INTEGRATION OF NONSMOOTH 2-FORMS: FROM YOUNG TO ITÔ AND STRATONOVICH

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Abstract. We show that geometric integrals of the type $\int_\Omega f \mathsf{d}g^1 \wedge \mathsf{d}g^2$ can be defined over a two-dimensional domain Ω when the functions $f, g^1, g^2 \colon \mathbb{R}^2 \to \mathbb{R}$ are just Hölder continuous with sufficiently large Hölder exponents and the boundary of Ω has sufficiently small dimension, by summing over a refining sequence of partitions the discrete Stratonovich or Itô type terms. This leads to a two-dimensional extension of the classical Young integral that coincides with the integral introduced recently by R. Züst. We further show that the Stratonovich-type summation allows to weaken the requirements on Hölder exponents of the map $g = (g^1, g^2)$ when $f(x) = F(x, g(x))$ with F sufficiently regular. The technique relies upon an extension of the sewing lemma from Rough paths theory to alternating functions of two-dimensional oriented simplices, also proven in the paper.

1. Introduction

The scope of the present paper is constructing explicitly, via the appropriate discrete approximations, the extension of the classical notion of the integral of the differential 2-form $f \mathsf{d}g^1 \wedge \mathsf{d}g^2$ over any sufficiently nice oriented planar domain $\Omega \subset \mathbb{R}^2$ (one might think for simplicity of Ω being just an oriented polygon, or even simpler, a triangle) to the case when the maps $f \colon \mathbb{R}^2 \to \mathbb{R}$, $g := (g_1, g_2) \colon \mathbb{R}^2 \to \mathbb{R}^2$ are only Hölder continuous, so that one might only put the word "differential" above in quotation marks, because g might have no derivatives. If g is sufficiently smooth and f just continuous, then $f \mathsf{d}g^1 \wedge \mathsf{d}g^2$ can be understood in the modern differential geometry language as $f g^*(\mathsf{d}x^1 \wedge \mathsf{d}x^2)$, where $\mathsf{d}x^i$ are coordinate 1-forms, $i = 1, 2$, and g^* stands for the pull-back via g, or, alternatively, in a more analytic language,

$$\int_\Omega f \mathsf{d}g^1 \wedge \mathsf{d}g^2 := \int_\Omega f(x) \det \begin{pmatrix} \partial_1 g^1(x) & \partial_2 g^1(x) \\ \partial_1 g^2(x) & \partial_2 g^2(x) \end{pmatrix} dx, \tag{1.1}$$

∂_i standing for partial derivatives in the coordinate direction x_i, $i = 1, 2$. The latter integral is the natural building block for integrals of classical (smooth) differential 2-forms over smooth parameterized 2-dimensional surfaces in $\mathbb{R}^n$ via pull-back.

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One comes therefore inevitably to the problem posed when trying to integrate even a very smooth differential 2-form ω in $\mathbb{R}^n$ over a parameterized Hölder surface $\varphi\colon \Omega \subset \mathbb{R}^2 \to \mathbb{R}^n$, $\varphi(x) = (\varphi^i(x))_{i=1}^n$, letting formally

$$\int_{\varphi(\Omega)} \omega := \int_\Omega \varphi^*\omega,$$

where $\varphi^*\omega$ stands for pull-back of ω via φ, i.e. $\varphi^*\omega := \sum_{i,j}(a_{ij}\circ\varphi)\mathsf{d}\varphi^i \wedge \mathsf{d}\varphi^j$ when $\omega = \sum_{i,j} a_{ij}\mathsf{d}x^i \wedge \mathsf{d}x^j$.

1.1. **History.**

1.1.1. *One-dimensional integrals.* The one-dimensional prototype of this problem, that is, extending the integral of a differential 1-form $u\mathsf{d}v$ over an interval $[a,b]$ of the real axis to the maps $u, v\colon \mathbb{R} \to \mathbb{R}$ that are only Hölder continuous, has been solved by L.C. Young [22] and independently by V. Kondurar [10]. They defined the respective integral $\int_a^b u\mathsf{d}v$ as a limit in k of a converging sequence of Riemann sums of the type $\sum_{i=0}^{k-1} u(a_i)(v(a_{i+1}) - v(a_i))$ over an appropriate sequence of refining partitions of the interval $[a,b]$ by consequtive points $a_0 := a < a_1 < \ldots < a_k := b$, thus mimicking the definition of the classical Riemann integral. This provides an extension of the latter to the case $u \in C^\alpha(\mathbb{R})$, $v \in C^\beta(\mathbb{R})$ when $\alpha + \beta > 1$ (later several generalizations of this result for wider classes of functions were provided, see e.g. [23] as well as the recent paper [21] and references therein). It is worth remarking that the original proof of Young [22] was quite "handmade", just by the repetitive use of Hölder inequality. Rather, nowadays it is a custom to do it in a more "automated" way by using the so-called one-dimensional *sewing lemma* [4, lemma 2.1], which together with the construction of this integral, now usually called *Young integral*, is one of the basic pillars of the modern theory of *Rough paths* [5, 6] [1].

Note that in the summands $u(a_i)(v(a_{i+1}) - v(a_i))$ one could replace $u(a_i)$ by, for instance,

$$\bar{u}_{[a_i a_{i+1}]} := \frac{1}{2}(u(a_i) + u(a_{i+1})),$$

thus leading to a different notion of integral. Minding the obvious analogy with stochastic *Itô* (resp. *Stratonovich*) integration, we will further call these two constructions Itô (resp. Stratonovich) summation. The general conditions on functions u and v for the limits in each of these cases to exist have been studied in [15] (in the subsequent paper [19] even more general weighted averages of u in place of $\bar{u}_{[a_i a_{i+1}]}$ were considered). Finally, V. Matsaev and M. Solomyak constructed in [12] a similar integral substituting $\bar{u}_{[a_i a_{i+1}]}$ by the integral average $⨍_{[a_i, a_{i+1}]} u$, which extends the classical integral of a smooth differential 1-form $u\mathsf{d}v$ over an interval to the case when $v \in C^\beta(\mathbb{R})$ is Hölder continuous and u belongs to the Besov space $B^\alpha_{1,1}$ with $\alpha + \beta \geq 1$. In all the mentioned cases the result is the same for $u \in C^\alpha(\mathbb{R})$, $v \in C^\beta(\mathbb{R})$ with $\alpha + \beta > 1$, but may be different for more general functions.

[1] A historic curiosity: the modern construction of the Young integral via sewing lemma is closer to the original one used by Kondurar in [10] although his contribution to the subject seems to be unfortunately not so well-known.

1.1.2. *Multidimensional integrals.* Subsequently, several ways were proposed to extend the above mentioned one-dimensional constructions to multidimensional cases, notably [17, 3], which however lack the very important geometric property of the classical integral of multidimensional forms, namely, that of being *alternating*, i.e. changing sign with the change of domain orientation (although we also have to mention quite a different and purely geometric approach of [8] allowing to treat integration of smooth differential forms over nonsmooth domains, e.g. having fractal boundary, and a quite curious recent construction of [20], reducing the multidimensional integral to a one-dimensional one involving a Peano-like curve).

A different approach to the definition of a multidimensional integral of nonsmooth "differential forms" has been taken by R. Züst [24]. Applied to the 2D situation which is of interest in the present paper, it shows that the integral (1.1) defined over smooth maps, admits the unique extension by continuity with respect to the natural topology of pointwise convergence with bounded Hölder constants to a multilinear continuous functional

$$(f, g^1, g^2) \in C^\alpha(\mathbb{R}^2) \times C^{\beta_1}(\mathbb{R}^2) \times C^{\beta_2}(\mathbb{R}^2) \mapsto I(f, g^1, g^2)$$

vanishing over degenerate rectangles and triangles (namely, those having zero area) and alternating in the last two entries, if $\alpha + \beta_1 + \beta_2 > 2$. This functional can be therefore naturally called an integral

$$\int_\Omega f \mathsf{d}g^1 \wedge \mathsf{d}g^2 := I(f, g^1, g^2),$$

and can be approximated by sums over triangles forming the sufficiently fine dyadic decomposition of Ω of the functions of three variables (which can be better thought as functions of a triangle) $(p, q, r) \in (\mathbb{R}^2)^3 \mapsto \eta_{pqr}$ defined by

$$\eta_{pqr} := f_p \int_{\partial[pqr]} g^1 \mathsf{d}g^2, \tag{1.2}$$

where $f_p := f(p)$, the integral above being intended in the sense of Young (note that in [24] a slightly different language was used with rectangles instead of triangles; the current language is taken from [16] where a unified approach for integration of multidimensional nonsmooth "differential forms" called *"rough differential forms"* up to dimensions 1 and 2 was suggested). R. Züst himself has further successfully employed this integral in several remarkable geometric problems in [25].

It is easy to observe that the definition of the integral of $f \mathsf{d}g^1 \wedge \mathsf{d}g^2$ through the limit of sums of terms (1.2) over sequences of refining partitions, is a clear generalization of the construction of the one-dimensional Young integral described above. It is inherently based upon integration by parts, i.e. is made so that the Stokes theorem

$$\int_\Omega \mathsf{d}g^1 \wedge \mathsf{d}g^2 = \int_{\partial\Omega} g^1 \mathsf{d}g^2$$

almost automatically be satisfied for appropriate $\Omega \subset \mathbb{R}^2$ (rectangle in [24] or triangle in [16]). This is however not how one usually expects the integral to be defined: in fact, the Young integrals over the sides of the triangle $[pqr]$ in (1.2) have themselves to be defined either indirectly as continuous extensions of integrals of smooth differential forms approximating the "rough differential form" $g^1 \mathsf{d}g^2$ or as a limit of sums of appropriate discrete approximations (on the contrary, the abstract extension of (1.1) from spaces of smooth functions to Sobolev or Besov spaces can be done via the techniques from [13, 2, 9] dealing with weak Jacobians).

1.2. **Our contribution.** It seems therefore more natural to define the integral of the "rough differential forms" $f\,\mathsf{d}g^1 \wedge \mathsf{d}g^2$ by *purely discrete* approximations. To this aim for $f \in C^\alpha(\mathbb{R}^2)$, $g^i \in C^{\beta_i}(\mathbb{R}^2)$, $i = 1, 2$, with $\alpha + \beta_1 + \beta_2 > 2$, we write

$$\begin{aligned} \mathsf{strat}_{pqr} &:= \frac{1}{2}\left(\frac{f_p + f_q + f_r}{3}\right)\det\begin{pmatrix} \delta g^1_{pq} & \delta g^1_{pr} \\ \delta g^2_{pq} & \delta g^2_{pr} \end{pmatrix}, \\ \mathsf{ito}_{pqr} &:= \frac{1}{2} f_p \det\begin{pmatrix} \delta g^1_{pq} & \delta g^1_{pr} \\ \delta g^2_{pq} & \delta g^2_{pr} \end{pmatrix} \quad \text{for } [pqr] \subset \mathbb{R}^2, \end{aligned} \tag{1.3}$$

where we write f_u instead of $f(u)$ and $\delta g^i_{uv} := g^i(v) - g^i(u)$, $i = 1, 2$. We refer to strat and ito seen as functions of three variables (better viewed as functions of a two-dimensional simplex) as *Stratonovich germ* and to the latter one as *Itô germ* because of their obvious similarity with discrete constructions of the respective integrals in stochastic calculus. The terminology of "germs", meaning just functions of finite-dimensional simplices, is borrowed from *"germs of rough differential forms"* [16], which is in turn inherited from the Rough Paths theory [6].

In this paper we show that

(A) if Ω is an oriented simplex (i.e. a triangle), then summing either Itô or Stratonovich germs over any sufficiently nice family of its refining triangular partitions (in particular, dyadic ones) with the appropriately chosen orientation will still lead to the same integral defined by Züst, and estimate the rate of convergence (Theorems 4.4, 5.1). The respective integral may be called both Itô and Stratonovich, and in fact generalizes the one-dimensional Young integral.

It is worth emphasizing that this result might seem counterintuitive. In fact the integral should clearly vanish over degenerate triangles Ω (i.e. those having zero area), while neither the Stratonovich nor the Itô germ possess this property (which we will further call *nonatomicity*), as opposed to the germ η defined by (1.2), nor they are in some obvious way asymptotically close to some nonatomic germ (unless of course the functions g_1 and g_2 are differentiable). It is therefore not at all clear how can one expect to be nonatomic a limit of sums of germs which are essentially not so;

(B) the integral defined in such a way can be extended to a large class of bounded open sets $\Omega \subset \mathbb{R}^2$ having sufficiently small box-counting dimension of the topological boundary (Theorem 6.2), and in particular can be defined in a very natural way for Ω a simple polygon (Proposition 6.1).

These results give a partial answer to the curious and important question that can be termed informally as follows: along what kind of "surfaces" (or, more generally, against which de Rham currents) can one integrate the "rough forms" of the type $f\mathsf{d}g^1 \wedge \mathsf{d}g^2$ with just Hölder f, g^1, g^2. It is clearly inherently related to the recent work of R. Züst on "functions of fractional bounded variation" [26] and of J. Harrison on continuity of integrals with respect to the domain [7], though is essentially beyond the scope of the present paper. As a pure speculation however we may suggest that further investigation in this direction would surely lead to extension of Stokes' theorem to weak classes of surfaces/currents which may be helpful e.g. in extending the classical Frobenius intagrability theorem and Chow-Rachevsky theorem to irregular vector fields or forms like e.g. in [14, 11, 18];

(C) when f has a particular form $f(x) = F(x, g(x))$, then the conditions of the existence of the integral extending the classical one (for smooth forms), i.e. the requirements on Hölder exponents of g^i, may be significantly relaxed at the price of requiring $F\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ to be sufficiently regular (Theorem 7.1) by employing Stratonovich germs. This is however a very particular feature of Stratonovich but not of Itô summation as can be seen also in the one-dimensional situation (Remark 7.4). The resulting Stratonovich type integral is shown to satisfy the classical chain rule (Proposition 7.6) and may be identified with the "second order Riemann-Stieltjes" integral introduced in [24], the respective identification leading to a curious continuity estimate for the degree of Hölder maps (Remark 7.10).

We also give an interpretation of these results in geometric terms of the existence of continuous extensions of De Rham currents associated with the graphs of smooth maps $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ to those associated with graphs of Hölder maps with sufficiently large Hölder exponents, the continuity being intended in the weak (pointwise) topology of currents (Proposition 7.7).

The key role in the proofs will be played by the observation that both the integral and the Stratonovich germ are *alternating*, i.e. they change sign when the triangle over which they are defined changes the orientation. In fact, our basic instrument will be the natural generalization of the two-dimensional sewing lemma and stability theorem from [16] to abstract alternating germs (Lemmata A.1 and A.3 respectively).

An open, though im our opinion rather technical question is extending the above results to n-forms of the type $f\mathsf{d}g^1 \wedge \ldots \wedge \mathsf{d}g^n$ with arbitrary $n \in \mathbb{N}$. The major technical difficulty one encouters here is the absense of natural nice subdivisions of n-dimensional simplices with generic $n \in \mathbb{N}$ similar to dyadic subdivisons of segments and triangles (i.e. 1- and 2-dimensional simplices) that we successfully employ in the present paper.

2. Notation and preliminaries

Spaces. Let $D \subset \mathbb{R}^n$ be an open set. For an $\alpha \in (0,1)$ we will write $C^\alpha(\bar{D})$ (abbreviated just to C^α when there is no possibility of confusion) for the Hölder space with exponent α. For an $f \in C^\alpha(\bar{D})$ we denote by $[\delta f]_\alpha$ its Hölder seminorm, and $\|f\|_\alpha := \|f\|_\infty + [\delta f]_\alpha$ its Hölder norm, where $\|\cdot\|_\infty$ stands for the usual supremum norm in the space of continuous function $C(\bar{D})$ (usually abbreviated to C). The notation $C^1(\bar{D})$ (or just C^1 for brevity) will stand for the usual space of continuously differentiable functions.

Simplices, chains, germs and rough differential forms. For an ordered $(k+1)$-uple of points $S = [p_0 p_1 \ldots p_k] \in D^{k+1}$ we write $\operatorname{conv} S := \operatorname{conv}\{p_0 p_1 \ldots p_k\}$ and $\operatorname{diam} S$ for the convex envelope and the diameter of the set of points $\{p_0, \ldots, p_k\}$ respectively, and call S an (oriented) k-*simplex* in D, if $\operatorname{conv} S \subset D$, the set of such simplices being defnoted by $\operatorname{Simp}^k(D)$. For a k-simplex $S \in \operatorname{Simp}^k(D)$ we denote by $|S|$ its k-dimensional volume. A (real polyhedral) k-*chain* in D is an element of the real vector space $\mathrm{Chain}^k(\mathrm{D})$ generated by k-simplices in D. A k-simplex can be identified with the "geometric" simplex $\operatorname{conv} S$ with a chosen base point p_0 and the chosen orientation given by the order of the points in the list, so that 0-simplices correspond to points, 1-simplices to oriented segments and 2-simplices are pointed oriented triangles.

A *k-germ* (of a k-differential form in D) is a function $\omega\colon \mathrm{Simp}^k(D) \to \mathbb{R}$,

$$S = [p_0 p_1 \dots p_k] \mapsto \omega_S = \omega_{p_0 p_1 \dots p_k}.$$

We also often write $\langle S, \omega\rangle$ instead of ω_S. A *k-cochain* in D is a linear functional $\omega\colon \mathrm{Chain}^k(D) \to \mathbb{R}$,

$$C \mapsto \langle C, \omega\rangle\,.$$

For instance, 0-germs are just functions $p_0 \mapsto f(p_0) = f_{p_0} = \langle [p_0], f\rangle$.

The *boundary* ∂S of an $S \in \mathrm{Simp}^k(D)$ is the $(k-1)$-chain defined by

$$\partial[p_0 p_1 \dots, p_k] := \sum_{i=0}^{k} (-1)^i [p_0 \dots \hat{p}_i \dots p_k],$$

the notation $\hat{p}_i$ standing for removal of the respective element from the list. The operator ∂ is naturally extended by linearity to k-chains. The *coboundary* of a k-germ ω is the $(k+1)$-germ $\delta\omega$ defined by duality with the boundary of simplices, namely,

$$\langle S, \delta\omega\rangle := \langle \partial S, \omega\rangle\,.$$

For instance, for a 0-germ f one has $(\delta f)_{pq} = f_q - f_p$, and for a 1-germ ω one has $(\delta\omega)_{pqr} = \omega_{qr} - \omega_{pr} + \omega_{pq}$.

A k-germ ω is called

- *nonatomic*, if it vanishes on degenerate k-simplices S (i.e. on those having zero k-dimensional volume $|S| = 0$). For instance, the germ η defined by (1.2) is nonatomic, while the germs strat and ito defined by (1.3) are not;
- *alternating*, if

$$\langle [p_0 p_1 \dots p_k], \delta\omega\rangle := (-1)^\sigma \langle [\sigma(p_0)\sigma(p_1)\dots\sigma(p_k)], \omega\rangle\,.$$

  for every permutation of vertices $\sigma\colon \{p_0, p_1 \dots p_k\} \to \{p_0 p_1 \dots p_k\}$, $(-1)^\sigma$ standing for the sign of permutation (positive for even and negative for odd permutations). For instance, among the germs defined by (1.2) and (1.3), strat is alternating, while η and ito are not.

Finally, a k-germ ω is called a *rough differential k-form*, if it is continuous (as a function of vertices of a simplex), and both ω and $\delta\omega$ are nonatomic. An example of a rough differential 1-form (written $g^1 \mathsf{d} g^2$ for $g^i \in C^{\beta_i}$, $i = 1, 2$, with $\beta_1 + \beta_2 > 1$) is given by the Young integral over the line segment $[pq]$, that is,

$$\left\langle [pq], g^1 \mathsf{d} g^2 \right\rangle := \int_{[pq]} g^1 \mathsf{d} g^2.$$

An example of a rough differential 2-form (written $f \mathsf{d} g^1 \wedge \mathsf{d} g^2$ for $f \in C^\alpha$, $g^i \in C^{\beta_i}$, $i = 1, 2$, with $\alpha + \beta_1 + \beta_2 > 2$) is given by the integral defined by R. Züst in [24], namely,

$$\left\langle [pqr], f \mathsf{d} g^1 \wedge \mathsf{d} g^2 \right\rangle := \int_{[pqr]} f \mathsf{d} g^1 \wedge \mathsf{d} g^2.$$

The *cup product* (called external product in [6]) between a k-germ ω and a h-germ $\tilde{\omega}$ is the $(k+h)$-germ $\omega \cup \tilde{\omega}$ defined by

$$\langle [p_0 p_1 \ldots p_k p_{k+1} \ldots p_{k+h}], \omega \cup \tilde{\omega} \rangle := \langle [p_0 p_1 \ldots p_k], \omega \rangle \, \langle [p_k p_{k+1} \ldots p_{k+h}], \tilde{\omega} \rangle \, .$$

The cup product is associative but in general not commutative, and the following Leibniz rule holds [16]: for $\omega \in \mathrm{Germ}^k(D)$, $\tilde{\omega} \in \mathrm{Germ}^h(D)$ one has

$$\delta(\omega \cup \tilde{\omega}) = (\delta\omega) \cup \tilde{\omega} + (-1)^k \omega \cup (\delta\tilde{\omega}). \tag{2.1}$$

3. Estimates on germs

We start with the following useful algebraic lemma.

Lemma 3.1. *One has*

$$\begin{aligned} \frac{1}{2}\det \begin{pmatrix} \delta g^1_{pq} & \delta g^1_{pr} \\ \delta g^2_{pq} & \delta g^2_{pr} \end{pmatrix} = \frac{1}{2}\det \begin{pmatrix} \delta g^1_{pq} & \delta g^1_{qr} \\ \delta g^2_{pq} & \delta g^2_{qr} \end{pmatrix} = \frac{1}{2}\det \begin{pmatrix} \delta g^1_{rq} & \delta g^1_{pr} \\ \delta g^2_{rq} & \delta g^2_{pr} \end{pmatrix} \\ = \mathcal{A}(\delta g^1 \cup \delta g^2)_{pqr}, \end{aligned} \tag{3.1}$$

where $\mathcal{A}$ stands for the antisymmetrization operator

$$\mathcal{A}(\phi \cup \psi) := \frac{1}{2}\left(\phi \cup \psi - \psi \cup \phi\right).$$

In particular,

$$\mathsf{ito}_{pqr} = (f \cup \mathcal{A}(\delta g^1 \cup \delta g^2))_{pqr}. \tag{3.2}$$

Proof. It suffices to calculate

$$\begin{aligned} \det \begin{pmatrix} \delta g^1_{pq} & \delta g^1_{pr} \\ \delta g^2_{pq} & \delta g^2_{pr} \end{pmatrix} - \det \begin{pmatrix} \delta g^1_{pq} & \delta g^1_{qr} \\ \delta g^2_{pq} & \delta g^2_{qr} \end{pmatrix} &= \det \begin{pmatrix} \delta g^1_{pq} & \delta g^1_{pr} - \delta g^1_{qr} \\ \delta g^2_{pq} & \delta g^2_{pr} - \delta g^2_{qr} \end{pmatrix} \\ &= \det \begin{pmatrix} \delta g^1_{pq} & \delta g^1_{pq} \\ \delta g^2_{pq} & \delta g^2_{pq} \end{pmatrix} = 0 \end{aligned}$$

to show the first equality in (3.1); the third one follows then from the definition of $\mathcal{A}$. The second equality is quite analogous from

$$\begin{aligned} \det \begin{pmatrix} \delta g^1_{pq} & \delta g^1_{pr} \\ \delta g^2_{pq} & \delta g^2_{pr} \end{pmatrix} - \det \begin{pmatrix} \delta g^1_{rq} & \delta g^1_{pr} \\ \delta g^2_{rq} & \delta g^2_{pr} \end{pmatrix} &= \det \begin{pmatrix} \delta g^1_{pq} - \delta g^1_{rq} & \delta g^1_{pr} \\ \delta g^2_{pq} - g^2_{rq} & \delta g^2_{pr} \end{pmatrix} \\ &= \det \begin{pmatrix} \delta g^1_{qp} & \delta g^1_{pq} \\ \delta g^2_{qp} & \delta g^2_{pq} \end{pmatrix} = 0, \end{aligned}$$

concluding the proof. $\square$

Notice that $\mathcal{A}(\delta g^1 \cup \delta g^2) = \delta\eta$ with $\eta = \frac{1}{2}\left(g^1 \delta g^2 - g^2 \delta g^1\right)$.

Lemma 3.2. *One has*

$$|\mathsf{ito}_{pqr} - \mathsf{strat}_{pqr}| \le 2[\delta f]_\alpha [\delta g^1]_{\beta_1} [\delta g^2]_{\beta_2} \operatorname{diam}([pqr])^{\alpha+\beta_1+\beta_2} \tag{3.3}$$

$$|\mathsf{strat}_{pqr}| \le \|f\|_\infty \, [\delta g^1]_{\beta_1} [\delta g^2]_{\beta_2} \operatorname{diam}([pqr])^{\beta_1+\beta_2} \tag{3.4}$$

$$|\delta \mathsf{strat}_{pqrs}| \le 8[\delta f]_\alpha [\delta g^1]_{\beta_1} [\delta g^2]_{\beta_2} \operatorname{diam}([pqrs])^{\alpha+\beta_1+\beta_2} \tag{3.5}$$

and strat *is alternating, namely,*

$$\mathsf{strat}_{pqr} = \mathsf{strat}_{rpq} = \mathsf{strat}_{qrp} = -\mathsf{strat}_{rqp} = -\mathsf{strat}_{prq} = -\mathsf{strat}_{prq}.$$

Remark 3.3. Clearly, (3.4) holds even for every $f \in M$, where M stands for the space of bounded (not necessarily measurable) functions over $\mathbb{R}^2$ equipped with the supremum norm (still denoted by $\| \cdot \|_\infty$).

Proof. The estimate (3.4) as well as the alternating property of strat is immediate from the definition of strat. To show (3.3), we calculate

$$\begin{aligned}|\mathsf{ito}_{pqr} - \mathsf{strat}_{pqr}| &= \frac{1}{2}\left|\left(\frac{f_p+f_q+f_r}{3} - f_p\right)\det\begin{pmatrix} \delta g^1_{pq} & \delta g^1_{pr} \\ \delta g^2_{pq} & \delta g^2_{pr}\end{pmatrix}\right| \\ &\le [\delta f]_\alpha [\delta g^1]_{\beta_1}[\delta g^2]_{\beta_2}\operatorname{diam}(pqr)^{\alpha+\beta_1+\beta_2}\end{aligned}$$

as claimed. Thus, (3.5) would follow once one proves

$$|\delta \mathsf{ito}_{pqrs}| \le [\delta f]_\alpha [\delta g^1]_{\beta_1}[\delta g^2]_{\beta_2}\operatorname{diam}([pqrs])^{\alpha+\beta_1+\beta_2}. \tag{3.6}$$

To show the latter inequality, we use Lemma 3.1: namely, by (3.2) one has

$$\mathsf{ito} = \frac{1}{2}\left((f\cup\delta g^1\cup\delta g^2) - (f\cup\delta g^2\cup\delta g^1)\right). \tag{3.7}$$

Therefore, using the fact that

$$\delta(\delta g^1\cup\delta g^2) = \delta g^1\cup\delta(\delta g^2) - \delta(\delta g^1)\cup\delta g^2 = 0,$$

and analogously $\delta(\delta g^2\cup\delta g^1) = 0$, from (3.7) we get

$$\begin{aligned}\delta\mathsf{ito} &= \frac{1}{2}\left(\delta(f\cup\delta g^1\cup\delta g^2) - \delta(f\cup\delta g^2\cup\delta g^1)\right) \\ &= \frac{1}{2}\left((\delta f\cup\delta g^1\cup\delta g^2) - (\delta f\cup\delta g^2\cup\delta g^1)\right).\end{aligned} \tag{3.8}$$

Since clearly,

$$\begin{aligned}|(\delta f\cup\delta g^1\cup\delta g^2)_{pqrs}| &= |(\delta f)_{pq}|\cdot|(\delta g^1)_{qr}|\cdot|(\delta g^2)_{rs}| \\ &\le [\delta f]_\alpha [\delta g^1]_{\beta_1}[\delta g^2]_{\beta_2}\operatorname{diam}([pqrs])^{\alpha+\beta_1+\beta_2},\end{aligned}$$

and analogously

$$|(\delta f\cup\delta g^2\cup\delta g^1)_{pqrs}| \le [\delta f]_\alpha [\delta g^1]_{\beta_1}[\delta g^2]_{\beta_2}\operatorname{diam}([pqrs])^{\alpha+\beta_1+\beta_2},$$

from (3.8) we get (3.6), and therefore (3.5), hence concluding the proof. □

Later in section 7 we will need also the following curious algebraic identity which is a peculiar property of only the Stratonovich germ strat and not of the Itô germ ito, and could have been also used for an alternative proof of (3.5) in Lemma 3.2.

Lemma 3.4. *One has*

$$(\delta\mathsf{strat})_{pqrs} = \frac{1}{6}\det\begin{pmatrix} \delta f_{pq} & \delta f_{pr} & \delta f_{ps} \\ \delta g^1_{pq} & \delta g^1_{pr} & \delta g^1_{ps} \\ \delta g^2_{pq} & \delta g^2_{pr} & \delta g^2_{ps}\end{pmatrix}.$$

Proof. By Lemma 3.1 one has

$$\begin{aligned}&6\mathsf{strat}_{pqr} \\ &\quad= f_p\det\begin{pmatrix} \delta g^1_{pq} & \delta g^1_{qr} \\ \delta g^2_{pq} & \delta g^2_{qr}\end{pmatrix} + f_q\det\begin{pmatrix} \delta g^1_{pq} & \delta g^1_{qr} \\ \delta g^2_{pq} & \delta g^2_{qr}\end{pmatrix} + f_r\begin{pmatrix} \delta g^1_{pq} & \delta g^1_{qr} \\ \delta g^2_{pq} & \delta g^2_{qr}\end{pmatrix} \\ &\quad= (f\cup\delta g^1\cup\delta g^2 - f\cup\delta g^2\cup\delta g^1)_{pqr} + (\delta g^1\cup f\cup\delta g^2 - \delta g^2\cup f\cup\delta g^1)_{pqr} \\ &\qquad+ (\delta g^1\cup\delta g^2\cup f - \delta g^2\cup\delta g^1\cup f)_{pqr}.\end{aligned}$$

Hence,

$$
\begin{aligned}
6(\delta\mathsf{strat})_{pqrs} &= (\delta f \cup \delta g^1 \cup \delta g^2 - \delta f \cup \delta g^2 \cup \delta g^1)_{pqrs} + \\
&\quad (-\delta g^1 \cup \delta f \cup \delta g^2 + \delta g^2 \cup \delta f \cup \delta g^1)_{pqrs} + \\
&\quad (\delta g^1 \cup \delta g^2 \cup \delta f - \delta g^2 \cup \delta g^1 \cup \delta f)_{pqrs} \\
&= \det \begin{pmatrix} \delta f_{pq} & \delta f_{qr} & \delta f_{rs} \\ \delta g^1_{pq} & \delta g^1_{qr} & \delta g^1_{rs} \\ \delta g^2_{pq} & \delta g^2_{qr} & \delta g^2_{rs} \end{pmatrix} = \frac{1}{6} \det \begin{pmatrix} \delta f_{pq} & \delta f_{pr} & \delta f_{ps} \\ \delta g^1_{pq} & \delta g^1_{pr} & \delta g^1_{ps} \\ \delta g^2_{pq} & \delta g^2_{pr} & \delta g^2_{ps} \end{pmatrix},
\end{aligned}
$$

where the latter identity follows by adding the first column to the second one and subsequently the second column to the third one. □

4. Riemann summation over dyadic partitions

Recall [16] the *dyadic decomposition* of a 2-simplex $[p_0p_1p_2] \in \mathrm{Simp}^2(D)$

$$\mathrm{dya}[p_0p_1p_2] := [q_0q_1q_2] + [q_1q_0p_2] + [q_2p_1q_0] + [p_0q_2q_1],$$

where $q_i := (p_j + p_\ell)/2$ for $\{i, j, \ell\} = \{0, 1, 2\}$. Write also $\mathrm{cut}[p_0p_1] := [p_0q] + [qp_1]$ and $\mathrm{fill}[p_0p_1] := [p_0qp_1]$, with $q := (p_0 + p_1)/2$ (naturally extended to chains).

For $n \in \mathbb{N}$ define the n-th *Stratonovich sum* strat^n, the *side corrector* S^n as well as the *Itô sum* ito^n respectively by the formulae

$$
\begin{aligned}
&\mathsf{strat}^n_{pqr} := \langle \mathrm{dya}^n[pqr], \mathsf{strat}\rangle, \quad S^n_{pq} := \sum_{i=0}^{n-1} \left\langle \mathrm{fill}\,\mathrm{cut}^i[pq], \mathsf{strat}\right\rangle, \\
&\mathsf{ito}^n_{pqr} := \langle \mathrm{dya}^n[pqr], \mathsf{ito}\rangle.
\end{aligned} \tag{4.1}
$$

Lemma 4.1. *One has*

$$|S^{n+1}_{pq} - S^n_{pq}| \le C\,\|f\|_\infty [\delta g^1]_{\beta_1} [\delta g^2]_{\beta_2} \operatorname{diam}([pq])^{\beta_1+\beta_2} 2^{n(1-\beta_1-\beta_2)}, \tag{4.2}$$

$$
\begin{aligned}
&|\langle [pqr], (\mathsf{strat}^n - \delta S^n) - (\mathsf{strat}^{n+1} - \delta S^{n+1})\rangle| \\
&\qquad \le C[\delta f]_\alpha [\delta g^1]_{\beta_1} [\delta g^2]_{\beta_2} \operatorname{diam}([pqr])^{\alpha+\beta_1+\beta_2} 2^{n(2-\alpha-\beta_1-\beta_2)}
\end{aligned} \tag{4.3}
$$

with $C > 0$ a universal constant. In particular, if $\alpha + \beta_1 + \beta_2 > 2$, then

$$
\begin{aligned}
S_{pq} &:= \lim_{n\to\infty} S^n_{pq}, \\
V_{pqr} &:= \lim_{n\to\infty} \mathsf{strat}^n_{pqr} = \lim_{n\to\infty} (\mathsf{strat}^n_{pqr} - \delta S^n_{pqr}) + \delta S^n_{pqr}
\end{aligned} \tag{4.4}
$$

are well defined continuous alternating germs with

$$S_{pq} \colon C^0 \times C^{\beta_1} \times C^{\beta_2} \to \mathbb{R}, \qquad V_{pqr} \colon C^\alpha \times C^{\beta_1} \times C^{\beta_2} \to \mathbb{R}$$

continuous and

$$|S^n_{pq} - S_{pq}| \le \|f\|_\infty [\delta g^1]_{\beta_1} [\delta g^2]_{\beta_2} \operatorname{diam}([pq])^{\beta_1+\beta_2} 2^{n(1-\beta_1-\beta_2)}, \tag{4.5}$$

$$
\begin{aligned}
&|\mathsf{strat}^n_{pqr} - V_{pqr} - \delta(S^n - S)_{pqr}| \le \\
&\qquad C[\delta f]_\alpha [\delta g^1]_{\beta_1} [\delta g^2]_{\beta_2} \operatorname{diam}([pqr])^{\alpha+\beta_1+\beta_2} 2^{n(2-\alpha-\beta_1-\beta_2)},
\end{aligned} \tag{4.6}
$$

$$|\mathsf{strat}^n_{pqr} - V_{pqr}| \le C\|f\|_\alpha [\delta g^1]_{\beta_1} [\delta g^2]_{\beta_2} \operatorname{diam}([pq])^{\beta_1+\beta_2} 2^{n(1-\beta_1-\beta_2)}. \tag{4.7}$$

Remark 4.2. As one easily deduces from the proof, in view of the Remark 3.3, one has, with the notation of the latter, that in fact S_{pq} itself may be defined over the larger space $M \times C^{\beta_1} \times C^{\beta_2}$ and is continuous there when just $\beta_1 + \beta_2 > 1$.

Proof. We apply Lemma A.1 to our germ strat (which is continuous and alternating by construction) recalling that it satisfies both (A.1) and (A.2) with

$$\gamma_1 := \beta_1 + \beta_2 > 1, \quad C_1 := \|f\|_\infty [\delta g^1]_{\beta_1} [\delta g^2]_{\beta_2},$$
$$\gamma_2 := \alpha + \beta_1 + \beta_2 > 2, \quad C_2 := 8[f]_\alpha [\delta g^1]_{\beta_1} [\delta g^2]_{\beta_2}$$

in view of Lemma 3.2. This gives (4.2) and (4.3), as well as the existence of limit germs alternating continuous S and V in (4.4) satisfying (4.5), (4.6) and (4.7). Finally, the continuity of S_{pq} (with fixed $[pq]$) as a functional follows from (4.2) and implies the continuity of $\delta S_{pqr} \colon C^0 \times C^{\beta_1} \times C^{\beta_2} \to \mathbb{R}$. Continuity of

$$V_{pqr} - \delta S_{pqr} := \lim_{n\to\infty} (\mathsf{strat}^n_{pqr} - \delta S^n_{pqr}) \colon C^\alpha \times C^{\beta_1} \times C^{\beta_2} \to \mathbb{R}$$

follows from (4.3), hence implying the continuity of V, and therefore concluding the proof. □

We will need also the following Lemma already formulated in [16, example 4.7].

Lemma 4.3. *If $\beta_1 = \beta_2 = 1$, then*

$$V_{pqr} = \int_{[pqr]} f \mathsf{d}g^1 \wedge \mathsf{d}g^2 = \int_{[pqr]} f \det(\nabla g^1, \nabla g^2).$$

We are now at a position to prove the first principal result of this paper.

Theorem 4.4. *If $\alpha + \beta_1 + \beta_2 > 2$, then*

$$\lim_{n\to\infty} \mathsf{strat}^n_{pqr} = \int_{[pqr]} f \mathsf{d}g^1 \wedge \mathsf{d}g^2 \tag{4.8}$$
$$= \lim_{n\to\infty} \mathsf{ito}^n_{pqr}. \tag{4.9}$$

In particular, the latter integral is

(A) *nonatomic, i.e.*

$$\int_{[pqr]} f \mathsf{d}g^1 \wedge \mathsf{d}g^2 = 0 \quad \textit{when } |[pqr]| = 0,$$

(B) *continuous and alternating in $[pqr]$, and*

(C) *additive, in the sense that when*

$$[pqr] = \sum_{i=1}^k \Delta_i + N + \partial R,$$

where Δ_i are oriented 2-simplices, N is a polyhedral 2-chain consisting of degenerate 2-simplices (i.e. having area zero), and R is a polyhedral 3-chain in $\mathbb{R}^2$, then

$$\int_{[pqr]} f \mathsf{d}g^1 \wedge \mathsf{d}g^2 = \sum_{i=1}^k \int_{\Delta_i} f \mathsf{d}g^1 \wedge \mathsf{d}g^2.$$

Moreover,

$$\left| \mathsf{strat}^n_{pqr} - \int_{[pqr]} f \mathsf{d}g^1 \wedge \mathsf{d}g^2 \right| \le C \|f\|_\alpha [\delta g^1]_{\beta_1} [\delta g^2]_{\beta_2} \operatorname{diam}([pqr])^{\beta_1+\beta_2} 2^{n(1-\beta_1-\beta_2)}, \tag{4.10}$$

(4.11)
$$\left|\mathsf{ito}^n_{pqr} - \int_{[pqr]} f \mathsf{d}g^1 \wedge \mathsf{d}g^2\right| \le C \left\|f\right\|_\alpha [\delta g^1]_{\beta_1} [\delta g^2]_{\beta_2} \operatorname{diam}([pqr])^{\beta_1+\beta_2} 2^{n(1-\beta_1-\beta_2)}$$

for some $C = C(\alpha, \beta_1, \beta_2) > 0$.

Proof. By Lemma 4.1, the limit

$$V_{pqr} := \lim_{n\to\infty} \mathsf{strat}^n_{pqr}$$

exists and is a continuous multilinear functional over $C^\alpha \times C^{\beta_1} \times C^{\beta_2}$, and

$$V_{pqr}(f, g^1, g^2) = \int_{[pqr]} f \mathsf{d}g^1 \wedge \mathsf{d}g^2 := \int_{[pqr]} f \det(\nabla g^1, \nabla g^2)\, dx$$

when $f \in C^0$, $g^i \in C^1$, $i = 1, 2$. However the unique continuous extension of the latter functional defined over $C^0 \times C^1 \times C^1$ to $C^\alpha \times C^{\beta_1} \times C^{\beta_2}$ is the Züst integral, which implies the claim (4.8), (4.10). Properties (A), (B) and (C) are now in fact the properties of the Züst integral (theorem 4.10 of [16] where they are stated by saying that the Züst germ (1.2) is *sewable*).

The claims (4.9), (4.11) follow now from (3.3). □

Remark 4.5. One also has the inequality (4.6) which can be rewritten, in view of the above Theorem 4.4 as

$$\begin{aligned}(4.12)\quad &\left|\mathsf{strat}^n_{pqr} - \int_{[pqr]} f \mathsf{d}g^1 \wedge \mathsf{d}g^2 - \delta(S^n - S)_{pqr}\right| \\ &\qquad\le C \left\|f\right\|_\alpha [\delta g^1]_{\beta_1} [\delta g^2]_{\beta_2} \operatorname{diam}([pqr])^{\alpha+\beta_1+\beta_2} 2^{n(2-\alpha-\beta_1-\beta_2)}\end{aligned}$$

with $C = C(\alpha, \beta_1, \beta_2) > 0$. Thus, in order to improve the convergence rate one should better approximate $S^n - S$. This is the case e.g. when on the boundary of $[pqr]$ either f is null or one of the g^i is constant: in fact, in these cases $S^n = 0$ and hence also $S = 0$.

Remark 4.6. The 2-germ $f \cup \delta g^1 \cup \delta g^2$ in general does not provide an integral even when f, g^1 and g^2 are smooth. In fact, let $f = 1$, $g^i(x_1, x_2) := x_i$, $i = 1, 2$, $p = (0, 0)$, $q = (1, 0)$, $r = (0, 1)$. Then $\langle \mathrm{dya}^n[pqr], f \cup \delta g^1 \cup \delta g^2\rangle \to 2|[pqr]|$ while $\langle \mathrm{dya}^n[pqr], f \cup \delta g^2 \cup \delta g^1\rangle \to 0$ as $n \to \infty$, i.e. the limit is not alternating.

Remark 4.7. As mentioned in the introduction, the above theorem allows to define the integral of a differential 2-form $\omega = f \mathsf{d}g^1 \wedge \mathsf{d}g^2$ on $\mathbb{R}^n$ over a parameterized Hölder surface $\varphi\colon \Omega \to \mathbb{R}^n$, $\varphi(x) = (\varphi^i(x))_{i=1}^n$, letting

$$\int_{\varphi([pqr])} f \mathsf{d}g^1 \wedge \mathsf{d}g^2 := \int_{[pqr]} (f \circ \varphi) \mathsf{d}(g^1 \circ \varphi) \wedge \mathsf{d}(g^2 \circ \varphi),$$

provided that $f \in C^\alpha(\mathbb{R}^n)$, $g^i \in C^{\beta_i}(\mathbb{R}^n)$, $i = 1, 2$, $\varphi \in C^\gamma(\mathbb{R}^2; \mathbb{R}^n)$ with

$$\gamma(\alpha + \beta_1 + \beta_2) > 2.$$

Notice however that the above integral differs from the integral obtained partitioning the triangle $[\varphi(p)\varphi(q)\varphi(r)]$ with an order $\operatorname{diam}([pqr])^{\gamma(\beta_1+\beta_2)}$ and not $\operatorname{diam}([pqr])^{\gamma(\alpha+\beta_1+\beta_2)}$, see [16, proposition 4.29].

Corollary 4.8. *If there is an $h \in C^{\beta_3}$, $\beta_3 \in (0,1]$, such that both g^1 and g^2 are h-differentiable in the sense*

$$(\delta g^i)_{pq} = a_p^i (\delta h)_{pq} + o(|p-q|), \quad i = 1,2$$

for every $p \in D$ as $q \to p$, and, moreover,

$$|(\delta g^i)_{pq} - a_p^i (\delta h)_{pq}| \le C|p-q|^{1+\gamma_i} \tag{4.13}$$

for some $\gamma_i > 1 - \beta_3$, $i = 1,2$, and $C > 0$, then $\mathsf{d}g^1 \wedge \mathsf{d}g^2 = 0$ in the sense

$$\int_{[pqr]} f \mathsf{d}g^1 \wedge \mathsf{d}g^2 = 0$$

for every $f \in C^\alpha$ with $\alpha + \beta_1 + \beta_2 > 2$ and every $[pqr] \subset D$.

Proof. Let

$$\rho_{pq}^i := (\delta g^i)_{pq} - a_p^i (\delta h)_{pq}.$$

Then

$$\begin{aligned} \frac{1}{2}\det\begin{pmatrix} \delta g_{pq}^1 & \delta g_{pr}^1 \\ \delta g_{pq}^2 & \delta g_{pr}^2 \end{pmatrix} &= \frac{1}{2} a_p^1 a_p^2 \det\begin{pmatrix} \delta h_{pq} & \delta h_{pr} \\ \delta h_{pq} & \delta h_{pr} \end{pmatrix} + \frac{1}{2} a_p^1 a_p^2 \det\begin{pmatrix} \rho_{pq}^1 & \delta h_{pr} \\ \rho_{pq}^2 & \delta h_{pr} \end{pmatrix} \\ &\quad + \frac{1}{2} a_p^1 a_p^2 \det\begin{pmatrix} \delta h_{pq} & \rho_{pr}^1 \\ \delta h_{pq} & \rho_{pr}^2 \end{pmatrix} + \frac{1}{2}\det\begin{pmatrix} \rho_{pq}^1 & \rho_{pr}^1 \\ \rho_{pq}^2 & \rho_{pr}^2 \end{pmatrix}. \end{aligned} \tag{4.14}$$

Letting $\gamma := \gamma_1 \wedge \gamma_2$, from (4.13) we get

$$\begin{aligned} \left|\det\begin{pmatrix} \rho_{pq}^1 & \delta h_{pr} \\ \rho_{pq}^2 & \delta h_{pr} \end{pmatrix}\right| &\le 2C[h]_{\beta_3} \operatorname{diam}([pqr])^{1+\gamma+\beta_3}, \\ \left|\det\begin{pmatrix} \delta h_{pq} & \rho_{pr}^1 \\ \delta h_{pq} & \rho_{pr}^2 \end{pmatrix}\right| &\le 2C[h]_{\beta_3} \operatorname{diam}([pqr])^{1+\gamma+\beta_3}, \\ \left|\det\begin{pmatrix} \rho_{pq}^1 & \rho_{pr}^1 \\ \rho_{pq}^2 & \rho_{pr}^2 \end{pmatrix}\right| &\le 2C \operatorname{diam}([pqr])^{2+\gamma_1+\gamma_2}, \end{aligned}$$

so that by (4.14) one has

$$|\mathsf{strat}_{pqr}| \le 2C\|f\|_\infty (\|a^1\|_\infty \|a^2\|_\infty [h]_{\beta_3} \operatorname{diam}([pqr])^{1+\gamma+\beta_3} + \operatorname{diam}([pqr])^{2+\gamma_1+\gamma_2}),$$

which concludes the proof since $1 + \gamma + \beta_3 > 2$ and $2 + \gamma_1 + \gamma_2 > 2$. □

Remark 4.9. In particular, if g^1 is g^2-differentiable and, moreover,

$$|(\delta g^1)_{pq} - a_p (\delta g^2)_{pq}| \le C|p-q|^{1+\gamma} \tag{4.15}$$

for some $\gamma > 1 - \beta_2$ and $C > 0$, then $\mathsf{d}g^1 \wedge \mathsf{d}g^2 = 0$.

5. General partitions

Theorem 4.4 shows that the integral $\int_{[pqr]} f \mathsf{d}g^1 \wedge \mathsf{d}g^2$ can be obtained as a limit of sums of the Stratonovich germs over dyadic partitions of the the simplex $[pqr]$. Here we show that it can be obtained by a similar summation of such germs over more general partitions.

Theorem 5.1. *Assume that the simplex $[pqr]$ be partitioned in a finite number of disjoint simplices $\{\Delta_i\}_{i=1}^N$ not belonging to the sides of $[pqr]$ so that*

$$[pqr] - \sum_{i=1}^{N} \Delta_i = \partial P + \sum_{j=1}^{M} Q_j, \tag{5.1}$$

where $P \in \mathrm{Chain}^3(D)$ and each $Q_j \in \mathrm{Simp}^2(D)$ is a degenerate simplex reduced to a line segment belonging to some side of $[pqr]$ such that two sides of each Q_j are sides of some Δ_i (with opposite direction). Then

$$\begin{aligned} &\left| \sum_{i=1}^{N} \langle \Delta_i, \mathsf{strat} \rangle - \int_{[pqr]} f \mathsf{d} g^1 \wedge \mathsf{d} g^2 \right| \\ &\qquad \leq C \, \|f\|_\alpha \, [\delta g^1]_{\beta_1} [\delta g^2]_{\beta_2} \left(\sum_{i=1}^{N} \mathrm{diam}(\Delta_i)^{\alpha+\beta_1+\beta_2} + \sum_{j=1}^{M} \mathrm{diam}(Q_j)^{\beta_1+\beta_2} \right). \end{aligned} \tag{5.2}$$

Proof. The estimate (4.12) applied to each Δ_i with $n := 0$ gives

$$\begin{aligned} &\left| \langle \Delta_i, \mathsf{strat} \rangle - \int_{\Delta_i} f \mathsf{d} g^1 \wedge \mathsf{d} g^2 \; - \big\langle \Delta_i, \delta(S^0 - S) \big\rangle \right| \\ &\qquad \leq C \, \|f\|_\alpha \, [\delta g^1]_{\beta_1} [\delta g^2]_{\beta_2} \, \mathrm{diam}(\Delta_i)^{\alpha+\beta_1+\beta_2}. \end{aligned}$$

Summing the latter estimates over $i = 1, \ldots, N$, and recalling that

$$\int_{[pqr]} f \mathsf{d} g^1 \wedge \mathsf{d} g^2 = \sum_{i=1}^{N} \int_{\Delta_i} f \mathsf{d} g^1 \wedge \mathsf{d} g^2$$

in view of (5.1), we get

$$\begin{aligned} &\left| \sum_{i=1}^{N} \langle \Delta_i, \mathsf{strat} \rangle - \int_{[pqr]} f \mathsf{d} g^1 \wedge \mathsf{d} g^2 \; - \sum_{j=1}^{M} \big\langle q_j, S^0 - S \big\rangle \right| \\ &\qquad \leq C \, \|f\|_\alpha \, [\delta g^1]_{\beta_1} [\delta g^2]_{\beta_2} \sum_{i=1}^{N} \mathrm{diam}(\Delta_i)^{\alpha+\beta_1+\beta_2}, \end{aligned} \tag{5.3}$$

where $q_j \in \mathrm{Simp}^1(D)$ is the side of Q_j which is not a side of any Δ_i: in fact, when summing the terms

$$\big\langle \Delta_i, \delta(S^0 - S) \big\rangle = \big\langle \partial \Delta_i, S^0 - S \big\rangle$$

over i, we have that every side of some simplex of the partition which is not one of q_j (i.e. does not belong to a side of $[pqr]$) appears in this sum twice and in opposite directions, and hence is cancelled out from this sum. Moreover, from (4.5) applied with q_j instead of $[pq]$ and $n := 0$ we get

$$| \big\langle q_j, S^0 - S \big\rangle | \leq \|f\|_\infty \, [\delta g^1]_{\beta_1} [\delta g^2]_{\beta_2} \, \mathrm{diam}(q_j)^{\beta_1+\beta_2},$$

which together with (5.3) gives (5.2) since $\mathrm{diam}\, q_j = \mathrm{diam}\, Q_j$. □

6. Integration over general domains

In section 4 we defined the integral of the "rough differential form" $f\mathsf{d}g^1 \wedge \mathsf{d}g^2$ over an arbitrary oriented simplex $[pqr]$ in the domain of definition of f and g. Here we show how the latter can be naturally extended to more general domains $\Omega \subset \mathbb{R}^2$.

First, consider the case when Ω is an oriented simple (i.e. not self-intersecting) polygon with vertices $a_0, \ldots, a_k$, enumerated according to the orientation of Ω (say, counterclockwise). We will write in this case $\Omega = [a_0 \ldots a_k]$. Consider the triangulation of Ω in two-dimensional simplices $\{\Delta_i\}_{i=1}^m$ oriented in the same direction of Ω. We set then by definition

$$\int_{[a_0 \ldots a_k]} f\mathsf{d}g^1 \wedge \mathsf{d}g^2 := \sum_{i=1}^m \int_{\Delta_i} f\mathsf{d}g^1 \wedge \mathsf{d}g^2. \tag{6.1}$$

The following statement is valid.

Proposition 6.1. *Under conditions of Theorem 4.4 for every $b \in \mathbb{R}^2$ one has*

$$\int_{[a_0 \ldots a_k]} f\mathsf{d}g^1 \wedge \mathsf{d}g^2 = \sum_{j=0}^k \int_{[a_j a_{j+1} b]} f\mathsf{d}g^1 \wedge \mathsf{d}g^2, \tag{6.2}$$

where $k+1 := 0$. In particular, the definition (6.1) is correct (i.e. independent on the particular triangulation $\{\Delta_i\}$), the above integral is alternating (i.e. preserves/resp. changes sign with odd/resp. even permutation of the vertices), nonatomic (i.e. zero on polygons of zero area), and the map

$$(f, g^1, g^2) \mapsto \int_{[a_0 \ldots a_k]} f\mathsf{d}g^1 \wedge \mathsf{d}g^2$$

is a continuous multilinear functional over $C^\alpha \times C^{\beta_1} \times C^{\beta_2}$ continuous also in the vertices $a_0, \ldots, a_k$ (i.e. continuous with respect to the simultaneous convergence of both functions involved and of the vertices).

Proof. Writing $\Delta_i := [\alpha_i^1 \alpha_i^2 \alpha_i^3]$, one has

$$\sum_{i=1}^m \partial[b\alpha_i^1\alpha_i^2\alpha_i^3] = \sum_{i=1}^m [\alpha_i^1\alpha_i^2\alpha_i^3] - \sum_{i=1}^m [b\alpha_i^2\alpha_i^3] + \sum_{i=1}^m [b\alpha_i^1\alpha_i^3] - \sum_{i=1}^m [b\alpha_i^1\alpha_i^2],$$

so that taking into account (6.1), and recalling that

$$\left\langle \partial[pqrs], f\mathsf{d}g^1 \wedge \mathsf{d}g^2 \right\rangle = 0,$$

we get

$$\begin{aligned}\int_{[a_0 \ldots a_k]} & f\mathsf{d}g^1 \wedge \mathsf{d}g^2 = \\ & \sum_{i=1}^m \left(\int_{[b\alpha_i^2\alpha_i^3]} f\mathsf{d}g^1 \wedge \mathsf{d}g^2 - \int_{[b\alpha_i^1\alpha_i^3]} f\mathsf{d}g^1 \wedge \mathsf{d}g^2 + \int_{[b\alpha_i^1\alpha_i^2]} f\mathsf{d}g^1 \wedge \mathsf{d}g^2 \right) = \\ & \sum_{i=1}^m \left(\int_{[\alpha_i^1\alpha_i^2 b]} f\mathsf{d}g^1 \wedge \mathsf{d}g^2 + \int_{[\alpha_i^2\alpha_i^3 b]} f\mathsf{d}g^1 \wedge \mathsf{d}g^2 + \int_{[\alpha_i^3\alpha_i^1 b]} f\mathsf{d}g^1 \wedge \mathsf{d}g^2 \right),\end{aligned}$$

the latter equality being due to the alternating property of the integral. Every one-dimensional edge $[pq]$ of the triangulation not belonging to the boundary of Ω

belongs to exactly two simplices of the triangulation leading to two terms in the right-hand side of the latter equality, $\int_{[pqb]} f\mathsf{d}g^1 \wedge \mathsf{d}g^2$ and $\int_{[qpb]} f\mathsf{d}g^1 \wedge \mathsf{d}g^2$ which cancel out due to the alternating property of the integral. Therefore, the right-hand side of the latter equality contains only terms of the type $\int_{[pqb]} f\mathsf{d}g^1 \wedge \mathsf{d}g^2$ with $[pq]$ belonging to the boundary of Ω; due to the additivity property of the integral they all sum up to the right-hand side of (6.2). The rest of the statement follows now immediately from (6.2) together with the respective properties of the integral over simplices. □

If Ω is a finite union of disjoint simple oriented polygons $\Omega_1, \ldots, \Omega_l$ then it is natural to set

$$\int_\Omega f\mathsf{d}g^1 \wedge \mathsf{d}g^2 := \sum_{i=1}^{l} \int_{\Omega_i} f\mathsf{d}g^1 \wedge \mathsf{d}g^2, \tag{6.3}$$

so that the above integral clearly exists under the conditions of Theorem 4.4.

Finally, we able to define naturally the $\int_\Omega f\mathsf{d}g^1 \wedge \mathsf{d}g^2$ for quite general bounded open sets $\Omega \subset \mathbb{R}^2$ with a chosen orientation. To this aim for every $k \in \mathbb{N}$ let P_k be the union of open squares with vertices in $2^{-k}\mathbb{Z}^2$ contained in Ω. Clearly this is a bounded open set which is a finite union of simple polygons. We assume all P_k to be oriented in the same way as Ω. The following result holds true.

Theorem 6.2. *Under conditions of Theorem 4.4, if additionally $\Omega \subset \mathbb{R}^2$ is a bounded open set satisfying*

$$\overline{\dim}_{\mathrm{box}} \partial\Omega < \beta_1 + \beta_2, \tag{6.4}$$

where $\overline{\dim}_{\mathrm{box}}$ stands for the upper box-counting dimension, there is the limit

$$\int_\Omega f\mathsf{d}g^1 \wedge \mathsf{d}g^2 := \lim_k \int_{P_k} f\mathsf{d}g^1 \wedge \mathsf{d}g^2. \tag{6.5}$$

In this case the map

$$(f, g^1, g^2) \mapsto \int_\Omega f\mathsf{d}g^1 \wedge \mathsf{d}g^2$$

is a continuous multilinear functional over $C^\alpha \times C^{\beta_1} \times C^{\beta_2}$.

Proof. Take a $d \in (\overline{\dim}_{\mathrm{box}} \partial\Omega, \beta_1 + \beta_2)$. The set $P_{k+m} \setminus P_k$ can be naturally covered by triangles by dividing along the diagonal each of the squares of sidelength $2^{-(k+m)}$ with disjoint interiors composing it. The total number of such squares is estimated from above by the number of squares with vertices in $2^{-k}\mathbb{Z}^2$ touching $\partial\Omega$, hence by $C(2^k)^d$ where $C > 0$ depends only on $\partial\Omega$. Hence the number of triangles in the chosen cover of $P_{k+m} \setminus P_k$ is estimated by $2C(2^k)^d(2^m)^2$. Each triangle Δ in this cover has diameter $D := 2^{-(k+m)}$, and therefore by (4.10) together with (3.4) one has

$$\left| \int_\Delta f\mathsf{d}g^1(x) \wedge \mathsf{d}g^2(x) \right| \le C' D^{\beta_1+\beta_2},$$

where $C' > 0$ depends only on $\|f\|_\alpha$, $[g^1]_{\beta_1}$, $[g^2]_{\beta_2}$. Thus

$$\left| \int_{P_{k+m}} f\mathsf{d}g^1(x) \wedge \mathsf{d}g^2(x) - \int_{P_k} f\mathsf{d}g^1(x) \wedge \mathsf{d}g^2(x) \right| \le 2C(2^k)^d(2^m)^2 C' 2^{-(k+m)(\beta_1+\beta_2)}$$
$$\to 0 \quad \text{as } k \to +\infty$$

(even uniformly over bounded sets of $C^\alpha \times C^{\beta_1} \times C^{\beta_2}$) because of the assumption $\beta_1+\beta_2>d$. This shows that the sequence of integrals $\{\int_{P_k} f\mathsf{d}g^1\wedge\mathsf{d}g^2\}_k$ is Cauchy, and hence the existence of the limit as claimed. This limit is clearly multilinear on (f,g^1,g^2) since so is the integral over simple polygons, and its continuity over $C^\alpha \times C^{\beta_1} \times C^{\beta_2}$ follows from that of the integral over polygons and of the fact that the above convergence is uniform over bounded sets of $C^\alpha \times C^{\beta_1} \times C^{\beta_2}$. □

Remark 6.3. Clearly under the condition (6.4) the integral $\int_\Omega f\mathsf{d}g^1\wedge\mathsf{d}g^2$ coincides with the classical one if f, g^1 and g^2 are smooth.

Remark 6.4. Combining Theorem 5.1 and Proposition 6.1, we have that the integral $\int_\Omega f\mathsf{d}g^1\wedge\mathsf{d}g^2$ in Theorem 6.2 may be also approximated directly by sums of either Stratonovich or Itô germs over sufficiently fine triangulations of P_k (for sufficiently large k).

Remark 6.5. If in the construction used in Theorem 6.2 one substitutes the dyadic grids $2^{-k}\mathbb{Z}^2$ with some other ones (e.g. rotated and/or with sidelength of the cubes converging to zero with different speed), one would obtain under conditions of Theorem 6.2 in exactly the same way the existence of the limit in (6.5) (but now with different meaning of P_k), and its continuity and multilinearity over $C^\alpha\times C^{\beta_1}\times C^{\beta_2}$. Since this limit for smooth f, g^1 and g^2 still coincides with the classical integral, we get therefore that it also coincides with $\int_\Omega f\mathsf{d}g^1\wedge\mathsf{d}g^2$ over the whole $C^\alpha\times C^{\beta_1}\times C^{\beta_2}$, and hence the role of the particular sequence of grids in the definition (6.5) is not essential.

7. Stratonovich type integrals of more irregular forms

We consider in this section the integrals of the type

$$\int_\Omega F(x,g(x))\mathsf{d}g^1(x)\wedge\mathsf{d}g^2(x)$$

defined for Hölder functions $g:=(g^1,g^2)\colon\mathbb{R}^2\to\mathbb{R}^2$ when $F\colon\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}$. In fact, it happens that if one uses a Stratonovich-type construction, i.e. employs alternating germs strat_{pqr} defined for $f(x):=F(x,g(x))$, then the above integral may be defined under much less restrictive requirements than those given by Theorem 4.4. In particular, we are able to trade regularity of g for the higher regularity of F. Here we only limt ourselves to the case when the domain of integration $\Omega\subset\mathbb{R}^2$ is an oriented simplex (i.e. triangle $[pqr]$), since the case of more general domains can be easily treated as in section 6.

Theorem 7.1. *Let $F\colon\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}$ such that*

(i) $u\mapsto F(u,\cdot)\in C(\mathbb{R}^2;C^{1,\gamma}(\mathbb{R}^2))$, $\gamma\in(0,1]$,
(ii) $u\mapsto F(\cdot,u)\in C(\mathbb{R}^2;C^\alpha)$,

and let $f(x):=F(x,g(x))$, where $g(x):=(g^1(x),g^2(x))$. If $\beta_1+\beta_2>1$ and

$$\begin{gathered}\alpha+\beta_1+\beta_2>2,\\ (1+\gamma)\beta_i+\beta_1+\beta_2>2,\quad i=1,2,\end{gathered}\tag{7.1}$$

then, with the notation of (4.1) the limit

$$V_{pqr}(g):=\lim_{n\to\infty}\mathsf{strat}^n_{pqr}$$

exists. Moreover, it is continuous and alternating as a function of $[pqr]$ *fixed* g^1 *and* g^2*, nonatomic in the sense that*

$$V_{pqr}(g) = 0 \quad \textit{when } |[pqr]| = 0,$$

and continuous as the functional of g*, so that it is reasonable to denote*

$$\int_{[pqr]} F(x, g(x))\mathsf{d}g^1(x) \wedge \mathsf{d}g^2(x) := V_{pqr}(g).$$

Remark 7.2. It is worth observing that (7.1) implies $\beta_i > 1/3$, $i = 1, 2$. In fact, assuming without loss of generality $\beta_1 < \beta_2$, we get from (7.1) $(2+\gamma)\beta_1 + \beta_2 > 2$, and hence

$$\beta_1 > \frac{2 - \beta_2}{2 + \gamma} \geq \frac{1}{3}.$$

On the other hand, $\beta_i > 1/2$, $i = 1, 2$, is clearly sufficient for the second inequality in (7.1) to hold. Note also that if $\beta_1 = \beta_2 = \beta$, and $F(x, y) := F(y)$ for every $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2$, then the first inequality of (7.1) is automatically satisfied since we may take α to be arbitrarily close to 1, and therefore (7.1) is equivalent to $\beta > 2/(3+\gamma)$ (e.g. $\beta > 1/2$ when $F \in C^{1,1}$), which is far less restrictive than what is asserted in Theorem 4.4 (the latter requires in this case $\beta > 2/3$, since $f \in C^\beta$).

Remark 7.3. It follows from the proof that the limit germ

$$V_{pqr} := \int_{[pqr]} F(x, g(x))\mathsf{d}g^1(x) \wedge \mathsf{d}g^2(x))$$

is continuous also with respect to F (with respect to a topology compatible with (i) and (ii)).

Remark 7.4. We notice that an analogous result is easy to obtain in the one-dimensional case. Namely, roughly speaking, if $g \in C^\beta(\mathbb{R})$ is Hölder continuous and $F\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is $C^\alpha(\mathbb{R})$ in the first variable and $C^{1,\gamma}(\mathbb{R})$ in the second one, then the Stratonovich-type sums

$$\sum_i \frac{1}{2}\left(F(x_i, g(x_i)) + F(x_{i+1}, g(x_{i+1}))\right)(\delta g)_{x_i x_{i+1}}$$

over a sequence of partitions $(x_i)_i$ of $[a, b]$ converge as $\sup_i |x_{i+1} - x_i| \to 0$ when

$$\alpha + \beta > 1 \text{ and } \beta(2+\gamma) > 1. \tag{7.2}$$

This can be deduced at once starting from the calculation

$$\delta\theta_{pqr} = \frac{1}{2}\det\begin{pmatrix} \delta f_{pq} & \delta f_{pr} \\ \delta g_{pq} & \delta g_{pr} \end{pmatrix}$$

with $f_p := F(p, g_p)$ and $\theta_{pq} := \frac{1}{2}(f_p + f_q)\,\delta g_{pq}$. The assumptions on f give the Taylor expansion

$$\delta f_{pq} = a_p \delta g_{pq} + O(|q-p|^\alpha + |q-p|^{\beta(1+\gamma)})$$

so that a cancelation occurs in the determinant providing $|\delta\theta_{pqr}| = O(|q-p|^{\alpha+\beta} + |q-p|^{\beta(2+\gamma)})$, which gives the possibility to apply the one-dimensional sewing lemma [4, lemma 2.1] if (7.2) holds. In particular, we notice that if $\alpha = \gamma = 1$, then $\beta > 1/3$ is allowed, which is well below the threshold of Hölder exponents for the existence the Young integral (defined for $\beta > 1/2$). It is worth emphasizing that this is the peculiar feature of the Stratonovich integral, not of the Itô one. In fact, if we take just $F(x,y) := y$, then the integral reduces to $\int_{[pq]} g \mathsf{d} g$, and for $g \in C^\beta(\mathbb{R})$ with $\beta \in (1/3, 1/2]$ it is a limit of the sum of Stratonovich germs but in general not of Itô germs. This is the case for instance when g has infinite total quadratic variation, because the difference between the two germs over $[pq]$ is $(\delta g)^2_{pq}/2$, so that if the integral existed as the limit of sums of either of the germs, then the total quadratic variation of g had to be finite.

Proof. Let $f_u(t) := F(u, x + t(y-x))$ for $\{u, x, y\} \in \mathbb{R}^2$. Writing

$$\begin{aligned} F(u,y) &= f_u(1) \\ &= f_u(0) + \int_0^1 (f_u)'(s)\, ds = f_u(0) + (f_u)'(0) + \int_0^1 ((f_u)'(s) - (f_u)'(0))\, ds \\ &= F(u,x) + \nabla F(u,\cdot)(x) \cdot (y-x) \\ &\quad + \int_0^1 (\nabla F(u,\cdot)(x + s(y-x)) - \nabla F(u,\cdot)(x)) \cdot (y-x)\, ds, \end{aligned}$$

we get with $x := g_u$, $y := g_v$ the relationship

$$\tag{7.3} \begin{aligned} (\delta F)_{uv} &= (\delta F(\cdot, g_v))_{uv} + (\delta F(u,\cdot))_{g_u g_v} \\ &= (\delta F(\cdot, g_v)_{uv} + \delta g^1_{uv} \partial_1 F(u,\cdot)(g^1_u, g^2_u) + \delta g^2_{uv} \partial_2 F(u,\cdot)(g^1_u, g^2_u) + R_{uv}, \end{aligned}$$

where

$$\begin{aligned} R_{uv} := \delta g^1_{uv} \int_0^1 &\left(\partial_1 F(u,\cdot)(g^1_u + s\delta g^1_{uv}, g^2_u + s\delta g^2_{uv}) - \partial_1 F(u,\cdot)(g^1_u, g^2_u)\right)\, ds \\ + \delta g^2_{uv} \int_0^1 &\left(\partial_2 F(u,\cdot)(g^1_u + s\delta g^1_{uv}, g^2_u + s\delta g^2_{uv}) - \partial_2 F(u,\cdot)(g^1_u, g^2_u)\right)\, ds, \end{aligned}$$

so that

$$\begin{aligned} |(\delta F(\cdot, g_v)_{uv}| &\le C|v-u|^\alpha, \\ |R_{uv}| &\le C\left(|\delta g^1_{uv}| + |\delta g^2_{uv}|\right)\left((\delta g^1_{uv})^2 + (\delta g^2_{uv})^2\right)^{\gamma/2} \end{aligned}$$

for (u,v) in a bounded set (the constant $C > 0$ depending on this set). From Lemma 3.4 one gets therefore

$$\tag{7.4} \begin{aligned} (\delta\mathsf{strat})_{pqrs} = &\frac{1}{6} \det \begin{pmatrix} (\delta F(\cdot, g_q))_{pq} & (\delta F(\cdot, g_r))_{pr} & (\delta F(\cdot, g_s))_{ps} \\ \delta g^1_{pq} & \delta g^1_{pr} & \delta g^1_{ps} \\ \delta g^2_{pq} & \delta g^2_{pr} & \delta g^2_{ps} \end{pmatrix} + \\ &\frac{1}{6} \det \begin{pmatrix} R_{pq} & R_{pr} & R_{ps} \\ \delta g^1_{pq} & \delta g^1_{pr} & \delta g^1_{ps} \\ \delta g^2_{pq} & \delta g^2_{pr} & \delta g^2_{ps} \end{pmatrix}, \end{aligned}$$

and hence

$$\tag{7.5} \begin{aligned} |(\delta\mathsf{strat})_{pqrs}| &\le C\left(\operatorname{diam}([pqrs])^{\alpha+\beta_1+\beta_2} + \operatorname{diam}([pqrs])^{(1+\gamma)(\beta_1\wedge\beta_2)+\beta_1+\beta_2}\right) \\ &\le C \operatorname{diam}([pqrs])^d \end{aligned}$$

with $d := (\alpha \wedge (1+\gamma)(\beta_1 \wedge \beta_2)) + \beta_1 + \beta_2$ and $C > 0$ depending continuously on F (with respect to the topology compatible with (i) and (ii)) and on $[\delta g^i]_{\beta_i}$, $i = 1, 2$.

Recalling (3.4) from Lemma 3.2, and that strat is alternating by the same Lemma, while $d > 2$ because of (7.1), we have that Lemma A.1 applies with

$$\gamma_1 := \beta_1 + \beta_2 > 1, \quad C_1 := \|f\|_\infty [\delta g^1]_{\beta_1} [\delta g^2]_{\beta_2},$$
$$\gamma_2 := d > 2, \quad C_2 := C,$$

yielding the existence of continuous alternating germs

$$S_{pq} := \lim_{n\to\infty} S^n_{pq},$$
$$V_{pqr} := \lim_{n\to\infty} \mathsf{strat}^n_{pqr} = \lim_{n\to\infty} (\mathsf{strat}^n_{pqr} - \delta S^n_{pqr}) + \delta S^n_{pqr}.$$

It remains now to prove that fixed $[pqr]$, the map

$$g \in C^{\beta_1} \times C^{\beta_2} \mapsto V_{pqr}(g)$$

is continuous. To this aim let $\{g_k\} \subset C^{\beta_1} \times C^{\beta_2}$, converging to g pointwise as $k \to \infty$, and $[\delta g^1_k]_{\beta_1} + [\delta g^2_k]_{\beta_2} < C < +\infty$ for all $k \in \mathbb{N}$. Let f_k, R_k, S^n_k, S_k, strat^n_k, strat_k, V_k be the same as f, R, S^n, S, strat^n, strat, V respectively but with g^1_k, g^2_k instead of g^1, g^2. Clearly, as in (7.5) we have

$$|(\delta \mathsf{strat}_k)_{pqrs}| \leq C \operatorname{diam}([pqrs])^d. \tag{7.6}$$

The claim follows now by Lemma A.3 with $\gamma_2 := d$, $\gamma_1 = \beta_1 + \beta_2$ (in fact, (A.2) is given by (7.6), and (A.1) is just (3.4) from Lemma 3.2). □

Remark 7.5. One could strengthen the above Theorem 7.1 by proving the existence and continuity with respect to the data of a more general Stratonovich type integral

$$\int_{[pqr]} F(x, h(x)) \mathsf{d}g^1(x) \wedge \mathsf{d}g^2(x),$$

where F is as in Theorem 7.1, $\psi \in C^{2,\gamma}(\mathbb{R}^2; \mathbb{R}^2)$, $\gamma \in (0,1]$, $h^i \in C^{\beta_i}(\mathbb{R}^2)$, $g^i := \psi^i \circ h$, $i = 1,2$ with $h := (h^1, h^2)$, $\psi := (\psi^1, \psi^2)$ and $\beta_i > 1/2$, $i = 1,2$ and satisfy the first inequality of (7.1). In fact, letting $f(x) := F(x, h(x))$, and using the notation of (4.1) we would have the existence of the limit

$$\lim_{n\to\infty} \mathsf{strat}^n_{pqr} =: \int_{[pqr]} F(x, h(x)) \mathsf{d}g^1(x) \wedge \mathsf{d}g^2(x).$$

To show this, we adapt the arguments of the proof of the above Theorem 7.1, changing (7.4) with

$$\begin{aligned}
(\delta \mathsf{strat})_{pqrs} = & \frac{1}{6} \det \begin{pmatrix} (\delta F(\cdot, g_q))_{pq} & (\delta F(\cdot, g_r))_{pr} & (\delta F(\cdot, g_s))_{ps} \\ \delta g^1_{pq} & \delta g^1_{pr} & \delta g^1_{ps} \\ \delta g^2_{pq} & \delta g^2_{pr} & \delta g^2_{ps} \end{pmatrix} + \\
& \frac{1}{6} \det \begin{pmatrix} R_{pq} & R_{pr} & R_{ps} \\ \delta g^1_{pq} & \delta g^1_{pr} & \delta g^1_{ps} \\ \delta g^2_{pq} & \delta g^2_{pr} & \delta g^2_{ps} \end{pmatrix} + \\
& \frac{1}{6} \det \begin{pmatrix} \nabla_h F(p, h_p) \cdot \delta h_{pq} & \nabla_h F(p, h_p) \cdot \delta h_{pr} & \nabla_h F(p, h_p) \cdot \delta h_{ps} \\ r^1_{pq} & r^1_{pr} & r^1_{ps} \\ \nabla \psi^2_{h_p} \cdot \delta h_{pq} & \nabla \psi^2_{h_p} \cdot \delta h_{pr} & \nabla \psi^2_{h_p} \cdot \delta h_{ps} \end{pmatrix} + \\
& \frac{1}{6} \det \begin{pmatrix} \nabla_h F(p, h_p) \cdot \delta h_{pq} & \nabla_h F(p, h_p) \cdot \delta h_{pr} & \nabla_h F(p, h_p) \cdot \delta h_{ps} \\ \nabla \psi^1_{h_p} \cdot \delta h_{pq} & \nabla \psi^1_{h_p} \cdot \delta h_{pr} & \nabla \psi^1_{h_p} \cdot \delta h_{ps} \\ r^2_{pq} & r^2_{pr} & r^2_{ps} \end{pmatrix},
\end{aligned} \tag{7.7}$$

where

$$r^i_{uv} := \delta g^i_{uv} - (\nabla\psi_i)_{h_u} \cdot \delta h_{uv}, \quad i = 1, 2.$$

Then the first two terms in (7.7) are estimated by $C\,\mathrm{diam}([pqrs])^{d_1}$ with $d_1 > 2$ as in (7.5) because of (7.1) (the second inequality of which is automatically satisfied in view of Remark 7.2 due to the requirement $\beta_i > 1/2$, $i = 1, 2$), while the other two are estimated by $C\,\mathrm{diam}([pqrs])^{d_2}$ with $d_2 := 4(\beta_1 \wedge \beta_2) > 2$, because

$$|r^i_{uv}| \le C|u - v|^{2(\beta_1\wedge\beta_2)},$$

and thus $|\delta\mathsf{strat}_{pqrs}| \le C\,\mathrm{diam}([pqrs])^d$, the constants in all the above estimates depending continuously on the data. This allows to proceed as in the proof of Theorem 7.1 showing the existence and continuity with respect to the data of the above integral.

Proposition 7.6 (chain rule)**.** *Let F be as in Theorem 7.1, $\psi \in C^{2,\gamma}(\mathbb{R}^2; \mathbb{R}^2)$, $\gamma \in (0, 1]$, $h^i \in C^{\beta_i}(\mathbb{R}^2)$, and $g^i := \psi^i \circ h$, $i = 1, 2$, where $h := (h^1, h^2)$, $\psi := (\psi^1, \psi^2)$. If $\beta_i > 1/2$, $i = 1, 2$ and the first inequality of (7.1) holds, then*

$$\begin{aligned}\int_{[pqr]} & F(x, h(x))\mathsf{d}g^1(x) \wedge \mathsf{d}g^2(x)\\ &= \int_{[pqr]} F(x, h(x)) \det D\psi(h^1(x), h^2(x))\mathsf{d}h^1(x) \wedge \mathsf{d}h^2(x)\end{aligned} \tag{7.8}$$

Note that the integral on the right-hand side of (7.8) exists, is continuous and alternating as a function of $[pqr]$ fixed h^1 and h^2, and continuous as the functional of h^1, h^2 by Theorem 7.1.

Proof. The equality (7.8) is true when g^i are smooth. The general case follows from continuity of the integrals on the left and righthand sides of (7.8) with respect to the pointwise convergence of g^i, $i = 1, 2$ with uniformly bounded Hölder constants. □

We may give an interpretation of the above results in the spirit of theorem 3.2 from [1]. Namely, a smooth (say, C^1) function $g = (g_1, g_2)\colon [pqr] \subset \mathbb{R}^2 \to \mathbb{R}^2$ can be naturally identified with the smooth surface representing its graph, and therefore, with the De Rham 2-current T_g over $[pqr] \times \mathbb{R}^2$ (endowed with orthogonal coordinates $(x, y) := (x^1, x^2, y^1, y^2)$) defined by

$$T_g(F\mathsf{d}x^1 \wedge \mathsf{d}x^2) := \int_{[pqr]} F(x, g(x))\,\mathsf{d}x^1 \wedge \mathsf{d}x^2, \tag{7.9}$$

$$T_g(F\mathsf{d}x^i \wedge \mathsf{d}y^j) := \int_{[pqr]} F(x, g(x))\,\mathsf{d}x^i \wedge \mathsf{d}g^j(x), \tag{7.10}$$

$$T_g(F\mathsf{d}y^1 \wedge \mathsf{d}y^2) := \int_{[pqr]} F(x, g(x))\,\mathsf{d}g^1(x) \wedge \mathsf{d}g^2(x), \tag{7.11}$$

for every $f \in C^2([pqr] \times \mathbb{R}^2)$.

Proposition 7.7. *If $g^i \in C^{\beta_i}$, $i = 1, 2$, with*

$$3\beta_1 + \beta_2 > 2, \quad 3\beta_2 + \beta_1 > 2, \tag{7.12}$$

then the map $g \mapsto T_g$ between $C^1([pqr]; \mathbb{R}^2)$ and the space $D_2([pqr] \times \mathbb{R}^2)$ of 2-currents in $[pqr] \times \mathbb{R}^2$ endowed with its weak (pointwise) topology admits the unique continuous extension to the space $C^{\beta_1} \times C^{\beta_2}$ (the continuity being intended, as usual, with respect to pointwise convergence with uniformly bounded Hölder constants).

Proof. If $g^i \in C^{\beta_i}$, $i = 1, 2$, then the formulae (7.9), (7.10) and (7.11) still make sense for an $F \in C^2([pqr] \times \mathbb{R}^2)$ if one interprets the integrals involved in the sense of Stratonovich. Namely, one defines the integral

(A) in (7.9), say, in the usual Riemann (or Lebesgue) sense (which in this case is equivalent to the Stratonovich integral),
(B) in (7.11) in the sense of Theorem 7.1 (with $\alpha := 1$, $\gamma := 1$), and
(C) in (7.10) again in the sense of Theorem 7.1 but with x^i in place of g^1, g^j in place of g^2, and $\bar{F}$ in place of F, where $\bar{F}$ is defined by

$$\bar{F}(x^1, x^2, y^1, y^2) := \begin{cases} F(x^1, x^2, g^1(x), y^2), & i = 1, j = 2, \\ F(x^1, x^2, y^1, g^2(x)), & i = 2, j = 1, \end{cases}$$

and with $\gamma := 1$, $\alpha := \beta_1$ and 1 in place of β_1 for the case $i = 1$, $j = 2$ or $\alpha := \beta_2$ and 1 in place of β_2 for the case $i = 2$, $j = 1$.

Note that (7.12) makes Theorem 7.1 to be applicable with such data.

Continuity of the map $g \mapsto T_g$ between $C^{\beta_1} \times C^{\beta_2}$ and the space of currents endowed with its weak (pointwise) topology is given by Theorem 7.1. The fact that it is the unique continuous extension of its restriction to $C^1 \times C^1$ follows from the density of C^1 in any Hölder space (with respect to the uniform convergence with bounded Hölder constants). □

Remark 7.8. The proof of Proposition 7.7 shows that the formulae (7.9), (7.10) and (7.11) still make sense for the current T_g with $g \in C^{\beta_1} \times C^{\beta_2}$ when $F \in C^2([pqr] \times \mathbb{R}^2)$ (in fact, even for $F \in C^{1,1}$), if one interprets the integrals appearing there in the sense of Stratonovich, i.e. as in Theorem 7.1 (in particular, in (7.9) it may be interpreted as the usual Riemann or Lebesgue integral).

Remark 7.9. Theorem 3.2 from [1] says that the map $g \mapsto T_g$ defined by the formulae (7.9), (7.10) and (7.11) between $C^1([pqr]; \mathbb{R}^2)$ and the space of currents endowed with its weak topology admits a unique continuous extension to the Sobolev space $W^{1,1}_{loc}([pqr]; \mathbb{R}^2)$ (even sequentially weakly continuous one). It is worth noting that the extended current may be then defined for continuous differential forms (i.e. with F just continuous), while here we have to require that the forms be smoother (in fact, requesting F to be C^2, we are guaranteed only that the extended current T_g be defined over twice continuously differential forms). One may weaken the regularity requirement for forms (e.g. requesting that F might be less regular than C^2), but this will inevitably strengthen the requirement of (7.12) on the regularity of T_g.

Remark 7.10. In order to identify the extension with the "second order Riemann-Stieltjes" integral introduced in [24], we extend by continuity the identity

$$\int_{\mathbb{R}^2} f(x) \deg\left((h^1, h^2), [pqr], x\right) \mathsf{d}x = \int_{[pqr]} f(h^1, h^2) \mathsf{d}h^1 \wedge \mathsf{d}h^2 \tag{7.13}$$

for every $f \in C^{1,\gamma}$ from smooth functions (h^1, h^2) to $h_1 \in C^{\beta_1}$, $h_2 \in R^{\beta_2}$. In combination with [24, theorem 4.3] this identifies the two integrals. Formula (7.13) follows by continuity and approximation.

We also notice that continuity of the right hand side in (7.13) gives the following quantitative continuity of degree of Hölder maps:

$$\int_{\mathbb{R}^2} f(x) \left(\deg\left((h^1, h^2), [pqr], x\right) - \deg\left((k^1, k^2), [pqr], x\right)\right) \mathsf{d}x \le \|f\|_{1,\gamma} \|h - k\|_{\beta}$$

Appendix A. Existence, uniqueness and stability of integrals

In this section we assume that ω be an abstract 2-germ in $D \subset \mathbb{R}^2$ (i.e. not necessarily the one defined by (1.3) satisfying

$$|\omega_{pqr}| \le C_1 \operatorname{diam}([pqr])^{\gamma_1}, \tag{A.1}$$

$$|(\delta\omega)_{pqrs}| \le C_2 \operatorname{diam}([pqrs])^{\gamma_2}, \tag{A.2}$$

with positive constants γ_1, γ_2, C_1, C_2 independent on $[pqr]$ and $[pqrs]$. We define then ω^n and S^n by

$$\omega^n_{pqr} := \langle \mathrm{dya}^n[pqr], \omega\rangle\,, \quad S^n_{pq} := \sum_{i=0}^{n-1} \left\langle \mathrm{fill}\,\mathrm{cut}^i[pq], \omega\right\rangle. \tag{A.3}$$

We prove here the existence of limits $\lim_n \omega^n$ and $\lim_n S^n$ and their basic stability properties. Note that we do not prove here that the respective germs are nonatomic and additive (although in fact this could be proven), as it is usually done in the sewing lemma.

Lemma A.1. *Under the conditions (A.1) and (A.2) if ω is alternating, then*

$$|S^{n+1}_{pq} - S^n_{pq}| \le C \operatorname{diam}([pq])^{\gamma_1} 2^{n(1-\gamma_1)}, \tag{A.4}$$

$$|\langle [pqr], (\omega^n - \delta S^n) - (\omega^{n+1} - \delta S^{n+1})\rangle| \le C \operatorname{diam}([pqr])^{\gamma_2} 2^{n(2-\gamma_2)} \tag{A.5}$$

with $C > 0$. In particular, if $\gamma_1 > 1$ and $\gamma_2 > 2$, then the germs

$$S_{pq} := \lim_{n\to\infty} S^n_{pq},$$
$$V_{pqr} := \lim_{n\to\infty} \omega^n_{pqr} = \lim_{n\to\infty} (\omega^n_{pqr} - \delta S^n_{pqr}) + \delta S^n_{pqr}$$

are well defined, continuous (if so is ω), alternating and

$$|S^n_{pq} - S_{pq}| \le C \operatorname{diam}([pq])^{\gamma_1} 2^{n(1-\gamma_1)}, \tag{A.6}$$

$$|\omega^n_{pqr} - V_{pqr} - \delta(S^n - S)_{pqr}| \le C \operatorname{diam}([pqr])^{\gamma_2} 2^{n(2-\gamma_2)}, \tag{A.7}$$

$$|\omega^n_{pqr} - V_{pqr}| \le C \operatorname{diam}([pqr])^{\gamma_1\wedge\gamma_2} 2^{n(1-\gamma_1\wedge\gamma_2)}. \tag{A.8}$$

Proof. For the readers' convenience we organize the proof in several steps.

Step 1. To prove (A.5), observe that for some geometric map $\rho\colon \mathrm{Simp}^2(D) \to \mathrm{Chain}^3(D)$ one has

$$\begin{aligned}
\omega^1_{p_0p_1p_2} - \omega^0_{p_0p_1p_2} &= \langle \mathrm{dya}[p_0p_1p_2], \omega\rangle - \langle [p_0p_1p_2], \omega\rangle \\
&= \langle \partial\rho([p_0p_1p_2]), \omega\rangle - \langle \mathrm{fill}[p_0p_1], \omega\rangle + \langle \mathrm{fill}[p_1p_2], \omega\rangle + \langle \mathrm{fill}[p_2p_0], \omega\rangle \\
&= \langle \rho([p_0p_1p_2]), \delta\omega\rangle + \langle \mathrm{fill}\,\partial[p_0p_1p_2], \omega\rangle\,.
\end{aligned} \tag{A.9}$$

Moreover,

$$\rho([p_0p_1p_2]) = \sum_{i=1}^4 Q_i, \quad Q_i \in \mathrm{Simp}^3(D), \operatorname{diam} Q_i \le \operatorname{diam}([p_0p_1p_2]), i = 0, \ldots, 2,$$

and therefore by (A.2) we have

$$|\,\langle \rho([p_0p_1p_2]), \delta\omega\rangle\,| \le C \operatorname{diam}([p_0p_1p_2])^{\gamma_2}, \tag{A.10}$$

with $C := 4C_1$. Writing then $\mathrm{dya}[pqr] = \sum_{i=1}^{2^{2n}} \Delta_i$ with $\Delta_i \in \mathrm{Simp}^2(D)$ being dyadic simplices equal up to translations to to $2^{-n}_{\#}[pqr]$, we get from (A.9)

$$\left\langle \Delta_i, \omega^1\right\rangle - \left\langle \Delta_i, \omega^0\right\rangle = \langle \rho(\Delta_i), \delta\omega\rangle + \langle \mathrm{fill}\,\partial\Delta_i, \omega\rangle\,,$$

and summing the latter expressions over $i = 1, \ldots, 2^{2n}$, we arrive at

$$
\begin{aligned}
\omega^{n+1}_{pqr} - \omega^n_{pqr} &= \sum_{i=1}^{2^{2n}} \left\langle \Delta_i, \omega^1 - \omega^0 \right\rangle \\
&= \sum_{i=1}^{2^{2n}} \langle \rho(\Delta_i), \delta\omega \rangle + \langle \operatorname{fill} \operatorname{cut}^n \partial[pqr], \omega \rangle \,,
\end{aligned}
\tag{A.11}
$$

since if Δ_i and Δ_j have a common couple of vertices, say, p_0 and p_1, then by alternating property of ω one has

$$\langle \operatorname{fill}[p_0p_1], \omega \rangle = - \langle \operatorname{fill}[p_1p_0], \omega \rangle \,,$$

i.e. the respective terms cancel out from the above sum, while the terms coming from the sides of dyadic simplices belonging to the boundary of $[pqr]$ remain, their sum giving rise to $\langle \operatorname{fill} \operatorname{cut}^n \partial[pqr], \omega \rangle$. Observing that

$$\langle \operatorname{fill} \operatorname{cut}^n \partial[pqr], \omega \rangle = \left\langle [pqr], \delta S^{n+1} - \delta S^n \right\rangle$$

and rewriting (A.11) with this help, we arrive at

$$
(\omega^{n+1}_{pqr} - (\delta S^{n+1})_{pqr}) - (\omega^n_{pqr} - (\delta S^n)_{pqr}) = \sum_{i=1}^{2^{2n}} \langle \rho(\Delta_i), \delta\omega \rangle \,.
\tag{A.12}
$$

Therefore,

$$
\begin{aligned}
|(\omega^{n+1}_{pqr} - (\delta S^{n+1})_{pqr}) - (\omega^n_{pqr} - (\delta S^n)_{pqr})| &\le \sum_{i=1}^{2^{2n}} | \langle \rho(\Delta_i), \delta\omega \rangle | \\
&\le C \sum_{i=1}^{2^{2n}} \operatorname{diam}(\Delta_i)^{\gamma_2} \quad \text{by (A.10)} \\
&\le C 2^{2n} \left(\frac{\operatorname{diam}([pqr])}{2^n} \right)^{\gamma_2}
\end{aligned}
$$

as claimed.

Step 2. The estimate (A.4) follows with $C := C_1$ just observing that

$$S^{n+1}_{pq} - S^n_{pq} = \langle \operatorname{fill} \operatorname{cut}^n [pq], \omega \rangle \,,$$

while in view of (A.1) and of the definition of $\operatorname{fill} \operatorname{cut}^n$ one has

$$| \langle \operatorname{fill} \operatorname{cut}^n [pq], \omega \rangle | \le C_1 2^n \left(\frac{\operatorname{diam}([pq]}{2^n} \right)^{\gamma_1} .$$

Step 3. Existence of S and V follow now from (A.4) and (A.5) respectively. Since ω is alternating, then so are ω^n and S^n, and therefore also V and S. Now, the continuity of ω implies that of S^n and ω^n for each fixed $n \in \mathbb{N}$, and hence the continuity of S and V follow from (A.6) and (A.8) respectively once they are proven. E.g. to prove continuity of S, for $[pq] \subset D$ and $[rs] \subset D$ with D bounded, given an $\varepsilon > 0$, we choose an $n \in \mathbb{N}$ such that $C \operatorname{diam} D 2^{n(1-\gamma_1)} < \varepsilon/3$, so that

$$
\begin{aligned}
|S_{pq} - S_{rs}| &\le |S_{pq} - S^n_{pq}| + |S^n_{pq} - S^n_{rs}| + |S^n_{rs} - S_{rs}| \\
&\le 2\varepsilon/3 + |S^n_{pq} - S^n_{rs}| \quad \text{by (A.6) and the choice of } \varepsilon,
\end{aligned}
$$

so that it is enough to find a $\delta = \delta(n, \varepsilon) > 0$ such that $|S^n_{pq} - S^n_{rs}| < \varepsilon/3$ once $|p-q|+|r-s| < \delta$ to get $|S_{pq} - S_{rs}| < \varepsilon$. The proof of continuity of V is completely analogous (with the use of (A.8) instead of (A.6)).

Step 3. Finally, we prove (A.6), (A.7) and (A.8). The inequality (4.5) is proven by the chain of estimates

$$\begin{aligned}|S^n_{pq} - S_{pq}| &= \left|\sum_{k=n+1}^{\infty} (S^k_{pq} - S^{k-1}_{pq})\right| \leq C \operatorname{diam}([pq])^{\gamma_1} \sum_{k=n+1}^{\infty} 2^{k(1-\gamma_1)} \quad \text{by (A.4)}\\ &\leq C \frac{2^{n(1-\gamma_1)}}{1-2^{1-\gamma_1}} \operatorname{diam}([pq])^{\gamma_1}.\end{aligned}$$

Analogously, (A.7) follows from

$$\begin{aligned}|\omega^n_{pqr} - V_{pqr} - \delta(S^n - S)_{pqr}| &= |(\omega^n_{pqr} - \delta S^n_{pqr}) - (V_{pqr} - \delta S_{pqr})|\\ &= \left|\sum_{k=n+1}^{\infty} \left((\omega^k_{pqr} - \delta S^k_{pqr}) - (\omega^{k-1}_{pqr} - \delta S^{k-1}_{pqr})\right)\right|\\ &\leq C \operatorname{diam}([pqr])^{\gamma_2} \sum_{k=n+1}^{\infty} 2^{k(2-\gamma_2)} \quad \text{by (A.5)}\\ &\leq C \frac{2^{n(2-\gamma_2)}}{1-2^{2-\gamma_2}} \operatorname{diam}([pqr])^{\gamma_2}.\end{aligned}$$

Finally, (A.6) gives

$$|\delta(S^n - S)_{pqr}| \leq C \operatorname{diam}([pqr])^{\gamma_1} 2^{n(1-\gamma_1)},$$

which together with (A.7) implies (A.8) for $\operatorname{diam}([pqr]) < 1$ (which is enough since D is assumed bounded), thus concluding the proof. ☐

As a result of Lemma A.1 we have that V and S satisfy

$$\begin{aligned}|S_{pq}| &\leq C \operatorname{diam}([pq])^{\gamma_1},\\ |\omega_{pqr} - (V - \delta S)_{pqr}| &\leq C \operatorname{diam}([pqr])^{\gamma_2}.\end{aligned}$$

In particular, if $\gamma_1 > 1$ and $\gamma_2 > 2$ this implies

$$|S_{pq}| \leq o(\operatorname{diam}([pq])) \quad \text{as } \operatorname{diam}([pq]) \to 0, \tag{A.13}$$

$$|\omega_{pqr} - (V - \delta S)_{pqr}| \leq o(\operatorname{diam}([pqr])^2) \quad \text{as } \operatorname{diam}([pqr]) \to 0. \tag{A.14}$$

Moreover, since

$$S_{pq} = \sum_{i=0}^{\infty} \left\langle \operatorname{fill} \operatorname{cut}^i [pq], \omega \right\rangle,$$

then one has

$$(\delta S)_{prq} = \omega_{prq} \quad \text{when } r = \tfrac{p+q}{2}. \tag{A.15}$$

Finally,

$$\langle \operatorname{dya}[pqr], V \rangle = \langle [pqr], V \rangle. \tag{A.16}$$

The following curious result, though not used elsewhere in this paper, gives the uniqueness of such a couple (S, V) for a given ω.

Lemma A.2. *Given an $\omega \in \operatorname{Germ}^2(D)$, the couple of germs $(S, V) \in \operatorname{Germ}^1(D) \times \operatorname{Germ}^2(D)$ satisfying (A.13), (A.14), (A.15) and (A.16) is unique.*

Proof. Suppose that there are two couples $(S_i, V_i) \in \mathrm{Germ}^1(D)\times\mathrm{Germ}^2(D)$, $i = 1, 2$ satisfying (A.13), (A.14), (A.15) and (A.16). Then for $S := S_1 - S_2$ and $V := V_1 - V_2$ we get

$$|S_{pq}| \le o(\mathrm{diam}([pq]) \quad \text{as } \mathrm{diam}([pq]) \to 0, \tag{A.17}$$

$$|(V - \delta S)_{pqr}| = o\left(\mathrm{diam}([pqr])^2\right), \quad \text{as } \mathrm{diam}([pqr]) \to 0, \text{ and} \tag{A.18}$$

$$(\delta S)_{prq} = 0. \quad \text{when } r = \tfrac{p+q}{2}. \tag{A.19}$$

For each $n \in \mathbb{N}$ dividing dyadically the line segment $[pq]$ by consecutive points

$$r_j := \left(1 - \frac{j}{2^n}\right) p + \frac{j}{2^n} q, \quad j = 0, \dots, 2^n,$$

we get

$$S_{pq} = \sum_{j=0}^{2^n} S_{r_j r_{j+1}}$$

by (A.19), and hence,

$$|S_{pq}| \le \sum_{j=0}^{2^n} |S_{r_j r_{j+1}}| \le 2^n o\left(\frac{|pq|}{2^n}\right) = |pq| o\,(1)$$

as $n \to 0$, by (A.17), and taking the limit in the above inequality as $n \to \infty$, we get $S_{pq} = 0$. Then (A.18) is reduced to

$$|V_{pqr}| = o\left(\mathrm{diam}([pqr])^2\right) \quad \text{as } \mathrm{diam}([pqr]) \to 0. \tag{A.20}$$

Recalling that $\langle (\mathrm{dya})^n[pqr], V\rangle = \langle [pqr], V\rangle$ for every $n \in \mathbb{N}$ (because both V_1 and V_2 are assumed to satisfy (A.16)), we get using (A.20) the estimate

$$\begin{aligned} |V_{pqr}| = |\,\langle \mathrm{dya}^n[pqr], V\rangle\,| &= 2^{2n} o\left(\frac{\mathrm{diam}([pqr])^2}{2^{2n}}\right) \\ &= \mathrm{diam}([pqr])^2 o(1) \to 0 \end{aligned}$$

as $n \to \infty$. This implies $V = 0$ concluding the proof. $\square$

Consider now a sequence of continuous alternating germs $\{\omega_k\} \subset \mathrm{Germ}^2(D)$ satisfying

$$|(\omega_k)_{pqr}| \le C_1 \,\mathrm{diam}([pqr])^{\gamma_1}, \tag{A.21}$$

$$|(\delta\omega_k)_{pqrs}| \le C_2 \,\mathrm{diam}([pqr])^{\gamma_2}, \tag{A.22}$$

with positive constants $\gamma_1 > 1$, $\gamma_2 > 2$, C_1, C_2 independent on $[pqr]$, $[pqrs]$ and k.

$$(\omega_k^n)_{pqr} := \langle \mathrm{dya}^n[pqr], \omega_k\rangle\,, \quad (S_k^n)_{pq} := \sum_{i=0}^{n-1} \left\langle \mathrm{fill}\,\mathrm{cut}^i[pq], \omega_k\right\rangle. \tag{A.23}$$

Lemma A.1 guarantees the existence for each $k \in \mathbb{N}$ of continuous alternating germs

$$\begin{aligned} (S_k)_{pq} &:= \lim_{n\to\infty} (S_k^n)_{pq}, \\ (V_k)_{pqr} &:= \lim_{n\to\infty} (\omega_k^n)_{pqr} = \lim_{n\to\infty} ((\omega_k^n)_{pqr} - \delta(S_k^n)_{pqr}) + (\delta S_k^n)_{pqr}. \end{aligned}$$

Suppose further that $\omega_k \to \omega$ pointwise. Then clearly the latter satisfy (A.1) and (A.1) and thus Lemma A.1 provides the existence of continuous alternating germs

$$\begin{aligned} S_{pq} &:= \lim_{n\to\infty} S^n_{pq}, \\ V_{pqr} &:= \lim_{n\to\infty} \omega^n_{pqr} = \lim_{n\to\infty} (\omega^n_{pqr} - \delta S^n_{pqr}) + \delta S^n_{pqr}, \end{aligned}$$

where ω^n and S^n are defined by (A.3). The following stability statement is valid.

Lemma A.3. *Under the above conditions one has $S = \lim_k S_k$ and $V = \lim_k V_k$ pointwise.*

Proof. We note first that

$$|(S^n_k)_{pq} - S^n_{pq}| = |\langle \text{fill}\,\text{cut}^n[pq], \omega - \omega_k\rangle| \le C_1 2^n \left(\frac{\text{diam}([pq])}{2^n}\right)^{\gamma_1} \to 0$$

as $n \to \infty$ uniformly in k, which implies $S = \lim_k S_k$ pointwise via the standard estimate

$$|(S_k)_{pq} - S_{pq}| \le |(S_k)_{pq} - (S^n_k)_{pq}| + |(S^n_k)_{pq} - S^n_{pq}| + |S_{pq} - S^n_{pq}|.$$

Writing

$$\begin{aligned} (V_k - \delta S_k) - (V - \delta S) = &- (\omega^n_k - V_k - \delta(S^n_k - S_k)) + \\ &(\omega^n - V - \delta(S^n - S)) - (\omega^n - \omega^n_k - \delta(S^n - S^n_k)), \end{aligned}$$

and evaluating the latter relationship at $[pqr]$, using

$$\begin{aligned} |(\omega^n_k)_{pqr} - (V_k)_{pqr} - \delta(S^n_k - S_k)_{pqr}| &\le C 2^{n(2-\gamma_2)}, \\ |\omega^n_{pqr} - V_{pqr} - \delta(S^n - S)_{pqr}| &\le C 2^{n(2-\gamma_2)} \end{aligned}$$

with $C > 0$ independent of n and k, we arrive at the estimate

$$|(V_k - \delta S_k)_{pqr} - (V - \delta S)_{pqr}| \le 2C 2^{n(2-\gamma_2)} + \left|\omega^n_{pqr} - (\omega^n_k)_{pqr} - \delta(S^n - S^n_k)_{pqr}\right|. \tag{A.24}$$

Given an $\varepsilon > 0$ we fix an $n = n(\varepsilon) \in \mathbb{N}$ such that the first term on the right-hand side of (A.24) does not exceed $\varepsilon/2$, and since $\lim_k S^n_k = S^n$ and $\lim_k \omega^n_k = \omega^n$ pointwise, we get that also the second term on the does not exceed $\varepsilon/2$ for all sufficiently large k. This means

$$V - \delta S = \lim_k (V_k - \delta S_k)$$

pointwise and therefore $V = \lim_k V_k$ pointwise since $\lim_k S_k = S$, concluding the proof. □

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