The Bernstein problem in Heisenberg groups

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1 Introduction

These notes aim to review results and open problems concerning the so-called *Bernstein's problem* for entire area minimizing graphs of (topological) codimension 1, in the setting of sub-Riemannian Heisenberg groups. The results and examples we will introduce are well-known and then we will only state them and quote the references where they can be

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found. The example introduced in Section 4.1 is instead new to our knowledge, and then we will carry out its complete construction.

Let us briefly recall the well-known results concerning the Euclidean Bernstein problem for entire minimal graphs of codimension 1; we refer to these classical references for their proof: [Ber17; Ber27; Fle62; Alm66; DeG65; Sim68; BDG69]. Let us also recall the following classical monographs dealing with the Bernstein problem in the Euclidean case: [Giu84; MM84; Mag12].

Theorem 1.1 (Euclidean Bernstein Theorem)**.** *Let $U \subset \mathbb{R}^n$ be a (non-empty) set of least perimeter in the whole $\mathbb{R}^n$. Then, if $n \leq 8$, ∂U is a hyperplane.*

We recall that a set U, of locally finite perimeter in $\mathbb{R}^n$, is said to be a *set of least perimeter* in an open set $A \subset \mathbb{R}^n$ whenever the following holds: for every bounded open subset A' of A and for every $V \subset \mathbb{R}^n$ satisfying $U \triangle V \Subset A'$ (*i.e.*, $U \triangle V$ is compactly contained in A'), it holds that the perimeter of V in A' is greater or equal to the perimeter of U in A'.

Definition 1.2 (Classical minimal surface equation)**.** Let $\psi : \Omega \to \mathbb{R}$ be a $\mathbf{C}^2$ function on the open set $\Omega \subset \mathbb{R}^{n-1}$. We say that ψ solves the **(classical) minimal surface equation** in Ω if

$$\operatorname{div}\left(\frac{\nabla\psi}{\sqrt{1+|\nabla\psi|^2}}\right) = 0 \qquad \text{in } \Omega. \tag{MSE}$$

Theorem 1.3 (Euclidean Bernstein Theorem for graphs)**.** *Let $n \geq 2$. The following statements hold:*

(i) If $n \leq 8$, then any solution $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ to the (classical) minimal surface equation (MSE) *is affine.*

(ii) If $n \geq 9$, then there exist analytic solutions $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ to the (classical) minimal surface equation (MSE) *which are not affine.*

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2 Preliminaries: the Heisenberg group, intrinsic regular hypersurfaces and their area, graphs

2.1 The Heisenberg group $\mathbb{H}^n$

In this article, we will mean the Heisenberg group $\mathbb{H}^n$ as the set $\mathbb{C}^n \times \mathbb{R} \equiv \mathbb{R}^{2n+1}$, and represent its points as $p = [z,t] = [\mathbf{x} + i\mathbf{y}, t] = (\mathbf{x}, \mathbf{y}, t)$, $z \in \mathbb{C}^n$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $t \in \mathbb{R}$. The group operation on $\mathbb{H}^n$ will be defined as follows: whenever $p = [z,t] \in \mathbb{H}^n$ and $q = [\zeta, \tau] \in \mathbb{H}^n$,

$$p \cdot q := \Big[z + \zeta, t + \tau + 2\operatorname{Im}\Big(\Big\langle z, \bar{\zeta}\Big\rangle\Big)\Big]. \tag{2.1}$$

As a consequence, it is easy to verify that the group identity is the origin 0 and the inverse of a point is given by $[z,t]^{-1} = [-z, -t]$. Moreover, the following family of *non-isotropic* *dilations* is defined: if $p=[z,t]\in\mathbb{H}^n$ and $\lambda>0$,

$$\delta_\lambda(p) := [\lambda z, \lambda^2 t]. \tag{2.2}$$

Notice the $\delta_\lambda:\mathbb{H}^n\to\mathbb{H}^n$, $\lambda>0$, is a family of automorphisms. The Heisenberg group $\mathbb{H}^n$ admits the structure of a Lie group of topological dimension $2n+1$. Its Lie algebra $\mathfrak{h}_n$ of left invariant vector fields is (linearly) generated by

$$X_j=\frac{\partial}{\partial x_j}+2y_j\frac{\partial}{\partial t},\qquad Y_j=\frac{\partial}{\partial y_j}-2x_j\frac{\partial}{\partial t},\qquad \text{for } j=1,\dots,n;\qquad T=\frac{\partial}{\partial t}; \tag{2.3}$$

and the only non-trivial commutator relation is

$$[X_j,Y_j]=-4T, \tag{2.4}$$

valid for any $j=1,\dots,n$. The following notation will be also used: $X_j := Y_{j-n}$ for $j=n+1,\dots,2n$; so that the set of left invariant vector fields which generates $\mathfrak{h}_n$ can be listed as $X_1,\dots,X_{2n}$. The vector fields $X_1,\dots,X_{2n}$ called *horizontal vector fields*. The group $\mathbb{H}^n$ endowed with this algebraic structure is usually called a *Carnot group* [Gro96].

Moreover, an intrinsic differentiable structure can be introduced within the tangent space of $\mathbb{H}^n$.

Definition 2.1 (Horizontal bundle)**.** We call **horizontal bundle** of $\mathbb{H}^n$ the subbundle $H\mathbb{H}^n$ of the tangent bundle $T\mathbb{H}^n$ which is spanned by the left-invariant vector fields $X_1,\dots,X_{2n}$. We say that vectors of

$$H\mathbb{H}^n_x=\operatorname{span}\{X_1(x),\dots,X_{2n}(x)\} \tag{2.5}$$

are **horizontal vectors**.

It is also well-known that the $(2n+1)$-dimensional Lebesgue measure $\mathcal{L}^{2n+1}$ on $\mathbb{H}^n\equiv\mathbb{R}^{2n+1}$ is the so-called *Haar measure* of the group, and the integer

$$Q := 2n+2$$

is the *homogeneous dimension* of $\mathbb{H}^n$. Indeed it holds that, if $E\subset\mathbb{R}^{2n+1}$,

$$\mathcal{L}^{2n+1}(p\cdot E)=\mathcal{L}^{2n+1}(E)\ \text{ for each } p\in\mathbb{R}^{2n+1}, \tag{2.6}$$

and

$$\mathcal{L}^{2n+1}(\delta_\lambda(E))=\lambda^Q\,\mathcal{L}^{2n+1}(E)\ \text{ for each } \lambda>0, \tag{2.7}$$

where $p\cdot E:=\{p\cdot q:\,q\in E\}$ and $\delta_\lambda(E):=\{\delta_\lambda(q):\,q\in E\}$.

The group $\mathbb{H}^n$ is typically endowed with a *left-invariant homogeneous* metric d, that is a metric $d:\mathbb{R}^{2n+1}\times\mathbb{R}^{2n+1}\to[0,\infty)$, which is continuous with respect to the Euclidean topology and satisfies:

$$d(p\cdot q_1,p\cdot q_2)=d(q_1,q_2)\quad \forall\, p,q_1,\,q_2\in\mathbb{R}^{2n+1}, \tag{2.8}$$

and

$$d\left(\delta_\lambda(q_1),\delta_\lambda(q_2)\right)=\lambda\,d(q_1,q_2)\quad \forall\, q_1,\,q_2\in\mathbb{R}^{2n+1},\ \lambda>0. \tag{2.9}$$

It is also well-known that any left-invariant homogeneous metric is equivalent to the (left-invariant) *Carnot-Carathéodory* (also called *sub-Riemannian*) metric d_c on $\mathbb{R}^{2n+1}$, associated to the horinzontal subbundle $H\mathbb{H}^n$ (see, for instance, [Gro96; Mon02]). The group $\mathbb{H}^n$ endowed with a left-invariant metric inherits the structure of a Carnot-Carathéodory (or sub-Riemannian) metric space. Moreover, by (2.6)-(2.9), the metric measure space $(\mathbb{H}^n, d_c, \mathcal{L}^{2n+1})$ turns out to be an *Ahlfors metric measure space* of dimension Q. Thus, by a well-known result about Ahlfors metric measure spaces (see, for instance, [Ser16, Theorem 2.26]), the *metric* (or *Hausdorff*) *dimension* of $\mathbb{H}^n$ is Q and then it is greater than its *topological dimension* $Q-1$: that is the typical gap between the metric and topological dimension in a sub-Riemannian metric structure, which does not take place in a Riemannian metric structure, where they agree.

2.2 Intrinsic regular hypersurfaces

Ambrosio and Kirchheim in 2001 proved that the classical notion of rectifiability, due to Federer, by means of classical (Euclidean) $\mathbf{C}^1$-regular hypersurfaces does not work in the sub-Riemannian Heisenberg group $\mathbb{H}^1$ [Ser16, Theorem 4.99]. Thus an alternative notion of intrinsic rectifiability was proposed in sub-Riemannian Heisenberg groups $\mathbb{H}^n$, by replacing the class of $\mathbf{C}^1$-regular hypersurfaces with one of more intrinsic $\mathbf{C}^1$-regular hypersurfaces which better fits the new geometry [FSS01]. In order to give this definition, we first need to introduce a notion of continuously differentiable function which is regular along the horizontal vector fields.

Definition 2.2 (Horizontally $\mathbf{C}^1$ functions)**.** Let f be a real measurable function defined on an open set $\mathcal{U} \subset \mathbb{H}^n$. We call **horizontal gradient** of f the distribution $\nabla_{\mathbb{H}} f := (X_1 f, \dots, X_{2n} f)$. Moreover, f is said to be of class $\mathbf{C}^1_{\mathbb{H}}(\mathcal{U})$ if f is continuous and its horizontal gradient $\nabla_{\mathbb{H}} f$ is represented by a continuous function.

Thanks to the definition just given, one can naturally extend to $\mathbb{H}^n$ the notion of "regular surface" by considering level sets of $\mathbf{C}^1_{\mathbb{H}}$ maps:

Definition 2.3 ($\mathbb{H}$-regular surface)**.** We shall say that $S \subset \mathbb{H}^n$ is an **$\mathbb{H}$-regular surface** if for every $p \in S$ there exist a neighborhood $\mathcal{U} \subset \mathbb{H}^n$ and a function $f \in \mathbf{C}^1_{\mathbb{H}}(\mathcal{U})$ such that $\nabla_{\mathbb{H}} f \neq 0$ in $\mathcal{U}$ and $S \cap \mathcal{U} = \{q \in \mathcal{U} : f(q) = 0\}$.

Once this definition is given, one can naturally define a notion of "normal to a surface" which suits the sub-Riemannian structure of $\mathbb{H}^n$:

Definition 2.4 (Horizontal normal)**.** Let S be an $\mathbb{H}$-regular surface, and $p \in S$ a point. Let $f \in \mathbf{C}^1_{\mathbb{H}}(\mathcal{U})$ be a function as in the definition of $\mathbb{H}$-regular surface, defined in a neighborhood $\mathcal{U}$ of p. We define the **horizontal normal** to S at p as

$$\nu_S(p) := -\frac{\nabla_{\mathbb{H}} f(p)}{|\nabla_{\mathbb{H}} f(p)|}. \tag{2.10}$$

Remark. We will see below by means of an implicit function theorem (see Theorem 2.17 (iii)) that the horizontal normal is well-defined (up to orientation), in the sense that it is independent of the defining function f.

Remark (Relation between Euclidean and $\mathbb{H}$-surfaces)*.* It is clear that not all $\mathbb{H}$-regular surfaces are $\mathbf{C}^1$-smooth surfaces in the Euclidean sense, since the definition of $\mathbf{C}^1_{\mathbb{H}}$ gives no information about the differentiability along the vector field T; what's more, $\mathbb{H}$-regular surfaces can behave very badly from a Euclidean point of view: Kirchheim and Serra Cassano gave in [KS04] an example of a $\mathbb{H}$-regular surface in $\mathbb{H}^1$ having Euclidean Hausdorff dimension $\frac{5}{2}$.

The converse inclusion is false as well: in general, it's not true that a Euclidean surface S can be (locally) defined as the level set of a map f with non-zero *intrinsic* gradient. For instance, the horizontal plane $S := \{t = 0\}$, near the origin, already gives an example of this situation. More generally, when considering a Euclidean hypersurface, the only problems arise at points where $H\mathbb{H}^n_x \subset T_xS$, where T_xS denotes the tangent plane of S at x. These points are, in many aspects, irregular points in the setting of sub-Riemannian geometry of $\mathbb{H}^n$. We give a name to these points.

Definition 2.5 (Characteristic set)**.** Let S be a (classical) $\mathbf{C}^1$-hypersurface in $\mathbb{H}^n$. We set

$$\mathrm{Char}(S) := \{x \in S \mid H\mathbb{H}^n_x \subset T_xS\}, \tag{2.11}$$

and we call $\mathrm{Char}(S)$ the **characteristic set** of S.

It is trivial to see that if $\mathrm{Char}(S) = \varnothing$, then S is also a $\mathbb{H}$-surface. Moreover it is also well-known that the "size" of $\mathrm{Char}(S)$ is small with respect to the $(Q-1)$-dimensional Hausdorff measure induced by distance d_c, which we will denote $\mathcal{H}_c^{Q-1}$. Indeed, Balogh proved in 2003 that, if S is a $\mathbf{C}^1$-regular hypersurface then $\mathcal{H}_c^{Q-1}(\mathrm{Char}(S)) = 0$ (see, for instance, [Ser16, Theorem 4.23]).

2.3 The intrinsic area: horizontal perimeter

In this Section, we are going to introduce a notion of intrinsic area for hypersurfaces in the setting of the sub-Riemannian Heisenberg group. We will use the notion of *horizontal perimeter* introduced in [CDG94] and inspired by the classical (Euclidean) notion of De Giorgi's perimeter . Whenever Ω is an open subset of $\mathbb{H}^n \equiv \mathbb{R}^{2n+1}$ and $\varphi = (\varphi_1, \dots, \varphi_{2n}) \in \mathbf{C}^1_c(\Omega; \mathbb{R}^{2n})$, we set

$$\mathrm{div}_{\mathbb{H}}\, \varphi := -\sum_{j=1}^{2n} X_j^* \varphi_j, \tag{2.12}$$

where X_j^* is the adjoint operator of X_j in $L^2(\mathbb{R}^{2n+1})$. Given a measurable subset $E \subset \mathbb{R}^n$ we define the $\mathbb{H}$-perimeter of E in Ω as

$$|\partial E|_{\mathbb{H}}(\Omega) := \sup\left\{ \int_E \mathrm{div}_{\mathbb{H}}\, \varphi \;\middle|\; \varphi \in \mathbf{C}^1_c(\Omega, \mathbb{R}^{2n}), |\varphi|_{\mathbb{R}^{2n}} \leq 1 \right\}; \tag{2.13}$$

alternatively, we can define $|\partial E|_{\mathbb{H}}(\Omega)$ as the (horizontal) total variation in Ω of χ_E, where χ_E denotes the characteristic function of set E.

We say that a measurable set $E \subset \mathbb{H}^n$ is of *locally finite* (respectively, *finite*) $\mathbb{H}$-perimeter in Ω, if $|\partial E|_{\mathbb{H}}(\Omega') < \infty$ for each open set $\Omega' \Subset \Omega$ (respectively, $|\partial E|_{\mathbb{H}}(\Omega) < \infty$).

It is well known that, if E is a set of locally finite $\mathbb{H}$-perimeter in Ω, by the Riesz representation Theorem we can identify $|\partial E|_{\mathbb{H}}$ as a Radon measure on Ω for which there exists a unique Borel measurable function $\nu_E : \Omega \to \mathbb{R}^{2n}$ such that

$$|\nu_E|_{\mathbb{R}^{2n}} = 1 \qquad \text{for } |\partial E|_{\mathbb{H}}\text{-a.e. in } \Omega \tag{2.14}$$

$$\int_E \mathrm{div}_{\mathbb{H}}\, \varphi \, d\mathscr{L}^{2n+1} = -\int_\Omega \langle \varphi, \nu_E \rangle_{\mathbb{R}^{2n}} \, d|\partial E|_{\mathbb{H}} \quad \text{for all } \varphi \in \mathbf{C}^1_c(\Omega, \mathbb{R}^{2n}). \tag{2.15}$$

In the following we will call ν_E the *horizontal inward normal to* E (see [CDG94; FSS96]). Moreover, if the set E has a regular boundary, we can give an explicit representation of the $\mathbb{H}$-perimeter with respect to the $2n$-dimensional (Euclidean) Hausdorff measure (see [CDG94; FSS96]).

Theorem 2.6. *Let $E \subset \mathbb{H}^n$ and suppose that its boundary ∂E is (Euclidean) $\mathbf{C}^1$ regular and let N_E denotes its outward unit normal. Then E is a set of locally finite $\mathbb{H}$-perimeter and, for each open set $\Omega \subset \mathbb{H}^n$, it holds that*

$$|\partial E|_{\mathbb{H}}(\Omega) = \int_{\Omega\cap\partial E} \sqrt{\sum_{j=1}^{2n} \langle X_j, N_E\rangle^2_{\mathbb{R}^{2n+1}}}\, d\mathcal{H}^{2n} \tag{2.16}$$

where $\mathcal{H}^{2n}$ denotes the (Euclidean) $2n$-dimensional Hausdorff measure in $\mathbb{R}^{2n+1}$. Moreover the horizontal inward unit normal ν_E can be represented as

$$\nu_E(p) = -\frac{(\langle X_1, N_E\rangle_{\mathbb{R}^{2n+1}}, \dots, \langle X_{2n}, N_E\rangle_{\mathbb{R}^{2n+1}})}{\sqrt{\sum_{j=1}^{2n} \langle X_j, N_E\rangle^2_{\mathbb{R}^{2n+1}}}}(p) \qquad \textit{for } |\partial E|_{\mathbb{H}}\textit{-a.e. } p \in \mathbb{H}^n\,.$$

Moreover, by representation formula (2.16), we can introduce a notion of $\mathbb{H}$-area for (Euclidean) regular hypersurfaces in $\mathbb{H}^n$.

Definition 2.7. Let $S \subset \mathbb{H}^n$ be a (Euclidean) $\mathbf{C}^1$-regular hypersurface and let N_E denote its unit normal. Then we call $\mathbb{H}$-area of S in an open set $\Omega \subset \mathbb{H}^n$ the nonnegative value (possibly infinite)

$$\mathcal{A}_{\mathbb{H}}(S)(\Omega) = \int_{\Omega\cap S} \sqrt{\sum_{j=1}^{2n} \langle X_j, N_E\rangle^2_{\mathbb{R}^{2n+1}}}\, d\mathcal{H}^{2n}\,. \tag{2.17}$$

We are in order to introduce a notion of minimality for sets of locally finite $\mathbb{H}$-perimeter and regular hypersurfaces.

Definition 2.8 ($\mathbb{H}$-minimality)**.**
(i) We will say that a set E of locally finite $\mathbb{H}$-perimeter is a minimizer (or perimeter minimizing) for the $\mathbb{H}$-perimeter in Ω if

$$|\partial E|_{\mathbb{H}}(\Omega') \le |\,\partial F|_{\mathbb{H}}(\Omega') \tag{2.18}$$

for any open set $\Omega' \Subset \Omega$ and any measurable $F \subset \mathbb{H}^n$ such that $E\Delta F \Subset \Omega'$.

(ii) We will say that a $\mathbf{C}^1$-regular hypersurface $S \subset \mathbb{H}^n$ is area minimizing for the $\mathbb{H}$-area in Ω if

$$\mathcal{A}_{\mathbb{H}}(S)(\Omega') \le \mathcal{A}_{\mathbb{H}}(S_*)(\Omega') \tag{2.19}$$

for any open set $\Omega' \Subset \Omega$ and any $\mathbf{C}^1$-regular hypersurface $S_* \subset \mathbb{H}^m$ such that $S\Delta S_* \Subset \Omega'$.

By adapting the classical calibration method, one can obtain the following sufficient condition for perimiter minimality in $\mathbb{H}^n$. Notice that an analogous result holds more in general for Carnot groups, and an even more general one holds in Carnot-Carathéodory spaces (see [BSV07, Section 2]).

Theorem 2.9. *Let E, Ω be respectively a measurable and open set of $\mathbb{H}^n$. Let us assume*

(i) E has locally finite $\mathbb{H}$-perimeter in Ω;

(ii) *if* $\nu_E : \Omega \to \mathbb{R}^{2n}$ *denotes the horizontal inward normal to* E *in* Ω*, then* $\mathrm{div}_{\mathbb{H}}(\nu_E) = 0$ *in* Ω *in distributional sense;*

(iii) *there exists an open set* $\tilde{\Omega} \subset \Omega$ *such that* $|\partial E|_{\mathbb{H}}(\Omega \setminus \tilde{\Omega}) = 0$ *and* $\nu_E \in \mathbf{C}^0(\tilde{\Omega}; \mathbb{R}^{2n})$*.*

Then E *is a minimizer of the* $\mathbb{H}$*-perimeter in* Ω*.*

2.4 t-graphs

A t-graph in $\mathbb{H}^n$ is a graph with respect to the (non horizontal) vector field T . We identify

$$\Pi := \Big\{(\mathbf{x}, \mathbf{y}, t) \in \mathbb{R}^{2n+1} : \, t = \, 0\Big\} \equiv \mathbb{R}^{2n}$$

and we use the coordinates

$$(x_1, \dots, x_n, y_1, \dots, y_n) = (\mathbf{x}, \mathbf{y}) \tag{2.20}$$

on it.

Definition 2.10 (t-graph and t-subgraph)**.** If $\phi : \mathcal{U} \to \mathbb{R}$ is an $\mathbb{R}$-valued function defined on an open subset $\mathcal{U} \subset \mathbb{R}^{2n}$, then we define its t**-graph** to be the following subset of $\mathbb{H}^n = \mathbb{R}^{2n+1}$:

$$\begin{aligned} S^t(\phi) &:= \{(x_1, \dots, x_n, y_1, \dots, y_n, t) \in \mathcal{U} \times \mathbb{R} \mid t = \phi(x_1, \dots, x_n, y_1, \dots, y_n)\} \\ &= \{(\mathbf{x}, \mathbf{y}, \phi(\mathbf{x}, \mathbf{y})) \in \mathbb{H}^n \mid (\mathbf{x}, \mathbf{y}) \in \mathcal{U}\}. \end{aligned} \tag{2.21}$$

Analogously, we define its t**-subgraph** to be:

$$\begin{aligned} E^t(\phi) &:= \{(x_1, \dots, x_n, y_1, \dots, y_n, t) \in \mathcal{U} \times \mathbb{R} \mid t < \phi(x_1, \dots, x_n, y_1, \dots, y_n)\} \\ &= \{(\mathbf{x}, \mathbf{y}, t) \in \mathbb{H}^n \mid (\mathbf{x}, \mathbf{y}) \in \mathcal{U}, \, t < \phi(\mathbf{x}, \mathbf{y})\}. \end{aligned} \tag{2.22}$$

Remark. Notice that, if $e_{2n+1} := (0, \dots, 0, 1) \in \mathbb{R}^{2n+1}$, then a t-graph $S^t(\phi)$ can be also represented as

$$S^t(\phi) = \, \{A \cdot \phi(A) \, e_{2n+1} : \, A \in \mathcal{U}\}$$

Observe also that Π is not a subgroup of $\mathbb{H}^n$.

It is clear that if ϕ is a (Euclidean) $\mathbf{C}^1$ function then $S^t(\phi)$ is a (classical) $\mathbf{C}^1$-surface. With an abuse of notation, we will write

$$\mathrm{Char}(\phi) := \Big\{(\mathbf{x}, \mathbf{y}) \in \mathcal{U} \, \Big| \, (\mathbf{x}, \mathbf{y}, \phi(\mathbf{x}, \mathbf{y})) \in \mathrm{Char}(S^t(\phi))\Big\}, \tag{2.23}$$

where $\mathrm{Char}(S^t(\phi))$ is the characteristic set defined in Definition 2.5. One can easily verify that

$$\mathrm{Char}(\phi) = \{(\mathbf{x}, \mathbf{y}) \in \mathcal{U} \mid \nabla_{\mathbf{x}} \phi(\mathbf{x}, \mathbf{y}) - 2\mathbf{y} = \nabla_{\mathbf{y}} \phi(\mathbf{x}, \mathbf{y}) + 2\mathbf{x} = 0\}; \tag{2.24}$$

more compactly, if we define

$$\mathbf{X}^*(\mathbf{x}, \mathbf{y}) := (-2\mathbf{y}, 2\mathbf{x}) \tag{2.25}$$

then

$$\mathrm{Char}(\phi) = \{(\mathbf{x}, \mathbf{y}) \in \mathcal{U} \mid \nabla \phi + \mathbf{X}^* = 0\}. \tag{2.26}$$

Definition 2.11 (Area for a t-graph)**.** Let $\mathcal{U} \subset \mathbb{R}^{2n}$. We define the **area functional** $\mathcal{A}^t$ in $\mathcal{U}$ as

$$\mathcal{A}^t(\phi) := |\partial E^t(\phi)|_{\mathbb{H}}(\mathcal{U} \times \mathbb{R}) \tag{2.27}$$

for any $\phi \in W^{1,1}(\mathcal{U})$.

By some standard computations (see [CDG94]), one can prove the following:

Proposition 2.12. *For any $\phi \in W^{1,1}(\mathcal{U})$, it holds*

$$\mathcal{A}^t(\phi) = \int_{\mathcal{U}} |\nabla\phi + \mathbf{X}^*| \, d\mathscr{L}^{2n}. \tag{2.28}$$

Remark. The Definition 2.11 makes sense for a larger class of functions: one could define the space $BV^t(\mathcal{U})$ to as the space of maps $\phi \in L^1(\mathcal{U})$ such that $|\partial E^t(\phi)|_{\mathbb{H}}(\mathcal{U} \times \mathbb{R}) < \infty$. By [SV14, Theorem 1.2], this space actually coincides with the space $BV(\mathcal{U})$ of functions with bounded variation in the standard Euclidean sense.

For local minimizers of $\mathcal{A}^t$, the following holds (see [CHY07, Section 2]):

Proposition 2.13 (First variation formula)**.** *Let $\phi \in \mathbf{C}^2(\mathcal{U})$ be a local minimizer for $\mathcal{A}^t$, and let $(\mathbf{x}, \mathbf{y}) \in \mathcal{U} \setminus \mathrm{Char}(\phi)$. Then at $(\mathbf{x}, \mathbf{y})$ we have*

$$\operatorname{div} \frac{\nabla\phi + \mathbf{X}^*}{|\nabla\phi + \mathbf{X}^*|} = 0. \tag{t-MSE}$$

In the sequel, we will write for simplicity

$$N(\mathbf{x}, \mathbf{y}) := \frac{\nabla\phi(\mathbf{x}, \mathbf{y}) + \mathbf{X}^*(\mathbf{x}, \mathbf{y})}{|\nabla\phi(\mathbf{x}, \mathbf{y}) + \mathbf{X}^*(\mathbf{x}, \mathbf{y})|}. \tag{2.29}$$

Definition 2.14. Let $\mathcal{U}$ be an open bounded domain in $\mathbb{R}^{2n}$, and $\phi \in W^{1,1}(\mathcal{U})$. We say that ϕ is a solution to the **weak minimal surface equation** for t-graphs if

$$\int_{\mathcal{U}} \langle N, \nabla\psi \rangle_{\mathbb{R}^{2n}} d\mathscr{L}^{2n} = 0 \tag{WMSE}$$

holds for every test function $\psi \in \mathbf{C}_0^\infty(\mathcal{U})$.

Then, by [CHY07, Theorem 3.3], we also have:

Proposition 2.15. *A map $\phi \in W^{1,1}(\mathcal{U})$ is a minimizer for the area functional if and only if it is a solution to the weak minimal surface equation for t-graphs* (WMSE)*.*

Remark. Observe that functional $\mathcal{A}^t : W^{1,1}(\mathcal{U}) \to \mathbb{R}$ is convex. Thus it is not surprising that a stationary type point $\phi \in W^{1,1}(\mathcal{U})$ of functional $\mathcal{A}^t$, that is, a function $\phi \in W^{1,1}(\mathcal{U})$ satisfying (WMSE), turns out to be a minimizer for $\mathcal{A}^t$.

Notice that, from Theorem 2.9, Propostion 2.15 can be strengthened. Indeed one can prove the following result (see [Ser16, Example 5.29]):

Theorem 2.16. *If $\phi \in W^{1,1}(\mathcal{U})$ satisfies* (WMSE)*, then its subgraph $E^t(\phi)$ is a minimizer for the $\mathbb{H}$-perimeter in $\Omega = \mathcal{U} \times \mathbb{R}$, according to Definition 2.8.*

2.5 Intrinsic graphs

An intrinsic graph of $\mathbb{H}^n$ is a graph along a horizontal vector field. Without loss of generality, we will always consider X_1-graphs, i.e. intrinsic graphs along the X_1-direction. The notion of intrinsic graph arose in the setting of the intrinsic rectifiability in the sub-Riemannian $\mathbb{H}^n$ [FSS01]. Indeed, by means of an implicit function theorem, an intrinsic regular hypersurface can be locally represented as an intrinsic graph. Let us recall the implicit function theorem [FSS01]. Let us first introduce some preliminary notation. If $n \geq 2$, we identify the maximal subgroup

$$\mathbb{W} := \{(\mathbf{x}, \mathbf{y}, t) \in \mathbb{H}^n : x_1 = 0\}$$

with $\mathbb{R}^{2n}$ by writing $(x_2, \dots, x_n, y_1, \dots, y_n, t)$ instead of $(0, x_2, \dots, x_n, y_1, \dots, y_n, t)$; similarly $\mathbb{W} := \{(0, y, t) \in \mathbb{H}^1 : y, t \in \mathbb{R}\} \equiv \mathbb{R}^2_{y,t}$ if $n = 1$. We identify the 1-dimensional horizontal subgroup

$$\mathbb{V} := \{(s, 0, \dots, 0)) \in \mathbb{H}^n : s \in \mathbb{R}\}$$

with $\mathbb{R}$ by writing s instead of $(s, 0, \dots, 0)$.

Remark. Observe that the subgroups $\mathbb{W}$ and $\mathbb{V}$ are also *homogeneous*, i.e.,

$$\delta_\lambda(\mathbb{W}) \subset \mathbb{W} \text{ and } \delta_\lambda(\mathbb{V}) \subset \mathbb{V} \text{ for each } \lambda > 0\,.$$

Moreover they are *complementary*, i.e.,

$$\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}.$$

If $\delta > 0$ and $h > 0$ let us denote

$$I_\delta := [-\delta, \delta]^{2n} \subset \mathbb{R}^{2n} \equiv \mathbb{W} \text{ and } J_h := [-h, h] \subset \mathbb{R} \equiv \mathbb{V}.$$

Let ω denote a fixed open bounded subset of $\mathbb{W}$; the intrinsic cylinder $\omega \cdot \mathbb{R}$ is defined by

$$\omega \cdot \mathbb{R} := \{A \cdot s \in \mathbb{H}^n : A \in \omega, s \in \mathbb{R}\}\,,$$

where, for $A \in \mathbb{W}$ and $s \in \mathbb{R}$ we write $A \cdot s$ to denote the Heisenberg product $A \cdot (s, 0, \dots, 0)$. In this way $I \cdot J = \{A \cdot s : A \in I, s \in J\}$ for any $I \subset \mathbb{W},\ J \subset \mathbb{R}$. Similarly, we will write $s \cdot A$ to denote $(s, 0, \dots, 0) \cdot A$.

Theorem 2.17. (Implicit function theorem) *Let Ω be an open set in $\mathbb{H}^n$, $0 \in \Omega$, and let $f \in \mathbf{C}^1_{\mathbb{H}}(\Omega)$ be such that $f(0) = 0$ and $X_1 f(0) > 0$. Define*

$$E = \{p \in \Omega : f(p) < 0\}, \quad S = \{p \in \Omega : f(p) = 0\}\,.$$

Denote now by $\gamma(s, A)$ the integral curve of the vector field X_1 at the time s issued from $A \in \mathbb{W} \equiv \mathbb{R}^{2n}$, i.e.

$$\gamma(s, A) = \exp(sX_1)(A) = A \cdot s.$$

Then there exist $\delta, h > 0$ such that the map $(s, A) \to \gamma(s, A)$ is a diffeomorphism of a neighborhood of $J_h \times I_\delta$ onto an open subset of $\mathbb{H}^n \equiv \mathbb{R}^{2n+1}$, and, if we denote by $\mathcal{U} \subset\subset \Omega$ the image of $\mathrm{Int}(J_h \times I_\delta)$ through this map, we have

$$E \textit{ has finite } \mathbb{H}\textit{-perimeter in } \mathcal{U}; \tag{i}$$

$$\partial E \cap \mathcal{U} = S \cap \mathcal{U}; \tag{ii}$$

$$\nu_E(p) = -\frac{\nabla_{\mathbb{H}} f(p)}{|\nabla_{\mathbb{H}} f(p)|_{\mathbb{R}^{2n}}} \textit{ for all } p \in S \cap \mathcal{U}, \tag{iii}$$

where ν_E is the generalized inward unit normal defined by (2.14) *and* (2.15) *, that can be identified with a section of $H\mathbb{H}^n$ with $|\nu_E(p)|_{\mathbb{R}^{2n}} = 1$ for $|\partial E|_{\mathbb{H}}$-a.e. $x \in \mathcal{U}$. In particular, ν_E can be identified with a continuous function and $|\nu_E| \equiv 1$. Moreover, there exists a unique function*

$$\phi :\, I_\delta \subset \mathbb{W} \to J_h \subset \mathbb{V}$$

such that the following parametrization holds: if $A \in I_\delta$, put $\Phi(A) := \gamma(\phi(A), A)$, then

$$S \cap \bar{\mathcal{U}} = \{p \in \bar{\mathcal{U}} : p = \Phi(A), A \in I_\delta\}; \tag{iv}$$

$$\phi \textit{ is continuous;} \tag{v}$$

the $\mathbb{H}$-perimeter has an integral representation (area formula):

$$|\partial E|_{\mathbb{H}}(\mathcal{U}) = \int_{I_\delta} \frac{|\nabla_{\mathbb{H}} f(\Phi(A))|_{\mathbb{R}^{2n}}}{X_1 f(\Phi(A))}\, d\mathcal{L}^{2n}(A). \tag{vi}$$

Inspired by the previous implicit function theorem, let us now recall the definitions of intrinsic graph and subgraph (in the horizontal direction X_1) which were introduced in [FSS01]:

Definition 2.18 (Intrinsic graphs)**.** Let $\omega \subset \mathbb{W} \equiv \mathbb{R}^{2n}$, and let $\phi : \omega \subset \mathbb{W} \equiv \mathbb{R}^{2n} \to \mathbb{V} \equiv \mathbb{R}$ be a function. We define the **(intrinsic) X_1-graph S^{X_1}** of ϕ as the set:

$$\begin{aligned} S^{X_1}(\phi) &:= \{A \cdot \phi(A) \mid A \in \omega\} \\ &= \{(\phi(A), x_2, \dots, x_n, y_1, y_2, \dots, y_{2n}, t + 2y_1\phi(A)) \mid A \in \omega\}, \end{aligned} \tag{2.30}$$

and the **(intrinsic) X_1-subgraph E^{X_1}** of ϕ as the set:

$$E^{X_1}(\phi) := \{A \cdot s :\, A \in \omega,\, s < \phi(A)\} \tag{2.31}$$

Remark (Intrinsic graphs as $\mathbb{H}$-surfaces). By writing the X_1-graph of $\phi : \Omega \to \mathbb{R}$ as

$$S^{X_1}(\phi) = \{(\mathbf{x}, \mathbf{y}, t) \in \omega \cdot \mathbb{R} \mid g(\mathbf{x}, \mathbf{y}, t) = 0\} \tag{2.32}$$

with $g(\mathbf{x}, \mathbf{y}, t) = x_1 - \phi(x_2, \dots, y_n, t - 2x_1y_1)$, it is clear that if ϕ is $\mathbf{C}^1$ then the intrinsic X_1-graph of ϕ is a $\mathbb{H}$-regular surface (in addition to being a classical surface). Equivalently, when seen as a $\mathbf{C}^1$ surface, $S^{X_1}(\phi)$ does not have characteristic points. These are not the only $\mathbb{H}$-regular surfaces, however: we'll soon clarify this observation.

We now introduce an intrinsic notion of *gradient of a function defined on* $\mathbb{R}^{2n}$, which was first defined in [ASV06].

Definition 2.19 (Intrinsic gradient)**.** Let $\phi : \mathbb{R}^{2n} \to \mathbb{R}$ be a continuous function. The **intrinsic gradient** of ϕ is defined as

$$\nabla^\phi \phi := \begin{cases} \Big(X_2\phi, \dots, X_n\phi, W^\phi\phi, Y_2\phi, \dots, Y_n\phi\Big) & \text{if } n \geq 2 \\ W^\phi\phi & \text{if } n = 1 \end{cases}, \tag{2.33}$$

where

$$X_j\phi := \frac{\partial\phi}{\partial x_j} + 2x_{j+n}\frac{\partial\phi}{\partial t}, \qquad j = 2, \dots, n \tag{2.34}$$

$$Y_j\phi := \frac{\partial\phi}{\partial x_{j+n}} - 2x_j\frac{\partial\phi}{\partial\tau}, \qquad j = 2, \dots, n \tag{2.35}$$

$$W^\phi\phi := \frac{\partial\phi}{\partial y_1} - 2\frac{\partial(\phi^2)}{\partial t}, \tag{2.36}$$

and the derivatives are to be meant in the distributional sense.

Notice that the notion of intrinsic gradient ∇^ϕ is substantially different from the horizontal gradient $\nabla_{\mathbb{H}}$ introduced in Definition 2.2: the latter is defined for functions on subsets of $\mathbb{H}^n$, while the former acts on functions on the subgroup $\mathbb{W} \equiv \mathbb{R}^{2n}$.

The intrinsic gradient just defined provides a useful tool for characterizing $\mathbb{H}$-regular surfaces, as in the Eucldean case. Indeed, the following theorem can be obtained combining [ASV06] and [BC10]:

Theorem 2.20. *Let $\omega \subset \mathbb{R}^{2n}$ be open, and let $\phi : \omega \to \mathbb{R}$ be a continuous function. Let $S = S^{X_1}(\phi)$ be the intrinsic graph of ϕ. Then the following are equivalent:*

(a) S is an $\mathbb{H}$-regular surface, and $\nu_S^1(p) < 0$ for all $p \in S$, where $\nu_S = (\nu_S^1, \dots, \nu_S^{2n})$ is the horizontal normal to S.

(b) There exists $\nabla^\phi \phi \in \mathbf{C}^0(\omega; \mathbb{R}^{2n})$.

We can now give the following definition (which, again, differs substantially from the definition of $\mathbf{C}^1_{\mathbb{H}}$ given in Definition 2.2):

Definition 2.21 (Intrinsic $\mathbf{C}^1$)**.** Let $\omega \subset \mathbb{R}^{2n}$ be an open set, and let $\phi : \omega \to \mathbb{R}$. We say that ϕ is **intrisically $\mathbf{C}^1$**, and we write $\phi \in \mathbf{C}^1_{\mathbb{W}}(\omega)$, whenever one of the (equivalent) conditions (a), (b) of Theorem 2.20 is satisfied.

When ϕ is intrinsically $\mathbf{C}^1$, we have convenient formulas for the normal to the intrinsic graph and for its $\mathbb{H}$-perimeter, which are basically adapted version of the Euclidean ones. We refer again to [ASV06] for a proof.

Theorem 2.22 (Area of an intrisic graph)**.** *Let $\omega \subset \mathbb{R}^{2n}$ be open, and let $\phi \in \mathbf{C}^1_{\mathbb{W}}(\omega)$ be an intrinsically regular function. Let $S = S^{X_1}(\phi)$ and $E = E^{X_1}(\phi)$ be the intrinsic graph and the subgraph of ϕ, respectibely and let $\Phi : \omega \to S$, defined as $\Phi(A) := A \cdot \phi(A)$ for all $A \in \omega$. Then E is a set of locally finite $\mathbb{H}$-perimeter in $\omega \cdot \mathbb{R}$ and we have*

$$\nu_E(p) = \left(-\frac{1}{\sqrt{1 + |\nabla^\phi \phi|^2}}, \frac{\nabla^\phi \phi}{\sqrt{1 + |\nabla^\phi \phi|^2}} \right) \left(\Phi^{-1}(p) \right) \text{ for all } p \in S \tag{2.37}$$

$$|\partial E|_{\mathbb{H}}(\omega \cdot \mathbb{R}) = c(n) \int_\omega \sqrt{1 + |\nabla^\phi \phi|^2}\, d\mathscr{L}^{2n} \tag{2.38}$$

where $c(n)$ is a dimensional constant.

Thanks to Theorem 2.22, the following definition makes sense:

Definition 2.23 (Intrinsic area functional)**.** If $\omega \subset \mathbb{R}^{2n}$ is an open set and $\phi : \omega \to \mathbb{R}$ is a $\mathbf{C}^1_{\mathbb{W}}$, we define the **area functional** for intrinsic graphs as

$$\mathcal{A}^{X_1}(\phi) := \int_\omega \sqrt{1 + |\nabla^\phi \phi|^2}\, d\mathscr{L}^{2n}. \tag{2.39}$$

Definition 2.24 (H-minimal functions)**.** Let $\omega \subset \mathbb{R}^{2n}$, and let $\phi : \omega \to \mathbb{R}$ be a $\mathbf{C}^2$ function. We say that ϕ (or the X_1-graph of ϕ) is **H-minimal** if it satisfies the ntrinsic minimal surface equation (IMSE):

$$\nabla^\phi \cdot \left(\frac{\nabla^\phi \phi}{\sqrt{1 + |\nabla^\phi \phi|^2}} \right) = 0 \text{ in } \omega\,. \tag{IMSE}$$

It is easy to prove that *a* $\mathbf{C}^2$ *function is* H*-minimal if and only if its graph is stationary*, in the sense that the first variation of the perimeter vanishes.

Remark. Observe that the intrinsic area functional $\mathcal{A}^{X_1} : \mathrm{Lip}(\omega) \to \mathbb{R}$ is no longer convex ([Dan+09; SV14]).

Let us conclude this section by stressing a relevant class of intrinsic graphs, i.e. the so-called *vertical hyperplanes* of $\mathbb{H}^n$.

Definition 2.25. A set $S \subset \mathbb{H}^n$ is said to be a vertical hyperplane, if

$$S = \Big\{(\mathbf{x}, \mathbf{y}, t) \in \mathbb{R}^{2n+1} : \langle a, \mathbf{x}\rangle_{\mathbb{R}^n} + \langle b, \mathbf{y}\rangle_{\mathbb{R}^n} = c\Big\} \tag{2.40}$$

for some $a,\, b \in \mathbb{R}^n$, $c \in \mathbb{R}$ with $|a|^2 + |b|^2 = 1$.

It is well-known that a vertical hyperplane S plays the role of an intrinsic hyperplane in $\mathbb{H}^n$. Indeed, if $c = 0$ in (2.40), then S is a maximal subgroup of $\mathbb{H}^n$, and one can easily prove that each maximal subgroup of $\mathbb{H}^n$ agrees with the set S in (2.40) with $c = 0$ and for suitable $a,\, b \in \mathbb{R}^n$. Moreover each S in (2.40) can be represented as some left-coset of a maximal subgroup of $\mathbb{H}^n$.

Remark. Notice that if, for instance, $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ with $a_1 \neq 0$, then the vertical plane S can be represented as (Euclidean) regular entire X_1-graph. Indeed $S = S^{X_1}(\phi)$ with $\phi : \mathbb{W} \equiv \mathbb{R}^{2n} \to \mathbb{V} \equiv \mathbb{R}$, where

$$\phi(A) := \frac{1}{a_1}\left(c - b_1 y_1 - \sum_{i=2}^{n}(a_i x_i + b_i y_i)\right) \text{ if } A = (x_2, \dots, x_n, y_1, \dots, y_n, t) \text{ and } n \geq 2$$

and

$$\phi(A) := \frac{1}{a_1}\,(c - b_1 y) \text{ if } A = (y, t) \text{ and } n = 1\,.$$

Moreover

$$\nabla^{\phi}\phi(A) = \text{constant for each } A \in \mathbb{W}. \tag{2.41}$$

2.6 The Bernstein problem for graphs in the sub-Riemannian $\mathbb{H}^n$

We are in order to state an approach to the Bernstein problem for t- and intrinsic graphs in $\mathbb{H}^n$, by means of the $\mathbb{H}$-perimeter. This approach is due to Miranda in 1964 in the Eucldean case [Mir64].

Definition 2.26. (i) A function $\phi : \mathcal{U} \subset \Pi \equiv \mathbb{R}^{2n} \to \mathbb{R}$, is said to be *$t$-graph area minimizing* in $\Omega := \mathcal{U} \times \mathbb{R}$ if its associated subgraph $E^t(\phi)$ is a minimizer for the $\mathbb{H}$-perimeter in Ω.

(ii) A function $\phi : \omega \subset \mathbb{W} \equiv \mathbb{R}^{2n} \to \mathbb{V} \equiv \mathbb{R}$ is said to be *X_1-graph area minimizing* in $\Omega := \omega \cdot \mathbb{R}$ if its associated subgraph $E^{X_1}(\phi)$ is a minimizer for the $\mathbb{H}$-perimeter in Ω.

Remark. Notice that, by Theorem 2.9, one can prove that each vertical hyperplane is area minimizing for $\mathbb{H}$-area in $\Omega = \mathbb{H}^n$ (see [BSV07]). In particular, each vertical hyperplane, that can be represented as an entire X_1-graph, is an area minimizing X_1-graph in $\Omega = \mathbb{H}^n$.

The Bernstein problem for graphs in $\mathbb{H}^n$: The main goal is the characterization of functions $\phi : \mathbb{R}^{2n} \to \mathbb{R}$ which are (entire) area minimizing t- or X_1-graph in $\Omega = \mathbb{R}^{2n+1}$. In particular we are looking for positive/negative answers to the following Bernstein-type rigidity problem for intrinsic graphs: finding out classes of functions $\mathcal{X}$, defined on the whole $\mathbb{R}^{2n}$, such that, if $\phi \in \mathcal{X}$ and its X_1-graph S is area minimizing in $\Omega \equiv \mathbb{H}^n$, then S is an intrinsic hyperplane of $\mathbb{H}^n$, that is a vertical hyperplane.

3 Results in $\mathbb{H}^1$

Area minimizing surfaces have mainly been studied in the first Heisenberg group $\mathbb{H}^1$. Let us recall here the most significant results as well as some open problems.

For the sake of simplicity, we will denote a point $p \in \mathbb{H}^1$ as $p = [z, t] = [x+iy, t] = (x, y, t)$.

3.1 Classification of area minimizing $\mathbf{C}^m$ surfaces

In this section, we state a few results concerning *general* minimal surfaces in the first Heisenberg group $\mathbb{H}^1$ with enough regularity. As we will remark later, the surfaces treated here are classical regular hypersurfaces; we recall that whenever it is required that the characteristic set of S is empty, this is the same as requiring that S is also a $\mathbb{H}$-regular surface. When we restrict ourselves to consider surfaces with $\mathbf{C}^2$-regularity, a complete characterization of *stable* surfaces is available in [HRR10]. For the sake of simplicity, we restrict the result to the case of $\mathbf{C}^2$-regular, area minimizing surfaces (see Definition 2.7 (ii)), which, in particular, are stable.

Theorem 3.1. *Let S be a $\mathbf{C}^2$ complete, oriented, connected surface in $\mathbb{H}^1$. Then S is area minimizing if and only if it is a Euclidean plane* or *it is congruent to the hyperbolic paraboloid $t = 2\,xy$.*

If we keep stability but we also ask for the characteristic set to be empty we get [GR15]:

Theorem 3.2. *Let S be a $\mathbf{C}^1$ complete, oriented, connected area minimizng surface in $\mathbb{H}^1$ with empty singular set. Then S is a vertical plane.*

3.2 t-graphs in $\mathbb{H}^1$

A first study of the Bernstein problem for t-graphs was carried out in [GP02]. The classification of all the complete $\mathbf{C}^2$ solutions to the minimal surface equation (WMSE) for t-graphs in $\mathbb{H}^1$ was studied in [Che+05] . This classification was refined in [HRR10] by means of Theorem 3.1. In particular, from this result, we can infer that there is no Bernstein rigidity for t-graphs: (Euclidean) planes are not the only area minimizing t-graphs in $\mathbb{H}^1$.

It is interesting to note, however, that by lowering the allowed regularity, and thus enlarging the class of functions we consider, the family of functions with area-minimizing t-graph grows considerably. Indeed, several examples of this phenomenon have been constructed in [Rit09]: for any non-decreasing continuous function $\beta : \mathbb{R} \to \mathbb{R}$, consider the function

$$f_\beta(x, y, t) := t + xy + y|y|\beta\left(-\frac{t}{y}\right); \tag{3.1}$$

let then $\phi_\beta(x, y)$ be the unique solution to $f_\beta(x, y, \phi_\beta(x, y)) = 0$. Then the regularity of ϕ_β is in general no better then locally Euclidean Lipschitz, while it was shown through a calibration argument that the t-graph of ϕ_β is area minimizing. For example (see again [Rit09]), with $\beta(x)=x$ one gets $\phi_\beta(x,y)=-\frac{xy}{1+|y|}$, which is Euclidean $\mathbf{C}^{1,1}$; with $\beta(x)=x\chi_{\{x\geq 0\}}$ one gets

$$\phi_\beta(x,y)=\begin{cases}-xy & \text{if } x<0\\ -\frac{xy}{1+|y|} & \text{if } x\geq 0\end{cases}, \tag{3.2}$$

which is only locally Lipschitz.

3.3 Intrinsic graphs in $\mathbb{H}^1$

As a first observation, we notice here that in $\mathbb{H}^1$ the intrinsic minimal surface equation (IMSE) becomes quite simpler: a first computation shows that it is equivalent to

$$\frac{(W^\phi)^2\phi}{(1+|W^\phi\phi|^2)^{\frac{3}{2}}}=0 \qquad \text{in } \mathbb{R}^2, \tag{3.3}$$

which in turn holds if and only if

$$(W^\phi)^2\phi=0 \qquad \text{in } \mathbb{R}^2\,. \tag{IMSE-1}$$

We recall here that the operator W^ϕ acts on $\mathbf{C}^1$ functions as $W^\phi=\frac{\partial}{\partial y}-4\phi\frac{\partial}{\partial t}$; here we are using the coordinates (y,t) on $\mathbb{R}^2$ (see Section 2.5).

In this Section, we first show that the correspondence between (IMSE-1) and perimeter minimizers no longer holds, differently from what happens with the Euclidean case and the t-graph case (Section 3.3.1). Then (Section 3.3.2) we show that the Bernstein-type rigidity result fails in general for area minimizing intrinsic graphs in the class $\mathcal{X}=C^{0,\alpha}(\mathbb{R}^2)\cap W^{1,p}_{\text{loc}}(\mathbb{R}^2)$. Finally, we show in Section 3.3.3 that the Berstein-type rigidity result holds instead for entire area minimizing intrinsic graphs in the class $\mathcal{X}=\text{Lip}_{\text{loc}}(\mathbb{R}^2)$.

3.3.1 Smooth solutions to the intrinsic minimal surface equation not area minimizing

We give here an example of an entire function $\phi:\mathbb{R}^2_{y,t}\to\mathbb{R}$ such that ϕ is a solution to the equation Equation (IMSE-1), but:

- $E^{X_1}(\phi)$ is not perimeter minimizing in $\Omega=\mathbb{H}^1$;
- $S^{X_1}(\phi)$ is not a vertical plane in $\mathbb{H}^1$.

This shows that *area stationary points* of the intrinsic area functional, that is functions satisfying (IMSE-1), need not be area minimizing. This is not surprising since, as we pointed out before, the intrinsic area functional $\mathcal{A}^{X_1}: Lip(\omega)\to\mathbb{R}$ is not convex.

To see this one only needs to define, for an arbitrary $\alpha>0$,

$$\phi_\alpha(y,t):=-\frac{\alpha yt}{1+2\alpha y^2} \text{ if } (y,t)\in\mathbb{R}^2\,. \tag{3.4}$$

A very elementary computation shows that ϕ_α satisfies (IMSE-1). However, it was proved in [DGN08] that the intrinsic graph $S=S^{X_1}(\phi)$ is *unstable*. More precisely, there exists $\varphi\in\mathbf{C}^\infty_c(\mathbb{R}^2)$ such that

$$\frac{d^2}{d\varepsilon^2}\mathcal{A}^{X_1}(\phi_\alpha+\varepsilon\,\varphi)\bigg|_{\varepsilon=0}<0. \tag{3.5}$$

This in particular shows that S is not area minimizing in $\Omega=\mathbb{H}^1$.

3.3.2 Area minimizing intrinsic graphs which are intrinsic cones

Even when the intrinsic graph of a function is area minimizing (not only stationary), we cannot in general infer that it is a vertical plane. As shown in [MSV08], one can build a function whose regularity is not better than $\mathbf{C}^{0,\frac{1}{2}}(\mathbb{R}^2) \cap W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2)$ with $1 \leq p < 2$ and the associated X_1-graph $S^{X_1}(\phi)$ is area minimizing in $\Omega = \mathbb{H}^1$. For example, the map

$$\phi(y,t) := -\operatorname{sgn} t\sqrt{|t|} \text{ if } (y,t) \in \mathbb{R}^2\,, \tag{3.6}$$

satisfies the aforesaid condition. In order to prove it is area minimizing, one can represent the X_1-graph of ϕ as the t-graph of a suitable function and use a calibration argument by means of Theorem 2.9.

Let us stress that $S = S^{X_1}(\phi)$ is an *intrinsic cone*, that is, it is invariant by the intrinsic dilations of $\mathbb{H}^1$ (see (2.2)). Indeed, it is easy to see

$$\delta_\lambda(S) = S \text{ for each } \lambda > 0\,.$$

3.3.3 A positive result for Lipschitz functions

Let us recall the main positive answers to the Bernstein-type rigidity problem in different classes of functions.

A first positive answer to the Bernstein-type rigidity problem for intrinsic graphs was proved in [BSV07]. Indeed, here, it was proved that any entire stable $\mathbf{C}^2$ X_1-graph must be a vertical plane . This result was extended, in [HRR10] and [Dan+10], to more general $\mathbf{C}^2$ embedded surfaces in $\mathbb{H}^1$ without characteristic points. It was then improved to $\mathbf{C}^1$-regularity, as a consequence of Theorem 3.2 in [GR15]).

Finally, to our knowledge, the current most general positive answer to the Bernstein-type rigidity problem is given in [NS19]:

Theorem 3.3. *Let $\phi \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^2)$. Assume that $S^{X_1}(\phi)$ is* stable*, that is, for each $\varphi \in \mathbf{C}^\infty_c(\mathbb{R}^2)$ it holds that*

$$\frac{d}{d\varepsilon}\mathcal{A}^{X_1}(\phi+\varepsilon\,\varphi)\Big|_{\varepsilon=0} = 0 \quad \textit{and} \quad \frac{d^2}{d\varepsilon^2}\mathcal{A}^{X_1}(\phi+\varepsilon\,\varphi)\Big|_{\varepsilon=0} \geq 0\,. \tag{3.7}$$

Then $S^{X_1}(\phi)$ is a vertical plane. In particular, if $\phi \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^2)$ is area minimizing in $\Omega = \mathbb{H}^1$, then $S^{X_1}(\phi)$ is a vertical plane.

3.3.4 Open problems

It is unknown whether the Bernstein-type rigidity result holds true for intrinsic graphs when $\mathcal{X} = \mathbf{C}^1_{\mathbb{W}}(\mathbb{R}^2)$ or $\mathcal{X} = W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2)$, with exponent p finite but high enough. In [NS19], two examples were given of *stable* intrinsic graphs, that is, satisfying condition (3.7) ,with associated functions in $\mathbf{C}^1_{\mathbb{W}}(\mathbb{R}^2) \cap W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$ and $\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{0\}) \cap W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2)$ with $1 \leq p < 3$, respectively; however, we are currently not aware whether they are area minimizing in $\Omega = \mathbb{H}^1$.

4 Results in $\mathbb{H}^n$ with $n \geq 2$

There are only very few results about the Bernstein problem for graphs in the Heisenberg group $\mathbb{H}^n$ with $n \geq 2$, to our knowledge. They are mainly negative results and we collect them below.

4.1 t-graphs in $\mathbb{H}^n$

In this section, we show again there is no Bernstein-type rigidity result for t-graphs also for $n \geq 2$, even in the class of space of functions $\mathcal{X} = \mathbf{C}^2(\mathbb{R}^{2n})$. We do this by simply recovering the counterexamples of $\mathbb{H}^1$ and extending them "cylindrically" to higher Heisenberg groups.

Consider a set $\mathcal{V} = \tilde{\mathcal{V}} \times \hat{\mathcal{V}}$ of $\mathbb{R}^{2h}$, with $1 \leq h < n$, with $\tilde{\mathcal{V}}$ and $\hat{\mathcal{V}}$ open sets of $\mathbb{R}^h$. Assume $v \in C^2(\mathcal{V})$ is a function defined on $\mathcal{V}$: we aim to define (and analyze) a cylindrical extension of v on a subset of $\mathbb{R}^{2n}$.

If $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let us represent $\mathbf{x}$ as

$$\mathbf{x} = (\tilde{\mathbf{x}}, \hat{\mathbf{x}}) \text{ with } \tilde{\mathbf{x}} := (x_1, \ldots, x_h),\ \hat{\mathbf{x}} := (x_{h+1}, \ldots, x_n)\,.$$

According to definition (2.25), we write $\mathbf{X}_h^*$ with the subscript to stress that we are considering the vector field $(-2\mathbf{y}, 2\mathbf{x})$ in $\mathbb{R}^{2h}$. By this notation, we can then consider the function $u : \mathcal{U} := \tilde{\mathcal{V}} \times \mathbb{R}^{n-h} \times \hat{\mathcal{V}} \times \mathbb{R}^{n-h} \subset \mathbb{R}^{2n} \to \mathbb{R}$ defined by

$$u(\mathbf{x}, \mathbf{y}) := v(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\,. \tag{4.1}$$

Notice that in this case, we have:

$$\nabla v = (\nabla_{\tilde{\mathbf{x}}} v, \nabla_{\tilde{\mathbf{y}}} v) \text{ and } \nabla u = (\nabla_{\tilde{\mathbf{x}}} v, 0, \nabla_{\tilde{\mathbf{y}}} v, 0)$$

where

$$\nabla_{\tilde{\mathbf{x}}} v := (\partial_{x_1} v, \ldots, \partial_{x_h} v) \text{ and } \nabla_{\tilde{\mathbf{y}}} v := (\partial_{y_1} v, \ldots, \partial_{y_h} v).$$

This implies that

$$\nabla u + \mathbf{X}^* = (\nabla_{\tilde{\mathbf{x}}} v - 2\tilde{\mathbf{y}}, -2\hat{\mathbf{y}}, \nabla_{\tilde{\mathbf{y}}} v + 2\tilde{\mathbf{x}}, 2\hat{\mathbf{x}})$$

In particular, the characteristic set of u is given by the points

$$\mathrm{Char}(u) = \{(\tilde{\mathbf{x}}, 0, \tilde{\mathbf{y}}, 0) \in \mathcal{U} : (\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \in \mathrm{Char}(v)\}\,. \tag{4.2}$$

Lemma 4.1. *In this context,*

$$\operatorname{div} \frac{\nabla u + \mathbf{X}^*}{|\nabla u + \mathbf{X}^*|}(\mathbf{x}, \mathbf{y}) = \frac{4\Delta v(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})}{|\nabla u + \mathbf{X}^*|^3(\mathbf{x}, \mathbf{y})}(|\hat{\mathbf{x}}|^2 + |\hat{\mathbf{y}}|^2) + \operatorname{div} \frac{\nabla v + \mathbf{X}_h^*}{|\nabla v + \mathbf{X}_h^*|}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \tag{4.3}$$

whenever $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \in \mathcal{V} \setminus \mathrm{Char}(v)$*; moreover*

$$\operatorname{div} \frac{\nabla u + \mathbf{X}^*}{|\nabla u + \mathbf{X}^*|}(\mathbf{x}, \mathbf{y}) = \frac{4\Delta v(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})}{|\nabla u + \mathbf{X}^*|^3(\mathbf{x}, \mathbf{y})}(|\hat{\mathbf{x}}|^2 + |\hat{\mathbf{y}}|^2) \tag{4.4}$$

whenever $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \in \mathrm{Char}(v)$ *and* $(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \neq (0, 0)$.

Proof. Let $(\mathbf{x},\mathbf{y}) \in \mathcal{U} \setminus \mathrm{Char}(u)$. Through straightforward computations we obtain, at the point $(\mathbf{x},\mathbf{y})$:

$$|\nabla u + \mathbf{X}^*|^2 = |\nabla v + \mathbf{X}_h^*|^2 + 4\sum_{i=h+1}^{n}(x_i^2+y_i^2) \tag{4.5}$$

$$\begin{aligned}\mathrm{div}(\nabla u + \mathbf{X}^*) &= \sum_{i=1}^{n}\left(\frac{\partial^2}{\partial x_i^2}u - \frac{\partial}{\partial x_i}2y_i + \frac{\partial^2}{\partial y_i^2}u + \frac{\partial}{\partial y_i}2x_i\right) = \\ &= \mathrm{div}(\nabla v + \mathbf{X}_h^*) = \Delta v,\end{aligned} \tag{4.6}$$

where v and X_h^* are meant to be computed at $(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$. Moreover, one can notice that:

$$\begin{aligned}\frac{\partial}{\partial x_i}|\nabla u + \mathbf{X}^*|^2 &= \frac{\partial}{\partial x_i}\left(|\nabla v + \mathbf{X}_h^*|^2 + \sum_{j=h+1}^{n}4(x_j^2+y_j^2)\right) = \\ &= \begin{cases}\frac{\partial}{\partial x_i}|\nabla v + \mathbf{X}_h^*|^2 & \text{if } 1 \le i \le h \\ 8x_i & \text{if } h+1 \le i \le n\end{cases};\end{aligned} \tag{4.7}$$

$$\frac{\partial}{\partial y_i}|\nabla u + \mathbf{X}^*|^2 = \begin{cases}\frac{\partial}{\partial y_i}|\nabla v + \mathbf{X}_h^*|^2 & \text{if } 1 \le i \le h \\ 8y_i & \text{if } h+1 \le i \le n\end{cases}. \tag{4.8}$$

This in turn implies that

$$\begin{aligned}\Big\langle \nabla u + \mathbf{X}^*, \nabla|\nabla u + \mathbf{X}^*|^2\Big\rangle &= \Big\langle \nabla v + \mathbf{X}_h^*, \nabla|\nabla v + \mathbf{X}_h^*|^2\Big\rangle + \\ &\qquad + 16\sum_{i=h+1}^{n}(-y_i x_i + x_i y_i) \\ &= \Big\langle \nabla v + \mathbf{X}_h^*, \nabla|\nabla v + \mathbf{X}_h^*|^2\Big\rangle.\end{aligned} \tag{4.9}$$

Notice now that, if $F : \mathbb{R}^m \to \mathbb{R}^m$ is a C^1 vector field, and $p \in \mathbb{R}^m$ is a point where F is not zero, then in p the following equality holds:

$$\mathrm{div}\,\frac{F}{|F|} = \frac{1}{|F|^3}\left(|F|^2\,\mathrm{div}\,F - \frac{1}{2}\Big\langle F, \nabla|F|^2\Big\rangle\right). \tag{4.10}$$

Hence, by applying (4.10) to the vector field $\nabla u + \mathbf{X}^*$, we get:

$$\begin{aligned}\mathrm{div}\,\frac{\nabla u + \mathbf{X}^*}{|\nabla u + \mathbf{X}^*|} &= \frac{1}{|\nabla u + \mathbf{X}^*|^3}\Big(|\nabla u + \mathbf{X}^*|^2\,\mathrm{div}(\nabla u + \mathbf{X}_n^*) + \\ &\qquad - \frac{1}{2}\Big\langle \nabla u + \mathbf{X}^*, \nabla|\nabla u + \mathbf{X}^*|^2\Big\rangle\Big) = \\ &= \frac{1}{|\nabla u + \mathbf{X}^*|^3}\Big(|\nabla v + \mathbf{X}_h^*|^2\,\mathrm{div}(\nabla v + \mathbf{X}_h^*) + \\ &\qquad - \frac{1}{2}\Big\langle \nabla v + \mathbf{X}_h^*, \nabla|\nabla v + \mathbf{X}_h^*|^2\Big\rangle + \\ &\qquad + 4\,\mathrm{div}(\nabla v + \mathbf{X}_h^*)\sum_{i=h+1}^{n}(x_i^2+y_i^2)\Big)\end{aligned} \tag{4.11}$$

Now this allows to reach both the conclusions: if $(\tilde{\mathbf{x}},\tilde{\mathbf{y}}) \in \mathrm{Char}(v)$ then the first two terms are zero; if instead $(\tilde{\mathbf{x}},\tilde{\mathbf{y}}) \notin \mathrm{Char}(v)$, then we can apply (4.10) on $\nabla v + \mathbf{X}_h^*$ to obtain (4.3). ☐

Let's see now what this means in our context, when $v \in C^2(\mathcal{V})$ is a (classical) solution of the (t-MSE) in $\mathcal{V} \setminus \mathrm{Char}(v)$:

$$\operatorname{div} \frac{\nabla v + \mathbf{X}_h^*}{|\nabla v + \mathbf{X}_h^*|} = 0 \qquad \text{in } \mathcal{V} \setminus \mathrm{Char}(v). \tag{MSE$_h$}$$

Corollary 4.2. *Assume v solves (MSE$_h$) and u is defined as in (4.1). Then u satisfies (t-MSE) (in $\mathbb{R}^{2n}$) if and only if $\Delta v = 0$.*

Corollary 4.3. *For any subset of indices $J \subset \{1, \dots, n\}$, the map*

$$\tilde{u}_J(x_1, \dots, x_n, y_1, \dots, y_n) := 2 \sum_{j \in J} x_j y_j \tag{4.12}$$

satisfies the minimal surface equation in any open set $\mathcal{U}$ that has empty intersection with

$$\mathrm{Char}(\tilde{u}_J) = \left\{ (x, y) \in \mathbb{R}^{2n} \;\middle|\; \begin{matrix} x_i = 0 & \forall i = 1, \dots n \\ y_i = 0 & \forall i \notin J \end{matrix} \right\}. \tag{4.13}$$

Now, by [CHY07, Corollary F], the following holds:

Proposition 4.4. *Let $\mathcal{U} \subset \mathbb{R}^{2n}$ be bounded, with $n \geq 2$. Assume $\phi \in \mathbf{C}^0(\bar{\mathcal{U}}) \cap \mathbf{C}^2(\mathcal{U})$ and $\phi \in W^{1,1}(\mathcal{U})$. If ϕ satisfies the minimal surface equation (t-MSE) out of the set* $\mathrm{Char}(\phi)$*, then it is a solution to the weak minimal surface equation for t-graphs (WMSE).*

Thanks to Corollary 4.3, we now have a family of smooth maps on $\mathcal{U} := \mathbb{R}^{2n}$ which satisfy the minimal surface equation for t-graphs *out of the singular set* without being affine. Thus, by combining Theorem 2.16 and Proposition 4.4 (the latter being only valid when $n \geq 2$) we get that the t-subgraph $E^t(\phi)$ of any such function ϕ is a minimizer for the $\mathbb{H}$-perimeter in $\Omega = \mathbb{H}^n = \mathbb{R}^{2n+1}$. In particular, such a ϕ is area minimizing in $\Omega = \mathbb{H}^n$.

4.2 Intrinsic graphs: a negative answer in $n \geq 5$

In this section, we show that the Bernstein-type rigidity result fails for intrinsic graphs also for $n \geq 2$, even if we restrict the space of functions to $\mathbf{C}^2(\mathbb{R}^{2n})$. This was proved in [BSV07] (see, also, [Dan+09]). Let us briefly recall the strategy for constructing the example. Recall that a $\mathbf{C}^2$ function ϕ on $\mathbb{R}^{2n}$ with area-stationary intrinsic graph satisfies (IMSE) introduced in Definition 2.24, which we can write as

$$\sum_{j=1}^{n} \left(X_j \frac{X_j \phi}{\sqrt{1 + |\nabla^\phi \phi|^2}} + Y_j \frac{Y_j \phi}{\sqrt{1 + |\nabla^\phi \phi|^2}} \right) + W^\phi \frac{W^\phi \phi}{\sqrt{1 + |\nabla^\phi \phi|^2}} = 0. \tag{4.14}$$

Notice that, if one looks for solutions ϕ which do not depend on the t variable, i.e. such that $\phi(x_2, \dots, x_n, y_1, \dots, y_n, t) = \psi(x_2, \dots, x_n, y_1, \dots, y_n)$ for some $\psi : \mathbb{R}^{2n-1} \to \mathbb{R}$, then (4.14) reduces to the classic minimal surface equation we introduced in Definition 1.2:

$$\operatorname{div} \left(\frac{\nabla \psi}{\sqrt{1 + |\nabla \psi|^2}} \right) = 0 \qquad \text{in } \mathbb{R}^{2n-1}. \tag{MSE}$$

By the classical Bernstein Theorem (Theorem 1.3), we know that if $2n-1 \geq 9$ we can find non-affine analytic solutions to this equation. In particular, if $n \geq 5$, this strategy provides a function $\phi(x_2, \dots, x_n, y_1, \dots, y_n, t) = \psi(x_2, \dots, x_n, y_1, \dots, y_n)$ which solves (4.14) and whose intrinsic graphs is not an intrinsic plane.

We also notice that X_1-graphs of such functions $\phi(x_2, \dots, x_n, y_1, \dots, y_n, t)$ are actually minimizers of the $\mathbb{H}$-perimeter; in fact it is easy to check that the smooth section $\nu : \mathbb{H}^n \to H\mathbb{H}^n$ defined by

$$\begin{aligned}\nu(x, y, t) &= \left(-\frac{1}{\sqrt{1+|\nabla^\phi \phi|^2}}, \frac{\nabla^\phi \phi}{\sqrt{1+|\nabla^\phi \phi|^2}}\right)(x_2, \dots, x_n, y_1, \dots, y_n, 0) \\ &= \left(-\frac{1}{\sqrt{1+|\nabla \psi|^2}}, \frac{\nabla \psi}{\sqrt{1+|\nabla \psi|^2}}\right)(x_2, \dots, x_n, y_1, \dots, y_n)\end{aligned} \tag{4.15}$$

is a calibration for the X_1-graph of ϕ (see Theorem 2.9), i.e.

- $\operatorname{div}_X \nu = 0$ in $\mathbb{H}^n$;
- $|\nu(p)| = 1$ for all $p \in \mathbb{H}^n$;
- ν coincides with the horizontal inward normal to the X_1-graph of ϕ (see Theorem 2.22).

Observe that in this argument (which is basically the same used to prove the minimality of any entire graph solution of (MSE) in the classical case) it was essential the non-dependence of ϕ on the vertical variable t: as we have seen in Section 3.3.1, in general it is not true that an entire solution of (IMSE) parametrizes a minimizer.

4.3 Comments for the remaining cases

To our knowledge, what happens for intrinsic graphs in dimensions $n = 2, 3, 4$ remains unknown. One important observation can be made: by the same argument used in Section 4.2, it is clear that any (smooth) counterexample to the Bernstein-type rigidity result must depend on the variable t: otherwise, the projected function $\psi(x_2, \dots, x_n, y_1, \dots, y_n) = \phi(x_2, \dots, x_n, y_1, \dots, y_n, t)$ solves the classical minimal surface equation, thus it is (Euclidean) affine and the its X_1-graph is a vertical hyperplane. Eventually it is easy to see that the strategy exploited in Section 4.1 for t-graphs, of extending cylindrically to higher Heisenberg, does not work for intrinsic graphs.

References

[Alm66] F. J. Almgren Jr. "Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem". In: *Ann. of Math. (2)* 84 (1966), pp. 277–292.

[ASV06] L. Ambrosio, F. Serra Cassano, and D. Vittone. "Intrinsic regular hypersurfaces in Heisenberg groups". In: *J. Geom. Anal.* 16.2 (2006), pp. 187–232.

[BC10] F. Bigolin and F. S. Cassano. "Distributional solutions of Burgers' equation and intrinsic regular graphs in Heisenberg groups". In: *J. Math. Anal. Appl.* 366.2 (2010), pp. 561–568.

[BDG69] E. Bombieri, E. De Giorgi, and E. Giusti. "Minimal cones and the Bernstein problem". In: *Invent. Math.* 7 (1969), pp. 243–268.

[Ber17] S. Bernstein. “Sur un thèoréme de géometrie et ses applications aux équations dérivées partielles du type elliptique”. In: *Comm. Soc. Math. Karkhov* 15 (1915-17), pp. 38–45.

[Ber27] S. Bernstein. “Über ein geometrisches Theorem und seine Anwendung auf die partiellen Differentialgleichungen vom elliptischen Typus”. In: *Mathematische Zeitschrift* 26.1 (Dec. 1927), 551–558 (Transleted from French).

[BSV07] V. Barone Adesi, F. Serra Cassano, and D. Vittone. “The Bernstein problem for intrinsic graphs in Heisenberg groups and calibrations”. In: *Calc. Var. Partial Differential Equations* 30.1 (2007), pp. 17–49.

[CDG94] L. Capogna, D. Danielli, and N. Garofalo. “The geometric Sobolev embedding for vector fields and the isoperimetric inequality”. In: *Comm. Anal. Geom.* 2.2 (1994), pp. 203–215.

[Che+05] J.-H. Cheng et al. “Minimal surfaces in pseudohermitian geometry”. In: *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* 4.1 (2005), pp. 129–177.

[CHY07] J.-H. Cheng, J.-F. Hwang, and P. Yang. “Existence and uniqueness for p-area minimizers in the Heisenberg group”. In: *Math. Ann.* 337.2 (2007), pp. 253–293.

[Dan+09] D. Danielli et al. “Instability of graphical strips and a positive answer to the Bernstein problem in the Heisenberg group $\mathbb{H}^1$”. In: *J. Differential Geom.* 81.2 (2009), pp. 251–295.

[Dan+10] D. Danielli et al. “The Bernstein problem for embedded surfaces in the Heisenberg group $\mathbb{H}^1$”. In: *Indiana Univ. Math. J.* 59.2 (2010), pp. 563–594.

[DeG65] E. De Giorgi. “Una estensione del teorema di Bernstein”. In: *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)* 19 (1965), pp. 79–85.

[DGN08] D. Danielli, N. Garofalo, and D. M. Nhieu. “A notable family of entire intrinsic minimal graphs in the Heisenberg group which are not perimeter minimizing”. In: *Amer. J. Math.* 130.2 (2008), pp. 317–339.

[Fle62] W. H. Fleming. “On the oriented Plateau problem”. In: *Rend. Circ. Mat. Palermo (2)* 11 (1962), pp. 69–90.

[FSS01] B. Franchi, R. Serapioni, and F. Serra Cassano. “Rectifiability and perimeter in the Heisenberg group”. In: *Math. Ann.* 321.3 (2001), pp. 479–531.

[FSS96] B. Franchi, R. Serapioni, and F. Serra Cassano. “Meyers-Serrin type theorems and relaxation of variational integrals depending on vector fields”. In: *Houston J. Math.* 22.4 (1996), pp. 859–890.

[Giu84] E. Giusti. *Minimal surfaces and functions of bounded variation.* Vol. 80. Monographs in Mathematics. Birkhäuser Verlag, Basel, 1984, pp. xii+240.

[GP02] N. Garofalo and S. D. Pauls. “The Bernstein Problem in the Heisenberg Group”. In: *arXiv Mathematics e-prints* (Sept. 2002).

[GR15] M. Galli and M. Ritoré. “Area-stationary and stable surfaces of class C^1 in the sub-Riemannian Heisenberg group $\mathbb{H}^1$”. In: *Adv. Math.* 285 (2015), pp. 737–765.

[Gro96] M. Gromov. “Carnot-Carathéodory spaces seen from within”. In: *Sub-Riemannian geometry.* Vol. 144. Progr. Math. Birkhäuser, Basel, 1996, pp. 79–323.

[HRR10] A. Hurtado, M. Ritoré, and C. Rosales. “The classification of complete stable area-stationary surfaces in the Heisenberg group $\mathbb{H}^1$”. In: *Adv. Math.* 224.2 (2010), pp. 561–600.

[KS04] B. Kirchheim and F. Serra Cassano. “Rectifiability and parameterization of intrinsic regular surfaces in the Heisenberg group”. In: *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* 3.4 (2004), pp. 871–896.

[Mag12] F. Maggi. *Sets of finite perimeter and geometric variational problems.* Vol. 135. Cambridge Studies in Advanced Mathematics. An introduction to geometric measure theory. Cambridge University Press, Cambridge, 2012, pp. xx+454.

[Mir64] M. Miranda. “Superfici cartesiane generalizzate ed insiemi di perimetro localmente finito sui prodotti cartesiani”. In: *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)* 18 (1964), pp. 515–542.

[MM84] U. Massari and M. Miranda. *Minimal surfaces of condimension* 1. Vol. 91. Mathematical Studies. North-Holland, Amsterdam-New York-Oxford, 1984, pp. xii+242.

[Mon02] R. Montgomery. *A tour of subriemannian geometries, their geodesics and applications.* Vol. 91. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2002, pp. xx+259.

[MSV08] R. Monti, F. Serra Cassano, and D. Vittone. “A negative answer to the Bernstein problem for intrinsic graphs in the Heisenberg group”. In: *Boll. Unione Mat. Ital. (9)* 1.3 (2008), pp. 709–727.

[NS19] S. Nicolussi Golo and F. Serra Cassano. “The Bernstein problem for Lipschitz intrinsic graphs in the Heisenberg group”. In: *Calc. Var. Partial Differential Equations* 58.4 (2019), Art. 141, 28.

[Rit09] M. Ritoré. “Examples of area-minimizing surfaces in the sub-Riemannian Heisenberg group $\mathbb{H}^1$ with low regularity”. In: *Calc. Var. Partial Differential Equations* 34.2 (2009), pp. 179–192.

[Ser16] F. Serra Cassano. “Some topics of geometric measure theory in Carnot groups”. In: *Geometry, analysis and dynamics on sub-Riemannian manifolds. Vol. 1.* EMS Ser. Lect. Math. Eur. Math. Soc., Zürich, 2016, pp. 1–121.

[Sim68] J. Simons. “Minimal varieties in riemannian manifolds”. In: *Ann. of Math. (2)* 88 (1968), pp. 62–105.

[SV14] F. Serra Cassano and D. Vittone. “Graphs of bounded variation, existence and local boundedness of non-parametric minimal surfaces in Heisenberg groups”. In: *Adv. Calc. Var.* 7.4 (2014), pp. 409–492.