MINIMALITY OF THE BALL FOR A MODEL OF CHARGED LIQUID DROPLETS

EKATERINA MUKOSEEVA AND GIULIA VESCOVO

Abstract. We deduce that charged liquid droplets minimizing *Debye-Hückel-type free energy* are spherical in the small charge regime. The variational model was proposed by Muratov and Novaga in 2016 to avoid the ill-posedness of the classical one. By combining a recent (partial) regularity result with Selection Principle of Cicalese and Leonardi, we prove that the ball is the unique minimizer in the small charge regime.

1. Introduction

1.1. **Background and description of the model.** In this paper we deal with a variational model describing the shape of charged liquid droplets. We investigate the droplets minimizing a suitable free energy composed by an attractive term, coming from surface tension forces, and a repulsive one, due to the electric forces generated by the interaction between charged particles. Thanks to the particular structure of the energy, one may expect that for small values of the total charge the attractive part is predominant, forcing in this way the spherical shape.

The experiments agree with this guess - one observes the following phenomenon: the shape of the liquid droplet is spherical in a small charge regime. Then, as soon as the value of the total charge increases, the droplet gradually deforms into an ellipsoid, it develops conical singularities, the so-called Taylor cones, [T64], and finally, the liquid starts emitting a thin jet ([DMV64],[DAMHL03],[RPH89], [WT25]). The first experiments were conducted by Zeleny in 1914, [Z], but in a slightly different context.

Several mathematical models of charged liquid droplets have been studied over the years. A difficulty is that contrary to the numerical and experimental observations these models are in general mathematically ill-posed, see [GNR15]. For a more exhaustive discussion we refer the reader to [MN16].

The main issue with the variational model studied in [GNR15] comes from the tendency of charges to concentrate at the interface of the liquid. To restore the well-posedness one should consider a physical regularizing mechanism in the functional. With this purpose in mind, Muratov and Novaga in [MN16] integrate the entropic effects associated with the presence of free ions in the liquid. The advantage of this model is that the charges are now distributed inside of the droplet. More precisely, they suggest considering the following *Debye-Hückel-type free energy* (in every dimension):

$$\mathcal{F}(E,u,\rho) := P(E) + Q^2\left\{ \int_{\mathbb{R}^n} a_E|\nabla u|^2\,dx + K\int_E \rho^2\,dx \right\}. \tag{1.1}$$

Here $E \subset \mathbb{R}^n$ represents the droplet, $P(E)$ is the De Giorgi perimeter, [M, Chapter 12], the constant $Q > 0$ is the total charge enclosed in E and

$$a_E(x) := \mathbf{1}_{E^c} + \beta\mathbf{1}_E,$$

where $\mathbf{1}_F$ is the characteristic function of a set F and $\beta > 1$ [1] is the permittivity of the liquid.

The normalized density of charge $\rho \in L^2(\mathbb{R}^n;\mathbb{R}^n)$ satisfies

$$\rho \mathbf{1}_{E^c} = 0 \qquad \text{and} \qquad \int \rho = 1, \tag{1.2}$$

and the electrostatic potential u is such that $\nabla u \in L^2(\mathbb{R}^n)$ and

$$-\operatorname{div}\big(a_E \nabla u\big) = \rho \qquad \text{in } \mathcal{D}'(\mathbb{R}^n). \tag{1.3}$$

For a fixed set E we define the set of admissible pairs of functions u and ρ:

$$\mathcal{A}(E) := \big\{(u,\rho) \in D^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\colon u \text{ and } \rho \text{ satisfy (1.3) and (1.2)}\big\}, \tag{1.4}$$

where

$$D^1(\mathbb{R}^n) = \overline{C_c^\infty(\mathbb{R}^n)}^{\mathring{W}^{1,2}(\mathbb{R}^n)}, \qquad \|\varphi\|_{\mathring{W}^{1,2}(\mathbb{R}^n)} = \|\nabla \varphi\|_{L^2(\mathbb{R}^n)}.$$

Note that the class of admissible couples $\mathcal{A}(E)$ is non-empty only if $n \geq 3$ (see [DPHV19, Remark 2.2]). For this reason the assumption $n \geq 3$ will be in force throughout the whole paper. The variational problem proposed in [MN16] is the following:

$$\min\big\{\mathcal{F}(E,u,\rho) : |E| = V, E \subset B_R, (u,\rho) \in \mathcal{A}(E)\big\}. \tag{1.5}$$

The a-priori boundedness assumption $E \subseteq B_R$ ensures the existence of a minimizer in the class of sets of finite perimeter with a prescribed volume, [MN16, Theorem 3].

For convenience we introduce the following notation:

$$\mathcal{G}_{\beta,K}(E) := \inf_{(u,\rho)\in\mathcal{A}(E)} \left\{ \int_{\mathbb{R}^n} a_E |\nabla u|^2 + K \int_E \rho^2 \right\}. \tag{1.6}$$

For $E \subset \mathbb{R}^n$ we set

$$\mathcal{F}_{\beta,K,Q}(E) := P(E) + Q^2 \mathcal{G}_{\beta,K}(E).$$

By scaling (see the introduction of [DPHV19]), we can reduce the problem to the case $|E| = |B_1|$ and so in the rest of the paper we will work with the following problem:

$$\min\big\{\mathcal{F}_{\beta,K,Q}(E) : |E| = |B_1|,\ E \subset B_R\big\}. \tag{$\mathcal{P}_{\beta,K,Q,R}$}$$

We will often omit the subscripts β and K as those are fixed physical parameters.

We note that the model we investigate can be seen as "interpolation" between Gamow model and the free interface problems arising in optimal design (see, for example, [FJ15]). For the former, it has been recently shown ([KM14],[Jul14]) that for small enough charges the unique minimizers are balls. However, in Gamow model the non-local term is Lipschitz with respect to symmetric difference between sets, implying that on small scales the perimeter dominates the non-local part of the energy. This is not the case for the energy $\mathcal{F}$ defined in (1.1) and our analysis is thus more complicated.

1.2. **Main results.**

As we mentioned above, one can expect that the shape of the droplet in a small charge regime is spherical. We confirm this intuition by proving that the ball is the unique minimizer of the functional $\mathcal{F}$ for small values of the total charge Q. Precisely, we obtain the following result.

Theorem 1.1. *Fix $K > 0$, $\beta > 1$. Then there exists $Q_0 > 0$ such that for all $Q < Q_0$ and any $R \geq 1$ the only minimizers of ($\mathcal{P}_{\beta,K,Q,R}$) are the balls of radius* 1.

---

[1] Mathematically, considering $\beta \leq 1$ amounts to considering the complement of the set E in place of E. Our proof would work without change also for the case $\beta \leq 1$. However, some changes would be needed in [DPHV19], so we wouldn't be able to use their regularity results directly.

The condition $E \subset B_R$ in the minimizing problem $(\mathcal{P}_{\beta,K,Q,R})$ is required to have existence of minimizers. However, thanks to Theorem 1.1 it can be dropped for small enough charges.

Corollary 1.2. *Fix $K > 0$, $\beta > 1$. Then there exists $Q_0 > 0$ such that for all $Q < Q_0$ the infimum in the problem*

$$(\mathcal{P}_{\beta,K,Q}) \qquad \inf\left\{\mathcal{F}_{\beta,K,Q}(E) : |E| = |B_1|\right\}$$

is attained. Moreover, the only minimizers are the balls of radius 1.

Remark 1.3. In the proof we provide the constant Q_0 is the same as in Theorem 1.1. However, almost the same proof would give existence of minimizers (but not the fact that they are balls) for $Q < Q_c$ where Q_c is such that the minimizers for $\mathcal{P}_{\beta,K,Q,R}$ are close enough to the ball in L^1 norm. A priori Q_c might be bigger than Q_0 and indeed we expect that existence fails later than minimizers cease to be spherical.

For the proof of Theorem 1.1 we combine an improved version of (partial) regularity results for the minimizers of [DPHV19, Theorem 1.2] with second variation techniques. The first step is to obtain the partial $C^{2,\vartheta}$-regularity of minimizers. In fact, we are able to prove the following partial C^∞-regularity of minimizers, a result that is interesting in itself.

Theorem 1.4 (C^∞-regularity)**.** *Given $n \geq 3$, $A > 0$ and $\vartheta \in (0, 1/2)$, there exists $\varepsilon_{\mathrm{reg}} = \varepsilon_{\mathrm{reg}}(n, A, \vartheta) > 0$ such that if E is a minimizer of $(\mathcal{P}_{\beta,K,Q,R})$ with $Q+\beta+K+\frac{1}{K} \leq A$,*

$$x_0 \in \partial E \quad \textit{and} \quad r + \mathbf{e}_E(x_0, r) + Q^2\, D_E(x_0, r) \leq \varepsilon_{\mathrm{reg}},$$

then $E \cap \mathbf{C}(x_0, r/2)$ coincides with the epi-graph of a C^∞-function f. In particular, we have that $\partial E \cap \mathbf{C}(x_0, r/2)$ is a C^∞ $(n-1)$-dimensional manifold. Moreover [2],

$$[f]_{C^{k,\vartheta}(\mathbf{D}(x_0', r/2))} \leq C(n, A, k, \vartheta) \tag{1.7}$$

for every $k \in \mathbb{N}$ with $k \geq 2$.

We refer the reader to Notation 2.1 for the definition of $\mathbf{e}_E(x_0, r)$, $D_E(x_0, r)$ and $\mathbf{C}(x_0, r/2)$.

1.3. **Strategy of the proof and structure of the paper.** We use Selection Principle, the technique introduced by Cicalese and Leonardi in [CL12] for the proof of quantitative isoperimetric inequality (see also [AFM13], where the authors use a similar approach to investigate a nonlocal isoperimetric problem).

To prove Theorem 1.1 we first reduce our problem to the so-called *nearly-spherical sets*. Those are the sets which can be described as subgraphs of smooth functions defined over the boundary of the unitary ball. The advantage is that for this particular class of sets we are able to deduce a Taylor expansion for the energy near the ball B_1.

In the first part of the paper (from Section 3 to Section 6) we show that a minimizer is nearly-spherical whenever the total charge is small enough. We argue by contradiction

[2] Let $\Omega \subset \mathbb{R}^m$ be an open and bounded set, $f \in C(\overline{\Omega})$. Then

$$[f]_{C^{0,\vartheta}(\overline{\Omega})} := \sup_{x \neq y, x, y \in \overline{\Omega}} \frac{|f(x) - f(y)|}{|x - y|^\vartheta}.$$

Moreover, if $f \in C^k(\overline{\Omega})$ then

$$[f]_{C^{k,\vartheta}(\overline{\Omega})} := \sum_{|\alpha| \leq k} \|D^\alpha f\|_{C(\overline{\Omega})} + \sum_{|\alpha| = k} [D^\alpha f]_{C^{0,\vartheta}(\overline{\Omega})}.$$

and get a sequence of minimizers with corresponding total charge going to zero. In Section 3 we prove the L^1-convergence of the minimizers to the unitary ball and the convergence of the perimeters as the charge goes to zero. Thanks to *uniform* density estimates for the volume and the perimeter of a minimizer we obtain the Kuratowski convergence of sets as well as their boundaries.

Now we need to improve the convergence deduced in Section 3. For this purpose it is crucial to enhance the regularity result obtained in [DPHV19]. Hence, Section 4 is dedicated to the higher regularity of minimizers. By exploiting the Euler-Lagrange equation and the $C^{1,\eta}$-regularity of u up to the boundary ∂E, we deduce the partial $C^{2,\vartheta}$-regularity of minimizers.

In Section 5, by a bootstrap argument, we obtain the partial smooth regularity of minimizers.

Since for each Q small enough the corresponding minimal set E_Q has $C^{2,\vartheta}$-regular boundary (with uniform bounds), by Ascoli-Arzelà, up to extracting a subsequence, we get that E_Q converges to B_1 in a stronger $C^{2,\vartheta'}$-sense for every $\vartheta' < \vartheta$. This is the content of Section 6.

In Sections 7 and 8 we prove Theorem 1.1 for nearly spherical sets. To this end, we write Taylor expansion of the energy $\mathcal{G}$ using shape derivatives and providing a bound for the "Hessian". A direct computation provides a similar bound for the perimeter and this allows us to conclude.

In Appendix A we provide a sharp bound for the second variation of the energy $\mathcal{G}$ at the ball. We don't need it for the main results but we think it might be of some interest.

Acknowledgements. We warmly thank our advisor Guido De Philippis for introducing us to the problem and for many fruitful discussions.

The work of the authors is supported by the INDAM-grant "Geometric Variational Problems".

2. Notation and preliminary results

In this section we fix the notation and collect some results obtained in [DPHV19] which will be useful in the proof of regularity.

Notation 2.1. Let $E \subset \mathbb{R}^n$ be a set of finite perimeter, $x \in \mathbb{R}^n$, $\nu \in \mathbb{S}^{n-1}$ and $r > 0$.

- We call $\mathbf{p}^\nu(x) := x - (x \cdot \nu)\,\nu$ and $\mathbf{q}^\nu(x) := (x \cdot \nu)\,\nu$, respectively, the *orthogonal projection* onto the plane $\nu^\perp$ and the *projection* on ν. For simplicity we write $\mathbf{p}(x) := \mathbf{p}^{e_n}(x)$ and $\mathbf{q}(x) := \mathbf{q}^{e_n}(x) = x_n$.
- We define the *cylinder* with center at $x_0 \in \mathbb{R}^n$ and radius $r > 0$ with respect to the direction $\nu \in \mathbb{S}^{n-1}$ as

$$\mathbf{C}(x_0, r, \nu) := \big\{x \in \mathbb{R}^n \,:\, |\mathbf{p}^\nu(x - x_0)| < r\,, |\mathbf{q}^\nu(x - x_0)| < r\big\},$$

  and write $\mathbf{C}_r := \mathbf{C}(0, r, e_n)$, $\mathbf{C} := \mathbf{C}_1$.
- We denote the $(n-1)$-dimensional *disk* centered at $y_0 \in \mathbb{R}^{n-1}$ and of radius r by

$$\mathbf{D}(y_0, r) := \big\{y \in \mathbb{R}^{n-1} : |y - y_0| < r\big\}.$$

  We let $\mathbf{D}_r := \mathbf{D}(0, r)$ and $\mathbf{D} := \mathbf{D}(0, 1)$.
- We define

$$\mathbf{e}_E(x, r) := \inf_{\nu \in \mathbb{S}^{n-1}} \frac{1}{r^{n-1}} \int_{\partial^* E \cap B_r(x)} \frac{|\nu_E(y) - \nu|^2}{2}\, d\mathcal{H}^{n-1}(y).$$

We call $\mathbf{e}_E(x,r)$ the *spherical excess*. Note that from from the definition it follows that

$$\mathbf{e}_E(x,\lambda r) \le \frac{1}{\lambda^{n-1}}\mathbf{e}_E(x,r)$$

for any $\lambda \in (0,1)$.

- Let $(u,\rho) \in \mathcal{A}(E)$ be the minimizer of $\mathcal{G}_{\beta,K}(E)$. We define the *normalized Dirichlet energy* at x as

$$D_E(x,r) := \frac{1}{r^{n-1}}\int_{B_r(x)} |\nabla u|^2\,dx.$$

Convention 2.2 (Universal constants)**.** Let $A > 0$ be a positive constant. We say that

- the parameters β, K, Q with $\beta \ge 1$ are *controlled by* A if
$$\beta + K + \frac{1}{K} + Q \le A;$$
- a constant is *universal* if it depends only on the dimension n and on A.

Note that in particular universal constants *do not depend* on the size of the container where the minimization problem is set.

In the following theorem we collect some properties of minimizers. For the proofs we refer the reader to [DPHV19].

Theorem 2.3. *Let $E \subset \mathbb{R}^n$ be a set of finite measure. Then*

(i) *there exists a unique pair $(u_E,\rho_E) \in \mathcal{A}(E)$ minimizing $\mathcal{G}_{\beta,K}(E)$. Moreover,*
$$u_E + K\rho_E = \mathcal{G}_{\beta,K}(E) \qquad \text{in } E,$$
and
$$0 \le u_E \le \mathcal{G}_{\beta,K}(E), \qquad 0 \le K\rho_E \le \mathcal{G}_{\beta,K}(E)\mathbf{1}_E.$$
In particular, $\rho_E \in L^p$ for all $p \in [1,\infty]$ with
$$\|\rho_E\|_p \le C(n,\beta,K,1/|E|).$$

(ii) (Euler-Lagrange equation) *If E is a minimizer of $(\mathcal{P}_{\beta,K,Q,R})$, then*
$$\int_{\partial^* E} \mathrm{div}_E\eta\,d\mathcal{H}^{n-1} - Q^2\int_{\mathbb{R}^n} a_E\Big(|\nabla u_E|^2\mathrm{div}\eta - 2\nabla u_E\cdot(\nabla\eta\,\nabla u_E)\big)\,dx$$
$$- Q^2\,K\int_{\mathbb{R}^n}\rho_E^2\mathrm{div}\eta\,dx = 0$$
for all $\eta \in C_c^1(B_R;\mathbb{R}^n)$ with $\int_E \mathrm{div}\eta\,dx = 0$.

(iii) (Compactness) *Let $K_h, Q_h \in \mathbb{R}$, $\beta_h \ge 1$ and $R_h \ge 1$ be such that*
$$K_h \to K > 0\,, \quad \beta_h \to \beta \ge 1\,, \quad R_h \to R \ge 1\,, \quad Q_h \to Q \ge 0,$$
when $h \to \infty$. For every $h \in \mathbb{N}$ let E_h be a minimizer of $(\mathcal{P}_{\beta_h,K_h,Q_h,R_h})$.
Then, up to a non relabelled subsequence, there exists a set of finite perimeter E such that
$$\lim_{h\to\infty}|E\Delta E_h| = 0.$$
Moreover, E is a minimizer of $(\mathcal{P}_{\beta,K,Q,R})$ and
$$\mathcal{F}_{\beta,K,Q}(E) = \lim_{h\to\infty}\mathcal{F}_{\beta_h,K_h,Q_h}(E_h), \qquad \lim_{h\to\infty}P(E_h) = P(E).$$

Let $A > 0$. For the following properties we require that β, K and Q are controlled by A.

(iv) (Boundedness of the normalized Dirichlet energy) *There exists a universal constant $C_{\mathrm{e}} > 0$ such that, if E is a minimizer of $(\mathcal{P}_{\beta,K,Q,R})$, then for all $x \in \overline{B_R}$,*

$$Q^2 D_E(x,r) = \frac{Q^2}{r^{n-1}} \int_{B_r(x)} |\nabla u|^2 \, dx \le C_{\mathrm{e}}.$$

(v) (Density estimates) *There exist universal constants C_{o}, $C_{\mathrm{i}} > 0$ and $\bar r > 0$ such that, if E is a minimizer of $(\mathcal{P}_{\beta,K,Q,R})$, then*[3]

$$\frac{1}{C_{\mathrm{i}}} r^{n-1} \le P(E, B_r(x)) \le C_{\mathrm{o}} r^{n-1} \qquad \text{for all } x \in \partial E \text{ and } r \in (0, \bar r),$$

and

$$\frac{1}{C_{\mathrm{i}}} \le \frac{|B_r(x) \cap E|}{|B_r(x)|} \le C_{\mathrm{o}} \qquad \text{for all } x \in E \text{ and } r \in (0, \bar r).$$

(vi) (Excess improvement) *There exists a universal constant $C_{\mathrm{dec}} > 0$ such that for all $\lambda \in (0, 1/4)$ there exists $\varepsilon_{\mathrm{dec}} = \varepsilon_{\mathrm{dec}}(n, A, \lambda) > 0$ satisfying the following: if E is a minimizer of $(\mathcal{P}_{\beta,K,Q,R})$ and*

$$x \in \partial E, \quad r + Q^2 D_E(x,r) + \mathbf{e}_E(x,r) \le \varepsilon_{\mathrm{dec}},$$

then

$$Q^2 D_E(x, \lambda r) + \mathbf{e}_E(x, \lambda r) \le C_{\mathrm{dec}} \lambda \Big(\mathbf{e}_E(x,r) + Q^2 D_E(x,r) + r \Big).$$

(vii) (Decay of the Dirichlet energy) *There exists a universal constant $C_{\mathrm{dir}} > 0$ such that for all $\lambda \in (0, 1/2)$ there exists $\varepsilon_{\mathrm{dir}} = \varepsilon_{\mathrm{dir}}(n, A, \lambda)$ satisfying the following: if E is a minimizer of $(\mathcal{P}_{\beta,K,Q,R})$, $x \in \partial E$ and*

$$r + \mathbf{e}_E(x, r, e_n) \le \varepsilon_{\mathrm{dir}},$$

then

$$D_E(x, \lambda r) \le C_{\mathrm{dir}} \lambda \Big(D_E(x,r) + r \Big).$$

Proof. The proofs of (i), (iii), (iv), (v), (vi) and (vii) can be found respectively in [DPHV19, Proposition 2.1, Proposition 5.1, Lemma 6.5, Proposition 6.4, Proposition 6.6, Theorem 7.1, Proposition 7.6]. There is no detailed proof of (ii) in [DPHV19]. Moreover, the formula given in [DPHV19, Corollary 3.3] has a sign mistake and thus we give a proof of (ii) here.

We start by showing the following identity for any $\rho \in L^2(\mathbb{R}^n)$:

$$\begin{aligned} \inf \left\{ \int_{\mathbb{R}^n} a_E |\nabla u|^2 : u \in D^1(\mathbb{R}^n), -\operatorname{div}(a_E \nabla u) = \rho \right\} \\ = \inf \left\{ \int_{\mathbb{R}^n} \frac{|V|^2}{a_E} : V \in L^2(\mathbb{R}^n; \mathbb{R}^n), -\operatorname{div}(V) = \rho \right\}. \end{aligned} \tag{2.1}$$

[3] Here and in the sequel we will always work with the representative of E such that

$$\partial E = \left\{ x : \frac{|B_r(x) \setminus E|}{|B_r(x)|} \cdot \frac{|B_r(x) \cap E|}{|B_r(x)|} > 0 \quad \text{for all } r > 0 \right\},$$

see [M, Proposition 12.19].

Right-hand side is trivially not larger than the left-hand side, as we can take $V = a_E\nabla u$ as a competitor. So we only need to show that

$$\inf\left\{\int_{\mathbb{R}^n} a_E|\nabla u|^2 : u\in D^1(\mathbb{R}^n), -\operatorname{div}(a_E\nabla u)=\rho\right\} \le \inf\left\{\int_{\mathbb{R}^n}\frac{|V|^2}{a_E} : V\in L^2(\mathbb{R}^n;\mathbb{R}^n), -\operatorname{div}(V)=\rho\right\}.$$

We use that the infimum is achieved in both cases by convexity. Hence, the right-hand side has a minimizer V_0 and it satisfies the corresponding Euler-Langrange equation, that is

$$\frac{V_0}{a_E}\cdot X = 0 \qquad \text{for any } X\in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n) \text{ such that } \operatorname{div} X = 0.$$

But that gives us that $\frac{V_0}{a_E} = \nabla u_0$ for some $u_0 \in D^1(\mathbb{R}^n)$. Since $\operatorname{div}(V_0)=\rho$, we get $-\operatorname{div}(a_E\nabla u_0)=\rho$ and thus

$$\inf\left\{\int_{\mathbb{R}^n} a_E|\nabla u|^2 : u\in D^1(\mathbb{R}^n), -\operatorname{div}(a_E\nabla u)=\rho\right\} \le \int_{\mathbb{R}^n} a_E|\nabla u_0|^2$$
$$\le \int_{\mathbb{R}^n}\frac{|V_0|^2}{a_E} = \inf\left\{\int_{\mathbb{R}^n}\frac{|V|^2}{a_E} : V\in L^2(\mathbb{R}^n;\mathbb{R}^n), -\operatorname{div}(V)=\rho\right\},$$

which finishes the proof of (2.1).

Suppose now that E is a minimizer of $(\mathcal{P}_{\beta,K,Q,R})$ and $(u,\rho)\in\mathcal{A}$ is the pair minimizing $\mathcal{G}(E)$. We fix a vector field $\eta\in C_c^\infty(B_R;\mathbb{R}^n)$ with $\int_E \operatorname{div}\eta\,dx = 0$ and, following [DPHV19, Lemma 3.1], we define

$$\varphi_t(x) := x + t\eta, \qquad u_t := u\circ\varphi_t^{-1}, \qquad \tilde\rho_t := \det(\nabla\varphi_t^{-1})\rho\circ\varphi_t^{-1}.$$

By [DPHV19, Lemma 3.1] we have

$$-\operatorname{div}(a_{E_t}A_t\nabla u_t) = \rho_t, \tag{2.2}$$

where $E_t = \varphi_t(E)$, $A_t = \det(\nabla\varphi_t^{-1})(\nabla\varphi_t^{-1})^{-t}(\nabla\varphi_t^{-1})^{-1}$. Note that $|E_t| = |E| + o(t) = |B_1| + o(t)$ since $\int_E \operatorname{div}\eta\,dx = 0$.

Now we recall that E is a minimizer and we use (2.1) to get that

$$\begin{aligned}
&P(E) + Q^2\left\{\int_{\mathbb{R}^n} a_E|\nabla u|^2\,dx + K\int_E \rho^2\,dx\right\}\\
&= \min\left\{P(E) + Q^2\inf_{(u,\rho)\in\mathcal{A}(E)}\left\{\int_{\mathbb{R}^n} a_E|\nabla u|^2 + K\int_E\rho^2\right\} : |E| = |B_1|, E\subset B_R\right\}\\
&= \min\left\{P(E) + Q^2\inf_{\rho\in L^2(\mathbb{R}^n)}\left\{\inf_{-\operatorname{div}(a_E\nabla u)=\rho}\int_{\mathbb{R}^n} a_E|\nabla u|^2 + K\int_E\rho^2\right\} : |E| = |B_1|, E\subset B_R\right\}\\
&= \min\left\{P(E) + Q^2\inf_{\rho\in L^2(\mathbb{R}^n)}\left\{\inf_{-\operatorname{div}(V)=\rho}\int_{\mathbb{R}^n}\frac{|V|^2}{a_E} + K\int_E\rho^2\right\} : |E| = |B_1|, E\subset B_R\right\}\\
&\le P(E_t) + Q^2\left\{\int_{\mathbb{R}^n} a_{E_t}|A_t\nabla u_t|^2 + K\int_{E_t}\rho_t^2\right\},
\end{aligned}$$

where for the last inequality we used (2.2) and the fact that $|E_t| = |B_1| + o(t)$ by the choice of η. To get Euler-Lagrange equation it remains to expand the last quantity in terms of t. It is well known (see, for example, [M, Theorem 17.8]) that

$$P(E_t) = P(E) + t\int_{\partial^* E}\operatorname{div}_E\eta + o(t), \tag{2.3}$$

where $\operatorname{div}_E \eta = \operatorname{div} \eta - \nu_E \cdot (\nabla\eta\, \nu_E)$. As for the other part of the energy, using change of variables, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} a_{E_t} |A_t \nabla u_t|^2 + K \int_{E_t} \rho_t^2 &= \int_{\mathbb{R}^n} a_E |\det(\nabla\varphi_t^{-1})(\nabla\varphi_t)^t (\nabla\varphi_t)(\nabla\varphi_t)^{-t} \nabla u|^2 \det(\nabla\varphi_t) \\
&\quad + K \int_E \rho^2 \det{}^2(\nabla\varphi_t^{-1}) \det(\nabla\varphi_t) \\
&= \int_{\mathbb{R}^n} a_E |(1 - t \operatorname{div} \eta)(\mathrm{Id} + t(\nabla\eta)^t)(\mathrm{Id} + t\nabla\eta)(\mathrm{Id} - t(\nabla\eta)^t)\nabla u|^2 (1 + t \operatorname{div} \eta) \\
&\quad + K \int_E \rho^2 (1 - t \operatorname{div} \eta) + o(t) \\
&= \int_{\mathbb{R}^n} a_E |(1 - t \operatorname{div} \eta)(\mathrm{Id} + t\nabla\eta)\nabla u|^2 (1 + t \operatorname{div} \eta) + K \int_E \rho^2 (1 - t \operatorname{div} \eta) + o(t) \\
&= \int_{\mathbb{R}^n} a_E |(1 + t\,(-\operatorname{div} \eta\, \mathrm{Id} + \nabla\eta))\nabla u|^2 (1 + t \operatorname{div} \eta) + K \int_E \rho^2 (1 - t \operatorname{div} \eta) + o(t) \\
&= \int_{\mathbb{R}^n} a_E \left(|\nabla u|^2 + t \left(-2 \operatorname{div} \eta |\nabla u|^2 + 2\nabla u \cdot (\nabla\eta \nabla u) \right) \right) (1 + t \operatorname{div} \eta) \\
&\quad + K \int_E \rho^2 (1 - t \operatorname{div} \eta) + o(t) \\
&= \int_{\mathbb{R}^n} a_E |\nabla u|^2 + K \int_E \rho^2 + t \left(\int_{\mathbb{R}^n} a_E \left(-\operatorname{div} \eta |\nabla u|^2 + 2\nabla u \cdot (\nabla\eta \nabla u) \right) - K \int_E \rho^2 \operatorname{div} \eta \right) \\
&\quad + o(t),
\end{aligned}$$

where for the first equality we used that $\nabla\varphi_t = \left(\nabla\varphi_t^{-1}\right)^{-1} \circ \varphi_t$. Bringing it all together we get that for any $\eta \in C_c^\infty(B_R; \mathbb{R}^n)$ with $\int_E \operatorname{div}\eta \, dx = 0$ for any t we have

$$\begin{aligned}
P(E) + Q^2 \left\{ \int_{\mathbb{R}^n} a_E |\nabla u|^2 \, dx + K \int_E \rho^2 \, dx \right\} &\le P(E) + Q^2 \left\{ \int_{\mathbb{R}^n} a_E |\nabla u|^2 + K \int_E \rho^2 \right\} \\
+ t \left(\int_{\partial^* E} \operatorname{div}_E \eta + Q^2 \left(\int_{\mathbb{R}^n} a_E \left(-\operatorname{div} \eta |\nabla u|^2 + 2\nabla u \cdot (\nabla\eta \nabla u) \right) - K \int_E \rho^2 \operatorname{div} \eta \right) \right) &+ o(t),
\end{aligned}$$

which gives us (ii). □

We state now the ε-regularity theorem.

Theorem 2.4 ([DPHV19, Theorem 1.2])**.** *Given $n \ge 3$, $A > 0$ and $\vartheta \in (0, 1/2)$, there exists $\varepsilon_{\mathrm{reg}} = \varepsilon_{\mathrm{reg}}(n, A, \vartheta) > 0$ such that if E is minimizer of $(\mathcal{P}_{\beta,K,Q,R})$ with $Q + \beta + K + \frac{1}{K} \le A$, $x \in \partial E$ and*

$$r + \mathbf{e}_E(x, r) + Q^2\, D_E(x, r) \le \varepsilon_{\mathrm{reg}},$$

then $E \cap \mathbf{C}(x, r/2)$ coincides with the epi-graph of a $C^{1,\vartheta}$ function. In particular, $\partial E \cap \mathbf{C}(x, r/2)$ is a $C^{1,\vartheta}$ $(n-1)$-dimensional manifold.

3. Closeness to the ball

In this section we deduce the L^∞-closeness of minimizers to the unitary ball in the small charge regime. Let us start with the following proposition.

Proposition 3.1 (L^1-closeness to the ball)**.** *Let $\{Q_h\}_{h\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ such that $Q_h > 0$ and $Q_h \to 0$ when $h \to \infty$. Let $\{E_h\}_{h\in\mathbb{N}}$ be a sequence of minimizers of $(\mathcal{P}_{\beta,K,Q_h,R})$. Then, up to translations, $E_h \to B_1$ in L^1 and $P(E_h) \to P(B_1)$ when $h \to \infty$.*

Proof. By the quantitative isoperimetric inequality, [FMP08, Theorem 1.1], for every $h \in \mathbb{N}$ there exists a point $x_h \in \mathbb{R}^n$ such that

$$|E_h \Delta B_1(x_h)|^2 \le C\,(P(E_h) - P(B_1))$$

for some constant $C = C(n) > 0$ which depends only on n. By translating each set E_h we can assume without loss of generality that the following inequality holds:

$$|E_h \Delta B_1|^2 \le C\,(P(E_h) - P(B_1))\,. \tag{3.1}$$

By the minimality of E_h we have

$$\begin{aligned}\mathcal{F}_{\beta,K,Q_h,R}(E_h) &= P(E_h) + Q_h^2\,\mathcal{G}_{\beta,K}(E_h)\\ &\le P(B_1) + Q_h^2\,\mathcal{G}_{\beta,K}(B_1) = \mathcal{F}_{\beta,K,Q_h,R}(B_1)\,, \quad \forall\, h \in \mathbb{N}.\end{aligned}$$

Hence, (3.1) yields

$$|E_h \Delta B_1|^2 \le C\,(P(E_h) - P(B_1)) \le C\,Q_h^2\,\mathcal{G}_{\beta,K}(B_1) \quad \forall\, h \in \mathbb{N},$$

for some constant $C = C(n) > 0$ which depends only on the dimension n.

Then $Q_h \to 0$ implies $E_h \to B_1$ in L^1 and $P(E_h) \to P(B_1)$ when $h \to \infty$. □

Thanks to the density estimates (see Theorem 2.3 (v)), we can improve the convergence of Proposition 3.1.

Proposition 3.2 (L^∞-closeness to the ball)**.** *Let $\{Q_h\}_{h\in\mathbb{N}}$ be a sequence such that $Q_h > 0$ and $Q_h \to 0$ when $h \to \infty$. Let $\{E_h\}_{h\in\mathbb{N}}$ be a sequence of minimizers of $(\mathcal{P}_{\beta,K,Q_h,R})$. Then, up to translations, $E_h \to \overline{B}_1$ and $\partial E_h \to \partial B_1$ in the Kuratowski sense.*

Proof. By Proposition 3.1 we know that up to translations $E_h \to B_1$ in L^1. First, we prove the Kuratowski convergence of E_h to the ball $\overline{B}_1$, i.e.

(i) $x_h \to x$, $x_h \in E_h \;\Rightarrow\; x \in \overline{B}_1$,

(ii) $x \in \overline{B}_1 \Rightarrow \exists x_h \in E_h$ such that $x_h \to x$.

In order to prove (i) let $x_h \to x$ and $x_h \in E_h$. Assume by contradiction that $x \notin \overline{B}_1$. Then there exits $B_s(x) \subset \mathbb{R}^n$ such that $B_s(x) \cap B_1 = \emptyset$. By Theorem 2.3 (v), for Q_h small enough there exist a radius $\bar{r} > 0$ and a constant $C > 0$, both independent of Q_h, such that

$$|B_r(x_h) \cap E_h| \ge C\,r^n \quad \forall r \le \bar{r}. \tag{3.2}$$

Since $x_h \to x$, for any $r > 0$ we can define $h(r) \in \mathbb{N}$ such that $B_{\frac{r}{2}}(x_h) \subset B_r(x)$ for every $h \ge h(r)$. Then, for any $r \le \overline{r}$, $h \ge h(r)$,

$$|B_r(x) \cap E_h| \ge |B_{\frac{r}{2}}(x_h) \cap E_h| \ge C\,r^n. \tag{3.3}$$

By the L^1-convergence of E_h to B_1 and (3.3) we deduce $|B_r(x) \cap B_1| > 0$ for any $r \le \overline{r}$, a contradiction with $B_s(x) \cap B_1 = \emptyset$.

The proof of (ii) follows by arguing similarly as above, exploiting the L^1-convergence. Analogously, by using density estimates for the perimeter of E_h and the convergence of perimeters $P(E_h) \to P(B_1)$, one can prove that $\partial E_h \to \partial B_1$ in the Kuratowski sense. □

4. Higher regularity

In this section we improve Theorem 2.4. To be more precise, we deduce the partial $C^{2,\vartheta}$ regularity of minimizers. The first step is to obtain better regularity for a couple $(u, \rho) \in \mathcal{A}(E)$, where $E \subset \mathbb{R}^n$ is a minimizer of the problem $(\mathcal{P}_{\beta,K,Q,R})$: we prove that u is $C^{1,\eta}$-regular up to the boundary of E. We start with some preliminary results.

Notation 4.1. Let $E \subset \mathbb{R}^n$ be such that $\partial E \cap \mathbf{C}(x_0, r)$ is described by the graph of a regular function f.

- If $x \in \mathbb{R}^n$, we write $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$.
- We denote by ν_E the outer-unit normal to ∂E. Moreover, we extend ν_E at every point in the following way
$$\nu_E(x', x_n) = \nu_E(x', f(x')) \quad \forall x = (x', x_n) \in \mathbf{C}(x_0, r).$$
- Let u be a solution of
$$-\mathrm{div}(a_E \nabla u) = \rho_E \quad \text{in } \mathcal{D}'\left(B_r(x_0)\right),$$
where
$$\rho_E \in L^\infty\left(B_r(x_0)\right) \quad \text{and} \quad a_E = \beta \mathbf{1}_E + \mathbf{1}_{E^c}.$$
We denote by
$$T_E u := \partial_{\nu_E^\perp} u + (1 + (\beta - 1)\mathbf{1}_E)\partial_{\nu_E} u,$$
where
$$\partial_{\nu_E^\perp} u := \nabla u - (\nabla u \cdot \nu_E)\, \nu_E \quad \text{and} \quad \partial_{\nu_E} u := (\nabla u \cdot \nu_E)\, \nu_E.$$
- We denote by
$$[g]_{x,r} := \frac{1}{|B_r|} \int_{B_r(x)} g\, dx$$
the *mean value* of $g \in L^1(B_r(x))$. We simply write $[g]_r := [g]_{0,r}$.
- We denote the restrictions of a function v to E and E^c by v^+ and v^- respectively:
$$v^+ := v\, \mathbf{1}_E, \quad v^- := v\, \mathbf{1}_{E^c}.$$

Let us recall the following integral characterization of Hölder continuous functions.

Lemma 4.2 (Campanato's lemma, see [AFP, Theorem 7.51])**.** *Let $p \geq 1$ and $g \in L^p(B_{2R}(x_0))$. Assume that there exist $\sigma \in (0,1)$ and $A > 0$ such that for every $x \in B_R(x_0)$*

$$\frac{1}{|B_r|} \int_{B_r(x)} |g(y) - [g]_{x,r}|^p\, dy \leq A^p \left(\frac{r}{R}\right)^{p\sigma}, \quad \forall B_r(x) \subset B_R(x_0). \tag{4.1}$$

Then there exists a constant $C = C(n, p, \sigma)$ such that g is σ-Hölder continuous in $B_R(x_0)$ with a constant $C\frac{A}{R^\sigma}$ and

$$\max_{x \in B_R(x_0)} |g(x)| \leq CA + |[g]_{x_0,R}|.$$

We also recall a simple iteration lemma.

Lemma 4.3 ([AFP, Lemma 7.54])**.** *Let $0 < q < p$, $s > 0$. Suppose that $h : (0, a) \to [0, +\infty)$ is an increasing function such that*

$$h(r) \leq c_1 \left(\left(\frac{r}{R}\right)^p + R^s\right) h(R) + c_2\, R^q \quad \textit{for every } 0 < r < R \leq a,$$

where c_1 and c_2 are positive constants. Then there exists $R_0 = R_0(p, q, s, c_1) > 0$, $c = c(p, q, s) > 0$ such that

$$h(r) \leq c \left\{\left(\frac{r}{R}\right)^q h(R) + c_2\, r^q\right\} \quad \textit{for every } 0 < r < R \leq \min(R_0, a).$$

We are going to use the following lemma.

Lemma 4.4 ([AFP, Theorem 7.53]). *Let v be a solution of*

$$-\mathrm{div}(a_H \nabla v) = \rho_H \quad \text{in } \mathcal{D}'(B_1(x_0)),$$

where $\rho_H \in L^\infty(B_1(x_0))$ and

$$H := \{y \in \mathbb{R}^n \,:\, (y - x_0)\cdot e_n \le 0\}, \qquad a_H = \beta \mathbf{1}_H + \mathbf{1}_{H^c}.$$

Then there exist $\gamma \in (0,1)$ and a constant $C_0 = C_0(n, \beta, \|\rho_H\|_\infty) > 0$ such that

$$\int_{B_{\lambda r}(x_0)} |T_H v - [T_H v]_{x_0,\lambda r}|^2\,dx \le C_0 \lambda^{n+2\gamma} \int_{B_r(x_0)} |T_H v - [T_H v]_{x_0,r}|^2\,dx + C_0\, r^{n+1},$$

for all $\lambda \in (0,1)$ small enough. Note that $T_H v := \big(\partial_1 v, \ldots, \partial_{n-1} v, (1 + (\beta-1)\mathbf{1}_H)\partial_n v\big)$.

We argue similarly to the proof of Theorem 7.53 in [AFP] to show the following lemma.

Lemma 4.5. *Let $H \subset \mathbb{R}^n$ be the half space. Let $v \in W^{1,2}(B_1)$ be a solution of*

$$-\mathrm{div}(A\nabla v) = \mathrm{div}\, G \quad \text{in } \mathcal{D}'(B_1), \tag{4.2}$$

where

$$G^+ := G\,\mathbf{1}_H \in C^{0,\alpha}(H), \quad G^- := G\,\mathbf{1}_{H^c} \in C^{0,\alpha}(H^c),$$

A is an elliptic matrix and $A^+ = A\,\mathbf{1}_H$, $A^- = A\,\mathbf{1}_{H^c}$ have coefficients respectively in $C^{0,\alpha}(B_r \cap \overline{H})$ and $C^{0,\alpha}(B_1 \cap \overline{H^c})$. Then

$$v^+ := v\,\mathbf{1}_H \in C^{1,\alpha}(B_{1/2} \cap \overline{H}), \quad v^- := v\,\mathbf{1}_{H^c} \in C^{1,\alpha}(B_{1/2} \cap \overline{H^c}).$$

Moreover, there exists a constant $C = C\left(\|G^+\|_{C^{0,\alpha}}, \|G^-\|_{C^{0,\alpha}}, \|A^+\|_{C^{0,\alpha}}, \|A^-\|_{C^{0,\alpha}}\right) > 0$ such that

$$[\nabla v^+]_{C^{0,\alpha}(\overline{H}\cap B_{1/2})} \le C \quad \text{and} \quad [\nabla v^-]_{C^{0,\alpha}(\overline{H^c}\cap B_{1/2})} \le C. \tag{4.3}$$

Proof. Fix $x_0 \in B_{1/2}$, and let r be such that $B_r(x_0) \subset B_1$. We denote by a^+ and a^- the averages of A in $B_r(x_0) \cap H$ and $B_r(x_0) \cap H^c$ respectively. In an analogous way we define g^+ and g^- as the averages of G in $B_r(x_0) \cap H$ and $B_r(x_0) \cap H^c$. For $x \in B_r(x_0)$ we set

$$\overline{A} := \begin{cases} a^+ \text{ if } x_n > 0 \\ a^- \text{ if } x_n < 0 \end{cases} \quad \text{and} \quad \overline{G} := \begin{cases} g^+ \text{ if } x_n > 0 \\ g^- \text{ if } x_n < 0 \end{cases}.$$

By the assumptions of the lemma,

$$|A(x) - \overline{A}(x)| \le cr^\alpha \quad \text{and} \quad |G(x) - \overline{G}(x)| \le cr^\alpha. \tag{4.4}$$

Let w be the solution of

$$\begin{cases} -\mathrm{div}(\overline{A}\nabla w) = \mathrm{div}\,\overline{G} \text{ in } B_r, \\ w = v \text{ on } \partial B_r(x_0). \end{cases}$$

Note that the last equation can be rewritten as

$$\begin{cases} -\mathrm{div}(a^+\nabla w^+) = 0 \text{ in } H \cap B_r(x_0), \\ -\mathrm{div}(a^-\nabla w^-) = 0 \text{ in } H^c \cap B_r(x_0), \\ w^+ = w^- \text{ on } \partial H \cap B_r(x_0), \\ a^+\nabla w^+ \cdot e_n - a^-\nabla w^- \cdot e_n = g^+\cdot e_n - g^- \cdot e_n \text{ on } \partial H \cap B_r(x_0), \\ w = v \text{ on } \partial B_r(x_0), \end{cases} \tag{4.5}$$

where $w^+ := w\,\mathbf{1}_{H\cap B_r(x_0)}$, $w^- := w\,\mathbf{1}_{H^c\cap B_r(x_0)}$. For a function u set

$$\overline{D}_c u(x) = \sum_{i=1}^n \overline{A}_{i,n} \nabla_i u(x) + \overline{G}\cdot e_n; \tag{4.6}$$

$$D_c u(x) = \sum_{i=1}^{n} A_{i,n} \nabla_i u(x) + G \cdot e_n. \tag{4.7}$$

The reason for such a definition is that $D_c v$ and $\overline{D}_c w$ have no jumps on the boundary thanks to the transmission condition in (4.5). We are going to estimate the decay of $D_\tau w$ and $\overline{D}_c w$, which will lead to Hölder continuity of $D_\tau v$ and $D_c v$, yielding the desired estimate on ∇v.

Step 1: tangential derivatives of w. Since both $\overline{A}$ and $\overline{G}$ are constant along the tangential directions, the classical difference quotient method (see, for example, [GM12, Section 4.3]) gives us that $D_\tau w \in W^{1,2}_{loc}(B_r(x_0))$ and $\operatorname{div}(\overline{A}\nabla(D_\tau w)) = 0$ in $B_r(x_0)$. Hence, Caccioppoli's inequality holds:

$$\int_{B_\rho(x)} |\nabla(D_\tau w)|^2 dy \le C\rho^{-2} \int_{B_{2\rho}(x)} |D_\tau w - (D_\tau w)_{x,2\rho}|^2 dy \tag{4.8}$$

for all balls $B_{2\rho}(x) \subset B_r(x_0)$ and by De Giorgi's regularity theorem (see, for example, [AFP, Theorem 7.50]), $D_\tau w$ is Hölder-continuous and, thus, if $B_{\rho'}(x) \subset B_r(x_0)$,

$$\int_{B_\rho(x)} |D_\tau w - (D_\tau w)_{x,\rho}|^2 dy \le c \left(\frac{\rho}{\rho'}\right)^{n+2\gamma} \int_{B_{\rho'}(x)} \left|D_\tau w - (D_\tau w)_{x,\rho'}\right|^2 dy \tag{4.9}$$

for any $\rho \in (0, \rho'/2)$ and

$$\max_{B_{\rho'/2}(x)} |D_\tau w|^2 \le \frac{C}{(\rho')^n} \int_{B_{\rho'}(x)} |D_\tau w|^2 dy. \tag{4.10}$$

Step 2: regularity of $\overline{D}_c w$. First let us show that the distributional gradient of $\overline{D}_c w$ is given by the gradient of $\overline{D}_c w$ on the upper half ball plus the one on the lower, i.e. that there is no contribution on the hyperplane. For that, we need to check that

$$-\int_{B_r(x_0)} \overline{D}_c w \operatorname{div} \varphi \, dx = \int_{B_r(x_0)^+} \nabla \overline{D}_c w \cdot \varphi \, dx + \int_{B_r(x_0)^-} \nabla \overline{D}_c w \cdot \varphi \, dx$$

for any $\varphi \in C_c^\infty(B_r(x_0); \mathbb{R}^n)$. Indeed, if we perform integration by parts on the left hand side, we get

$$\begin{aligned} -\int_{B_r(x_0)} \overline{D}_c w \operatorname{div} \varphi \, dx &= \int_{B_r(x_0)^+} \nabla \overline{D}_c w \cdot \varphi \, dx + \int_{B_r(x_0)^-} \nabla \overline{D}_c w \cdot \varphi \, dx \\ &+ \int_{\partial H \cap B_r(x_0)} \left(\sum_{i=1}^{n} a^+_{i,n} \nabla_i w(x) + g^+ \cdot e_n - \sum_{i=1}^{n} a^-_{i,n} \nabla_i w(x) - g^- \cdot e_n \right) (\varphi \cdot e_n) \, d\mathcal{H}^{n-1} \end{aligned}$$

for any $\varphi \in C_c^\infty(B_r(x_0); \mathbb{R}^n)$ and the last term vanishes thanks to the transmission condition in (4.5). Thus, the distributional gradient of $\overline{D}_c w$ coincides with the point-wise one.

Since $D_\tau(\overline{D}_c w) = \overline{D}_c(D_\tau w) - \overline{G} \cdot e_n$, the tangential derivatives of $\overline{D}_c w$ are in L^2_{loc}. As for the normal derivative, by the definition (4.6)

$$\left| \frac{\partial \overline{D}_c w}{\partial \nu}(x) \right| \le C|\nabla D_\tau w| + 2\|\overline{G}\|_{L^\infty}.$$

It implies

$$\left|\nabla \overline{D}_c w(x)\right| \le C \left(|\nabla D_\tau w| + \|\overline{G}\|_{L^\infty}\right).$$

and thus $\overline{D}_c w$ is in $W^{1,2}_{loc}$. Now, using Poincaré's inequality and (4.8), we have

$$\int_{B_\rho(x)} \left|\overline{D}_c w - (\overline{D}_c w)_{x,\rho}\right|^2 dy \leq C\rho^2 \int_{B_\rho(x)} \left|\nabla(\overline{D}_c w)\right|^2 dy$$
$$\leq C\rho^2 \int_{B_\rho(x)} |\nabla(D_\tau w)|^2 dy + C\rho^{n+2} \leq C \int_{B_{2\rho}(x)} |D_\tau w - (D_\tau w)_{x,2\rho}|^2 dy + C\rho^{n+2}$$

for any $B_{2\rho}(x) \subset B_r(x_0)$. Remembering (4.9), we obtain

$$\int_{B_\rho(x)} \left|\overline{D}_c w - (\overline{D}_c w)_{x,\rho}\right|^2 dy \leq C \left(\frac{\rho}{r}\right)^{n+2\gamma} \int_{B_{r/2}(x)} \left|D_\tau w - (D_\tau w)_{x,r/2}\right|^2 dy + C\rho^{n+2}$$
$$\leq C \left(\frac{\rho}{r}\right)^{n+2\gamma} \int_{B_r(x_0)} |D_\tau w|^2 dy + C\rho^{n+2} \tag{4.11}$$

for any $x \in B_{r/4}(x_0)$, $\rho \leq r/4$. Hence, by Lemma 4.2, $\overline{D}_c w$ is Hölder-continuous and

$$\max_{B_{r/4}(x_0)} \left|\overline{D}_c w\right|^2 \leq \frac{C}{r^n} \int_{B_r(x_0)} \left|\overline{D}_c w\right|^2 dy + C. \tag{4.12}$$

Step 3: compairing v and w. Subtracting the equation for w from the equation for v we get

$$\int_{B_r(x_0)} \overline{A}_{i,j}(y) \left(\frac{\partial v}{\partial y_i} - \frac{\partial w}{\partial y_i}\right) \frac{\partial \varphi}{\partial y_j} dy = \int_{B_r(x_0)} \left(\overline{A}_{i,j}(y) - A_{i,j}(y)\right) \frac{\partial v}{\partial y_i}\frac{\partial \varphi}{\partial y_j} dy$$
$$+ \int_{B_r(x_0)} \left(\overline{G}_i - G_i\right) \frac{\partial \varphi}{\partial y_i} dy \tag{4.13}$$

for any $\varphi \in W^{1,2}_0(B_r(x_0))$. We test (4.13) with $\varphi = v - w$ to get

$$\int_{B_r(x_0)} |\nabla v - \nabla w|^2 dy \leq Cr^{2\alpha} \int_{B_r(x_0)} |\nabla v|^2 dy + Cr^{n+2\alpha}, \tag{4.14}$$

which in turn gives us

$$\int_{B_\rho(x_0)} |\nabla v|^2 dy \leq 2\int_{B_\rho(x_0)} |\nabla w|^2 dy + 2\int_{B_\rho(x_0)} |\nabla v - \nabla w|^2 dy$$
$$\leq 2\omega_n \rho^n \sup_{B_{r/4}(x_0)} |\nabla w|^2 + Cr^{2\alpha} \int_{B_r(x_0)} |\nabla v|^2 dy + Cr^{n+2\alpha}$$

for $\rho \leq r/4$. Recalling (4.10) and (4.12), we obtain

$$\int_{B_\rho(x_0)} |\nabla v|^2 dy \leq C\left(\frac{\rho}{r}\right)^n \int_{B_r(x_0)} |\nabla w|^2 dy + C\rho^n + Cr^{2\alpha}\int_{B_r(x_0)} |\nabla v|^2 dy + Cr^{n+2\alpha}$$
$$\leq C\left(\frac{\rho}{r}\right)^n \int_{B_r(x_0)} |\nabla v|^2 dy + Cr^{2\alpha}\int_{B_r(x_0)} |\nabla v|^2 dy + Cr^n.$$

Now we can apply Lemma 4.3 and get that there exists $r_0 > 0$ such that for $\rho < r/4 < r_0$

$$\int_{B_\rho(x_0)} |\nabla v|^2 dy \leq C\left(\frac{\rho}{r}\right)^{n-\alpha} \int_{B_r(x_0)} |\nabla v|^2 dy + C\rho^{n-\alpha}.$$

In particular, for $\rho < r_0$ we have

$$\int_{B_\rho(x_0)} |\nabla v|^2 dy \leq C\rho^{n-\alpha}, \tag{4.15}$$

where $C = C\left(\|G^+\|_{C^{0,\alpha}}, \|G^-\|_{C^{0,\alpha}}, \|A^+\|_{C^{0,\alpha}}, \|A^-\|_{C^{0,\alpha}}\right)$. Note that the L^2 norm of ∇v in B_1 is bounded by some constant depending only on L^∞ norms of A and G, as can be seen by testing the equation (4.2) with v.

Step 4: Hölder-continuity of ∇v. We show local Hölder continuity of $D_c v$ and $D_\tau v$, Hölder-continuity of ∇v in $B_{1/2} \cap \overline{H}$ and in $B_{1/2} \cap \overline{H^c}$ follows immediately.

Take $\rho < r_0$, where r_0 is from the previous step. Let d be any real number. Using the definitions (4.6) and (4.7), we get

(4.16)
$$\begin{aligned}
&\int_{B_\rho(x_0)} |D_c v - d|^2 dy = \int_{B_\rho(x_0)} \left| \overline{D}_c v - d + \sum_{i=1}^n (A_{i,n} - \overline{A}_{i,n}) \nabla_i v + (G - \overline{G}) e_n \right|^2 dy \\
&\leq 2 \int_{B_\rho(x_0)} |\overline{D}_c v - d|^2 dy + 4 \int_{B_\rho(x_0)} \left| \sum_{i=1}^n (A_{i,n} - \overline{A}_{i,n}) \nabla_i v \right|^2 dy + 4 \int_{B_\rho(x_0)} \left|(G - \overline{G}) e_n\right|^2 dy \\
&\leq 2 \int_{B_\rho(x_0)} |\overline{D}_c w - d + \sum_{i=1}^n \overline{A}_{i,n} (\nabla_i v - \nabla_i w)|^2 dy + 4 \int_{B_\rho(x_0)} C r^{2\alpha} |\nabla v|^2 dy + 4 \int_{B_\rho(x_0)} C r^{2\alpha} dy \\
&\leq 4 \int_{B_\rho(x_0)} |\overline{D}_c w - d|^2 dy + C r^{n+\alpha},
\end{aligned}$$

where we used inequalities (4.4) for the second to last inequality, and inequalities (4.15) and (4.14) for the last inequality. Thus, using (4.11) we have for $\rho < r/4$, $r < r_0$

(4.17)
$$\begin{aligned}
&\int_{B_\rho(x_0)} |D_c v - (D_c v)_{x_0,\rho}|^2 dy \leq \int_{B_\rho(x_0)} |D_c v - (\overline{D}_c w)_{x_0,\rho}|^2 dy \\
&\leq 4 \int_{B_\rho(x_0)} |\overline{D}_c w - (\overline{D}_c w)_{x_0,\rho}|^2 dy + C r^{n+\alpha} \leq C \left(\frac{\rho}{r}\right)^{n+2\gamma} \int_{B_r(x_0)} |D_\tau w|^2 dy + C r^{n+\alpha},
\end{aligned}$$

where we used the fact that $\int_\Omega (f(x) - t)\, dx$ is minimized by $t^* = \fint_\Omega f$ for the first inequality and the inequality (4.16) with $d = (\overline{D}_c w)_{x_0,\rho}$ for the second inequality. Similarly, using (4.9) instead of (4.11), we get

$$\int_{B_\rho(x_0)} |D_\tau v - (D_\tau v)_{x_0,\rho}|^2 dy \leq C \left(\frac{\rho}{r}\right)^{n+2\gamma} \int_{B_r(x_0)} |D_\tau w|^2 dy + C r^{n+\alpha}. \tag{4.18}$$

Applying Lemma 4.3 to (4.17) and (4.18), we deduce that $D_c v$ and $D_\tau v$ are Hölder by Lemma 4.2.

□

Lemma 4.6. *Given a minimizer E of $(\mathcal{P}_{\beta,K,Q,R})$, let $(u, \rho) \in \mathcal{A}(E)$ be the minimizing pair of $\mathcal{G}_{\beta,K}(E)$. Assume that $\partial E \cap \mathbf{C}(x_0, r)$ is a $C^{1,\vartheta}$-manifold. Then for every $\gamma \in (0,1)$ there exist $0 < \bar{r} \leq r$ and $C > 0$ such that the following inequality holds true*

$$Q^2 \int_{B_{\tilde{r}}(x_0)} |\nabla u|^2 \, dx \leq C\, \tilde{r}^{n-\gamma}$$

for every $\tilde{r} \leq \bar{r}$.

Proof. Fix $\gamma \in (0,1)$. Choose $\lambda \in (0, 1/4)$ such that

$$(1 + C_{\text{dec}})\, \lambda \leq \lambda^{1-\gamma},$$

where C_{dec} is as in Theorem 2.3 (vi). Let $s = s(\lambda) < \frac{1}{2}$ be such that

$$C_{\text{dir}}(C_{\text{e}} + 1)\, s(\lambda) \le \frac{\varepsilon_{\text{dec}}(\lambda)}{2}, \tag{4.19}$$

where ε_{dec} , C_{dir} and C_{e} are as in Theorem 2.4 and Theorem 2.3 (vii), (iv). Define

$$\varepsilon(\lambda) := \min\left\{ s^{n-1}\frac{\varepsilon_{\text{dec}}(\lambda)}{2}, \varepsilon_{\text{dir}}(\lambda)\right\}.$$

Since $\partial E \cap \mathbf{C}(x_0, r)$ is regular, we can take a radius $0 < \bar r < \min\left(r, 1, \frac{1}{Q^2}\right)$ such that

$$\bar r + \mathbf{e}_E(x_0, \bar r) \le \varepsilon(\lambda).$$

Then, thanks to the definition of $\varepsilon(\lambda)$, Theorem 2.3 (vii), (iv), and (4.19) we have

$$Q^2\, D_E(x_0, s\bar r) \le C_{\text{dir}} s\, (Q^2\, D_E(x_0, \bar r) + Q^2 \bar r) \le C_{\text{dir}}(C_{\text{e}} + 1)s \le \frac{\varepsilon_{\text{dec}}(\lambda)}{2}. \tag{4.20}$$

Furthermore, notice that

$$s\bar r + \mathbf{e}_E(x_0, s\bar r) \le \bar r + \frac{1}{s^{n-1}}\mathbf{e}_E(x_0, \bar r) \le \frac{\varepsilon_{\text{dec}}(\lambda)}{2}. \tag{4.21}$$

Combining (4.20) and (4.21), we have

$$s\bar r + Q^2\, D_E(x_0, s\bar r) + \mathbf{e}_E(x_0, s\bar r) \le \varepsilon_{\text{dec}}(\lambda).$$

The hypothesis of Theorem 2.3 (vi) is satisfied, hence (recall that $\lambda s\bar r \le \varepsilon_{\text{dec}}(\lambda)$)

$$\begin{aligned} &Q^2\, D_E(x_0, \lambda s\bar r) + \mathbf{e}_E(x_0, \lambda s\bar r) + \lambda s\bar r \le \lambda^{1-\gamma}\, \left(\mathbf{e}_E(x_0, s\bar r) + Q^2\, D_E(x_0, s\bar r) + s\bar r\right) \\ &\le \lambda^{1-\gamma}\varepsilon_{\text{dec}}(\lambda) \le \varepsilon_{\text{dec}}(\lambda). \end{aligned}$$

Exploiting again Theorem 2.3, we obtain

$$\begin{aligned} Q^2\, D_E(x_0, \lambda^2 s\bar r) + \mathbf{e}_E(x_0, \lambda^2 s\bar r) + \lambda^2 s\bar r &\le \lambda^{(1-\gamma)}\, \left(\mathbf{e}_E(x_0, \lambda s\bar r) + Q^2\, D_E(x_0, \lambda s\bar r) + \lambda s\bar r\right) \\ &\le \lambda^{2(1-\gamma)}\, \left(\mathbf{e}_E(x_0, s\bar r) + Q^2\, D_E(x_0, s\bar r) + s\bar r\right) \\ &\le \lambda^{2(1-\gamma)}\varepsilon_{\text{dec}}(\lambda) \le \varepsilon_{\text{dec}}(\lambda). \end{aligned}$$

Iterating this argument k times, we conclude that

$$Q^2\, D_E(x_0, \lambda^k s\bar r) + \mathbf{e}_E(x_0, \lambda^k s\bar r) + \lambda^k s\bar r \le \lambda^{k(1-\gamma)}\varepsilon_{\text{dec}}(\lambda), \quad \forall k \in \mathbb{N}.$$

In particular, the inequality above yields

$$Q^2\, D_E(x_0, \lambda^k s\bar r) \le \lambda^{k(1-\gamma)}\varepsilon_{\text{dec}}(\lambda), \quad \forall k \in \mathbb{N}.$$

Therefore,

$$Q^2 \int_{B_{\lambda^k s\bar r}(x_0)} |\nabla u|^2\, dx \le C\, (\lambda^k s\bar r)^{(n-\gamma)}, \quad \forall k \in \mathbb{N}$$

for some constant $C > 0$. Now if we take any $\tilde r \le \lambda s\bar r$, there exists an integer $k > 0$ such that $\lambda^{k+1}s\bar r < \tilde r \le \lambda^k s\bar r$, hence

$$Q^2 \int_{B_{\tilde r}(x_0)} |\nabla u|^2\, dx \le Q^2 \int_{B_{\lambda^k s\bar r}(x_0)} |\nabla u|^2\, dx \le C\, (\lambda^k s\bar r)^{(n-\gamma)} \le \frac{C}{\lambda^{n-\gamma}}\, \tilde r^{(n-\gamma)}.$$

□

Proposition 4.7. *Let E be a minimizer of $(\mathcal{P}_{\beta,K,Q,R})$, let $(u,\rho)\in\mathcal{A}(E)$ be the minimizing pair of $\mathcal{G}_{\beta,K}(E)$, $x_0\in\partial E$, and $f\in C^{1,\vartheta}(\mathbf{D}(x_0',r))$. Suppose that $Q\le 1$ and*

$$E\cap\mathbf{C}(x_0,r)=\big\{x=(x',x_n)\in\mathbf{D}(x_0',r)\times\mathbb{R}\,:\,x_n<f(x')\big\}\cap\mathbf{C}(x_0,r),$$

for some $0<r\le\min\{\bar r,1\}$, where $\bar r$ is as in Lemma 4.6. Then there exist $\alpha=\alpha(\vartheta)\in(0,1)$ and a constant $C=C(n,\beta,\vartheta,\|\rho\|_\infty)>0$ such that

$$Q^2\int_{B_{\lambda r}(x_0)}|T_Eu-[T_Eu]_{x_0,\lambda r}|^2\,dx\le C\,Q^2\lambda^{n+2\alpha}\int_{B_r(x_0)}|T_Eu-[T_Eu]_{x_0,r}|^2\,dx+C\,r^{n+\alpha}. \tag{4.22}$$

Proof. Without loss of generality assume $0\in\partial E$, $x_0=0$. Let $\lambda\in(0,1/2)$ be given and let v be the solution of

$$\begin{cases}-\mathrm{div}(a_H\nabla v)=\rho & \text{in } B_{r/2},\\ v=u & \text{on }\partial B_{r/2},\end{cases}$$

where H is the half-space $\{x=(x',x_n):x_n<0\}$. In particular, $w=v-u\in W_0^{1,2}(B_{r/2})$ and

$$-\mathrm{div}(a_H\nabla w)=-\mathrm{div}\left((a_E-a_H)\nabla u\right). \tag{4.23}$$

Since $[T_Eg]_s$ minimizes the functional $m\mapsto\int_{B_s}|T_Eg-m|^2\,dx$, we have

$$\begin{aligned}\int_{B_{\lambda r}}|T_Eu-[T_Eu]_{\lambda r}|^2\,dx&\le\int_{B_{\lambda r}}|T_Eu-[T_Hu]_{\lambda r}|^2\,dx\\&\le 2\left(\int_{B_{\lambda r}}|T_Hu-[T_Hu]_{\lambda r}|^2\,dx+\int_{B_{\lambda r}}|T_Eu-T_Hu|^2\,dx\right).\end{aligned} \tag{4.24}$$

We want now to estimate the first term in the right hand side of (4.24). Notice that, since $u=v-w$, by linearity of T_H we have

$$|T_Hu-[T_Hu]_{\lambda r}|^2\le 2\left(|T_Hv-[T_Hv]_{\lambda r}|^2+|T_Hw-[T_Hw]_{\lambda r}|^2\right).$$

Hence, integrating the above inequality on $B_{\lambda r}$ we obtain

$$\begin{aligned}\int_{B_{\lambda r}}|T_Hu-[T_Hu]_{\lambda r}|^2\,dx&\le 2\left(\int_{B_{\lambda r}}|T_Hv-[T_Hv]_{\lambda r}|^2\,dx+\int_{B_{\lambda r}}|T_Hw-[T_Hw]_{\lambda r}|^2\,dx\right)\\&\le 2\left(\int_{B_{\lambda r}}|T_Hv-[T_Hv]_{\lambda r}|^2\,dx+\int_{B_{\lambda r}}|T_Hw|^2\,dx\right)\\&\le C\,\left(\int_{B_{\lambda r}}|\nabla w|^2\,dx+\int_{B_{\lambda r}}|T_Hv-[T_Hv]_{\lambda r}|^2\,dx\right).\end{aligned} \tag{4.25}$$

To estimate the second term in the right hand side of (4.24), recall the Notation 4.1

$$\partial_{\nu_E^\perp}u=\nabla u-(\nabla u\cdot\nu_E)\,\nu_E\quad\text{and}\quad\partial_{e_n^\perp}u=\nabla u-(\nabla u\cdot e_n)e_n.$$

Hence,

$$
\begin{aligned}
|T_E u - T_H u| &= |\nabla u - (\nabla u\cdot\nu_E)\nu_E + (1+(\beta-1)\mathbf{1}_E)(\nabla u\cdot\nu_E)\,\nu_E \\
&\qquad - (\nabla u - (\nabla u\cdot e_n)e_n + (1+(\beta-1)\mathbf{1}_H)(\nabla u\cdot e_n)\,e_n)\,| \\
&= |(\nabla u\cdot e_n)e_n - (\nabla u\cdot\nu_E)\nu_E + (1+(\beta-1)\mathbf{1}_E)(\nabla u\cdot\nu_E)\,\nu_E - ((1+(\beta-1)\mathbf{1}_H)(\nabla u\cdot e_n)\,e_n)\,| \\
&\le (1+\beta)|(\nabla u\cdot e_n)e_n - (\nabla u\cdot\nu_E)\nu_E| + \big|\big((1+(\beta-1)\mathbf{1}_E) - (1+(\beta-1)\mathbf{1}_H)\big)(\nabla u\cdot e_n)\,e_n\big| \\
&= (1+\beta)\big|\big((\nabla u\cdot e_n) - (\nabla u\cdot\nu_E)\big)e_n + (\nabla u\cdot\nu_E)(e_n-\nu_E)\big| + (\beta-1)\mathbf{1}_{E\Delta H}|\nabla u\cdot e_n| \\
&\le \big(2(1+\beta)|\nu_E - e_n| + (\beta-1)\mathbf{1}_{E\Delta H}\big)|\nabla u|.
\end{aligned}
$$

Therefore,

$$
\int_{B_{\lambda r}} |T_E u - T_H u|^2\,dx \le C\left(\int_{B_{\lambda r}} |\nabla u|^2\,|\nu_E - e_n|^2\,dx + \int_{B_{\lambda r}} |\nabla u|^2\,\mathbf{1}_{E\Delta H}\,dx\right). \tag{4.26}
$$

Combining (4.24), (4.25) and (4.26) we obtain

$$
\begin{aligned}
\int_{B_{\lambda r}} |T_E u - [T_E u]_{\lambda r}|^2\,dx &\le C\int_{B_{\lambda r}} |T_H v - [T_H v]_{\lambda r}|^2\,dx \\
&\quad + C\int_{B_{r/2}} |\nabla w|^2\,dx + C\left(\int_{B_{\lambda r}} |\nabla u|^2\,|\nu_E - e_n|^2\,dx + \int_{B_{\lambda r}} |\nabla u|^2\,\mathbf{1}_{E\Delta H}\,dx\right).
\end{aligned}
$$

By Lemma 4.4 we have

$$
\int_{B_{\lambda r}} |T_H v - [T_H v]_{\lambda r}|^2\,dx \le C\,\lambda^{n+2\gamma}\int_{B_{r/2}} |T_H v - [T_H v]_{r/2}|^2\,dx + C\,r^{n+1}. \tag{4.27}
$$

By arguing as above one can easily see that

$$
\begin{aligned}
\int_{B_{r/2}} |T_H v - [T_H v]_{r/2}|^2\,dx &\le C\int_{B_{r/2}} |T_E u - [T_E u]_{r/2}|^2\,dx \\
&\quad + C\int_{B_{r/2}} |\nabla w|^2\,dx + C\left(\int_{B_{r/2}} |\nabla v|^2\,|\nu_E - e_n|^2\,dx + \int_{B_{r/2}} |\nabla v|^2\,\mathbf{1}_{E\Delta H}\,dx\right).
\end{aligned}
$$

We note that

$$
\begin{aligned}
&\int_{B_{r/2}} |\nabla v|^2\,|\nu_E - e_n|^2\,dx + \int_{B_{r/2}} |\nabla v|^2\,\mathbf{1}_{E\Delta H}\,dx \\
&\qquad \le 2\int_{B_{r/2}} |\nabla w|^2\,|\nu_E - e_n|^2\,dx + \int_{B_{r/2}} |\nabla u|^2\,|\nu_E - e_n|^2\,dx \\
&\qquad\quad + 2\int_{B_{r/2}} |\nabla w|^2\,\mathbf{1}_{E\Delta H}\,dx + 2\int_{B_{r/2}} |\nabla u|^2\,\mathbf{1}_{E\Delta H}\,dx \\
&\qquad \le 8\int_{B_{r/2}} |\nabla w|^2\,dx + \int_{B_{r/2}} |\nabla u|^2\,|\nu_E - e_n|^2\,dx \\
&\qquad\quad + 2\int_{B_{r/2}} |\nabla w|^2\,dx + 2\int_{B_{r/2}} |\nabla u|^2\,\mathbf{1}_{E\Delta H}\,dx.
\end{aligned}
$$

Bringing it all together, we get

$$
\begin{aligned}
\int_{B_{\lambda r}} |T_E u - [T_E u]_{\lambda r}|^2\,dx &\le C\,\lambda^{n+2\gamma}\int_{B_{r/2}} |T_E u - [T_E u]_{r/2}|^2\,dx \\
&\quad + C\int_{B_{r/2}} |\nabla u|^2\,|\nu_E - e_n|^2\,dx + C\int_{B_{r/2}} |\nabla u|^2\,\mathbf{1}_{E\Delta H}\,dx + C\int_{B_{r/2}} |\nabla w|^2\,dx.
\end{aligned} \tag{4.28}
$$

We need to estimate the last three terms in the right hand side of the above inequality. Since E is parametrised by $f \in C^{1,\vartheta}(\mathbf{D}_r)$ in the cylinder $\mathbf{C}(x_0, r)$, there exists a constant $C > 0$ such that

$$\frac{|(E\Delta H) \cap B_r|}{|B_r|} \le C\, r^{\vartheta}. \tag{4.29}$$

We will estimate the last two terms together. By testing (4.23) with w we deduce

$$\int_{B_{r/2}} |\nabla w|^2 \, dx \le \int_{B_{r/2}} a_H \, |\nabla w|^2 \, dx = \int_{B_{r/2}} (a_E - a_H)\, \nabla u \cdot \nabla w \, dx. \tag{4.30}$$

By applying Hölder inequality in (4.30) we obtain

$$\begin{aligned} C \int_{B_{r/2}} |\nabla u|^2 \, \mathbf{1}_{E\Delta H} \, dx + C \int_{B_{r/2}} |\nabla w|^2 \\ \le C \int_{B_{r/2}} |\nabla u|^2 \, \mathbf{1}_{E\Delta H} \, dx + C \int_{B_{r/2}} (a_E - a_H)^2 \, |\nabla u|^2 \, dx \\ \le C \int_{(E\Delta H)\cap B_{r/2}} |\nabla u|^2 \, dx. \end{aligned} \tag{4.31}$$

By the higher integrability [DPHV19, Lemma 6.1], there exists $p > 1$ such that

$$\left(\frac{1}{|B_{r/2}|} \int_{B_{r/2}} |\nabla u|^{2p} \, dx \right)^{\frac{1}{p}} \le C \, \frac{1}{|B_r|} \int_{B_r} |\nabla u|^2 \, dx + C\, r^{n+2} \, \|\rho\|_\infty^2. \tag{4.32}$$

Hence by exploiting Hölder inequality, (4.29), and (4.32) we have

$$\begin{aligned} \int_{(E\Delta H)\cap B_{r/2}} |\nabla u|^2 \, dx &\le \left|(E\Delta H) \cap B_{r/2}\right|^{1-\frac{1}{p}} \left(\int_{B_{r/2}} |\nabla u|^{2p} \, dx \right)^{\frac{1}{p}} \\ &\le C\, |B_r| \left(\frac{|(E\Delta H)\cap B_r|}{|B_r|} \right)^{1-\frac{1}{p}} \left(\frac{1}{|B_{r/2}|} \int_{B_{r/2}} |\nabla u|^{2p} \, dx \right)^{\frac{1}{p}} \\ &\le C\, r^{\vartheta\left(1-\frac{1}{p}\right)} \left\{ \int_{B_r} |\nabla u|^2 \, dx + r^{n+2} \, \|\rho\|_\infty^2 \right\}. \end{aligned} \tag{4.33}$$

Therefore, (4.31) together with (4.33) (recall $r < 1$) yield

$$C \int_{B_{r/2}} |\nabla u|^2 \, \mathbf{1}_{E\Delta H} \, dx + C \int_{B_{r/2}} |\nabla w|^2 \le C \left\{ r^{\vartheta\left(1-\frac{1}{p}\right)} \int_{B_r} |\nabla u|^2 + r^{n+2} \|\rho\|_\infty^2 \right\}. \tag{4.34}$$

On the other hand, by Lemma 4.6 we have

$$Q^2 \int_{B_s} |\nabla u|^2 \, dx \le C\, s^{n-\gamma} \quad \forall\, s < \bar{r}. \tag{4.35}$$

Hence, combining (4.34) and (4.35), we obtain

$$Q^2 \left(C \int_{B_{r/2}} |\nabla u|^2 \, \mathbf{1}_{E\Delta H} \, dx + C \int_{B_{r/2}} |\nabla w|^2 \right) \le C \left\{ r^{\vartheta\left(1-\frac{1}{p}\right)+n-\gamma} + r^{n+2} \|\rho\|_\infty^2 \right\}.$$

Finally, we estimate the second term in (4.28). Notice that

$$\int_{B_{r/2}} |\nabla u|^2 \, |\nu_E - e_n|^2 \, dx = \int_{B_{r/2}} |\nabla u(x',x_n)|^2 \, |\nu_E(x',x_n) - e_n|^2 \, dx$$
$$= \int_{B_{r/2}} |\nabla u|^2 \, |\nu_E(x',f(x')) - e_n|^2 \, dx.$$

Since $\sqrt{1+t} \le 1 + \frac{t}{2}$ for every $t > 0$,
(4.36)

$$\left|\nu_E(x',f(x')) - e_n\right|^2 = 2 - \frac{2}{\sqrt{1+|\nabla f(x')|^2}} \le 2\left(\frac{\sqrt{1+|\nabla f(x')|^2}-1}{\sqrt{1+|\nabla f(x')|^2}}\right) \le |\nabla f(x')|^2.$$

Thanks to (4.35) and (4.36), and using that ∇f is ϑ-Hölder, we deduce

$$Q^2 \int_{B_{r/2}} |\nabla u|^2 \, |\nu_E - e_n|^2 \, dx \le C \, r^{n+2\vartheta-\gamma}. \tag{4.37}$$

Let

$$\alpha := \min\left\{\gamma, \vartheta\,(1-1/p) - \gamma, 2\vartheta - \gamma\right\}.$$

Therefore, by multiplying (4.28) and (4.34) with Q^2 and by recalling that $Q < 1$ we have that (4.37) implies (4.22). □

We are now ready to prove that u is regular up to the boundary. Recall that $u^+ = u\,\mathbf{1}_E$ and $u^- = u\,\mathbf{1}_{E^c}$.

Theorem 4.8. *Let E be a minimizer of $(\mathcal{P}_{\beta,K,Q,R})$, let $(u,\rho) \in \mathcal{A}(E)$ be the minimizing pair of $\mathcal{G}_{\beta,K}(E)$ and $f \in C^{1,\vartheta}(\mathbf{D}(x_0',r))$. Suppose $Q \le 1$ and*

$$E \cap \mathbf{C}(x_0,r) = \left\{x = (x',x_n) \in \mathbf{D}(x_0',r) \times \mathbb{R} \,:\, x_n < f(x')\right\} \cap \mathbf{C}(x_0,r)$$

for some $0 < r \le \min\{\bar r, 1\}$, where $\bar r$ is as in Lemma 4.6. Then there exists $\eta = \eta(\vartheta) \in (0,1)$ such that $u^+ \in C^{1,\eta}(\overline{E} \cap \mathbf{C}_{r/2}(x_0))$ and $u^- \in C^{1,\eta}(\overline{E}^c \cap \mathbf{C}_{r/2}(x_0))$. Furthermore, let $A > 0$ and let β, K, Q be controlled by A and $R \ge 1$. Then there exists a universal constant $C = C(n,A) > 0$ such that

$$\|Q\,u^+\|_{C^{1,\eta}(\overline{E}\cap\mathbf{C}_{r/2}(x_0))} \le C \quad \text{and} \quad \|Q\,u^-\|_{C^{1,\eta}(\overline{E}^c\cap\mathbf{C}_{r/2}(x_0))} \le C. \tag{4.38}$$

Proof. Let $u_Q := Q\,u$. By Proposition 4.7 there exists $C = C(n,\beta,\gamma,\|\rho\|_\infty) > 0$ such that
(4.39)

$$\int_{B_{\lambda r}(x_0)} |T_E u_Q - [T_E u_Q]_{x_0,\lambda r}|^2 \, dx \le C\lambda^{n+2\alpha} \int_{B_r(x_0)} |T_E u_Q - [T_E u_Q]_{x_0,r}|^2 \, dx + C\,r^{n+\alpha},$$

where $\alpha \in (0,1)$ is as in Proposition 4.7. Therefore, Lemma 4.3 implies that there exists a universal constant $C = C(n,A) > 0$ such that

$$\frac{1}{|B_r|}\int_{B_r(x_0)} |T_E u_Q - [T_E u_Q]_{x,r}|^2 \, dy \le C\left(\frac{r}{R}\right)^{2\eta}, \quad \forall B_r(x_0) \subset B_R. \tag{4.40}$$

for some $\eta = \eta(\vartheta) \in (0,1)$. Hence, by Lemma 4.2, recalling the definition of T_E, we get $u_Q\mathbf{1}_E \in C^{1,\eta}(\overline{E} \cap \mathbf{C}_{r/2}(x_0))$ and $u_Q\mathbf{1}_{E^c} \in C^{1,\eta}(\overline{E}^c \cap \mathbf{C}_{r/2}(x_0))$ and (4.38). □

In the next proposition we rewrite the Euler-Lagrange equation (see Theorem 2.3 (ii)) in a more convenient form by exploiting the regularity of ∂E.

Proposition 4.9 (Euler-Lagrange equation)**.** *Let E be a minimizer for $(\mathcal{P}_{\beta,K,Q,R})$ and $(u,\rho)\in\mathcal{A}(E)$. Assume that $f\in C^{1,\vartheta}(\mathbf{D}(x_0',r))$ and*

$$E\cap\mathbf{C}(x_0,r)=\big\{x=(x',x_n)\in\mathbf{D}(x_0',r)\times\mathbb{R}\,:\,x_n<f(x')\big\}\cap\mathbf{C}(x_0,r).$$

Then there exists a constant $C>0$ such that

$$\begin{aligned}-\mathrm{div}\left(\frac{\nabla f(x')}{\sqrt{1+|\nabla f(x')|^2}}\right)&=Q^2\left(\beta|\nabla u^+|^2-|\nabla u^-|^2+K\,\rho^2\right)(x',f(x'))\\&\quad-2Q^2\left(\beta\partial_n u^+\,\nabla u^+-\partial_n u^-\,\nabla u^-\right)(x',f(x'))\cdot(-\nabla f(x'),1)+C\end{aligned}\tag{4.41}$$

weakly in $\mathbf{D}(x_0',r)$.

Proof. Let $E\subset\mathbb{R}^n$ be a minimizer of $(\mathcal{P}_{\beta,K,Q,R})$ and let $(u,\rho)\in\mathcal{A}(E)$.

Notice that $E\cap\mathbf{C}(x_0,r)$ is an open set of $\mathbb{R}^n$. Moreover, by an approximation argument, we can integrate over $E\cap\mathbf{C}(x_0,r)$ the following identity,

$$\begin{aligned}|\nabla u^+|^2\,\mathrm{div}\,\eta&=\mathrm{div}(|\nabla u^+|^2\eta)-\nabla|\nabla u^+|^2\cdot\eta\\&=\mathrm{div}(|\nabla u^+|^2\eta)-2\,\mathrm{div}(\nabla u^+(\nabla u^+\cdot\eta))+2\Delta u^+\,\nabla u^+\cdot\eta+2\nabla u^+\cdot(\nabla\eta\nabla u^+)\end{aligned}$$

for every $\eta\in C_c^\infty(\mathbf{C}(x_0,r),\mathbb{R}^n)$. Therefore,

$$\begin{aligned}\int_{E\cap\mathbf{C}(x_0,r)}\left(|\nabla u^+|^2\,\mathrm{div}\,\eta-2\nabla u^+\cdot(\nabla\eta\nabla u^+)\right)dx&=\int_{E\cap\mathbf{C}(x_0,r)}\mathrm{div}(|\nabla u^+|^2\eta)\,dx\\&\quad-\int_{E\cap\mathbf{C}(x_0,r)}2\,\mathrm{div}(\nabla u^+(\nabla u^+\cdot\eta))\,dx\\&\quad+\int_{E\cap\mathbf{C}(x_0,r)}2\Delta u^+\,\nabla u^+\cdot\eta\,dx.\end{aligned}\tag{4.42}$$

On the other hand, since $(u,\rho)\in\mathcal{A}(E)$, we have

$$-\beta\Delta u^+=\rho\quad\text{in}\quad\mathcal{D}'(E\cap\mathbf{C}(x_0,r)).$$

Moreover, by Theorem 2.3 (i) we deduce

$$\nabla u^+=-K\,\nabla\rho\quad\text{in}\quad E\cap\mathbf{C}(x_0,r).$$

Then, by multiplying equation (4.42) by β, we have

$$\begin{aligned}\int_{E\cap\mathbf{C}(x_0,r)}\beta\left(|\nabla u^+|^2\,\mathrm{div}\,\eta-2\nabla u^+\cdot(\nabla\eta\nabla u^+)\right)dx&=\int_{E\cap\mathbf{C}(x_0,r)}\beta\,\mathrm{div}(|\nabla u^+|^2\eta)\,dx\\&\quad-\int_{E\cap\mathbf{C}(x_0,r)}2\beta\,\mathrm{div}(\nabla u^+(\nabla u^+\cdot\eta))\,dx\\&\quad+K\int_{E\cap\mathbf{C}(x_0,r)}2\rho\,\nabla\rho\cdot\eta\,dx.\end{aligned}\tag{4.43}$$

Integrating by parts the first and the second term in the right hand side of (4.43), we can write

(4.44)
$$\begin{aligned}\int_{E\cap\mathbf{C}(x_0,r)} \beta\left(|\nabla u^+|^2 \operatorname{div}\eta - 2\nabla u^+\cdot(\nabla\eta\nabla u^+)\right)\,dx &= \int_{\partial E\cap\mathbf{C}(x_0,r)} \beta|\nabla u^+|^2\eta\cdot\nu_E\,d\mathcal{H}^{n-1}\\ &\quad-\int_{\partial E\cap\mathbf{C}(x_0,r)} 2\beta\,(\nabla u^+\cdot\eta)(\nabla u^+\cdot\nu_E)\,d\mathcal{H}^{n-1}\\ &\quad+K\int_{E\cap\mathbf{C}(x_0,r)} 2\rho\,\nabla\rho\cdot\eta\,dx.\end{aligned}$$

By arguing similarly as above, one can also prove

(4.45)
$$\begin{aligned}\int_{E^c\cap\mathbf{C}(x_0,r)} \left(|\nabla u^-|^2 \operatorname{div}\eta - 2\nabla u^-\cdot(\nabla\eta\nabla u^-)\right)\,dx &= \int_{E^c\cap\mathbf{C}(x_0,r)} \operatorname{div}(|\nabla u^-|^2\eta)\,dx\\ &\quad-\int_{E^c\cap\mathbf{C}(x_0,r)} 2\operatorname{div}(\nabla u^-(\nabla u^-\cdot\eta))\,dx.\end{aligned}$$

Integrating by parts the right hand side of (4.45), we can write

(4.46)
$$\begin{aligned}\int_{E^c\cap\mathbf{C}(x_0,r)} \left(|\nabla u^-|^2 \operatorname{div}\eta - 2\nabla u^-\cdot(\nabla\eta\nabla u^-)\right)\,dx &= -\int_{\partial E\cap\mathbf{C}(x_0,r)} |\nabla u^-|^2\eta\cdot\nu_E\,d\mathcal{H}^{n-1}\\ &\quad+\int_{\partial E\cap\mathbf{C}(x_0,r)} 2\,(\nabla u^-\cdot\eta)\,(\nabla u^-\cdot\nu_E)\,d\mathcal{H}^{n-1}.\end{aligned}$$

Therefore, combining (4.44) and (4.46), we get

$$\begin{aligned}\int_{\mathbb{R}^n} a_E\Big(\operatorname{div}\eta\,|\nabla u|^2 - 2\nabla u\cdot(\nabla\eta\nabla u)\Big)\,dx &= \int_{\partial E}\left(\beta|\nabla u^+|^2-|\nabla u^-|^2\right)\eta\cdot\nu_E\,d\mathcal{H}^{n-1}\\ -\int_{\partial E\cap\mathbf{C}(x_0,r)} 2\big(\beta\,(\nabla u^+\cdot\eta)(\nabla u^+\cdot\nu_E) &- (\nabla u^-\cdot\eta)\,(\nabla u^-\cdot\nu_E)\big)\,d\mathcal{H}^{n-1}\\ +K\int_{E\cap\mathbf{C}(x_0,r)} 2\rho\,\nabla\rho\cdot\eta\,dx.\end{aligned} \tag{4.47}$$

Notice that the following identity holds true

$$\begin{aligned}K\int_{\mathbb{R}^n}\rho^2\operatorname{div}\eta &= K\int_{E\cap\mathbf{C}(x_0,r)}\operatorname{div}(\rho^2\eta)\,dx - K\int_{E\cap\mathbf{C}(x_0,r)} 2\rho\,\nabla\rho\,\cdot\eta\,dx\\ &= K\int_{\partial E\cap\mathbf{C}(x_0,r)}\rho^2\eta\cdot\nu_E\,d\mathcal{H}^{n-1} - K\int_{E\cap\mathbf{C}(x_0,r)} 2\rho\,\nabla\rho\,\cdot\eta\,dx.\end{aligned} \tag{4.48}$$

Combining the Euler-Lagrange equation of Theorem 2.3 (ii), (4.47) and (4.48), we find

$$\begin{aligned}\int_{\partial E}\operatorname{div}_E\eta\,d\mathcal{H}^{n-1} &= Q^2\int_{\partial E}\left(\beta|\nabla u^+|^2-|\nabla u^-|^2+K\,\rho^2\right)\,\eta\cdot\nu_E\,d\mathcal{H}^{n-1}\\ &\quad-2Q^2\int_{\partial E}\beta(\eta\cdot\nabla u^+)\,(\nabla u^+\cdot\nu_E)-(\eta\cdot\nabla u^-)\,(\nabla u^-\cdot\nu_E)\,d\mathcal{H}^{n-1}\end{aligned} \tag{4.49}$$

for every $\eta\in C^1_c(B_r(x_0),\mathbb{R}^n)$ with $\int_E \operatorname{div}\eta\,dx = 0$.

Now we are ready to prove (4.41). The tangential divergence of η on ∂E is

$$\mathrm{div}_E\eta := \mathrm{div}\eta - \sum_{i,j=1}^{n} (\nu_E)_i\,(\nu_E)_j\,\partial_j\eta_i \quad \text{on } \partial E, \tag{4.50}$$

where $\nu_E : \partial E \to \mathbb{S}^{n-1}$ is the normal vector to ∂E:

$$\nu_E := \frac{1}{\sqrt{1+|\nabla f|^2}}(-\nabla f, 1).$$

Let $\eta := (0, \ldots, 0, \eta_n)$, then by (4.50) we have

$$\mathrm{div}_E\eta := \partial_n\eta_n + \frac{1}{1+|\nabla f|^2}\left\{\sum_{j=1}^{n-1}\partial_j\eta_n\,\partial_j f - \partial_n\eta_n\right\} \quad \text{on } \partial E. \tag{4.51}$$

Choose $\eta_n(x) := \varphi(\mathbf{p}x)\,s(x_n)$, where $\varphi \in C^1_c(\mathbf{D}(x'_0, r))$ is such that $\int_{\mathbf{D}(x'_0,r)}\varphi = 0$ and $s : (-1, 1) \to \mathbb{R}^n$ is such that $s(t) = 1$ for every $|t| \le \|f\|_\infty$. Since now η_n does not depend on the n-th component on ∂E, we have

$$\eta\cdot\nu_E = \frac{\varphi(\mathbf{p}x)}{\sqrt{1+|\nabla f|^2}} \quad \text{on } \partial E \cap \mathbf{C}(x_0, r), \tag{4.52}$$

and the above equation (4.51) reads as

$$\mathrm{div}_E\eta := \frac{1}{1+|\nabla f|^2}\,\nabla\varphi\cdot\nabla f \quad \text{on } \partial E\cap\mathbf{C}(x_0, r). \tag{4.53}$$

Moreover,

$$\begin{aligned}
\int_E \mathrm{div}\eta\,dx &= \int_{\partial E}(\eta\cdot\nu_E)\,d\mathcal{H}^{n-1} = \int_{\partial E\cap\mathbf{C}(x_0,r)}\eta_n(\nu_E\cdot e_n)\,d\mathcal{H}^{n-1}\\
&= \int_{\partial E\cap\mathbf{C}(x_0,r)}\varphi(\mathbf{p}x)s(f(x))\,(\nu_E\cdot e_n)\,d\mathcal{H}^{n-1}\\
&= \int_{\partial E\cap\mathbf{C}(x_0,r)}\frac{\varphi(\mathbf{p}x)}{\sqrt{1+|\nabla f(\mathbf{p}x)|}}\,d\mathcal{H}^{n-1} = \int_{\mathbf{p}(\partial E\cap\mathbf{C}(x_0,r))}\varphi\,dx = 0.
\end{aligned}$$

This implies that η is admissible in (4.49). Hence by using η as a test function in (4.49), by combining (4.52) and (4.53), we have

$$\begin{aligned}
\int_{\mathbf{D}(x'_0,r)}\frac{\nabla f}{1+|\nabla f|^2}\cdot\nabla\varphi\,dx' &= Q^2\int_{\mathbf{D}(x'_0,r}\left(\beta|\nabla u^+|^2 - |\nabla u^-|^2 + K\rho^2\right)(x', f(x'))\,\frac{\varphi(x')}{\sqrt{1+|\nabla f|^2}}\,dx'\\
&\quad - 2Q^2\int_{\mathbf{D}(x'_0,r}\left(\beta\partial_n u^+\nabla u^+ - \partial_n u^-\nabla u^-\right)(x', f(x'))\cdot\frac{(-\nabla f, 1)}{\sqrt{1+|\nabla f|^2}}\,\varphi(x')\,dx'
\end{aligned}$$

for any $\varphi \in C^1_c(\mathbf{D}(x'_0, r))$ with $\int_{\mathbf{D}(x'_0,r)}\varphi = 0$. It remains to multiply this equality by $\sqrt{1+|\nabla f|^2}$ and use divergence theorem on the left-hand side.

□

Corollary 4.10. *Let E be a minimizer for $(\mathcal{P}_{\beta,K,Q,R})$ and $(u, \rho) \in \mathcal{A}(E)$. Assume that $f \in C^{1,\vartheta}(\mathbf{D}(x'_0, r))$ and*

$$E\cap\mathbf{C}(x_0, r) = \left\{x = (x', x_n) \in \mathbf{D}(x'_0, r)\times\mathbb{R} \,:\, x_n < f(x')\right\}\cap\mathbf{C}(x_0, r).$$

Then there exists a vector field $M : \mathbb{R}^n \to \mathbb{R}^n$ such that the matrix $\nabla M(\nabla f)$ is uniformly elliptic and Hölder continuous and a Hölder continuous function G such that

$$-\mathrm{div}\,(\nabla M(\nabla f)\,\nabla\partial_i f) = \partial_i G \quad \textit{weakly on } \partial E\cap\mathbf{C}(x_0, r/2)$$

for every $i = 1, \dots, n$.

Proof. Exploiting Proposition 4.9, we have

$$-\mathrm{div}\left(\frac{\nabla f(x')}{\sqrt{1+|\nabla f(x')|^2}}\right) = G(x', f(x')) \quad \text{for a.e. } x' \in \mathbf{D}(x_0', r/2), \tag{4.54}$$

where

$$G(x', f(x')) = Q^2 \left(\beta|\nabla u^+|^2 - |\nabla u^-|^2 + K\rho^2\right)(x', f(x'))$$
$$- 2Q^2 \left(\beta\partial_n u^+ \nabla u^+ - \partial_n u^- \nabla u^-\right)(x', f(x')) \cdot (-\nabla f(x'), 1) + C, \quad x' \in \mathbf{D}(x_0', r/2).$$

Hence, (4.54) is equivalent to

$$-\mathrm{div}\,(M(\nabla f)) = G \quad \text{a.e. on } \partial E \cap \mathbf{C}(x_0, r/2), \tag{4.55}$$

where

$$M(\xi) := \frac{\xi}{\sqrt{1+|\xi|^2}}, \quad \forall \xi \in \mathbb{R}^n.$$

By [M, Theorem 27.1] we can take the derivatives of (4.55). Then,

$$-\mathrm{div}\,(\nabla M(\nabla f)\,\nabla\partial_i f) = \partial_i G \quad \text{a.e. on } \partial E \cap \mathbf{C}(x_0, r/2)$$

for every $i = 1, \dots, n$. Notice that

$$\nabla M(\xi) = \frac{1}{\sqrt{1+|\xi|^2}}\left(\mathrm{Id} - \frac{\xi \otimes \xi}{1+|\xi|^2}\right) \quad \forall \xi \in \mathbb{R}^n,$$

meaning that the matrix $\nabla M(\nabla f)$ is uniformly elliptic, more precisely

$$|\eta|^2 \geq \nabla M(\nabla f)\eta \cdot \eta \geq (1 + \|\nabla f\|_\infty)^{-3/2}\,|\eta|^2 \quad \forall \eta \in \mathbb{R}^n.$$

It follows from Theorem 4.8 that G is Hölder continuous. By the definition of M and by the regularity of f we also have that $\nabla M(\nabla f)$ is Hölder continuous. ☐

We prove now the partial $C^{2,\vartheta}$-regularity of minimizers.

Theorem 4.11 ($C^{2,\vartheta}$-regularity)**.** *Given* $n \geq 3$, $A > 0$ *and* $\vartheta \in (0, 1/2)$, *there exists* $\varepsilon_{\mathrm{reg}} = \varepsilon_{\mathrm{reg}}(n, A, \vartheta) > 0$ *such that if* E *is minimizer of* $(\mathcal{P}_{\beta,K,Q,R})$, $Q + \beta + K + \frac{1}{K} \leq A$, $x_0 \in \partial E$, *and*

$$r + \mathbf{e}_E(x_0, r) + Q^2\,D_E(x_0, r) \leq \varepsilon_{\mathrm{reg}},$$

then $E \cap \mathbf{C}(x_0, r/2)$ *coincides with the epi-graph of a* $C^{2,\vartheta}$*-function* f*. In particular, we have that* $\partial E \cap \mathbf{C}(x_0, r/2)$ *is a* $C^{2,\vartheta}$ $(n-1)$*-dimensional manifold and*

$$[f]_{C^{2,\vartheta}(\mathbf{D}(x_0', r/2))} \leq C\,(n, A, r, \vartheta)\,. \tag{4.56}$$

Proof. Choose $\varepsilon_{\mathrm{reg}}$ as in Theorem 2.4. Then there exists $f \in C^{1,\vartheta}(\mathbf{D}(x_0', r/2))$ such that

$$E \cap \mathbf{C}(x_0, r/2) = \left\{x = (x', x_n) \in \mathbf{D}(x_0', r/2) \times \mathbb{R} \,:\, x_n < f(x')\right\}.$$

By Corollary 4.10 we have

$$-\mathrm{div}\,(\nabla M(\nabla f)\,\nabla\partial_i f) = \partial_i G \quad \text{a.e. on } \partial E \cap \mathbf{C}(x_0, r/2)$$

for every $i = 1, \dots, n$, with $\nabla M(\nabla f)$ uniformly elliptic and G - Hölder continuous. We also have that $\nabla M(\nabla f)$ is Hölder continuous. Hence the following Schauder estimates hold in this case

$$[\nabla\partial_i f]_{C^{0,\vartheta}(\mathbf{D}(x_0', r/2))} \leq C\,\{\|\partial_i f\|_{L^2(\mathbf{D}(x_0', r/2))} + [G]_{C^{0,\eta}(\mathbf{D}(x_0', r/2))}\} \quad \forall i = 1, \dots, n,$$

for some constant C depending on r. In particular, f is $C^{2,\vartheta}$ and

$$[f]_{C^{2,\vartheta}(\mathbf{D}(x_0', r/2))} \leq C\,\{\|\nabla f\|_{L^2(\mathbf{D}(x_0', r/2))} + [G]_{C^{0,\eta}(\mathbf{C}(x_0, r/2))}\}.$$

By the definition of G, recalling (4.38) and Theorem 2.3 (i), using Poincaré inequality and since f is Lipschitz, one can easily see that there exists $C = C(n, A, \vartheta, r) > 0$ such that

$$[G]_{C^{0,\vartheta}(\mathbf{C}(x_0,r/2))} \leq C(n, A, \vartheta, r).$$

By the Lipschitz approximation theorem it follows that

$$\frac{1}{r^{n-1}} \int_{\mathbf{D}(x_0',r/2)} |\nabla f|^2 \, dz \leq C_L \, \mathbf{e}_E(x_0, r) \leq C_L \, \varepsilon_{\text{reg}}, \tag{4.57}$$

which implies (4.56). □

Remark 4.12. A minimizer E_Q of the problem $(\mathcal{P}_{\beta,K,Q,R})$ satisfies the hypothesis of Theorems 4.11 and 5.3 whenever $Q > 0$ is small enough. Indeed, assume $x_0 \in \partial B_1$. Then, by the regularity of ∂B_1, there exists a radius $r = r(n) > 0$ such that

$$r + \mathbf{e}_{B_1}(x_0, 2r) \leq \frac{\varepsilon_{\text{reg}}}{2}, \tag{4.58}$$

where ε_{reg} is as in Theorem 5.3. On the other hand, by Proposition 3.2 we have that E_Q converges to B_1 in the Kuratowski sense when $Q \to 0$. Hence, by properties of the excess function, $\mathbf{e}_{E_Q}(x_0, 2r) \to \mathbf{e}_{B_1}(x_0, 2r)$ when $Q \to 0$. By Theorem 2.3 (iii) we also have $Q^2 \, D_{E_Q}(x_0, 2r) \to 0$ when $Q \to 0$. Therefore,

$$r + \mathbf{e}_{E_Q}(x_0, 2r) + Q^2 \, D_{E_Q}(x_0, 2r) \leq \varepsilon_{\text{reg}}, \tag{4.59}$$

when $Q > 0$ is small enough.

5. C^∞ regularity

In this section, by a bootstrap argument, we obtain the C^∞ partial regularity of minimizers. Since this result is not necessary for the proof of the main theorem, the reader may skip it unless interested.

Improving the regularity from $C^{2,\eta}$ to C^∞ is easier than from $C^{1,\eta}$ to $C^{2,\eta}$, because we can straighten the boundary in a nice way once it is C^2. More precisely, we have the following lemma.

Lemma 5.1. *Let $k \in \mathbb{N}$, $k \geq 2$ and f is $C^{k,\vartheta}(\mathbf{D})$. There exists $\varepsilon > 0$ such that if*

$$\|f\|_{C^{2,\vartheta}(\mathbf{D})} \leq \varepsilon \quad \text{and} \quad f(0) = 0,$$

then there exists a diffeomorphism $\Phi \in C^{k-1,\vartheta}$, $\Phi : \mathbf{C}_{1-\varepsilon} \to \mathbf{C}_{1-\varepsilon}$, such that

$$\Phi(\Gamma_f \cap \mathbf{C}_{1-\varepsilon}) = \{x = (x', x_n) \in \mathbf{D}_{1-\varepsilon} \times \mathbb{R} \, : \, x_n = 0\},$$

where Γ_f is the graph of f. Moreover,

$$\left(\nabla\Phi(\Phi^{-1}(x)) \, (\nabla\Phi(\Phi^{-1}(x)))^T\right)_{jn} = 0 \ \ \forall j \neq n \quad \text{and} \quad \left(\nabla\Phi(\Phi^{-1}(x)) \, (\nabla\Phi(\Phi^{-1}(x)))^T\right)_{nn} \neq 0. \tag{5.1}$$

Proof. Define

$$\Psi(x', x_n) := (x', f(x')) + x_n \frac{(-\nabla f(x'), 1)}{\sqrt{1 + |\nabla f(x')|^2}} \quad \forall x = (x', x_n) \in \mathbf{C}_{1-\varepsilon},$$

then $\Phi := \Psi^{-1}$ is the desired diffeomorphism. □

Lemma 5.2. *Let k be a positive integer and let f be a $C^{k+1,\vartheta}$-Hölder continuous function defined on $\mathbf{D}(x_0, r)$ such that $\|f\|_{C^{k+1,\vartheta}} \leq \varepsilon$ for some $\varepsilon > 0$ and*

$$E \cap \mathbf{C}(x_0, r) = \left\{x = (x', x_n) \in \mathbf{D}(x_0', r) \times \mathbb{R} \, : \, x_n < f(x')\right\} \cap \mathbf{C}(x_0, r).$$

Suppose v is a solution of

$$-\operatorname{div}(a_E\,\nabla v) = h \quad \text{in } \mathcal{D}'\left(B_r(x_0)\right), \quad a_E := \mathbf{1}_{E^c} + \beta\,\mathbf{1}_E,$$

with h^+ and h^- $C^{k,\eta}$-Hölder continuous respectively on $\overline{E} \cap \mathbf{C}(x_0,r)$ and $\overline{E^c} \cap \mathbf{C}(x_0,r)$, where $h^+ = h\,1_E$, $h^- = h\,1_{E^c}$. Then v^+, v^- are $C^{k+1,\eta}$-Hölder continuous respectively on $\overline{E} \cap \mathbf{C}(x_0,r)$ and $\overline{E^c} \cap \mathbf{C}(x_0,r)$.

Moreover,

$$\|v_1\|_{C^{k+1,\eta}(\overline{E}^c \cap \mathbf{C}(x_0,r))} \le C \quad \text{and} \quad \|v_\beta\|_{C^{k+1,\eta}(\overline{E} \cap \mathbf{C}(x_0,r))} \le C \tag{5.2}$$

for some constant $C \ge 0$ which depends on the $C^{k,\eta}$- Hölder norms of h^+ and h^- and on the $C^{k+1,\vartheta}$ norm of f.

Proof. Assume $x_0 = 0$. Let $H := \{x \in \mathbb{R}^n \,:\, x_n = x \cdot e_n \le 0\}$ be the half space in $\mathbb{R}^n$. By Lemma 5.1, we can assume that

$$\Gamma_f \cap \mathbf{C}_r = \partial H \cap \mathbf{C}_r,$$

where $\Gamma_f \cap \mathbf{C}_{r/2} := \{(x', f(x')) \,:\, x' \in \mathbf{D}_r\}$, $f(0) = 0$ and that v solves the following equation

$$-\operatorname{div}(a_H A\,\nabla v) = h, \tag{5.3}$$

where by (5.1), A is a $C^{k-1,\vartheta}$-continuous elliptic matrix such that $A_{jn} = 0$ for every $j \ne n$, $A_{nn} \ne 0$.

We continue the proof by induction on k. For clarity, we do the detailed computations for the case $k = 1$ and we explain how the formulas look like for bigger k.

Case $\mathbf{k = 1}$. By taking the derivatives with respect to the tangential coordinates $j \ne n$ of (5.3) we deduce

$$\begin{aligned} -\operatorname{div}\big(a_H A\,\nabla \partial_j v\big) &= \partial_j h + \operatorname{div}\big(\partial_j(a_H A)\,\nabla v\big) \\ &= \operatorname{div}(h\,e_j + \partial_j(a_H A)\,\nabla v) \qquad \text{in } \mathcal{D}'(\mathbb{R}^n). \end{aligned} \tag{5.4}$$

Notice that a_H is constant along tangential directions and that $(a_H\,A)^+$, $(a_H\,A)^-$ have coefficients respectively in $C^{0,\eta}(\overline{H}^c \cap \mathbf{C}_r)$ and $C^{0,\eta}(\overline{H} \cap C_r)$. Furthermore,

$$(h\,e_j + \partial_j(a_H A)\,\nabla v)^+ \in C^{0,\eta}(\overline{H}^c \cap \mathbf{C}_r) \ \text{ and } \ (h\,e_j + \partial_j(a_H A)\,\nabla v)^- \in C^{0,\eta}(\overline{H} \cap \mathbf{C}_r).$$

Hence, exploiting Lemma 4.5 we deduce

$$\partial_j v^+ \in C^{1,\eta}(\overline{H} \cap \mathbf{C}_r) \quad \text{and} \quad \partial_j v^- \in C^{1,\eta}(\overline{H^c} \cap \mathbf{C}_r) \qquad \forall j \ne n. \tag{5.5}$$

Furthermore, by (5.3) we have

$$-\sum_{i,j=1}^{n} \{a_H\,A_{ij}\partial_{ij} v + \partial_i(a_H\,A_{ij})\partial_j v\} = h.$$

Thanks to the form of the matrix A we obtain

$$-a_H\,A_{nn}\partial_{nn} v = \sum_{i,j \ne n} \{a_H\,A_{ij}\partial_{ij} v + \partial_i(a_H\,A_{ij})\,\partial_j v\} + h. \tag{5.6}$$

Since the right hand side of the previous equation is Hölder continuous, we have

$$\partial_{nn} v^+ \in C^{0,\eta}(\overline{H}^c \cap \mathbf{C}_r) \quad \text{and} \quad \partial_{nn} v^- \in C^{0,\eta}(\overline{H} \cap \mathbf{C}_r).$$

Moreover, (5.5) implies

$$\partial_{nj} v^+ \in C^{0,\eta}(\overline{H}^c \cap \mathbf{C}_r) \quad \text{and} \quad \partial_{nj} v^- \in C^{0,\eta}(\overline{H} \cap \mathbf{C}_r)$$

for every $j \ne n$. Therefore,

$$v^+ \in C^{2,\eta}(\overline{H}^c \cap \mathbf{C}_r) \quad \text{and} \quad v^- \in C^{2,\eta}(\overline{H}^c \cap \mathbf{C}_r).$$

By Lemma 4.5 we deduce also that

$$\|\nabla v^+\|_{C^{1,\eta}(\overline{H}\cap\mathbf{C}_r)} \quad \text{and} \quad \|\nabla v^-\|_{C^{1,\eta}(\overline{H^c}\cap\mathbf{C}_r)}$$

are bounded by a constant which depends on the Hölder norms of ∇h^+, ∇h^-, the coefficients of $(a_H A)^+$ and $(a_H A)^-$.

General k. As in the case $k = 1$, we start by taking the derivatives of (5.3) with respect to the tangential coordinates $j \neq n$. We get an equation similar to (5.4):

$$-\mathrm{div}\big(a_H A\,\nabla\partial_{i_1,i_2,\dots,i_k} v\big) = \mathrm{div}\left(\partial_{i_2,\dots,i_k} h\, e_{i_1} + \sum \partial_{j_1,j_2,\dots,j_l}(a_H A)\nabla\left(\partial_{\{i_1,i_2,\dots,i_k\}\setminus\{j_1,j_2,\dots,j_l\}} v\right)\right)$$

in $\mathcal{D}'(\mathbb{R}^n)$. This gives us

$$\partial_{i_1,i_2,\dots,i_k} v^+ \in C^{1,\eta}(\overline{H}\cap\mathbf{C}_r) \quad \text{and} \quad \partial_{i_1,i_2,\dots,i_k} v^- \in C^{1,\eta}(\overline{H^c}\cap\mathbf{C}_r) \tag{5.7}$$

for all $i_1 \neq n$, $i_2 \neq n$, $\dots$, $i_k \neq n$.

By (5.7)

$$\partial_{i_1,i_2,\dots,i_k,n} v^+ \in C^{0,\eta}(\overline{H}\cap\mathbf{C}_r) \quad \text{and} \quad \partial_{i_1,i_2,\dots,i_k,n} v^- \in C^{0,\eta}(\overline{H^c}\cap\mathbf{C}_r)$$

for all $i_1 \neq n$, $i_2 \neq n$, $\dots$, $i_k \neq n$, and thus, taking derivatives of (5.6) in tangential directions, we get

$$\partial_{i_1,i_2,\dots,i_{k-1},n,n} v^+ \in C^{0,\eta}(\overline{H}\cap\mathbf{C}_r) \quad \text{and} \quad \partial_{i_1,i_2,\dots,i_{k-1},n,n} v^- \in C^{0,\eta}(\overline{H^c}\cap\mathbf{C}_r).$$

Induction on the number of normal directions yields

$$v^+ \in C^{k+1,\eta}(\overline{H}^c\cap\mathbf{C}_r) \quad \text{and} \quad v^- \in C^{k+1,\eta}(\overline{H}^c\cap\mathbf{C}_r).$$

□

Theorem 5.3 (C^∞-regularity)**.** *Given $n \geq 3$ and $A > 0$, there exists $\varepsilon_{\mathrm{reg}} = \varepsilon_{\mathrm{reg}}(n, A) > 0$ such that if E is minimizer of $(\mathcal{P}_{\beta,K,Q,R})$ with $Q + \beta + K + \frac{1}{K} \leq A$, $x_0 \in \partial E$, and*

$$r + \mathbf{e}_E(x_0, r) + Q^2\, D_E(x_0, r) \leq \varepsilon_{\mathrm{reg}},$$

then $E \cap \mathbf{C}(x_0, r/2)$ coincides with the epi-graph of a C^∞-function f. In particular, we have that $\partial E \cap \mathbf{C}(x_0, r/2)$ is a C^∞ $(n-1)$-dimensional manifold. Moreover, for every $\vartheta \in (0, \frac{1}{2})$ there exists a constant $C(n, A, k, r, \vartheta) > 0$ such that

$$[f]_{C^{k,\vartheta}(\mathbf{D}(x_0', r/2))} \leq C(n, A, k, r, \vartheta) \tag{5.8}$$

for every $k \in \mathbb{N}$.

Proof. If we choose $\varepsilon_{\mathrm{reg}}$ as in Theorem 4.11, then there exists $f \in C^{2,\vartheta}(\mathbf{D}(x_0', r/2))$ such that

$$E \cap \mathbf{C}(x_0, r/2) = \left\{x = (x', x_n) \in \mathbf{D}(x_0', r/2) \times \mathbb{R} \,:\, x_n < f(x')\right\}.$$

By Corollary 4.10 we have

$$-\mathrm{div}\,(\nabla M(\nabla f)\,\nabla\partial_i f) = \partial_i G \quad \text{a.e. on } \partial E \cap \mathbf{C}(x_0, r/2) \tag{5.9}$$

for every $i = 1, \dots, n$, with $\nabla M(\nabla f)$ uniformly elliptic and Hölder continuous and G - Hölder continuous.

Now we argue by induction on k. The induction step is divided into two parts:

Claim 1:

f is C^k-Hölder continuous $\Longrightarrow u^+, u^-$ are C^k-Hölder continuous respectively on $\overline{E} \cap \mathbf{C}(x_0, r/2)$ and $\overline{E^c} \cap \mathbf{C}(x_0, r/2)$.

Moreover, there exists a universal constant $C = C(n, A) > 0$ and $\eta \in (0, \frac{1}{2})$ such that

$$\|Q\,u^+\|_{C^{k,\eta}(\overline{E}\cap\mathbf{C}(x_0,r/2))} \leq C \quad \text{and} \quad \|Q\,u^-\|_{C^{k,\eta}(\overline{E^c}\cap\mathbf{C}(x_0,r/2))} \leq C. \tag{5.10}$$

Claim 2:

$$f \text{ is } C^k\text{-Hölder continuous} \implies f \text{ is } C^{k+1}\text{-Hölder continuous.}$$

To proof Claim 1, we apply Lemma (5.2) to $v = Qu$ and $h = Q\rho$. By (4.38) the norms

$$\|Q\,\nabla u^+\|_{C^{0,\eta}(\overline{H}\cap \mathbf{C}_{r/2})} \quad \text{and} \quad \|Q\,\nabla u^-\|_{C^{0,\eta}(\overline{H^c}\cap \mathbf{C}_{r/2})}$$

and bounded by a universal constant. That gives us (5.10).

As for Claim 2, notice that by the definition of M, since f is C^k-Hölder continuous, we have that $\nabla M(\nabla f)$ in (5.9) is C^{k-1}-Hölder continuous. By Claim 1 we deduce that G is C^{k-1}-Hölder continuous with its norm uniformly bounded. Then, using Schauder estimates for (5.9), we get that f is C^{k+1}-Hölder continuous.

□

6. Reduction to nearly spherical sets

In this section, by combining Proposition 3.2 with the higher regularity (Theorem 4.11), we prove that for small enough values of the total charge the minimizers are *nearly-spherical sets.* Recall the following definition.

Definition 6.1 ($C^{2,\gamma}$-nearly spherical set)**.** An open bounded set $\Omega \subset \mathbb{R}^n$ is called *nearly-spherical* of class $C^{2,\gamma}$ parametrized by φ, if there exists $\varphi \in C^{2,\gamma}$ with $\|\varphi\|_{L^\infty} < \frac{1}{2}$ such that

$$\partial\Omega = \{(1+\varphi(x))x : x \in \partial B_1\}.$$

Theorem 6.2. *Let $\{Q_h\}_{h\in\mathbb{N}}$ be a sequence such that $Q_h > 0$ and $Q_h \to 0$ when $h \to \infty$. Let $\{E_h\}_{h\in\mathbb{N}}$ be a sequence of minimizers of $(\mathcal{P}_{\beta,K,Q_h,R})$. Then for h big enough E_h is nearly spherical of class C^∞, i.e. there exists $\varphi_h \in C^\infty$ with uniform bounds and $\|\varphi_h\|_{L^\infty} < \frac{1}{2}$ such that*

$$\partial E_h = \{(1+\varphi_h(x))x : x \in \partial B_1\}.$$

Moreover, $\|\varphi_h\|_{C^k} \to 0$ when $h \to \infty$, for every $k \in \mathbb{N}$.

Proof. Fix a point $\bar{x} \in \partial B_1$. By Remark 4.12 there exists $\bar{r} > 0$ and a smooth function g such that

(6.1)
$$\partial B_1 \cap \mathbf{C}\left(\bar{x}, r, \nu_{B_1}(\bar{x})\right) = \partial\left\{x \in \mathbb{R}^n \,:\, \mathbf{q}^{\nu_{B_1}(\bar{x})}(x-\bar{x}) < g(\mathbf{p}^{\nu_{B_1}(\bar{x})}(x-\bar{x}))\right\} \cap \mathbf{C}\left(\bar{x}, r, \nu_{B_1}(\bar{x})\right)$$

for every $0 < r \le \bar{r}$. Furthermore, there exist $r_0 \le \bar{r}$ small enough and $f_h \in C^\infty\left(\mathbf{D}\left(\bar{x}, r, \nu_{B_1}(\bar{x})\right)\right)$ such that

(6.2)
$$\partial E_h \cap \mathbf{C}\left(\bar{x}, r, \nu_{B_1}(\bar{x})\right) = \partial\left\{x \in \mathbb{R}^n \,:\, \mathbf{q}^{\nu_{B_1}(\bar{x})}(x-\bar{x}) < f_h(\mathbf{p}^{\nu_{B_1}(\bar{x})}(x-\bar{x}))\right\} \cap \mathbf{C}\left(\bar{x}, r, \nu_{B_1}(\bar{x})\right)$$

for every h big enough and $r \le r_0$. Define $\varphi_h^{\bar{x}}(x) := f_h(g^{-1}(x))$ for every $x \in \partial B_1$. Then $\{\varphi_h^{\bar{x}}\}_{h\in\mathbb{N}}$ is a family of C^∞ functions with $\|\varphi_h^{\bar{x}}\|_{C^k}$ uniformly bounded (by Theorem 1.4) such that

$$\partial E_h \cap \mathbf{C}\left(\bar{x}, r, \nu_{B_1}(\bar{x})\right) = \{(1+\varphi_h^{\bar{x}}(x))x : x \in \partial B_1\}.$$

Hence, by a covering argument we obtain a family $\{\varphi_h\}_{h\in\mathbb{N}}$ of C^∞ functions with $\|\varphi_h\|_{C^k}$ uniformly bounded such that

$$\partial E_h = \{(1+\varphi_h(x))x : x \in \partial B_1\}.$$

By Ascoli-Arzelà and the convergence of ∂E_h to ∂B_1 in the sense of Kuratowski we obtain that $\varphi_h \to 0$ in $C^{k-1}(\partial B_1)$ for every $k \in \mathbb{N}$. □

7. Theorem 1.1 for nearly spherical sets

To prove Theorem 1.1 for nearly spherical sets we are going to write Taylor expansion for the energy. We only need to deal with the repulsive term $\mathcal{G}$, as the expansion for perimeter is well-known. To this end, we need to compute shape derivatives of the energy $\mathcal{G}$ near the ball and get a bound on the second derivative. For the convenience of the reader we make these calculations later in Section 8 as they are rather technical.

In this section, we first replace our problem with an equivalent one and write Euler-Lagrange equations for it. We do it to facilitate the computations of Section 8. Then we conclude the proof of Theorem 1.1 for nearly spherical sets given Taylor expansion. Thanks to the quantitative isoperimetric inequality for nearly-spherical sets, we see that we can be crude in the bounds of Section 8 as we have a small parameter in front of the disaggregating term.

7.1. **Changing minimization problem.** For a fixed domain E we are solving the following minimization problem.

$$\mathcal{G}(E) = \inf_{\substack{u\in H^1(\mathbb{R}^n)\\ \rho\mathbf{1}_{E^c}=0}} \left\{ \frac{1}{2}\int_{\mathbb{R}^n} \left(a_E|\nabla u|^2 + K\rho^2\right) dx : -\operatorname{div}(a_E\nabla u) = \rho, \int_{\mathbb{R}^n} \rho dx = 1 \right\}.$$

We want to get rid of the constraints and make it a minimization problem over single functions rather than over pairs. More precisely, we prove the following lemma.

Lemma 7.1. *For any $E \subset \mathbb{R}^n$ the energy $\mathcal{G}$ can be represented in the following way:*
(7.1)

$$\mathcal{G}(E) = \frac{K}{2|E|} - \inf_{\psi\in H^1(\mathbb{R}^n)} \left(\frac{1}{2}\int_{\mathbb{R}^n} a_E|\nabla\psi|^2 dx + \frac{1}{|E|}\int_E \psi dx - \frac{1}{2|E|K}\left(\int_E \psi dx\right)^2 + \frac{1}{2K}\int_E \psi^2 dx \right).$$

Proof. We use an "infinite dimension Lagrange multiplier":

$$\begin{aligned}
\mathcal{G}(E) = &\inf_{\substack{u\in H^1(\mathbb{R}^n)\\ \rho\mathbf{1}_{E^c}=0}} \Big\{ \frac{1}{2}\int_{\mathbb{R}^n} a_E|\nabla u|^2 dx + \frac{1}{2}\int_E K\rho^2 dx \\
&+ \sup_{\psi\in H^1(\mathbb{R}^n)} \left[\int_{\mathbb{R}^n} (a_E\nabla u\cdot\nabla\psi - \rho\psi) dx\right] : \int_E \rho dx = 1 \Big\} \\
= &\inf_{\substack{u\in H^1(\mathbb{R}^n)\\ \rho\mathbf{1}_{E^c}=0}} \sup_{\psi\in H^1(\mathbb{R}^n)} \left\{ \frac{1}{2}\int_{\mathbb{R}^n} a_E(|\nabla u|^2 + 2\nabla u\cdot\nabla\psi) dx + \frac{1}{2}\int_E (K\rho^2 - 2\rho\psi) dx : \int_E \rho dx = 1 \right\}.
\end{aligned}$$

The convexity of the problem allows us to use Sion minimax theorem ([Sion58, Corollary 3.3]) and interchange the infimum and the supremum:

$$\begin{aligned}
\mathcal{G}(E) &= \sup_{\psi\in H^1(\mathbb{R}^n)} \inf_{\substack{u\in H^1(\mathbb{R}^n)\\ \rho\mathbf{1}_{E^c}=0}} \left\{ \frac{1}{2}\int_{\mathbb{R}^n} a_E(|\nabla u|^2 + 2\nabla u\cdot\nabla\psi) dx + \frac{1}{2}\int_E (K\rho^2 - 2\rho\psi) dx : \int_E \rho dx = 1 \right\} \\
&= \sup_{\psi\in H^1(\mathbb{R}^n)} \left\{ \inf_{u\in H^1(\mathbb{R}^n)} \frac{1}{2}\int_{\mathbb{R}^n} a_E(|\nabla u|^2 + 2\nabla u\cdot\nabla\psi) dx + \inf_{\substack{\rho\mathbf{1}_{E^c}=0\\ \int_E \rho dx=1}} \frac{1}{2}\int_E (K\rho^2 - 2\rho\psi) dx \right\}.
\end{aligned}$$

We denote the infimums inside by I and II, that is

$$\mathrm{I} := \inf_{u \in H^1(\mathbb{R}^n)} \left\{ \frac{1}{2} \int_{\mathbb{R}^n} a_E(|\nabla u|^2 + 2\nabla u \cdot \nabla \psi) dx \right\};$$
$$\mathrm{II} := \inf_{\rho} \left\{ \frac{1}{2} \int_E (K\rho^2 - 2\rho\psi) dx : \int_E \rho dx = 1 \right\}.$$

We want to compute both I and II in terms of ψ.
For I it is immediate. Since a_E is positive we get that

$$\begin{aligned}\mathrm{I} &= \inf_{u \in H^1(\mathbb{R}^n)} \left\{ \frac{1}{2} \int_{\mathbb{R}^n} a_E(|\nabla u|^2 + 2\nabla u \cdot \nabla \psi) dx \right\} \\ &= \inf_{u \in H^1(\mathbb{R}^n)} \left\{ \frac{1}{2} \int_{\mathbb{R}^n} a_E \left(|\nabla u + \nabla \psi|^2 - |\nabla \psi|^2\right) dx \right\} \\ &= -\frac{1}{2} \int_{\mathbb{R}^n} a_E |\nabla \psi|^2 dx.\end{aligned}$$

We note that the corresponding minimizing u equals to $-\psi$.
To compute II, note that

$$\begin{aligned}\mathrm{II} &= \inf_{\rho} \left\{ \frac{1}{2} \int_E (K\rho^2 - 2\rho\psi) dx : \int_E \rho dx = 1 \right\} \\ &= \inf_{\rho} \left\{ \frac{1}{2} \int_E \left(\sqrt{K}\rho - \frac{\psi}{\sqrt{K}} \right)^2 dx : \int_E \rho dx = 1 \right\} - \frac{1}{2K} \int_E \psi^2 dx \\ &= \frac{K}{2} \inf_f \left\{ \int_E \left(f - \left(\frac{\psi}{K} - \frac{1}{|E|} \right) \right)^2 dx : \int_E f dx = 0 \right\} - \frac{1}{2K} \int_E \psi^2 dx.\end{aligned}$$

Then the minimizing function f^* is the projection in $L^2(E)$ of a function $\left(\frac{\psi}{K} - \frac{1}{|E|}\right)$ onto the linear space $\left\{f : \int_E f dx = 0\right\}$. Thus, $f^* = \left(\frac{\psi}{K} - \frac{1}{|E|}\right) - c$, where c is the constant such that $\int_E f^* = 0$, i.e. $c = \frac{\int_E \left(\frac{\psi}{K} - \frac{1}{|E|}\right)}{|E|}$. The corresponding minimizing ρ equals to $\mathbf{1}_E \frac{1}{K} \left(\psi + \frac{\left(1 - \frac{1}{K} \int_E \psi dx\right) K}{|E|} \right)$.

Bringing it all together,
(7.2)

$$\begin{aligned}\mathcal{G}(E) &= \frac{K}{2|E|} + \sup_{\psi \in H^1(\mathbb{R}^n)} \left(-\frac{1}{2} \int_{\mathbb{R}^n} a_E |\nabla \psi|^2 dx - \frac{1}{|E|} \int_E \psi dx + \frac{1}{2|E|K} \left(\int_E \psi dx \right)^2 - \frac{1}{2K} \int_E \psi^2 dx \right) \\ &= \frac{K}{2|E|} - \inf_{\psi \in H^1(\mathbb{R}^n)} \left(\frac{1}{2} \int_{\mathbb{R}^n} a_E |\nabla \psi|^2 dx + \frac{1}{|E|} \int_E \psi dx - \frac{1}{2|E|K} \left(\int_E \psi dx \right)^2 + \frac{1}{2K} \int_E \psi^2 dx \right).\end{aligned}$$

□

7.2. **Euler-Lagrange.** We now consider the following minimization problem:
(7.3)

$$\mathcal{J}(E) = \inf_{\psi \in H^1(\mathbb{R}^N)} \left(\frac{1}{2} \int_{\mathbb{R}^n} a_E |\nabla \psi|^2 dx + \frac{1}{|E|} \int_E \psi dx - \frac{1}{2|E|K} \left(\int_E \psi dx \right)^2 + \frac{1}{2K} \int_E \psi^2 dx \right).$$

Remark 7.2. Note that $\mathcal{J}(E) \le 0$. By Lemma 7.1

$$\mathcal{G}(E) = \frac{K}{2|E|} - \mathcal{J}(E).$$

By the inequality (2.1) in [DPHV19], $\mathcal{G}(E) \le C(n, K, \beta, |E|)$. This implies that

$$|\mathcal{J}(E)| \le C(n, K, \beta, |E|). \tag{7.4}$$

A minimizer for this problem exists, and it is unique by convexity. Indeed, to see coercivity of the functional note that

$$-\frac{1}{2|E|K}\left(\int_E \psi dx\right)^2 + \frac{1}{2K}\int_E \psi^2 dx \ge 0$$

by Jensen inequality. As for convexity, we use that

$$-\frac{1}{2|E|K}\left(\int_E \psi dx\right)^2 + \frac{1}{2K}\int_E \psi^2 dx = \frac{1}{2K}\int_E \left(\psi - \fint_E \psi\right)^2.$$

Note that the minimizers in the definitions of $\mathcal{J}$ and $\mathcal{G}$ coincide since the set is fixed. We denote the minimizer by ψ_E. We would also need the interior and exterior restrictions of the function ψ_E, i.e.

$$\psi_E^+ := \psi_E|_E, \quad \psi_E^- := \psi_E|_{E^c}.$$

Proposition 7.3. *The following identities hold for* ψ_E*:*

(i) (Euler-Lagrange equation, integral form) for any $\Psi \in D^1(\mathbb{R}^n)$

$$\begin{aligned} &\int_{\mathbb{R}^n} a_E \nabla\psi_E \cdot \nabla\Psi dx + \frac{1}{K}\int_E \psi_E \Psi dx + \frac{1}{|E|}\left(\int_E \Psi dx\right)\left(1 - \frac{1}{K}\int_E \psi_E dx\right) \\ &= \int_{\mathbb{R}^n} \Psi \left(\frac{\mathbf{1}_E \psi_E}{K} - \operatorname{div}(a_E \nabla\psi_E)\right) dx + \int_{\partial E} \left(\beta\nabla\psi_E^+ - \nabla\psi_E^1\right)\cdot \nu \Psi \, d\mathcal{H}^{n-1} \\ &+ \frac{1}{|E|}\left(\int_E \Psi dx\right)\left(1 - \frac{1}{K}\int_E \psi_E dx\right) = 0. \end{aligned} \tag{7.5}$$

(ii) (Euler-Lagrange equation)

$$\begin{cases} -\beta\Delta\psi_E = -\frac{1}{K}\psi_E + \frac{2}{K}\mathcal{J}(E) - \frac{1}{|E|} \text{ in } E, \\ \Delta\psi_E = 0 \text{ in } E^c, \\ \psi_E^+ = \psi_E^- \text{ on } \partial E, \\ \beta\nabla\psi_E^+ \cdot \nu = \nabla\psi_E^- \cdot \nu \text{ on } \partial E. \end{cases} \tag{7.6}$$

(iii)

$$\mathcal{J}(E) = \frac{1}{2|E|}\int_E \psi_E dx. \tag{7.7}$$

(iv) There exists a constant $C = C(n, K, \beta, |E|)$ *such that*

$$\int_{\mathbb{R}^n} a_E |\nabla\psi_E|^2 dx \le C. \tag{7.8}$$

Proof. To prove (7.7) we use ψ_E as a test function in (7.5).

To see (7.8), we use ψ_E as a test function in (7.5) and Cauchy-Schwarz inequality to get

$$\int_{\mathbb{R}^n} a_E |\nabla\psi_E|^2 dx \le -\frac{1}{|E|}\left(\int_E \psi_E dx\right).$$

Now we apply (7.7) and (7.4) to obtain

$$\int_{\mathbb{R}^n} a_E |\nabla \psi_E|^2 dx \le -2\mathcal{J}(E) \le 2C(n, K, \beta, |E|).$$

□

Proposition 7.4. *Let ψ_0 be the minimizer for $\mathcal{J}(B_1)$. Then ψ_0 is radial.*

Proof. Let $R : \mathbb{R}^n \to \mathbb{R}^n$ be any rotation. Since $R(B_1) = B_1$, $\psi_0 \circ R$ is also a minimizer for $\mathcal{J}(B_1)$. But the minimizer is unique, so we got that $\psi_0 \circ R = \psi_0$ for any rotation R. This implies that ψ_0 is radial.

□

7.3. **Proof of Theorem 1.1.** We will use the following notation.

Definition 7.5. For an open set Ω, x_Ω denotes the barycenter of Ω, namely

$$x_\Omega = \frac{1}{|\Omega|} \int_\Omega x dx.$$

We want to prove that for Q small enough the only minimizer of $\mathcal{F}(\Omega) = P(\Omega) + Q^2 \mathcal{G}(\Omega)$ for Ω nearly spherical is a ball.

We will use the following theorem proved by Fuglede.

Theorem 7.6. *(*[Fug89, Theorem 1.2]*) There exists a constant $c = c(N)$ such that for any Ω — nearly spherical set parametrized by φ with $|\Omega| = |B_1|$, $x_\Omega = 0$, the following inequality holds*

$$P(\Omega) - P(B_1) \ge c\|\varphi\|^2_{H^1(\partial B_1)}.$$

We will also need the following bound on the energy $\mathcal{J}$, see Section 8 for the proof.

Lemma 7.7. *Given $\vartheta \in (0, 1]$, there exists $\delta = \delta(N, \vartheta) > 0$ and a bounded function g such that for every nearly spherical set E parametrized by φ with $\|\varphi\|_{C^{2,\vartheta}(\partial B_1)} < \delta$ and $|E| = |B_1|$, we have*

$$\mathcal{J}(E) \ge \mathcal{J}(B_1) - g(\|\varphi\|_{C^{2,\vartheta}})\|\varphi\|^2_{H^1(\partial B_1)}.$$

Finally, we are ready to prove the main result of the paper.

Proof of Theorem 1.1. Argue by contradiction. Suppose there exists a sequence of minimizers E_h corresponding to $Q_h \to 0$ such that E_h are not balls. By Theorem 6.2 we have that starting from a certain h the sets (possibly, translated) are nearly-spherical parametrized by φ_h with $\|\varphi_h\|_{C^{2,\gamma}(\partial B_1)} < \delta$, where δ is the one of Lemma 7.7.

To apply Theorem 7.6 and Lemma 7.7 we need the sets to have barycenters at the origin. It is not necessarily true for the sequence E_h, however, we can exploit the fact that nearly-spherical sets have barycenters close to the origin. Indeed, suppose that E is the nearly-spherical set parametrized by φ_E. Then, using that the barycenter of the ball B_1 is at the origin, we have

$$\begin{aligned}
x_E &= \frac{1}{|E|} \int_{\partial B_1} x \frac{(1 + \varphi_E(x))^{n+1}}{n+1} d\mathcal{H}^{n-1}(x) \\
&= \frac{1}{|E|} \int_{\partial B_1} x \frac{(1 + \varphi_E(x))^{n+1}}{n+1} d\mathcal{H}^{n-1}(x) - \frac{1}{|E|} \int_{\partial B_1} x \, d\mathcal{H}^{n-1}(x) \\
&= \frac{1}{|E|} \int_{\partial B_1} x \frac{\sum_{i=1}^{n+1} \binom{N+1}{i} \varphi_E(x)^i}{n+1} d\mathcal{H}^{n-1}(x).
\end{aligned}$$

If $\|\varphi_E\|_{L^\infty(\partial B_1)} < 1$, then the last computation yields

$$|x_E| < C\|\varphi_E\|_{L^\infty(\partial B_1)}.$$

We chose the sequence E_h so that $E_h \to B_1$ in L^∞, thus, $x_{E_h} \to 0$. So if we now look at the sequence of sets $\tilde{E}_h = \{x - x_{E_h} : x \in E_h\}$, we see that $\tilde{E}_h \to B_1$ in L^∞ and $x_{\tilde{E}_h} = 0$. It remains to apply Theorem 6.2 to the sequence $\{E_h\}$ to see that these new translated sets are still nearly-spherical. For the sake of simplicity let us not rename the sequence and assume that the sequence $\{E_h\}$ is such that $x_{E_h} = 0$.

Now we can apply Theorem 7.6 and Lemma 7.7. We want to show that $\mathcal{F}(E_h) > F(B_1)$ for h big enough. Indeed, if Q_h is small enough, we have

$$\begin{aligned}
\mathcal{F}(E_h) = P(E_h) + Q_h^2\,\mathcal{G}(E_h) &\geq P(B_1) + c\|\varphi_h\|^2_{H^1(\partial B_1)} + Q_h^2\left(\frac{K}{2|B_1|} - \mathcal{J}(E_h)\right)\\
&\geq P(B_1) + c\|\varphi_h\|^2_{H^1(\partial B_1)} + Q_h^2\left(\frac{K}{2|B_1|} - \mathcal{J}(B_1) - c'\|\varphi_h\|^2_{H^1(\partial B_1)}\right)\\
&> P(B_1) + Q_h^2\left(\frac{K}{2|B_1|} - \mathcal{J}(B_1)\right) = \mathcal{F}(B_1).
\end{aligned}$$

□

We can now prove Corollary 1.2, which follows from Theorem 1.1 and properties of minimizers established in [DPHV19].

Proof of Corollary 1.2. Let Q_0 be the one of Theorem 1.1. Let E be an open set such that $|E| = |B_1|$. Let us show that $\mathcal{F}(E) \geq \mathcal{F}(B_1)$. If E is bounded, then $\mathcal{F}(E_h) \geq F(B_1)$ by Theorem 1.1. Assume now that E is unbounded.

We can assume that E is of finite perimeter, since otherwise $\mathcal{F}(E) = \infty$. Then, by [M, Remark 13.12], there exists a sequence $R_h \to \infty$ such that $E \cap B_{R_h} \to E$ in L^1, $P(E \cap B_{R_h}) \to P(E)$. Rescale the sets so that their volumes are the same as the one of the ball, i.e.

$$\Omega_h = \alpha_h\,(E \cap B_{R_h}) \text{ with } \alpha_h = \left(\frac{|B_1|}{|E \cap B_{R_h}|}\right)^{1/n}.$$

Note that since $|E| = |B_1|$, $\alpha_h \to 1$, so also for Ω_h we have $|\Omega_h \Delta E| \to 0$, $P(\Omega_h) \to P(E)$. Now, by the continuity of the functional $\mathcal{G}$ in L^1 (see [DPHV19, Proposition 2.6]), we get

$$F(\Omega_h) = P(\Omega_h) + \mathcal{G}(\Omega_h) \to P(E) + \mathcal{G}(E) = \mathcal{F}(E). \tag{7.9}$$

On the other hand, $\Omega_h \subset \alpha_h B_{R_h}$, so it is bounded and hence, by Theorem 1.1,

$$\mathcal{F}(\Omega_h) \geq \mathcal{F}(B_1) \text{ for every } h.$$

Combining the last inequality with (7.9), we get $\mathcal{F}(E) \geq \mathcal{F}(B_1)$. Thus, the infimum in the problem $(\mathcal{P}_{\beta,K,Q})$ is achieved on balls.

Let us show that the only minimizers are the balls. Let E be a minimizer for $(\mathcal{P}_{\beta,K,Q})$. If E is bounded, then by Theorem 1.1 it should be a ball of radius 1. We now explain why E cannot be unbounded. Indeed, suppose the contrary holds. Then there we can find a sequence of points x_k such that $x_k \in E$, $|x_k - x_j| \geq 1$ for $k \neq j$ (for example, we can define $x_k := E \backslash B_{\max\{|x_1|,|x_2|,\ldots,|x_{k-1}|\}+1}$). Now, by density estimates for minimizers (Theorem 2.3 (v)), we have

$$\frac{|B_r(x) \cap E|}{|B_r|} \geq \frac{1}{C} \text{ for } x \in E,\ r \in (0, \overline{r}). \tag{7.10}$$

Note that even though Theorem 2.3 (v) deals with minimizers of $(\mathcal{P}_{\beta,K,Q,R})$, the constants C and $\overline{r}$ do not depend on R, so it applies in our case. It remains to use (7.10) for $x = x_k$ and $r = \min(1/2\overline{r}, 1/2)$ to see that

$$|E| \geq \sum_{k=1}^{\infty} |B_r(x_k) \cap E| \geq \sum_{k=1}^{\infty} \frac{|B_r|}{C} = \infty,$$

which contradicts the fact that $|E| = |B_1|$. Thus, E is bounded and it is a ball of radius 1. □

8. Proof of Lemma 7.7

We will need the following technical lemma, which is almost identical to [BDPV15, Lemma A.1]. Since we need a slightly different conclusion than in [BDPV15], we repeat the proof here. Throughout this section we will be using the following notation.

Notation 8.1. We denote by $J_{\Phi_t}(x)$ the jacobian of Φ_t at x:

$$J_{\Phi_t}(x) = \det \nabla\Phi_t(x).$$

Lemma 8.2. *Given $\vartheta \in (0,1]$ there exists $\delta = \delta(n,\gamma) > 0$, a modulus of continuity ω, and a bounded function g such that for every nearly spherical set E parametrized by φ with $\|\varphi\|_{C^{2,\vartheta}(\partial B_1)} < \delta$ and $|\Omega| = |B_1|$, we can find an autonomous vector field X_φ for which the following holds true:*

(i) $\operatorname{div} X_\varphi = 0$ *in a δ-neighborhood of ∂B_1;*

(ii) if $\Phi_t := \Phi(t,x)$ is the flow of X_φ, i.e.

$$\partial_t \Phi_t = X_\varphi(\Phi_t), \qquad \Phi_0(x) = x,$$

then $\Phi_1(\partial B_1) = \partial E$ and $|\Phi_t(B_1)| = |B_1|$ for all $t \in [0,1]$;

(iii) denote $E_t := \Phi_t(B_1)$, then

$$\|\Phi_t - Id\|_{C^{2,\vartheta}} \leq \omega(\|\varphi\|_{C^{2,\vartheta}(\partial B_1)}) \text{ for every } t \in [0,1], \tag{8.1}$$

$$|J_\Phi| \leq g(\|\varphi\|_{C^{2,\vartheta}(\partial B_1)}) \text{ in a neighborhood of } B_1, \tag{8.2}$$

$$\|X \cdot \nu\|_{H^1(\partial E_t)} \leq g(\|\varphi\|_{C^{2,\vartheta}(\partial B_1)})\, \|\varphi\|_{H^1(\partial B_1)}, \tag{8.3}$$

and for the tangential part of X, defined as $X = X - (X \cdot \nu)\nu$, there holds

$$|X^\tau| \leq \omega(\|\varphi\|_{C^{2,\vartheta}(\partial B_1)})\, |X \cdot \nu| \text{ on } \partial E_t. \tag{8.4}$$

Proof. Such a vector field can be constructed for any smooth set, see for example [Dam02]. However, for the ball one can write an explicit expression in a neighborhood of ∂B_1. The proof for the case of the ball can be found in [BDPV15, Lemma A.1]. For the convenience of the reader we provide the expression here, as well as brief explanation of how to get the needed bounds. In polar coordinates, $\rho = |x|$, $\theta = x/|x|$ the field looks like this:

$$X_\varphi(\rho,\theta) = \frac{(1+\varphi(\theta))^n - 1}{n\rho^{n-1}}\theta,$$

$$\Phi_t(\rho,\theta) = \left(\rho^n + t\left((1+\varphi(\theta))^n - 1\right)\right)^{\frac{1}{N}}\theta$$

for $|\rho - 1| \ll 1$. Then we extend this vector field globally in order to satisfy (8.1). Notice that (8.2) is a direct consequence of (8.1).

By direct computation we get

$$(X\cdot\theta)\circ\Phi_t - X\cdot\nu_{\partial B_1} = (X\cdot\nu_{\partial B_1})f \text{ on } \partial B_1, \tag{8.5}$$

with $\|f\|_{C^{2,\vartheta}(\partial B_1)} \le \omega\left(\|\varphi\|_{C^{2,\vartheta}(\partial B_1)}\right)$. Now we can get the bound (8.3). Indeed, (8.5) together with (8.2) gives us

$$\|X\cdot\nu\|_{H^1(\partial E_t)} \le g\left(\|\varphi\|_{C^{2,\vartheta}(\partial B_1)}\right)\|X\cdot\nu\|_{H^1(\partial B_1)}.$$

From the definition of X, on ∂B_1 we have

$$\varphi - X\cdot\nu = \frac{1}{n}\sum_{i=2}^{n}\binom{n}{i}\varphi^i$$

and thus,

$$\|\varphi - X\cdot\nu\|_{H^1(\partial B_1)} \le \omega\left(\|\varphi\|_{C^{2,\vartheta}(\partial B_1)}\right)\|X\cdot\nu\|_{H^1(\partial B_1)},$$

yielding the inequality (8.3).

To see (8.4) we use that by definition X is parallel to θ close to ∂B_1. Thus,

$$\begin{aligned}
|X^\tau\circ\Phi_t| &= |((X\cdot\theta)\,\theta)\circ\Phi_t - ((X\cdot\nu)\,\nu)\circ\Phi_t| \\
&= \Big|\,(X\cdot\nu_{\partial B_1})\left(1+\omega\left(\|\varphi\|_{C^{2,\vartheta}(\partial B_1)}\right)\right)\nu_{\partial B_1}\left(1+\omega\left(\|\varphi\|_{C^{2,\vartheta}(\partial B_1)}\right)\right) \\
&\quad - (X\cdot\nu_{\partial B_1})\left(1+\omega\left(\|\varphi\|_{C^{2,\vartheta}(\partial B_1)}\right)\right)\nu_{\partial B_1}\left(1+\omega\left(\|\varphi\|_{C^{2,\vartheta}(\partial B_1)}\right)\right)\Big| \\
&= \omega\left(\|\varphi\|_{C^{2,\vartheta}(\partial B_1)}\right)|(X\cdot\nu)\circ\Phi_t|\,.
\end{aligned}$$

□

In what follows we omit the subscript φ for brevity.

8.1. **First derivative.** We want to compute $\frac{d}{dt}\mathcal{J}(E_t)$.

Let ψ_t be the minimizer in the minimization problem (7.3) for E_t. Recall that by (7.6) it means that ψ_t satisfies

$$\begin{cases}
-\beta\Delta\psi_t = -\frac{1}{K}\psi_t + \frac{2}{K}\mathcal{J}(E_t) - \frac{1}{|B_1|} \text{ in } E_t,\\
\Delta\psi_t = 0 \text{ in } E_t^c,\\
\psi_t^+ = \psi_t^- \text{ on } \partial E_t,\\
\beta\nabla\psi_t^+\cdot\nu = \nabla\psi_t^-\cdot\nu \text{ on } \partial E_t.
\end{cases} \tag{8.6}$$

First we notice that ψ_t is regular since it is a solution to a transmission problem. More precisely, by Lemma 5.2, the following holds.

Proposition 8.3. *There exists $\delta > 0$ such that if $\|\varphi\|_{C^{2,\vartheta}(\partial B_1)} < \delta$, then*

$$\|\psi_t\|_{C^2(\overline{E_t})} \le g(\|\varphi\|_{C^{2,\vartheta}(\partial B_1)}) \text{ for every } t\in[0,1],$$

where g is a bounded function.

To compute the derivative of $\mathcal{J}(E_t)$ we would like to use Hadamard formula (see [HP, Chapter 5]). For that, we first need to prove the following proposition.

Proposition 8.4. *The function $t\mapsto\psi_t$ is differentiable in t and its derivative $\dot\psi_t$ satisfies*

$$\begin{cases}
-\beta\Delta\dot\psi_t = -\frac{1}{K}\dot\psi_t + \frac{2}{K}\dot{\mathcal{J}}(E_t) \text{ in } E_t,\\
\Delta\dot\psi_t = 0 \text{ in } E_t^c,\\
\dot\psi_t^+ - \dot\psi_t^- = -\left(\nabla\psi_t^+ - \nabla\psi_t^-\right)\cdot\nu(X\cdot\nu) \text{ on } \partial E_t,\\
\beta\nabla\dot\psi_t^+\cdot\nu - \nabla\dot\psi_t^-\cdot\nu = -\left(\left(\beta\nabla[\nabla\psi_t^+] - \nabla[\nabla\psi_t^-]\right)X\right)\cdot\nu \text{ on } \partial E_t.
\end{cases} \tag{8.7}$$

Proof. The proof is standard, see ([HP, Chapter 5]) for the general strategy and ([ADK07, Theorem 3.1]) for a different kind of a transmission problem. We were unable to find a result covering our particular case in the literature, so we provide a proof here.

We first deal with material derivative of the function ψ, i.e. we shall look at the function $t \mapsto \tilde{\psi}_t := \psi_t(\Phi_t(x))$. The advantage is that its derivative in time is in H^1 as we will see. Note that the time derivative of ψ_t itself is not in H^1 as it has a jump on ∂E_t.

Step 1: moving everything to a fixed domain.

We introduce the following notation:

$$A_t(x) := D\Phi_t^{-1}(x)\left(D\Phi_t^{-1}\right)^t(x) J_{\Phi_t}(x).$$

Note that A_t is symmetric and positive definite and for t small enough it is elliptic with a constant independent of t.

Now we perform a change of variables in Euler-Lagrange equation for ψ_t (7.5) to get Euler-Lagrange equation for $\tilde{\psi}_t$:

(8.8)
$$\int_{\mathbb{R}^n} \nabla\Psi\left(a_B A_t \nabla\tilde{\psi}_t\right)dx + \frac{1}{K}\int_B \Psi\tilde{\psi}_t J_{\Phi_t}(x)dx + \frac{1}{|B|}\left(\int_B \Psi J_{\Phi_t}(x)dx\right)\left(1 - \frac{1}{K}\int_B \tilde{\psi}_t J_{\Phi_t}(x)dx\right) = 0$$

for any $\Psi \in D^1(\mathbb{R}^n)$.

Step 2: convergence of the material derivative.

Let us for convenience denote

$$f(t) := \frac{1}{|B|}\left(1 - \frac{1}{K}\int_B \tilde{\psi}_t J_{\Phi_t}(x)dx\right).$$

We write the difference of equations (8.8) for $\tilde{\psi}_{t+h}$ and $\tilde{\psi}_t$ and divide it by h to get

$$\begin{aligned}
&\int_{\mathbb{R}^n} \nabla\Psi\left(a_B \frac{A_{t+h}\nabla\tilde{\psi}_{t+h} - A_t\nabla\tilde{\psi}_t}{h}\right)dx + \frac{1}{K}\int_B \Psi\left(\frac{\tilde{\psi}_{t+h} - \tilde{\psi}_t}{h}\right)J_{\Phi_t}(x)dx \\
&+ \frac{1}{K}\int_B \Psi\tilde{\psi}_{t+h}\frac{J_{\Phi_{t+h}} - J_{\Phi_t}}{h}dx + \left(\int_B \Psi J_{\Phi_t}(x)dx\right)\frac{f(t+h) - f(t)}{h} \\
&+ \left(\int_B \Psi \frac{J_{\Phi_{t+h}}(x) - J_{\Phi_t}(x)}{h}dx\right)f(t+h) = 0
\end{aligned}$$

for any $\Psi \in D^1(\mathbb{R}^n)$.

Now, introducing also $g_h(x) := \frac{\tilde{\psi}_{t+h} - \tilde{\psi}_t}{h}$ for convenience, we get

$$\begin{aligned}
&\int_{\mathbb{R}^n} \nabla\Psi\left(a_B A_{t+h}\nabla g_h\right)dx + \frac{1}{K}\int_B \Psi g_h J_{\Phi_t}(x)dx + \int_{\mathbb{R}^n}\nabla\Psi\left(a_B \frac{A_{t+h} - A_t}{h}\nabla\tilde{\psi}_t\right)dx \\
&+ \frac{1}{K}\int_B \Psi\tilde{\psi}_{t+h}\frac{J_{\Phi_{t+h}} - J_{\Phi_t}}{h}dx + \left(\int_B \Psi J_{\Phi_t}(x)dx\right)\frac{f(t+h) - f(t)}{h} \qquad (8.9)\\
&+ \left(\int_B \Psi \frac{J_{\Phi_{t+h}}(x) - J_{\Phi_t}(x)}{h}dx\right)f(t+h) = 0
\end{aligned}$$

for any $\Psi \in D^1(\mathbb{R}^n)$.

Now we want to get a uniform bound on g_h in $D^1(\mathbb{R}^n)$. To do that we argue in a way similar to the proof of (7.8). We use g_h as a test function in (8.9) and get

$$\int_{\mathbb{R}^n} a_B \nabla g_h \cdot (A_{t+h}\nabla g_h)\,dx + \frac{1}{K}\int_B g_h^2 J_{\Phi_t}(x)dx + \int_{\mathbb{R}^n} a_B \nabla g_h \cdot \left(\frac{A_{t+h}-A_t}{h}\nabla\tilde{\psi}_t\right)dx$$
$$+\frac{1}{K}\int_B g_h\tilde{\psi}_{t+h}\frac{J_{\Phi_{t+h}}-J_{\Phi_t}}{h}\,dx + \left(\int_B g_h J_{\Phi_t}(x)dx\right)\frac{f(t+h)-f(t)}{h}$$
$$+\left(\int_B g_h \frac{J_{\Phi_{t+h}}(x)-J_{\Phi_t}(x)}{h}\,dx\right)f(t+h) = 0.$$

Since $\frac{A(t+h,x)-A(t,x)}{h}$ is bounded in L^∞ and A_t is uniformly elliptic we know that there exist some positive constant c independent of h such that

$$\int_{\mathbb{R}^n} a_B\nabla g_h\cdot(A_{t+h}\nabla g_h)\,dx+\int_{\mathbb{R}^n} a_B\nabla g_h\cdot\left(\frac{A_{t+h}-A_t}{h}\nabla\tilde{\psi}_t\right)dx \geq c\int_{\mathbb{R}^n}|\nabla g_h|^2\,dx - C\int_{\mathbb{R}^n}|\nabla\psi_t|^2\,dx.$$

Thus,

(8.10)
$$c\int_{\mathbb{R}^n}|\nabla g_h|^2\,dx + \frac{1}{K}\int_B g_h^2 J_{\Phi_t}(x)dx \leq C\int_{\mathbb{R}^n}|\nabla\psi_t|^2\,dx + \frac{1}{K}\int_B\left|g_h\tilde{\psi}_{t+h}\frac{J_{\Phi_{t+h}}-J_{\Phi_t}}{h}\right|dx$$
$$+\left|\frac{f(t+h)-f(t)}{h}\right|\int_B|g_hJ_{\Phi_t}(x)|dx + |f(t+h)|\int_B\left|g_h\frac{J_{\Phi_{t+h}}(x)-J_{\Phi_t}(x)}{h}\right|dx$$
$$\leq C + C\int_B|g_h|\,dx + \left|\frac{f(t+h)-f(t)}{h}\right|\int_B|g_h|dx + |f(t+h)|\int_B|g_h|\,dx,$$

where in the last inequality we used the inequality (7.8), Proposition 8.3, and (8.1). We want to show now that f is bounded and Lipschitz. Indeed, we recall the definition of f and use the definition of $\tilde{\psi}_t$ and (7.7)

$$f(t) = \frac{1}{|B|}\left(1-\frac{1}{K}\int_B\tilde{\psi}_tJ_{\Phi_t}(x)dx\right) = \frac{1}{|B|}-\frac{2}{K}\mathcal{J}(E_t).$$

We get that f is bounded by (7.4). To get Lipschitz continuity, we notice that by direct computation in Lagrangian coordinates one can get that $\frac{\mathcal{J}(E_{t+h})-\mathcal{J}(E_t)}{h}$ is uniformly bounded, see [DPHV19, Lemma 3.2]. Plugging this information into (8.10), we get

$$c\int_{\mathbb{R}^n}|\nabla g_h|^2\,dx + \frac{1}{K}\int_B g_h^2J_{\Phi_t}(x)dx \leq C + C\int_B|g_h|\,dx.$$

Finally, we use Young's inequality and (8.1) to obtain

$$c\int_{\mathbb{R}^n}|\nabla g_h|^2\,dx + \frac{1}{2K}\int_B g_h^2J_{\Phi_t}(x)dx \leq C. \tag{8.11}$$

Thus, g_h is uniformly bounded in $D^1(\mathbb{R}^n)$ and up to a subsequence, there exists a weak limit g_0 as h goes to zero. Note that g_0 satisfies

$$\begin{aligned}&\int_{\mathbb{R}^n}\nabla\Psi\,(a_BA_t\nabla g_0)\,dx + \int_{\mathbb{R}^n}\nabla\Psi\left(a_B\frac{d}{dt}A_t\nabla\tilde{\psi}_t\right)dx + \frac{1}{K}\int_B\Psi g_0J_{\Phi_t}(x)\,dx\\ &+\frac{1}{K}\int_B\Psi\tilde{\psi}_t\dot{J}_{\Phi_t}\,dx - \frac{1}{|B|K}\left(\int_B\Psi J_{\Phi_t}(x)dx\right)\left(\int_B g_0J_{\Phi_t}(x)\,dx - \int_B\tilde{\psi}_t\dot{J}_{\Phi_t}\,dx\right)\\ &+\frac{1}{|B|}\left(\int_B\Psi\dot{J}_{\Phi_t}(x)\,dx\right)\left(1-\frac{1}{K}\int_B\tilde{\psi}_tJ_{\Phi_t}(x)\,dx\right) = 0\end{aligned} \tag{8.12}$$

for any $\Psi \in D^1(\mathbb{R}^n)$. Let us show that the equation (8.12) has unique solution. To that end, assume that both g_0 and g_0' are solutions of (8.12). Then their difference $w = g_0 - g_0'$ satisfies the following equation

(8.13)
$$\int_{\mathbb{R}^n} \nabla\Psi\,(a_B A_t \nabla w)\,dx + \frac{1}{K}\int_B \Psi w J_{\Phi_t}(x)\,dx - \frac{1}{|B|K}\int_B \Psi J_{\Phi_t}(x)dx \int_B w J_{\Phi_t}(x)\,dx = 0$$

for any $\Psi \in D^1(\mathbb{R}^n)$. Since $w \in D^1(\mathbb{R}^n)$, we can test (8.13) with w and get

$$\int_{\mathbb{R}^n} \nabla w\,(a_B A_t \nabla w)\,dx + \frac{1}{K}\int_B w^2 J_{\Phi_t}(x)\,dx - \frac{1}{|B|K}\left(\int_B w J_{\Phi_t}(x)\,dx\right)^2 = 0.$$

By Cauchy-Schwartz inequality, it yields

$$\int_{\mathbb{R}^n} \nabla w\,(a_B A_t \nabla w)\,dx \le 0,$$

which in turn gives us $w = 0$ by ellipticity of A_t. Thus, the solution of (8.12) is unique and thus the whole sequence g_h converges to g_0.

To get the strong convergence of the material derivative, we observe that using g_h as a test function in its Euler-Lagrange equation, we get the convergence of the norm in H^1 to the norm of g_0. That, together with weak convergence, gives us strong convergence of g_h.

Step 3: existence of the shape derivative.

We want to show that

$$\dot{\psi}_t = \frac{d}{dt}\tilde{\psi}_t - X \cdot \nabla\psi_t$$

in $D^1(E_t) \cap D^1(E_t^c)$. Indeed, since $\psi_t(x) = \dot{\psi}_t(\Phi_t^{-1}(x))$, we have

$$\frac{\psi_{t+h}(x) - \psi_{t+h}(x)}{h} = \frac{\psi_{t+h}(\Phi_{t+h}^{-1}(x)) - \psi_t(\Phi_{t+h}^{-1}(x))}{h} + \frac{\psi_t(\Phi_{t+h}^{-1}(x)) - \psi_t(\Phi_t^{-1}(x))}{h}. \tag{8.14}$$

The first term on the right-hand side converges strongly to $\frac{d}{dt}\psi_t(\Phi_t^{-1}(x))$ as h goes to 0 by Step 2 and continuity of Φ_t. As for the second term, by Proposition 8.3 and the definition of Φ, it converges to $-\nabla\psi_t(\Phi_t^{-1}(x)) \cdot X$. strongly in $D^1(E_t) \cap D^1(E_t^c)$.

Step 4: the equation for the shape derivative.

Now that we know that $t \mapsto \psi_t$ is differentiable, we can differentiate the Euler-Lagrange equation for ψ_t given by (8.6) and we get

$$\begin{cases} -\beta\Delta\dot{\psi}_t = -\frac{1}{K}\dot{\psi}_t + \frac{2}{K}\dot{\mathcal{J}}(E_t) \text{ in } E_t \\ \Delta\dot{\psi}_t = 0 \text{ in } E_t^c \\ \dot{\psi}_t^+ - \dot{\psi}_t^- = -\left(\nabla\psi_t^+ - \nabla\psi_t^-\right)\cdot X \text{ on } \partial E_t \\ \beta\nabla\dot{\psi}_t^+ \cdot \nu - \nabla\dot{\psi}_t^1 \cdot \nu = -\left(\left(\beta\nabla[\nabla\psi_t^+] - \nabla[\nabla\psi_t^-]\right)X\right)\cdot\nu \text{ on } \partial E_t \end{cases}.$$

Now we can use the boundary conditions in (8.6) to get rid of the tangential part in the right-hand side. Indeed,

$$-\left(\nabla\psi_t^+ - \nabla\psi_t^-\right)\cdot X = -\left(\nabla^\tau\psi_t^+ - \nabla^\tau\psi_t^-\right)\cdot X^\tau - \left(\nabla\psi_t^+ - \nabla\psi_t^-\right)\cdot\nu(X\cdot\nu)$$

and $\nabla^\tau\psi_t^+ = \nabla^\tau\psi_t^-$ by differentiating the equality $\psi_t^+ = \psi_t^-$ on the boundary of E_t. □

The following observation, which is a consequence of equality for $\dot{\psi}_t$ will be useful for us.

Lemma 8.5. *There exists $f \in H^{3/2}(E_t) \cap H^{3/2}(E_t^c)$ such that*

$$f^\pm = \nabla\psi_t^\pm \cdot X \text{ on } \partial E_t, \quad \|f^\pm\|_{H^{3/2}} \le C\|\nabla\psi_t^\pm \cdot X\|_{H^1(\partial E_t)}. \tag{8.15}$$

Consider a function $v := \dot\psi_t + f$*, Then* v *satisfies the equation*

$$\begin{cases} -\beta\Delta v = -\frac{1}{K}v + \frac{2}{K}\dot{\mathcal{J}}(E_t) - \beta\Delta f + \frac{1}{K}f \ \text{ in } E_t, \\ \Delta v = \Delta f \ \text{ in } E_t^c, \\ v^+ - v^- = 0 \ \text{ on } \partial E_t, \\ \beta\nabla v^+\cdot\nu - \nabla v^-\cdot\nu = \left(-\left(\beta\nabla[\nabla\psi_t^+] - \nabla[\nabla\psi_t^-]\right)X + \beta\nabla f^+ - \nabla f^-\right)\cdot\nu \ \text{ on } \partial E_t. \end{cases}$$

$$v = \dot\psi_t^\pm + \nabla\psi_t^\pm\cdot X \ \text{ on } \partial E_t. \tag{8.16}$$

Moreover, the following bounds hold:

$$\|v\|_{W^{1,2}(E_t)} + \|v\|_{D^{1,2}(E_t^c)} \le C\left(|\dot{\mathcal{J}}(E_t)| + \|X\cdot\nu\|_{H^1(\partial E_t)}\right); \tag{8.17}$$

$$\|v\|_{L^{2^*}(\mathbb{R}^n)} \le C\left(|\dot{\mathcal{J}}(E_t)| + \|X\cdot\nu\|_{H^1(\partial E_t)}\right). \tag{8.18}$$

Proof. The function f exists since $\nabla\psi_t^\pm\cdot X\in H^1(\partial E_t)$. The equation for v follows from the equation for $\dot\psi_t$ and the definition of f. Using divergence theorem, we get

$$\int_{E_t}\frac{1}{K}v^2dx + \int_{E_t}\beta|\nabla v|^2dx + \int_{E_t^c}|\nabla v|^2dx = \int_{E_t}\left(\frac{2}{K}\dot{\mathcal{J}}(E_t) - \beta\Delta f + \frac{1}{K}f\right)v\,dx$$
$$-\int_{E_t^c}\Delta f\,v\,dx + \int_{\partial E_t}\left(\left(-\left(\beta\nabla[\nabla\psi_t^+] - \nabla[\nabla\psi_t^-]\right)X + \beta\nabla f^+ - \nabla f^-\right)\cdot\nu\right)v\,dx,$$

which by Young, Cauchy-Schwarz, and trace inequalities, recalling (8.15), implies that

$$\|v\|_{W^{1,2}(E_t)} + \|v\|_{D^1(E_t^c)} \le C\left(|\dot{\mathcal{J}}(E_t)| + \|\nabla\psi_t\cdot X\|_{H^1(\partial E_t)}\right),$$

which in turn implies by Proposition 8.3 and (8.4)

$$\|v\|_{W^{1,2}(E_t)} + \|v\|_{D^1(E_t^c)} \le C\left(|\dot{\mathcal{J}}(E_t)| + \|X\cdot\nu\|_{H^1(\partial E_t)}\right).$$

Moreover, we also can bound the L^{2^*} norm of v. Indeed, since v doesn't have a jump on the boundary of E_t, we know by (8.17) that it belongs to the space $D^1(\mathbb{R}^n)$. Thus, employing Gagliardo-Nirenberg-Sobolev inequality we get (8.18).

□

Proposition 8.6. *For any* $t\in[0,1]$,

$$\begin{aligned}\dot{\mathcal{J}}(E_t) &= \left(1 - \frac{1}{K}\int_{E_t}\psi_t dx\right)\frac{1}{|E_t|}\int_{\partial E_t}\psi_{E_t}(X\cdot\nu)d\mathcal{H}^{n-1} \\ &\quad + \frac{1}{2}\int_{\partial E_t}\left(\beta|\nabla\psi_t^+|^2 - |\nabla\psi_t^-|^2\right)(X\cdot\nu)d\mathcal{H}^{n-1} + \frac{1}{2K}\int_{\partial E_t}\psi_t^2(X\cdot\nu)d\mathcal{H}^{n-1} \\ &\quad - \int_{\partial E_t}\left(\nabla\psi_t^-\cdot\nu\right)\left(\left(\nabla\psi_t^+ - \nabla\psi_t^-\right)\cdot\nu\right)(X\cdot\nu)d\mathcal{H}^{n-1} \\ &= \left(1 - \frac{1}{K}\int_{E_t}\psi_t dx\right)\frac{1}{|E_t|}\int_{E_t}\operatorname{div}(\psi_t X)dx + \frac{1}{2}\int_{\mathbb{R}^n}\operatorname{div}(a_{E_t}|\nabla\psi_t|^2X)dx \\ &\qquad + \frac{1}{2K}\int_{E_t}\operatorname{div}(\psi_t^2X)dx - \int_{\mathbb{R}^n}\operatorname{div}\left(a_{E_t}\left(\nabla\psi_t\cdot\nu\right)^2X\right)dx.\end{aligned}$$

In particular,

$$\dot{\mathcal{J}}(B_1) = 0.$$

Proof. We note that by (7.7)

$$\frac{d}{dt}\mathcal{J}(E_t) = \frac{1}{2|E_t|}\int_{E_t}\dot{\psi}_t dx + \frac{1}{2|E_t|}\int_{\partial E_t}\psi_{E_t}(X\cdot\nu)d\mathcal{H}^{n-1}.$$

Now we use the definition of $\mathcal{J}$ to get

$$\begin{aligned}\dot{\mathcal{J}}(E_t) = &\int_{\mathbb{R}^n} a_{E_t}\nabla\psi_t\cdot\nabla\dot{\psi}_t dx + \frac{1}{2}\int_{\partial E_t}\left(\beta|\nabla\psi_t^+|^2 - |\nabla\psi_t^-|^2\right)(X\cdot\nu)d\mathcal{H}^{n-1} \\ &+ 2\dot{\mathcal{J}}(E_t) - \frac{2}{K}|E|2\dot{\mathcal{J}}(E_t)\mathcal{J}(E_t) + \frac{1}{K}\int_{E_t}\dot{\psi}_t\psi_t dx + \frac{1}{2K}\int_{\partial E_t}\psi_t^2(X\cdot\nu)dx.\end{aligned}$$

Using (8.7), we obtain

$$\begin{aligned}\dot{\mathcal{J}}(E_t) &= -\frac{1}{|E_t|}\left(\int_{E_t}\dot{\psi}_t dx\right)\left(1 - \frac{1}{K}\int_{E_t}\psi_t dx\right) + 2\dot{\mathcal{J}}(E_t)\left(1 - \frac{1}{K}\int_{E_t}\psi_t dx\right) \\ &\quad + \frac{1}{2}\int_{\partial E_t}\left(\beta|\nabla\psi_t^+|^2 - |\nabla\psi_t^-|^2\right)(X\cdot\nu)d\mathcal{H}^{n-1} + \frac{1}{2K}\int_{\partial E_t}\psi_t^2(X\cdot\nu)dx \\ &\quad + \int_{\partial E_t}\left(\beta\dot{\psi}^+\nabla\psi_t^+\cdot\nu - \dot{\psi}^-\nabla\psi_t^-\cdot\nu\right)d\mathcal{H}^{n-1} \\ &= \left(1 - \frac{1}{K}\int_{E_t}\psi_t dx\right)\frac{1}{|E_t|}\int_{\partial E_t}\psi_{E_t}(X\cdot\nu)d\mathcal{H}^{n-1} \\ &\quad + \frac{1}{2}\int_{\partial E_t}\left(\beta|\nabla\psi_t^+|^2 - |\nabla\psi_t^-|^2\right)(X\cdot\nu)d\mathcal{H}^{n-1} + \frac{1}{2K}\int_{\partial E_t}\psi_t^2(X\cdot\nu)d\mathcal{H}^{n-1} \\ &\quad - \int_{\partial E_t}(\nabla\psi_t^-\cdot\nu)\left((\nabla\psi_t^+ - \nabla\psi_t^-)\cdot\nu\right)(X\cdot\nu)d\mathcal{H}^{n-1} \\ &= \left(1 - \frac{1}{K}\int_{E_t}\psi_t dx\right)\frac{1}{|E_t|}\int_{E_t}\operatorname{div}(\psi_t X)dx + \frac{1}{2}\int_{\mathbb{R}^n}\operatorname{div}(a_{E_t}|\nabla\psi_t|^2 X)dx \\ &\qquad + \frac{1}{2K}\int_{E_t}\operatorname{div}(\psi_t^2 X)dx - \int_{\mathbb{R}^n}\operatorname{div}\left(a_{E_t}(\nabla\psi_t\cdot\nu)^2 X\right)dx.\end{aligned}$$

Note that from the second to last expression it is easy to see that $\dot{\mathcal{J}}(B_1) = 0$ as ψ_0 is radial by Proposition 7.4 and the volume of E_t is constant (hence $\int_{\partial B_1}(X\cdot\nu)d\mathcal{H}^{n-1} = 0$). □

8.2. **Second derivative.** Now we differentiate again to get

$$
\begin{aligned}
\ddot{\mathcal{J}}(E_t) = &-\frac{2}{K}\dot{\mathcal{J}}(E_t)\int_{E_t}\operatorname{div}(\psi_t X)dx \\
&+\frac{1-\frac{2}{K}|E_t|\mathcal{J}(E_t)}{|E_t|}\left(\int_{E_t}\operatorname{div}(\dot{\psi}_t X)dx + \int_{\partial E_t}\operatorname{div}(\psi_t X)(X\cdot\nu)d\mathcal{H}^{n-1}\right) \\
&+\int_{\partial E_t}(\beta\nabla\psi_t^+\cdot\nabla\dot{\psi}_t^+ - \nabla\psi_t^-\cdot\nabla\dot{\psi}_t^-)(X\cdot\nu)d\mathcal{H}^{n-1} \\
&+\frac{1}{2}\int_{\partial E_t}\nabla\left[\beta|\nabla\psi_t^+|^2 - |\nabla\psi_t^-|^2\right]\cdot X(X\cdot\nu)d\mathcal{H}^{n-1} \\
&+\frac{1}{K}\int_{\partial E_t}\psi_t\dot{\psi}_t(X\cdot\nu)d\mathcal{H}^{n-1} + \frac{1}{K}\int_{\partial E_t}\psi_t\nabla\psi_t\cdot X(X\cdot\nu)d\mathcal{H}^{n-1} \\
&-2\int_{\partial E_t}\left(\beta\left(\nabla\dot{\psi}_t^+\cdot\nu\right)\left(\nabla\psi_t^+\cdot\nu\right) - \left(\nabla\dot{\psi}_t^-\cdot\nu\right)\left(\nabla\psi_t^-\cdot\nu\right)\right)(X\cdot\nu)d\mathcal{H}^{n-1} \\
&-\int_{\partial E_t}\nabla\left[\beta(\nabla\psi_t^+\cdot\nu)^2 - (\nabla\psi_t^-\cdot\nu)^2\right]\cdot X(X\cdot\nu)d\mathcal{H}^{n-1}.
\end{aligned}
$$

Using that the vector field X is divergence-free in the neighborhood of ∂B_1 we get for t small enough

(8.19)

$$
\begin{aligned}
\ddot{\mathcal{J}}(E_t) = &-\frac{2}{K}\dot{\mathcal{J}}(E_t)\int_{\partial E_t}\psi_t(X\cdot\nu)d\mathcal{H}^{n-1} \\
&+\frac{1-\frac{2}{K}|E_t|\mathcal{J}(E_t)}{|E_t|}\left(\int_{\partial E_t}\dot{\psi}_t(X\cdot\nu)d\mathcal{H}^{n-1} + \int_{\partial E_t}(\nabla\psi_t^+\cdot X)(X\cdot\nu)d\mathcal{H}^{n-1}\right) \\
&+\int_{\partial E_t}(\beta\nabla\psi_t^+\cdot\nabla\dot{\psi}_t^+ - \nabla\psi_t^-\cdot\nabla\dot{\psi}_t^-)(X\cdot\nu)d\mathcal{H}^{n-1} \\
&+\frac{1}{2}\int_{\partial E_t}\nabla\left[\beta|\nabla\psi_t^+|^2 - |\nabla\psi_t^-|^2\right]\cdot X(X\cdot\nu)d\mathcal{H}^{n-1} \\
&+\frac{1}{K}\int_{\partial E_t}\psi_t\dot{\psi}_t^+(X\cdot\nu)d\mathcal{H}^{n-1} + \frac{1}{K}\int_{\partial E_t}\psi_t\nabla\psi_t^+\cdot X(X\cdot\nu)d\mathcal{H}^{n-1} \\
&-2\int_{\partial E_t}\left(\beta\left(\nabla\dot{\psi}_t^+\cdot\nu\right)\left(\nabla\psi_t^+\cdot\nu\right) - \left(\nabla\dot{\psi}_t^-\cdot\nu\right)\left(\nabla\psi_t^-\cdot\nu\right)\right)(X\cdot\nu)d\mathcal{H}^{n-1} \\
&-\int_{\partial E_t}\nabla\left[\beta(\nabla\psi_t^+\cdot\nu)^2 - (\nabla\psi_t^-\cdot\nu)^2\right]\cdot X(X\cdot\nu)d\mathcal{H}^{n-1}.
\end{aligned}
$$

Now to prove Lemma 7.7 we only need the following bound on the second derivative.

Lemma 8.7. *There exist $\delta > 0$ and a bounded function g such that if $\|\varphi\|_{C^{2,\vartheta}} < \delta$, then*

$$
\left|\ddot{\mathcal{J}}(E_t)\right| \le g\left(\|\varphi\|_{C^{2,\vartheta}}\right)\|X\cdot\nu\|^2_{H^1(\partial B_1)}.
$$

We will need the following proposition.

Proposition 8.8.

$$
\|\dot{\psi}_t^+\|_{H^1(\partial E_t)} + \|\dot{\psi}_t^-\|_{H^1(\partial E_t)} \le C\left(\|X\cdot\nu\|_{H^1(\partial E_t)} + \left|\dot{\mathcal{J}}(E_t)\right|\right).
$$

To prove the proposition we will use the following theorem concerning Sobolev bounds.

Theorem 8.9. *([McL, Theorem 4.20]) Let G_1 and G_2 be bounded open subsets of $\mathbb{R}^n$ such that $\overline{G}_1 \Subset G_2$ and G_1 intersects an $(n-1)$-dimensional manifold Γ, and put*

$$\Omega_j^{\pm} = G_j \cap \Omega^{\pm} \text{ and } \Gamma_j = G_j \cap \Gamma \text{ for } j = 1, 2.$$

Suppose, for an integer $r \geq 0$, that Γ_2 is $C^{r+1,1}$, and consider two equations

$$\mathcal{P}u^{\pm} = f^{\pm} \text{ on } \Omega_2^{\pm},$$

where $\mathcal{P}$ is strongly elliptic on G_2 with coefficients in $C^{r,1}(\overline{\Omega_2^{\pm}})$. If $u \in L^2(G_2)$ satisfies

$$u^{\pm} \in H^1(\Omega_2^{\pm}), \quad [u]_\Gamma \in H^{r+\frac{3}{2}}(\Gamma_2), \quad [\mathcal{B}_\nu u]_\Gamma \in H^{r+\frac{1}{2}}(\Gamma_2)^4,$$

and if $f^{\pm} \in H^r(\Omega_2^{\pm})$, then $u^{\pm} \in H^{r+2}(\Omega_1^{\pm})$ and

$$\begin{aligned} \|u^+\|_{H^{r+2}(\Omega_1^+)} + \|u^+\|_{H^{r+2}(\Omega_1^-)} &\leq C\left(\|u^+\|_{H^1(\Omega_2^+)} + \|u^-\|_{H^1(\Omega_2^-)}\right) \\ &+ C\left(\|[u]_{\Gamma_2}\|_{H^{r+\frac{3}{2}}(\Gamma_2)} + \|[\mathcal{B}_\nu u]_{\Gamma_2}\|_{H^{r+\frac{1}{2}}(\Gamma_2)}\right) \\ &+ C\left(\|f^+\|_{H^r(\Omega_2^+)} + \|f^-\|_{H^r(\Omega_2^-)}\right). \end{aligned}$$

We need an analogue of the above theorem for $r = -\frac{1}{2}$. To get it, we are going to interpolate between $r = 0$ and $r = -1$. We first prove the following lemma.

Lemma 8.10. *Let E be a set with the boundary in $C^{1,1}$ and let $R > 0$ be such that $B_R \supset \overline{E}$. Consider the equation*

$$\begin{cases} \beta\Delta u^+ = f^+ \text{ in } E, \\ \Delta u^- = f^- \text{ in } B_R\backslash E, \\ u^+ = u^- \text{ on } \partial E, \\ \beta\nabla u^+ \cdot \nu - \nabla u^- \cdot \nu = g \text{ on } \partial E, \\ u^- = 0 \text{ on } \partial B_R, \end{cases} \tag{8.20}$$

where $f^+ \in H^{-1}(E)$, $f^- \in H^{-1}(B_R\backslash E)$, and $g \in H^{-1/2}(\partial E)$ are given. Then there exists u - the solution of (8.20) in $W_0^{1,2}(B_R)$ and it satisfies

$$\|u\|^2_{H^1(B_R)} \leq C\left(\|f^+\|^2_{H^{-1}(E)} + \|f^-\|^2_{H^{-1}(B_R\backslash E)} + \|g\|^2_{H^{-1/2}(\partial E)}\right) \tag{8.21}$$

with $C = C(n, R) > 0$. Moreover, if $f^+1 \in H^{-1/2}(E)$, $f^- \in H^{-1/2}(B_R\backslash E)$, and $g \in L^2(\partial E)$, then

$$\|u\|^2_{H^{3/2}(B_R)} \leq C\left(\|f^+\|^2_{H^{-1/2}(E)} + \|f^-\|^2_{H^{-1/2}(B_R\backslash E)} + \|g\|^2_{L^2(\partial E)}\right) \tag{8.22}$$

with $C = C(n, R) > 0$.

Proof. First we observe that the solution in H^1 exists since it is a minimizer of the following convex functional:

$$\int_{E_t}\left(\frac{1}{2}\beta\left|\nabla u^+\right|^2 - f^+u^+\right) + \int_{E_t^c}\left(\frac{1}{2}\left|\nabla u^-\right|^2 - f^-u^-\right) + \int_{\partial E_t} g(u^+ - u^-).$$

Note that if we test the equation with the solution itself, we get

$$\int_{E_t}\frac{1}{2}\beta\left|\nabla u^+\right|^2 dx + \int_{E_t^c}\frac{1}{2}\left|\nabla u^-\right|^2 dx = -\int_{E_t} f^+u^+ dx - \int_{E_t} f^-u^- dx + \int_{\partial E_t} u^+ g\, d\mathcal{H}^{n-1}.$$

By Poincaré, Cauchy-Schwarz, Young, and the trace inequality we obtain (8.21).

[4] $\mathcal{B}_\nu$ here denotes conormal derivative. In our case it reduces to $a_E\partial_\nu$ since we deal with Laplacian.

Now we consider an operator that takes the functions of the right-hand side and returns the solution of the corresponding transmission problem, i.e. we define $T(f_1, f_2, g)$ for $f_1 \in H^r(E_t)$, $f_2 \in H^r(E_t^c)$, $g \in H^{r+\frac{1}{2}}(\partial E_t)$ as the only H^1 solution of (8.20).

By (8.21), $T : H^r \times H^r \times H^{r+\frac{1}{2}} \to H^{r+2}$ for $r = -1$. Moreover, (8.21) together with Theorem 8.9 yields $T : H^r \times H^r \times H^{r+\frac{1}{2}} \to H^{r+2}$ for $r \geq 0$ - integer. Thus, interpolating between $r = 0$ and $r = -1$ we get that $T : H^{-\frac{1}{2}} \times H^{-\frac{1}{2}} \times L^2 \to H^{\frac{3}{2}}$, so (8.22) holds for appropriately regular right-hand side. □

Proof. (Proposition 8.8) Since we are interested only in the value of $\dot{\psi}_t$ on ∂E_t, we multiply it by a cut-off function η. The function $\eta \in C_c^\infty(\mathbb{R}^n)$ is such that

$$0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ in } B_2, \quad \eta \equiv 0 \text{ outside of } B_3, \quad |\nabla \eta| \leq 2, \quad |\Delta \eta| \leq C(n).$$

We would also like to eliminate the jump on the boundary in order to use Lemma 8.10, so we consider a function $u := v\eta$, where v is as in Lemma 8.5 (we recall that $v = \dot{\psi}_t + f$, where f is a $H^{3/2}$ continuation of $\nabla \psi_t \cdot X$ from ∂E_t inside and outside). For δ small enough, all sets E_t lie inside of B_2, so

$$u = \dot{\psi}_t + \nabla \psi_t \cdot X \text{ on } \partial E_t. \tag{8.23}$$

Note that u satisfies

$$\begin{cases} -\beta \Delta u = -\frac{1}{K} v + \frac{2}{K} \dot{\mathcal{J}}(E_t) + \Delta f \text{ in } E_t, \\ \Delta u = \nabla v \cdot \nabla \eta + \left(\dot{\psi}_t + f\right) \Delta \eta \text{ in } E_t^c, \\ u^+ - u^- = 0 \text{ on } \partial E_t, \\ \beta \nabla u^+ \cdot \nu - \nabla u^- \cdot \nu = \left(-\left(\beta \nabla[\nabla \psi_t^+] - \nabla[\nabla \psi_t^-]\right) X + \beta \nabla f^+ - \nabla f^-\right) \cdot \nu \text{ on } \partial E_t, \\ u = 0 \text{ on } \partial B_3. \end{cases}$$

By Lemma 8.10,

$$\begin{aligned} \|u^+\|_{H^{\frac{3}{2}}(E_t)} + \|u^-\|_{H^{\frac{3}{2}}(E_t^c)} \leq\; & C \left(\left\| \left(\beta \nabla[\nabla \psi_t^+ \cdot X]\right) \cdot \nu \right\|_{L^2(\Gamma_2)} + \left\| \left(\nabla[\nabla \psi_t^- \cdot X]\right) \cdot \nu \right\|_{L^2(\Gamma_2)} \right) \\ & + C \left(\left\| \left(\beta \nabla[\nabla \psi_t^+] \cdot X\right) \cdot \nu \right\|_{L^2(\Gamma_2)} + \left\| \left(\nabla[\nabla \psi_t^-] \cdot X\right) \cdot \nu \right\|_{L^2(\Gamma_2)} \right) \\ & + C \left(\left\| \frac{1}{K} v \right\|_{H^{-\frac{1}{2}}(E_t)} + \left\| \frac{2}{K} \dot{\mathcal{J}}(E_t) \right\|_{H^{-\frac{1}{2}}(E_t)} + \|\Delta f\|_{H^{-\frac{1}{2}}(E_t)} \right) \\ & + C \left(\|\nabla v \cdot \nabla \eta\|_{H^{-\frac{1}{2}}(E_t^c)} + \|v \Delta \eta\|_{H^{-\frac{1}{2}}(E_t^c)} \right). \end{aligned}$$

Now we employ Proposition 8.3, inequality (8.4), and the definition of f to get

$$\begin{aligned} \|u^+\|_{H^{\frac{3}{2}}(E_t)} + \|u^-\|_{H^{\frac{3}{2}}(E_t^c)} \leq\; & C \left(\|X \cdot \nu\|_{H^1(\partial E_t)} + \left| \dot{\mathcal{J}}(E_t) \right| \right) \\ & + C \left(\|\nabla v \cdot \nabla \eta\|_{H^{-\frac{1}{2}}(E_t^c)} + \|v \Delta \eta\|_{H^{-\frac{1}{2}}(E_t^c)} \right). \end{aligned}$$

Remembering (8.23), using trace inequality and properties of η, we have

$$\begin{aligned}\|\dot\psi_t^+\|_{H^1}(\partial E_t)+\|\dot\psi_t^-\|_{H^1(\partial E_t)} &\le C\left(\|X\cdot\nu\|_{H^1(\partial E_t)}+\left|\dot{\mathcal{J}}(E_t)\right|\right)\\ &\quad + C\left(\|\nabla v\cdot\nabla\eta\|_{H^{-\frac12}(E_t^c)}+\|v\Delta\eta\|_{H^{-\frac12}(E_t^c)}\right)\\ &\le C\left(\|X\cdot\nu\|_{H^1(\partial E_t)}+\left|\dot{\mathcal{J}}(E_t)\right|\right)+C\left(\|\nabla v\cdot\nabla\eta\|_{L^2(E_t^c)}+\|v\Delta\eta\|_{L^2(E_t^c)}\right)\\ &\le C\left(\|X\cdot\nu\|_{H^1(\partial E_t)}+\left|\dot{\mathcal{J}}(E_t)\right|\right)+C\left(\|\nabla v\|_{L^2(E_t^c)}+\|v\|_{L^2(B_3\setminus B_2)}\right).\end{aligned}$$

Now it remains to recall the bounds (8.17) and (8.18) and notice that $\|\cdot\|_{L^2(B_3\setminus B_2)} \le C\|\cdot\|_{L^{2^*}(B_3\setminus B_2)}$. □

Proof. (Lemma 8.7)

Let us first show that the lemma is implied by the following claim.

Claim: $\left|\ddot{\mathcal{J}}(E_t)\right| \le C\left(\|X\cdot\nu\|^2_{H^1(\partial B_1)}+\dot{\mathcal{J}}(E_t)\,\|X\cdot\nu\|_{H^1(\partial B_1)}\right).$

Indeed, suppose we proved the claim. Denote $\dot{\mathcal{J}}(E_t)$ by $f(t)$. Then we know the following:

$$\begin{cases}|f'(t)|\le C\left(\|X\cdot\nu\|^2_{H^1(\partial B_1)}+f(t)\,\|X\cdot\nu\|_{H^1(\partial B_1)}\right),\\ f(0)=0.\end{cases}$$

Let us show that

$$|f(t)|\le\|X\cdot\nu\|_{H^1(\partial B_1)}, \tag{8.24}$$

then the lemma will follow immediately. Suppose that there exists a time $t\in(0,1]$ such that the inequality (8.24) fails. We denote by t^* the first time when it happens, i.e.

$$t^* := \inf_{t\in[0,1]}\{t : (8.24) \text{ fails}\}.$$

Since inequality (8.24) is true for $t=0$, the following holds:

$$|f(t^*)| = \|X\cdot\nu\|_{H^1(\partial B_1)}, \quad |f(t)|\le\|X\cdot\nu\|_{H^1(\partial B_1)} \text{ for } t\in[0,t^*].$$

Now, as $f(0)=0$, we can write

$$f(t^*)=\int_0^{t^*} f'(t)dt$$

and thus

$$\begin{aligned}\|X\cdot\nu\|_{H^1(\partial B_1)} = |f(t^*)| &\le \int_0^{t^*}|f'(t)|dt\\ &\le \int_0^{t^*} C\left(\|X\cdot\nu\|^2_{H^1(\partial B_1)}+f(t)\,\|X\cdot\nu\|_{H^1(\partial B_1)}\right)dt \le 2C\,\|X\cdot\nu\|^2_{H^1(\partial B_1)}.\end{aligned}$$

However, that cannot hold for $\|X\cdot\nu\|_{H^1(\partial B_1)}$ small enough. That means that (8.24) holds for all times t.

Proof of the claim.

By (8.19) we have

$$
\begin{aligned}
\ddot{\mathcal{J}}(E_t) &= -\frac{2}{K}\dot{\mathcal{J}}(E_t)\int_{\partial E_t}\psi_t(X\cdot\nu)d\mathcal{H}^{n-1}\\
&+\frac{1}{2}\int_{\partial E_t}\nabla\left[\beta|\nabla\psi_t^+|^2-|\nabla\psi_t^-|^2\right]\cdot X(X\cdot\nu)d\mathcal{H}^{n-1}\\
&+\int_{\partial E_t}\left(\frac{1-\frac{2}{K}|B_1|\mathcal{J}(E_t)}{|B_1|}+\frac{1}{K}\psi_t\right)(\nabla\psi_t^+\cdot X)(X\cdot\nu)d\mathcal{H}^{n-1}\\
&+\int_{\partial E_t}\left(\left(\frac{1-\frac{2}{K}|B_1|\mathcal{J}(E_t)}{|B_1|}\right)+\frac{1}{K}\psi_t\right)\dot{\psi}_t^+(X\cdot\nu)d\mathcal{H}^{n-1}\\
&+\int_{\partial E_t}(\beta\nabla\psi_t^+\cdot\nabla\dot{\psi}_t^+-\nabla\psi_t^-\cdot\nabla\dot{\psi}_t^-)(X\cdot\nu)d\mathcal{H}^{n-1}\\
&-2\int_{\partial E_t}\left(\beta\left(\nabla\dot{\psi}_t^+\cdot\nu\right)\left(\nabla\psi_t^+\cdot\nu\right)-\left(\nabla\dot{\psi}_t^-\cdot\nu\right)\left(\nabla\psi_t^-\cdot\nu\right)\right)(X\cdot\nu)d\mathcal{H}^{n-1}\\
&-\int_{\partial E_t}\nabla\left[\beta(\nabla\psi_t^+\cdot\nu)^2-(\nabla\psi_t^-\cdot\nu)^2\right]\cdot X(X\cdot\nu)d\mathcal{H}^{n-1}\\
&=: I_1(t)+I_2(t)+I_3(t)+I_4(t)+I_5(t)+I_6(t)+I_7(t).
\end{aligned}
$$

We start with I_1. Using the expression for $\dot{\mathcal{J}}(E_t)$ obtained in Proposition 8.6, we get

$$
\begin{aligned}
-\frac{K}{2}I_1(t) =&\dot{\mathcal{J}}(E_t)\int_{\partial E_t}\psi_t(X\cdot\nu)d\mathcal{H}^{n-1} = \left(1-\frac{1}{K}\int_{E_t}\psi_t dx\right)\frac{1}{|B_1|}\left(\int_{\partial E_t}\psi_t(X\cdot\nu)d\mathcal{H}^{n-1}\right)^2\\
&+\frac{1}{2}\int_{\partial E_t}\left(\beta|\nabla\psi_t^+|^2-|\nabla\psi_t^-|^2\right)(X\cdot\nu)d\mathcal{H}^{n-1}\int_{\partial E_t}\psi_t(X\cdot\nu)d\mathcal{H}^{n-1}\\
&+\frac{1}{2K}\int_{\partial E_t}\psi_t^2(X\cdot\nu)d\mathcal{H}^{n-1}\int_{\partial E_t}\psi_t(X\cdot\nu)d\mathcal{H}^{n-1}.
\end{aligned}
$$

Thus,

$$
|I_1(t)|\le g(\|\psi_t\|_{C^1(\overline{E_t})})\|X\cdot\nu\|^2_{L^1(\partial E_t)}
$$

for some bounded function g.

To prove the bounds for I_2, I_3, and I_7, we rewrite X as $(X\cdot\nu)\nu+X^\tau$ and use that

$$
|X^\tau\circ\Phi_t|\le\omega(\|\varphi\|_{C^{2,\vartheta}})|X\cdot\nu_{B_1}|.
$$

Indeed,

$$
\begin{aligned}
I_2(t)&=\frac{1}{2}\int_{\partial E_t}\nabla\left[\beta|\nabla\psi_t^+|^2-|\nabla\psi_t^-|^2\right]\cdot X(X\cdot\nu)d\mathcal{H}^{n-1}\\
&=\frac{1}{2}\int_{\partial E_t}\nabla\left[\beta|\nabla\psi_t^+|^2-|\nabla\psi_t^-|^2\right]\cdot\nu(X\cdot\nu)^2d\mathcal{H}^{n-1}\\
&+\frac{1}{2}\int_{\partial E_t}\nabla\left[\beta|\nabla\psi_t^+|^2-|\nabla\psi_t^-|^2\right]\cdot X^\tau(X\cdot\nu)d\mathcal{H}^{n-1}
\end{aligned}
$$

and thus

$$
|I_2(t)|\le g(\|\psi_t\|_{C^2(\overline{E_t})})\|X\cdot\nu\|^2_{L^2(\partial E_t)}
$$

for some bounded function g. I_3 and I_7 are treated in the same way.

To bound I_4, I_5, and I_6 we use Proposition 8.8 and Proposition 8.3. Let us show the inequality for I_5, I_4 and I_6 can be treated in a similar way.

$$
\begin{aligned}
&\left|\int_{\partial E_t}\left(\beta\nabla\psi_t^+\cdot\nabla\dot\psi_t^+ - \nabla\psi_t^-\cdot\nabla\dot\psi_t^-\right)(X\cdot\nu)d\mathcal{H}^{n-1}\right| \\
&\le \int_{\partial E_t}\left(\left|\beta\nabla\psi_t^+\cdot\nabla\dot\psi_t^+\right| + \left|\nabla\psi_t^-\cdot\nabla\dot\psi_t^-\right|\right)|X\cdot\nu|d\mathcal{H}^{n-1} \\
&\le \left(\left(\int_{\partial E_t}\left|\beta\nabla\psi_t^+\cdot\nabla\dot\psi_t^+\right|^2\right)^{\frac12} + \left(\int_{\partial E_t}\left|\nabla\psi_t^-\cdot\nabla\dot\psi_t^-\right|^2\right)^{\frac12}\right)\|X\cdot\nu\|_{L^2(\partial E_t)} \qquad (8.25)\\
&\le g(\|\psi_t\|_{C^2(\overline{E_t})})\left(\left(\int_{\partial E_t}\left|\nabla\dot\psi_t^+\right|^2\right)^{\frac12} + \left(\int_{\partial E_t}\left|\nabla\dot\psi_t^-\right|^2\right)^{\frac12}\right)\|X\cdot\nu\|_{L^2(\partial E_t)} \\
&\le g(\|\psi_t\|_{C^2(\overline{E_t})})\left(\|X\cdot\nu\|_{H^1(\partial E_t)} + \left|\dot{\mathcal{J}}(E_t)\right|\right)\|X\cdot\nu\|_{L^2(\partial E_t)}
\end{aligned}
$$

□

Now we are ready to prove Lemma 7.7.

Proof. (Lemma 7.7)

$$\mathcal{J}(E) = \mathcal{J}(B_1) + \dot{\mathcal{J}}(B_1) + \int_0^1 (1-s)\ddot{\mathcal{J}}(E_s)ds.$$

By Proposition 8.6 we know that $\dot{\mathcal{J}}(B_1) = 0$. Now use Lemma 8.7 to bound the integral. □

Appendix A. Second derivative on the ball

We want to show that the second variation of the energy which we know is bounded by $\|\varphi\|_{H^1}^2$ is actually bounded by a stronger $H^{1/2}$ norm on the ball. We don't need this for our main results but it is a sharp bound so we prove it for the sake of completeness.

We first show the following proposition.

Proposition A.1. *Given $\vartheta \in (0,1]$ there exists $\delta = \delta(n,\vartheta)$ such that for any $\varphi \in C^{2,\vartheta}$ with $\|\varphi\|_{C^{2,\vartheta}} < \delta$ we have*

$$\partial^2\mathcal{G}(B_1)[\varphi,\varphi] := \hat c_1\int_{\partial B_1}\varphi^2 d\mathcal{H}^{n-1} + \int_{\partial B_1}\hat c_2 H(\varphi)^+ + \hat c_3(\nabla H(\varphi)^-\cdot\nu)\varphi d\mathcal{H}^{n-1},$$

where $H(\varphi)$ is the unique solution of

$$\begin{cases}\beta\Delta u = \frac{1}{K}u \text{ in } B_1,\\ \Delta u = 0 \text{ in } B_1^c,\\ u^+ - u^- = c_1\varphi \text{ on } \partial B_1,\\ \beta\nabla u^+\cdot\nu - \nabla u^-\cdot\nu = c_2\varphi \text{ on } \partial B_1\end{cases}$$

and $\hat c_1$, $\hat c_2$, $\hat c_3$, c_1, and c_2 are constants depending only on β, K and dimension n.

Proof. We have

$$
\begin{aligned}
\ddot{\mathcal{J}}(B_1) &= -\frac{2}{K}\dot{\mathcal{J}}(B_1)\int_{\partial B_1}\psi_0(X\cdot\nu)d\mathcal{H}^{n-1} \\
&+ \frac{1-\frac{2}{K}|B_1|\mathcal{J}(B_1)}{|B_1|}\left(\int_{\partial B_1}\dot{\psi}_0^+(X\cdot\nu)d\mathcal{H}^{n-1}+\int_{\partial B_1}(\nabla\psi_0^+\cdot X)(X\cdot\nu)d\mathcal{H}^{n-1}\right) \\
&+ \int_{\partial B_1}(\beta\nabla\psi_0^+\cdot\nabla\dot{\psi}_0^+-\nabla\psi_0^-\cdot\nabla\dot{\psi}_0^-)(X\cdot\nu)d\mathcal{H}^{n-1} \\
&+ \frac{1}{2}\int_{\partial B_1}\nabla\left[\beta|\nabla\psi_0^+|^2-|\nabla\psi_0^-|^2\right]\cdot X(X\cdot\nu)d\mathcal{H}^{n-1} \\
&+ \frac{1}{K}\int_{\partial B_1}\psi_0\dot{\psi}_0^+(X\cdot\nu)d\mathcal{H}^{n-1}+\frac{1}{K}\int_{\partial B_1}\psi_0\nabla\psi_0^+\cdot X(X\cdot\nu)d\mathcal{H}^{n-1} \\
&- 2\int_{\partial B_1}\left(\beta\left(\nabla\dot{\psi}_0^+\cdot\nu\right)\left(\nabla\psi_0^+\cdot\nu\right)-\left(\nabla\dot{\psi}_0^-\cdot\nu\right)\left(\nabla\psi_0^-\cdot\nu\right)\right)(X\cdot\nu)d\mathcal{H}^{n-1} \\
&- \int_{\partial B_1}\nabla\left[\beta(\nabla\psi_0^+\cdot\nu)^2-(\nabla\psi_0^-\cdot\nu)^2\right]\cdot X(X\cdot\nu)d\mathcal{H}^{n-1},
\end{aligned}
$$

where ψ_0 is the minimizer of the energy of the ball B_1, meaning it solves the equation

$$
\begin{cases}
-\beta\Delta\psi_0 = -\frac{1}{K}\psi_t+\frac{2}{K}\mathcal{J}(B_1)-\frac{1}{|B_1|} \text{ in } B_1, \\
\Delta\psi_0 = 0 \text{ in } B_1^c, \\
\psi_0^+ = \psi_0^- \text{ on } \partial B_1, \\
\beta\nabla\psi_0^+\cdot\nu = \nabla\psi_0^-\cdot\nu \text{ on } \partial B_1.
\end{cases}
$$

We recall that $\dot{\mathcal{J}}(B_1)=0$ by Proposition 8.6 and ψ_0 is radial by Proposition 7.4. This allows us to say that $\psi_0(x)=\psi(|x|)$ for some function $\psi:\mathbb{R}^+\to\mathbb{R}$ and we get

$$
\begin{aligned}
\ddot{\mathcal{J}}(B_1) &= \frac{1-\frac{2}{K}|B_1|\mathcal{J}(B_1)}{|B_1|}\left(\int_{\partial B_1}\dot{\psi}_0^+(X\cdot\nu)d\mathcal{H}^{n-1}+(\psi^+)'(1)\int_{\partial B_1}(X\cdot\nu)^2d\mathcal{H}^{n-1}\right) \\
&- \beta(\psi^+)'(1)\int_{\partial B_1}(\nabla\dot{\psi}_0^+-\nabla\dot{\psi}_0^-)\cdot\nu(X\cdot\nu)d\mathcal{H}^{n-1} \\
&- \big(\beta(\psi^+)'(1)(\psi^+)''(1)-(\psi^-)'(1)(\psi^-)''(1)\big)\int_{\partial B_1}(X\cdot\nu)^2d\mathcal{H}^{n-1} \\
&+ \frac{1}{K}\psi(1)\int_{\partial B_1}\dot{\psi}_0^+(X\cdot\nu)d\mathcal{H}^{n-1}+\frac{1}{K}\psi(1)(\psi^+)'(1)\int_{\partial B_1}(X\cdot\nu)^2d\mathcal{H}^{n-1} \\
&= \Big(\frac{1-\frac{2}{K}|B_1|\mathcal{J}(B_1)}{|B_1|}(\psi^+)'(1)-\big(\beta(\psi^+)'(1)(\psi^+)''(1)-(\psi^-)'(1)(\psi^-)''(1)\big) \\
&\qquad\qquad+\frac{1}{K}\psi(1)(\psi^+)'(1)\Big)\int_{\partial B_1}(X\cdot\nu)^2d\mathcal{H}^{n-1} \\
&+ \left(\frac{1-\frac{2}{K}|B_1|\mathcal{J}(B_1)}{|B_1|}+\frac{1}{K}\psi(1)\right)\int_{\partial B_1}\dot{\psi}_0^+(X\cdot\nu)d\mathcal{H}^{n-1} \\
&+ (\beta-1)(\psi^+)'(1)\int_{\partial B_1}\nabla\dot{\psi}_0^-\cdot\nu(X\cdot\nu)d\mathcal{H}^{n-1},
\end{aligned}
$$

where $\dot\psi_0$ satisfies

$$\begin{cases} -\beta\Delta\dot\psi_0 = -\frac{1}{K}\dot\psi_0 \text{ in } B_1, \\ \Delta\dot\psi_0 = 0 \text{ in } B_1^c, \\ \dot\psi_0^+ - \dot\psi_0^- = -\left((\psi^+)'(1) - (\psi^-)'(1)\right)(X\cdot\nu) \text{ on } \partial B_1, \\ \beta\nabla\dot\psi_0^+\cdot\nu - \nabla\dot\psi_0^-\cdot\nu = -\left(\beta(\psi^+)''(1) - (\psi^-)''(1)\right)(X\cdot\nu) \text{ on } \partial B_1. \end{cases}$$

Remembering that $\mathcal{G}(E) = \frac{K}{2|E|} - \mathcal{J}(E)$, we get

$$\partial^2\mathcal{G}(B_1)[\varphi,\varphi] := \hat c_1\int_{\partial B_1}\varphi^2 d\mathcal{H}^{n-1} + \int_{\partial B_1}\hat c_2 H(\varphi)^+ + \hat c_3(\nabla H(\varphi)^-\cdot\nu)\varphi d\mathcal{H}^{n-1}$$

where

$$\hat c_1 = -\frac{1-\frac{2}{K}|B_1|\mathcal{J}(B_1)}{|B_1|}(\psi^+)'(1) + \big(\beta(\psi^+)'(1)(\psi^+)''(1) - (\psi^-)'(1)(\psi^-)''(1)\big) - \frac{1}{K}\psi(1)(\psi^+)'(1),$$

$$\hat c_2 = -\frac{1-\frac{2}{K}|B_1|\mathcal{J}(B_1)}{|B_1|} - \frac{1}{K}\psi(1),$$

$$\hat c_3 = -(\beta-1)(\psi^+)'(1)$$

and $H(\varphi)$ is the unique solution of

$$\begin{cases} \beta\Delta u = \frac{1}{K}u \text{ in } B_1, \\ \Delta u = 0 \text{ in } B_1^c, \\ u^+ - u^- = c_1\varphi \text{ on } \partial B_1, \\ \beta\nabla u^+\cdot\nu - \nabla u^-\cdot\nu = c_2\varphi \text{ on } \partial B_1, \end{cases}$$

where $c_1 = -\left((\psi^+)'(1) - (\psi^-)'(1)\right)$, $c_2 = -\left(\beta(\psi^+)''(1) - (\psi^-)''(1)\right)$.

□

We are now ready to show the following lemma.

Lemma A.2. *Given $\vartheta \in (0,1]$ there exists $\delta = \delta(n,\vartheta)$, $c = c(n,\beta,K)$ such that for any $\varphi \in C^{2,\vartheta}$ with $\|\varphi\|_{C^{2,\vartheta}} < \delta$ we have*

$$\partial^2\mathcal{G}(B_1)[\varphi,\varphi] \geq -c\|\varphi\|^2_{H^{\frac12}(\partial B_1)}.$$

Proof. Consider φ in the basis of spherical harmonics,

$$\varphi = \sum_{m=0}^{\infty}\sum_{i=1}^{N(m,n)}\alpha_{m,i}Y_{m,i}.$$

First, we would like to bound $\partial^2\mathcal{G}(B_1)[Y_{m,i},Y_{m,i}]$. One can easily see that $H(Y_{m,i}) = R(r)Y_{m,i}$, where $R(r)$ is the only solution of the following system:

$$\begin{cases} R_1''(r) + \frac{n-1}{r}R_1'(r) + \left(-\frac{1}{\beta K} + \frac{\lambda_{m,i}}{r^2}\right)R_1(r) = 0 \text{ for } r \leq 1, \\ R_2''(r) + \frac{n-1}{r}R_2'(r) + \frac{\lambda_{m,i}}{r^2}R_2(r) = 0 \text{ for } r \geq 1, \\ R_1(1) - R_2(1) = c_1, \\ \beta R_1'(1) - R_2'(1) = c_2, \end{cases}$$

where $\lambda_{m,i} = -m(m+n-2)$.

A straightforward computation gives us that $R_2(r) = Ar^{-(m+n-2)}$ for some constant A.

Let us search for R_1 in the form $R_1(r) = \sum_{k=0}^{\infty} a_k r^k$. The equation for R_1 then will take the following form:

$$\sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)r^k + \frac{n-1}{r}\sum_{k=0}^{\infty} a_{k+1}(k+1)r^k + \Big(-\frac{1}{\beta K} + \frac{\lambda_{m,i}}{r^2}\Big)\sum_{k=0}^{\infty} a_k r^k = 0 \text{ for } r \le 1$$

If $m \ge 2$, it means that

$$a_0 = 0, \quad a_1 = 0, \quad a_k\big(k(n+k-2) - m(n+m-2)\big) = \frac{1}{\beta K} a_{k-2} \text{ for } k \ge 2.$$

The recurrent condition can be rewritten as

$$a_k(k-m)(k+n+m-2) = -a_{k-2} \text{ for } k \ge 2.$$

Hence,

$$\begin{cases} a_m = C, \\ a_{m+2i} = \beta K a_{m+2(i-1)} \frac{1}{2i(2i+2m+n-2)} \text{ for } i \ge 1, \\ a_k = 0 \text{ for all other } k, \end{cases}$$

where C is a constant. So, the coefficients a_k decrease as $\frac{(\beta K)^k}{(k!)^2}$ and the series is absolutely converging. Note that $a_k = C b_k$, where $\{b_k\}_k$ is the following fixed sequence:

$$\begin{cases} b_m = 1, \\ b_{m+2i} = (\beta K)^i \Pi_{j=1}^{i} \frac{1}{2j(2j+2m+n-2)} \text{ for } i \ge 1, \\ b_k = 0 \text{ for all other } k. \end{cases}$$

Our system for R then becomes

$$\begin{cases} R_1(r) = C\sum_{i=0}^{\infty} b_{m+2i} r^{m+2i} \text{ for } r \le 1, \\ R_2(r) = A r^{-(m+N-2)} \text{ for } r \ge 1, \\ C\sum_{i=0}^{\infty} b_{m+2i} - A = c_1, \\ \beta C\sum_{i=0}^{\infty}(m+2i)b_{m+2i} + A(m+n-2) = c_2 \end{cases}$$

with A and C unknowns. We are interested in the value of $|R_2'(1)|$:

$$\begin{aligned} |R_2'(1)| &= |A(N+m-2)| \\ &= \left| \frac{c_1(m+N-2) + c_2}{\sum_{i=0}^{\infty} b_{m+2i} + \beta\sum_{i=0}^{\infty}(m+2i)b_{m+2i}} \beta \sum_{i=0}^{\infty}(m+2i)b_{m+2i} - c_2 \right| \sim m. \end{aligned}$$

Thus,

$$|\partial^2 \mathcal{G}(B_1)[Y_{m,i}, Y_{m,i}]| = |\hat{c}_1 + \hat{c}_2 A + \hat{c}_3 A(n+m-2)| \sim m. \tag{A.1}$$

Now recall that φ is such that $|\Omega| = |B_1|$ and $x_\Omega = 0$. It means that

$$|B_1| = |\Omega| = \int_{\partial B_1} \frac{(1+\varphi(x))^n}{n} d\mathcal{H}^{n-1};$$

$$0 = x_\Omega = \int_{\partial B_1} y \frac{(1+\varphi(x))^{n+1}}{n+1} d\mathcal{H}^{n-1}.$$

Hence,

$$\left|\int_{\partial B_1} \varphi(x) d\mathcal{H}^{n-1}\right| = \left|\int_{\partial B_1} \sum_{i=2}^{n} \binom{n}{i} \frac{\varphi(x)^i}{n} d\mathcal{H}^{n-1}\right|$$
$$\leq C(n) \int_{\partial B_1} \varphi(x)^2 d\mathcal{H}^{n-1} \leq C(n)\delta \|\varphi\|_{L^2}$$

and

$$\left|\int_{\partial B_1} x_i \varphi(x) d\mathcal{H}^{n-1}\right| \leq \int_{\partial B_1} \sum_{j=2}^{n} \binom{n}{j} \left|\frac{\varphi(x)^j}{n+1}\right| d\mathcal{H}^{n-1} \leq C(n)\delta\|\varphi\|_{L^2}.$$

Thus, for δ sufficiently small we have

$$a_0^2 + \sum_{i=1}^{n} a_{1,i}^2 \leq 2 \sum_{m=2}^{\infty} \sum_{i=1}^{N(m,n)} a_{m,i}^2,$$

which in turn implies

$$\partial^2 \mathcal{G}(B_1)[\varphi,\varphi] \geq -c\|\varphi\|^2_{H^{\frac{1}{2}}(\partial B_1)},$$

thanks to (A.1). □

References

[ADK07] L. Afraites, M. Dambrine and D. Kateb, *Shape Methods for the Transmission Problem with a Single Measurement*, Numerical Functional Analysis and Optimization, 28(5-6):519–551, 2007.

[AFM13] E. Acerbi, N. Fusco and M. Morini, *Minimality via Second Variation for a Nonlocal Isoperimetric Problem*, Communications in Mathematical Physics, 322(2):515–557, 2013.

[AFP] L. Ambrosio, N. Fusco, D. Pallara, *Functions of Bounded Variations and Free Discontinuity Problems*, Calderon Press, Oxford, 2000.

[BDPV15] L. Brasco, G. De Philippis, and B. Velichkov, *Faber–krahn inequalities in sharp quantitative form*, Duke Math. J., 164(9):1777–1831, 2015.

[CL12] M. Cicalese and G. P. Leonardi, *A selection principle for the sharp quantitative isoperimetric inequality*, Arch. Ration. Mech. Anal., 206:617–643, 2012.

[Dam02] M. Dambrine *On variations of the shape Hessian and sufficient conditions for the stability of critical shapes*, RACSAM. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat., 96(1):pp. 95–121, 2002.

[DPHV19] G. D. Philippis, J. Hirsch, G. Vescovo, *Regularity of minimizers for a model of charged droplets*, Accepted Paper, CMP, 2019.

[DMV64] A. Doyle, D. Moffett, and B. Vonnegut, *Behavior of evaporating electrically charged droplets*, Journal of Colloid Science, 19 (1964), pp. 136 – 143.

[DAMHL03] D. Duft, T. Achtzehn, R. Müller, B. A. Huber, and T. Leisner, *Rayleigh jets from levitated microdroplets*, Nature, 421 (2003), pp. 128.

[Fug89] B. Fuglede, *Stability in the isoperimetric problem for convex or nearly spherical domains in $\mathbb{R}^n$*, Trans. Amer. Math. Soc., 314 (1989), pp. 619–638.

[FMP08] N. Fusco, F. Maggi, and A. Pratelli, *The sharp quantitative isoperimetric inequality*, Ann. Math., 168 (2008), pp. 941–980.

[FJ15] N. Fusco and V. Julin, *On the regularity of critical and minimal sets of a free interface problem*, Interfaces Free Bound., 17 (2015), pp. 117—142.

[GM12] M. Giaquinta and L. Martinazzi, *An Introduction to the Regularity Theory for Elliptic Systems, Harmonic Maps and Minimal Graphs*, volume 11 of Lecture Notes (Scuola Normale Superiore), Edizioni della Normale, 2012.

[GNR15] M. Goldman, M. Novaga, and B. Ruffini, *Existence and stability for a non-local isoperimetric model of charged liquid drops*, Arch. Ration. Mech. Anal., 217 (2015), pp. 1–36.

[HP] A. Henrot, M. Pierre, *Variation et optimisation de formes*, Math. Appl. (Berlin) **48**, Springer, Berlin, 2005.

[Jul14] V. Julin, *Isoperimetric problem with a Coulombic repulsive term*, Indiana Univ. Math. J., 63 (2014), pp. 77—89.

[KM14] H. Knüpfer and C. B. Muratov, *On an isoperimetric problem with a competing non-local term. II. The general case*, Commun. Pure Appl. Math., 67 (2014), pp. 1974—1994.

[Lin93] F.-H. Lin, *Variational problems with free interfaces*, Calc. Var. Partial Differential Equations, 1 (1993), pp. 149–168.

[M] F. Maggi, *Sets of finite perimeter and geometric variational problems: an introduction to Geometric Measure Theory*, vol. 135 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2012.

[McL] W. McLean, *Strongly elliptic systems and boundary integral equations*, Cambridge University Press, 2002.

[MN16] C. B. Muratov and M. Novaga, *On well-posedness of variational models of charged drops*, Proc. A., 472 (2016).

[RPH89] C. B. Richardson, A. L. Pigg, and R. L. Hightower, *On the stability limit of charged droplets*, Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences, 422 (1989), pp. 319–328.

[Sion58] M. Sion, *On general minimax theorems*, Pacific J. Math. 8, no. 1 (1958), pp. 171–176.

[T64] G. Taylor, *Disintegration of water drops in electric field*, Proceedings of the Royal Society of London Series a-Mathematical and Physical Sciences, 280 (1964).

[WT25] C. T. R. Wilson and G. I. Taylor, *The bursting of soap-bubbles in a uniform electric field*, Mathematical Proceedings of the Cambridge Philosophical Society, 22 (1925), pp. 728–730.

[Z] J. Zeleny, *Instability of electrified liquid surfaces*, Phys. Rev., 10 (1917), pp. 1–6.

E.M: Department of Mathematics and Statistics, P.O. Box 68 (Gustaf Hällströmin katu 2), FI-00014 University of Helsinki, Finland
E-mail address: ekaterina.mukoseeva@helsinki.fi

G.V.: SISSA, Via Bonomea 265, 34136 Trieste, Italy
E-mail address: giulia.vescovo88@gmail.com