt}{t(t^2-1)^s},$$

which from Proposition A.1 gives that

$$\lim_{s\nearrow 1} C_{s,p}J_\varepsilon(x) = 0.$$

On the other hand, we have that

$$\begin{aligned}
I_\varepsilon(x) &= \int_{B_{1+\varepsilon}\setminus B_1} \frac{u(x+ry)+u(x-ry)}{|y|^n(|y|^2-1)^s}\chi_{[c_p,1]}\left(\frac{y}{|y|}\cdot z(x)\right)dy\\
&= \int_{\mathbb{S}^{n-1}}\left(\int_1^{1+\varepsilon}\frac{u(x+r\rho\omega)+u(x-r\rho\omega)}{\rho(\rho^2-1)^s}\,d\rho\right)\chi_{[c_p,1]}\left(\omega\cdot z(x)\right)d\omega
\end{aligned}$$

and integrating by parts, that

$$\int_1^{1+\varepsilon}\frac{u(x+r\rho\omega)+u(x-r\rho\omega)}{\rho(\rho^2-1)^s}\,d\rho = \frac{\varepsilon^{1-s}}{1-s}\frac{u(x+r(1+\varepsilon)\omega)+u(x-r(1+\varepsilon)\omega)}{(1+\varepsilon)(2+\varepsilon))^s} - I_\varepsilon^o(x)$$

with

$$I_\varepsilon^o(x) := \int_1^{1+\varepsilon}\frac{(\rho-1)^{1-s}}{1-s}\frac{d}{d\rho}\left(\frac{u(x+r\rho\omega)+u(x-r\rho\omega)}{\rho(\rho+1)^s}\right)d\rho.$$

We notice that

$$|I_\varepsilon^o(x)| \le C\frac{\varepsilon^{2-s}}{1-s},$$

hence we get

$$\lim_{s\nearrow 1} C_{s,p}I_\varepsilon^o(x) = \mathcal{O}(\varepsilon).$$

Therefore we obtain

$$\lim_{s\nearrow 1} M_r^{s,p}u(x) = \frac{1}{(1+\varepsilon)(2+\varepsilon)}\int_{\mathbb{S}^{n-1}}\left(u(x+r\omega)+u(x-r\omega)\right)\chi_{[c_p,1]}\left(\omega\cdot z(x)\right)dy + \mathcal{O}(\varepsilon),$$

and (3.3) follows by sending $\varepsilon \to 0$. ☐

3.2. **The infinity fractional Laplacian.** In this section, we deal with the infinity fractional Laplacian, arising in a nonlocal tug-of-war game, as introduced in [4]. Therein, the authors deal with viscosity solutions of a Dirichlet monotone problem and a monotone double obstacle problem, providing a comparison principle on compact sets and Hölder regularity of solutions.

The infinity Laplacian in the non-divergence form is defined by omitting the term $|\nabla u|^2$. Precisely, by convention, denoting for any $A \in \mathscr{M}^{n\times n}(\mathbb{R})$ and $\xi \in \mathbb{R}^n$,

$$\langle A\xi,\xi\rangle = \sum_{i,j=1}^n a_{ij}\xi_i\xi_j.$$

we define when $\nabla u = 0$,

$$\Delta_{\infty,+}u := \sup_{\xi\in\mathbb{S}^{n-1}}\langle D^2u\,\xi,\xi\rangle \quad \text{and} \quad \Delta_{\infty,-}u := \inf_{\xi\in\mathbb{S}^{n-1}}\langle D^2u\,\xi,\xi\rangle,$$

whereas when $\nabla u = 0$,

$$\Delta_\infty u := \Delta_{\infty,\pm} u = \langle D^2 u\, z(x), z(x)\rangle, \qquad \text{where} \qquad z(x) = \frac{\nabla u(x)}{|\nabla u(x)|}.$$

The definition in the fractional case is well posed for $s \in (1/2, 1)$, given in [4, Definition 1.1].

Definition 3.8. *Let $s \in (\frac{1}{2}, 1)$. The infinity fractional Laplacian $(-\Delta)^s_\infty \colon C^{1,1}(x) \cap BC(\mathbb{R}^n)$ at a point x is defined in the following way:*

- *If $\nabla u(x) \neq 0$ then*

$$(-\Delta)^s_\infty u(x) = \int_0^\infty \frac{2u(x) - u(x+\rho z(x)) - u(x - \rho z(x))}{\rho^{1+2s}} d\rho, \tag{3.5}$$

  where $z(x) = \frac{\nabla u(x)}{|\nabla u(x)|} \in \mathbb{S}^{n-1}$.

- *If $\nabla u(x) = 0$ then*

$$(-\Delta)^s_\infty u(x) = \sup_{\omega \in \mathbb{S}^{n-1}} \inf_{\zeta \in \mathbb{S}^{n-1}} \int_0^\infty \frac{2u(x) - u(x+\rho\omega) - u(x-\rho\zeta)}{\rho^{1+2s}} d\rho. \tag{3.6}$$

In the above,

$$BC(\mathbb{R}^n) := \{u \colon \mathbb{R}^n \to \mathbb{R} \,|\, u \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\}$$

and $u \in C^{1,1}(x)$ if there exists a vector $p \in \mathbb{R}^n$ and numbers $M, \eta_0 > 0$ such that

$$|u(x+y) - u(x) - p \cdot y| \leq M|y|^2$$

for $|y| < \eta_0$. We define $\nabla u(x) := p$.

As an example, it is proved in [4] that the function

$$C(x) = A|x - x_0|^{2s-1} + B$$

satisfies

$$(-\Delta)^s_\infty u(x) = 0 \text{ for any } x \neq x_0.$$

We denote

$$\mathcal{L}u(x, \omega, \zeta) := \int_0^\infty \frac{2u(x) - u(x+\rho\omega) - u(x-\rho\zeta)}{\rho^{1+2s}}\, d\rho$$

and for $r > 0$

$$M^s_r u(x, y, z) := c_s r^{2s} \int_r^\infty \frac{u(x+\rho\omega) + u(x-\rho\zeta)}{(\rho^2 - r^2)^s \rho} d\rho,$$

with

$$c_s := \left(\int_1^\infty \frac{d\rho}{\rho(\rho^2-1)^s}\right)^{-1} = \frac{2\sin \pi s}{\pi}.$$

We define the operators

- If $\nabla u(x) \neq 0$

$$\mathscr{M}^{s,\infty}_r u(x) = M^s_r u(x, z(x), z(x)), \qquad \text{with } z(x) = \frac{\nabla u(x)}{|\nabla u(x)|}.$$

- If $\nabla u(x) = 0$

$$\mathscr{M}^{s,\infty}_r u(x) = \sup_{\omega \in \mathbb{S}^{n-1}} \inf_{\zeta \in \mathbb{S}^{n-1}} M^s_r u(x, \omega, \zeta)$$

We obtain the asymptotic mean value property for smooth functions, as follows.

Theorem 3.9. *Let $u \in C^{1,1}(x) \cap BC(\mathbb{R}^n)$. Then*

$$u(x) = \mathscr{M}^{s,\infty}_r u(x) + c(s) r^{2s} (-\Delta)^s_\infty u(x) + \mathcal{O}(r^{2+2s})$$

as $r \to 0$.

Proof. Since $u \in C^2_{\text{loc}}(\mathbb{R}^n)$ we have that for any $\bar\varepsilon > 0$ there exists $r = r(\bar\varepsilon) > 0$ such that (2.8) is satisfied. We have that

$$u(x) - M^s_r u(x,y,z) = c_s r^{2s} \int_r^\infty \frac{2u(x) - u(x+\rho\omega) - u(x-\rho\zeta)}{\rho(\rho^2-r^2)^s} d\rho,$$

hence

$$\begin{aligned} u(x) - M^s_r u(x,y,z) = c_s \Bigg[& r^{2s}\mathcal{L}u(x,\omega,\zeta) - \int_{B_1} \frac{2u(x) - u(x+r\rho\omega) - u(x-r\rho\zeta)}{\rho^{1+2s}} d\rho \\ & + \int_{\mathcal{C}B_1} \frac{2u(x) - u(x+r\rho\omega) - u(x-r\rho\zeta)}{\rho^{1+2s}} \left(\frac{\rho^{2s}}{(\rho^2-1)^{2s}} - 1 \right) d\rho \Bigg] \\ =: {} & c_s \left(r^{2s}\mathcal{L}u(x,\omega,\zeta) - I_r + J_r \right). \end{aligned}$$

Then

$$\begin{aligned} J_r = & \int_{B_{\frac{1}{r}} \setminus B_1} \frac{2u(x) - u(x+r\rho\omega) - u(x-r\rho\zeta)}{\rho^{1+2s}} \left(\frac{\rho^{2s}}{(\rho^2-1)^{2s}} - 1 \right) d\rho \\ & + \int_{\mathcal{C}B_{\frac{1}{r}}} \frac{2u(x) - u(x+r\rho\omega) - u(x-r\rho\zeta)}{\rho^{1+2s}} \left(\frac{\rho^{2s}}{(\rho^2-1)^{2s}} - 1 \right) d\rho \\ =: {} & J^1_r + J^r_2. \end{aligned}$$

We proceed as in the proof of Theorem 3.3, using also (2.11) and Proposition A.1, and obtain that

$$J^2_r = \mathcal{O}(r^{2s+2s}) \qquad \text{and} \qquad J^1_r - I_r = \mathcal{O}(r^{2s+2s}).$$

This concludes the proof of the Theorem. ☐

The main result of this section, which follows from Theorem 3.9, is stated next.

Theorem 3.10. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $u \in BC(\mathbb{R}^n)$. The asymptotic expansion*

$$u(x) = \mathscr{M}^{s,\infty}_r u(x) + o(r^{2s}), \qquad \text{as } r \to 0 \tag{3.7}$$

holds for all $x \in \Omega$ in the viscosity sense if and only if

$$(-\Delta)^s_\infty u(x) = 0$$

in the viscosity sense.

We investigate also the limit case $s \nearrow 1$.

Proposition 3.11. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $u \in C^1(\Omega) \cap L^\infty(\mathbb{R}^n)$. Then*

$$\lim_{s \nearrow 1} \mathscr{M}^s_r u(x) = \begin{cases} \dfrac{1}{2}\Big(u\big(x + rz(x)\big) + u\big(x - rz(x)\big) \Big) & \text{when } \nabla u(x) \neq 0, \\ \dfrac{1}{2}\left(\sup\limits_{\omega \in \mathbb{S}^{n-1}} u(x+r\omega) + \inf\limits_{\zeta \in \mathbb{S}^{n-1}} u(x-r\zeta) \right) & \text{when } \nabla u(x) = 0, \end{cases}$$

for every $x \in \Omega, r > 0$ such that $B_{2r}(x) \subset \Omega$.

Proof. For some $\varepsilon > 0$ small enough, we have that

$$\begin{aligned} M^s_r u(x,\omega,\zeta) = & c_s \left(\int_{1+\varepsilon}^\infty \frac{u(x+r\rho\omega) + u(x-r\rho\zeta)}{(\rho^2-1)^s \eta} d\rho + \int_1^{1+\varepsilon} \frac{u(x+r\rho\omega) + u(x-r\rho\zeta)}{(\eta^2-1)^s \rho} d\rho \right) \\ =: {} & I^1_s + I^2_s. \end{aligned} \tag{3.8}$$

Using Proposition A.1, we get that

$$\lim_{s \to 1} c_s I^1_s = 0.$$

Integrating by parts in I_s^2, we have

$$\left|\int_1^{1+\varepsilon} \frac{u(x+r\rho\omega)}{(\eta^2-1)^s\rho}d\rho - \frac{\varepsilon^{1-s}u\left(x+r(1+\varepsilon)\omega\right)}{(1-s)(\varepsilon+2)^s(1+\varepsilon)}\right| \leq C|y|\frac{\varepsilon^{2-s}}{1-s},$$

thus

$$\left|\mathcal{I}_s^2 - \frac{\varepsilon^{1-s}}{(1-s)(\varepsilon+2)^s(1+\varepsilon)}\Big(u\left(x+r(1+\varepsilon)\omega\right)+u\left(x-r(1+\varepsilon)\zeta\right)\Big)\right| \leq C\frac{\varepsilon^{2-s}}{1-s}.$$

We get that

$$\lim_{s\nearrow 1} c_s\mathcal{I}_s^2 = \frac{1}{(\varepsilon+2)(\varepsilon+1)}\Big(u(x+r(1+\varepsilon)y)+u(x-r(1+\varepsilon)z)\Big) + C\varepsilon.$$

Sending $\varepsilon \to 0$ we get the conclusion. $\square$

For completeness, we show the following, already known, result.

Proposition 3.12. *Let $u \in C^2_{\text{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. For all $x \in \mathbb{R}^n$ for which $|\nabla u(x)| \neq 0$ it holds that*

$$\lim_{s\nearrow 1}(1-s)(-\Delta)^s_\infty u(x) = -\Delta_\infty u(x).$$

Proof. Since $u \in C^2_{\text{loc}}(\mathbb{R}^n)$ we have that for any $\bar{\varepsilon} > 0$ there exists $r = r(\bar{\varepsilon}) > 0$ such that (2.8) holds. We prove the result for $\nabla u(x) \neq 0$ (the other case can be proved in the same way). We have that

$$\begin{aligned}(-\Delta)^s_\infty = &\int_0^r \frac{2u(x) - u\left(x+\rho z(x)\right) - u\left(x-\rho z(x)\right)}{\rho^{1+2s}}\,d\rho \\ &+ \int_r^\infty \frac{2u(x) - u\left(x+\rho z(x)\right) - u\left(x-\rho z(x)\right)}{\rho^{1+2s}}\,d\rho = I_r + J_r.\end{aligned}$$

We have that

$$|J_r| \leq C\|u\|_{L^\infty(\mathbb{R}^n)}\frac{r^{-2s}}{2s}, \qquad \text{and} \qquad \lim_{s\nearrow 1}(1-s)J_r = 0.$$

On the other hand, using (2.11) we have that

$$I_r = -\int_0^r \frac{\langle D^2u(x)z(x), z(x)\rangle}{\rho}^{1-2s}\,d\rho + I_r^o = -\langle D^2u(x)z(x), z(x)\rangle\frac{r^{2-2s}}{2(1-s)} + I_r^o,$$

with

$$\lim_{s\nearrow 1}(1-s)I_r^o = \mathcal{O}(\bar{\varepsilon}).$$

The conclusion follows by sending $\bar{\varepsilon} \to 0$. $\square$

Appendix A. Useful asymptotics

Proposition A.1. *Let $s \in (0,1)$. There exists $\tilde{r} > 0$ such that for all $r < \tilde{r}$ the following hold*

$$\int_1^{\frac{1}{r}} t\left(\frac{1}{(t^2-1)^s} - \frac{1}{t^{2s}}\right)dt = \mathcal{O}(1),$$

$$\int_0^r t^{1-2s} - \int_1^{\frac{1}{r}} \frac{t}{(t^2-1)^s}\,dt = \mathcal{O}(r^{2s}),$$

$$\int_{\frac{1}{r}}^\infty \frac{1}{t}\left(\frac{t^{2s}}{(t^2-1)^s} - 1\right)dt = \mathcal{O}(r^2).$$

Furthermore,

$$\lim_{s\nearrow 1}(1-s)\int_{1+r}^\infty \frac{dt}{t(t^2-1)^s}\,dt = 0.$$

Proof. Integrating, we have that

$$\int_1^{\frac{1}{r}} t\left(\frac{1}{(t^2-1)^s}-\frac{1}{t^{2s}}\right)dt=\frac{1}{2(1-s)}\left(\frac{(1-r^2)^{1-s}-1}{r^{2(1-s)}}+1\right)=\frac{1}{2(1-s)}\left(\mathcal{O}(r^{2s})+1\right).$$

Since $\frac{1}{t}<r<1$, with a Taylor expansion we get that

$$\frac{1}{(1-\frac{1}{t^2})^s}-1=s\frac{1}{t^2}+o\left(\frac{1}{t^2}\right).$$

Integrating, we obtain the second result. Furthermore

$$\int_{1+r}^{\infty}\frac{dt}{t(t^2-1)^s}\,dt=\int_{1+r}^{2}\frac{dt}{t(t^2-1)^s}\,dt+\int_2^{\infty}\frac{dt}{t(t^2-1)^s}\,dt\le\frac{c(1-r^{1-s})}{1-s}+\frac{c}{s}.$$

The conclusion follows by multiplying by $(1-s)$ and taking the limit. □

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