HOMOGENISATION OF HIGH-CONTRAST BRITTLE MATERIALS

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Abstract. This paper is an overview on some recent results concerning the variational analysis of static fracture in the so-called high-contrast brittle composite materials. The paper is divided into two main parts. The first part is devoted to establish a compactness result for a general class of free-discontinuity functionals with degenerate (or high-contrast) integrands. The second part is focussed on some specific examples which show that the degeneracy of the integrands may lead to non-standard limit effects, which are specific to this high-contrast setting.

Keywords: Γ-convergence, homogenisation, free-discontinuity problems, high-contrast materials, porous materials, brittle fracture.

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1. Introduction

In this note we analyse the large-scale behaviour of *high-contrast* composite materials which can undergo fracture. In a variational setting, the microscopic behaviour of high-contrast composites is typically described by means of scale-dependent energy functionals with "degenerate" integrands. For *brittle* materials the scale-dependent energies are of the general form

$$\mathcal{F}_\varepsilon(u) = \int_\Omega f_\varepsilon(x, \nabla u)\, dx + \int_{S_u} g_\varepsilon(x, \nu_u)\, d\mathcal{H}^{n-1}, \tag{1.1}$$

where $\varepsilon > 0$ describes both the composite-microstructure and the degeneracy of the mechanical properties of the material (cf. (1.3)). In (1.1) the variable $u\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}$ belongs to $SBV(\Omega)$, the space of special functions of bounded variation in Ω. In this simplified scalar setting, u represents an anti-plane displacement and Ω is the cross-section of an infinite cylindrical body. Being u an SBV-function, discontinuities are allowed and the discontinuity set of u, denoted by S_u, models the cracks in the material. The deformation gradient Du can be decomposed into the sum of a bulk part $\nabla u\, dx$ and a surface part $(u^+ - u^-)\nu_u \mathcal{H}^{n-1} \llcorner S_u$, where ∇u denotes the approximate gradient of u, u^+ and u^- the traces of u on both sides of S_u, and ν_u denotes the (generalised) normal to S_u. The volume term in $\mathcal{F}_\varepsilon$ represents the elastic energy stored in the unfractured part of the material, whereas the surface term in $\mathcal{F}_\varepsilon$ accounts for the presence of cracks. According to the Griffith criterion, in brittle materials, already for the smallest crack-amplitude, there is no interaction between the two lips of the crack, so that the corresponding fracture energy does not depend on $[u] = u^+ - u^-$.

For *finite-contrast* brittle materials, the limit behaviour of energies of type (1.1) is by-now well-understood and the corresponding theory provides a rigorous micro-to-macro upscaling for brittle fracture. In fact, if f_ε and g_ε satisfy (mild regularity assumptions and) standard growth and coercivity conditions of type

$$c_1|\xi|^p \le f_\varepsilon(x,\xi) \le c_2(1+|\xi|^p) \quad \text{and} \quad c_3 \le g_\varepsilon(x,\nu) \le c_4, \tag{1.2}$$

for every $\varepsilon > 0$, $x, \xi \in \mathbb{R}^n$, $\nu \in \mathbb{S}^{n-1}$, for some $p > 1$, and $0 < c_1 \le c_2 < +\infty$, $0 < c_3 \le c_4 < +\infty$, then in [18] Giacomini and Ponsiglione showed, among other, that the limit behaviour of $\mathcal{F}_\varepsilon$ is captured by a scale-independent free-discontinuity functional of the same type as $\mathcal{F}_\varepsilon$; *i.e.,*

$$\mathcal{F}_0(u) = \int_\Omega f_0(x, \nabla u)\, dx + \int_{S_u} g_0(x, \nu_u)\, d\mathcal{H}^{n-1},$$

with f_0 and g_0 also satisfying (1.2). Under these assumptions, Giacomini and Ponsiglione also showed that volume and surface energy *decouple* in the limit, so that the energy density f_0 is not affected by the presence of the surface term in $\mathcal{F}_\varepsilon$, whereas the surface energy density g_0 is not affected by the volume term in $\mathcal{F}_\varepsilon$. In a recent work, Cagnetti, Dal Maso, Scardia, and Zeppieri [12] generalised the asymptotic analysis carried out in [18] and devised (nearly optimal) sufficient conditions which ensure a macroscopic bulk-surface energy decoupling for a wide class of *finite-contrast* vectorial free-discontinuity functionals which may also depend on $[u]$. The class of periodic free-discontinuity functionals originally analysed by Braides, Defranceschi and Vitali [9] satisfy the sufficient conditions provided in [12]. Moreover, random free-discontinuity functionals with stationary finite-contrast integrands can be also seen as a special instance of those treated in [12], as shown by Cagnetti, Dal Maso, Scardia, and Zeppieri in [13]. Therefore, a volume-surface interaction can be ruled out for a large class of finite-contrast free-discontinuity functionals. In this setting, in particular, microscopic brittle energies always converge to macroscopic brittle energies. However, the general theory established in [9, 12, 13, 18] is not well-suited for studying the large-scale behaviour of those brittle composites whose different constituents have very different mechanical properties from one another. Indeed, in this case the integrands f_ε and g_ε in (1.1) may exhibit a so-called *high-contrast* behaviour and satisfy (1.2) only in a subset Ω_ε of Ω.

In the last decade there has been an ever increasing interest in the study of high-contrast free-discontinuity functionals and in the derivation of their effective properties. In particular, the case where (at least) one of the conditions in (1.2) is violated in "many small" periodically distributed regions inside Ω has been considered (see, *e.g.*, [3, 4, 5, 6, 11, 14, 16, 17, 19, 20, 21]). Depending on the type of degeneracy of f_ε and g_ε, nonstandard limit effects have been also observed. These nonstandard effects are typical of the high-contrast setting and arise from a nontrivial volume-surface limit interaction, which cannot be excluded in this degenerate setting. In fact, in the two companion papers [4, 16], Barchiesi, Dal Maso, and Zeppieri show that when only g_ε is degenerate, already for very simple free-discontinuity functionals of Mumford-Shah type, a bulk-surface interaction cannot be ruled out. Namely, a volume-surface coupling can be observed when homogenising a material made of "many" purely brittle inclusions periodically distributed in a connected unbreakable structure, whose fracture-resistance is assumed to be infinite. This coupling produces a homogeneous material whose overall behaviour is of *ductile* (or cohesive) type; in other words, the homogenised surface energy explicitly depends on $[u]$. A similar phenomenon is also observed by Barchiesi, Lazzaroni, and Zeppieri [6] who show that a ductile behaviour can be seen as the macroscopic effect of a nontrivial volume-surface interaction in the homogenisation of two purely brittle materials with a high-contrast bulk energy. Moreover, in the recent work [19] Pellet, Scardia, and Zeppieri prove, instead, that nonstandard constitutive laws may arise when homogenising two purely brittle materials with a high-contrast surface energy. The functionals analysed in [6] and [19] are both of type

$$\mathcal{F}_\varepsilon^{\alpha_\varepsilon,\beta_\varepsilon}(u) = \int_\Omega a_\varepsilon\Big(\frac{x}{\varepsilon}\Big)|\nabla u|^2\,dx + \int_{S_u} b_\varepsilon\Big(\frac{x}{\varepsilon}\Big)\,d\mathcal{H}^{n-1} \tag{1.3}$$

where the elastic modulus a_ε and the fracture resistance (or fracture thoughness) b_ε are Q-periodic functions and in the unit periodicity cell $Q := (-1/2, 1/2)^n$ are defined as

$$a_\varepsilon(y) = \begin{cases} \alpha_\varepsilon & \text{if } y \in \overline{Q}_r \\ 1 & \text{if } y \in Q \setminus \overline{Q}_r \end{cases} \qquad b_\varepsilon(y) = \begin{cases} \beta_\varepsilon & \text{if } y \in \overline{Q}_r \\ 1 & \text{if } y \in Q \setminus \overline{Q}_r \end{cases}$$

with $\alpha_\varepsilon, \beta_\varepsilon \in [0,1]$, $r \in (0,1)$, and $Q_r := (-r/2, r/2)^n$. Since $\alpha_\varepsilon, \beta_\varepsilon$ are not bounded away from zero, the functions a_ε and b_ε can be degenerate. In their turn, the integrands $f_\varepsilon(y,\xi) = a_\varepsilon(y)|\xi|^2$ and $g_\varepsilon(y,\nu) = b_\varepsilon(y)$ in (1.3) will not satisfy, in general, the coercivity conditions in (1.2).

The limit case $\alpha_\varepsilon = \beta_\varepsilon = 0$ corresponds to the case of periodically perforated brittle materials studied by Cagnetti and Scardia [14] and by Focardi, Gelli, and Ponsiglione [17] (see also Barchiesi and Focardi [5] for more general free-discontinuity functionals). In spite of the strong degeneracy of the coefficients a_ε and b_ε, which in this case are equal to zero in a "large" portion of Ω, in this case it can be proven that the functionals $\mathcal{F}_\varepsilon^{0,0}$ exhibit a limit behaviour which is qualitatively similar to that of free-discontinuity functionals with coercive integrands. Namely, in this case bulk and surface terms *do not* interact in the limit.

The aim of this note is to show that, contrary to the coercive case, where general homogenisation results can be proven to describe the limit behaviour of a large class of free-discontinuity functionals, in the non-coercive setting, already for special functionals of type (1.3), a unified homogenisation theory cannot be established. In fact, the limit behaviour of $\mathcal{F}_\varepsilon^{\alpha_\varepsilon,\beta_\varepsilon}$ is highly sensitive both to the choice of the parameters α_ε and β_ε and to their vanishing rate compared to the period of the microstructure ε.

This note is divided into two main parts and organised as follows. In first part we will deal with sequences of general free-discontinuity functionals of type (1.1) whose coefficients f_ε and g_ε are "weakly coercive" or "degenerate"; *i.e.,* they satisfy the lower bounds in (1.2) only in a set Ω_ε which is obtained removing from Ω many small periodically distributed connected regions. We will use the localisation method of Γ-convergence [8, 15] to prove that these kind of functionals are (pre)compact. That is, up to subsequences, they always Γ-converge to a free-discontinuity functional of type

$$\int_\Omega f(x,\nabla u)\,dx + \int_{S_u} g(x,[u],\nu_u)\,d\mathcal{H}^{n-1}.$$

Moreover, the limit integrands f and g are *non-degenerate* and satisfy coercivity conditions of type (1.2) for some positive constants $\hat{c}_1, \hat{c}_3$ which are strictly smaller than c_1, c_3, respectively. In this part of the analysis a pivotal role is played by an extension result for SBV-functions defined in periodically perforated domains, proved by Cagnetti and Scardia [14] (see also the later variant in [5]).

In the second part of this note we will specialise the general theory to some prototypical and yet relevant model cases. Namely, we will briefly review the case of *perforated (or porous) brittle materials* studied by Cagnetti and Scardia [14] and by Focardi, Gelli, and Ponsiglione [17] (see also [5]), the case of *high-contrast brittle materials with soft inclusions* treated by Barchiesi, Lazzaroni, and Zeppieri in [6], and eventually the case of *high-contrast brittle materials with weak inclusions* analysed by Pellet, Scardia, and Zeppieri in [19]. In particular we will show that the choice of the integrands f_ε and g_ε in (1.1) strongly affects the form of the Γ-limit which can give rise to macroscopic models accounting for damage as well as to models accounting for cohesive fracture.

2. Part I: A compactness result for high-contrast free-discontinuity functionals

In this part we will use the localisation method of Γ-convergence [8, 15] to prove a convergence result for a general class of free-discontinuity functionals of brittle type, with degenerate coefficients.

In the choice of the convergence to compute the Γ-limit, a crucial role will be played by an extension result for SBV-functions defined in periodically perforated domains due to Cagnetti and Scardia [14, Theorem 1.3] and by a later variant due to Barchiesi and Focardi [5, Theorem 1].

2.1. **Notation and setting of the problem.** We list below a few notation which will be used throughout the paper.

- $\Omega \subset \mathbb{R}^n$ denotes an open and bounded set with Lipschitz boundary. The set $\mathcal{A}(\Omega)$ denotes the collection of all open subsets of Ω;
- Q denotes the open unit cube of $\mathbb{R}^n$ centred at the the origin, whereas for $x \in \mathbb{R}^n$ and $r > 0$ we set $Q_r(x) := rQ + x$;
- for $\nu \in \mathbb{S}^{n-1}$ we denote with Q^ν the open unit cube of $\mathbb{R}^n$ centred at the the origin, with one face orthogonal to ν and for $x \in \mathbb{R}^n$ and $r > 0$ we set $Q^\nu_r(x) := rQ^\nu + x$;
- for $x \in \mathbb{R}^n$ and $\nu \in \mathbb{S}^{n-1}$ we denote by $\Pi^\nu(x)$ the hyperplane through x and perpendicular to ν; *i.e.,* $\Pi^\nu(x) := \{y \in \mathbb{R}^n \colon (y - x) \cdot \nu = 0\}$. If $x = 0$ we simply write Π^ν;
- For $u \in L^1(\Omega)$ and $m > 0$ the function u^m denotes the truncated function of u at level m; *i.e.,* $u^m := (u \wedge m) \vee (-m)$;
- For $\xi \in \mathbb{R}^n$ we denote by u_ξ the linear function with gradient equal to ξ; *i.e.,* $u_\xi(x) := \xi \cdot x$, for every $x \in \mathbb{R}^n$;
- For $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, and $\nu \in \mathbb{S}^{n-1}$ we denote with $u^{\nu,t}_x$ the piecewise constant function taking values $0, t$ and jumping across the hyperplane $\Pi^\nu(x)$; *i.e.,*

$$u^{\nu,t}_x(y) := \begin{cases} t & \text{if } (y - x) \cdot \nu \geq 0, \\ 0 & \text{if } (y - x) \cdot \nu < 0. \end{cases}$$

The functional setting we are going to consider in this note is that of SBV, the space of special functions of bounded variation. We recall here only the definition of the spaces which are relevant for our analysis and we refer the reader to [2] for a comprehensive treatment on the subject. We set

$$SBV(\Omega) := \{u \in BV(\Omega) : \; Du = \nabla u \mathcal{L}^n + (u^+ - u^-)\nu_u d\mathcal{H}^{n-1} \llcorner S_u\}.$$

Here S_u denotes the approximate discontinuity set of u, ν_u is the generalised normal to S_u, u^+ and u^- are the traces of u on both sides of S_u. In this paper we work with the following vector subspace of $SBV(\Omega)$

$$SBV^p(\Omega) := \{u \in SBV(\Omega) : \; \nabla u \in L^p(\Omega) \text{ and } \mathcal{H}^{n-1}(S_u) < +\infty\},$$

where $p > 1$. We consider also the larger space of generalised special functions of bounded variation in Ω,

$$GSBV(\Omega) := \{u \in L^1(\Omega) : \; u^m \in SBV(\Omega) \text{ for all } m \in \mathbb{N}\},$$

as well as

$$GSBV^p(\Omega) := \{u \in GSBV(\Omega) : \; \nabla u \in L^p(\Omega) \text{ and } \mathcal{H}^{n-1}(S_u) < +\infty\}.$$

We consider also

$$SBV^{\mathrm{pc}}(\Omega) := \{u \in SBV(\Omega) : \nabla u = 0 \; \mathcal{L}^n\text{-a.e.}, \mathcal{H}^{n-1}(S_u) < +\infty\};$$

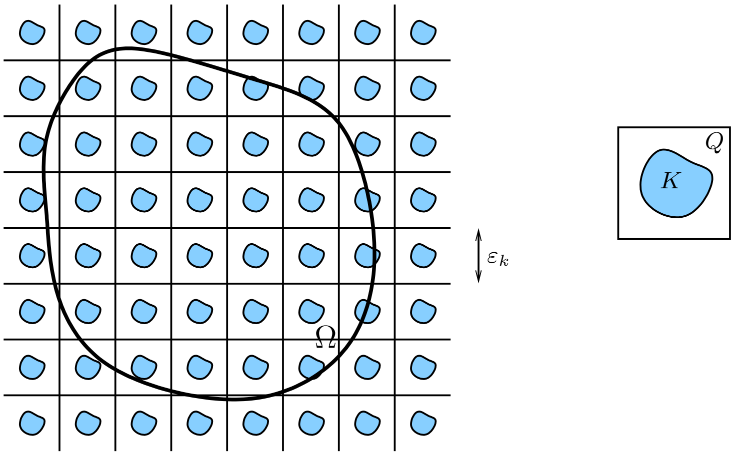

FIGURE 1. Schematic of a high-contrast composite material.

it is known (see [2, Theorem 4.23]) that every u in $SBV^{\rm pc}(\Omega) \cap L^\infty(\Omega)$ is piecewise constant in the sense of [2, Definition 4.21], namely there exists a Caccioppoli partition (E_i) of Ω such that u is constant $\mathcal{L}^n$-a.e. in each set E_i. Moreover, we set

$$\mathcal{P}(\Omega) := \{u \in SBV^{\rm pc}(\Omega)\colon u(x) \in \{0,1\} \ \mathcal{L}^n\text{- a.e. in } \Omega\}.$$

For $u, w \in L^1(U)$, in what follows, by "$u = w$ near ∂U" we mean that there exists a neighbourhood V of ∂U in $\mathbb{R}^n$ such that $u = w$ $\mathcal{L}^n$-a.e. in $V \cap U$.

Let $f_k\colon \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ be Carathéodory functions such that

(H1) there exist $p > 1$ and $0 < c_1 \le c_2 < +\infty$ such that for every $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and for every $k \in \mathbb{N}$

$$c_1|\xi|^p \le f_k(x, \xi) \le c_2(1 + |\xi|^p); \tag{2.1}$$

(H2) $f_k(x, 0) = 0$ for every $x \in \mathbb{R}^n$ and for every $k \in \mathbb{N}$.

Let moreover $g_k\colon \mathbb{R}^n \times \mathbb{S}^{n-1} \to (0, +\infty)$ be Borel functions such that

(H3) there exist $0 < c_3 \le c_4 < +\infty$ such that for every $(x, \nu) \in \mathbb{R}^n \times \mathbb{S}^{n-1}$ and every $k \in \mathbb{N}$

$$c_3 \le g_k(x, \nu) \le c_4; \tag{2.2}$$

(H4) $g_k(x, \nu) = g_k(x, -\nu)$, for every $(x, \nu) \in \mathbb{R}^n \times \mathbb{S}^{n-1}$ and every $k \in \mathbb{N}$.

Let $\Omega \subset \mathbb{R}^n$ be open bounded and with Lipschitz boundary and let $K \subset Q$ be compact and such that $Q \setminus K$ has a Lipschitz boundary. We define

$$E := \mathbb{R}^n \setminus \bigcup_{i \in \mathbb{Z}^n} (K + i);$$

the set E is open, connected, Q-periodic, and has a Lipschitz boundary. Let ε_k be a sequence of positive numbers such that $\varepsilon_k \searrow 0$ as $k \to +\infty$ and denote by Ω_k the $\varepsilon_k Q$-periodic set defined as $\Omega_k := \Omega \cap \varepsilon_k E$ (see Figure 1).

Let moreover $\alpha_k, \beta_k \in [0,1]$ and consider the sequence of functionals $\mathcal{F}_k\colon L^1(\Omega) \longrightarrow [0, +\infty]$ defined as

$$\mathcal{F}_k(u) := \begin{cases} \displaystyle\int_{\Omega_k} f_k(x, \nabla u)\, dx + \alpha_k \int_{\Omega \setminus \Omega_k} f_k(x, \nabla u)\, dx + \int_{S_u \cap \Omega_k} g_k(x, \nu_u)\, d\mathcal{H}^{n-1} + \beta_k \int_{S_u \cap (\Omega \setminus \Omega_k)} g_k(x, \nu_u)\, d\mathcal{H}^{n-1} & \text{if } u \in SBV^p(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases} \tag{2.3}$$

We observe that thanks to assumption (H2) the functionals $\mathcal{F}_k$ decrease by truncation, wherease they do not satisfy the standard coercivity conditions required, *e.g.*, in [9, 12, 18] since the coefficients α_k, β_k are not bounded away from zero.

2.2. **Equi-coercivity and choice of the convergence.** Due to the possible degeneracy of the coefficients α_k and β_k, the functionals $\mathcal{F}_k$ are not, in general, equi-coercive with respect to the strong $L^1(\Omega)$-convergence. Similarly as in [5, 11, 17, 19], in what follows we give a notion of convergence on $L^1(\Omega)$ which is weaker that the $L^1(\Omega)$-convergence and ensures the equi-coercivity of the functionals $\mathcal{F}_k$. This will be done by appealing to [5, Theorem 1]. For the readers' convenience we recall here a slightly simplified version of this result which is useful for our purposes.

Theorem 2.1 (cf. Theorem 1 in [5]). *Let $(u_k) \subset SBV^p(\Omega_k)$ be such that*

$$\sup_k \Big(\int_{\Omega_k} |u_k|^p\, dx + \int_{\Omega_k} |\nabla u_k|^p\, dx + \mathcal{H}^{n-1}(S_{u_k} \cap \Omega_k) \Big) < +\infty. \tag{2.4}$$

Then, there exist $(\tilde{u}_k) \subset SBV^p(\Omega)$, with $\tilde{u}_k = u_k$ a.e. in Ω_k, and a function $u \in GSBV^p(\Omega) \cap L^p(\Omega)$ such that (up to subsequences) $\tilde{u}_k \to u$ in $L^1(\Omega)$.

If moreover $\sup_k \|u_k\|_{L^\infty(\Omega_k)} < +\infty$ then $u \in SBV^p(\Omega) \cap L^\infty(\Omega)$ and $\tilde{u}_k \to u$ in $L^p(\Omega)$.

Let $(u_k) \subset L^1(\Omega)$ be a sequence satisfying

$$\sup_k \|u_k\|_{L^p(\Omega_k)} < +\infty \quad \text{and} \quad \sup_k \mathcal{F}_k(u_k) < +\infty.$$

Then, clearly $(u_k) \subset SBV^p(\Omega_k)$; moreover in view of (H1) and (H3) the sequence (u_k) satisfies the uniform bound (2.4). Therefore invoking Theorem 2.1 immediately yields the existence of a function $u \in GSBV^p(\Omega) \cap L^p(\Omega)$ and a sequence $(\tilde{u}_k) \subset SBV^p(\Omega)$ with $\tilde{u}_k = u_k$ a.e. in Ω_k, such that (up to subsequences not relabelled) $\tilde{u}_k \to u$ in $L^1(\Omega)$.

This observation motivates the choice of the following notion of convergence on $L^1(\Omega)$.

Definition 2.2 (Convergence). Let (u_k) be a sequence in $L^1(\Omega)$. We say that (u_k) converges to a function $u \in L^1(\Omega)$, and we write $u_k \rightsquigarrow u$, if there exists a sequence $(\tilde{u}_k) \subset L^1(\Omega)$ such that $\tilde{u}_k = u_k$ a.e. in Ω_k, and $\tilde{u}_k$ converges to u in $L^1(\Omega)$.

Remark 2.3 (Uniqueness of the limit). We observe that since $C(K) := \mathcal{L}^n(Q \setminus K) > 0$, then the limit in the sense of Definition 2.2 is well-defined. Indeed, assume that $u_k \rightsquigarrow u_1$ and $u_k \rightsquigarrow u_2$. Then by definition there exist $(\tilde{u}_{1,k}), (\tilde{u}_{2,k}) \subset L^1(\Omega)$ such that $\tilde{u}_{1,k} = \tilde{u}_{2,k} = u_k$ in Ω_k and $\tilde{u}_{1,k} \to u_1$ and $\tilde{u}_{2,k} \to u_2$ in $L^1(\Omega)$. Therefore

$$0 = \lim_{k\to+\infty} \int_{\Omega_k} |\tilde{u}_{1,k} - \tilde{u}_{2,k}|\, dx = \lim_{k\to+\infty} \int_{\Omega} |\tilde{u}_{1,k} - \tilde{u}_{2,k}| \chi_{\Omega_k}\, dx = C(K) \int_{\Omega} |u_1 - u_2|\, dx,$$

where the last inequality follows by the Riemann-Lebesgue Theorem applied to the $\varepsilon_k Q$-periodic function χ_{Ω_k}. Then, since $C(K) > 0$ we necessarily have $u_1 = u_2$ a.e. in Ω.

We notice moreover that the convergence $u_k \rightsquigarrow u$ readily implies

$$\lim_{k\to+\infty} \|u_k - u\|_{L^1(\Omega_k)} = 0.$$

Remark 2.4 (Convergence of truncated functions). Let $(u_k) \subset L^1(\Omega)$ be such that $u_k \rightsquigarrow u$ for some $u \in L^1(\Omega)$. Let $m \in \mathbb{N}$ and denote by (u_k^m) the sequence of truncated functions of u_k at level m, then $u_k^m \rightsquigarrow u^m$ where u^m denotes the truncated function of u at level m. Indeed, set $v_k := (\tilde{u}_k)^m$, then $v_k = u_k^m$ a.e. in Ω_k, moreover since $\tilde{u}_k \to u$ in $L^1(\Omega)$ then $(\tilde{u}_k)^m \to u^m$ in $L^1(\Omega)$, and actually in any $L^p(\Omega)$.

In what follows we study the Γ-convergence of the functionals $\mathcal{F}_k$ with respect to the convergence as in Definition 2.2. To this end we give the following sequential notion of Γ-convergence.

Definition 2.5 (Sequential Γ-convergence). Let $\mathcal{F}_k, \mathcal{F} \colon L^1(\Omega) \longrightarrow [0, +\infty]$; we say that the functionals $\mathcal{F}_\varepsilon$ Γ-converge to $\mathcal{F}$ with respect to the convergence as in Definition 2.2 if for every $u \in L^1(\Omega)$ the two following conditions are satisfied:

(i) (Ansatz-free lower bound) For every $(u_k) \subset L^1(\Omega)$ with $u_k \rightsquigarrow u$ we have

$$\mathcal{F}(u) \leq \liminf_{k\to+\infty} \mathcal{F}_k(u_k);$$

(ii) (Existence of a recovery sequence) There exists $(\bar{u}_k) \subset L^1(\Omega)$ with $\bar{u}_k \rightsquigarrow u$ such that

$$\mathcal{F}(u) \geq \limsup_{k\to+\infty} \mathcal{F}_k(\bar{u}_k).$$

Remark 2.6. It is standard to show that $\mathcal{F}$ is lower semicontinuous with respect to the convergence as in Definition 2.2 and hence with respect to the strong $L^1(\Omega)$-convergence.

For every $u \in L^1(\Omega)$ we consider the functionals

$$\Gamma\text{-}\liminf_{k\to+\infty} \mathcal{F}_k(u) := \inf\Big\{ \liminf_{k\to+\infty} \mathcal{F}_k(u_k) \colon u_k \rightsquigarrow u \Big\} \tag{2.5}$$

and

$$\Gamma\text{-}\limsup_{k\to+\infty} \mathcal{F}_k(u) := \inf\Big\{ \limsup_{k\to+\infty} \mathcal{F}_k(u_k) \colon u_k \rightsquigarrow u \Big\}. \tag{2.6}$$

It is easy to show that the infima in (2.5) and (2.6) are actually attained.

In what follows we also use the compact notation

$$\mathcal{F}'(u) := \Gamma\text{-}\liminf_{k\to+\infty} \mathcal{F}_k(u) \quad\text{and}\quad \mathcal{F}''(u) := \Gamma\text{-}\limsup_{k\to+\infty} \mathcal{F}_k(u). \tag{2.7}$$

It is immediate to see that Definition 2.5 is equivalent to $\mathcal{F}' = \mathcal{F}'' = \mathcal{F}$ in $L^1(\Omega)$.

Remark 2.7 (The case $\alpha_k, \beta_k = 0$). In the case of porous brittle materials [5, 14, 17], which corresponds to the parameter choice $\alpha_k, \beta_k = 0$, the Γ-convergence of the functionals $\mathcal{F}_k$ can be equivalently studied with respect to the strong $L^1(\Omega)$-convergence. Indeed, in this case a sequence (u_k) with equibounded energy can be replaced by the $L^1(\Omega)$-converging sequence $(\tilde{u}_k)$ given by Theorem 2.1, without changing the energy.

The following proposition shows that the domain of the Γ-limit of $\mathcal{F}_k$ (if it exists) is $GSBV^p(\Omega)$.

Proposition 2.8 (Domain of the Γ-limit)**.** *Let $\mathcal{F}'$ and $\mathcal{F}''$ be as in* (2.7)*; then*

$$\operatorname{dom}\mathcal{F}' = \operatorname{dom}\mathcal{F}'' = GSBV^p(\Omega).$$

Proof. We first show that $GSBV^p(\Omega) \subset \operatorname{dom}\mathcal{F}''$. By the growth conditions (2.1) and (2.2) we have $\mathcal{F}_k(u) \le \mathcal{G}(u)$ where

$$\mathcal{G}(u) := \begin{cases} c_2 \displaystyle\int_\Omega (1 + |\nabla u|^p)\,dx + c_3 \mathcal{H}^{n-1}(S_u \cap \Omega) & \text{in } GSBV^p(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega) \end{cases} \tag{2.8}$$

The functional $\mathcal{G}$ is lower semicontinuous with respect to the strong $L^1(\Omega)$-convergence, hence we have

$$\inf\Big\{ \limsup_{k\to+\infty} \mathcal{F}_k(u_k) \colon u_k \to u \text{ in } L^1(\Omega) \Big\} \le \mathcal{G}(u).$$

Since the convergence in Definition 2.2 is weaker than the $L^1(\Omega)$-convergence we then have $\mathcal{F}'' \le \mathcal{G}$, and thus the desired inclusion.

We now prove that $\operatorname{dom}\mathcal{F}' \subset GSBV^p(\Omega)$. To this end, let $u \in \operatorname{dom}\mathcal{F}'$ then there exists $(u_k) \subset L^1(\Omega)$ with $u_k \rightsquigarrow u$ such that $\liminf_k \mathcal{F}_k(u_k) = \mathcal{F}(u) < +\infty$. Then, up to subsequences (not relabelled) we have $\sup_k \mathcal{F}(u_k) < +\infty$, thus in particular $(u_k) \subset SBV^p(\Omega)$.

Let $m \in \mathbb{N}$ and let u_k^m be the truncated function of u_k at level m; then $(u_k^m) \subset SBV^p(\Omega) \cap L^\infty(\Omega)$. Since the functionals $\mathcal{F}_k$ decrease by truncation, for every fixed $m \in \mathbb{N}$ it also holds $\sup_k \mathcal{F}_k(u_k^m) < +\infty$. Therefore, for $m \in \mathbb{N}$ fixed we can appeal to Theorem 2.1 to deduce the existence of a sequence $(v_k) \subset L^1(\Omega)$ such that $v_k = u_k^m$ a.e. in Ω_k and of a function $v \in SBV^p(\Omega)$ such that up to subsequence (not relabelled) $v_k \to v$ in $L^1(\Omega)$. Since $u_k \rightsquigarrow u$ we have

$$0 = \lim_{k\to+\infty} \int_{\Omega_k} |v_k - u_k^m|\,dx = \lim_{k\to+\infty} \int_\Omega |v_k - (\tilde{u}_k)^m| \chi_{\Omega_k}\,dx = C(K) \int_\Omega |v - u^m|\,dx,$$

therefore $v = u^m$ a.e. in Ω. Eventually, the arbitrariness of $m \in \mathbb{N}$ yields $u \in GSBV^p(\Omega)$. $\square$

2.3. **Γ-convergence and integral representation.** In this section we show that, up to subsequences, the functionals $\mathcal{F}_k$ Γ-converge to a free-discontinuity functional of the form

$$\mathcal{F}(u) = \int_\Omega f_\infty(x, \nabla u)\,dx + \int_{S_u} g_\infty(x, u^+ - u^-, \nu_u)\,d\mathcal{H}^{n-1}.$$

for some f_∞ and g_∞. Moreover, we show that, despite the degeneracy of the coefficients α_k, β_k, the limit integrands f_∞ and g_∞ satisfy standard coercivity conditions similar to (2.1) and (2.2), respectively.

If not otherwise specified, in what follows the Γ-convergence of the functionals $\mathcal{F}_k$ is always understood in the sense of Definition 2.5.

To prove the existence of a Γ-convergent subsequence of $\mathcal{F}_k$ we make use of the so-called localisation method [8, 15] which we adapt to the sequential notion of Γ-convergence as in Definition 2.5.

We start by localising the functionals $\mathcal{F}_k$; that is we consider $\mathcal{F}_k : L^1(\Omega) \times \mathcal{A}(\Omega) \longrightarrow [0,+\infty]$ defined as

$$\mathcal{F}_k(u,U) := \begin{cases} \displaystyle\int_{U_k} f_k(x,\nabla u)\,dx + \alpha_k \int_{U\setminus U_k} f_k(x,\nabla u)\,dx + \int_{S_u\cap U_k} g_k(x,\nu_u)\,d\mathcal{H}^{n-1} + \beta_k \int_{S_u\cap(U\setminus U_k)} g_k(x,\nu_u)\,d\mathcal{H}^{n-1} & \text{if } u\in SBV^p(U), \\ +\infty & \text{otherwise in } L^1(\Omega), \end{cases} \tag{2.9}$$

where $U_k := U \cap \varepsilon_k E$.

We also define the localised versions of (2.5) and (2.6); *i.e.,* for every $U \in \mathcal{A}(\Omega)$ we consider the functionals defined as

$$\mathcal{F}'(\cdot,U) := \Gamma\text{-}\liminf_{k\to+\infty} \mathcal{F}_k(\cdot,U), \qquad \mathcal{F}''(\cdot,U) := \Gamma\text{-}\limsup_{k\to+\infty} \mathcal{F}_k(\cdot,U). \tag{2.10}$$

Remark 2.9 (Properties of $\mathcal{F}',\mathcal{F}''$). It is easy to show that $\mathcal{F}'$ and $\mathcal{F}''$ are lower semicontinuous with respect to the convergence in Definition 2.2, local, and that they decrease by truncation. Moreover, as set functions they are both increasing, whereas $\mathcal{F}'$ is also superadditive.

Remark 2.10 (On assumption (H2)). If we drop assumption (H2) the functionals $\mathcal{F}_k$ will not decrease by truncation, but rather satisfy

$$\mathcal{F}_k(u^m,U) \le \mathcal{F}_k(u,U) + c_2\mathcal{L}^n(U_k\cap\{|u|\ge m\}) + \alpha_k\, c_2\mathcal{L}^n(U\setminus U_k\cap\{|u|\ge m\}). \tag{2.11}$$

If α_k is infinitesimal, the inequality in (2.11) implies

$$\mathcal{F}'(u^m,U) \le \mathcal{F}'(u,U) + \frac{c_2}{m}\|u\|_{L^1(\Omega)}, \tag{2.12}$$

(and analogously for $\mathcal{F}''$). In fact, by definition of Γ-liminf there exists a sequence $(u_k) \subset L^1(\Omega)$ such that $u_k \rightsquigarrow u$ and $\mathcal{F}'(u,U) = \liminf_{k\to+\infty} \mathcal{F}_k(u_k,U)$. Then if u_k^m is the truncated function of u_k at level m, by (2.11) we get

$$\begin{aligned} \mathcal{F}_k(u_k^m,U) &\le \mathcal{F}_k(u_k,U) + c_2\mathcal{L}^n(U_k\cap\{|u_k|\ge m\}) + \alpha_k\, c_2\mathcal{L}^n(U\setminus U_k\cap\{|u_k|\ge m\}) \\ &\le \mathcal{F}_k(u_k,U) + c_2\mathcal{L}^n(U_k\cap\{|\tilde u_k|\ge m\}) + \alpha_k\, c_2\mathcal{L}^n(\Omega) \\ &\le \mathcal{F}_k(u_k,U) + c_2\mathcal{L}^n(U\cap\{|\tilde u_k|\ge m\}) + \alpha_k\, c_2\mathcal{L}^n(\Omega) \end{aligned}$$

where $\tilde u_k$ is as in Definition 2.2 and thus $\tilde u_k \to u$ in $L^1(\Omega)$. Therefore, taking the liminf as $k\to+\infty$ gives

$$\liminf_{k\to+\infty} \mathcal{F}_k(u_k^m,U) \le \mathcal{F}'(u,U) + c_2\mathcal{L}^n(U\cap\{|u|\ge m\}),$$

hence (2.12) follows by the definition of Γ-liminf, taking into account that $u_k^m \rightsquigarrow u^m$, and by the Chebyshev inequality. Therefore, (2.12) ensures that $\mathcal{F}'$ "almost" decreases by truncation up to an error which becomes small for m large. Inequality (2.12) is then enough to to carry out the Γ-convergence analysis below (cf. [12]). Hence, if α_k is infinitesimal assumption (H2) can be dropped.

However, if the sequence α_k is uniformly bounded from below, we have no control on the term $\alpha_k\, c_2\mathcal{L}^n(U\setminus U_k\cap\{|u_k|\ge m\})$, therefore from (2.11) we cannot infer (2.12). Since with we want to study the Γ convergence of $\mathcal{F}_k$ for any choices of $\alpha_k\in[0,1]$, assumption (H2) is actually necessary.

In general the set functions $\mathcal{F}'(u,\cdot)$ and $\mathcal{F}''(u,\cdot)$ are not inner regular. Then we consider their inner regular envelopes defined as:

$$\mathcal{F}'_-(u,U) := \sup\big\{\mathcal{F}'(u,V)\colon V\subset\subset U,\ V\in\mathcal{A}(\Omega)\big\}.$$

and

$$\mathcal{F}''_-(u,U) := \sup\big\{\mathcal{F}''(u,V)\colon V\subset\subset U,\ V\in\mathcal{A}(\Omega)\big\}.$$

Remark 2.11 (Properties of $\mathcal{F}'_-,\mathcal{F}''_-$). The functionals $\mathcal{F}'_-$ and $\mathcal{F}''_-$ are lower semicontinuous with respect to the convergence in Definition 2.2 [15, Remark 15.10], local [15, Remark 15.25], and it is immediate to check that they decrease by truncation. Furthermore, as set functions, they are both increasing and $\mathcal{F}'_-$ is superadditive [15, Remark 15.10].

The following compactness result is the analogue of [15, Theorem 16.9], when the sequential notion of Γ-convergence in Definition 2.5 is considered. We omit its proof since it is standard.

Proposition 2.12 (Compactness by Γ-convergence)**.** *Let $\mathcal{F}_k$ be the localised functionals as in* (2.9)*. Then there exists a subsequence $(\mathcal{F}_{k_j}) \subset (\mathcal{F}_k)$ such that the corresponding functionals $\mathcal{F}'$ and $\mathcal{F}''$ defined in* (2.10) *satisfy $\mathcal{F}'_- = \mathcal{F}''_-$.*

We now set

$$\mathcal{F} := \mathcal{F}'_- = \mathcal{F}''_-. \tag{2.13}$$

In what follows we show that actually $\mathcal{F}$ coincides with the Γ-limit of the subsequence $(\mathcal{F}_{k_j})$. To this end we start noticing that by monotonicity we always have $\mathcal{F}''_- = \mathcal{F}'_- \leq \mathcal{F}' \leq \mathcal{F}''$. Therefore, if we show that $\mathcal{F}'' = \mathcal{F}''_-$; *i.e.*, that $\mathcal{F}''$ is inner regular, we immediately get $\mathcal{F}' = \mathcal{F}'' = \mathcal{F}$ and therefore that $\mathcal{F}_{k_j}(\cdot, U)$ Γ-converges to $\mathcal{F}(\cdot, U)$ for every $U \in A(\Omega)$, as desired.

A crucial preliminary result needed to prove the inner-regularity of $\mathcal{F}''$ is the so-called fundamental estimate, which has to hold uniformly in k. Since the Γ-limit is computed with respect to the convergence in Definition 2.2, the fundamental estimate we need is non-standard. Namely, we have to prove that the error in the fundamental estimate tends to zero when $u_k \rightsquigarrow u$. This is achieved by first showing that the error goes like $\|u_k - u\|_{L^p(\Omega_k)}$ and then by resorting to a truncation argument.

We notice that an analogous estimate for degenerate functionals defined in Sobolev spaces can be found in [10, Proposition 3.3]. Whereas in the SBV-setting, for functionals of Mumford-Shah type with degenerate surface energy it can be found in the recent [19, Lemma 4.4].

Following [10] we start showing how to construct suitable cut-off functions which are constant in $\bigcup_{i\in\mathbb{Z}^n} \varepsilon_k(K+i)$. To this end let $\delta > 0$ be small enough so that the set $K_\delta := \{x \in \mathbb{R}^n : \operatorname{dist}(x, K) < \delta\}$ satisfies $K_\delta \subset\subset Q$. Let $\psi \in C_0^\infty(Q)$ be a cut-off function between K and K_δ (that is $0 \leq \psi \leq 1$, $\psi \equiv 1$ on K, and $\operatorname{spt}\psi \subset K_\delta$) such that $|\nabla\psi| \leq \frac{2}{\eta}$.

For $k \in \mathbb{N}$ and $i \in \mathbb{Z}^n$, we define the operator $R_i^k : W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n) \to W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$ as

$$R_i^k(\phi)(x) := \left(1 - \psi\left(\frac{x}{\varepsilon_k} - i\right)\right)\phi(x) + \psi\left(\frac{x}{\varepsilon_k} - i\right) \fint_{\varepsilon_k K_\delta + \varepsilon_k i} \phi(y)\,dy.$$

By definition we have that

$$R_i^k(\phi)(x) = \phi(x) \quad \text{if } x \notin \varepsilon_k K_\delta + \varepsilon_k i,$$

while R_i^k is constant in $\varepsilon_k K + \varepsilon_k i$, namely we have

$$R_i^k(\phi)(x) = \fint_{\varepsilon_k K_\delta + \varepsilon_k i} \phi(y)\,dy \quad \text{if } x \in \varepsilon_k K + \varepsilon_k i.$$

Finally, we consider the operator $\mathcal{R}^k : W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n) \to W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$ defined as

$$\mathcal{R}^k(\phi)(x) := \begin{cases} R_i^k(\phi)(x) & \text{if } x \in \varepsilon_k K_\delta + \varepsilon_k i,\ i \in \mathbb{Z}^n, \\ \phi(x) & \text{otherwise.} \end{cases}$$

Let $U \subset \mathbb{R}^n$ be open and bounded and let $\phi \in W^{1,\infty}(U)$ then $\nabla\mathcal{R}^k(\phi)$ is uniformly bounded in k. More precisely, we have

$$\|\nabla\mathcal{R}^k(\phi)\|_{L^\infty(U;\mathbb{R}^n)} \leq \left(\frac{2}{\delta}d + 1\right)\|\nabla\phi\|_{L^\infty(U;\mathbb{R}^n)}, \tag{2.14}$$

where d denotes the diameter of K_δ. In fact,

$$\|\nabla\mathcal{R}^k(\phi)\|_{L^\infty(U;\mathbb{R}^n)} \leq \frac{2}{\varepsilon_k\delta} \sup_i \left\|\phi - \fint_{\varepsilon K_\delta + \varepsilon i} \phi(y)\,dy\right\|_{L^\infty(\varepsilon_k K_\delta + \varepsilon_k i;\mathbb{R}^n)} + \|\nabla\phi\|_{L^\infty(U;\mathbb{R}^n)}$$

and

$$\left\|\phi - \fint_{\varepsilon_k K_\delta + \varepsilon_k i} \phi(y)\,dy\right\|_{L^\infty(\varepsilon_k K_\delta + \varepsilon_k i;\mathbb{R}^n)} \leq \varepsilon_k d\|\nabla\phi\|_{L^\infty(U;\mathbb{R}^n)}.$$

In the next proposition we make use of the operator $\mathcal{R}^k$ to construct cut-off functions whose gradient vanishes in $\mathbb{R}^n \setminus \varepsilon_k E$; these cut-off functions are then used to prove the desired fundamental estimate.

Proposition 2.13 (Fundamental estimate)**.** *For every $\eta > 0$, and for every U', U'', $V \in \mathcal{A}(\Omega)$, with $U' \subset\subset U''$, there exist two constants $M(\eta) > 0$ and $k_\eta \in \mathbb{N}$ satisfying the following property: for every $k > k_\eta$, for every $u \in L^1(\Omega)$ with $u \in SBV^p(U'')$, and for every $v \in L^1(\Omega)$ with $v \in SBV^p(V)$, there exists a function $\varphi \in C_0^\infty(\Omega)$ with $\varphi = 1$ in a neighbourhood of U', $\operatorname{spt}\varphi \subset U''$ and $0 \leq \varphi \leq 1$ such that*

$$\mathcal{F}_k(\varphi u + (1-\varphi)v, U' \cup V) \leq (1+\eta)\big(\mathcal{F}_k(u, U'') + \mathcal{F}_k(v, V)\big) + M(\eta)||u - v||_{L^p(S\cap\varepsilon_k E)} \tag{2.15}$$

where $S := (U'' \setminus U') \cap V$.

Proof. Let $U', U'', V \in \mathcal{A}(\Omega)$ be as in the statement. Let $\eta > 0$ be fixed and choose $N \in \mathbb{N}$ in a way such that

$$\frac{1}{N} \max\left\{c_2\mathcal{L}^n((U'' \setminus U') \cap V), 3^{p-1} \max\left\{\frac{c_2}{c_1}, \frac{c_4}{c_3}\right\}\right\} < \eta. \tag{2.16}$$

Let moreover $U \in \mathcal{A}(\Omega)$ be such that $U' \subset\subset U \subset\subset U''$ and consider the open sets

$$U' \subset\subset U_1 \subset\subset \ldots \subset\subset U_{3N} \subset\subset U''$$

where

$$U_l := \left\{x \colon \mathrm{dist}(x, U') < \frac{\mathrm{dist}(U', \partial U)}{3N} l\right\}, \quad \text{for every } l = 1, \ldots, 3N.$$

We notice that by definition of U_l we have that

$$\mathrm{dist}(U_l, \partial U_{l+1}) = \frac{1}{3N}, \quad \text{for every } l = 1, \ldots, 3N-1. \tag{2.17}$$

For every $j = 0, \ldots, N-1$ let ϕ_j be a cut-off function between U_{3j+1} and U_{3j+2} with $|\nabla \phi_j| < 4N$.

Let $k_\eta \in \mathbb{N}$ be such that

$$\varepsilon_k d < \frac{1}{3N} \quad \text{for every } k > k_\eta, \tag{2.18}$$

where $d := \mathrm{diam}(K_\delta) < \sqrt{2}$.

If $i \in \mathbb{Z}^n$ is such that $(\varepsilon_k K + \varepsilon_k i) \cap U_l \neq \emptyset$ for every $k > k_\eta$, then thanks to (2.17)-(2.18) we can deduce that $(\varepsilon_k K + \varepsilon_k i) \cap (\mathbb{R}^n \setminus U_{l+1}) = \emptyset$. Therefore the functions $\varphi_j := \mathcal{R}^k(\phi_j)$ are cut-off functions between the sets U_{3j} and $U_{3(j+1)}$, for every $j = 0, \ldots, N-1$ (where we have set $U_0 := U'$).

Now let $u \in SBV^p(U'')$ and $v \in SBV^p(V)$; for every $j = 0, \ldots, N-1$ fixed we have

$$\begin{aligned}
\mathcal{F}_k(\varphi_j u + (1-\varphi_j)v, U' \cup V) &= \mathcal{F}_k(u, (U' \cup V) \cap \overline{U}_{3j}) + \mathcal{F}_k(v, V \setminus U_{3(j+1)}) \\
&+ \mathcal{F}_k(\varphi_j u + (1-\varphi_j)v, V \cap (U_{3(j+1)} \setminus \overline{U}_{3j})) \leq \mathcal{F}_k(u, U'') + \mathcal{F}_k(v, V) \\
&+ \mathcal{F}_k(\varphi_j u + (1-\varphi_j)v, V \cap (U_{3(j+1)} \setminus \overline{U}_{3j})).
\end{aligned}$$

We set

$$w_j := \varphi_j u + (1-\varphi_j)v, \quad S_j := V \cap (U_{3(j+1)} \setminus \overline{U}_{3j})$$

and estimate the term $\mathcal{F}_k(w_j, S_j)$. We clearly have

$$\mathcal{F}_k(w_j, S_j) = \mathcal{F}_k(w_j, S_j \setminus \varepsilon_k E) + \mathcal{F}_k(w_j, S_j \cap \varepsilon_k E). \tag{2.19}$$

By construction $\nabla \varphi_j = 0$ in $\mathbb{R}^n \setminus \varepsilon_k E$, therefore appealing to (2.1) and (2.2) we deduce

$$\begin{aligned}
\mathcal{F}_k(w_j, S_j \setminus \varepsilon_k E) &= \alpha_k \int_{S_j \setminus \varepsilon_k E} f_k(x, \varphi_j \nabla u + (1-\varphi_j)\nabla v)\, dx + \beta_k \int_{(S_j \setminus \varepsilon_k E) \cap S_{w_j}} g_k(x, \nu_{w_j})\, d\mathcal{H}^{n-1} \\
&\leq c_2 \alpha_k \left(\mathcal{L}^n(S_j \setminus \varepsilon_k E) + \int_{S_j \setminus \varepsilon_k E} |\nabla u|^p\, dx + \int_{S_j \setminus \varepsilon_k E} |\nabla v|^p\, dx\right) \\
&+ c_4 \beta_k \left(\mathcal{H}^{n-1}((S_j \setminus \varepsilon_k E) \cap S_u) + \mathcal{H}^{n-1}((S_j \setminus \varepsilon_k E) \cap S_v)\right) \\
&\leq c_2 \alpha_k \mathcal{L}^n(S_j \setminus \varepsilon_k E) + \frac{c_2}{c_1}\left(\alpha_k \int_{S_j \setminus \varepsilon_k E} f_k(x, \nabla u)\, dx + \alpha_k \int_{S_j \setminus \varepsilon_k E} f_k(x, \nabla v)\, dx\right) \\
&+ \frac{c_4}{c_3}\left(\beta_k \int_{(S_j \setminus \varepsilon_k E) \cap S_u} g_k(x, \nu_u)\, d\mathcal{H}^{n-1} + \beta_k \int_{(S_j \setminus \varepsilon_k E) \cap S_v} g_k(x, \nu_v)\, d\mathcal{H}^{n-1}\right) \\
&\leq c_2 \alpha_k \mathcal{L}^n(S_j \setminus \varepsilon_k E) + \max\left\{\frac{c_2}{c_1}, \frac{c_4}{c_3}\right\}\Big(\mathcal{F}_k(u, S_j \setminus \varepsilon_k E) + \mathcal{F}_k(v, S_j \setminus \varepsilon_k E)\Big).
\end{aligned} \tag{2.20}$$

Moreover, again invoking (2.1) and (2.2), in $\varepsilon_k E$ we have

$$\begin{aligned}
\mathcal{F}_k(w_j, S_j \cap \varepsilon_k E) &= \int_{S_j\cap\varepsilon_k E} f_k(x, \nabla w_j)\, dx + \int_{S_j\cap\varepsilon_k E\cap S_{w_j}} g_k(x, \nu_{w_j})\, d\mathcal{H}^{n-1} \\
&\le c_2\Big(\mathcal{L}^n(S_j\cap\varepsilon_k E) + \int_{S_j\cap\varepsilon_k E} |\nabla\varphi_j(u-v) + \varphi_j\nabla u + (1-\varphi_j)\nabla v|^p\, dx\Big) \\
&\quad + c_4\Big(\mathcal{H}^{n-1}((S_j\cap\varepsilon_k E)\cap S_u) + \mathcal{H}^{n-1}((S_j\cap\varepsilon_k E)\cap S_v)\Big) \\
&\le c_2\mathcal{L}^n(S_j\cap\varepsilon_k E) + c_2 3^{p-1}\|\nabla\varphi_j\|^p_{L^\infty(U;\mathbb{R}^n)}\int_{S_j\cap\varepsilon_k E}|u-v|^p\, dx \\
&\quad + 3^{p-1}\max\Big\{\frac{c_2}{c_1}, \frac{c_4}{c_3}\Big\}\Big(\mathcal{F}_k(u, S_j\cap\varepsilon_k E) + \mathcal{F}_k(v, S_j\cap\varepsilon_k E)\Big).
\end{aligned} \tag{2.21}$$

Since $|\nabla\phi_j| \le 4N$, combining the definition of φ_j with (2.14) gives

$$\|\nabla\varphi_j\|_{L^\infty(U;\mathbb{R}^n)} \le \Big(\frac{2}{\delta}d + 1\Big)4N. \tag{2.22}$$

In view of (2.19), by gathering (2.20)-(2.22) we then obtain for every $j = 0, \dots, N-1$

$$\mathcal{F}_k(w_j, S_j) \le c_2\mathcal{L}^n(S_j) + 3^{p-1}\max\Big\{\frac{c_2}{c_1}, \frac{c_4}{c_3}\Big\}\Big(\mathcal{F}_k(u, S_j) + \mathcal{F}_k(v, S_j)\Big) + M(\eta)\int_{S_j\cap\varepsilon_k E}|u-v|^p\, dx,$$

where

$$M(\eta) := c_2 3^{p-1}\Big(\frac{2}{\delta}d + 1\Big)^p (4N)^p.$$

Therefore there exists $j^* \in \{0, \dots, N-1\}$ such that

$$\begin{aligned}
\mathcal{F}_k(w_{j^*}, S_{j^*}) &\le \frac{1}{N}\sum_{i=0}^{N-1}\mathcal{F}_k(w_j, S_j) \le \frac{c_2}{N}\mathcal{L}^n((U''\setminus U')\cap V) \\
&+ \frac{3^{p-1}}{N}\max\Big\{\frac{c_2}{c_1}, \frac{c_4}{c_3}\Big\}\Big(\mathcal{F}_k(u, U'') + \mathcal{F}_k(v, V)\Big) + M(\eta)\int_{(U''\setminus U')\cap V\cap\varepsilon_k E}|u-v|^p\, dx.
\end{aligned}$$

Finally the thesis follows from (2.16) by choosing φ_{j^*} as a cut-off function and setting $S := (U''\setminus U')\cap V$. □

Thanks to the fundamental estimate Proposition 2.13 we are now able to prove the following abstract Γ-convergence result for the sequence of localised functionals $\mathcal{F}_{k_j}(\cdot, U)$.

Theorem 2.14 (Abstract Γ-convergence and properties of the Γ-limit)**.** *Let $\mathcal{F}$ be as in* (2.13)*, then:*

1. (locality and lower semicontinuity) *for every $U \in \mathcal{A}(\Omega)$, the functional $\mathcal{F}(\cdot, U)$ is local and lower semicontinuous with respect to the $L^1(\Omega)$-convergence;*
2. (measure property) *for every $u \in GSBV^p(\Omega)$, the set function $\mathcal{F}(u, \cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure on Ω;*
3. (Γ-convergence) *for every $U \in \mathcal{A}(\Omega)$ it holds $\mathcal{F}(\cdot, U) = \mathcal{F}'(\cdot, U) = \mathcal{F}''(\cdot, U)$ on $GSBV^p(\Omega)$;*
4. (translational invariance in u) *for every $u \in L^1(\Omega)$ and $U \in \mathcal{A}(\Omega)$ there holds $\mathcal{F}(u + s, U) = \mathcal{F}(u, U)$ for every $s \in \mathbb{R}$.*

Proof. Since the $L^1(\Omega)$-convergence implies the convergence in the sense of Definition 2.2, property 1 immediately follows from Remark 2.11. In view of Remark 2.11, property 2 follows by the De Giorgi and Letta criterion (see, *e.g.*, [15, Theorem 14.23]) once we show that for every $u \in GSBV^p(\Omega)$ the set function $\mathcal{F}(u, \cdot)$ is subadditive. In its turn, the subadditivity of $\mathcal{F}(u, \cdot)$ follows from Proposition 2.13. Since in our setting this proof is not entirely standard, we discuss it in detail for the readers' convenience.

We start observing that on $GSBV^p(\Omega)$ the functional $\mathcal{F}$ satisfies the following limsup-type inequality: For every $u \in GSBV^p(\Omega)$ and for every $U, U' \in \mathcal{A}(\Omega)$ with $U' \subset\subset U$, there exists a sequence $(u_j) \subset GSBV^p(U') \cap L^1(\Omega)$ with $u_j \rightsquigarrow u$ such that

$$\limsup_{j\to+\infty}\mathcal{F}_{k_j}(u_j, U') \le \mathcal{F}(u, U)$$

(see, *e.g.*, [15, Proposition 16.4 and Remark 16.5] also recalling that the infimum in the definition of $\mathcal{F}''$ is actually attained).

Now let $U, V \in \mathcal{A}(\Omega)$ and let $u \in GSBV^p(\Omega) \cap L^\infty(\Omega)$. Fix any $U' \subset\subset U$, $V' \subset\subset V$, $U', V' \in \mathcal{A}(\Omega)$. Choose an open set U'' such that $U' \subset\subset U'' \subset\subset U$ and two sequences $(u_j) \subset GSBV^p(U'') \cap L^1(\Omega)$ and $(v_j) \subset GSBV^p(V') \cap L^1(\Omega)$, with $u_j \rightsquigarrow u$ and $v_j \rightsquigarrow u$ such that

$$\limsup_{j\to+\infty} \mathcal{F}_{k_j}(u_j, U'') \le \mathcal{F}(u, U), \quad \limsup_{j\to+\infty} \mathcal{F}_{k_j}(v_j, V') \le \mathcal{F}(u, V). \tag{2.23}$$

Since the functionals $\mathcal{F}_k$ decrease by truncation, we can additionally assume that $\|u_j\|_{L^\infty(\Omega)}, \|v_j\|_{L^\infty(\Omega)} \le \|u\|_{L^\infty(\Omega)}$; clearly,

$$\lim_{j\to+\infty} \|u_j - u\|_{L^p(\Omega_{k_j})} = \lim_{j\to+\infty} \|v_j - u\|_{L^p(\Omega_{k_j})} = 0. \tag{2.24}$$

Let $\eta > 0$ be fixed and arbitrary. The fundamental estimate Proposition 2.13 provides us with constants $M(\eta) > 0$ and $j_\eta \in \mathbb{N}$ and with a sequence (φ_j) of cut-off functions between U' and U'' such that

$$\begin{aligned} &\mathcal{F}_{k_j}(\varphi_j u_j + (1-\varphi_j)v_j, U' \cup V') \\ &\le (1+\eta)\Big(\mathcal{F}_{k_j}(u_j, U'') + \mathcal{F}_{k_j}(v_j, V')\Big) + M(\eta)\|u_j - v_j\|^p_{L^p(\Omega_{k_j})} \end{aligned}$$

for every $j \ge j_\eta$. Hence appealing to (2.23), to the convergence $\varphi_j u_j + (1-\varphi_j)v_j \rightsquigarrow u$, and to the obvious inequality $\mathcal{F} \le \mathcal{F}'$, by taking the limit as $j \to +\infty$, we get

$$\mathcal{F}(u, U' \cup V') \le (1+\eta)\Big(\mathcal{F}(u, U) + \mathcal{F}(u, V)\Big).$$

Now letting $\eta \to 0$, and then $U' \nearrow U$, $V' \nearrow V$ in view of the inner-regularity of $\mathcal{F}$ we get

$$\mathcal{F}(u, U \cup V) \le \mathcal{F}(u, U) + \mathcal{F}(u, V), \tag{2.25}$$

hence the subadditivity of $\mathcal{F}(u, \cdot)$ for $u \in GSBV^p(\Omega) \cap L^\infty(\Omega)$.

Now let $u \in GSBV^p(\Omega)$ and, for every $m \in \mathbb{N}$, set $u^m := (u \wedge m) \vee (-m)$. Then, since $\mathcal{F}$ decreases by truncation (2.25) immediately gives

$$\mathcal{F}(u^m, U \cup V) \le \mathcal{F}(u, U) + \mathcal{F}(u, V).$$

Then, taking the limit as $m \to +\infty$, in view of the convergence $u^m \to u$ in $L^1(\Omega)$ and the lower semicontinuity of $\mathcal{F}$ we obtain

$$\mathcal{F}(u, U \cup V) \le \liminf_{m\to+\infty} \mathcal{F}(u^m, U \cup V) \le \mathcal{F}(u, U) + \mathcal{F}(u, V),$$

and thus the subadditivity of $\mathcal{F}(u, \cdot)$ for every $u \in GSBV^p(\Omega)$.

The proof of property 3 is achieved by showing that $\mathcal{F}''$ is inner-regular. Indeed, this is equivalent to $\mathcal{F}'' = \mathcal{F}''_-$, which by definition of $\mathcal{F}$ implies $\mathcal{F}'' \le \mathcal{F} \le \mathcal{F}'$. Since clearly $\mathcal{F}' \le \mathcal{F}''$, we actually deduce that $\mathcal{F}$ is the Γ-limit of $\mathcal{F}_{k_j}$.

The inner regularity of $\mathcal{F}''$ follows from the fundamental estimate Proposition 2.13. To see this, for every $U \in \mathcal{A}(\Omega)$ let $\mathcal{G}(\cdot, U)$ be the localised version of the functional $\mathcal{G}$ defined in (2.8); *i.e.,*

$$\mathcal{G}(u, U) := \begin{cases} c_2 \displaystyle\int_U (1 + |\nabla u|^p)\,dx + c_4 \mathcal{H}^{n-1}(S_u \cap U) & \text{if } u \in GSBV^p(U), \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases} \tag{2.26}$$

Now fix $W \in \mathcal{A}(\Omega)$ and $u \in GSBV^p(\Omega)$; since $\mathcal{G}(u, \cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure, for every $\eta > 0$ there exists a compact set $\widetilde{W} \subset W$ such that and $MS(u, W \setminus \widetilde{W}) < \eta$.

Now choose $U, U' \in \mathcal{A}(\Omega)$ satisfying $\widetilde{W} \subset U' \subset\subset U \subset\subset W$ and set $V := W \setminus \widetilde{W}$. Recalling that $\mathcal{F}''(u, \cdot)$ is increasing, appealing to Proposition 2.13 easily gives

$$\mathcal{F}''(u, W) \le \mathcal{F}''(u, U' \cup V) \le \mathcal{F}''(u, U) + \mathcal{F}''(u, V) = \mathcal{F}''(u, U) + \mathcal{F}''(u, W \setminus \widetilde{W}).$$

Recalling that $\mathcal{F}'' \le \mathcal{G}$, by taking the sup on $U \subset\subset W$ we get

$$\mathcal{F}''(u, W) \le \mathcal{F}''_-(u, W) + \mathcal{G}(u, W \setminus \widetilde{W}) \le \mathcal{F}''_-(u, W) + \eta.$$

Hence, by the arbitrariness of $\eta > 0$ we get $\mathcal{F}''(u, W) \le \mathcal{F}''_-(u, W)$ for every $W \in \mathcal{A}(\Omega)$ and every $u \in GSBV^p(\Omega)$. Since the opposite inequality is always satisfied, we readily deduce the inner regularity of $\mathcal{F}''(u, \cdot)$, as desired.

Eventually, the proof of property 4 is standard and follows as in, *e.g.*, [9, Lemma 3.7]. □

In the following theorem we show that the Γ-limit $\mathcal{F}$ can be represented in an integral form as a free-discontinuity functional. Moreover, thanks to [5, Theorem 4] the functional $\mathcal{F}$ turns out to be non-degenerate, unlike the functionals $\mathcal{F}_k$.

Theorem 2.15 (Integral representation of the Γ-limit)**.** *Let $\mathcal{F}$ be as in Theorem 2.14. Then, there exist a Carathéodory function $f_\infty\colon \mathbb{R}^n \times \mathbb{R}^n \to [0,+\infty)$ and a Borel function $g_\infty\colon \mathbb{R}^n \times \mathbb{S}^{n-1} \to (0,+\infty)$ such that*

$$\mathcal{F}(u,U) = \int_U f_\infty(x,\nabla u)\,dx + \int_{S_u\cap U} g_\infty(x,[u],\nu_u)\,d\mathcal{H}^{n-1} \tag{2.27}$$

for every $u \in GSBV^p(\Omega)$ and every $U \in \mathcal{A}(\Omega)$.

Furthermore, the function $f_\infty\colon \mathbb{R}^n \times \mathbb{R}^n \to [0,+\infty)$ satisfies the following properties:

i) (convexity in ξ) *for a.e. $x \in \mathbb{R}^n$, $f_\infty(x,\cdot)$ is convex;*

ii) (p growth and coercivity) *there exists $\tilde{c}_1 > 0$ such that for a.e. $x \in \mathbb{R}^n$ and for every $\xi \in \mathbb{R}^n$ it holds*

$$\tilde{c}_1|\xi|^p \le f_\infty(x,\xi) \le c_2(1+|\xi|^p), \tag{2.28}$$

where c_2 is as in (2.1)*.*

The function $g_\infty\colon \mathbb{R}^n \times \mathbb{R} \times \mathbb{S}^{n-1} \to [0,+\infty)$ satisfies the following properties:

iii) (monotonicity in t and symmetry) *for a.e. $x \in \mathbb{R}^n$ and for every $\nu \in \mathbb{S}^{n-1}$, $g_\infty(x,\cdot,\nu)$ is nondecreasing on $(0,+\infty)$ and satisfies the symmetry condition $g_\infty(x,-t,-\nu) = g_\infty(x,t,\nu)$ for every $t \in \mathbb{R}$;*

iv) (subadditivity in t) *for a.e. $x \in \mathbb{R}^n$ and for every $\nu \in \mathbb{S}^{n-1}$*

$$g_\infty(x,t_1+t_2,\nu) \le g_\infty(x,t_1,\nu) + g_\infty(x,t_2,\nu),$$

for every $t_1,t_2 \in \mathbb{R}$;

v) (convexity in ν) *for a.e. $x \in \mathbb{R}^n$ and for every $t \in \mathbb{R}$, the 1-homogeneous extension of $g_\infty(x,t,\cdot)$ to $\mathbb{R}^n$ is convex. Equivalently, for a.e. $x \in \mathbb{R}^n$ and for every $t \in \mathbb{R}$ the function g_∞ satisfies*

$$g_\infty(x,t,\nu) \le \lambda_1 g_\infty(x,t,\nu_1) + \lambda_2 g_\infty(x,t,\nu_2),$$

for every $\nu,\nu_1,\nu_2 \in \mathbb{S}^{n-1}$, $\lambda_1,\lambda_2 \ge 0$ such that $\lambda_1\nu_1 + \lambda_2\nu_2 = \nu$;

vi) (bounds) *there exists $\tilde{c}_3 > 0$ such that for a.e. $x \in \mathbb{R}^n$, for every $t \in \mathbb{R}$, and every $\nu \in \mathbb{S}^{n-1}$ it holds*

$$\tilde{c}_3 \le g_\infty(x,t,\nu) \le c_4, \tag{2.29}$$

where c_4 is as in (2.2)*.*

Proof. Let $\mathcal{E}_k : L^1(\Omega) \times \mathcal{A}(\Omega) \longrightarrow [0,+\infty]$ be the functionals defined as

$$\mathcal{E}_k(u,U) := \begin{cases} c_1 \displaystyle\int_{U_k} |\nabla u|^p\,dx + c_3\mathcal{H}^{n-1}(S_u \cap U_k) & \text{if } u \in SBV^p(U) \\ +\infty & \text{otherwise in } L^1(\Omega), \end{cases} \tag{2.30}$$

with c_1 and c_3 as in (2.1) and (2.2), respectively. Appealing to [5, Theorem 4] and also noticing that the L^p-convergence in the statement can be equivalently replaced by the convergence in Definition 2.2, we deduce that $\mathcal{E}_k(\cdot,U)$ Γ-converges to $\mathcal{E}(\cdot,U)$ for every $U \in \mathcal{A}(\Omega)$, where

$$\mathcal{E}(u,U) = \int_U \hat{f}(\nabla u)\,dx + \int_{S_u\cap U} \hat{g}(\nu)\,d\mathcal{H}^{n-1}$$

with $\hat{f}$ and $\hat{g}$ as in [5, Theorem 4] formulas (40) and (41), respectively. Moreover $\hat{f}$ and $\hat{g}$ satisfy

$$\tilde{c}_1|\xi|^p \le \hat{f}(\xi) \text{ for every } \xi \in \mathbb{R}^n \quad \text{and} \quad \tilde{c}_3 \le \hat{g}(\nu) \text{ for every } \nu \in \mathbb{S}^{n-1},$$

for some $\tilde{c}_1, \tilde{c}_3 > 0$. Then, since $\mathcal{E}_k \le \mathcal{F}_k$, we may deduce that for every $u \in SBV^p(\Omega)$ and every $U \in \mathcal{A}(\Omega)$ we have

$$\mathcal{E}(u,U) \le \mathcal{F}(u,U). \tag{2.31}$$

We recall that for every $u \in SBV^p(\Omega)$ and every $U \in \mathcal{A}(\Omega)$ we also have

$$\mathcal{F}(u,U) \le \mathcal{G}(u,U), \tag{2.32}$$

where $\mathcal{G}$ is as in (2.26).

Now let $\sigma > 0$ and for every $u \in SBV^p(\Omega)$ and $U \in \mathcal{A}(\Omega)$ set

$$\mathcal{F}^\sigma(u,U) := \mathcal{F}(u,U) + \sigma \int_{S_u\cap U} |[u]|\,d\mathcal{H}^{n-1}.$$

For every fixed $\sigma > 0$ the functional $\mathcal{F}^\sigma$ satisfies properties 1, 2, and 4 in Theorem 2.14. Moreover, in view of (2.31)-(2.32) it holds

$$\tilde{c}_1 \int_U |\nabla u|^p \, dx + \int_{S_u \cap U} (\tilde{c}_3 + \sigma|[u]|) \, d\mathcal{H}^{n-1} \le \mathcal{F}^\sigma(u, U)$$
$$\le c_2 \int_U (1 + |\nabla u|^p) \, dx + \int_{S_u \cap U} (c_4 + \sigma|[u]|) \, d\mathcal{H}^{n-1}$$

Therefore, we can invoke the integral representation result [7, Theorem 1] to deduce that for every $u \in SBV^p(\Omega)$ and every $U \in \mathcal{A}(\Omega)$ we have

$$\mathcal{F}^\sigma(u, U) = \int_U f^\sigma_\infty(x, \nabla u) \, dx + \int_{S_u \cap U} g^\sigma_\infty(x, [u], \nu_u) \, d\mathcal{H}^{n-1},$$

where f^σ_∞ and g^σ_∞ are given by the following derivation formulas

$$f^\sigma_\infty(x, \xi) := \limsup_{\rho \to 0^+} \frac{1}{\rho^n} \inf \big\{ \mathcal{F}^\sigma(u, Q_\rho(x)) \colon u \in SBV^p(Q_\rho(x)), u = u_\xi \text{ near } \partial Q_\rho(x) \big\} \tag{2.33}$$

and

$$g^\sigma_\infty(x, t, \nu) := \limsup_{\rho \to 0^+} \frac{1}{\rho^{n-1}} \inf \big\{ \mathcal{F}^\sigma(u, Q^\nu_\rho(x)) \colon u \in SBV^p(Q^\nu_\rho(x)), u = u^{t,\nu}_x \text{ near } \partial Q^\nu_\rho(x) \big\}. \tag{2.34}$$

By (2.33) and (2.34) the sequences $(f^\sigma_\infty)_{\sigma>0}$ and $(g^\sigma_\infty)_{\sigma>0}$ are decreasing as σ decreases, therefore by setting $f_\infty := \lim_{\sigma \to 0^+} f^\sigma_\infty$ and $g_\infty := \lim_{\sigma \to 0^+} g^\sigma_\infty$, by the pointwise convergence of $(\mathcal{F}^\sigma)_{\sigma>0}$ to $\mathcal{F}$ and the Monotone Convergence Theorem, we get

$$\mathcal{F}(u, U) = \int_U f_\infty(x, \nabla u) \, dx + \int_{S_u \cap U} g_\infty(x, [u], \nu_u) \, d\mathcal{H}^{n-1},$$

for every $u \in SBV^p(\Omega)$ and $U \in \mathcal{A}(\Omega)$. Eventually, a standard truncation and continuity argument allows to extend this integral representation to the whole space $GSBV^p(\Omega)$ and thus to get exactly (2.27).

The measurability properties of f_∞ and g_∞ follow from the derivation formulas (2.33) and (2.34), arguing as in the appendix of [12]. The convexity of f_∞ in ξ, the subadditivity of g_∞ in t, and the convexity in ν of its 1-homogeneous extension are immediate consequences of the $L^1(\Omega)$-lower semicontinuity of $\mathcal{F}$.

To show that f_∞ and g_∞ satisfy, respectively, the lower bounds as in *ii)* and *vi)* we argue as follows. Set

$$\Phi^\sigma(u, U) := \begin{cases} \tilde{c}_1 \int_U |\nabla u|^p \, dx + \int_{S_u \cap U} (\tilde{c}_3 + \sigma|[u]|) \, d\mathcal{H}^{n-1} & \text{if } u \in SBV^p(U) \\ +\infty & \text{otherwise in } L^1(\Omega), \end{cases}$$

and for every $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$ define

$$\phi^\sigma(x, \xi) := \limsup_{\rho \to 0^+} \frac{1}{\rho^n} \inf \big\{ \Phi^\sigma(u, Q_\rho(x)) \colon u \in SBV^p(Q_\rho(x)), u = u_\xi \text{ near } \partial Q_\rho(x) \big\},$$

while for every $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, and $\nu \in \mathbb{S}^{n-1}$ set

$$\psi^\sigma(x, t, \nu) := \limsup_{\rho \to 0^+} \frac{1}{\rho^{n-1}} \inf \big\{ \Phi^\sigma(u, Q^\nu_\rho(x)) \colon u \in SBV^p(Q^\nu_\rho(x)), u = u^{\nu,t}_x \text{ near } \partial Q^\nu_\rho(x) \big\}.$$

Since $\Phi^\sigma \le \mathcal{F}^\sigma$ on $SBV^p(\Omega)$ we clearly have both $\phi^\sigma \le f^\sigma_\infty$ and $\psi^\sigma \le g^\sigma_\infty$. We now show that $\phi^\sigma(x, \xi) = \tilde{c}_1|\xi|^p$ for every $x \in \mathbb{R}^n$ and every $\xi \in \mathbb{R}^n$ and $\psi^\sigma(x, t, \nu) = \tilde{c}_3 + \sigma t$. To do so we notice that by the homogeneity in x of Φ^σ, we have both $\phi^\sigma(x, \xi) = \phi^\sigma(0, \xi)$ for every $x \in \mathbb{R}^n$ and every $\xi \in \mathbb{R}^n$ and $\psi^\sigma(x, t, \nu) = \psi^\sigma(0, t, \nu)$. We can now apply the integral representation result [7, Theorem 1] to Φ^σ so that choosing $u = u_\xi$ and $U = Q$ we obtain

$$\tilde{c}_1|\xi|^p = \Phi^\sigma(u_\xi, Q) = \int_Q \phi^\sigma(y, \xi) \, dy = \phi^\sigma(0, \xi) = \phi^\sigma(x, \xi),$$

while choosing $u = u^{t,\nu}_0$ and $U = Q^\nu$ we obtain

$$\tilde{c}_3 + \sigma t = \Phi^\sigma(u^{t,\nu}_0, Q^\nu) = \int_{\Pi^\nu \cap Q^\nu} \psi^\sigma(y, t, \nu) \, d\mathcal{H}^{n-1} = \psi^\sigma(0, t, \nu) = \psi^\sigma(x, t, \nu),$$

and hence the desired equalities. Therefore we deduce

$$\tilde{c}_1|\xi|^p = \phi^\sigma(x, \xi) \le f^\sigma_\infty(x, \xi) \quad \text{for every } x, \xi \in \mathbb{R}^n$$

which immediately gives the lower bound of f_∞; moreover there holds

$$\tilde{c}_3 \le \tilde{c}_3 + \sigma t = \psi^\sigma(x, t, \nu) \le g^\sigma_\infty(x, t, \nu) \quad \text{for every } x \in \mathbb{R}^n, t \in \mathbb{R}, \nu \in \mathbb{S}^{n-1}$$

hence, taking the inf on $\sigma > 0$ yields the the lower bound on g_∞.

The upper bound in *ii)* immediately follows from (2.33) and the obvious inquality $\mathcal{F}^\sigma(u_\xi, Q_\rho(x)) \leq \rho^n c_2(1+|\xi|^p)$, while the upper bound in *vi)* follows from (2.34) and

$$\mathcal{F}(u_x^{t,\nu}, Q_\rho^\nu(x)) \leq \mathcal{F}^\sigma(u_x^{t,\nu}, Q_\rho^\nu(x)) \leq \rho^{n-1}(c_4 + \sigma t),$$

which holds true for every $\sigma > 0$ and hence also in the limit as $\sigma \to 0^+$.

Finally, the monotonicity in t and the symmetry of g_∞ easily follow from (2.34). $\square$

Theorem 2.16 (Γ-convergence)**.** *Let $\mathcal{F}_k$ be the functionals defined in* (2.3)*. Then, there exists a subsequence $k_j \to +\infty$ such that $(\mathcal{F}_{k_j})$ Γ-converges to the functional $\mathcal{F}$ given by* (2.27)*, for some Carathéodory function $f_\infty \colon \mathbb{R}^n \times \mathbb{R}^n \to [0,+\infty)$ and some Borel function $g_\infty \colon \mathbb{R}^n \times \mathbb{S}^{n-1} \to (0,+\infty)$ satisfying properties $(i) - (vi)$ as in Theorem 2.15.*

Proof. The proof is an immediate consequence of Theorem 2.14 and Theorem 2.15. $\square$

Corollary 2.17 (Γ-convergence of porous brittle materials)**.** *Let $\alpha_k = \beta_k = 0$ and let $\mathcal{F}_k$ be the corresponding functionals given by* (2.3)*. Then, there exists a subsequence $k_j \to +\infty$ such that $(\mathcal{F}_{k_j})$ Γ-converges with respect to the $L^1(\Omega)$-convergence to the functional $\mathcal{F}$ given by* (2.27)*, for some Carathéodory function $f_\infty \colon \mathbb{R}^n \times \mathbb{R}^n \to [0,+\infty)$ and some Borel function $g_\infty \colon \mathbb{R}^n \times \mathbb{S}^{n-1} \to (0,+\infty)$ satisfying properties $(i) - (vi)$ as in Theorem 2.15.*

Proof. Since the $L^1(\Omega)$-convergence implies the convergence in Definition 2.2, the proof of the liminf inequality is immediate from Theorem 2.16. Now let $u \in GSBV^p(\Omega)$, then by Theorem 2.16 there exists $(u_j) \subset L^1(\Omega)$ such that $u_j \rightsquigarrow u$ and $\lim_j \mathcal{F}_{k_j}(u_j) = \mathcal{F}(u)$. In view of Definition 2.2 this means that there exists a sequence $(\tilde{u}_j) \subset L^1(\Omega)$ such that $\tilde{u}_j = u_j$ a.e. in Ω_{k_j} and $\tilde{u}_j \to u$ in $L^1(\Omega)$. Then, since the choice $\alpha_{k_j} = \beta_{k_j} = 0$ implies the equality $\mathcal{F}_{k_j}(\tilde{u}_j) = \mathcal{F}_{k_j}(u_j)$, the sequence $(\tilde{u}_j)$ is the desired recovery sequence. $\square$

2.4. **Convergence of minimisation problems.** On account of the Γ-convergence result Theorem 2.14 in this section we establish a convergence result for minimisation problems associated to a suitable perturbation of the functionals $\mathcal{F}_k$. To this end, let $h \in L^\infty(\Omega)$ and for every k set

$$M_k := \inf\big\{\mathcal{F}_k(u) + \|u-h\|^p_{L^p(\Omega_k)} : u \in L^1(\Omega)\big\}.$$

By a standard truncation argument it is immediate to show that

$$M_k = \inf\big\{\mathcal{F}_k(u) + \|u-h\|^p_{L^p(\Omega_k)} : u \in SBV^p(\Omega),\ \|u\|_{L^\infty(\Omega)} \leq \|h\|_{L^\infty(\Omega)}\big\}. \tag{2.35}$$

Proposition 2.18. *Let $\mathcal{F} = \Gamma\text{-}\lim_j \mathcal{F}_{k_j}$ and let $(u_j) \subset SBV^p(\Omega)$ be such that*

$$\lim_{j\to+\infty} \big(\mathcal{F}_{k_j}(u_j) + \|u_j - h\|^p_{L^p(\Omega_{k_j})} - M_j\big) = 0. \tag{2.36}$$

Then, up to subsequences (not relabelled), u_j converges in the sense of Definition 2.2 to a function $\bar{u} \in SBV^p(\Omega) \cap L^\infty(\Omega)$ which solves

$$M := \min\big\{\mathcal{F}(u) + C(K)\|u-h\|^p_{L^p(\Omega)} : u \in SBV^p(\Omega),\ \|u\|_{L^\infty(\Omega)} \leq \|h\|_{L^\infty(\Omega)}\big\},$$

where $C(K) := \mathcal{L}^n(Q \setminus K)$. Moreover it holds $M_j \to M$, as $j \to +\infty$.

Proof. Let $(u_j) \subset SBV^p(\Omega)$ be as in (2.36). Then, in view of (2.35), (H1), and (H3) we have

$$\sup_j \Big(\|u_j\|_{L^\infty(\Omega)} + \int_{\Omega_{k_j}} |\nabla u_j|^p\,dx + \mathcal{H}^{n-1}(S_{u_j} \cap \Omega_{k_j})\Big) < +\infty.$$

Therefore Theorem 2.1 yields the existence of a function $\bar{u} \in SBV^p(\Omega) \cap L^\infty(\Omega)$ and of a sequence $(\tilde{u}_j) \subset SBV^p(\Omega)$ with $\tilde{u}_j = u_j$ a.e. in Ω_{k_j} such that (up to subsequences) $\tilde{u}_j \to \bar{u}$ in $L^p(\Omega)$, moreover $\|\bar{u}\|_{L^\infty(\Omega)} \leq \|h\|_{L^\infty(\Omega)}$. We have

$$C(K)\|\bar{u}-h\|^p_{L^p(\Omega)} = \lim_{j\to+\infty} \|(\tilde{u}_j - h)\chi_{\Omega_{k_j}}\|^p_{L^p(\Omega)} = \lim_{j\to+\infty} \|u_j - h\|^p_{L^p(\Omega_{k_j})},$$

thus by Theorem 2.14 we get

$$\mathcal{F}(\bar{u}) + C(K)\|\bar{u}-h\|^p_{L^p(\Omega)} \leq \liminf_{j\to+\infty} \Big(\mathcal{F}_{k_j}(u_j) + \|u_j - h\|^p_{L^p(\Omega_{k_j})}\Big).$$

Therefore, by definition of u_j we obtain

$$\mathcal{F}(\bar{u}) + C(K)\|\bar{u}-h\|^p_{L^p(\Omega)} \leq \liminf_{j\to+\infty} M_j. \tag{2.37}$$

Now let $w \in SBV^p(\Omega) \cap L^\infty(\Omega)$ be an arbitrary function such that $\|w\|_{L^\infty(\Omega)} \leq \|h\|_{L^\infty(\Omega)}$. Again appealing to Theorem 2.14 we can find $(w_j) \subset L^1(\Omega)$ such that $w_j \rightsquigarrow w$ and $\lim_j \mathcal{F}_j(w_j) = \mathcal{F}(w)$. Now let $\tilde{w}_j$ be as in Definition 2.2, let $m := \|h\|_{L^\infty(\Omega)}$ and denote with $(\tilde{w}_j^m)$ the sequence of truncated functions of $(\tilde{w}_j)$ at level m. We clearly have $\tilde{w}_j^m = w_j^m$ a.e. in Ω_{k_j} and $\tilde{w}_j^m \to w$ in $L^p(\Omega)$. Hence

$$\lim_{j\to+\infty} \|w_j^m - h\|^p_{L^p(\Omega_{k_j})} = \lim_{j\to+\infty} \|(\tilde{w}_j^m - h)\chi_{\Omega_{k_j}}\|^p_{L^p(\Omega)} = C(K)\|w-h\|^p_{L^p(\Omega)}.$$

Moreover, since $\limsup_j \mathcal{F}_j(w_j^m) \leq \mathcal{F}(w)$, we immediately deduce

$$\limsup_{j\to+\infty} M_j \leq \mathcal{F}(w) + C(K)\|w-h\|^p_{L^p(\Omega)}. \tag{2.38}$$

Finally, by gathering (2.37) and (2.38) we obtain

$$\begin{aligned}\mathcal{F}(\bar{u}) + C(K)\|\bar{u}-h\|^p_{L^p(\Omega)} &\leq \liminf_{j\to+\infty} M_j \leq \limsup_{j\to+\infty} M_j \\ &\leq \mathcal{F}(w) + C(K)\|w-h\|^p_{L^p(\Omega)},\end{aligned}$$

hence by the arbitrariness of w we deduce that $\bar{u}$ is a minimiser for $\mathcal{F} + C(K)\|\cdot - h\|^p_{L^p(\Omega)}$. Finally, taking $w = \bar{u}$ also implies $M_j \to M$. Since moreover this limit does not depend on the subsequence, the convergence holds true for the whole (M_j). □

3. Part II: Examples

In this section we restrict the analysis to the case of ε_k-periodic integrands f_k and g_k. That is, we consider the functionals $\mathcal{F}_k^{\alpha_k,\beta_k} : L^1(\Omega) \longrightarrow [0,+\infty]$ defined as

$$\mathcal{F}_k^{\alpha_k,\beta_k}(u) := \begin{cases} \displaystyle\int_{\Omega_k} f\Big(\frac{x}{\varepsilon_k}, \nabla u\Big)\,dx + \alpha_k \int_{\Omega\setminus\Omega_k} f\Big(\frac{x}{\varepsilon_k}, \nabla u\Big)\,dx \\ \qquad\displaystyle + \int_{S_u\cap\Omega_k} g\Big(\frac{x}{\varepsilon_k}, \nu_u\Big)\,d\mathcal{H}^{n-1} + \beta_k \int_{S_u\cap(\Omega\setminus\Omega_k)} g\Big(\frac{x}{\varepsilon_k}, \nu_u\Big)\,d\mathcal{H}^{n-1} & \text{if } u \in SBV^p(\Omega) \\ +\infty & \text{otherwise,}\end{cases} \tag{3.1}$$

where f and g are Q-periodic in the first variable and satisfy (H1)-(H2) and (H3)-(H4), respectively.

With the help of some specific examples, which correspond to some specific choices of f, g, and Ω_k, we show that the Γ-limit of $\mathcal{F}_k^{\alpha_k,\beta_k}$ is highly sensitive both to the choice of the coefficients α_k, β_k and to the asymptotic behaviour of α_k, β_k compared to the period of the microstructure ε_k. The examples we are going to discuss are taken from Barchiesi and Focardi [5] (see also Cagnetti and Scardia [14] and Focardi, Gelli, and Ponsiglione [17]), from Barchiesi, Lazzaroni, and Zeppieri [6], and from Pellet, Scardia, and Zeppieri [19]. For the corresponding proofs we refer the reader to the aforementioned papers.

3.1. Periodic brittle porous materials.

In this subsection we consider the limit case $\alpha_k = \beta_k = 0$; *i.e.,* we consider the functionals

$$\mathcal{F}_k^{0,0}(u) := \begin{cases} \displaystyle\int_{\Omega_k} f\Big(\frac{x}{\varepsilon_k}, \nabla u\Big)\,dx + \int_{S_u\cap\Omega_k} g\Big(\frac{x}{\varepsilon_k}, \nu_u\Big)\,d\mathcal{H}^{n-1} & \text{if } u \in SBV^p(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega).\end{cases} \tag{3.2}$$

Loosely speaking, in this case the soft or weak inclusions in the material are replaced by perforations [5, 14, 17].

Theorem 3.1 (Homogenisation of periodic porous brittle materials). *Let $\mathcal{F}_k^{0,0}$ be the functionals as in (3.2). Then $(\mathcal{F}_k^{0,0})$ Γ-converges both with respect to the convergence in Definition 2.2 and with respect to the $L^1(\Omega)$-convergence to the functional $\mathcal{F}^0$ which is finite on $GSBV^p(\Omega)$ and given by*

$$\mathcal{F}^0(u) = \int_\Omega f^0(\nabla u)\,dx + \int_{S_u} g^0(\nu_u)\,d\mathcal{H}^{n-1}, \tag{3.3}$$

where f^0 and g^0 are, respectively, given by the following homogenisation formulas

$$f^0(\xi) = \inf\left\{\int_{Q\cap E} f(y,\nabla u)\,dx \colon u \in W^{1,p}(Q\cap E),\ u = u_\xi \ \text{ near } \partial Q\right\}, \tag{3.4}$$

for every $\xi \in \mathbb{R}^n$*, whereas*

$$g^0(\nu) = \lim_{T\to+\infty} \frac{1}{T^{n-1}} \inf\Big\{ \int_{S_u\cap TQ^\nu\cap E} g(y,\nu_u)\, d\mathcal{H}^{n-1} : u \in \mathcal{P}(TQ^\nu \cap E),\ u = u_0^{\nu,1} \ \text{ near } \partial TQ \Big\}, \quad (3.5)$$

for every $\nu \in \mathbb{S}^{n-1}$.

Proof. Theorem 2.16 and Corollary 2.17 yield the existence of a subsequence $k_j \to +\infty$ such that the corresponding functionals $\mathcal{F}^{0,0}_{k_j}$ Γ-converge to $\mathcal{F}$ as in (2.27), both with respect to the convergence in Definition 2.2 and to the $L^1(\Omega)$-convergence. Then, the homogenisation formulas (3.4) and (3.5) together with the identity $\mathcal{F} = \mathcal{F}^0$ follow from [5, Theorem 4]. Finally, since (3.4) and (3.5) are subsequence-independent, invoking the Urysohn property [15, Proposition 8.3] readily implies the Γ-convergence of the whole sequence $(\mathcal{F}^{0,0}_k)$ to $\mathcal{F}^0$. □

The following result is an immediate consequence of Theorem 3.1 and of an adaptation of the Cagnetti and Scardia extension result [14, Theorem 1.3] to the case of a general exponent $p > 1$.

Corollary 3.2. *Let* $\alpha_k, \beta_k \to 0$ *and let* $\mathcal{F}^{\alpha_k,\beta_k}_k$ *be the corresponding functionals as in* (2.3)*. Then, the sequence* $(\mathcal{F}^{\alpha_k,\beta_k}_k)$ Γ*-converges to the functional* $\mathcal{F}^0$ *given by* (3.3)*.*

Proof. By Theorem 2.16 (up to subsequences not relabelled) the functionals $\mathcal{F}^{\alpha_k,\beta_k}_k$ Γ-converge to $\mathcal{F}$. Since $\mathcal{F}^{0,0}_k \le \mathcal{F}^{\alpha_k,\beta_k}_k$ by Theorem 3.1 we immediately get $\mathcal{F}^0 \le \mathcal{F}$.

We now prove the converse inequality. To this end let $u \in SBV^p(\Omega)\cap L^\infty(\Omega)$ and let $(u_k) \subset SBV^p(\Omega_k)$ be a recovery sequence for $\mathcal{F}^{0,0}_k$. That is $u_k \rightsquigarrow u$ and $\lim_k \mathcal{F}^{0,0}_k(u_k) = \mathcal{F}(u)$. Since the functionals $\mathcal{F}^{0,0}_k$ decrease by truncation it is not restrictive to assume that $\|u_k\|_{L^\infty(\Omega)} \le \|u\|_{L^\infty(\Omega)}$. Starting from (u_k) we now want to construct a sequence (v_k) which both satisfies $v_k \to u$ in $L^1(\Omega)$ and $\lim_k \mathcal{F}^{\alpha_k,\beta_k}_k(v_k) = \mathcal{F}^0(u)$. To this end, we start noticing that the bounds (2.1) and (2.2) readily imply

$$\sup_k \Big(\int_{\Omega_k} |\nabla u_k|^p\, dx + \mathcal{H}^{n-1}(S_{u_k}\cap\Omega_k)\Big) < +\infty. \quad (3.6)$$

For every fixed k let $v_k := T^k u_k \in SBV^p(\Omega)$ be the extended function of u_k to Ω whose existence is given by [14, Theorem 1.3]; *i.e.,* v_k is such that $v_k = u_k$ a.e. in Ω_k, $\|u_k\|_{L^\infty(\Omega)} \le \|u\|_{L^\infty(\Omega)}$, and

$$\int_\Omega |\nabla v_k|^p\, dx + \mathcal{H}^{n-1}(S_{v_k}\cap\Omega) \le C\Big(\int_{\Omega_k} |\nabla u_k|^p\, dx + \mathcal{H}^{n-1}(S_{u_k}\cap\Omega_k)\Big) \quad (3.7)$$

for some $C > 0$ independent of k. By definition of v_k, also invoking the Ambrosio compactness Theorem, it is immediate to check that $v_k \to u$ in $L^1(\Omega)$.

By (3.7) we get that

$$\begin{aligned}\alpha_k \int_{\Omega\setminus\Omega_k} |\nabla v_k|^p\, dx + \beta_k \mathcal{H}^{n-1}(S_{v_k}\cap(\Omega\setminus\Omega_k)) &\le \alpha_k \int_\Omega |\nabla v_k|^p\, dx + \alpha_k\mathcal{H}^{n-1}(S_{v_k}\cap\Omega)\\ &\le C\,\max\{\alpha_k,\beta_k\}\Big(\int_{\Omega_k} |\nabla u_k|^p\, dx + \mathcal{H}^{n-1}(S_{u_k}\cap\Omega_k)\Big), \quad (3.8)\end{aligned}$$

where (3.8) is infinitesimal thanks to (3.6), since $\max\{\alpha_k,\beta_k\} \to 0$ as $k \to +\infty$. Thus eventually

$$\lim_{k\to+\infty} \mathcal{F}^{\alpha_k,\beta_k}_k(v_k) = \lim_{k\to+\infty} \mathcal{F}^{0,0}_k(u_k) = \mathcal{F}^0(u),$$

hence (v_k) is the desired sequence. Therefore, by the Γ-convergence of $\mathcal{F}^{\alpha_k,\beta_k}_k$ to $\mathcal{F}$ we can deduce that for every $u \in SBV^p(\Omega)\cap L^\infty(\Omega)$ it holds $\mathcal{F}(u) \le \mathcal{F}^0(u)$.

Now let $u \in GSBV^p(\Omega)$ and denote with u^m its truncated function at level $m > 0$. We clearly have $\mathcal{F}(u^m) \le \mathcal{F}^0(u^m) \le \mathcal{F}^0(u)$, hence the desired inequality follows by the $L^1(\Omega)$ convergence u^m to u and by the lower semicontinuity of $\mathcal{F}$. □

The following remarks are in order.

Remark 3.3. In view of Remark 2.10, both in Theorem 3.1 and in Corollary 3.2 assumption (H2) on f can be dropped.

Remark 3.4 (On f^0). The homogenised volume energy density f^0 given by (3.4) is the same as that obtained by Acerbi, Chiadó-Piat, Dal Maso, and Percivale [1] in the case of elastic perforated materials. Moreover, it is easy to check that if f is p-homogeneous then the corresponding f^0 given by (3.4) is also p-homogeneous.

Remark 3.5 (Energy decoupling). In spite of the strong degeneracy of the integrands in (3.2) (resp. in (3.1)), which in this case are identically equal to zero (resp. both infinitesimal) in the ε_k-periodic set $\Omega \setminus \Omega_k$, Theorem 3.1 (resp. Corollary 3.2) shows that the functionals $\mathcal{F}_k^{0,0}$ (resp. $\mathcal{F}_k^{\alpha_k,\beta_k}$) exhibit a limit behaviour which is qualitatively similar to that of free-discontinuity functionals with coercive integrands [9, 13, 18]. Namely, in the homogenised limit there is no interaction between bulk and surface term. As a consequence the homogenised surface energy density g^0 does not depend on t, and therefore the Γ-limit is of brittle type.

3.2. **Periodic brittle high-contrast materials.** In this section we show that if only one of the coefficients α_k and β_k is infinitesimal (while the other stays uniformly bounded from below), then the asymptotic behaviour of the functionals $\mathcal{F}_k^{\alpha_k,\beta_k}$ can be very different from that of $\mathcal{F}_k^{0,0}$ (or of $\mathcal{F}_k^{\alpha_k\to 0,\beta_k\to 0}$). In particular, we show that in this case a volume-surface energy coupling cannot be excluded in general. To do so we exhibit coefficients α_k, β_k, integrands f, g and a geometry for the periodic set E which give rise to the desired limit coupling. This is done by resorting to the analysis of Barchiesi, Lazzaroni, and Zeppieri [6] and Pellet, Scardia, and Zeppieri [19], which is briefly reviewed in Subsection 3.2.1 and Subsection 3.2.2, respectively.

The functionals analysed in [6] and [19] are both of Mumford-Shah type and can be written in the form

$$\mathcal{MS}_k^{\alpha_k,\beta_k}(u) = \int_\Omega a_k\Big(\frac{x}{\varepsilon_k}\Big)|\nabla u|^2\,dx + \int_{S_u} b_k\Big(\frac{x}{\varepsilon_k}\Big)\,d\mathcal{H}^{n-1}, \quad u \in SBV^2(\Omega) \tag{3.9}$$

where $a_k, b_k \colon \mathbb{R}^n \to [0,1]$ are Q-periodic functions and in the periodicity cell Q are defined as

$$a_k(y) = \begin{cases} \alpha_k & \text{if } y \in \overline{Q}_r \\ 1 & \text{if } y \in Q \setminus \overline{Q}_r \end{cases} \qquad b_k(y) = \begin{cases} \beta_k & \text{if } y \in \overline{Q}_r \\ 1 & \text{if } y \in Q \setminus \overline{Q}_r \end{cases} \tag{3.10}$$

with $r \in (0,1)$. From (3.9)-(3.10) we infer that in this case $f = f(\xi) = |\xi|^2$, $g \equiv 1$, and $\Omega_k = \Omega \cap \varepsilon_k E$ with $E = \mathbb{R}^n \setminus \bigcup_{i\in\mathbb{Z}^n}(\overline{Q}_r + i)$.

Remark 3.6 (Mumford-Shah functional in perforated domains). The choice $\alpha_k = \beta_k = 0$ corresponds to the Mumford-Shah functionals in perforated domain. The functional $\mathcal{MS}_k^{0,0}$ is a special instance of (3.2) and its homogenised limit is treated in [14, 17] for general sets E. In this case the homogenised integrands (3.4) and (3.5) reduce, respectively, to

$$f^0(\xi) = \inf\left\{\int_{Q\setminus\overline{Q}_r} |\nabla u|^2\,dx \colon u \in W^{1,2}(Q\setminus\overline{Q}_r),\ u = u_\xi \ \text{ near } \ \partial Q\right\}, \tag{3.11}$$

for every $\xi \in \mathbb{R}^n$, and to

$$g^0(\nu) = \lim_{T\to+\infty} \frac{1}{T^{n-1}} \inf\left\{\mathcal{H}^{n-1}(S_u \cap TQ^\nu \cap E) \colon u \in \mathcal{P}(TQ^\nu \cap E),\ u = u_0^{\nu,1} \ \text{ near } \ \partial TQ\right\}, \tag{3.12}$$

for every $\nu \in \mathbb{S}^{n-1}$. From (3.11) and (3.12) it is easy to check that $f^0(\xi) = A^0\xi\cdot\xi$, for some $A^0 \in \mathbb{R}^{n\times n}$ which satisfies $\tilde{c}_1 I \le A^0 \le I$, in the sense of quadratic forms (cf. (2.29)). Hence, f^0 is a positive quadratic form. Moreover, it holds $g^0(e_i) = 1 - r^{n-1}$, for every $i = 1, \dots, n$.

3.2.1. *Soft inclusions.* We consider the case $\alpha_k \to 0$ and $\beta_k = 1$ which models the situation where the periodic set $\Omega \setminus \Omega_k$ is occupied by a brittle material with a very small elastic modulus. For this reason, we refer to the set $\Omega \setminus \Omega_k$ as the set of *soft* inclusions. With this choice the functionals in (3.9) become

$$\mathcal{MS}_k^{\alpha_k,1}(u) = \int_{\Omega_k} |\nabla u|^2\,dx + \alpha_k \int_{\Omega\setminus\Omega_k} |\nabla u|^2\,dx + \mathcal{H}^{n-1}(S_u), \quad u \in SBV^2(\Omega). \tag{3.13}$$

In [6] Barchiesi, Lazzaroni, and Zeppieri showed that the asymptotic behaviour of $\mathcal{MS}_k^{\alpha_k,1}$ heavily depends on the mutual vanishing rate of α_k and ε_k; that is, it depends on the parameter

$$\ell := \lim_{k\to+\infty} \frac{\alpha_k}{\varepsilon_k} \in [0,+\infty]. \tag{3.14}$$

For the proof of Theorem 3.7 below we refer the reader to [6, Theorems 1, 4, and Remark 6].

Theorem 3.7 (Homogenisation of periodic brittle materials with *soft* inclusions). *Let $\mathcal{MS}_k^{\alpha_k,1}$ be the functionals defined in* (3.13) *and let $\ell \in [0,+\infty]$ be as in* (3.14)*. Then, up to subsequences not relabelled,* $(\mathcal{MS}_k^{\alpha_k,1})$ *Γ-converges to the functional $\mathcal{F}^\ell$ which is finite on $GSBV^2(\Omega)$ and given by*

$$\mathcal{F}^\ell(u) = \int_\Omega f^0(\nabla u)\,dx + \int_{S_u} g^\ell([u],\nu_u)\,d\mathcal{H}^{n-1}, \tag{3.15}$$

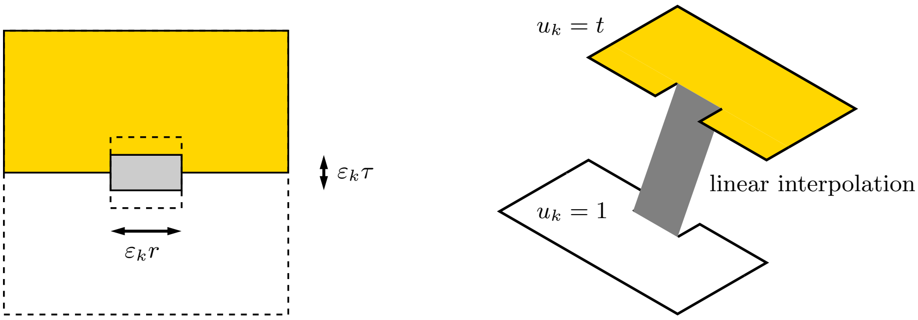

FIGURE 2. Construction of the recovery sequence in an ε_k-cell, across the interface $x_2 = 0$.

where f^0 *is as in* (3.11) *and for every* $t \in \mathbb{R}$, $\nu \in \mathbb{S}^{n-1}$

$$g^\ell(t,\nu) = \begin{cases} g^0(\nu) & \text{if } \ell = 0 \\ 1 & \text{if } \ell = +\infty. \end{cases}$$

Moreover for every $\ell \in (0,+\infty)$ *it holds*

$$\min\{g_0(e_i) + c_\ell t^2,\, 1\} \le g^\ell(t,e_i) \le \min\{g_0(e_i) + \hat{c}_\ell t,\, 1\} \tag{3.16}$$

for every $t > 0$, $i = 1,\dots,n$, *and for some* $c_\ell, \hat{c}_\ell > 0$, *with* $\lim_{\ell\to 0+} c_\ell = \lim_{\ell \to 0+} \hat{c}_\ell = 0$.

Remark 3.8. The following remarks are in order.

(i) As far as the homogenised volume energy is concerned, the soft inclusions are (energetically) equivalent to the perforations in the material.

(ii) For $\ell = 0$, which corresponds to $\alpha_k \ll \varepsilon_k$, the functionals $\mathcal{MS}_k^{\alpha_k,1}$ are equivalent to the functionals $\mathcal{MS}_k^{0,0}$, in the sense of Γ-convergence.

(iii) For $\ell \in (0,+\infty)$ the bounds in (3.16) imply that, along the coordinate directions, g^ℓ depends on t. Moreover it becomes constant (and equal to 1) above a certain threshold $t_0 > 0$; *i.e.*, g^ℓ is of *cohesive* type. Being the microscopic energies $\mathcal{MS}_k^{\alpha_k,1}$ of brittle type, the cohesive behaviour of g^ℓ can only be explained as the result of a non trivial bulk-surface coupling by homogenisation. This interaction is particularly apparent from the upper-bound construction in [6] which we briefly illustrate here in the case $n = 2$.

For $i = 1,2$ we have $g^\ell(t,e_i) = \mathcal{F}^\ell(u_0^{e_i,t}, Q)$, moreover it is immediate to check that $g^\ell(t,e_1) = g^\ell(t,e_2)$. Clearly $g^\ell(t,e_2) \le 1$ for every $t > 0$. Then, to get the upper bound in (3.16) it suffices to show that $g^\ell(t,e_2) \le g^0(e_2) + \hat{c}_\ell t$ for some $\hat{c}_\ell > 0$. Let $R \subset Q \subset \mathbb{R}^2$ be the open rectangle defined as

$$R := (-\tfrac{r}{2}, \tfrac{r}{2}) \times (-\tfrac{\tau}{2}, \tfrac{\tau}{2}),$$

with $\tau \in (0,r)$ to be determined. Set

$$R_k := Q \cap \bigcup_{i\in\mathbb{Z}} \Big(\varepsilon_k R + (i\varepsilon_k, 0)\Big)$$

and let $(u_k) \subset SBV^2(Q)$ be the sequence of functions defined as

$$u_k(x) := \begin{cases} t & \text{if } x \in Q \setminus R_k \text{ and } x_2 \ge 0, \\ \frac{t}{2} + \frac{t}{\tau\varepsilon_k}\, x_n & \text{if } x \in R_k, \\ 0 & \text{if } x \in Q \setminus R_k \text{ and } x_2 < 0, \end{cases}$$

(see Figure 2). We clearly have $u_k \to u_0^{e_2,t}$ in $L^1(Q)$; moreover

$$\int_{R_k} |\nabla u_k|^2\, dx \le \Big(\frac{1}{\varepsilon_k} + 1\Big)\frac{t^2}{\tau} \quad \text{and} \quad \mathcal{H}^1(S_{u_k}) \le \varepsilon_k\Big(\frac{1}{\varepsilon_k} + 1\Big)(1 - r + 2\tau),$$

therefore

$$g^\ell(t,e_2) = \mathcal{F}^\ell(u_0^{e_2,t}, Q) \le \limsup_{k\to+\infty} \mathcal{MS}_k^{\alpha_k,1}(u_k, Q). \le 1 - r + 2\tau + \ell\frac{t^2}{\tau}$$

Hence, by optimising on τ we get

$$g^\ell(t,e_2) \le 1 - r + 2\sqrt{2\ell}\, t \tag{3.17}$$

thus the desired estimate follows with $\hat{c}_\ell = 2\sqrt{2\ell}$, by recalling that $g^0(e_2) = 1 - r$.

Loosely speaking, the construction as above shows that, the cost of an elastic deformation of the soft inclusions is of the same order of the energy spent to create a microscopic crack. Since the former depends linearly on t (while the latter is constant in t) for small values of t, to approximate a macroscopic crack it can be convenient to combine microscopic deformations of the soft inclusions (with high gradients) and microscopic jumps.

(*iv*) Even if not immediately apparent from the homogenisation formulas, a volume-surface interaction takes place for $\ell = 0$, as well. Indeed, in this case $g^\ell = g^0$ whereas in $\mathcal{MS}_k^{\alpha_k,1}$ the surface energy density is identically equal to one. In this case in fact, the cost of an elastic deformation of the soft inclusions is negligible (cf. (3.17) for $\ell = 0$) so that to approximate a macroscopic crack it is never convenient to introduce microscopic cracks inside the soft material. On the contrary, in the regime $\ell = +\infty$, which corresponds to $\alpha_k \gg \varepsilon_k$, there is a complete volume-surface decoupling, as in the coercive case.

3.2.2. *Weak inclusions.* We consider the case $\alpha_k = 1$ and $\alpha_k \to 0$ which models the situation where the periodic set $\Omega \setminus \Omega_k$ is occupied by a brittle material with a very small fracture resistance. For this reason, we refer to the set $\Omega \setminus \Omega_k$ as the set of *weak* inclusions. With this choice the functionals in (3.9) become

$$\mathcal{MS}_k^{1,\beta_k}(u) = \int_\Omega |\nabla u|^2\,dx + \mathcal{H}^{n-1}(S_u \cap \Omega_k) + \beta_k \mathcal{H}^{n-1}(S_u \cap (\Omega \setminus \Omega_k)), \quad u \in SBV^2(\Omega). \tag{3.18}$$

In [19] Pellet, Scardia, and Zeppieri showed that the asymptotic behaviour of $\mathcal{MS}_k^{1,\beta_k}$ heavily depends on the mutual vanishing rate of β_k and ε_k, that is on the parameter

$$\ell' := \lim_k \frac{\beta_k}{\varepsilon_k} \in [0, +\infty]. \tag{3.19}$$

For the proof of Theorem 3.9 below we refer the reader to [19].

Theorem 3.9 (Homogenisation of periodic brittle materials with *weak* inclusions)**.** *Let $\mathcal{MS}_k^{1,\beta_k}$ be the functionals defined in* (3.18) *and let $\ell' \in [0, +\infty]$ be as in* (3.19)*. Then, up to subsequences not relabelled, $(\mathcal{MS}_k^{1,\beta_k})$ Γ-converges to the functional $\mathcal{F}^{\ell'}$ which is finite on $GSBV^2(\Omega)$ and given by*

$$\mathcal{F}^{\ell'}(u) = \int_\Omega f^{\ell'}(\nabla u)\,dx + \int_{S_u} g^0(\nu_u)\,d\mathcal{H}^{n-1}, \tag{3.20}$$

where g^0 is as in (3.12) *and for every $\xi \in \mathbb{R}^n$*

$$f^{\ell'}(\xi) = \begin{cases} f^0(\xi) & \text{if } \ell' = 0 \\ |\xi|^2 & \text{if } \ell' = +\infty. \end{cases}$$

Moreover for every $\ell' \in (0, +\infty)$ it holds

$$f^0(\xi) \leq f^{\ell'}(\xi) \leq \min\{|\xi|^2, f^0(\xi) + C\ell'\} \tag{3.21}$$

for every $\xi \in \mathbb{R}^n$ and for some $C > 0$.

Remark 3.10. The following remarks are in order.

(*i*) As far as the homogenised surface energy is concerned, the weak inclusions are (energetically) equivalent to the perforations in the material.

(*ii*) For $\ell' = 0$, which corresponds to $\beta_k \ll \varepsilon_k$, the functionals $\mathcal{MS}_k^{1,\beta_k}$ are equivalent to the functionals $\mathcal{MS}_k^{0,0}$, in the sense of Γ-convergence. Indeed, "removing the weak inclusions from the material" has an infinitesimal cost of order β_k/ε_k given by the perimeter of the weak inclusions (proportional to $\beta_k \varepsilon_k^{n-1}$) multiplied by ε_k^{-n} (the number of ε_k-cells contained in Ω). In this case a volume-surface energy coupling takes place since the elastic energy can be lowered by introducing cracks in the materials.

(*iii*) For $\ell' \in (0, +\infty)$ the bounds in (3.21) hold true (see [19, Lemma 6.1]). The bound from below is immediate and it is a consequence of the trivial bound $\mathcal{MS}_k^{0,0} \leq \mathcal{MS}_k^{1,\beta_k}$. The bound from above shows that for large deformations; *i.e.,* for large $|\xi|$, to approximate a macroscopic elastic deformation is energetically favourable to mix elastic deformations and jumps in the weak inclusions. Moreover, (3.21) implies that for $|\xi|$ large it holds $f^{\ell'}(\xi) < |\xi|^2$. The latter shows that a *stiffness degradation* occurs in the homogenised limit, and that the macroscopic energy $\mathcal{F}^{\ell'}$ describes a *damaged* material (the same being true for $\ell = 0$).

(*iv*) The bounds in (3.21) combined with an easy scaling argument show that in the regime $\ell' \in (0, +\infty)$ the homogenised volume energy density $f^{\ell'}$ is not 2-homogeneous. Indeed, assume by contradiction that this is not the case and let $\lambda \neq 0$. Taking into account that f^0 is 2-homogeneous (see Remark 3.4), we can replace in (3.21) ξ with $\lambda\xi$ and divide by λ^2 to get

$$f^0(\xi) \leq f^{\ell'}(\xi) \leq \min\left\{|\xi|^2, f^0(\xi) + \frac{C\ell'}{\lambda^2}\right\}.$$

Therefore by letting $|\lambda| \to +\infty$ we get $f^{\ell'} \equiv f^0$ which leads to a contradiction in view of [19, Proposition 6.10].

(v) In the regime $\ell' = +\infty$, which corresponds to $\beta_k \gg \varepsilon_k$, there is a complete volume-surface decoupling, as in the coercive case. Loosely speaking, in this case the fracture resistance of the weak inclusions is not small enough to make cracks energetically more convenient than (or at least comparable to) elastic deformations.

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