SHARP SUMMABILITY FOR MONGE TRANSPORT DENSITY VIA INTERPOLATION

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ABSTRACT. Using some results proved in [6] (and [5]) and a suitable interpolation tecnique, we show that the transport density relative to an L^p source is also an L^p function for any $1 \leq p \leq +\infty$.

This paper is concerned with the transport problem, which consists in minimizing

$$\int_\Omega |x - t(x)|\, df^+(x) \tag{1}$$

among the *transports*, which are the measurable functions $t : \mathrm{spt}\,(f^+) \longrightarrow \mathrm{spt}\,(f^-)$ such that $t_\# f^+ = f^-$, i.e. for any Borel set B it is $f^+(t^{-1}(B)) = f^-(B)$; here $f = f^+ - f^-$ is a L^1 function on Ω with $\int f = 0$, while Ω is a convex and bounded subset of $\mathbb{R}^N$ (to find more general descriptions of the transport problem, see [1, 9]). To each optimal –i.e. minimizing (1)– transport t it is possible to associate a positive measure σ on Ω defined by

$$\langle \sigma, \varphi \rangle := \int_\Omega \left(\int_\Omega \varphi(z)\; d\mathcal{H}^1_{xt(x)}(z) \right) df^+(x) \tag{2}$$

where φ is any function in $C_0(\Omega)$ and $\mathcal{H}^1_{xy}$ is the one-dimensional Hausdorff measure on the segment xy. It has been proved (see [1, 8]) that there always exist (in this setting) optimal transports and in particular there are invertible optimal transports whose inverse is also an optimal transport for $-f$. A fundamental result, due to [1, 8], is that, even if there can be many different optimal transports, all define via (2) the same measure σ, which is then called *transport density* relative to f. This measure is very interesting for the transport problem (for example it plays an important role in [7]), and moreover it represents the connection between this problem and some shape optimization problem (see [3, 4]), which can be reduced to the research of a positive measure σ and a $1-$Lipschitz function u solving

$$\begin{cases} -\mathrm{div}\ (\sigma D u) = f & \textit{on } \Omega \\ |Du| = 1 & \sigma - \textit{a.e.} \end{cases} . \tag{3}$$

The relationship between the two problems relies on the fact that the (unique) transport density is also the unique solution of (3) (see [1, 3, 6]); the functions u solving (3) together with σ are also meaningful in the context of the transport problem, they are referred to as *Kantorovich potentials*. Equation (3) is often referred to as *Monge–Kantorovich equation.* Thus the study of the regularity of σ is useful both for the transport problem and for the shape optimization problem. It was proved (see [1, 8, 6]) that the fact that $f \in L^1$ implies also that $\sigma \in L^1$.

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In this paper we will show some sharp relationship between the summability of f and that of σ. The problem to derive regularity of σ from that of f has already been studied in [6, 5] following two different methods: in [6] we used a geometric construction starting from the definition (2), while in [5] the proofs used PDE tools starting from the equivalent definition (3). In the first work it was proved that

$$\begin{aligned} &f \in L^1 \Longrightarrow \sigma \in L^1, \qquad f \in L^\infty \Longrightarrow \sigma \in L^\infty, \\ &f \in L^p \Longrightarrow \sigma \in L^{p-\epsilon} \text{ for any } \epsilon > 0, \end{aligned} \tag{4}$$

and some examples were given in which $f \in L^p$ and $\sigma \notin L^q$ for any $q > p$. Thus it was left open the problem whether or not it is true that $f \in L^p$ implies $\sigma \in L^p$ for $p \neq 1, +\infty$. In the second work this problem was partially solved, since it was proved that

$$f \in L^p \Longrightarrow \sigma \in L^p \text{ for any } 2 \leq p < +\infty. \tag{5}$$

Since the cases $p = 1, \infty$ had already been solved in the first work, it was left open only the case with $1 < p < 2$. In this work we will show how the classic Marcinkievicz interpolation result can be used to infer from the results already mentioned the general property for any p. Note that this is not trivial since the map associating the transport density σ to any function f with $\int f = 0$ is far from being linear or sublinear, as easy examples show; however, this map is $1-$homogeneous, as one can hope in view of (6).

The result of this paper is the following

Theorem A. *For any $1 \leq p \leq +\infty$, if $f \in L^p$ is a function with $\int f = 0$, then the associated transport density σ is also in L^p. More precisely, there exist a constant C_p, depending only on Ω, such that*

$$\|\sigma\|_{L^p} \leq C_p \|f\|_{L^p}. \tag{6}$$

To prove the theorem, we first of all recall the known results we use, the regularity results proved in [6, 5] and the Marcinkievicz interpolation result, which can be found for example in [10].

Theorem 1. *For $p = 1$ and for any $p \geq 2$ there exists a constant C_p depending on Ω such that, for any $f \in L^p$ with $\int f = 0$,*

$$\|\sigma\|_{L^p} \leq C_p \|f\|_{L^p} \tag{7}$$

where σ is the transport density associated to f.

Theorem 2 (Marcinkievicz)**.** *If $T : L^1 \to L^1$ is a linear mapping such that, for two suitable constants M_p and M_q with $1 \leq p < q \leq +\infty$,*

$$\|T(g)\|_{L^p} \leq M_p \|g\|_{L^p} \qquad \text{and} \qquad \|T(g)\|_{L^q} \leq M_q \|g\|_{L^q}, \tag{8}$$

then it is also true that for any $s \in (p, q)$

$$\|T(g)\|_{L^s} \leq C\, M_p^{\frac{p(q-s)}{s(q-p)}}\, M_q^{\frac{q(s-p)}{s(q-p)}}\, \|g\|_{L^s}, \tag{9}$$

where C is a geometric constant depending only on Ω.

To prove our result, let us fix now a function $f \in L^p$ with $\int f = 0$ and $1 \leq p \leq +\infty$. We can assume $p \neq 1, +\infty$, since otherwise we already know that $\sigma \in L^p$. Fix also an invertible optimal transport t for f (as we said, this always exists when $f \in L^1$, even though it is not unique). For any function $g \in L^1$, there are of course two uniquely determined measurable functions λ and ν supported respectively on $\mathrm{spt}\,(f^+)$ and on $\Omega \setminus \mathrm{spt}\,(f^+)$ such that

$$g = \lambda f^+ + \nu. \tag{10}$$

Let us finally define the operator $T : L^1 \longrightarrow \mathcal{M}(\Omega)$ to which we will apply later the Marcinkievicz theorem: given any $g \in L^1$ and following the notations of (10), we define

$$T(g) := \int_\Omega \lambda(x) f^+(x) \mathcal{H}^1_{xt(x)}\, dx, \tag{11}$$

which can also be rewritten as

$$\langle T(g), \varphi\rangle = \int_{x\in\Omega} \left(\int_{z\in\Omega} \varphi(z)\, d\mathcal{H}^1_{xt(x)}(z) \right) \lambda(x)\, f^+(x)\, dx$$

for any $\varphi \in C_0(\Omega)$. Notice that $T(g)$ is a priori a measure, and that the definition of T depends on the function f we fixed; moreover, we point out that of course $T(f)$ is the transport density σ associated to f (just recall (2) and (11)).

We define now σ_1, σ_2 and f_1, f_2 (depending on f and g) as follows:

$$\begin{array}{ll} \sigma_1 := \int_\Omega \lambda^+(x) f^+(x) \mathcal{H}^1_{xt(x)}\, dx & f_1 := \lambda^+ f^+ - (\lambda^+ \circ t^{-1}) f^- \\ \sigma_2 := \int_\Omega \lambda^-(x) f^+(x) \mathcal{H}^1_{xt(x)}\, dx & f_2 := \lambda^- f^+ - (\lambda^- \circ t^{-1}) f^-; \end{array} \tag{12}$$

note that also these definitions depend on f and g, and that $T(g) = \sigma_1 + \sigma_2$. First we prove the

Lemma 3. *The function $t : \mathrm{spt}\,(f^+) \longrightarrow \mathrm{spt}\,(f^-)$ is defined f_i^+−a.e. and it is an optimal transport for the functions f_i, $i = 1, 2$ defined in (12); moreover, each σ_i is the transport density associated to f_i.*

Proof. The optimality of a transport is equivalent to the cyclical monotonicity of its graph (see [9, 2] to find the definition of the cyclical monotonicity and the proof of this assert). Then the fact that t is optimal for f assures that its graph is monotonically cyclic; thus, given any function h on Ω with 0 mean and such that $h^+ \ll f^+$, t is defined h^+−a.e. and it is an optimal transport for h if and only if it is a transport. Then to prove the first part of the assert it is enough to check that $t_\# f_i^+ = f_i^-$ for $i = 1, 2$, which is a straightforward consequence of the fact that $t_\# f^+ = f^-$ and of the properties of the push-forward. Finally, the fact that each σ_i is the transport density associated to f_i follows comparing (12) with the definition (2) of the transport density (replace f and σ in (2) by f_i and σ_i). □

We can then prove the following

Lemma 4. *$T : L^1 \longrightarrow L^1$ is a linear operator.*

Proof. The fact that T is linear follows immediately from the definition (11); moreover $T(g)$ is a L^1 function (recall that a priori we knew it only to be a measure) since it is the sum of the two transport densities σ_i thanks to the preceding Lemma, and thanks to Theorem 1 each of these densities is in L^1 since so is each f_i –recall (12) and that $g \in L^1$. □

We prove now the validity of (8) with $p = 1$ and $q = +\infty$ in order to apply the Marcinkievicz Theorem.

Lemma 5. *The inequalities (8) hold for T with $p = 1$ and $q = +\infty$; in particular, $M_1 = 2\,C_1$ and $M_\infty = 2\,C_\infty$, where the C_i's are the constants of (7).*

Proof. In view of Lemma 3 and Theorem 1, σ_1 is the transport density relative to f_1 and then $\|\sigma_1\|_{L^1} \leq C_1 \|f_1\|_{L^1}$; but since $t_\# f_1^+ = f_1^-$, then $\|f_1^+\|_{L^1} = \|f_1^-\|_{L^1}$ and we infer

$$\|\sigma_1\|_{L^1} \leq 2\,C_1 \|f_1^+\|_{L^1}.$$

In the same way we deduce also $\|\sigma_2\|_{L^1} \leq 2\,C_1 \|f_2^-\|_{L^1}$. Using now the fact that the supports of the f_i's are essentially disjoint –that is clear from (12)–, we have

$$\begin{aligned}\|T(g)\|_{L^1} &= \|\sigma_1+\sigma_2\|_{L^1} \leq \|\sigma_1\|_{L^1} + \|\sigma_2\|_{L^1} \leq 2\,C_1 \left(\|f_1^+\|_{L^1} + \|f_2^+\|_{L^1}\right) \\ &= 2\,C_1 \|f_1^+ + f_2^+\|_{L^1} = 2\,C_1 \|\lambda^+ f^+ + \lambda^- f^+\|_{L^1} = 2\,C_1 \|\lambda f^+\|_{L^1} \\ &\leq 2\,C_1 \|g\|_{L^1},\end{aligned}$$

which gives the first estimate.

On the other hand, to show the L^∞ inequality we note that, thanks to (10) and (12), it is $\|f_i\|_{L^\infty} \leq \|g\|_{L^\infty}$ for each i. Since $T(g) = \sigma_1 + \sigma_2$, from Lemma 3 and Theorem 1 we infer

$$\|T(g)\|_{L^\infty} \leq 2\,C_\infty \|g\|_{L^\infty},$$

and then also the L^∞ inequality follows. □

Thanks to Lemmas 4 and 5, we can apply Theorem 2 to prove (6), recalling that $T(f)$ is the transport density σ associated to f. Recall now that the function $f \in L^p$ was fixed at the beginning, but the constants C_p we obtained do not depend on f, but only on p and Ω. Then the estimate (6) is true, with the same constants, for any function $f \in L^p$.

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