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GENERATION OF ANALYTIC SEMIGROUPS AND DOMAIN CHARACTERIZATION FOR DEGENERATE ELLIPTIC OPERATORS WITH UNBOUNDED COEFFICIENTS ARISING IN FINANCIAL MATHEMATICS, PART I

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Abstract. In this paper we study the generation of analytic semigroup in the space $L^2(\mathbb{R}^d)$, and the characterization of the domain, for a family of degenerate elliptic operators with unbounded coefficients, which includes some well-known operators arising in Mathematical Finance. To prove the generation of analytic semigroups, the operators of the family are assumed to satisfy suitable growth and compensation conditions. Under stronger assumptions, we obtain also the characterization of the domain. Finally, various consequences of the obtained results are considered in connection with some applications (see, e.g., [6] for financial applications). In a forthcoming paper, part II of this work, we shall examine the problem of the generation of analytic semigroup in $L^p(\mathbb{R}^d)$, where $p \in (2, +\infty]$, and the characterization of the domain, for the same family of operators.

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1. Introduction

In this paper we study the generation of analytic semigroup in the space $L^2(\mathbb{R}^d)$, and the characterization of the domain, for a family of degenerate elliptic operators with unbounded coefficients, which includes some well-known operators arising in Mathematical Finance, and also in Mathematical Physics. These results can be employed to obtain existence, uniqueness, and regularity estimates for the solutions of the associated (linear or semilinear) parabolic problems, through the well-known theory of analytic semigroups (e.g., [37]).

This has been done partly in [6] for some so-called "no-arbitrage" operators arising in contingent claim pricing.

We consider the following differential operator in $\mathbb{R}^d$

$$\mathcal{A} \equiv \sum_{i,j=1}^{d} \psi_i(x)\psi_j(x)a_{i,j}(x)D_{i,j} + \sum_{i=1}^{d} b_i(x)D_i - \gamma^2(x), \tag{1.1}$$

where $\{a_{i,j}(x)\}_{i,j=1}^d$ is a uniformly elliptic bounded and measurable real-valued d-order matrix on $\mathbb{R}^d$, and the coefficients $\psi_i(x)$'s, $b_i(x)$'s and $\gamma(x)$ are measurable (not bounded) real-valued functions on $\mathbb{R}^d$, with the $\psi_i(x)$'s allowed to vanish in a negligible set Z at most.

It is well known that, in general, these operators do not generate analytic or strongly continuous semigroup (e.g. consider the one-dimensional Ornstein–Uhlenbeck operator $\sigma D^2 u + xDu$, where σ is a positive constant). Indeed, the main difficulty to overcome, to obtain some generation result, is to manage both the possible unboundedness of all coefficients, and the possible degeneration of the ones of the second-order terms. However, under suitable growth and compensation conditions on the coefficients, we can obtain the generation result and the characterization of the domain. The settlement of these conditions plays a central role in this paper. Actually they come out as a compromise between our attempt of covering, in a unitary approach, several operators interesting for the applications, and to avoid as many technicalities as possible in the proofs.

To give the reader an insight on the above-mentioned conditions and their application, we anticipate here the simple case of dimension $d = 1$. In this case we shall ask for the existence of two suitable constants B_1 and B_2 with $B_1 + B_2 < 2$ such that

$$|b(x)| \leq B_1 E^{1/2} |\psi(x)| \gamma(x) \quad \forall x \in \mathbb{R}, \tag{1.2}$$

and

$$\left|D\left(\psi(x)^2 a(x)\right)\right| \le B_2 E^{1/2} \left|\psi(x)\right| \gamma(x) \quad \forall x \in \mathbb{R}, \tag{1.3}$$

where E is the modulus of ellipticity of $\{a_{i,j}(x)\}_{i,j=1}^d$. For example, (1.2) and (1.3) are easily verified by the second-order differential operator

$$x^2D^2 + xD - (x^2 + 4),$$

by choosing $B_1 = 1/2$ and $B_2 = 1$. This clearly implies the generation result for the whole class of the modified Bessel operators

$$x^2D^2 + xD - (x^2 + \nu^2),$$

where $\nu \in \mathbb{R}$.

The above simple example also shows how to exploit the results of this paper. Namely, the verification of our assumptions for a suitably chosen operator actually yields the desired generation result, and the characterization of the domain, for a wider family of operators, which is deducible from the chosen one by some analytic perturbation (see [35]).

We shall show that other classical operators, as the generalized Schrödinger operator, and a wide class of diffusion generators, among which the Black and Scholes operator, can be handled by the approach outlined above.

Elliptic operators, that possibly degenerate somewhere in the domain, or have unbounded coefficients, have been studied by many authors. Among them we recall Baouendi and Goulaouic [3], [4], Clement and Timmerman [18], and Vespri [50], who have proved the generation of analytic semigroups for some operators whose coefficients are strongly elliptic in the interior of their bounded domain, but possibly degenerating in the boundary. We recall also Aronson and Besala [2], [7], and Cannarsa, Lunardi and Vespri [15], [16], [38], [39], who have obtained the generation of analytic or strongly continuous semigroups for wide families of operators with unbounded coefficients in $\mathbb{R}^d$, but satisfying everywhere the strong ellipticity condition. Finally, we recall Cerrai [17], who uses stochastic methods to handle operators similar to ours, but with different conditions and results, Campiti and Metafune [14], [43], who also studied operators similar to ours, but in the one-dimensional case and with the Ventcel's boundary conditions, and Colombo, Giuli and Vespri [20], who obtained the generation of strongly continuous semigroup for differential operators in $\mathbb{R}^d$, very similar to ours, but in divergence form and with different compensation conditions.

In this paper we assume that the coefficients of the operator $\mathcal{A}$, given by (1.1), are defined in all $\mathbb{R}^d$, and we prove first the generation of analytic semigroups in the space $L^2(\mathbb{R}^d)$. This is obtained by a straightforward application of standard Hilbert-space techniques, relying on suitable preliminary a priori estimates, which are allowed by a natural choice of the compensation conditions on the coefficients of $\mathcal{A}$. Then we obtain the characterization of the domain of $\mathcal{A}$ by a localization procedure adapted to the growth rate of the weights $\psi_i(x)$'s at infinity and close to Z.

In a forthcoming paper, part II of this work, we show how to pass from $L^2(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$, for $p > 2$, by applying a suitable modification of Stewart's method [47, 48, 50].

Our main aim is to provide a unitary framework in which we obtain the generation of analytic semigroups and the characterization of the domain for some elliptic operators arising in Mathematical Finance. Indeed, in the no-arbitrage pricing theory, which plays a central role in modern Mathematical Finance, it is important to provide general existence, uniqueness and regularity results for some parabolic equations. The semigroup techniques are then useful tools to get this kind of results, provided one can establish the semigroup generation and the characterization of the domain for the elliptic operator associated to the parabolic equation considered. In addition, a remark after Garman (see [28]) lightens the financial role played by the semigroup property in the framework of no-arbitrage prices. This gives one more motivation for applying the semigroup theory to problems coming from Mathematical Finance (see Section 4).

Finally, we point out that the analysis of several operators connected with financial applications calls for a restriction of the domain of the coefficients in $\mathbb{R}^d_+$, or even in a subset of it. Thus, an appropriate setting of the Cauchy problem associated to the operator $\mathcal{A}$ is usually the positive orthant $\mathbb{R}^d_+$. In the above-mentioned forthcoming paper we will show how to overcome this difficulty in some interesting cases, by imposing suitable boundary conditions. This will allows us to apply our results to prove existence and uniqueness of the solution of the Cauchy problem associated to the operator $\mathcal{A}$ and to give estimates of the solution of the no-arbitrage price and of its derivatives for a large class of contingent-claim contracts (see [6]).

The paper is organized as follows. In Section 2 we introduce the notation and recall some results about weighted spaces and analytic semigroups. Section 3 is the heart of the paper and is divided into four parts: the first two on the generation of analytic semigroup, the third on the characterization of the domain, and the fourth on the weighted case. Finally, in Section 4 we show some examples of operators which fit our setting.

2. Preliminary material and notation

Let Ω be an open subset of the d-dimensional Euclidean space $\mathbb{R}^d$. We denote by $C^\infty(\Omega)$ the linear space of all infinitely differentiable complex-valued functions on Ω, and we write $C_c^\infty(\Omega)$ for the linear submanifold of $C^\infty(\Omega)$ of all functions with compact support in Ω. For every integer $k \geq 0$, and every real $1 \leq p < \infty$, we denote by $C^{\infty,n,p}(\Omega)$ the linear submanifold of $C^\infty(\Omega)$ of all functions φ such that $\sum_{|\alpha|\leq n}(\int_\Omega |D^\alpha\varphi(x)|^p\,dx)^{1/p} < \infty$. Here we are employing the usual multi-index differential notation, $\alpha \equiv (\alpha_1,\dots,\alpha_d)$, $|\alpha| \equiv \sum_{i=1}^d \alpha_i$, $D^\alpha \equiv D^{\alpha_1}\cdots D^{\alpha_d}$, $D^{\alpha_i} \equiv \partial^{\alpha_i}/\partial x_i^{\alpha_i}$. Such a notation will be dropped in the sequel, writing $\int_\Omega |\varphi(x)|^p\,dx$, $\sum_{i=1}^d \int_\Omega |D_i\varphi(x)|^p\,dx$, and $\sum_{i,j=1}^d \int_\Omega |D_{i,j}\varphi(x)|^p\,dx$, rather than $\sum_{|\alpha|=n}\int_\Omega |D^\alpha\varphi(x)|^p\,dx$, when $n = 0,1,2$, respectively.

We denote by $W^{n,p}(\Omega)$ the usual *Sobolev space* (see, e.g., [1]), defined as the completion of $C_c^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W^{n,p}(\Omega)} \stackrel{def}{=} \sum_{|\alpha|\leq n}\Big(\int_\Omega |D^\alpha u(x)|^p\,dx\Big)^{1/p},$$

writing $L^p(\Omega)$ [resp. $H^n(\Omega)$] rather than $W^{0,p}(\Omega)$ [resp. $W^{n,2}(\Omega)$], and using the shorthands $W^{n,p}$, L^p and H^n for $W^{n,p}(\mathbb{R}^d)$, $L^p(\mathbb{R}^d)$ and $H^n(\mathbb{R}^d)$, respectively.

We denote by $W^{n,p}_{\mathrm{loc}}$ [resp. L^p_{loc}, H^n_{loc}] the linear space of all measurable complex-valued functions on $\mathbb{R}^d$ belonging to $W^{n,p}(\Omega)$ [resp. $L^p(\Omega)$, $H^n(\Omega)$] for every open subset Ω of $\mathbb{R}^d$ having compact closure, and, for any fixed real-valued function $\xi \in W^{n,p}_{\mathrm{loc}}$, we define the *weighted Sobolev space* $W^{n,p}_\xi$ as the completion of $C_c^\infty(\mathbb{R}^d)$ with respect to the *weighted norm*

$$\|u\|_{W^{n,p}_\xi} \stackrel{def}{=} \|\xi u\|_{W^{n,p}},$$

It is well known that $W^{n,p}_\xi$ can also be defined as the space of all measurable functions u such that $\xi u \in W^{n,p}$. Similarly, for any choice of the functions α, β_i, $i = 1,\dots,d$, $\delta_{i,j}$, $i,j = 1,\dots,d$ belonging to L^p_{loc}, with $\operatorname{ess\,inf}|\alpha| > 0$, we introduce the weighted Sobolev spaces $W^{1,p}_{(\alpha,\beta)}$ and $W^{2,p}_{(\alpha,\beta,\delta)}$ defined as the completion of $C_c^\infty(\mathbb{R}^d)$ with respect to the *weighted norm*

$$\|u\|_{W^{1,p}_{(\alpha,\beta)}} \stackrel{def}{=} \|\alpha u\|_{L^p} + \sum_{i=1}^d \|\beta_i D_i u\|_{L^p}$$

and

$$\|u\|_{W^{2,p}_{(\alpha,\beta,\delta)}} \overset{def}{=} \|\alpha u\|_{L^p} + \sum_{i=1}^{d} \|\beta_i D_i u\|_{L^p} + \sum_{i,j=1}^{d} \|\delta_{i,j} D_{i,j} u|_{L^p},$$

respectively, and we introduce also the spaces $W^{1,p}_{\xi,(\alpha,\beta)}$ [resp. $W^{2,p}_{\xi,(\alpha,\beta,\delta)}$] of all measurable functions u such that $\xi u \in W^{1,p}_{(\alpha,\beta)}$ [resp. $\xi u \in W^{2,p}_{(\alpha,\beta,\delta)}$], endowed with the norms

$$\|u\|_{W^{1,p}_{\xi,(\alpha,\beta)}} \overset{def}{=} \|\xi u\|_{W^{1,p}_{(\alpha,\beta)}} \quad [\text{resp. } \|u\|_{W^{2,p}_{\xi,(\alpha,\beta,\delta)}} \overset{def}{=} \|\xi u\|_{W^{2,p}_{(\alpha,\beta,\delta)}}].$$

Finally, we shall also write L^p_ξ, H^n_ξ, $H^n_{(\alpha,\beta)}$, $H^n_{(\alpha,\beta,\delta)}$, $H^n_{\xi,(\alpha,\beta)}$ and $H^n_{\xi,(\alpha,\beta,\delta)}$ in conformity with the notation introduced above.

3. Generation of analytic semigroups in $L^2(\mathbb{R}^d)$

In this section we consider the realization of the operator $\mathcal{A}$ in L^2. For clearness we divide the analysis into four subsections:

- first, we consider the easiest case, in which the operator is written in variational form, and we prove the generation of analytic semigroups, finding also the consequent estimates for the first derivatives;
- second, we examine the nonvariational case, again proving the generation of analytic semigroups and the estimates for the first derivatives;
- third, we characterize the domain of the realization $\mathcal{A}_2$ of $\mathcal{A}$ in L^2;
- fourth, we extend the previous results to the case of a weighted space L^2_ξ, for a suitable weight ξ.

3.1. **Operators in variational form.**

Let us consider the second-order differential operator in variational form

$$\widehat{\mathcal{A}}u \overset{def}{=} \sum_{i,j=1}^{d} D_j(\psi_i(x)\psi_j(x)a_{i,j}(x)D_i u) + \sum_{i=1}^{d} b_i(x)D_i u - \gamma^2(x)u, \tag{3.1}$$

which satisfies the assumption below.

Assumption 3.1. *Suppose that the following conditions hold true*

(1) *For all $i,j = 1,\dots,d$, the coefficients $a_{i,j}(x)$ are bounded measurable real-valued functions on $\mathbb{R}^d$ such that $a_{i,j}(x) = a_{j,i}(x)$, and satisfying*

the strong ellipticity condition

$$\operatorname{Re} \sum_{i,j=1}^{d} a_{i,j}(x) z_i \overline{z}_j \geq E\,|z|^2 \quad \forall z \in \mathbb{C}^d, \tag{3.2}$$

for a suitable modulus of ellipticity $E > 0$ independent of $x \in \mathbb{R}^d$;

(2) *for every $i = 1, \dots, d$, the coefficients $b_i(x)$ are measurable real-valued functions on $\mathbb{R}^d$, while $\gamma(x)$ and $\psi_i(x)$ are real-valued functions in L^2_{loc} with* $\operatorname{ess\,inf}\gamma \geq 1$; [1]

(3) *for every $i = 1, \dots, d$, we have*

$$|b_i(x)| \leq B E^{1/2} \eta_i(x)\,|\psi_i(x)|\,\gamma(x) \quad \forall x \in \mathbb{R}^d, \tag{3.3}$$

for a suitable constant $B < 2$ and measurable real-valued functions $\eta_i(x)$ on $\mathbb{R}^d$ such that $\sum_{i=1}^d \eta_i^2(x) = 1$.

Remark 3.2. Note that in the case $b_i(x) = 0$, for every $i = 1, \dots, d$, the operator $\widehat{\mathcal{A}}u$ reduces to a classical generalization of the Schrödinger operator (see [35]). In this case $\widehat{\mathcal{A}}u$ is clearly formally selfadjoint and it is easily checked that it is also negative. Hence, it is a classical problem to ask whether a suitable extension of $\widehat{\mathcal{A}}$ generates an analytic semigroup. On the other hand, in this particular case, (3.3) is trivially satisfied by choosing $B = 0$. Therefore, as a byproduct of our approach, we obtain the generation and the characterization of the domain for the classical family of generalized Schrödinger operators.

Remark 3.3. As mentioned in the introduction, we could also consider more general hypotheses on the coefficients γ and ψ_i. In particular, by minor modifications of the arguments below, we could deal with coefficients having isolated nonintegrable singularities. In addition, the arguments exploited in the proofs clearly show that Assumption 3.1.3, and the forthcoming 3.10.2, could be relaxed by requiring that they hold true, but a permutation of the occurring indices. Namely, we could rewrite 3.1.3 as follows:

$3'$. There exists a permutation σ of $\{1, \dots, d\}$ such that, for every $i = 1, \dots, d$, we have

$$|b_i(x)| \leq B E^{1/2} \eta_{\sigma(i)}(x)\,\left|\psi_{\sigma(i)}(x)\right|\,\gamma(x) \quad \forall x \in \mathbb{R}^d.$$

However, for the sake of simplicity, we have preferred to avoid such a more general formulation, except for the exhibition of a simple example when treating the nonvariational case.

[1] This condition could be replaced by the seemingly more general $\operatorname{ess\,inf}\gamma > 0$, provided to employ a standard normalization procedure.

In order to obtain a realization $\widehat{\mathcal{A}}_2 : D(\widehat{\mathcal{A}}_2) \to L^2$ of the formal operator $\widehat{\mathcal{A}}$ which generates an analytic semigroup, we begin by defining a suitable domain.

Write $\psi \equiv (\psi_1, \ldots, \psi_n)$ and consider the weighted Sobolev space $H^1_{(\gamma,\psi)}$ endowed with the norm $\|\cdot\|_{H^1_{(\gamma,\psi)}}$. This space becomes a dense linear submanifold of L^2 thanks to the condition ess inf$\gamma \geq 1$. Then define

$$D(\widehat{\mathcal{A}}_2) \stackrel{def}{=} \{u \in H^1_{(\gamma,\psi)} : \exists\, K(u) > 0 \,\text{s.t.}\ |\widehat{a}(u,\varphi)| \leq K(u)\|\varphi\|_{L^2}\ \forall \varphi \in C_c^\infty(\mathbb{R}^d)\},$$

where $\widehat{a}(\cdot,\cdot)$ is the sesquilinear form associated to $\widehat{\mathcal{A}}_2$ given by

$$\begin{aligned} \widehat{a}(u,v) \ \stackrel{def}{=}\ & -\int_{\mathbb{R}^d} \sum_{i,j=1}^d \psi_i(x)\psi_j(x)a_{i,j}(x)D_iu(x)D_j\overline{v}(x)dx \\ & + \int_{\mathbb{R}^d} \sum_{i=1}^d b_i(x)D_iu(x)\overline{v}(x)dx - \int_{\mathbb{R}^d} \gamma^2(x)u(x)\overline{v}(x)dx, \end{aligned}$$

for all $u, v \in L^2$ such that the above integrals make sense. Since $C_c^\infty(\mathbb{R}^d)$ is dense in L^2, for each $u \in D(\widehat{\mathcal{A}}_2)$ the antilinear functional

$$\varphi \in C_c^\infty(\mathbb{R}^d) \to \widehat{a}(u,\varphi) \in \mathbb{C}$$

may be continuously extended to the whole of L^2. Therefore, by virtue of Riesz' theorem, there exists a unique $f \in L^2$ such that

$$\widehat{a}(u,\varphi) = \langle f, \varphi \rangle\,,$$

for every $\varphi \in L^2$. This implies that if we choose

$$\widehat{\mathcal{A}}_2 u \stackrel{def}{=} f,$$

then the operator $\widehat{\mathcal{A}}_2 : D(\widehat{\mathcal{A}}_2) \to L^2$ is well defined. It follows that we can characterize a function $u \in D(\widehat{\mathcal{A}}_2)$ as a weak solution of the resolvent equation

$$(\lambda - \widehat{\mathcal{A}}_2)u = f, \tag{3.4}$$

for any fixed $\lambda \in \mathbb{C}$, if and only if we have

$$\begin{aligned} \int_{\mathbb{R}^d} f(x)\overline{\varphi}(x)dx = & \int_{\mathbb{R}^d} \sum_{i,j=1}^d \psi_i(x)\psi_j(x)a_{i,j}(x)D_iu(x)D_j\overline{\varphi}(x)dx \\ & - \int_{\mathbb{R}^d} \sum_{i=1}^d b_i(x)D_iu(x)\overline{\varphi}(x)dx + \int_{\mathbb{R}^d} (\lambda + \gamma^2(x))u(x)\overline{\varphi}(x)dx \end{aligned}$$

for every $\varphi \in C_c^\infty(\mathbb{R}^d)$.

Following Lunardi [37, Chapter 3], we establish the announced generation result for the differential operator $\widehat{\mathcal{A}}_2$ in two steps.

First we show that, for each $\lambda \in \mathbb{C}$ such that $\mathrm{Re}\,\lambda > 0$, the resolvent equation (3.4) has a unique solution in $D(\widehat{\mathcal{A}}_2)$ for every $f \in L^2$. [1] Then we prove that, for every such λ, the λ-resolvent of the operator $\widehat{\mathcal{A}}_2$ satisfies

$$\|\lambda R(\lambda; \widehat{\mathcal{A}}_2)\| \leq K,$$

for a suitable $K > 0$. The above procedure will be performed by applying some preliminary lemmas.

¿¿¿From now on we write A for $\max_{i,j=1,\ldots,d} \|a_{i,j}\|_\infty$, where

$$\|a_{i,j}\|_\infty \equiv \sup_{x \in \mathbb{R}^d} |a_{i,j}(x)|\,, \quad \text{for all } i, j = 1, \ldots, d.$$

Lemma 3.4. *Under Assumption 3.1, for all $u, v \in H^1_{(\gamma,\psi)}$, we have*

$$\int_{\mathbb{R}^d} \sum_{i,j=1}^d |\psi_i(x)\psi_j(x)a_{i,j}(x)D_i u(x) D_j \overline{v}(x)|\,dx \tag{3.5}$$
$$\leq A\Big(\sum_{i=1}^d \|\psi_i D_i u\|_{L^2}\Big)\Big(\sum_{i=1}^d \|\psi_i D_i v\|_{L^2}\Big),$$

$$\int_{\mathbb{R}^d} \sum_{i=1}^d |b_i(x) D_i u(x) \overline{v}(x)|\,dx \leq B E^{1/2} \|\gamma v\|_{L^2} \Big(\sum_{i=1}^d \|\psi_i D_i u\|_{L^2}\Big) \tag{3.6}$$

and

$$\int_{\mathbb{R}^d} \big|(\lambda + \gamma^2(x))u(x)\overline{v}(x)\big|\,dx \leq (|\lambda| + 1)\|\gamma u\|_{L^2}\|\gamma v\|_{L^2}. \tag{3.7}$$

Proof. The first claim follows by Schwarz's inequality, which gives

$$\int_{\mathbb{R}^d} \sum_{i,j=1}^d |\psi_i(x)\psi_j(x)a_{i,j}(x)D_i u(x) D_j \overline{v}(x)|\,dx \leq A \sum_{i,j=1}^d \|\psi_i D_i u\|_{L^2} \|\psi_j D_j \overline{v}\|_{L^2}.$$

Then, by virtue of (3.3), we have

$$\sum_{i=1}^d \int_{\mathbb{R}^d} |b_i(x) D_i u(x) \overline{v}(x)|\,dx$$

Notice that, as a consequence of this result, the resolvent set of the operator $\widehat{\mathcal{A}}_2$ contains the half plane $\{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda > 0\}$. This implies no loss of generality, thanks to the already-mentioned analytic perturbation techniques.

$$
\begin{aligned}
&\leq BE^{1/2}\sum_{i=1}^{d}\int_{\mathbb{R}^d}|\eta_i(x)\psi_i(x)\gamma(x)D_iu(x)\overline{v}(x)|\,dx\\
&\leq BE^{1/2}\sum_{i=1}^{d}\Big(\int_{\mathbb{R}^d}\eta_i^2(x)\,|\gamma(x)\overline{v}(x)|^2\,dx\Big)^{1/2}\Big(\int_{\mathbb{R}^d}|\psi_i(x)D_iu(x)|^2\,dx\Big)^{1/2}\\
&\leq BE^{1/2}\Big(\int_{\mathbb{R}^d}|\gamma(x)\overline{v}(x)|^2\,dx\Big)^{1/2}\sum_{i=1}^{d}\Big(\int_{\mathbb{R}^d}|\psi_i(x)D_iu(x)|^2\,dx\Big)^{1/2},
\end{aligned}
$$

which is the desired (3.6). Finally, since ess inf$\gamma \geq 1$ we can write

$$\left|(\lambda+\gamma^2(x))u(x)\overline{v}(x)\right| \leq (|\lambda|+1)\,|\gamma(x)u(x)|\,|\gamma(x)v(x)|\,,$$

for every $x\in\mathbb{R}^d$, and the latter implies (3.7), by applying again Schwarz's inequality. ☐

Lemma 3.5. *Under Assumption 3.1, for every* $u\in H^1_{(\gamma,\psi)}$, *we have*

$$\operatorname{Re}\int_{\mathbb{R}^d}\sum_{i,j=1}^{d}\psi_i(x)\psi_j(x)a_{i,j}(x)D_iu(x)D_j\overline{u}(x)dx \geq E\sum_{i=1}^{d}\|\psi_iD_iu\|_{L^2}^2 \tag{3.8}$$

and

$$\int_{\mathbb{R}^d}\sum_{i=1}^{d}|b_i(x)D_iu(x)\overline{u}(x)|\,dx \leq \frac{B}{2}\Big(\|\gamma u\|_{L^2}^2+E\sum_{i=1}^{d}\|\psi_iD_iu\|_{L^2}^2\Big). \tag{3.9}$$

Proof. Estimate (3.8) follows immediately from (3.2). Furthermore, (3.3) allows us to write

$$
\begin{aligned}
&\sum_{i=1}^{d}\int_{\mathbb{R}^d}|b_i(x)D_iu(x)\overline{u}(x)|\,dx\\
&\leq B\sum_{i=1}^{d}\int_{\mathbb{R}^d}\left|E^{1/2}\eta_i(x)\psi_i(x)\gamma(x)D_iu(x)\overline{u}(x)\right|dx\\
&\leq B\sum_{i=1}^{d}\Big(\int_{\mathbb{R}^d}\eta_i^2(x)\,|\gamma(x)\overline{u}(x)|^2\,dx\Big)^{1/2}\Big(\int_{\mathbb{R}^d}E\,|\psi_i(x)D_iu(x)|^2\,dx\Big)^{1/2}\\
&\leq B\sum_{i=1}^{d}\frac{1}{2}\Big(\int_{\mathbb{R}^d}\eta_i^2(x)\,|\gamma(x)\overline{u}(x)|^2\,dx+\int_{\mathbb{R}^d}E\,|\psi_i(x)D_iu(x)|^2\,dx\Big)\\
&=\frac{B}{2}\Big(\int_{\mathbb{R}^d}\Big(\sum_{i=1}^{d}\eta_i^2(x)\Big)\,|\gamma(x)\overline{u}(x)|^2\,dx+E\sum_{i=1}^{d}\int_{\mathbb{R}^d}|\psi_i(x)D_iu(x)|^2\,dx\Big),
\end{aligned}
$$

and this completes the proof. □

We are now in a position to carry on our procedure.

Proposition 3.6. *Under Assumption 3.1, for each* $\lambda \in \mathbb{C}$ *such that* $\mathrm{Re}\ \lambda > 0$*, Equation* (3.4) *has a unique solution* $u \in D(\widehat{\mathcal{A}}_2)$ *for every* $f \in L^2(\mathbb{R}^d)$*.*

Proof. For each $\lambda \in \mathbb{C}$, we introduce the sesquilinear form $\widehat{a}_\lambda(\cdot,\cdot)$ on the Hilbert space $H^1_{(\gamma,\psi)}$ given by

$$\begin{aligned}\widehat{a}_\lambda(u,v) \ &\overset{def}{=} \ \int_{\mathbb{R}^d}\sum_{i,j=1}^d \psi_i(x)\psi_j(x)a_{i,j}(x)D_iu(x)D_j\overline{v}(x)dx \\ &- \int_{\mathbb{R}^d}\sum_{i=1}^d b_i(x)D_iu(x)\overline{v}(x)dx + \int_{\mathbb{R}^d}(\lambda+\gamma^2(x))u(x)\overline{v}(x)dx.\end{aligned}$$

Then (3.5)–(3.7) imply that, for all $u, v \in H^1_{(\gamma,\psi)}$, we have

$$\begin{aligned}|\widehat{a}_\lambda(u,v)| &\le A\Big(\sum_{i=1}^d \|\psi_iD_iu\|_{L^2}\Big)\Big(\sum_{i=1}^d \|\psi_iD_iv\|_{L^2}\Big) \\ &+ BE^{1/2}\|\gamma v\|_{L^2}\Big(\sum_{i=1}^d\|\psi_iD_iu\|_{L^2}\Big) + (|\lambda|+1)\|\gamma u\|_{L^2}\|\gamma v\|_{L^2} \\ &\le H\|u\|_{H^1_{(\gamma,\psi)}}\|v\|_{H^1_{(\gamma,\psi)}},\end{aligned}$$

for a suitable $H > 0$. Moreover, if $\mathrm{Re}\ \lambda > 0$, thanks to (3.8) and (3.9), we can write

$$\begin{aligned}&\mathrm{Re}\ \widehat{a}_\lambda(u,u) \ge \mathrm{Re}\ \int_{\mathbb{R}^d}\sum_{i,j=1}^d \psi_i(x)\psi_j(x)a_{i,j}(x)D_iu(x)D_j\overline{u}(x)dx \\ &- \Big|\int_{\mathbb{R}^d}\sum_{i=1}^d b_i(x)D_iu(x)\overline{u}(x)dx\Big| + \mathrm{Re}\ \lambda\int_{\mathbb{R}^d}|u(x)|^2\,dx + \int_{\mathbb{R}^d}\gamma^2(x)\,|u(x)|^2\,dx \\ &\ge E\sum_{i=1}^d\|\psi_iD_iu\|^2_{L^2} - \frac{B}{2}\Big(\|\gamma u\|^2_{L^2} + E\sum_{i=1}^d\|\psi_iD_iu\|^2_{L^2}\Big) + \mathrm{Re}\ \lambda\|u\|^2_{L^2} + \|\gamma u\|^2_{L^2} \\ &\ge K\|u\|^2_{H^1_{(\gamma,\psi)}},\end{aligned}$$

for a suitable $K > 0$. Therefore, since for each $f \in L^2$ the map $v \in H^1_{(\gamma,\psi)} \to \langle f, v\rangle$ defines a continuous antilinear functional on $H^1_{(\gamma,\psi)}$, a standard application of the Lax–Milgram theorem allows us to conclude that for every

$f \in L^2$ there exists a unique $u \in H^1_{(\gamma,\psi)}$ satisfying the equation

$$\widehat{a}_\lambda(u,\varphi) = \langle f,\varphi\rangle\,,$$

for every $\varphi \in C_c^\infty(\mathbb{R}^d)$. Moreover, since

$$\widehat{a}(u,\varphi) = \lambda\,\langle u,\varphi\rangle - \widehat{a}_\lambda(u,\varphi),$$

it is clear that such a u belongs to $D(\widehat{\mathcal{A}}_2)$. Therefore, for each $\lambda \in \mathbb{C}$ such that $\mathrm{Re}\ \lambda > 0$, the operator $\lambda - \widehat{\mathcal{A}}_2 : D(\widehat{\mathcal{A}}_2) \to L^2$ is invertible, as desired. □

Proposition 3.7. *Under Assumption 3.1, for every $\lambda \in \mathbb{C}$ such that* $\mathrm{Re}\ \lambda >$ 0*, the λ-resolvent $R(\lambda;\widehat{\mathcal{A}}_2)$ of $\widehat{\mathcal{A}}_2$ satisfies*

$$\|\lambda R(\lambda;\widehat{\mathcal{A}}_2)\| \leq K,$$

for a suitable $K > 0$ independent of λ.

Proof. Given any fixed $\lambda \in \mathbb{C}$ such that $\mathrm{Re}\ \lambda > 0$, let $u \in D(\widehat{\mathcal{A}}_2)$ be the solution of (3.4) corresponding to some $f \in L^2$. Since

$$\widehat{a}_\lambda(u,u) = \langle f,u\rangle \tag{3.10}$$

holds true, we have

$$\begin{aligned}\|u\|_{L^2}\|f\|_{L^2} &\geq \mathrm{Re}\ \int_{\mathbb{R}^d}\sum_{i,j=1}^d \psi_i(x)\psi_j(x)a_{i,j}(x)D_iu(x)D_j\overline{u}(x)dx \\ &- \mathrm{Re}\ \int_{\mathbb{R}^d}\sum_{i=1}^d b_i(x)D_iu(x)\overline{u}(x)dx + \mathrm{Re}\ \int_{\mathbb{R}^d}(\lambda+\gamma^2(x))u(x)\overline{u}(x)dx \\ &\geq E\sum_{i=1}^d \|\psi_iD_iu\|^2_{L^2} - \left|\int_{\mathbb{R}^d}\sum_{i=1}^d b_i(x)D_iu(x)\overline{u}(x)dx\right| + \mathrm{Re}\ \lambda\|u\|^2_{L^2} + \|\gamma u\|^2_{L^2}.\end{aligned}$$

Therefore, thanks to (3.9), we obtain

$$\frac{2}{2-B}\|u\|_{L^2}\|f\|_{L^2} \geq \|\gamma u\|^2_{L^2} + E\sum_{i=1}^d \|\psi_iD_iu\|^2_{L^2}. \tag{3.11}$$

On the other hand, (3.10) also yields

$$\int_{\mathbb{R}^d}\lambda\,|u(x)|^2\,dx = -\int_{\mathbb{R}^d}\sum_{i,j=1}^d \psi_i(x)\psi_j(x)a_{i,j}(x)D_iuD_j\overline{u}(x)dx$$

$$+\int_{\mathbb{R}^d}\sum_{i=1}^{d} b_i(x)D_iu(x)\overline{u}(x)dx-\int_{\mathbb{R}^d}\gamma^2(x)\left|u(x)\right|^2dx+\int_{\mathbb{R}^d}f(x)\overline{u}(x)dx,$$

and the latter, by virtue of (3.5), (3.9) and (3.11), implies

$$\begin{aligned}|\lambda|\|u\|_{L^2}^2 &\le A(\sum_{i=1}^{d}\|\psi_iD_iu\|_{L^2})^2+\frac{B}{2}(\|\gamma u\|_{L^2}^2+E\sum_{i=1}^{d}\|\psi_iD_iu\|_{L^2}^2)\\ &\quad+\|\gamma u\|_{L^2}^2+\|f\|_{L^2}\|u\|_{L^2}\\ &\le Ad\sum_{i=1}^{d}\|\psi_iD_iu\|_{L^2}^2+\|\gamma u\|_{L^2}^2+\frac{2}{2-B}\|u\|_{L^2}\|f\|_{L^2}\le K\|u\|_{L^2}\|f\|_{L^2},\end{aligned}$$

for a suitable $K>0$ independent of λ. It then follows

$$|\lambda|\|u\|_{L^2}\le K\|f\|_{L^2}, \tag{3.12}$$

which is our claim. □

As a consequence of Propositions 3.6 and 3.7 it follows (see, e.g., [37, p. 37])

Theorem 3.8. *Under Assumption 3.1, the operator $\widehat{\mathcal{A}}_2 : D(\widehat{\mathcal{A}}_2)\to L^2$ generates an analytic semigroup on L^2.*

In addition, we have

Corollary 3.9. *Under Assumption 3.1, every solution $u\in D(\widehat{\mathcal{A}}_2)$ of* (3.4) *satisfies*

$$|\lambda|^{1/2}\|\gamma u\|_{L^2}\le K'\|f\|_{L^2}\quad \textit{and}\quad |\lambda|^{1/2}\|\psi_iD_iu\|_{L^2}\le K''\|f\|_{L^2},$$

for suitable $K',K''>0$ independent of λ.

Proof. The claim immediately follows combining (3.12) with (3.11). □

3.2. **Operators in nonvariational form.** Let us now consider the formal second-order differential operator

$$\mathcal{A}u\stackrel{def}{=}\sum_{i,j=1}^{d}\psi_i(x)\psi_j(x)a_{i,j}(x)D_{i,j}u+\sum_{i=1}^{d}b_i(x)D_iu-\gamma^2(x)u, \tag{3.13}$$

whose coefficients satisfy conditions similar to those on the coefficients of the operator (3.1), except for some slight modifications and some additional requirements. Namely we suppose

Assumption 3.10. *Conditions (2) and (3) of Assumption 3.1 hold true. In addition,*

(1) *for all* $i,j=1,\dots,d$ *the coefficients* $a_{i,j}(x)$ *are differentiable on* $\mathbb{R}^d$*;*
(2) *for all* $i=1,\dots,d$ *the coefficients* $\psi_i(x)$ *are differentiable, and we have*

$$\begin{aligned} |b_i(x)| &\le B_1 E^{1/2}\eta_{1,i}(x)\,|\psi_i(x)|\,\gamma(x) \quad \forall x\in\mathbb{R}^d, \\ |D_j(\psi_i(x)\psi_j(x)a_{i,j}(x))| &\le B_2 E^{1/2}\eta_{2,i,j}(x)\,|\psi_i(x)|\,\gamma(x) \quad \forall x\in\mathbb{R}^d, \end{aligned} \tag{3.14}$$

for suitable constants B_1 *and* B_2 *such that* $B_1+B_2<2$ *and measurable positive functions* $\eta_{1,i}(x)$ *and* $\eta_{2,i,j}(x)$ *satisfying*

$$\sum_{i=1}^d \eta_{1,i}^2(x) = d\sum_{i,j=1}^d \eta_{2,i,j}^2(x) = 1.$$

Remark 3.11. As anticipated in the Introduction, in the one-dimensional case (3.14) calls for the existence of constants B_1 and B_2 with $B_1+B_2<2$ such that

$$\begin{aligned} |b(x)| &\le B_1 E^{1/2}\,|\psi(x)|\,\gamma(x) \quad \forall x\in\mathbb{R}^d, \\ \left|D(\psi^2(x)a(x))\right| &\le B_2 E^{1/2}\,|\psi(x)|\,\gamma(x) \quad \forall x\in\mathbb{R}^d. \end{aligned} \tag{3.15}$$

The latter can be easily verified for several well-known operators, but a trivial analytic perturbation. Besides the *"modified"Bessel operator,* we mention also the infinitesimal generator of the geometric Brownian motion

$$\tfrac{1}{2}\alpha^2x^2D^2 + rxD$$

where $\alpha, r\in\mathbb{R}_+$.

Remark 3.12. The simplest nontrivial example in higher dimensions is perhaps the infinitesimal generator of the process which is often used to model the evolution in time of a basket of correlated risky assets. Suppose we have a basket of $d>1$ correlated risky assets whose evolution in time is described by the d-dimensional process $\left(X_t\right)_{t\ge0} \equiv \left(\left(X_t^{(1)}\right)_{t\ge0},\dots,\left(X_t^{(d)}\right)_{t\ge0}\right)$ which satisfies the system of differential equations

$$dX_t^{(k)} = r_kX_t^{(k)}dt + X_t^{(k)}\sum_{j=1}^d \alpha_{j,k}\,dW_t^{(j)}, \quad k=1,\dots d,$$

where $r_k,\alpha_{j,k}\in\mathbb{R}_+$, for all $j,k=1,\dots d$, and $\left(W_t^{(j)}\right)_{t\ge0}$, $j=1,\dots d$, are independent Wiener processes. Then, it is well known that the infinitesimal generator of the process $\left(X_t\right)_{t\geq 0}$ is the differential operator

$$\frac{1}{2}\sum_{j,k=1}^{d} x_j x_k \Big(\sum_{i=1}^{d} \alpha_{j,i}\alpha_{i,k}\Big) x_j x_k D_{j,k} + \sum_{k=1}^{d} r_k x_k D_k.$$

This operator clearly satisfies Assumption 3.10, but a trivial analytic perturbation.

Remark 3.13. As already mentioned, Condition (2) of 3.10 could be relaxed by requiring that it holds true, but a permutation of the occurring indices. For instance this applies to the infinitesimal generator of the *Brownian motion in the unit circle*, which is the bidimensional stochastic process $\big(X_t^{(k)}\big)_{t\geq 0}$, $k=1,2$ satisfying the system of differential equations

$$\begin{aligned} dX_t^{(1)} &= -\tfrac{1}{2}X_t^{(1)}dt - X_t^{(2)}dW_t,\\ dX_t^{(2)} &= -\tfrac{1}{2}X_t^{(2)}dt + X_t^{(1)}dW_t, \end{aligned}$$

where $(W_t)_{t\geq 0}$ is a standard Wiener process. The infinitesimal generator of the Brownian motion in the unit circle is then given by

$$\tfrac{1}{2}\left(x_2^2 D_1^2 - 2x_1x_2D_{1,2} + x_1^2 D_2^2 - x_1D_1 - x_2D_2\right),$$

and it satisfies Assumption 3.10, but a permutation of the occurring indices, and a trivial analytic perturbation.

Assumption 3.10 allows us to reduce the analysis of the nonvariational case to the analysis of the variational one. Indeed, introducing the sesquilinear form $a(\cdot,\cdot)$ associated to the operator $\mathcal{A}$, given by

$$a(u,v) \stackrel{def}{=} \widehat{a}(u,v) - \int_{\mathbb{R}^d}\sum_{i,j=1}^{d} D_j(\psi_i(x)\psi_j(x)a_{i,j}(x))D_iu(x)\overline{v}(x)dx,$$

for all $u,v\in H^1_{(\gamma,\psi)}$, and writing

$$D(\mathcal{A}_2) \stackrel{def}{=} \left\{u\in H^1_{(\gamma,\psi)} : \exists K(u)>0 \text{ s.t. } |a(u,\varphi)|\leq K(u)\|\varphi\|_2\ \forall\varphi\in C_c^\infty(\mathbb{R}^d)\right\},$$

we can study the realization $\mathcal{A}_2 : D(\mathcal{A}_2)\to L^2$ of $\mathcal{A}$ employing the results obtained for the operator $\widehat{\mathcal{A}}_2$. To this end, following the pattern of the previous section, we establish some preliminary lemmas.

Lemma 3.14. *Under Assumption 3.10, for all $u,v\in H^1_{(\gamma,\psi)}$, we have*

$$\int_{\mathbb{R}^d}\sum_{i,j=1}^{d} \left|D_j(\psi_i(x)\psi_j(x)a_{i,j}(x))D_iu(x)\overline{v}(x)\right| dx$$

$$\leq B_2 E^{1/2} d^{1/2} \|\gamma v\|_{L^2} \sum_{i=1}^{d} \|\psi_i D_i u\|_{L^2}. \tag{3.16}$$

Proof. Thanks to (3.14), a straightforward computation similar to that in the proof of Lemma 3.4 gives

$$\sum_{i,j=1}^{d} \int_{\mathbb{R}^d} |D_j(\psi_i(x)\psi_j(x)a_{i,j}(x))D_i u(x)\overline{v}(x)|\, dx$$

$$\leq B_2 E^{1/2} \sum_{i,j=1}^{d} \Big(\int_{\mathbb{R}^d} d\eta_{2,i,j}^2(x)|\gamma(x)\overline{v}(x)|^2 dx\Big)^{\frac{1}{2}} \Big(\frac{1}{d}\int_{\mathbb{R}^d} |\psi_i(x) D_i u(x)|^2\, dx\Big)^{\frac{1}{2}}$$

$$= B_2 E^{1/2} d^{1/2} \|\gamma v\|_{L^2} \sum_{i=1}^{d} \|\psi_i D_i u\|_{L^2},$$

which is our claim. □

Note that, if $\gamma(x)$ is locally bounded, then we have

$$\Big|\int_{\mathbb{R}^d} \sum_{i,j=1}^{d} D_j(\psi_i(x)\psi_j(x)a_{i,j}(x))D_i u(x)\overline{\varphi}(x)dx\Big|$$

$$\leq B_2 E^{1/2} d^{1/2} \sup_{x\in \text{supp}(\varphi)} |\gamma(x)|\, \|\varphi\|_{L^2} \sum_{i=1}^{d} \|\psi_i D_i u\|_{L^2},$$

for every $\varphi \in C_c^\infty(\mathbb{R}^d)$, and the above result yields $D(\mathcal{A}_2) = D(\widehat{\mathcal{A}}_2)$.

Lemma 3.15. *Under Assumption 3.10, for all $u \in H^1_{(\gamma,\psi)}$, we have*

$$\int_{\mathbb{R}^d} \sum_{i,j=1}^{d} |D_j(\psi_i(x)\psi_j(x)a_{i,j}(x))D_i u(x)\overline{u}(x)|\, dx$$

$$\leq \frac{B_2}{2}(\|\gamma u\|_{L^2}^2 + E \sum_{i=1}^{d} \|\psi_i D_i u\|_{L^2}^2). \tag{3.17}$$

Proof. By virtue of (3.14), an analysis similar to that in the proof of Lemma 3.5 yields

$$\sum_{i,j=1}^{d} \int_{\mathbb{R}^d} |D_j(\psi_i(x)\psi_j(x)a_{i,j}(x))D_i u(x)\overline{u}(x)|$$

$$\leq B_2 \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} \left| E^{1/2} \eta_{2,i,j}(x)\psi_i(x)\gamma(x) D_i u(x)\overline{u}(x) \right| dx$$

$$\leq \frac{B_2}{2}\Big(\int_{\mathbb{R}^d} d\Big(\sum_{i,j=1}^{d} \eta_{2,i,j}^2(x)\Big) |\gamma(x)\overline{u}(x)|^2 \, dx + \frac{E}{d} \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} |\psi_i(x) D_i u(x)|^2 \Big),$$

from which the desired result easily follows. □

Now we can prove

Proposition 3.16. *Under Assumption 3.10, for each $\lambda \in \mathbb{C}$ such that* $\mathrm{Re}\ \lambda > 0$*, the equation*

$$(\lambda - \mathcal{A}_2)u = f \tag{3.18}$$

has a unique solution $u \in D(\mathcal{A}_2)$ for every $f \in L^2(\mathbb{R}^d)$.

Proof. For each $\lambda \in \mathbb{C}$, we introduce the sesquilinear form $a_\lambda(\cdot,\cdot)$ on $H^1_{(\gamma,\psi)}$ given by

$$a_\lambda(u,v) \overset{def}{=} \widehat{a}_\lambda(u,v) + \int_{\mathbb{R}^d} \sum_{i,j=1}^{d} D_j(\psi_i(x)\psi_j(x)a_{i,j}(x)) D_i u(x)\overline{v}(x)dx.$$

Then, combining (3.5), (3.6), (3.9) and (3.16), for all $u, v \in H^1_{(\gamma,\psi)}$, we have

$$|a_\lambda(u,v)| \quad \leq \quad H\|u\|_{H^1_{(\gamma,\psi)}}\|v\|_{H^1_{(\gamma,\psi)}},$$

for a suitable constant $H > 0$. Moreover, if $\mathrm{Re}\ \lambda > 0$, from (3.8), (3.9) and (3.17) it follows that

$$\begin{aligned}
&\mathrm{Re}\ a_\lambda(u,u) \geq \mathrm{Re}\ \int_{\mathbb{R}^d} \sum_{i,j=1}^{d} \psi_i(x)\psi_j(x)a_{i,j}(x)D_i u(x) D_j\overline{u}(x)dx \\
&- \Big| \int_{\mathbb{R}^d} \sum_{i,j=1}^{d} D_j(\psi_i(x)\psi_j(x)a_{i,j}(x)) D_i u(x)\overline{u}(x)dx \Big| \\
&- \Big| \int_{\mathbb{R}^d} \sum_{i=1}^{d} b_i(x) D_i u(x)\overline{u}(x)dx \Big| + \mathrm{Re}\ \int_{\mathbb{R}^d} (\lambda + \gamma^2(x)) |u(x)|^2 \, dx \\
&\geq E \sum_{i=1}^{d} \|\psi_i D_i u\|^2_{L^2} - \frac{(B_1 + B_2)}{2}\Big(\|\gamma v\|^2_{L^2} + E \sum_{i=1}^{d} \|\psi_i D_i u\|^2_{L^2} \Big) \\
&\quad + \mathrm{Re}\ \lambda \|u\|^2_{L^2} + \|\gamma u\|^2_{L^2} \geq K\|u\|^2_{H^1_{(\gamma,\psi)}},
\end{aligned}$$

for a suitable constant $K > 0$. Therefore, the same argument as that in the proof of Proposition 3.6 shows that, for each $\lambda \in \mathbb{C}$ such that Re $\lambda > 0$, the operator $\lambda - \mathcal{A}_2 : D(\mathcal{A}_2) \to L^2$ is invertible. □

We have also

Proposition 3.17. *Under Assumption 3.10, for each $\lambda \in \mathbb{C}$ such that* Re $\lambda > 0$, *the λ-resolvent $R(\lambda; \mathcal{A}_2)$ of the operator $\mathcal{A}_2$ satisfies*

$$\|\lambda R(\lambda; \mathcal{A}_2)\| \leq K,$$

for a suitable $K > 0$ independent of λ.

Proof. For a fixed $\lambda \in \mathbb{C}$ such that Re $\lambda > 0$, let $u \in D(\mathcal{A}_2)$ be the solution of equation (3.18) corresponding to some $f \in L^2$. Since, as in the variational case, the equality

$$a_\lambda(u, u) = \langle f, u \rangle \tag{3.19}$$

holds true, we have

$$\begin{aligned}
&\|u\|_2 \|f\|_2 \geq \text{Re} \int_{\mathbb{R}^d} \sum_{i,j=1}^d \psi_i(x)\psi_j(x)a_{i,j}(x)D_i u(x)D_j \overline{u}(x)dx \\
&- \text{Re} \int_{\mathbb{R}^d} \sum_{i,j=1}^d D_j(\psi_i(x)\psi_j(x)a_{i,j}(x))D_i u(x)\overline{u}(x)dx \\
&- \text{Re} \int_{\mathbb{R}^d} \sum_{i=1}^d b_i(x)D_i u(x)\overline{u}(x)dx + \text{Re} \int_{\mathbb{R}^d} (\lambda + \gamma^2(x)) |u(x)|^2 dx \\
&\geq E \sum_{i=1}^d \|\psi_i D_i u\|_{L^2}^2 - \left| \int_{\mathbb{R}^d} \sum_{i,j=1}^d D_j(\psi_i(x)\psi_j(x)a_{i,j}(x))D_i u(x)\overline{u}(x)dx \right| \\
&- \left| \int_{\mathbb{R}^d} \sum_{i=1}^d b_i(x)D_i u(x)\overline{u}(x)dx \right| + \text{Re } \lambda \|u\|_{L^2}^2 + \|\gamma u\|_{L^2}^2.
\end{aligned}$$

Hence, from (3.17) and (3.9), it follows that

$$\|u\|_{L^2}\|f\|_{L^2} \geq \text{Re } \lambda \|u\|_{L^2}^2 + \frac{2-(B_1+B_2)}{2}\Big(\|\gamma u\|_{L^2}^2 + E\sum_{i=1}^d \|\psi_i D_i u\|_{L^2}^2\Big),$$

which implies

$$\frac{2}{2-(B_1+B_2)}\|u\|_{L^2}\|f\|_{L^2} \geq (\|\gamma u\|_{L^2}^2 + E\sum_{i=1}^d \|\psi_i D_i u\|_{L^2}^2). \tag{3.20}$$

Finally, a computation similar to that in the proof of Proposition 3.7, gives

$$|\lambda|\,\|u\|_{L^2}^2 \le K\|u\|_{L^2}\|f\|_{L^2}, \tag{3.21}$$

for a suitable $K > 0$ independent of λ, and this is the desired result. □

¿¿¿From the above results it follows

Theorem 3.18. *Under Assumption 3.10, the operator $\mathcal{A}_2 : D(\mathcal{A}_2) \to L^2$ generates an analytic semigroup on L^2.*

Moreover

Corollary 3.19. *Under Assumption 3.10, for every solution $u \in D(\mathcal{A}_2)$ of* (3.18)*, we have*

$$|\lambda|^{1/2}\,\|\gamma u\|_{L^2} \le K'\|f\|_{L^2} \quad \text{and} \quad |\lambda|^{1/2}\,\|\psi_i D_i u\|_{L^2} \le K''\|f\|_{L^2},$$

for suitably chosen $K', K'' > 0$ independent of λ.

Proof. As in the variational case, combining (3.21) with (3.20), the desired result immediately follows. □

3.3. **Domain characterization.** In this subsection we introduce some additional conditions on the coefficients of the operator $\mathcal{A}_2 : D(\mathcal{A}_2) \to L^2$ that allow us to characterize its domain. We divide the domain characterization in two parts: a first part in which we obtain the suitable estimates for the first-order derivatives and a second part devoted to the estimates for the second-order derivatives. This division is justified for better collating the assumptions needed to handle the two cases.

3.3.1. *Domain Characterization: first derivatives.* Throughout the sequel we make the following assumption:

Assumption 3.20. *Under (1) of Assumption 3.1, and (2) and (3) of Assumption 3.2, suppose in addition that γ is continuously differentiable and that, for all $i,j = 1,\dots,d$, we have*

$$|b_i(x)| \le B_1 E^{1/2}\eta_{1,i}(x)\,|\psi_i(x)|\,\gamma(x), \quad \forall x \in \mathbb{R}^d,$$

$$|D_j(\psi_i(x)\psi_j(x)a_{i,j}(x))| \le B_2 E^{1/2}\eta_{2,i,j}(x)\,|\psi_i(x)|\,\gamma(x), \quad \forall x \in \mathbb{R}^d, \tag{3.22}$$

$$2\,|\psi_j(x)D_j\gamma(x)a_{i,j}(x)| \le B_3 E^{1/2}\eta_{3,i,j}(x)\gamma^2(x), \quad \forall x \in \mathbb{R}^d,$$

for suitable constants B_1, B_2 and B_3 such that $B_1 + B_2 + B_3 < 2$ and suitable measurable functions $\eta_{1,i}(x)$, $\eta_{2,i,j}(x)$ and $\eta_{3,i,j}(x)$ on $\mathbb{R}^d$ satisfying $\sum_{i=1}^d \eta_{1,i}(x) = d\sum_{i,j=1}^d \eta_{2,i,j}^2(x) = d\sum_{i,j=1}^d \eta_{3,i,j}^2(x) = 1$.

Remark 3.21. Notice that

(i) Assuming γ continuously differentiable implies that the bounded measurable functions γ_n given by

$$\gamma_n(x) \stackrel{def}{=} \begin{cases} \gamma(x) & \text{if } \gamma(x) < n \\ n & \text{elsewhere} \end{cases}$$

for every $n \geq 1$, are differentiable almost everywhere in $\mathbb{R}^d$. Indeed, the set

$$L_n \stackrel{def}{=} \left\{ x \in \mathbb{R}^d : \gamma(x) \geq n \right\}$$

is closed for every $n \geq 1$, and γ_n is clearly differentiable in $\overset{\circ}{L}_n \cup L_n^c$. Furthermore, for every x belonging to the boundary ∂L_n of L_n we have $\gamma(x) = n$, and it is easily seen that γ_n is still differentiable in those $x \in \partial L_n$ where $(d\gamma)_x = 0$. Finally, the set $\{x \in \partial L_n : (d\gamma)_x \neq 0\}$, where γ_n is not differentiable, is a continuously differentiable $(d-1)$-dimensional submanifold of $\mathbb{R}^d$, which is negligible.

(ii) The assumption of continuous differentiability of γ could be weakened by recalling that the addition of a bounded measurable zero-order term does not influence the generation property of our operators. In fact, exploiting the same arguments, we could treat the case in which the function $\gamma(x)$ is written as $\gamma_0 + \widetilde{\gamma}$, where γ_0 is measurable and bounded, and $\widetilde{\gamma}$ is continuously differentiable in $\mathbb{R}^d$.

We have

Proposition 3.22. *Under Assumption 3.20, both $\gamma^2 u$ and $\psi_i \gamma D_i u$ belong to L^2, for every $i = 1, \ldots, d$. More precisely, u belongs to $H^1_{(\gamma^2, \gamma\psi)}$, and*

$$\|u\|_{H^1_{(\gamma^2,\gamma\psi)}} \leq K \|f\|_{L^2} \tag{3.23}$$

holds true for a suitable $K > 0$. In particular, for every $i = 1, \ldots, d$, also $b_i D_i u$ belongs to L^2, and we have

$$\sum_{i=1}^{d} \|b_i D_i u\|_{L^2} \leq d^2 B_1 E^{1/2} \sum_{i=1}^{d} \|\psi_i \gamma D_i u\|_{L^2}. \tag{3.24}$$

Proof. Let $u \in D(\mathcal{A}_2)$ be the solution of equation (3.18) corresponding to some $f \in L^2$. Choosing $u_n \stackrel{def}{=} \gamma_n^2 u$ we have clearly $u_n \in L^2$, and

$$a_\lambda(u, u_n) = \langle f, u_n \rangle_2 , \tag{3.25}$$

for every $n \geq 1$. Thanks to the derivation rule we can rewrite (3.25) as

$$\begin{aligned}
&\int_{\mathbb{R}^d} (\lambda + \gamma^2(x))\gamma_n^2(x)\,|u(x)|^2\,dx \qquad (3.26)\\
&+ \int_{\mathbb{R}^d} \sum_{i,j=1}^d \psi_i(x)\psi_j(x)a_{i,j}(x)\gamma_n^2(x)D_iu(x)D_j\overline{u}(x)dx\\
&= \int_{\mathbb{R}^d} f(x)\gamma_n^2(x)\overline{u}(x)dx + \int_{\mathbb{R}^d} \sum_{i=1}^d b_i(x)D_iu(x)\gamma_n^2(x)\overline{u}(x)dx\\
&- 2\int_{\mathbb{R}^d} \sum_{i,j=1}^d \psi_i(x)\psi_j(x)a_{i,j}(x)\gamma_n(x)D_j\gamma_n(x)D_iu(x)\overline{u}(x)dx\\
&- \int_{\mathbb{R}^d} \sum_{i,j=1}^d D_j(\psi_i(x)\psi_j(x)a_{i,j}(x))D_iu(x)\gamma_n^2(x)\overline{u}(x)dx,
\end{aligned}$$

which clearly implies

$$\begin{aligned}
&\mathrm{Re}\,\lambda \int_{\mathbb{R}^d} \gamma^2(x)\,|u(x)|^2\,dx + \int_{\mathbb{R}^d} \gamma_n^2(x)\gamma^2(x)\,|u(x)|^2\,dx\\
&+ \mathrm{Re}\,\int_{\mathbb{R}^d} \sum_{i,j=1}^d \psi_i(x)\psi_j(x)a_{i,j}(x)D_iu(x)\gamma_n^2(x)D_j\overline{u}(x)dx\\
&\leq \int_{\mathbb{R}^d} \left|f(x)\gamma_n^2(x)u(x)\right| dx + \int_{\mathbb{R}^d} \sum_{i=1}^d \left|b_i(x)D_iu(x)\gamma_n^2(x)\overline{u}(x)\right| dx\\
&+ 2\int_{\mathbb{R}^d} \sum_{i,j=1}^d |\psi_i(x)\psi_j(x)a_{i,j}(x)\gamma_n(x)D_j\gamma_n(x)D_iu(x)\overline{u}(x)|\,dx\\
&+ \int_{\mathbb{R}^d} \sum_{i,j=1}^d \left|D_j(\psi_i(x)\psi_j(x)a_{i,j}(x))D_iu(x)\gamma_n^2(x)\overline{u}(x)\right| dx. \qquad (3.27)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\mathrm{Re}\,\int_{\mathbb{R}^d} \sum_{i,j=1}^d \psi_i(x)\psi_j(x)a_{i,j}(x)D_iu(x)\gamma_n^2(x)D_j\overline{u}(x)dx\\
&= \int_{\mathbb{R}^d} \gamma_n^2(x)\,\mathrm{Re}\,(\sum_{i,j=1}^d \psi_i(x)\psi_j(x)a_{i,j}(x)D_iu(x)D_j\overline{u}(x))dx
\end{aligned}$$

$$\geq E \int_{\mathbb{R}^d} \gamma_n^2(x) \sum_{i=1}^d |\psi_i(x) D_i u(x)|^2 \, dx, \tag{3.28}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left| f(x)\gamma_n^2(x)u(x) \right| dx \leq (\int_{\mathbb{R}^d} |f(x)|^2 \, dx)^{1/2} (\int_{\mathbb{R}^d} \gamma_n^4(x) \, |u(x)|^2 \, dx)^{1/2} \\
&\leq \frac{1}{2\varepsilon} \int_{\mathbb{R}^d} |f(x)|^2 \, dx + \frac{\varepsilon}{2} \int_{\mathbb{R}^d} \gamma_n^4(x) \, |u(x)|^2 \, dx \\
&\leq \frac{1}{2\varepsilon} \|f\|_{L^2}^2 + \frac{\varepsilon}{2} \int_{\mathbb{R}^d} \gamma_n^2(x)\gamma^2(x) \, |u(x)|^2 \, dx, \qquad (3.29)
\end{aligned}$$

for every $\varepsilon > 0$. Furthermore, thanks to (3.22), computations similar to those in the proof of Lemma 3.15 give

$$\begin{aligned}
&\int_{\mathbb{R}^d} \sum_{i=1}^d \left| b_i(x) D_i u(x) \gamma_n^2(x) \overline{u}(x) \right| dx \\
&\leq B_1 \int_{\mathbb{R}^d} \sum_{i=1}^d E^{1/2} \eta_{1,i}(x)\gamma(x)\gamma_n^2(x) \, |\psi_i(x) D_i u(x) u(x)| \, dx \qquad (3.30) \\
&\leq \frac{B_1}{2} (\int_{\mathbb{R}^d} \gamma_n^2(x)\gamma^2(x) \, |u(x)|^2 \, dx + E \int_{\mathbb{R}^d} \gamma_n^2(x) \sum_{i=1}^d |\psi_i(x) D_i u(x)|^2 \, dx),
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^d} \sum_{i,j=1}^d \left| D_j(\psi_i(x)\psi_j(x)a_{i,j}(x)) D_i u(x) \gamma_n^2(x) \overline{u}(x) \right| dx \\
&\leq B_2 \int_{\mathbb{R}^d} \sum_{i,j=1}^d E^{1/2} \eta_{2,i,j}(x)\gamma(x)\gamma_n^2(x) \, |\psi_i(x) D_i u(x) u(x)| \, dx \qquad (3.31) \\
&\leq \frac{B_2}{2} (\int_{\mathbb{R}^d} \gamma_n^2(x)\gamma^2(x) \, |u(x)|^2 \, dx + E \int_{\mathbb{R}^d} \gamma_n^2(x) \sum_{i=1}^d |\psi_i(x) D_i u(x)|^2 \, dx).
\end{aligned}$$

Finally, since (3.22) clearly implies

$$2 \, |\psi_j(x) D_j \gamma_n(x) a_{i,j}(x)| \leq B_3 E^{1/2} \eta_{3,i,j}(x) \gamma_n(x) \gamma(x),$$

for every $x \in \mathbb{R}^d$, we have also

$$2 \int_{\mathbb{R}^d} \sum_{i,j=1}^d |\psi_i(x)\psi_j(x) a_{i,j}(x) \gamma_n(x) D_j \gamma_n(x) D_i u(x) \overline{u}(x)| \, dx$$

$$\leq B_3 \int_{\mathbb{R}^d} \sum_{i,j=1}^{d} E^{1/2}\eta_{3,i,j}(x)\gamma_n^2(x)\gamma(x) \left|\psi_i(x)D_i u(x)u(x)\right| dx \tag{3.32}$$

$$\leq \frac{B_3}{2}\Big(\int_{\mathbb{R}^d} \gamma_n^2(x)\gamma^2(x) \left|u(x)\right|^2 dx + E\int_{\mathbb{R}^d} \gamma_n^2(x) \sum_{i=1}^{d} \left|\psi_i(x)D_i u(x)\right|^2 dx\Big).$$

Hence, combining (3.28)–(3.32) with (3.27), it follows that

$$\begin{aligned}
&\mathrm{Re}\,\lambda \int_{\mathbb{R}^d} \gamma^2(x)\left|u(x)\right|^2 dx + \int_{\mathbb{R}^d} \gamma_n^2(x)\gamma^2(x)\left|u(x)\right|^2 dx \\
&+ E\int_{\mathbb{R}^d} \gamma_n^2(x)\sum_{i=1}^{d}\left|\psi_i(x)D_i u(x)\right|^2 dx \\
&\leq \frac{1}{2\varepsilon} + \|f\|_{L^2}^2 \frac{\varepsilon}{2}\int_{\mathbb{R}^d} \gamma_n^2(x)\gamma^2(x)\left|u(x)\right|^2 dx \\
&+ \frac{(B_1+B_2+B_3)}{2}\Big(\int_{\mathbb{R}^d} \gamma_n^2(x)\gamma^2(x)\left|u(x)\right|^2 dx \\
&+ E\int_{\mathbb{R}^d} \gamma_n^2(x)\sum_{i=1}^{d}\left|\psi_i(x)D_i u(x)\right|^2 dx\Big),
\end{aligned}$$

and this yields

$$\begin{aligned}
&\mathrm{Re}\,\lambda \int_{\mathbb{R}^d} \gamma^2(x)|u(x)|^2 dx + \frac{2-(B_1+B_2+B_3+\varepsilon)}{2}\int_{\mathbb{R}^d} \gamma_n^2(x)\gamma^2(x)|u(x)|^2 dx \\
&+ \frac{2-(B_1+B_2+B_3)}{2} E\int_{\mathbb{R}^d} \gamma_n^2(x)\sum_{i=1}^{d}\left|\psi_i(x)D_i u(x)\right|^2 dx \leq \frac{1}{2\varepsilon}\|f\|_{L^2}^2.
\end{aligned} \tag{3.33}$$

Now, choosing ε small enough, it is easily seen that each term on the left side of (3.33) is positive. Therefore, letting $n \to \infty$, the monotone convergence theorem allows us to conclude that both $\gamma^2(x)u(x)$ and $\gamma(x)\psi_i(x)D_i u(x)$ belong to L^2, and (3.23) holds true. Then, by virtue of (3.22), we can easily complete the proof. □

As an obvious consequence of the above proposition we have that the sum

$$\sum_{i,j=1}^{d} \psi_i(x)\psi_j(x)a_{i,j}(x)D_{i,j}u(x) \tag{3.34}$$

belongs to L^2.

3.3.2. *Domain Characterization: second derivatives.* Aiming to show that for all $i, j = 1, \ldots, d$ the single summand $\psi_i(x)\psi_j(x)D_{i,j}u(x)$ belongs to L^2, we need to strengthen our hypotheses on the coefficients $\psi_i(x)$'s. Therefore, having in mind our examples, we will assume then the negligibility of the set

$$Z \equiv \left\{x \in \mathbb{R}^d : \psi_i(x) = 0, \text{ for some } i = 1, \ldots, d\right\},$$

of all zeros of the $\psi_i(x)$'s, and the existence of a suitable countable covering of $\mathbb{R}^d - Z$ which allows us to perform a localization procedure. Such a covering will be made by rectangles of the type

$$R(x_0, r\psi) \equiv \left\{x \in \mathbb{R}^d : |x_i - x_i^{(0)}| \leq r|\psi_i(x_0)|, \; i = 1, \ldots, d\right\},$$

for $x_0 \equiv (x_1^{(0)}, \ldots, x_d^{(0)}) \in \mathbb{R}^d - Z$ and $r > 0$. More precisely, we assume

Assumption 3.23. *Under Assumption 3.20, suppose in addition that*

(i) *for every $i = 1, \ldots, d$, the differentiable function $\psi_i(x)$ belongs to H^1_{loc} and the set Z is negligible;*

(ii) *there exist real numbers $r_1 > 0$ and $L > 0$ such that for every $0 < r \leq r_1$ we can find a countable set $N_r \subset \mathbb{R}^d - Z$ such that*

   (a) *the family $\mathcal{F}_1 \equiv \{R(x, r\psi)\}_{x \in N_r}$ is a covering of $\mathbb{R}^d - Z$;*

   (b) *each rectangle of the family $\mathcal{F}_2 \equiv \{R(x, 2r\psi)\}_{x \in N_r}$ does not contain any element of Z and has a nonempty intersection with at most a fixed number n_r of other rectangles of $\mathcal{F}_2$ itself;*

   (c) *we have*

$$\frac{1}{L} \leq \min_{i=1,\ldots,d} \inf_{x \in R(x_0, 2r\psi)} \frac{|\psi_i(x)|}{|\psi_i(x_0)|} \leq \max_{i=1,\ldots,d} \sup_{x \in R(x_0, 2r\psi)} \frac{|\psi_i(x)|}{|\psi_i(x_0)|} \leq L,$$

   for each $x_0 \in N_r$.

Remark 3.24. Notice that

(i) Part (i) of Assumption 3.23 is necessary to get estimates for the second-order derivatives of the solution u of the problem $\mathcal{A}u = f$. In fact, if some of the $\psi_i(x)$'s vanish on some open set, then it is not possible, in general, to get estimates for $D_{i,i}u$ on this set.

(ii) Part (ii) of 3.23 allows us, via a change of variable that will be introduced later, to transform our degenerate global characterization problem into a set of nondegenerate local ones (one for every rectangle $R(x, r\psi)$). We will show that we can obtain local estimates for the $\psi_i(x)\psi_j(x)a_{i,j}(x)D_{i,j}u$'s, which turn out to be independent of the particular rectangle considered. Finally, thanks to the properties of the covering given in part (ii)–(a)/(b), these local estimates can be summed up giving the desired global result.

(iii) We point out that the real difficulty to overcome in this part is to get estimates in some open neighborhood $\mathcal{D}$ of Z and ∞. In fact, one can split our problem into two parts, one related to the neighborhood $\mathcal{D}$ and the other to the compact set $\mathbb{R}^d - \mathcal{D}$. The latter is a very classical problem, and it can be handled with well-known techniques (see [29]). For this reason we will concentrate on getting estimates on the neighborhood $\mathcal{D}$, while skipping this problem on $\mathbb{R}^d - \mathcal{D}$.

Assumption 3.23 is convenient for proving our characterization result (Proposition 3.29). However, it is not easy to check it for given operators. Therefore, before showing the characterization result, we will establish Lemma 3.27, which shows that 3.23 is verified under a more treatable assumption, befitted with our examples coming from Financial Mathematics (see Section 4). More precisely, we are show that the Assumption 3.25 below implies 3.23.

Assumption 3.25. *Under Assumption 3.20, suppose in addition that*

(i) *Part (i) of Assumption 3.23 holds true;*

(ii) *there exist $r_0 > 0$ (small), $R_0 > 0$ (large), and $\alpha > 0$ such that for every $x \in \left\{x \in \mathbb{R}^d : dist(x, Z) < r_0 \text{ or } dist(x, 0) > R_0\right\} \equiv \mathcal{D}\left(r_0, R_0\right)$ and every $i = 1, \ldots, d$ we have*

$$|D_j \psi_i(x)| \leq \alpha;$$

(iii) *for every $i = 1, \ldots, d$ the function $\psi_i(x)$ depends only on the variable x_i.*

Remark 3.26. Notice that

(i) Part (ii) of Assumption 3.25 states essentially that there exists a neighborhood of Z and of ∞ where all derivatives $D_j \psi_i$ are uniformly bounded. This yields a sublinear behavior of the ψ_i's close to Z and to ∞.

(ii) One can easily verify 3.25 when the operator $\mathcal{A}$ belongs to a family of operators arising in the pricing of a contingent claim in the multi-dimensional case (e.g., the d-dimensional Black and Scholes operator for every $d \geq 1$; see Section 4).

Lemma 3.27 below shows that 3.25 implies 3.23 and, actually, something more.

Lemma 3.27. *Under Assumption 3.25, there exist real numbers $r_1 > 0$ and $1 < L < 2$ such that*

(i) *for every $0 < r \le r_1$ and $x_0 \in \{x \in \mathbb{R}^d - Z : dist(x, Z) < r_0/2 \text{ or } dist(x, 0) > 2R_0\} \equiv \mathcal{D}\left(r_0/2, 2R_0\right)$ we have*

$$\frac{1}{L} \le \min_{i=1,\dots,d} \inf_{x \in R(x_0, 10r\psi)} \frac{|\psi_i(x)|}{|\psi_i(x_0)|} \le \max_{i=1,\dots,d} \sup_{x \in R(x_0, 10r\psi)} \frac{|\psi_i(x)|}{|\psi_i(x_0)|} \le L;$$

(ii) *for every $r < r_1$ there exists a countable set $N_r \subset \mathcal{D}\left(r_0/2, 2R_0\right)$ such that*
   (a) *the family of rectangles $\mathcal{F}_1 \equiv \{R(x_0, r\psi)\}_{x_0 \in N_r}$ is a covering of $\mathcal{D}\left(r_0/2, 2R_0\right)$,*
   (b) *every rectangle of the family $\mathcal{F}_2 \equiv \{R(x_0, 2r\psi)\}_{x_0 \in N_r}$ does not contain any point of Z, and has a nonempty intersection with at most a fixed number n_r of other rectangles of $\mathcal{F}_2$ itself.*

Proof.

Proof of (i).

Step (i)–1. We start by observing that, for $r_1 > 0$ sufficiently small, for every $0 < r < r_1$, and for each $x_0 \in \mathcal{D}(r_0/2, 2R_0)$, we have

$$R(x_0, 10r\psi) \subset \mathcal{D}(r_0, R_0).$$

Therefore, for each $x_0 \in \mathcal{D}(r_0/2, 2R_0)$ and every $x \in R(x_0, 10r\psi)$, we have

$$|D_j\psi_i(x)| \le \alpha,$$

for all $i, j = 1, \dots, d$.

Indeed, if $x_0 \in \mathcal{D}(r_0/2, 2R_0)$, we have either $\|x_0\| < 2R_0$, or $\|x_0\| \ge 2R_0$. Let $\|x_0\| < 2R_0$. We have then $d\left(x_0, Z\right) < r_0/2$, and taking $x \in R(x_0, 10r\psi)$ we can write

$$\begin{aligned} d\left(x, Z\right) &\le d\left(x_0, Z\right) + \|x - x_0\| \le r_0/2 + \Big(\sum_{i=1}^{d} |x_i - x_i^{(0)}|^2\Big)^{1/2} \qquad (3.35) \\ &\le r_0/2 + \Big(\sum_{i=1}^{d} |10r\psi_i(x_0)|^2\Big)^{1/2} \le r_0/2 + 10d^{1/2} r_1 \|\psi\|_{L^\infty(B(0,2R_0))}, \end{aligned}$$

where $\|\psi\|_{L^\infty(B(0,2R_0))} \equiv \max_{i=1,\dots,d} \sup_{x \in B(0,2R_0)} \{|\psi_i(x)|\}$. Now, (3.35) clearly implies that choosing r_1 small enough we obtain

$$d\left(x, Z\right) < r_0.$$

Suppose now $x_0 \in \mathcal{D}(r_0/2, 2R_0)$ and $\|x_0\| > 2R_0$. Taking $x \in R(x_0, 10r\psi)$ we have then $\|x\| > R_0$. Indeed,

$$\begin{aligned} \|x\| &\geq \|x_0\| - \|x - x_0\| = \|x_0\| - \Big(\sum_{i=1}^{d} |x_i - x_i^{(0)}|^2 \Big)^{1/2} \\ &\geq \|x_0\| - 10r \Big(\sum_{i=1}^{d} |\psi_i\,(x_0)\,|^2 \Big)^{1/2}. \end{aligned} \tag{3.36}$$

Now, the point $\tilde{x}_0 \equiv \frac{R_0}{\|x_0\|} x_0$ belongs to the boundary of $B\,(0, R_0)$, and the line segment between $\tilde{x}_0$ and x_0 is contained entirely in $\mathcal{D}(r_0, R_0)$. Then, by virtue of the differentiability of ψ_i and of 3.25 (iii), we get, for $i = 1, \ldots, d$,

$$\psi_i\,(x_0) = \psi_i\,(\tilde{x}_0) + D_i \psi_i\,(\xi) \left(x_i^{(0)} - \tilde{x}_i^{(0)} \right),$$

for a suitable $\xi \in [\tilde{x}_0, x_0] \subseteq \mathcal{D}(r_0, R_0)$, and it follows that

$$|\psi_i\,(x_0)\,| \leq |\psi_i\,(\tilde{x}_0)\,| + \alpha \|\tilde{x}_0 - x_0\| \leq \|\psi\|_{L^\infty(B(0,R_0))} + \alpha\,(\|x_0\| - R_0)\,. \tag{3.37}$$

Combining (3.36) and (3.37), we have then

$$\|x\| \geq \|x_0\| - 10r \left(\|\psi\|_{L^\infty(B(0,R_0))} + \alpha\,(\|x_0\| - R_0) \right) \sqrt{d},$$

namely

$$\|x\| \geq \|x_0\| \left(1 - 10r\alpha\sqrt{d} \right) - 10r \left(\|\psi\|_{L^\infty(B(0,R_0))} - \alpha R_0 \right) \sqrt{d}.$$

Finally, wiping off the term αR_0 and recalling that $\|x_0\| > 2R_0$, we have

$$\|x\| \leq 2R_0 \left(1 - 10r\alpha\sqrt{d} \right) - 10r\sqrt{d} \|\psi\|_{L^\infty(B(0,R_0))}.$$

¿¿¿From the latter it easily follows that choosing r_1 small enough we obtain the desired $\|x\| \geq R_0$.

Step (i)–2: Conclusion.

We are now in a position to prove (i) of the lemma. Let $x_0 \in \mathcal{D}(r_0/2, 2R_0)$, and choose $r_1 > 0$ sufficiently small. Thanks to (iii) of Assumption 3.25, for every $i = 1, \ldots, d$ and every $x \in R(x_0, 10r\psi)$, we can write

$$\psi_i(x) = \psi_i(x_0) + D_i \psi_i(\xi)(x_i - x_i^{(0)}), \tag{3.38}$$

for a suitable $\xi \in [x, x_0]$. It follows that

$$|\psi_i(x)| \leq |\psi_i(x_0)| + 10\alpha r |\psi_i(x_0)|,$$

which implies

$$\frac{|\psi_i(x)|}{|\psi_i(x_0)|} \leq 1 + 10\alpha r. \tag{3.39}$$

Similarly, from (3.38), we have also

$$|\psi_i(x)| \geq |\psi_i(x_0)| - |D_i\psi_i(\xi)||x_i - x_i^{(0)}| \geq |\psi_i(x_0)| - 10\alpha r|\psi_i(x_0)|,$$

and the latter yields

$$\frac{|\psi_i(x)|}{|\psi_i(x_0)|} \geq 1 - 10\alpha r. \tag{3.40}$$

Therefore the claim (i) follows by choosing again r_1 small enough.

Proof of (ii). Now we prove (ii). This proof is more technical and adapts some arguments used in the proof of the so-called Besicovitch lemma (see [8, 9, 10]). However, we remark that we cannot apply here the Besicovitch lemma straightforwardly, since

- the set $\mathbb{R}^d - Z$ to cover is not bounded;
- the family of sets $\mathcal{F} = \{R(x, r\psi)\}_{x\in\mathbb{R}^d - Z}$ is not uniformly regular in the sense used by Besicovitch (see, e.g., [10]).

We divide the proof into three steps:

Step (ii)–1: Extracting the "good" subfamily.

To extract the desired covering of $\mathcal{D}(r_0/2, 2R_0)$ we use a recursive argument. First, we consider the family $\mathcal{F} \equiv \{R(x, r\psi)\}_{x\in\mathcal{D}(r_0/2,2R_0)}$ of the rectangles having centers in all points of $\mathcal{D}(r_0/2, 2R_0)$, and, by slicing each side into three equal parts, we divide every rectangle $R(x, r\psi)$ of $\mathcal{F}$ into 3^d rectangles, each of which has the i-th side of length $\frac{1}{3}r\,|\psi_i(x)|$ for $i = 1, ..., d$. We write then $R(x, r\psi/3)$ for the internal rectangle. Next, among all rectangles of $\mathcal{F}$ we choose a subfamily $\mathcal{R}_0$ such that

(a) the smallest semiaxis is $\geq r \cdot 2^0 = r$;
(b) the internal rectangles $R(x, r\psi/3)$ are pairwise disjoint;
(c) the family is maximal with respect to the inclusion, namely, there is no subfamily of $\mathcal{F}$ enjoying (a) and (b) above and containing $\mathcal{R}_0$ strictly).

Clearly, if all $\psi(x)$'s in the set $\mathcal{D}(r_0/2, 2R_0)$ take values < 1 such a family is empty. Anycase $\mathcal{R}_0$ is at most countable, because of the two properties (a) and (b) above. If $\mathcal{R}_0$ covers $\mathcal{D}(r_0/2, 2R_0)$ we stop here. If not we choose another subfamily $\mathcal{R}_1$ of $\mathcal{F}$ satisfying

(1) the centers belong to $\mathcal{D}(r_0/2, 2R_0) - \cup\mathcal{R}_0$, where $\cup\mathcal{R}_0$ stands for the union of all rectangles belonging to $\mathcal{R}_0$;
(2) the smallest semiaxis is $\geq r \cdot 2^{-1} = r/2$;
(3) the internal rectangles $R(x, r\psi/3)$ are pairwise disjoint, and have also empty intersection with all internal rectangles of the family $\mathcal{R}_0$;

(4) the family is maximal with respect to inclusion; namely, there is no subfamily of $\mathcal{F}$ enjoying the properties above and containing $\mathcal{R}_1$ strictly).

As above, if $\mathcal{R}_0 \cup \mathcal{R}_1$ covers $\mathcal{D}\,(r_0/2, 2R_0)$ we stop here. If not we iterate the procedure. In any case we generate a sequence $\{\mathcal{R}_n\}_{n\geq 0}$ of subfamilies of $\mathcal{F}$ such that every $\mathcal{R}_n$ satisfies

(1) the centers belong to $\mathcal{D}\,(r_0/2, 2R_0) - \cup_{k=0}^{n-1}\,(\cup\mathcal{R}_k)$, where $\cup\mathcal{R}_k$ stands for the union of all sets belonging to $\mathcal{R}_k$);
(2) the smallest semiaxis is $\geq r\cdot 2^{-n}$;
(3) the internal rectangles $R(x, r\psi/3)$ are pairwise disjoint and have also empty intersection with all internal rectangles of the families $\mathcal{R}_k$ for $k = 1, \ldots, n-1$;
(4) the family is maximal with respect to inclusion.

If this procedure stops after a finite number of steps we have found a countable covering. If not, we consider the generated countable family $\cup_{n=0}^{\infty}\mathcal{R}_n$ and we prove below that it is a covering for $\mathcal{D}\,(r_0/2, 2R_0)$.

Step (ii)–2: The "good" subfamily is a covering.

We argue by contradiction. Let $x_0 \in \mathcal{D}\,(r_0/2, 2R_0) - \cup_{n=0}^{\infty}\,(\cup\mathcal{R}_n)$. Then consider the rectangle $R(x_0, r\psi/3)$, and let n be the first integer such that $\min_{i=1,\ldots,d}\psi_i\,(x_0) \geq 2^{-n}$. By the maximality of the families $\mathcal{R}_k$ for $0 \leq k \leq n$, there exist $k \in \{0, \ldots, n\}$ and $x_k \in \mathcal{D}\,(r_0/2, 2R_0)$ such that $R(x_k, r\psi) \in \mathcal{R}_k$ and $R(x_0, r\psi/3) \cap R(x_k, r\psi/3) \neq \emptyset$. This implies that we have

$$\left|x_i^{(0)} - x_i^{(k)}\right| \leq \frac{r}{3}\left(|\psi_i\,(x_0)\,| + |\psi_i\,(x_k)\,|\right). \tag{3.41}$$

for every $i = 1, \ldots, d$. On the other hand, since $x_0 \notin R(x_k, r\psi)$, there exists at least an index $i_0 \in \{1, \ldots, d\}$ such that

$$|x_{i_0}^{(0)} - x_{i_0}^{(k)}| > r|\psi_{i_0}\,(x_k)\,|. \tag{3.42}$$

Combining (3.41) and (3.42), we have then

$$|\psi_{i_0}\,(x_k)\,| < \frac{1}{3}\left(|\psi_{i_0}\,(x_k)\,| + |\psi_{i_0}\,(x_0)\,|\right),$$

or equivalently

$$\frac{|\psi_{i_0}\,(x_0)\,|}{|\psi_{i_0}\,(x_k)\,|} > 2. \tag{3.43}$$

But this is impossible. In fact, by Assumption (3.25)–(iii), we have also

$$|\psi_{i_0}\,(x_0) - \psi_{i_0}\,(x_k)\,| = |D_{i_0}\psi_{i_0}\,(\xi)\,||x_{i_0}^{(0)} - x_{i_0}^{(k)}|, \tag{3.44}$$

for suitable $\xi \in [x_0, x_k]$. Hence, combining (3.41) and (3.43), and taking into account Assumption 3.25–(ii), we obtain

$$||\psi_{i_0}(x_0)| - |\psi_{i_0}(x_k)|| \leq |D_{i_0}\psi_{i_0}(x_k)||x_{i_0}^{(0)} - x_{i_0}^{(k)}| \leq \alpha\frac{r}{3}(|\psi_{i_0}(x_0)| + |\psi_{i_0}(x_k)|),$$

or equivalently

$$\Big|\frac{|\psi_{i_0}(x_0)|}{|\psi_{i_0}(x_k)|} - 1\Big| \leq \alpha\frac{r}{3}\Big(\frac{|\psi_{i_0}(x_0)|}{|\psi_{i_0}(x_k)|} + 1\Big).$$

The latter yields

$$\frac{|\psi_{i_0}(x_0)|}{|\psi_{i_0}(x_k)|} \leq \frac{1+\frac{\alpha}{3}r}{1-\frac{\alpha}{3}r},$$

and choosing r_1 small enough, for every $0 < r < r_1$, we can make

$$\frac{1+\frac{\alpha}{3}r}{1-\frac{\alpha}{3}r} < 2,$$

which contradicts (3.43).

Step (ii)–3. The "good"subfamily enjoys the finite intersection property. We write $\mathcal{F}_1$ for the countable covering of rectangles defined above, and we write N_r for the set of all centers of the rectangles of $\mathcal{F}_1$. Hence, we are left to prove that the family $\mathcal{F}_2 \equiv \{R(x, 2r\psi)\}_{x\in N_r}$ enjoys the announced properties.

Notice first that the first two steps of our proof assure that the family $\mathcal{F}_2$ covers $\mathcal{D}(r_0, R_0)$ and has empty intersection with Z. Therefore, we need only to show that every rectangle of $\mathcal{F}_2$ has a nonempty intersection with at most a fixed number n_r of the others. To this end, we fix any $x_0 \in N_r$ and we claim that

- if $x \in N_r$ is such that $R(x, 2r\psi) \cap R(x_0, 2r\psi) \neq \emptyset$, then $x \in R(x_0, 10r\psi)$;
- there are at most 60^d points in $N_r \cap R(x_0, 10r\psi)$.

To prove the first claim, take $x \in N_r$ such that $R(x, 2r\psi)\cap R(x_0, 2r\psi) \neq \emptyset$. Then we have

$$|x_i - x_i^{(0)}| \leq 2r\left(|\psi_i(x)| + |\psi_i(x_0)|\right). \tag{3.45}$$

for every $i = 1, \ldots, d$. In addition, by virtue of the part (i) of the lemma, for any $y \in R(x, 2r\psi) \cap R(x_0, 2r\psi)$, we have

$$\frac{|\psi_i(y)|}{|\psi_i(x_0)|} < 2 \quad \text{and} \quad \frac{1}{2} < \frac{|\psi_i(y)|}{|\psi_i(x)|},$$

so that

$$|\psi_i(x)| < 2|\psi_i(y)| < 4|\psi_i(x_0)|. \tag{3.46}$$

Hence, combining (3.45) and (3.46), the desired claim immediately follows.

To prove the second claim, we recall that, by construction, we have

$$|x_i - y_i| > \frac{r}{3}\left(|\psi_i(x)| + |\psi_i(y)|\right).$$

for all $x, y \in N_r$ and every $i = 1, \ldots, d$. On the other hand, we have also

$$\frac{|\psi_i(z)|}{|\psi_i\left(x_0\right)|} > \frac{1}{2}.$$

for every $z \in R(x_0, 10r\psi)$. Thus, we can write

$$|x_i - y_i| > \frac{r}{3}|\psi_i\left(x_0\right)|,$$

for all $x, y \in N_r \cap R(x_0, 10r\psi)$ and every $i = 1, \ldots, d$. The latter implies, by suitably slicing the rectangle $R(x_0, 10r\psi)$, that the points $x \in N_r \cap R(x_0, 10r\psi)$ can be 60^d at most. □

The following lemma is also needed.

Lemma 3.28. *Suppose that Assumption 3.23 holds true, and let f and g be measurable functions satisfying*

$$\int_{B(x,r\psi)} |g(y)|^2\, dy \leq \int_{R(x,2r\psi)} |f(y)|^2\, dy$$

for every $x \in N_r$. Then g belongs to L^2 whenever f does.

Proof. Condition 3.27 (ii) clearly implies that for every $x \in N_r$ the corresponding rectangle $R\left(x, 2r\psi\right)$ of $\mathcal{F}_2$ can be covered by not more than n_r rectangles of the family $\mathcal{F}_1$, and that each rectangle of $\mathcal{F}_1$ can appear in not more than n_r coverings of different rectangles of the family $\mathcal{F}_2$. Therefore we can write

$$\sum_{x\in N_r} \int_{R(x,2r\psi)} |f\left(y\right)|^2\, dy \leq n_r^2 \sum_{x\in N_r} \int_{R(x,r\psi)} |f\left(y\right)|^2\, dy. \tag{3.47}$$

Moreover, since Condition 3.27 (ii) implies also that each rectangle of the family $\mathcal{F}_1$ has nonempty intersection with not more than n_r different rectangles of $\mathcal{F}_1$ itself, we have

$$\sum_{x\in N_r} \int_{R(x,r\psi)} |f\left(y\right)|^2\, dy \leq n_r \int_{\mathbb{R}^d} |f\left(y\right)|^2\, dy. \tag{3.48}$$

Combining (3.47) and (3.48), we obtain

$$\int_{\mathbb{R}^d} |g\left(y\right)|^2\, dy \leq \sum_{x\in N_r} \int_{R(x,r\psi)} |g\left(y\right)|^2\, dy \leq \sum_{x\in N_r} \int_{B(x,2r\psi)} |f\left(y\right)|^2\, dy$$

$$\leq n_r^3 \int_{\mathbb{R}^d} |f(y)|^2 \, dy,$$

which yields the lemma. □

We are now in a position to carry on our localization procedure, to obtain a full characterization of the domain.

For each $x_0 \equiv (x_1^{(0)}, \ldots, x_d^{(0)}) \in \mathbb{R}^d - Z$, consider the change of variables $T_{x_0,\psi} : \mathbb{R}^d \to \mathbb{R}^d$ given by

$$T_{x_0,\psi}(x) \stackrel{def}{=} \left((x_1 - x_1^{(0)})/|\psi_1(x_0)|, \ldots, (x_d - x_d^{(0)})/|\psi_d(x_0)|\right) \equiv (\widetilde{x}_1, \ldots, \widetilde{x}_d), \equiv \widetilde{x}$$

with inverse

$$T_{x_0,\psi}^{-1}(\widetilde{x}) = (x_1^{(0)} + |\psi_1(x_0)| \, \widetilde{x}_1, \ldots, x_d^{(0)} + |\psi_d(x_0)| \, \widetilde{x}_d) = (x_1, \ldots, x_d) = x.$$

Clearly $T_{x_0,\psi}(R(x_0, r\psi)) = R(0,r)$ and $T_{x_0,\psi}^{-1}(R(0,r)) = R(x_0, r\psi)$, where, for every $r > 0$, we write $R(0,r)$ for the d-dimensional cube of $\mathbb{R}^d$ centered at 0 with semiaxes of length r. Exploiting Lemmas 3.28 and 3.27, we obtain

Proposition 3.29. *Under Assumption 3.23, the functions*

$$\psi_i(x)\psi_j(x)D_{i,j}u(x)$$

belong to L^2 for all $i, j = 1, \ldots, d$. More precisely, we have $u \in H^2_{(\gamma^2, \gamma\psi, \psi^2)}$ and the estimate

$$|\lambda| \|u\|_{L^2} + |\lambda|^{1/2} \|u\|_{H^1_{(\gamma,\psi)}} + \|u\|_{H^2_{(\gamma^2,\gamma\psi,\psi^2)}} \leq K \|f\|_{L^2} \tag{3.49}$$

holds true for a suitable $K > 0$.

Proof. Writing

$$p(x) \stackrel{def}{=} \sum_{i,j=1}^{d} \psi_i(x)\psi_j(x)a_{i,j}(x)D_{i,j}u(x), \tag{3.50}$$

we know that p belongs to L^2, and it is easily seen that, for each $x_0 \in \mathbb{R}^d - Z$, the change of variables

$$x \stackrel{def}{=} T_{x_0,\psi}^{-1}(\widetilde{x})$$

allows us to rewrite (3.50) in the form

$$\widetilde{p}(\widetilde{x}) \stackrel{def}{=} p\left(T_{x_0,\psi}^{-1}(\widetilde{x})\right) = \sum_{i,j=1}^{d} \frac{\widetilde{\psi}_i(\widetilde{x})\widetilde{\psi}_j(\widetilde{x})}{|\psi_i(x_0)| \, |\psi_j(x_0)|} \widetilde{a}_{i,j}(\widetilde{x}) \widetilde{D}_{i,j} \widetilde{u}(\widetilde{x}), \tag{3.51}$$

with obvious meaning of the "tilda"-labeled functions, and with $\widetilde{p}$ still belonging to L^2. Now, given any smooth cut-off function on $\mathbb{R}^d$ such that

$$\theta_r(\widetilde{x}) = \begin{cases} 1 & \text{if } \widetilde{x} \in R(0,r) \\ 0 & \text{if } \widetilde{x} \notin R(0,2r) \end{cases},$$

we can consider the function

$$v(\widetilde{x}) \stackrel{def}{=} \theta_r(\widetilde{x})\widetilde{u}(\widetilde{x}).$$

Clearly $v \in L^2$ and, as a distribution, it satisfies

$$\sum_{i,j=1}^{d} \frac{\widetilde{\psi}_i(x)\widetilde{\psi}_j(x)}{|\psi_i(x_0)|\,|\psi_j(x_0)|}\widetilde{a}_{i,j}(x)D_{i,j}v = \theta_r(x)\widetilde{p} \tag{3.52}$$

$$+ \sum_{i,j=1}^{d} \frac{\widetilde{\psi}_i(x)\widetilde{\psi}_j(x)}{|\psi_i(x_0)|\,|\psi_j(x_0)|}\widetilde{a}_{i,j}(x)\left(\widetilde{u}D_{i,j}\theta_r(x) + D_i\theta_r(x)D_j\widetilde{u} + D_j\theta_r(x)D_i\widetilde{u}\right),$$ [1]

whose right hand side, say $\widetilde{q}$, fulfills $\widetilde{q} \in L^2(R(0,2r))$, and $\widetilde{q} = 0$ on $\partial R(0,2r)$. On the other hand, under our hypotheses, the second-order differential operator

$$v \to \sum_{i,j=1}^{d} \frac{\widetilde{\psi}_i(x)\widetilde{\psi}_j(x)}{|\psi_i(x_0)|\,|\psi_j(x_0)|}\widetilde{a}_{i,j}(x)D_{i,j}v$$

is strongly elliptic in $R(0,2r)$, and, thanks to well known regularity results (see [26, Theorem 17.2, p. 67], [29, 8.3, p. 173] and [19]), it follows that $v \in H^2(R(0,2r))$. In addition, using Assumption 3.10, a similar argument to that in the proof of [29, Theorem 8.8, p.173] assures us that there exists a suitable constant $K > 0$, independent of the particular rectangle $R(x, 2r\psi)$ considered, such that the estimate

$$\|v\|_{H^2(R(0,r))} \le K\Big(\|\widetilde{q}\|_{L^2(R(0,2r))} + \sum_{i=1}^{d} \frac{1}{|\psi_i(x_0)|}\|\widetilde{\gamma}\widetilde{\psi}_i D_i v\|_{L^2(R(0,2r))}\Big) \tag{3.53}$$

holds true.

---

[1] Here we dropped the cumbersome "tilda"label for the independent variables since no confusion can arise.

Indeed, to prove (3.53), observe first that, for every $\varphi \in C_c^\infty(R(0,2r))$, we have

$$\int_{R(0,2r)} \widetilde{q}(x)\varphi(x)dx = \int_{R(0,2r)} \sum_{i,j=1}^{d} \frac{\widetilde{\psi}_i(x)\widetilde{\psi}_j(x)}{|\psi_i(x_0)|\,|\psi_j(x_0)|}\widetilde{a}_{i,j}(x)D_{i,j}v(x)\varphi(x)dx. \tag{3.54}$$

In particular, choosing any $h \in \mathbb{R}$ such that $0 < |2h| < \operatorname{dist}(\operatorname{supp}(\varphi), \partial R(0,2r))$, and replacing φ in (3.54) with the difference quotient

$$\triangle^h \varphi \overset{def}{=} \frac{\varphi(x + he_k) - \varphi(x)}{h},$$

for some $k = 1, \ldots, d$, we obtain

$$\begin{aligned}
&\int_{R(0,2r)} \sum_{i,j=1}^{d} \frac{\widetilde{\psi}_i(x)\widetilde{\psi}_j(x)}{|\psi_i(x_0)|\,|\psi_j(x_0)|}\widetilde{a}_{i,j}(x)D_{i,j}v(x)\triangle^{-h}\varphi(x)dx \\
&= -\int_{R(0,2r)} \sum_{i,j=1}^{d} D_j\Big(\frac{\widetilde{\psi}_i(x)\widetilde{\psi}_j(x)}{|\psi_i(x_0)|\,|\psi_j(x_0)|}\widetilde{a}_{i,j}(x)\Big)D_i v(x)\triangle^{-h}\varphi(x)dx \\
&+ \int_{R(0,2r)} \sum_{i,j=1}^{d} \triangle^h\Big(\frac{\widetilde{\psi}_i(x)\widetilde{\psi}_j(x)}{|\psi_i(x_0)|\,|\psi_j(x_0)|}\widetilde{a}_{i,j}(x)D_i v(x)\Big)D_j\varphi(x)dx.
\end{aligned} \tag{3.55}$$

On the other hand, we have

$$\begin{aligned}
&\triangle^h\Big(\frac{\widetilde{\psi}_i(x)\widetilde{\psi}_j(x)}{|\psi_i(x_0)|\,|\psi_j(x_0)|}\widetilde{a}_{i,j}(x)D_j v(x)\Big) \\
&= \frac{\widetilde{\psi}_i(x+he_k)\widetilde{\psi}_j(x+he_k)}{|\psi_i(x_0)|\,|\psi_j(x_0)|}\widetilde{a}_{i,j}(x+he_k)D_j\triangle^h v(x) \\
&+ \triangle^h\Big(\frac{\widetilde{\psi}_i(x)\widetilde{\psi}_j(x)}{|\psi_i(x_0)|\,|\psi_j(x_0)|}\widetilde{a}_{i,j}(x)\Big)D_j v(x).
\end{aligned} \tag{3.56}$$

Therefore, combining (3.54), (3.55) and (3.56), it follows that

$$\begin{aligned}
&\int_{R(0,2r)} \sum_{i,j=1}^{d} \frac{\widetilde{\psi}_i(x+he_k)\widetilde{\psi}_j(x+he_k)}{|\psi_i(x_0)|\,|\psi_j(x_0)|}\widetilde{a}_{i,j}(x+he_k)D_i\triangle^h v(x)D_j\varphi(x)dx \\
&= \int_{R(0,2r)} \widetilde{q}(x)\triangle^{-h}\varphi(x)dx \\
&+ \int_{R(0,2r)} \sum_{i,j=1}^{d} D_j\Big(\frac{\widetilde{\psi}_i(x)\widetilde{\psi}_j(x)}{|\psi_i(x_0)|\,|\psi_j(x_0)|}\widetilde{a}_{i,j}(x)\Big)D_i v(x)\triangle^{-h}\varphi(x)dx
\end{aligned}$$

$$-\int_{R(0,2r)}\sum_{i,j=1}^{d}\triangle^{h}\Big(\frac{\widetilde{\psi}_i(x)\widetilde{\psi}_j(x)}{|\psi_i(x_0)|\,|\psi_j(x_0)|}\widetilde{a}_{i,j}(x)\Big)D_iv(x)D_j\varphi(x)dx. \tag{3.57}$$

Now, writing $\|D\varphi\|_{L^2(R(0,2r))}$ as a shorthand for $\sum_{i=1}^{d}\int_{R(0,2r)}|D_i\varphi(x)|^2\,dx$, and applying [29, Lemma 7.23, p. 161], we have

$$\int_{R(0,2r)}\Big|\widetilde{q}(x)\triangle^{-h}\varphi(x)\Big|\,dx\le\|\widetilde{q}\|_{L^2(R(0,2r))}\|D\varphi\|_{L^2(R(0,2r))}. \tag{3.58}$$

Moreover, since (3.14) of Assumption 3.10 implies

$$\Big|D_j(\widetilde{\psi}_i(x)\widetilde{\psi}_j(x)\widetilde{a}_{i,j}(x))\Big|\le B_2E^{1/2}\,|\psi_j(x_0)|\,|\widetilde{\psi}_i(x)|\widetilde{\gamma}(x),$$

we can write

$$\begin{aligned}&\int_{R(0,2r)}\sum_{i,j=1}^{d}\Big|D_j\Big(\frac{\widetilde{\psi}_i(x)\widetilde{\psi}_j(x)}{|\psi_i(x_0)|\,|\psi_j(x_0)|}\widetilde{a}_{i,j}(x)\Big)D_iv(x)\triangle^{-h}\varphi(x)\Big|dx\\&\le B_2E^{1/2}d\int_{R(0,2r)}\sum_{i=1}^{d}\frac{|\widetilde{\psi}_i(x)|\widetilde{\gamma}(x)}{|\psi_i(x_0)|}D_iv(x)\triangle^{-h}\varphi(x)dx\\&\le B_2E^{1/2}d\|D\varphi\|_{L^2(R(0,2r))}\sum_{i=1}^{d}\frac{1}{|\psi_i(x_0)|}\|\widetilde{\psi}_i\widetilde{\gamma}D_iv\|_{L^2(R(0,2r))},\end{aligned} \tag{3.59}$$

and similarly

$$\begin{aligned}&\int_{R(0,2r)}\sum_{i,j=1}^{d}\triangle^{h}\Big(\frac{\widetilde{\psi}_i(x)\widetilde{\psi}_j(x)}{|\psi_i(x_0)|\,|\psi_j(x_0)|}\widetilde{a}_{i,j}(x)\Big)D_iv(x)D_j\varphi(x)dx\\&\le B_2E^{1/2}d\|D\varphi\|_{L^2(R(0,2r))}\sum_{i=1}^{d}\frac{1}{|\psi_i(x_0)|}\|\widetilde{\psi}_i\widetilde{\gamma}D_iv\|_{L^2(R(0,2r)}.\end{aligned} \tag{3.60}$$

Hence, combining (3.57) with (3.58), (3.59) and (3.60), we obtain

$$\begin{aligned}&\int_{R(0,2r)}\sum_{i,j=1}^{d}\frac{\widetilde{\psi}_i(x+he_k)\widetilde{\psi}_j(x+he_k)}{|\psi_i(x_0)|\,|\psi_j(x_0)|}\widetilde{a}_{i,j}(x+he_k)D_i\triangle^{h}v(x)D_j\varphi(x)dx\\&\le\|D\varphi\|_{L^2(R(0,2r))}\Big(\|\widetilde{q}\|_{L^2(R(0,2r))}+2B_2E^{\frac{1}{2}}d\sum_{i=1}^{d}\frac{1}{|\psi_i(x_0)|}\|\widetilde{\psi}_i\widetilde{\gamma}D_iv\|_{L^2(R(0,2r))}\Big).\end{aligned} \tag{3.61}$$

¿¿¿From the ellipticity condition, we obtain that

$$E\int_{R(0,2r)}\sum_{i=1}^{d}|D_i\varphi(x)|^2\,dx \le \int_{R(0,2r)}\sum_{i,j=1}^{d}\widetilde{a}_{i,j}(x+he_k)D_i\varphi(x)D_j\varphi(x)dx. \tag{3.62}$$

On the other hand, since v has compact support, by the usual density argument, we can replace $\varphi(x)$ with $\triangle^h v$ in (3.62). Therefore, combining (3.61) with (3.62), and taking into account (i) of Lemma 3.27, we can write

$$\begin{aligned}&E\int_{R(0,2r)}\sum_{i=1}^{d}\left|D_i\triangle^h v(x)\right|^2 dx \le L^2\|D\triangle^h v\|_{L^2(R(0,2r))}\\ &\times\left(\|\widetilde{q}\|_{L^2(R(0,2r))}+2B_2Ed\sum_{i=1}^{d}\frac{1}{|\psi_i(x_0)|}\|\widetilde{\psi}_i\widetilde{\gamma}D_iv\|_{L^2(R(0,2r))}\right).\end{aligned} \tag{3.63}$$

Finally, thanks to Young's inequality, the desired (3.53) follows.

Now, since

$$\left|\det J(T^{-1}_{x_0,\psi})(x)\right| = \prod_{i=1}^{d}|\psi_i(x_0)|\,,$$

changing the variables back, and recalling the definition of $\widetilde{q}$, we have

$$\begin{aligned}&\sum_{i,j=1}^{d}\|\psi_i(x)\psi_j(x)D_{i,j}u(x)\|_{L^2(B(x_0,r\psi))}\\ &=\prod_{i=1}^{d}|\psi_i(x_0)|^{1/2}\sum_{i,j=1}^{d}\left\|\frac{\widetilde{\psi}_i(x)\widetilde{\psi}_j(x)}{|\psi_i(x_0)|\,|\psi_j(x_0)|}D_{i,j}\widetilde{u}(x)\right\|_{L^2(R(0,r))}\\ &\le L^2\prod_{i=1}^{d}|\psi_i(x_0)|^{1/2}\sum_{i,j=1}^{d}\|D_{i,j}\widetilde{u}(x)\|_{L^2(R(0,r))}\\ &\le KL^2\prod_{i=1}^{d}|\psi_i(x_0)|^{1/2}\left(\|\widetilde{q}\|_{L^2(R(0,2r))}+\sum_{i=1}^{d}\left\|\widetilde{\gamma}\frac{\widetilde{\psi}_i}{|\psi_i(x_0)|}D_i\widetilde{u}\right\|_{L^2(R(0,2r))}\right)\\ &\le KL^2\left(\|g\|_{L^2(B(x_0,2r\psi))}+\sum_{i=1}^{d}\|\gamma\psi_iD_iu\|_{L^2(B(x_0,2r\psi))}\right),\end{aligned} \tag{3.64}$$

where $g(x)\stackrel{def}{=}\widetilde{q}\left((T_{x_0,\psi}(x)\right)$ for every $x\in B(x_0,2r\psi)$. Moreover, since

$$\max_{i=1,\dots,d}\sup_{x\in B(x_0,2r\psi)}|D_i\theta_r\left(T_{x_0,\psi}(x)\right)| \le \frac{1}{r}\|D\theta\|_\infty,$$

and

$$\max_{i,j=1,\dots,d}\sup_{x\in B(x_0,2r\psi)}|D_{i,j}\theta_r\left(T_{x_0,\psi}(x)\right)|\le\frac{1}{r^2}\|D^2\theta\|_\infty,$$

it follows that

$$\begin{aligned}
&\Big|\sum_{i,j=1}^{d}\frac{\widetilde{\psi}_i(\widetilde{x})\widetilde{\psi}_j(\widetilde{x})}{|\psi_i(x_0)|\,|\psi_j(x_0)|}\widetilde{a}_{i,j}(\widetilde{x})\widetilde{u}(\widetilde{x})D_{i,j}\theta_r(\widetilde{x})\Big|\\
&=\Big|\sum_{i,j=1}^{d}\frac{\psi_i(x)\psi_j(x)}{|\psi_i(x_0)|\,|\psi_j(x_0)|}a_{i,j}(x)u(x)D_{i,j}\theta_r\left(T_{x_0,\psi}(x)\right)\Big| \qquad (3.65)\\
&\le\sum_{i,j=1}^{d}\Big|\frac{\psi_i(x)\psi_j(x)}{\psi_i(x_0)\psi_j(x_0)}\Big|\|a_{ij}\|_\infty\,|u(x)|\,\frac{1}{r^2}\|D^2\theta\|_\infty\le\frac{1}{r^2}d^2AL^2\|D^2\theta\|_\infty\,|u(x)|\,,
\end{aligned}$$

and

$$\begin{aligned}
&\Big|\sum_{i,j=1}^{d}\frac{\widetilde{\psi}_i(\widetilde{x})\widetilde{\psi}_j(\widetilde{x})}{|\psi_i(x_0)|\,|\psi_j(x_0)|}\widetilde{a}_{i,j}(\widetilde{x})D_i\theta_r(\widetilde{x})D_j\widetilde{u}(\widetilde{x})\Big|\\
&=\Big|\sum_{i,j=1}^{d}\frac{\psi_i(x)\psi_j(x)}{|\psi_i(x_0)|}a_{i,j}(x)D_i\theta_r\left(T_{x_0,\psi}(x)\right)D_ju(x)\Big|\\
&\le\sum_{i,j=1}^{d}\left|\frac{\psi_i(x)\psi_j(x)}{\psi_i(x_0)}\right|\|a_{ij}\|_\infty\frac{1}{r}\|D\theta\|_\infty\,|D_ju(x)|\\
&\le\frac{1}{r}dAL\|D\theta\|_\infty\sum_{j=1}^{d}|\psi_j(x)|\,|D_ju(x)|\,. \qquad (3.66)
\end{aligned}$$

Combining (3.52) with (3.65) and (3.66), we can write

$$\begin{aligned}
\|h\|_{L^2(B(x_0,2r\psi))}\le\|g\|_{L^2(B(x_0,2r\psi))}&+\frac{1}{r^2}d^2AL^2\|D^2\theta\|_\infty\|u\|_{L^2(B(x_0,2r\psi))}\\
&+\frac{2}{r}dAL\|D\theta\|_\infty\sum_{i=1}^{d}\|\psi_iD_iu\|_{L^2(B(x_0,2r\psi))}, \qquad (3.67)
\end{aligned}$$

and, by virtue of (3.64) and (3.67), it follows that

$$\sum_{i,j=1}^{d}\|\psi_i(x)\psi_j(x)D_{i,j}u(x)\|_{L^2(B(x_0,r\psi))}$$

$$\leq KL^2\Big((\|g\|_{L^2(B(x_0,2r\psi))} + \frac{1}{r^2}d^2AL^2\|D^2\theta\|_\infty\|u\|_{L^2(B(x_0,2r\psi))}) \tag{3.68}$$
$$+ \frac{2}{r}dAL\|D\theta\|_\infty \sum_{i=1}^d \|\psi_i D_i u\|_{L^2(B(x_0,2r\psi))} + \sum_{i=1}^d \|\gamma\psi_i D_i u\|_{L^2(B(x_0,2r\psi))}\Big).$$

Summing up to the covering, thanks to Lemma 3.28 and (3.23), we can conclude that $u \in H^2_{(\gamma^2,\gamma\psi,\psi^2)}$.

With regard to the proof of (3.49), notice that, rewriting (3.18) as

$$\sum_{i,j=1}^d \psi_i(x)\psi_j(x)a_{i,j}(x)D_{i,j}u = \left(\lambda+\gamma^2(x)\right)u - \sum_{i=1}^d b_i(x)D_iu - f(x),$$

we obtain

$$p(x) = \left(\lambda+\gamma^2(x)\right)u - \sum_{i=1}^d b_i(x)D_iu - f(x).$$

Hence

$$\|p\|_{L^2} \leq |\lambda|\|u\|_{L^2} + \|\gamma^2 u\|_{L^2} + \sum_{i=1}^d \|b_iD_iu\|_{L^2} + \|f\|_{L^2}.$$

Finally, combining the latter with the global estimate which arises from (3.68) by summing on the covering, and taking into account (3.24), we can complete the proof. □

3.4. **Generation in $L^2_\xi(\mathbb{R}^d)$.** In this subsection we consider the problem of the generation of analytic semigroups on a weighted Sobolev space for a suitable modification of the differential operator considered up to now. For this task we choose a weight function $\xi \in H^n_{\mathrm{loc}}$, and we invoke the related weighted Sobolev spaces L^2_ξ, $H^1_{\xi,(\gamma,\psi)}$ and $H^2_{\xi,(\gamma^2,\gamma\psi,\psi^2)}$ defined in Section 2. Our result is

Theorem 3.30. *Assume that Assumption 3.20 still holds true when replacing the first-order term of the operator $\mathcal{A}$ with*

$$\sum_{i=1}^d b_iD_i + \sum_{i,j=1}^d \psi_i\psi_j a_{i,j}\Big(\frac{D_i\xi}{\xi}D_j + \frac{D_j\xi}{\xi}D_i\Big)$$

and the zero-order term with

$$-\gamma^2 + \sum_{i,j=1}^d \psi_i\psi_j a_{i,j}\Big(\frac{D_{i,j}\xi}{\xi} + 2\frac{D_i\xi D_j\xi}{\xi^2}\Big) + \sum_{i=1}^d b_i\frac{D_i\xi}{\xi};$$

then the operator $\mathcal{A}$ has a realization $\mathcal{A}_{2,\xi} : D(\mathcal{A}_{2,\xi}) \to L^2_\xi$ which generates an analytic semigroup on L^2_ξ. Moreover, for each $\lambda \in \mathbb{C}$ such that $\mathrm{Re}\,\lambda > 0$, the resolvent equation $\lambda u - \mathcal{A}_{2,\xi} u = f$ has, for every $f \in L^2$, a unique solution $u \in D(\mathcal{A}_{2,\xi})$, which satisfies the estimate

$$|\lambda|\|u\|_{L^2_\xi} + |\lambda|^{1/2}\|u\|_{H^{1,2}_{\xi,(\gamma,\psi)}} + \|u\|_{H^{2,2}_{\xi,(\gamma^2,\gamma\psi,\psi^2)}} \le C\|f\|_{L^2_\xi}, \tag{3.69}$$

for a suitable constant $C > 0$. In particular we have $D(\mathcal{A}_{2,\xi}) = H^2_{\xi,(\gamma^2,\gamma\psi,\psi^2)}$.

Proof. Given $f \in L^2_\xi$, let us consider the formal resolvent equation

$$\lambda u - \mathcal{A}u = f.$$

Multiplying both the sides by the weight ξ, we can write

$$\lambda \xi u - \mathcal{A}(\xi u) + (\mathcal{A}(\xi u) - \xi \mathcal{A} u) = \xi f.$$

On the other hand, we have

$$\xi \mathcal{A} u - \mathcal{A}(\xi u) = \sum_{i,j=1}^d \psi_i \psi_j a_{i,j} \left((D_{i,j}\xi)u + D_j \xi D_i u + D_i \xi D_j u\right) + \sum_{i=1}^d b_i (D_i \xi) u,$$

and since

$$D_i \xi D_j u = \frac{D_i \xi}{\xi}(\xi D_j u) = \frac{D_i \xi}{\xi}\left(D_j(\xi u) - u D_j \xi\right),$$

we can rewrite

$$\begin{aligned}
&\mathcal{A}(\xi u) - \xi \mathcal{A} u = \sum_{i,j=1}^d \psi_i(x)\psi_j(x) a_{i,j}(x) \\
&\times \Big(\frac{D_{i,j}\xi(x)}{\xi(x)}\xi(x) u + \frac{D_i \xi}{\xi(x)} D_j(\xi u) + \frac{D_j \xi}{\xi(x)} D_i(\xi u) + 2\frac{D_i \xi D_j \xi}{\xi^2}(\xi u)\Big) \\
&+ \sum_{i=1}^d b_i \frac{D_i \xi}{\xi}(\xi u).
\end{aligned}$$

Therefore, if we consider the operator $\overline{\mathcal{A}}_2 : D(\overline{\mathcal{A}}_2) \to L^2$ given by

$$D(\overline{\mathcal{A}}_2) \stackrel{def}{=} D(\mathcal{A}_2),$$

and

$$\overline{\mathcal{A}}_2 v \quad \stackrel{def}{=} \quad \mathcal{A} v + \sum_{i,j=1}^d \psi_i(x)\psi_j(x) a_{i,j}(x)\Big(\frac{D_i \xi}{\xi(x)} D_j v + \frac{D_j \xi}{\xi(x)} D_i v\Big)$$

$$+\sum_{i,j=1}^{d} \psi_i(x)\psi_j(x)a_{i,j}(x)\frac{D_{i,j}\xi(x)}{\xi(x)}v + \sum_{i=1}^{d} b_i \frac{D_i\xi}{\xi}v,$$

our hypotheses assure that, for each $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > 0$, the resolvent equation

$$(\lambda - \overline{\mathcal{A}}_2)v = g$$

has, for every $g \in L^2$, a unique solution $v \in D(\overline{\mathcal{A}}_2)$, which satisfies the estimate

$$|\lambda|\|v\|_{L^2} + |\lambda|^{1/2}\|v\|_{H^1_{(\gamma,\psi)}} + \|v\|_{H^2_{(\gamma^2,\gamma\psi,\psi^2)}} \leq K\|g\|_{L^2}, \tag{3.70}$$

for a suitable $K > 0$. Now, writing

$$D(\mathcal{A}_{2,\xi}) \stackrel{def}{=} \left\{u \in L^2_\xi : u\xi \in D(\overline{\mathcal{A}}_2)\right\}, \quad \mathcal{A}_{2,\xi}u \stackrel{def}{=} \mathcal{A}u,$$

and multiplying by ξ, the resolvent equation $\lambda u - \mathcal{A}_{2,\xi}u = f$ becomes

$$\lambda(\xi u) - \overline{\mathcal{A}}_2(\xi u) = \xi f.$$

Therefore, (3.69) follows by a straightforward application of (3.70) to $v = \xi u$. Finally, we have

$$D(\mathcal{A}_{2,\xi}) = H^2_{\xi,(\gamma^2,\gamma\psi,\psi^2)},$$

as it easily follows from the definition of our weighted spaces (see Section 2).

Remark 3.31. The problem of determining the type of weights for which the assumptions of the above proposition are satisfied now arises. For instance, it can be checked that, choosing $\xi(x) = \left(c + |x|^{2d}\right)^{-1}$, where $c \in \mathbb{R}_+$, the contribution of the terms depending on ξ in the operator $\overline{\mathcal{A}}$ in Assumption 1 goes to zero as d increases. It follows that one can always find a good weight of the form described above, for which Proposition 3.30 holds true.

4. EXAMPLES

Example 4.1. We consider here the PDE for the price of a European contingent claim (see, e.g., [51]) in the multifactor case. Assume that the price process of the given underlying $d \geq 1$ assets $(X_t)_{t\geq 0}$, where $X_t = \left(X_t^1, \ldots, X_t^d\right)$ satisfies the SDE

$$dX_t = rX_t dt + \sigma\left(\operatorname{diag}X_t\right)dW_t, \quad \forall t \geq 0$$

where $(\operatorname{diag}X_t)$ is the diagonal matrix with the components of $(X_t)_{t\geq 0}$ on the main diagonal, r is the interest rate of a reference riskless asset in the market, σ is a given d-order matrix such that, writing σ^* for the transpose of σ, the matrix $\sigma^*\sigma$ is positive definite, and $(W_t)_{t\geq 0}$, is a d-dimensional Wiener process, $W_t = \left(W_t^1, \ldots, W_t^d\right)$.

Write $v \equiv v(x,t)$, where $x \equiv (x_1, \ldots, x_d)$, for the no-arbitrage price of a contingent claim having payoff $g \equiv g(x)$ at the expiration time T. Then, under the so-called no-arbitrage assumption, it is well known that v solves the backward Kolmogorov PDE

$$D_t v + \mathcal{A} v = 0,$$

with the terminal condition $v(x,T) = g(x)$, where

$$\mathcal{A} v = \frac{1}{2} \mathrm{Tr}\ \left((\sigma \mathrm{diag} x)\,(D_{i,j} v)\,(\sigma \mathrm{diag} x)^*\right) + r \sum_{i=1}^{d} x_i D_i v - rv,$$

and $(D_{i,j}v)$ denotes the d-order matrix having entries for $D_{i,j}v$, for $i,j = 1, \ldots, d$.

It can be easily verified that the second-order operator $\mathcal{A}$ fits all our assumptions. Of course, for financial applications we are interested in solving the above PDE on the positive orthant, not in all $\mathbb{R}^d$, but this does not affect our results. Indeed, it can be proved that the operator $\mathcal{A}$ generates a positivity-preserving semigroup (see the second part of this paper).

Notice that this equation can be also studied (and solved explicitly) by taking the change of variable (see [11, 51]) $y_i = \ln x_i$ for all $i = 1, \ldots, d$. Our approach has the advantage of proving a general generation property and a characterization of the domain which turns out to be useful when one considers some modifications of the basic model above. A first example is the case when σ depends on x, but preserves differentiability and strong ellipticity (see, e.g., [51]). Another example is discussed below.

Assume that the underlying assets pay dividends with a rate $\rho \equiv \rho(x,t)$, and that they are subjected to a tax rate $\epsilon \equiv \epsilon(x,t)$. Then we have the following no-arbitrage PDE

$$D_t v + \frac{1}{2} \mathrm{Tr}\ \left((\sigma \mathrm{diag} x)\,(D_{i,j} v)\,(\sigma \mathrm{diag} x)^*\right) + (r-\rho)(1-\epsilon) \sum_{i=1}^{d} x_i D_i v - rv,$$

with the terminal condition $v(x,T) = g(x)$. In the case $d = 1$, $\rho = 0$, $\epsilon = 0$ and $g(x) = (x-E)^+$, where E is the maturity price, we obtain the well-known Black and Scholes equation described in [11]. Also multifactor models, such as the ones appearing in [24], options on futures contracts, and swaps can be treated in our framework (see [6], [51]), along with the example below.

Example 4.2. We consider here the structure model of interest rate derivatives. For the so-called affine single-term structure model the interest rate is modeled by the stochastic process $(X_t)_{t\geq 0}$ satisfying the differential equation

$$dX_t = (\alpha_1(t) + \alpha_2(t)X_t)\,dt + (\beta_1(t) + \beta_2(t)X_t)\,dW_t. \tag{4.1}$$

Suitably choosing the coefficients $\alpha_1(t)$, $\alpha_2(t)$, $\beta_1(t)$ and $\beta_2(t)$, different term-structure models can be obtained. In particular (see part II of this work), two models fitting our framework can be obtained by choosing

(1) $\alpha_1 = \alpha_2 = \beta_1 = 0$ [23]
(2) $\beta_1 = 0$ [12].

The price of a zero-coupon bond maturing at date T is the solution of the Cauchy problem

$$D_t v + \mathcal{A}v = 0,$$

with the end terminal condition $v(x,T) = 1$, where

$$\mathcal{A}v = \tfrac{1}{2}(\beta_1 + \beta_2 x)^2 D_{x,x}v + (\alpha_1 + \alpha_2 x)D_x v.$$

We remark in addition that our results allow us to treat also multifactor models with time-dependent coefficients (see [24, 6]), and also semilinear perturbations of the above equations, as already shown in the previous example.

Example 4.3. The following equation, coming from nonlinear filtering, is considered in [5, 46]:

$$D_t = D_{x,x} + xD_x v - x^2 v, \quad t > 0, x \in \mathbb{R}$$

with initial condition $v(0,x) = g(x)$. It can be easily checked that the second order operator defined by the right-hand side of the above equation satisfies our assumptions.

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