Non-uniqueness of signed measure-valued solutions to the continuity equation in presence of a unique flow

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Abstract

We consider the continuity equation $\partial_t \mu_t + \operatorname{div}(\boldsymbol{b}\mu_t) = 0$, where $\{\mu_t\}_{t\in\mathbb{R}}$ is a measurable family of (possibily signed) Borel measures on $\mathbb{R}^d$ and $\boldsymbol{b}\colon \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^d$ is a bounded Borel vector field (and the equation is understood in the sense of distributions). If the measure-valued solution μ_t is non-negative, then the following *superposition principle* holds: μ_t can be decomposed into a superposition of measures concentrated along the integral curves of $\boldsymbol{b}$. For smooth $\boldsymbol{b}$ this result follows from the method of characteristics, and in the general case it was established by L. Ambrosio. A partial extension of this result for signed measure-valued solutions μ_t was obtained in [AB08], where the following problem was proposed: does the superposition principle hold for signed measure-valued solutions in presence of unique flow of homeomorphisms solving the associated ordinary differential equation? We answer to this question in the negative, presenting two counterexamples in which uniqueness of the flow of the vector field holds but one can construct non-trivial signed measure-valued solutions to the continuity equation with zero initial data.

KEYWORDS: *continuity equation, measure-valued solutions, uniqueness, Superposition Principle.*

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1 Introduction

In this paper we consider the initial value problem for the continuity equation

$$\begin{cases} \partial_t \mu_t + \operatorname{div}(\boldsymbol{b}\mu_t) = 0, \\ \mu_0 = \overline{\mu} \end{cases} \tag{PDE}$$

for finite Borel measures $\{\mu_t\}_{t\in[0,T]}$ on $\mathbb{R}^d$, where $\boldsymbol{b}\colon [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ is a given bounded Borel vector field, $T>0$ and $d\in\mathbb{N}$ and $\overline{\mu}\in\mathscr{M}(\mathbb{R}^d)$ is a given measure on $\mathbb{R}^d$. This class of measure-valued solutions arises naturally in the limit for weakly* converging subsequences of smooth solutions, and it appears in various applications including hyperbolic conservation laws, optimal transport and other areas, see e.g. [BJ98, AGS08, BPRS15].

We want to study the relationship between uniqueness of solutions to (PDE) and uniqueness to the ordinary differential equation drifted by $\boldsymbol{b}$, i.e.

$$\frac{d}{dt}\gamma(t) = \boldsymbol{b}(t,\gamma(t)), \quad t\in(0,T), \tag{ODE}$$

where $\gamma\in C([0,T];\mathbb{R}^d)$. As usual, a solution to (PDE) is intended in the sense of distributions, while a solution to (ODE) is defined to be a continuous curve $\gamma\in C([0,T];\mathbb{R}^d)$ such that

$$\gamma(\tau) = \gamma(s) + \int_s^\tau \boldsymbol{b}(r,\gamma(r))\,dr \qquad \text{for every } (s,\tau)\subset(0,T).$$

Note that this definition is sensitive to modifications of $\boldsymbol{b}$ in a Lebesgue-negligible set, therefore we underline that $\boldsymbol{b}$ is a function defined *everywhere* and not an equivalence class.

Given a solution $\gamma\in C([0,T];\mathbb{R}^d)$ of (ODE) one readily checks that $\mu_t := \delta_{\gamma(t)}$ solves (PDE), where δ_p denotes the Dirac measure concentrated at p. Therefore uniqueness for (PDE) implies uniqueness for (ODE). Hence it is natural to ask whether the converse implication holds.

In the class of non-negative measure-valued solutions the answer to this question is positive, and it was obtained in [AGS08] as a consequence of the so-called *superposition principle*. In order to formulate this principle, we will say that a family of Borel measures $\{\mu_t\}_{t\in[0,T]}$ is *represented by* a finite (possibly signed) Borel measure η on $C([0,T];\mathbb{R}^d)$ if

1. η is concentrated on $\Gamma_{\boldsymbol{b}}$;
2. $(e_t)_\sharp\eta = \mu_t$ for a.e. t,

where $e_t\colon C([0,T];\mathbb{R}^d)\to\mathbb{R}^d$ is the so-called *evaluation map* defined by $e_t(\gamma) := \gamma(t)$, $(e_t)_\sharp\eta$ denotes the image of η under e_t, and $\Gamma_{\boldsymbol{b}}$ denotes the set of solutions of (ODE) (note the $\Gamma_{\boldsymbol{b}}$ is a Borel subset of $C([0,T];\mathbb{R}^d)$ by [Ber08, Proposition 2]). For example, if $\gamma\in C([0,T];\mathbb{R}^d)$ solves (ODE) then $\eta := \delta_\gamma$ (as a measure on $C([0,T];\mathbb{R}^d)$) represents the solution $\mu_t := \delta_{\gamma(t)}$ of (PDE).

A straightforward computation shows that if $\{\mu_t\}_{t\in[0,T]}$ is represented by some (possibly signed) measure η then μ_t solves (PDE). In this case we will say that μ_t is a *superposition solution* of (PDE). Clearly uniqueness for (ODE) implies uniqueness for (PDE) in the class of superposition solutions. Indeed, by uniqueness for (ODE) the continuous mapping $e_0 \colon \Gamma_{\boldsymbol{b}} \to \mathbb{R}^d$ is injective, hence e_0^{-1} is Borel and thus $(e_0)_\sharp \eta = \mu_0$ is equivalent to $\eta = (e_0^{-1})_\sharp \mu_0$.

Therefore, when uniqueness holds for the Cauchy problem for (ODE), uniqueness for the Cauchy problem for (PDE) holds in the class of measure-valued solutions if and only if *any* measure-valued solution of such Cauchy problem is a superposition solution.

The superposition principle established in [AGS08] states that any *non-negative* solution μ_t of (PDE) can be represented by some non-negative measure η on $C([0,T];\mathbb{R}^d)$. However, without extra assumptions this result cannot be extended to *signed* solutions, because (PDE) can have a nontrivial signed solution even when $\Gamma_{\boldsymbol{b}} = \emptyset$ (see e.g. [Gus18] for the details).

Under Lipschitz bounds on the vector field $\boldsymbol{b}$ uniqueness for (PDE) within the class of signed measures is well known, see e.g. [AGS08, Prop. 8.1.7]. Out of the classical setting, the first (positive) result is contained in [BC94], where the authors considered log-Lipschitz vector fields. Later on, in the paper [AB08], the authors proved that the signed superposition principle holds provided that the vector field satisfies a quantitative two-sided diagonal Osgood condition. More precisely, in [AB08] the authors considered vector fields enjoying

(O) it holds

$$|\langle \boldsymbol{b}(t,x) - \boldsymbol{b}(t,y), x-y\rangle| \le C(t)\|x-y\|\rho(\|x-y\|) \qquad \forall x,y \in \mathbb{R}^d,\ \forall t \in (0,T),$$

where $C \in L^1(0,T)$ and $\rho \colon [0,1) \to [0,+\infty)$ is an Osgood modulus of continuity, i.e. a continuous, non-decreasing function with $\rho(0)=0$ and

$$\int_0^1 \frac{1}{\rho(s)}\, ds = +\infty.$$

(B) $|\boldsymbol{b}(t,x)| \le D(t)$ for some $D \in L^1(0,T)$ for every $t,x \in (0,T)\times\mathbb{R}^d$).

Their results is the following:

Theorem 1.1 (Thm. 1 in [AB08])**.** *If the vector field $\boldsymbol{b}$ satisfies (O) and (B), then there is uniqueness for* (PDE) *in the class of bounded signed measures, i.e. if μ_t is a solution of* (PDE) *such that $|\mu_t|(\mathbb{R}^d) \in L^\infty(0,T)$ then*

$$\mu_t = \boldsymbol{X}(t,\cdot)_\# \mu_0, \qquad \forall t \in (0,T),$$

where $\boldsymbol{X}(t,\cdot)$ is the flow of $\boldsymbol{b}$, i.e. the unique map solving

$$\begin{cases} \partial_t \boldsymbol{X}(t,x) = \boldsymbol{b}(t,\boldsymbol{X}(t,x)) & t \in [0,T], x \in \mathbb{R}^d \\ \boldsymbol{X}(0,x) = x & x \in \mathbb{R}^d \end{cases}$$

Notice that the Osgood assumption (O) is an assumption on $\boldsymbol{b}$, and it is much stronger than an implicit assumption of uniqueness for (ODE). For a simple example one can consider e.g. (for $d = 1$) $\boldsymbol{b}(t,x) = \mathbb{1}_{(-\infty,0]}(x) + 2 \cdot \mathbb{1}_{(0,+\infty)}(x)$. Moreover, according to a theorem of Orlicz [Orl32] (see also [Ber08, Thm. 1]), in the space of all continuous vector fields $\boldsymbol{b}$ (equiped with the topology of the uniform convergence on compact sets) those fields for which the differential equation (ODE) has at least one non-uniqueness point is of first category: this shows that in the generic situation Lipschitz/Osgood conditions are *not* necessary for uniqueness.

Let us mention some other generic uniqueness results for (PDE). The one-dimensional case was studied in [BJ98], where uniqueness of signed measure-valued solutions was obtained under the assumption that $\boldsymbol{b}$ satisfies a *one-sided Lipschitz condition*, i.e. there exists $\alpha \in L^1(0,T)$ such that $\partial_x \boldsymbol{b}(t,x) \leq \alpha(t)$ (in the sense of distributions). Still in $d = 1$, uniqueness in the class of absolutely continuous (with respect to Lebesgue measure) solutions was obtained in [Gus19] for nearly incompressible vector fields. In the multi-dimensional case uniqueness of absolutely continuous solutions was obtained in [BB17] for nearly incompressible vector fields with bounded variation. For generic solutions, besides [AB08], one can refer to [CJMO17], where uniqueness within the signed framework is shown for vector fields having an Osgood modulus of continuity.

The generic uniqueness results mentioned above require some *regularity* of $\boldsymbol{b}$ (e.g. some form of weak differentiability), but as discussed above one can ask if uniqueness for (ODE) is sufficient for uniqueness for (PDE). In particular, a natural question (raised in [AB08]) is whether uniqueness for (PDE) (in the class of signed measures) holds in the presence of a (unique) flow of homeomorphisms solving (ODE), without an explicit bound like (O) on the vector field. We show that the answer to this question in general is negative by constructing two counterexamples of bounded vector fields $\boldsymbol{b}\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ (for $d = 1$ and $d = 2$) such that for *any* $x \in \mathbb{R}^d$ only $\gamma(t) \equiv x$ ($\forall t \in [0,T]$) solves (ODE) but (PDE) with zero initial condition has a non-trivial measure-valued solution $\{\mu_t\}_{t\in[0,T]}$. More precisely, this is the main result of the present paper:

Main Theorem. *The following claims hold true.*

(i) *Let $d = 1$. Then there exist a vector field $\boldsymbol{b}\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ and a measurable measure-valued map $[0,T] \ni t \mapsto \mu_t \in \mathscr{M}(\mathbb{R})$ such that*

- *$\boldsymbol{b}$ is bounded and Borel (in particular it is defined everywhere);*
- *for* any *$x \in \mathbb{R}$ only $\gamma(t) \equiv x$ $\forall t \in [0,T]$ solves* (ODE)*, hence there exists a unique flow of homeomorphisms of $\boldsymbol{b}$;*
- *$[t \mapsto \mu_t] \in L^1([0,T];\mathscr{M}(\mathbb{R})) \setminus L^\infty([0,T];\mathscr{M}(\mathbb{R}))$ is a non-trivial solution of* (PDE) *with zero initial condition.*

(ii) *Let $d = 2$. Then there exist a vector field $\boldsymbol{b}\colon [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$ and a measurable measure-valued map $[0,T] \ni t \mapsto \mu_t \in \mathscr{M}(\mathbb{R}^2)$ such that*

- *$\boldsymbol{b}$ is bounded and Borel (in particular it is defined everywhere);*

- *for* any $x \in \mathbb{R}^2$ *only* $\gamma(t) \equiv x\ \forall t \in [0,T]$ *solves* (ODE)*, hence there exists a unique flow of homeomorphisms of* $\boldsymbol{b}$*;*
- $[t \mapsto \mu_t] \in L^\infty([0,T]; \mathscr{M}(\mathbb{R}^2))$ *is a non-trivial solution of* (PDE) *with zero initial condition.*

Remark 1.2. We stress the fact that in example related to Point (i) of the Main Theorem the map $[t \mapsto \mu_t] \notin L^\infty([0,T]; \mathscr{M}(\mathbb{R}))$, i.e. the measure μ_t is *not* bounded in time on every subinterval $I \subset [0,T]$. See also Lemma 3.7 below for a rigorous proof of this fact.

In the examples (i) and (ii) of the present paper the vector field $\boldsymbol{b}$ is only bounded, but not continuous. However all vector fields that satisfy (O) and (B) are continuous (see Proposition 5.1). It would therefore be interesting to understand whether for continuous vector fields uniqueness for (ODE) implies uniqueness for (PDE).

Note that our examples (i) and (ii) are based on a one-dimensional vector field that does not have integral curves and hence cannot be continuous (in view of Peano's theorem). And in fact for $d = 1$ it is possible to prove that if $\boldsymbol{b}$ is stationary and continuous then uniqueness for (ODE) implies uniqueness for (PDE) (see Proposition 5.2). It is interesting to note that such $\boldsymbol{b}$ can be very irregular and hence one cannot apply to it any of the generic uniqueness results discussed earlier.

Let us also mention that (still for $d = 1$) if $\boldsymbol{b}$ is continuous and for any t the function $x \mapsto \boldsymbol{b}(t,x)$ is non-strictly decreasing then uniqueness holds both for (PDE) (this follows from [BJ98]) and for (ODE) (this can be shown directly: if γ_1 and γ_2 are integral curves of $\boldsymbol{b}$ such that $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(t) < \gamma_2(t)$ for all sufficiently small $t > 0$ then $\partial_t(\gamma_1(t) - \gamma_2(t)) = \boldsymbol{b}(t, \gamma_1(t)) - \boldsymbol{b}(t, \gamma_2(t)) \geq 0$).

2 Preliminaries

In the following, we will denote by $\mathscr{B}(\mathbb{R}^d)$ the Borel σ-algebra on $\mathbb{R}^d$. We recall some basic definitions.

Definition 2.1. A family $\{\mu_t\}_{t\in[0,T]}$ of Borel measures on $\mathbb{R}^d$ is called *a Borel family* if for any $A \in \mathscr{B}(\mathbb{R}^d)$ the map $t \mapsto \mu_t(A)$ is Borel-measurable.

The following propositions are well-known (see, e.g. [AFP00, Prop. 2.26 and (2.16)]):

Proposition 2.2. *If* $\{\mu_t\}_{t\in[0,T]}$ *is a family of Borel measures on* $\mathbb{R}^d$ *such that* $t \mapsto \mu_t(A)$ *is Borel for any open set* $A \subset \mathbb{R}^d$ *then* $\{\mu_t\}_{t\in[0,T]}$ *is a Borel family.*

Proposition 2.3. *If* $\{\mu_t\}_{t\in[0,T]}$ *is a Borel family then* $\{|\mu_t|\}_{t\in[0,T]}$ *also is a Borel family.*

Proposition 2.4. *If* $\{\mu_t\}_{t\in[0,T]}$ *is a Borel family then for any bounded Borel function* $g\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}$ *the map* $t \mapsto \int_{\mathbb{R}^d} g(t,x)\, d\mu_t(x)$ *is Borel.*

In what follows we will write that $[t \mapsto \mu_t] \in L^1((0,T);\mathscr{M}(\mathbb{R}^d))$ if $\{\mu_t\}_{t\in[0,T]}$ is a a Borel family and

$$\int_0^T |\mu_t|(\mathbb{R}^d)\,dt < +\infty.$$

If, in addition,

$$\operatorname{ess\,sup}_{t\in[0,T]} |\mu_t|(\mathbb{R}^d) < +\infty,$$

then we will write $[t \mapsto \mu_t] \in L^\infty((0,T);\mathscr{M}(\mathbb{R}^d))$.

In view of Proposition 2.4 the distributional formulation of the continuity equation is well-defined:

Definition 2.5. A family $[t \mapsto \mu_t] \in L^1((0,T);\mathscr{M}(\mathbb{R}^d))$ is called a measure-valued solution of (PDE) if for any $\varphi \in C^1_c([0,T)\times\mathbb{R}^d)$

$$\int_0^T \int_{\mathbb{R}^d} (\partial_t\varphi + \boldsymbol{b}(t,x)\cdot\nabla_x\varphi(t,x))\,d\mu_t(x)\,dt + \int_{\mathbb{R}^d} \varphi(0,x)\,d\bar{\mu}(x) = 0. \tag{2.1}$$

Even though the distributional formulation of the Cauchy problem for (PDE) is well-defined for $[t \mapsto \mu_t] \in L^1((0,T);\mathscr{M}(\mathbb{R}^d))$, it is much more natural in the class $[t \mapsto \mu_t] \in L^\infty((0,T);\mathscr{M}(\mathbb{R}))$, because in this class the initial condition can be understood in the sense of traces, considering a weak* continuous representative of $[t \mapsto \mu_t]$. More precisely, we have the following Proposition (for a proof see e.g. [Bon17, Chapter 1, Prop. 1.6]).

Proposition 2.6 (Continuous representative)**.** *Let $\{\mu_t\}_{t\in[0,T]}$ be a Borel family of measures and assume $[t \mapsto \mu_t] \in L^\infty((0,T);\mathscr{M}(\mathbb{R}^d))$. Then there exists a* narrowly continuous *curve $[0,T] \ni t \mapsto \tilde{\mu}_t \in \mathscr{M}(\mathbb{R}^d)$ such that $\mu_t = \tilde{\mu}_t$ for a.e. $t \in [0,T]$.*

3 Non-uniqueness in the class $L^1((0,T);\mathscr{M}(\mathbb{R}))$

In this section we prove the following result:

Theorem 3.1. *There exist $T > 0$, a bounded Borel $\boldsymbol{b}\colon [0,T]\times\mathbb{R} \to \mathbb{R}$ and $[t \mapsto \mu_t] \in L^1((0,T);\mathscr{M}(\mathbb{R}))$ satisfying the following conditions:*

(i) $\boldsymbol{b}$ has only constant characteristics, i.e. $\gamma \in \Gamma_{\boldsymbol{b}}$ if and only if there exists $x \in \mathbb{R}$ such that $\gamma(t) = x$ for all $t \in [0,T]$;

(ii) $\{\mu_t\}_{t\in[0,T]}$ is not identically zero and solves (PDE) with zero initial condition.

3.1 Auxiliary result

We begin by the following auxiliary result: although it is well-known, we give a proof because some details will be used later.

Lemma 3.2. *There exist a Borel sets $P, N \subset \mathbb{R}$ such that*

1. $P \cap N = \emptyset$;

2. $P \cup N = \mathbb{R}$;

3. for any nonempty bounded open interval $I \subset \mathbb{R}$ it holds that $|I \cap P| > 0$ and $|I \cap N| > 0$,

where $|A|$ denotes the Lebesgue measure of $A \subset \mathbb{R}$.

Proof. Let $\{q_k\}_{k\in\mathbb{N}}$ be the set of all rational numbers. Let $f_0(x) := 1$ ($x \in \mathbb{R}$), $E_0 := \emptyset$ and $\varepsilon_0 := 1$.

Consider $k \in \mathbb{N}$, $k \geq 1$ and suppose that the set E_{k-1}, the number $\varepsilon_{k-1} > 0$ and the function f_{k-1} are already constructed. We assume that E_{k-1} is finite, $E_{k-1} \cap \mathbb{Q} = \emptyset$, hence $\mathbb{R} \setminus E_{k-1}$ is a union of finitely many open intervals. We also assume that f_{k-1} is either $+1$ or -1 on each of these intervals.

Since $\operatorname{dist}(q_k, E_{k-1}) > 0$ there exists $\varepsilon_k > 0$ such that

$$\varepsilon_k < 2^{-k}\varepsilon_{k-1}, \tag{3.1}$$

$$(q_k - \varepsilon_k, q_k + \varepsilon_k) \subset \mathbb{R} \setminus E_{k-1}, \tag{3.2}$$

and moreover

$$q_k \pm \frac{1}{2}\varepsilon_k \notin \mathbb{Q}. \tag{3.3}$$

We then define

$$I_k := \left(q_k - \frac{1}{2}\varepsilon_k, q_k + \frac{1}{2}\varepsilon_k\right) \subset \mathbb{R} \setminus E_{k-1} \tag{3.4}$$

and

$$f_k(x) := \begin{cases} f_{k-1}(x), & x \notin \overline{I_k}, \\ -f_{k-1}(x), & x \in I_k, \\ 0, & x \in \partial I_k \end{cases} \tag{3.5}$$

and $E_k := E_{k-1} \cup \partial I_k$. It is easy to see that E_k, ε_k and f_k satisfy the same assumptions as $E_{k-1}, \varepsilon_{k-1}$ and f_{k-1}. Therefore we can construct inductively the sequence $\{E_k, \varepsilon_k, f_k\}_{k\in\mathbb{N}}$.

Consider the set $R_k := \cup_{n=k+1}^{\infty} \overline{I_n}$ on which the function f_n ($n > k$) may differ from f_k. By (3.1)

$$|R_k| \leq \sum_{n=k+1}^{\infty} \varepsilon_n = \sum_{n=k}^{\infty} \varepsilon_{n+1} < \sum_{n=k}^{\infty} 2^{-(n+1)}\varepsilon_n < \varepsilon_k \sum_{n=k}^{\infty} 2^{-(n+1)} \leq \frac{1}{2}\varepsilon_k. \tag{3.6}$$

For any $x \in \mathbb{R} \setminus R_k$ it holds that $f_n(x) = f_k(x)$ for all $n > k$. Since $\varepsilon_k \to 0$ as $k \to \infty$, we conclude that f_k converges a.e. to some function $f\colon \mathbb{R} \to \mathbb{R}$ as $k \to \infty$. Moreover, on the complement of Lebesgue negligible set $\bigcap_{k\in\mathbb{N}} R_k \cup \bigcup_k \partial I_k$ the function f by construction takes only the values ± 1. We therefore set

$$P := f^{-1}(\{+1\}), \quad N := \mathbb{R} \setminus P. \tag{3.7}$$

Consider an arbitrary nonempty bounded open $I \subset \mathbb{R}$. There always exists a nonempty open interval J such that $\overline{J} \subset I$. Since J contains infinitely many rationals and $\varepsilon_k \to 0$ as $k \to \infty$, there exists $k_0 \in \mathbb{N}$ such that $(q_k - \varepsilon_k, q_k + \varepsilon_k) \subset I$ for some $k > k_0$.

Without loss of generality let us assume that $f_{k-1} = +1$ on $(q_k - \varepsilon_k, q_k + \varepsilon_k)$ (the argument is the same when this value is -1). Hence by construction

$$f_k(x) = \begin{cases} f_{k-1}(x) = +1, & x \in (q_k - \varepsilon_k, q_k + \varepsilon_k) \setminus \overline{I_k}, \\ -f_{k-1}(x) = -1, & x \in I_k. \end{cases}$$

Ultimately, by (3.6) the function f may differ from f_k only on the set R_k and $|R_k| < \frac{1}{2}\varepsilon_k$. Therefore

$$|I \cap P| \geq |(q_k - \varepsilon_k, q_k + \varepsilon_k) \setminus \overline{I_k}| - |R_k| \geq \varepsilon_k - \frac{\varepsilon_k}{2} = \frac{\varepsilon_k}{2}$$

and

$$|I \cap N| \geq |I_k| - |R_k| \geq \varepsilon_k - \frac{\varepsilon_k}{2} = \frac{\varepsilon_k}{2}.$$

□

3.2 The construction of the counterexample

Given the sets $P, N \subset \mathbb{R}$ constructed in Lemma 3.2 we now set

$$f(\tau) := 2 + \int_0^\tau (\mathbb{1}_P(r) - \mathbb{1}_N(r))\, dr \quad \text{and} \quad F(\tau) := (f(\tau), \tau) \tag{3.8}$$

where $\tau \in [0,1]$. Since the derivative of f is equal to $\mathbb{1}_P - \mathbb{1}_N$ a.e., for convenience we denote $f' := \mathbb{1}_P - \mathbb{1}_N$. Notice that since $N = \mathbb{R} \setminus P$ the function f' is defined everywhere and takes values in $\{\pm 1\}$ and it is a Borel representative of the derivative of the function f defined in (3.8).

We now set $T := 4$ and define

$$\boldsymbol{b}(t,x) := \mathbb{1}_{F[0,1]}(t,x) \cdot \frac{1}{f'(x)} \quad \text{and} \quad \tilde{\mu}_t := \sum_{x \in f^{-1}(t)} \operatorname{sign}(f'(x))\delta_x. \tag{3.9}$$

By definition $\boldsymbol{b}$ is Borel and bounded. Moreover by the area formula $\{\tilde{\mu}_t\}_{t \in [0,T]}$ is a measurable family of Borel measures (see also Figure 1).

Lemma 3.3. *For $\boldsymbol{b}$ and $\tilde{\mu}_t$ defined above*

$$\partial_t \tilde{\mu}_t + \operatorname{div}(\boldsymbol{b}\tilde{\mu}_t) = -\delta_{F(1)} + \delta_{F(0)} \qquad \text{in} \quad \mathscr{D}'((0,T) \times \mathbb{R}).$$

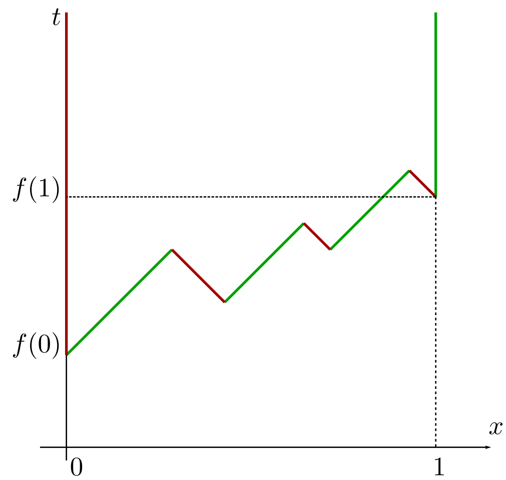


Figure 1: Graph of the function $t = f(x)$ (approximation step). At each $t \in [0,T]$ the measure $\tilde{\mu}_t$ is a superposition of Dirac masses with weight given by sign $f'(x)$, where $x \in f^{-1}(t)$ (notice the red/green parts).

Proof. Using the area formula (since $f([0,1]) \subset (0,T)$) we get

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} (\partial_t \varphi + \boldsymbol{b}\, \partial_x \varphi)\; d\tilde{\mu}_t(x)\, dt \\
&= \int_0^T \left(\sum_{x \in f^{-1}(t)} \left((\partial_t \varphi)(t,x) + \frac{1}{f'(x)}\, (\partial_x \varphi)(t,x) \right) \frac{f'(x)}{|f'(x)|} \right) dt \\
&= \int_0^1 \left((\partial_t \varphi)(f(x),x) + \frac{1}{f'(x)}\, (\partial_x \varphi)(f(x),x) \right) f'(x)\, dx \\
&= \int_0^1 \left(f'(x)(\partial_t \varphi)(f(x),x) + (\partial_x \varphi)(f(x),x) \right) dx \\
&= \int_0^1 \partial_x \left(\varphi(f(x),x) \right) dx = \varphi(f(1),1) - \varphi(f(0),0).
\end{aligned}$$

□

To get rid of the defect $-\delta_{F(1)} + \delta_{F(0)}$ we simply add to $\tilde{\mu}_t$ solutions concentrated on constant in time trajectories (since $\boldsymbol{b}$ is 0 outside $F([0,1])$). More precisely, one readily checks that

$$\mu_t := \tilde{\mu}_t + \mathbb{1}_{[f(1),+\infty)}(t)\, \delta_1 - \mathbb{1}_{[f(0),+\infty)}(t)\, \delta_0$$

solves (PDE).

To conclude the proof of Theorem 3.1, it remains to study the integral curves of $\boldsymbol{b}$. This issue is addressed in the following Lemma:

Lemma 3.4. *For any (t, x) there exists a unique characteristic of $\boldsymbol{b}$ passing through x.*

Proof. Clearly points $\gamma(t) = x$, $t > 0$, are characteristics of $\boldsymbol{b}$. Since the image of $[0,1]$ under F is closed, b vanishes identically in a neighbourhood of any $(t,x) \notin F([0,1])$. Therefore for $(t,x) \notin F([0,1])$ the claim is trivial.

Hence it is sufficient to prove that any characteristic $\gamma = \gamma(t)$ of $\boldsymbol{b}$ intersects $F([0,1])$ at most in one point. We argue by contradiction: suppose there exist $x < y$ such that

$$\gamma(f(x)) = x \quad \text{and} \quad \gamma(f(y)) = y. \tag{3.10}$$

Since $\gamma' = \boldsymbol{b}(t,\gamma)$ and $\|\boldsymbol{b}\|_\infty \leq 1$ it holds that

$$|x - y| = |\gamma(f(x)) - \gamma(f(y))| \leq |f(x) - f(y)| \tag{3.11}$$

On the other hand, by properties of the sets P and N

$$\begin{aligned} |f(y) - f(x)| &= \left| \int_x^y (\mathbb{1}_P(z) - \mathbb{1}_N(z))\, dz \right| \\ &= \Big| \big|[x,y] \cap P\big| - \big|[x,y] \cap N\big| \Big| < |x - y|. \end{aligned} \tag{3.12}$$

The inequalities (3.11) and (3.12) are not compatible, hence the proof is complete. □

Therefore we have constructed a vector field $\boldsymbol{b}$ for which the characteristics are unique, but there exists a nontrivial signed solution of the CE. Using a minor modification of the present construction one can construct a similar example of (μ_t, b) having compact support in spacetime.

Remark 3.5. The constructed solution $\{\mu_t\}$ is not a superposition solution (see Introduction).

As we in Section 2, the distributional formulation of the Cauchy problem for (PDE) is well-defined for $[t \mapsto \mu_t] \in L^1((0,T); \mathscr{M}(\mathbb{R}))$ but it is best suited in the class $[t \mapsto \mu_t] \in L^\infty((0,T); \mathscr{M}(\mathbb{R}))$, because of Proposition 2.6. Unfortunately for the present construction this bound on the solution $[t \mapsto \mu_t]$ does not hold, as the following Proposition shows.

Proposition 3.6. *The function $[t \mapsto \mu_t]$ is not bounded on any open subinterval $U \subset (0,T)$, i.e. $[t \mapsto \mu_t] \notin L^\infty(U; \mathscr{M}(\mathbb{R}))$.*

Before presenting the proof of Proposition 3.6 we need the following auxiliary

Lemma 3.7. *Let $g \in \mathrm{Lip}((0,1))$ be such that $g' \neq 0$ a.e. and let $O \subset g((0,1))$ be a non-empty open interval such that*

$$\operatorname{ess\,sup}_{t \in O} \#(g^{-1}(t)) < \infty. \tag{3.13}$$

Then there exists a nonempty open interval $I \subset (0,1)$ such that g is strictly monotone on I.

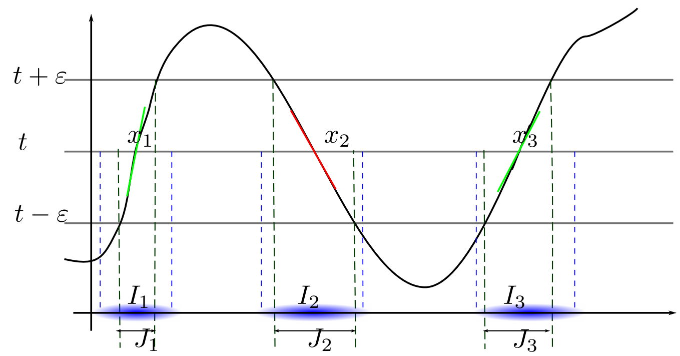


Figure 2: Situation described in the proof of Lemma 3.7. The intervals I_i are depicted in blue.

Proof (of Lemma 3.7). It is sufficient to prove the Lemma under the assumption

$$\operatorname{ess\,sup}_{t\in\mathbb{R}} \#(g^{-1}(t)) < \infty.$$

Indeed, being g Lipschitz continuous, the preimage $g^{-1}(O) \subset (0,1)$ is an open set and it can be written as countable union of disjoint, open intervals. Let (α,β) be one connected component of $g^{-1}(O)$ and consider the restriction $\tilde{g}$ of g to (α,β). Then it holds

$$\operatorname{ess\,sup}_{t\in\mathbb{R}} \#(\tilde{g}^{-1}(t)) < \infty,$$

because if $t \in O$ this is (3.13), while if $t \in \mathbb{R}\setminus O$ we have $\tilde{g}^{-1}(t) = \emptyset$. It is now clear that it is enough to prove the Lemma for $\tilde{g}$, because if we prove that $\tilde{g}$ is strictly monotone (on a subinterval of (α,β)) so is the function g. Let C denote the set of points $x \in (0,1)$ where g is not differentiable or $g'(x) = 0$. By the assumptions (and Rademacher's theorem) C has measure zero. Then by the area formula

$$0 = \int_C |g'(x)|\,dx = \int_{g(C)} \#(g^{-1}(t))\,dt,$$

hence $g(C)$ has zero Lebesgue measure (since $\#(g^{-1}(t)) \geq 1$ for all $t \in g(C)$).

Let

$$M := \operatorname{ess\,sup}_{t\in\mathbb{R}} \#(g^{-1}(t)).$$

Since for any $t \in \mathbb{R}$ we have $\#(g^{-1}(t)) \in \mathbb{N} \cup \{0\}$, there exists a set $R \subset \mathbb{R}$ with strictly positive measure such that $\#(g^{-1}(t)) = M$ for all $t \in R$. In particular, we can take $t \in R \setminus g(C)$. Then $g^{-1}(t) = \{x_1, x_2, \ldots, x_M\}$ and $g'(x_i) \neq 0$. Hence there exist disjoint open intervals I_i containing x_i such that $g(\cdot) - t$ has different signs on ∂I_i, where $i = 1, 2, \ldots, M$.

Using continuity of g we can always find an $\varepsilon > 0$ such that $[t-\varepsilon, t+\varepsilon] \subset \bigcap_{i=1}^M g(I_i)$. Hence, by the intermediate value property we can find nonempty open intervals $J_i \subset I_i$ (with $x_i \in J_i$) such that $g(\partial J_i) = \{t-\varepsilon, t+\varepsilon\}$ for each $i \in 1, 2 \ldots, M$.

By the intermediate value property for each $\tau \in [t-\varepsilon, t+\varepsilon]$ we have

$$\#(g^{-1}(\tau) \cap J_i) \geq 1, \qquad i \in 1, \dots, M. \tag{3.14}$$

On the other hand for *all* $\tau \in [t-\varepsilon, t+\varepsilon]$ we have

$$\sum_{i=1}^{M} \#(g^{-1}(\tau) \cap J_i) \leq M. \tag{3.15}$$

Indeed, by the definition of M the estimate (3.15) holds for a.e. τ, and if it fails for some τ, then at least for some i it holds that $\#(g^{-1}(\tau) \cap J_i) \geq 2$. Since $g' \neq 0$ a.e., by the intermediate value property this implies existence of $\xi > 0$ such that $\#(g^{-1}(s) \cap J_i) \geq 2$ for all $s \in [\tau, \tau+\xi)$ (or all $s \in (\tau-\xi, \tau]$), and in view of (3.14) this clearly contradicts the definition of M.

From the estimates (3.14) and (3.15) we conclude that for all $\tau \in [t-\varepsilon, t+\varepsilon]$ it holds that

$$\#(g^{-1}(\tau) \cap J_i) = 1, \qquad i \in 1, \dots, M.$$

Therefore for each $i \in 1, \dots, M$ the function g is injective on J_i, hence it is strictly monotone on J_i (by continuity). $\square$

Now we are in a position to prove Proposition 3.6.

Proof (of Proposition 3.6. We need to show that the map $[t \mapsto \mu_t]$ constructed in the proof of Theorem 3.1 is not bounded on any subinterval $U \subset (0, T)$. We argue by contradiction. Since by (3.9) for a.e. t

$$|\tilde{\mu}_t| = \#(f^{-1}(t)),$$

the inclusion $[t \mapsto \mu_t] \in L^\infty(U; \mathscr{M}(\mathbb{R}))$ is equivalent to the inequality $\operatorname{ess\,sup}_{t \in U} \#(f^{-1}(t)) < \infty$. From Lemma 3.7 it follows that the function f constructed above is monotone on some nonempty open interval $I \subset (0,1)$. But then $f' \geq 0$ a.e. on I, and this contradicts the construction of f (more specifically, the sets P and N). $\square$

Remark 3.8. We remark that if f were monotone on some interval I then uniqueness would fail for the Cauchy problem for (ODE) with $\boldsymbol{b}$ constructed in the proof of Theorem 3.1. Indeed, without loss of generality suppose that f is strictly increasing on I. Then for any $x \in I$ there exist at least two (actually, infinitely many) integral curves $\gamma \in \Gamma_{\boldsymbol{b}}$ such that $\gamma(0) = x$. Indeed, clearly $\gamma(t) := x$ ($\forall t \in [0, T]$) belongs to $\Gamma_{\boldsymbol{b}}$. On the other hand, for any $y \in I$ such that $y > x$ one can define γ by

$$\gamma(t) := \begin{cases} x, & t < f(x); \\ f^{-1}(t), & f(x) \leq t < f(y); \\ y, & t \geq f(y). \end{cases}$$

Then one readily checks that $\gamma \in \Gamma_{\boldsymbol{b}}$, since for a.e. $t \in (f(x), f(y))$ it holds that

$$\gamma'(t) = \frac{1}{f'(f^{-1}(t))} = \frac{1}{f'(\gamma(t))} = \boldsymbol{b}(t, \gamma(t)).$$

4 Non-uniqueness in the class $L^\infty((0,T);\mathscr{M}(\mathbb{R}^2))$

The aim of this final section is to show the following result:

Theorem 4.1. *There exist $T > 0$, a bounded Borel $\boldsymbol{b}\colon [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$ and $[t \mapsto \mu_t] \in L^\infty((0,T);\mathscr{M}(\mathbb{R}^2))$ satisfying the following conditions:*

(i) $\boldsymbol{b}$ has only constant characteristics, i.e. $\gamma \in \Gamma_{\boldsymbol{b}}$ if and only if there exists $x \in \mathbb{R}^2$ such that $\gamma(t) = x$ for all $t \in [0,T]$;

(ii) $\{\mu_t\}_{t\in[0,T]}$ is not identically zero and it solves (PDE) with zero initial condition.

Proof. The proof will consist in essentially two steps. We will first work in 2D, constructing an example very similar to the one discussed for the proof of Theorem 3.1. We will then suitably embed this into the three-dimensional euclidean space $\mathbb{R}^3$ in such a way that the path of measures resulting from this construction will be uniformly bounded.

Consider the three dimensional Euclidean space with the coordinates (x, y, t). Let (ξ, η, ζ) denote the coordinates in the Cartesian system with the origin $O' = (\frac{1}{2}, \frac{1}{2}, 0)$ and the axes $O'\xi$, $O'\eta$ and $O'\zeta$ having directions $\boldsymbol{e}_1' := \frac{1}{\sqrt{2}}(-1, 1, 0)$, $\boldsymbol{e}_2' := \frac{1}{\sqrt{6}}(-1,-1,2)$ and $\boldsymbol{e}_3' := \frac{1}{\sqrt{3}}(1,1,1)$ respectively (see Fig. 3a).

The 2D construction. Let us consider the plane $O'\xi\eta$ and work in the coordinates (ξ, η). Let f and f_k ($k \in \mathbb{N}$) be the functions constructed in the proof of Lemma 3.2. We set $P := f^{-1}(1)$, $N := \mathbb{R} \setminus P$, $P^k := (f^k)^{-1}(1)$ and $N^k := \mathbb{R} \setminus P^k$. Let

$$\boldsymbol{W}(\xi,\eta) := \alpha \cdot (1, \mathbb{1}_P(\xi) - \mathbb{1}_N(\xi)), \quad \boldsymbol{W}^k(\xi,\eta) := \alpha \cdot (1, \mathbb{1}_{P^k}(\xi) - \mathbb{1}_{N^k}(\xi)),$$

where $\alpha > 0$ is a geometrical constant to be specified later. Clearly $\operatorname{div}_{\xi,\eta}(\boldsymbol{W}) = 0$ and the η-component of $\boldsymbol{W}$ (and $\boldsymbol{W}^k$) takes only the values $\pm\alpha$.

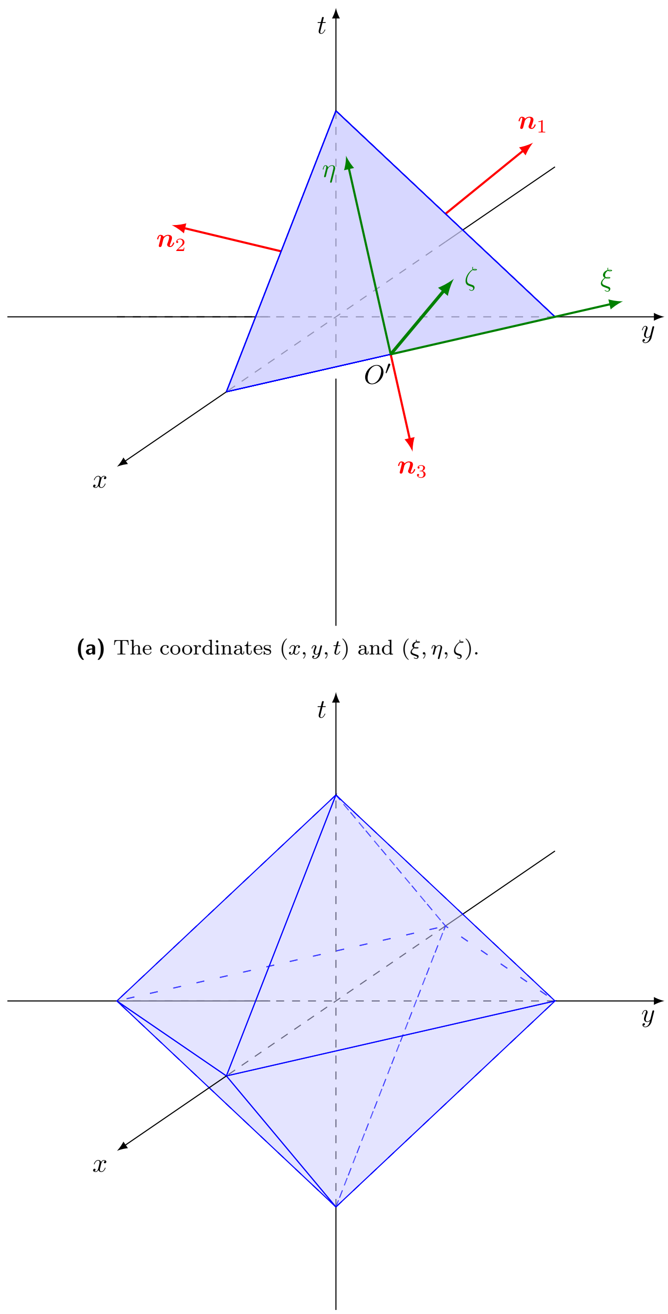
Let now $D \subset \mathbb{R}^2_{\xi,\eta}$ be an open, bounded set with piecewise smooth boundary ∂D and assume that ∂D does not contain vertical segments. We claim that

$$\operatorname{div}(\mathbb{1}_D \boldsymbol{W}) = \boldsymbol{W} \cdot \boldsymbol{\nu} \mathscr{H}^1 \llcorner_{\partial D} \quad \text{in } \mathscr{D}'(\mathbb{R}^2), \tag{4.1}$$

where $\boldsymbol{\nu}$ is the outer unit normal to ∂D and $\mathscr{H}^1 \llcorner_{\partial D}$ is the restriction of $\mathscr{H}^1$ to ∂D.

Indeed, $\boldsymbol{W}^k$ are piecewise constant inside D, so decomposing D into finitely many pieces, applying the classical Gauss–Green Theorem for each piece and summing the results we get that for any test function $\phi \in C_c^\infty(\mathbb{R}^2)$

$$\int_D \boldsymbol{W}^k \cdot \nabla\phi \, dx = \int_{\partial D} \phi \boldsymbol{W}^k \cdot \boldsymbol{\nu} d\mathscr{H}^1.$$



(a) The coordinates (x, y, t) and (ξ, η, ζ).

(b) Extension of the vector field using reflections.

Figure 3: Construction of the vector field $\boldsymbol{B}$ and the function u.

Since ∂D does not contain vertical segments, by construction of the sets P^k and N^k (see Lemma 3.2) we have $\boldsymbol{W}^k \to \boldsymbol{W}\quad \mathscr{H}^1$-a.e. on ∂D as $k \to \infty$. Passing to the limit by means of Dominated convergence Theorem we get (4.1).

Passage to 3D. We extend $\boldsymbol{W}$ to the whole space using the coordinates (ξ, η, ζ) as follows:

$$\boldsymbol{V}(\xi,\eta,\zeta) := \alpha \cdot (1, \mathbb{1}_P(\xi) - \mathbb{1}_N(\xi), 0).$$

Let us switch to the coordinates (x, y, t). Then $\boldsymbol{V}$ becomes a function of (x, y, t), which we still denote as $\boldsymbol{V}(x, y, t)$. Since at each point (x, y, t) we have $\boldsymbol{V}(x, y, t) = \alpha \boldsymbol{e}'_1 \pm \alpha \boldsymbol{e}'_2$ (where the sign depends on (x, y, t)), clearly $(0, 0, 1) \cdot \boldsymbol{V} = \pm\alpha \frac{2}{\sqrt{6}} = \pm\alpha\sqrt{2/3}$, hence fixing $\alpha = \sqrt{3/2}$ we achieve that the t-component of $\boldsymbol{V}$ is ± 1.

Let $\Sigma := \{(x, y, t) \mid x, y, t > 0,\ x + y + t = 1\}$. By (4.1) it holds that $\operatorname{div}(\mathbb{1}_\Sigma \boldsymbol{V} \mathscr{H}^2) = g_1 + g_2 + g_3$, where $g_i = \boldsymbol{V} \cdot \boldsymbol{n}_i \mathscr{H}^1 \llcorner E_i$ and

$$\begin{aligned} E_1 &= \{(0, y, t) \mid y, t > 0,\ y + t = 1\}, \quad &\boldsymbol{n}_1 = (-2, 1, 1)/\sqrt{6},\\ E_2 &= \{(x, 0, t) \mid x, t > 0,\ x + t = 1\}, \quad &\boldsymbol{n}_2 = (1, -2, 1)/\sqrt{6},\\ E_3 &= \{(x, y, 0) \mid x, y > 0,\ x + y = 1\}, \quad &\boldsymbol{n}_3 = (1, 1, -2)/\sqrt{6}. \end{aligned}$$

We define $\boldsymbol{U} \colon \mathbb{R}^3 \to \mathbb{R}^3$ as follows:

$$\boldsymbol{U}(x,y,t) = \sum_{s_1,s_2,s_3 \in \{\pm 1\}} \mathbb{1}_\Sigma(s_1 x, s_2 y, s_3 t)\, \boldsymbol{U}_{s_1,s_2,s_3}(s_1 x, s_2 y, s_3 t),$$

where

$$\boldsymbol{U}_{s_1,s_2,s_3}(x,y,t) = (s_2 s_3 \boldsymbol{V}_1(x,y,t),\ s_1 s_3 \boldsymbol{V}_2(x,y,t),\ s_1 s_2 \boldsymbol{V}_3(x,y,t)).$$

Observe that

$$\operatorname{div}(\boldsymbol{U}\mathscr{H}^2) = g \tag{4.2}$$

in the sense of distributions, being

$$g(x,y,t) = \sum_{i=1}^{3} \sum_{s_1,s_2,s_3 \in \{\pm 1\}} s_1 s_2 s_3\, g_i(s_1 x, s_2 y, s_3 t).$$

Notice that also $g_1(x, y, t) = g_1(-x, y, t)$, $g_2(x, y, t) = g_2(x, -y, t)$ and $g_3(x, y, t) = g_3(x, y, -t)$. Because of this symmetry the right hand side of (4.2) is zero. For instance, for $i = 1$ we have

$$\begin{aligned} &\sum_{s_1,s_2,s_3 \in \{\pm 1\}} s_1 s_2 s_3\, g_1(s_1 x, s_2 y, s_3 t)\\ &\quad = \sum_{s_2,s_3 \in \{\pm 1\}} s_2 s_3\, g_1(x, s_2 y, s_3 t) + \sum_{s_2,s_3 \in \{\pm 1\}} (-1) s_2 s_3\, g_1(-x, s_2 y, s_3 t) = 0. \end{aligned}$$

Consider the octahedron $\Delta := \{(x, y, t) : \boldsymbol{U}(x, y, t) \neq 0\}$ and let

$$u(x,y,t) := \begin{cases} \boldsymbol{U}_3(x,y,t), & (x,y,t) \in \Delta;\\ 0, & (x,y,t) \notin \Delta, \end{cases} \quad \boldsymbol{B}(x,y,t) := \begin{cases} \dfrac{\boldsymbol{U}(x,y,t)}{u(x,y,t)}, & (x,y,t) \in \Delta;\\ (0,0,1), & (x,y,t) \notin \Delta. \end{cases}$$

Then $\boldsymbol{B}_3 = 1$ (everywhere) and by (4.2) we have $\operatorname{div}(\mathbb{1}_\Delta u \boldsymbol{B} \mathscr{H}^2) = \operatorname{div}(\boldsymbol{U} \mathscr{H}^2) = 0$ (in the sense of distributions). Hence for any test function $\varphi \in C_c^\infty(\mathbb{R}^3)$

$$\int_\Delta u \boldsymbol{B} \cdot \nabla_{x,y,t} \varphi \, d\mathscr{H}^2 = 0. \tag{4.3}$$

Denoting with $S_t := \{x, y \in \mathbb{R} \mid (x,y,t) \in \Delta\}$ and disintegrating the measure $\mathscr{H}^2 \llcorner \Delta$ as

$$\mathscr{H}^2 \llcorner \Delta = \int \nu_t \, dt, \quad \text{where} \quad \nu_t = \alpha \mathscr{H}^1 \llcorner S_t,$$

(see e.g. [AFP00, Thm. 2.28]) we can rewrite (4.3) as

$$\int_\mathbb{R} \int_{\mathbb{R}^2} (u \boldsymbol{B} \cdot \nabla_{x,y,t} \varphi) d\nu_t \, dt = 0.$$

Then the family of measures

$$\mu_t := u \cdot \nu_t$$

satisfy (PDE) with

$$\boldsymbol{b}(x,y,t) := (\boldsymbol{B}_1(x,y,t), \boldsymbol{B}_2(x,y,t)).$$

The characteristics of $\boldsymbol{b}$. We claim that $\gamma \in C(\mathbb{R}; \mathbb{R}^2)$ is a characteristic of $\boldsymbol{b}$ if and only if $\gamma(t) = \gamma(0)$ for all t. This claim follows immediately if $\gamma(t) \notin \Delta$ for all t since outside of Δ the vector field $\boldsymbol{b}$ is zero. Therefore it is sufficient to show that γ can intersect each face of Δ at most once.

Suppose that γ intersects the face Σ (defined above) in two points. Since $\boldsymbol{b}$ is zero outside of Δ this is possible only if there exists some nonempty segment $[a,b]$ such that $(\gamma_1(t), \gamma_2(t), t) \in \Delta$ for all $t \in [a,b]$. Then in the coordinates (ξ, η, ζ) the ODE for γ can be written as

$$\dot{\xi} = v(\xi) := \alpha(\mathbb{1}_P(\xi) - \mathbb{1}_N(\xi)), \quad \dot{\eta} = \alpha, \quad \dot{\zeta} = 0.$$

But the first equation does not have solutions. (Indeed, suppose that $\xi = \xi(t)$ is a non-constant solution of $\dot{\xi} = v(\xi)$ such that $\xi(\tau) > \xi(0)$ for some $\tau > 0$. Then there would exist a Lebesgue point z for v such that $\xi(0) < z < \xi(\tau)$ and $v(z) < 0$. By continuity of ξ there exists $t_m := \min\{t : \xi(t) = z\}$. But then $\xi'(t_m) = v(\xi(t_m)) = v(z) < 0$, which contradicts the minimality of t_m. We refer e.g. to [Gus18] for the details). Hence we have obtained a contradiction.

The uniform bounds. Ultimately, by definition of ν_t

$$|\nu_t|(\mathbb{R}^2) = \alpha \cdot 4\sqrt{2} \cdot \begin{cases} 1 - t, & t \in [0,1]; \\ 1 + t, & t \in [-1, 0]; \\ 0, & t \notin [-1,1], \end{cases}$$

hence $|\mu_t| \leq \alpha \cdot 4\sqrt{2}$, i.e. the family of measures $\{\mu_t\}$ is uniformly bounded. □

5 Continuous vector fields

In this section we prove some partial results for continuous vector fields that were mentioned in the Introduction.

Proposition 5.1. *If a vector field* $\boldsymbol{b}\colon (0,T)\times\mathbb{R}^d \to \mathbb{R}^d$ *satisfies (O), then* $\boldsymbol{b}$ *is continuous in the space variable, i.e. for a.e.* $t \in (0,T)$ *the map* $\boldsymbol{b}_t(\cdot) = \boldsymbol{b}(t,\cdot)$ *is continuous.*

Proof. Let us first show that if $\boldsymbol{b}\colon (0,T)\times\mathbb{R}^d \to \mathbb{R}^d$ satisfies (O), then $\boldsymbol{b}_t$ is locally bounded in space for a.e. $t \in (0,T)$. Let $x \in \mathbb{R}^d$ be fixed and suppose by contradiction that there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^d$ such that $x_n \to x$ and $|\boldsymbol{b}_t(x_n)| \to +\infty$. In this case, up to subsequences,

$$\frac{\boldsymbol{b}_t(x_n)}{|\boldsymbol{b}_t(x_n)|} \to z$$

as $n \to +\infty$ for some $z \in \mathbb{R}^d$ with $|z| = 1$. By Osgood condition (O) for every $y \in \mathbb{R}^d$

$$\left|\left\langle \frac{\boldsymbol{b}_t(x_n)-\boldsymbol{b}_t(y)}{|\boldsymbol{b}_t(x_n)|}, x_n - y\right\rangle\right| \leq \frac{C(t)}{|\boldsymbol{b}_t(x_n)|}|x_n - y|\rho(|x_n-y|)$$

and passing to the limit (since $|\boldsymbol{b}_t(x_n)| \to +\infty$) we obtain

$$|\langle z, x-y\rangle| \leq 0$$

for every y, from which $z = 0$, a contradiction.

Now we can prove that $\boldsymbol{b}_t$ is (sequentially) continuous. Arguing again by contradiction, suppose that for some $t \in (0,T)$, $x \in \mathbb{R}^d$ and $\{x_n\}_{n\in\mathbb{N}}$ it holds that $x_n \to x$ and $\boldsymbol{b}_t(x_n) \not\to \boldsymbol{b}(x)$ as $n \to \infty$. Being $\boldsymbol{b}_t$ locally bounded, by passing if necessary to a subsequence, we may assume that $\boldsymbol{b}_t(x_n) \to \boldsymbol{b}_t(x) + z'$ for some $z' \in \mathbb{R}^d$ as $n \to \infty$. By (O) for any $y \in \mathbb{R}^d$

$$|\langle \boldsymbol{b}_t(x_n) - \boldsymbol{b}_t(y), x_n - y\rangle| \leq C(t)|x_n - y|\rho(|x_n - y|).$$

Passing to the limit in both sides of this inequality we get

$$|\langle z' + \boldsymbol{b}_t(x) - \boldsymbol{b}_t(y), x - y\rangle| \leq C(t)|x-y|\rho(|x-y|).$$

Hence by triangle inequality using (O) again we obtain

$$|\langle z', x-y\rangle| = |\langle z'+\boldsymbol{b}_t(x)-\boldsymbol{b}_t(y), x-y\rangle - \langle \boldsymbol{b}_t(x)-\boldsymbol{b}_t(y), x-y\rangle| \leq 2C(t)|x-y|\rho(|x-y|).$$

Taking $y = x + s\cdot z$ with $s > 0$ we get

$$|z'| \leq 2C(t)\rho(s|z'|).$$

Passing to the limit as $s \to 0$ we get $|z'| = 0$, and this concludes the proof. $\square$

Proposition 5.2. *Suppose that $\boldsymbol{b} \in C(\mathbb{R})$ and for any $(t,x) \in \mathbb{R}^2$ there exists a unique $\gamma \in \Gamma_{\boldsymbol{b}}$ such that $\gamma(t) = x$. Then for any $\bar{\mu} \in \mathscr{M}(\mathbb{R})$ the Cauchy problem for* (PDE) *with the initial condition $\mu_t|_{t=0} = \bar{\mu}$ has a unique solution $[t \mapsto \mu_t] \in L^1(0,T;\mathscr{M}(\mathbb{R}))$.*

Proof. Suppose that $[t \mapsto \mu_t] \in L^1(0,T;\mathscr{M}(\mathbb{R}))$ is a (signed) measure-valued solution to the continuity equation with $\bar{\mu} = 0$. Then there exists a Lebesgue negligible set $N \subset (0,T)$ such that for all $\tau \in (0,T) \setminus N$ for any $\Phi \in C_c^1([0,\tau] \times \mathbb{R})$ it holds that

$$\begin{aligned} &\int_{\mathbb{R}} \Phi(\tau,x)\, d\mu_\tau(x) - \int_{\mathbb{R}} \Phi(0,x) d\bar{\mu}(x) \\ &= \int_0^\tau \int_{\mathbb{R}} \left[\partial_t \Phi(t,x) + \boldsymbol{b} \cdot \partial_x \Phi(t,x)\right]\, d\mu_t(x)\, dt. \end{aligned} \tag{5.1}$$

(Indeed, first one can consider finite linear combinations of functions Φ having the form $\Phi(t,x) = \psi(t)\phi(x)$, where ϕ belong to some countable dense subset of $C_c^1(\mathbb{R})$ and $\psi \in C_c^1([0,T])$ are arbitrary. For such test functions (5.1) follows from (2.1), and in the general case one can apply an approximation argument.)

There are countably many open intervals where $\boldsymbol{b} > 0$ or $\boldsymbol{b} < 0$ (and $\boldsymbol{b} = 0$ on the complement of the union of all those intervals). Consider one of the intervals, i.e. suppose that $\boldsymbol{b}(\alpha) = \boldsymbol{b}(\beta) = 0$, $\alpha < \beta$ and $\boldsymbol{b} > 0$ on (α,β). Fix $x_0 \in (\alpha,\beta)$ and for all $x \in (\alpha,\beta)$ let

$$F(x) := \int_{x_0}^x \frac{dy}{\boldsymbol{b}(y)}. \tag{5.2}$$

(Note that $\frac{1}{\boldsymbol{b}} \in L^1[x_0,x]$ since $\min_{[x_0,x]} \boldsymbol{b} > 0$ by continuity.) Clearly $F \in C^1(\alpha,\beta)$. If $\beta = +\infty$ then $F(\beta - 0) \equiv F(+\infty) = +\infty$. Otherwise there exists $\xi \in \mathbb{R}$ such that $\int_\xi^{+\infty} \frac{dy}{\boldsymbol{b}(y)} < T$. This would mean that the solution γ of (ODE) with the initial condition ξ escapes to infinity in finite time, which contradicts the existence assumption that $\gamma \in \Gamma_{\boldsymbol{b}}$. Analogously, if $\alpha = -\infty$ then $F(\alpha+0) \equiv F(-\infty) = -\infty$. Finally, if $\alpha, \beta \in \mathbb{R}$ then by uniqueness of the integral curves $F(\alpha+0) = -\infty$ and $F(\beta - 0) = +\infty$.

Furthermore, F is strictly increasing and continuous, hence $F^{-1} \colon \mathbb{R} \to (\alpha,\beta)$ is continuous and strictly increasing as well. Since $F \in C^1(\alpha,\beta)$ we also have $F^{-1} \in C^1(\mathbb{R})$. Hence

$$X(t,x) := F^{-1}(F(x) + t)$$

belongs to $C^1(\mathbb{R} \times (\alpha,\beta))$ by the chain rule. Moreover, $X(\cdot,x)$ solves (ODE). Let now $\omega \in C_c^1(\alpha,\beta)$ be an arbitrary test function and fix $\tau \in (0,T)$. Then define

$$\varphi(t,x) := \omega(X(\tau - t, x)) \tag{5.3}$$

which belongs to $C_c^1([0,\tau] \times (\alpha,\beta))$. (Indeed, if $[u,v] \subset (\alpha,\beta)$ contains the support of ω, then the support of φ is contained in $[0,\tau] \times [X(-\tau,u),v]$.) Moreover, φ satisfies the transport equation $\partial_t \varphi + \boldsymbol{b} \partial_x \varphi = 0$ (pointwise) with the final condition $\varphi(\tau,x) = \omega(x)$.Using the test function $\Phi = \varphi$ in (5.1) we get

$$\int_{\mathbb{R}} \omega(x)\, d\mu_\tau(x) = 0$$

and by arbitrariness of ω this implies that $\mu_\tau = 0$ for a.e. $\tau \in (0,T)$ (more precisely, for all $\tau \in (0,T) \setminus N$). In particular, this implies that the solution $(\mu_t)\llcorner_{(\alpha,\beta)} = 0$ for a.e. t and hence μ_t vanishes on every connected component of the set $\{\boldsymbol{b} > 0\}$. Similarly, one can show that μ_t vanishes for a.e. t on every connected component of the set $\{\boldsymbol{b} < 0\}$. We have thus proved that μ_t is concentrated on $\{\boldsymbol{b} = 0\}$ and then it solves (PDE) with $\boldsymbol{b} \equiv 0$. Hence $\mu_t = 0$ globally for a.e. $t \in (0,T)$ and this concludes the proof. □

Remark 5.3. Currently it is not known to us whether Proposition 5.2 can be extended to more than one space dimensions, or for non-autonomous one-dimensional case. Our proof of Proposition 5.2 relies on C^1 regularity of the flow $X(t,x)$ of $\boldsymbol{b}$ on the set $[0,T] \times \{x \in \mathbb{R} : \boldsymbol{b}(x) \neq 0\}$. This allows us to construct C^1 solutions φ of the transport equation and use them as the test functions in the distributional formulation of the continuity equation.

But in the case when $d > 1$ (and also in the case when $d = 1$ and $\boldsymbol{b}$ is non-autonomous) the flow of $\boldsymbol{b}$ in general is not differentiable on the set where $\boldsymbol{b} \neq 0$.

Indeed, let $f \colon \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. Then the flow of $\boldsymbol{b} \colon \mathbb{R}^2 \ni x \mapsto (0, f(x_1)) \in \mathbb{R}^2$ is given by $X(t,x) = (x_1, x_2 + t \cdot f(x_1))$. It is evident that for all $t > 0$ the function $X(t,\cdot)$ is differentiable at $x \in \mathbb{R}^2$ if and only if $f(\cdot)$ is differentiable at x_1.

Similarly, in the one-dimensional non-autonomous setting one can show that if f is a strictly increasing biLipschitz function such that $f(0) = 0$ then there exists $T > 0$ (dependent on the Lipschitz constants for f and f^{-1}) such that the function $X(t,x) := x + t \cdot f(x)$ for every $t \in [0,T]$ is biLipschitz (as the function of $x \in \mathbb{R}$). Then denoting with $Y(t,\cdot)$ the inverse of $X(t,\cdot)$ one can show that X is the flow of continuous function $\boldsymbol{b}(t,x) := f(t, Y(t,x))$.

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