POSITIVE SOLUTIONS TO THE SUBLINEAR LANE-EMDEN EQUATION ARE ISOLATED

LORENZO BRASCO, GUIDO DE PHILIPPIS, AND GIOVANNI FRANZINA

Abstract. We prove that on a smooth bounded set, the positive least energy solution of the Lane-Emden equation with sublinear power is isolated. As a corollary, we obtain that the first $q-$eigenvalue of the Dirichlet-Laplacian is not an accumulation point of the $q-$spectrum, on a smooth bounded set. Our results extend to a suitable class of Lipschitz domains, as well.

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1. Introduction

1.1. Overview. For $N \geq 2$, we consider an open bounded set $\Omega \subset \mathbb{R}^N$, with its associated *homogeneous Sobolev space* $\mathcal{D}^{1,2}_0(\Omega)$. The latter is defined as the completion of $C^\infty_0(\Omega)$ with respect to the norm

$$\|\varphi\|_{\mathcal{D}^{1,2}_0(\Omega)} = \left(\int_\Omega |\nabla \varphi|^2\,dx\right)^{\frac{1}{2}}, \qquad \text{for } \varphi \in C^\infty_0(\Omega).$$

The notation $C^\infty_0(\Omega)$ stands for the set of C^∞ functions with compact support in Ω. We recall that an open bounded set Ω supports a Poincaré inequality of the type

$$\frac{1}{C}\int_\Omega |\varphi|^2\,dx \leq \int_\Omega |\nabla \varphi|^2\,dx, \qquad \text{for every } \varphi \in C^\infty_0(\Omega), \tag{1.1}$$

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thus the space $\mathcal{D}^{1,2}_0(\Omega)$ coincides with the closure of $C^\infty_0(\Omega)$ in the standard Sobolev space $W^{1,2}(\Omega)$ (for example, see [10, Théorème 4.2 and Remarque 4.1]).

It is well-known that in this setting the Dirichlet-Laplacian operator on Ω has a discrete spectrum, made of positive eigenvalues accumulating at $+\infty$. In other words, the boundary value problem

$$-\Delta u = \lambda\, u, \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega,$$

admits non-trivial solutions $u \in \mathcal{D}^{1,2}_0(\Omega)$ only for a discrete set of characteristic values λ, that we indicate with $0 < \lambda_1(\Omega) \le \lambda_2(\Omega) \le \dots$. If $u \in \mathcal{D}^{1,2}_0(\Omega)$ solves the above equation with $\lambda = \lambda_i(\Omega)$, it is called an *eigenfunction associated to* $\lambda_i(\Omega)$.

It is easy to see that these eigenvalues $\lambda_i(\Omega)$ can be characterized as the critical values of the Dirichlet integral

$$\varphi \mapsto \int_\Omega |\nabla \varphi|^2\, dx,$$

constrained to the manifold

$$\mathcal{S}_2(\Omega) = \Big\{\varphi \in \mathcal{D}^{1,2}_0(\Omega)\, :\, \|\varphi\|_{L^2(\Omega)} = 1\Big\}.$$

The associated critical points correspond to the eigenfunctions of the Dirichlet-Laplacian, normalized in order to have unit L^2 norm. In particular, the first eigenvalue corresponds to the global constrained minimum, i.e.

$$\lambda_1(\Omega) = \min_{\varphi \in \mathcal{D}^{1,2}_0(\Omega)} \left\{\int_\Omega |\nabla\varphi|^2\, dx\, :\, \|\varphi\|_{L^2(\Omega)} = 1\right\},$$

which in turn gives the sharp constant in (1.1).

If[1] $1 < q < 2^*$ and the constraint $\mathcal{S}_2(\Omega)$ is replaced by the more general one

$$\mathcal{S}_q(\Omega) = \Big\{u \in \mathcal{D}^{1,2}_0(\Omega)\, :\, \|u\|_{L^q(\Omega)} = 1\Big\},$$

then by the Lagrange's multipliers rule, the relevant elliptic equation is given by

$$-\Delta u = \lambda\, |u|^{q-2}\, u, \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega.$$

If we want to get rid of the normalization on the L^q norm, then the equation should be written in the following form

$$-\Delta u = \lambda\, \|u\|^{2-q}_{L^q(\Omega)}\, |u|^{q-2}\, u, \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega. \tag{1.2}$$

We define the *$q-$spectrum of the Dirichlet-Laplacian on* Ω as

$$\mathrm{Spec}(\Omega; q) = \Big\{\lambda \in \mathbb{R}\, :\, \text{equation (1.2) admits a solution in } \mathcal{D}^{1,2}_0(\Omega) \setminus \{0\}\Big\}.$$

Each element λ of this set is called a *$q-$eigenvalue*, while an associated solution of (1.2) will be called *$q-$eigenfunction*.

Some basic properties of this eigenvalue–type problem have recently been collected in the survey paper [5]. Let us briefly recall them, by referring to [5] for all the missing details.

[1] We use the usual notation 2^* for the critical Sobolev exponent, i.e.

$$2^* = \frac{2\,N}{N-2}, \quad \text{for } N \ge 3, \qquad 2^* = +\infty, \quad \text{for } N = 2.$$

First of all, by using standard variational techniques from Critical Point Theory, it is relatively easy to produce an infinite sequence of $q-$eigenvalues, diverging to $+\infty$. More precisely, for every $k \in \mathbb{N} \setminus \{0\}$, the $k-$th *Ljusternik-Schnirelmann variational eigenvalue* is defined by

$$\lambda_{k,LS}(\Omega; q) = \inf_{\mathcal{F} \in \Sigma_k(\Omega;q)} \left\{ \max_{\varphi \in \mathcal{F}} \int_\Omega |\nabla \varphi|^2 \, dx \right\}, \tag{1.3}$$

where

$$\Sigma_k(\Omega; q) = \Big\{ \mathcal{F} \subset \mathcal{S}_q(\Omega) \, : \, \mathcal{F} \text{ compact and symmetric with } \gamma(\mathcal{F}) \geq k \Big\},$$

and γ is the *Krasnosel'skiĭ genus*, defined by

$$\gamma(\mathcal{F}) = \inf \Big\{ k \in \mathbb{N} \setminus \{0\} \, : \, \exists \text{ a continuous odd map } \phi : \mathcal{F} \to \mathbb{S}^{k-1} \Big\}.$$

We recall that formula (1.3) is reminiscent of the celebrated *Courant-Fischer-Weyl min-max principle* for the eigenvalues of the Laplacian (see [16, equation (1.32)]).

For $k = 1$, it is not difficult to see that formula (1.3) reduces to the sharp Poincaré-Sobolev constant

$$\lambda_1(\Omega; q) = \min_{\varphi \in \mathcal{D}^{1,2}_0(\Omega)} \left\{ \int_\Omega |\nabla \varphi|^2 \, dx \, : \, \int_\Omega |\varphi|^q \, dx = 1 \right\},$$

and that this is the first $q-$eigenvalue of Ω, i.e.

$$\lambda_1(\Omega; q) \in \mathrm{Spec}(\Omega; q) \qquad \text{and} \qquad \lambda \geq \lambda_1(\Omega; q) \text{ for every } \lambda \in \mathrm{Spec}(\Omega; q).$$

Let us now specialize the discussion to the case $1 < q < 2$. In this case, for a generic open set we may have

$$\Big\{ \lambda_{k,LS}(\Omega; q) \Big\}_{k \in \mathbb{N} \setminus \{0\}} \neq \mathrm{Spec}(\Omega; q),$$

and $\mathrm{Spec}(\Omega; q)$ may not be discrete. Even worse, one can produce examples of sets Ω for which the first $q-$eigenvalue is not isolated, i.e. it is an accumulation point for $\mathrm{Spec}(\Omega; q)$ (see [6, Theorem 3.2]).

Examples of this last phenomenon are quite pathological, i.e. they are sets made of countably many connected components. It is thus reasonable to ask whether $\lambda_1(\Omega; q)$ is isolated or not, for connected sets or sets with a finite number of connected components.

This leads us to the question tackled in this paper: find classes of "good" sets such that the first $q-$eigenvalue is isolated, for $1 < q < 2$.

1.2. **The Lane-Emden equation.** The equation (1.2) may look weird at a first sight, but actually it is just a scaled version of the celebrated and well-studied *Lane-Emden equation*. More precisely, observe that equation (1.2) is no longer linear, but it is still $1-$homogeneous. This means that if $u \in \mathcal{D}^{1,2}_0(\Omega)$ is a solution, then $t\,u$ is still a solution for every $t \in \mathbb{R}$. Thus, if we take a $q-$eigenfunction u such that

$$\|u\|_{L^q(\Omega)} = \lambda^{\frac{1}{q-2}},$$

we get that this solves the usual Lane-Emden equation

$$-\Delta u = |u|^{q-2}\, u, \quad \text{in } \Omega. \tag{1.4}$$

This semilinear elliptic equation naturally arises in many fields, here we just want to mention that its solutions dictate the large time behavior of solutions to the Cauchy-Dirichlet problem for the

Porous Medium Equation, i.e.

$$\begin{cases} \Delta(|u|^{m-1}\,u) &=& u_t, & \text{in } \Omega\times(0,+\infty),\\ u &=& 0, & \text{on } \partial\Omega\times(0,+\infty),\\ u(\cdot,0) &=& u_0, & \text{in } \Omega, \end{cases} \tag{1.5}$$

where

$$m=\frac{1}{q-1}.$$

The reader can see for example [1, Theorem 3] and [23, Theorems 1.1 and 2.1]. We also refer to [8, Theorem 1.2] where the main result of this paper, i.e. Theorem A below, is used to show convergence of some sign-changing solutions to the ground state. In this respect, we can say that equation (1.4) plays the same role with respect to (1.5), as the usual eigenvalue equation does for the heat equation.

Now, it turns out that the question whether $\lambda_1(\Omega;q)$ is isolated or not is tightly connected with the question whether the positive least energy solution of (1.4) is isolated in the set of solutions or not. Again, for a general open bounded set, this is not true: as above, a counterexample is given by any set with countably many connected components.

1.3. **Main results.** We now present the main results of this paper, by postponing some comments on the assumptions to Remark 1.2 below.

Theorem A. *Let $1<q<2$, let $N\ge 2$, and let $\Omega\subset\mathbb{R}^N$ be a C^1 open bounded set, with a finite number of connected components. We indicate by $w_{\Omega,q}$ the positive least energy solution in Ω of (1.4), i.e. the unique positive minimizer in $\mathcal{D}^{1,2}_0(\Omega)$ of the energy*

$$\mathfrak{F}_q(\varphi)=\frac{1}{2}\int_\Omega|\nabla\varphi|^2\,dx-\frac{1}{q}\int_\Omega|\varphi|^q\,dx. \tag{1.6}$$

Then $w_{\Omega,q}$ is isolated *in the $L^1(\Omega)$ norm topology, i.e. there exists $\delta>0$ such that the neighborhood*

$$\mathcal{I}_\delta(w_{\Omega,q})=\Big\{\varphi\in\mathcal{D}^{1,2}_0(\Omega)\,:\,\|\varphi-w_{\Omega,q}\|_{L^1(\Omega)}<\delta\Big\},$$

does not contain any other solution of the Lane-Emden equation.

The previous result implies the following one, which has been announced in [5].

Theorem B. *Let $1<q<2$, let $N\ge 2$, and let $\Omega\subset\mathbb{R}^N$ be a C^1 open bounded set, with a finite number of connected components. Then the first $q-$eigenvalue $\lambda_1(\Omega;q)$ is isolated in* $\mathrm{Spec}(\Omega;q)$.

As a straightforward consequence of Theorem B and of the closedness of $\mathrm{Spec}(\Omega;q)$, we get the following

Corollary 1.1 (The second $q-$eigenvalue)**.** *Under the assumptions of Theorem B, if we set*

$$\lambda_2(\Omega;q):=\inf\{\lambda\in\mathrm{Spec}(\Omega;q)\,:\,\lambda>\lambda_1(\Omega;q)\},$$

then we have

$$\lambda_1(\Omega;q)<\lambda_2(\Omega;q)\qquad \textit{and}\qquad \lambda_2(\Omega;q)\in\mathrm{Spec}(\Omega;q).$$

Remark 1.2. Some comments are in order about our results:

(1) the C^1 regularity of Ω is not really needed, it is placed here just for ease of presentation. Indeed, our result is more general, as we can allow for Lipschitz sets (see Theorems 6.1 & 6.3 below). However, in this case, if the Lipschitz constant is too large, then the result is valid for a restricted range
$$q_\Omega < q < 2,$$
for a limit exponent $1 \leq q_\Omega < 2$ depending on the Lipschitz constant of Ω and degenerating to 2 as the Lipschitz constant blows-up (actually, the distinguishing condition is slightly more refined, as it depends only on the "interior" angles of the sets, see the discussions in Sections 2 and 5 below);

(2) in the statement of Theorem A, the precision *least energy solution* can be omitted when Ω is connected, since in this case the Lane-Emden equation has a unique positive solution, which is indeed the unique positive minimizer of the associated energy functional. On the contrary, when Ω has k connected components, the Lane-Emden equation has multiple non-negative solutions (see Remark 3.2). In this case, positive solutions not having least energy *are not isolated*, see Example 6.5;

(3) finally, the assumption on the number of connected components is optimal, as shown in [6, Theorem 3.2].

1.4. **Some comments on the case $q > 2$.** Our paper is focused on the case $1 < q < 2$ and our proofs and results do not extend to the super-homogeneous case $2 < q < 2^*$. The latter is indeed slightly different and some different phenomena may occur. We wish to comment on this case, in order to give a clearer picture.

We first point out that for $q > 2$ the concept of least energy solution is not well-defined, since the relevant energy functional (1.6) is now *unbounded from below*. Moreover, in general it is no more true that the Lane-Emden equation (1.4) has a unique positive solution, even on a connected set. Indeed, there has been an extensive study about existence of multiple positive solutions for equation (1.4) in the super-homogeneous regime $2 < q < 2^*$. We mention [2, Theorem B], [3], [12], and [14], just to name a few classical results.

The simplest example of this phenomenon is given by a sufficiently thin spherical shell. It is known that for every $2 < q < 2^*$ there exists a radius $0 < r < 1$ such that on
$$A_r = \{x \in \mathbb{R}^N : r < |x| < 1\},$$
any first $q-$eigenfunction (which must have constant sign) *is not* radial, see [21, Proposition 1.2]. This implies that on A_r there exist infinitely many positive solutions of (1.4), obtained by composing a solution with the group of symmetries of A_r. We point out that examples of multiplicity can be exhibited also in presence of a trivial topology, see for example [13] and [5, Example 4.7]. But the case of the spherical shell A_r has one interesting feature more: by construction, each positive solution constructed above *is not isolated*. As for the first $q-$eigenvalue, it is not known whether $\lambda_1(A_r; q)$ is isolated or not, in this case.

As observed in [15], our Theorem B holds for $2 < q < 2^*$ whenever $\lambda_1(\Omega; q)$ is *simple*, i.e. there exists a unique first $q-$eigenfunction with unit L^q norm, up to the choice of the sign. However, this condition does not always hold, as exposed above. By [11, Theorem 4.4], this is known to be true on sets for which a positive first $q-$eigenfunction u is *non-degenerate*. This means that the linearized operator
$$\varphi \mapsto -\Delta\varphi - (q-1)\, u^{q-2}\, \varphi,$$

does not contain 0 in its spectrum. Such a condition in general is quite difficult to be checked. To the best of our knowledge, this has been verified only in some special cases, that we list below:

- open bounded convex sets in $\mathbb{R}^2$;
- open bounded sets in $\mathbb{R}^2$, which are convex in the directions x_1 and x_2 and symmetric about the lines $\{x_1 = 0\}$ and $\{x_2 = 0\}$
- a ball in any dimension $N \geq 2$.

The first case is due to Lin (see [18, Lemma 2]), the other two cases are due to Dancer (see [14, Theorem 5]) and Damascelli, Grossi and Pacella (see [11, Theorem 4.1]).

For a general open bounded connected set Ω, the best result is due to Lin (see [18, Lemma 3] and also [5, Proposition 4.3]): this asserts that there always exists an exponent $q_0 = q_0(\Omega) \in (2, 2^*)$ such that $\lambda_1(\Omega; q)$ is simple (and thus isolated) for $2 < q < q_0$. The example of the spherical shell A_r shows that this result is optimal.

Remark 1.3. If $N = 1$, (1.2) reduces to an ODE whose solutions are completely classified for all $1 < q < 2^*$. In that case, if Ω is the union of a finite family of open bounded intervals, then the results stated in Subsection 1.3 hold valid: see, e.g., [5, Remarks 1.4 and 3.7].

1.5. **Strategy of the proof and plan of the paper.** The proof of our main result is still based on studying the linearized operator

$$\varphi \mapsto -\Delta\varphi - (q-1)\, w_{\Omega,q}^{q-2}\, \varphi, \tag{1.7}$$

around the positive least energy solution $w_{\Omega,q}$. However, with respect to the case $2 < q < 2^*$, in our setting difficulties are reversed. Indeed, by means of a suitable Hardy-type inequality proved in [7], it is quite simple to show that 0 does not belong to the spectrum of the operator (1.7). On the other hand, since $1 < q < 2$ this operator has a singular potential. Thus, rigorously establishing the reduction to the linearized problem requires a careful study of the weighted embedding

$$\mathcal{D}^{1,2}_0(\Omega) \hookrightarrow L^2(\Omega; w_{\Omega,q}^{q-2}). \tag{1.8}$$

We now briefly describe the structure of the paper: in Section 2 we recall some definitions and properties of C^1 and Lipschitz sets, needed to prove our main results.

Section 3 deals with some properties of the positive least energy solution $w_{\Omega,q}$ of equation (1.4). In particular, by appealing to some fine estimates for the Green function of a Lipschitz set obtained in [9], we obtain an estimate from below on $w_{\Omega,q}$, in terms of a suitable power of the distance from the boundary (see Corollary 3.5).

In Section 4 we prove an abstract version of our main result, namely that the positive least energy solution $w_{\Omega,q}$ is isolated whenever the weighted embedding (1.8) is compact, see Proposition 4.1. This is the cornerstone of Theorem A and Theorem B.

In Section 5, we analyze the embedding (1.8) and provide some sharp sufficient conditions for this to be compact, see Proposition 5.1 and Corollary 5.2.

Finally, in Section 6 we join the outcomes of the previous two sections, in order to prove Theorem A and Theorem B, in a larger generality.

The paper is complemented by three appendices, containing: a simple, yet crucial, pointwise inequality; a universal L^∞ bound for solutions of (1.4); some properties of solutions of (1.4) in convex cones, with an associated study of the embedding (1.8).

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2. Lipschitz sets

Throughout this paper, we shall assume that $N \in \mathbb{N}$, with $N \geq 2$. For an open bounded set $\Omega \subset \mathbb{R}^N$, in what follows we denote by d_Ω the distance function from the boundary, i.e.

$$d_\Omega(x) = \min_{y\in\partial\Omega} |x-y|, \qquad \text{for every } x \in \Omega.$$

We will need some fine comparison estimates for solutions of elliptic PDEs. These will be taken from [9]. In order to be consistent, we adopt the same definition of *Lipschitz sets* as in [9].

Definition 2.1. Let $r, h > 0$ be two positive numbers, then we set

$$C(r,h) = \{x = (x', x_N) \in \mathbb{R}^N \,:\, |x'| < r \ \text{ and } \ |x_N| < h\}.$$

An open bounded set $\Omega \subset \mathbb{R}^N$ is called *Lipschitz* if for any $x_0 \in \partial\Omega$, there exist $r, h > 0$ and a Lipschitz function $\varphi : \mathbb{R}^{N-1} \to \mathbb{R}$ such that, up to a rigid movement taking x_0 to the origin, we have

$$\partial\Omega \cap C(r,h) = \{(x', x_N) \,:\, |x'| < r,\, x_N = \varphi(x_1, \dots, x_{N-1})\},$$

and

$$\Omega \cap C(r,h) = \{(x', x_N) \,:\, |x'| < r,\, \varphi(x_1, \dots, x_{N-1}) < x_N < h\}.$$

We call an *atlas for* $\partial\Omega$ any finite collection of cylinders $\{C(r_k, h_k)\}_{1\leq k\leq m}$, with associated Lipschitz maps $\{\varphi_k\}_{1\leq k\leq m}$, which covers $\partial\Omega$. Then we define the *Lipschitz constant of* Ω as

$$\kappa_\Omega = \inf \Big(\max\{\|\nabla\varphi_k\|_{L^\infty} \,:\, 1 \leq k \leq m\}\Big), \tag{2.1}$$

where the infimum is taken over all atlases for $\partial\Omega$.

Remark 2.2. Similarly, as in [9], we say that Ω *is of class* C^1 if the functions φ in the previous definition are of class C^1. Then, for Ω of class C^1 we have

$$\kappa_\Omega = 0,$$

see [9, equation (3.25)].

Definition 2.3 (Cones)**.** For $0 \leq \beta < 1$, we consider the spherical cap

$$\mathcal{S}(\beta) = \{\omega \in \mathbb{S}^{N-1} \,:\, \beta < \langle \omega, \mathbf{e}_1\rangle\},$$

where $\mathbf{e}_1 = (1, 0, \dots, 0)$. We also consider the axially symmetric cone

$$\Gamma(\beta, R) = \left\{ x \in \mathbb{R}^N \,:\, 0 < |x| < R \text{ and } \frac{x}{|x|} \in \mathcal{S}(\beta)\right\}.$$

We indicate by $\lambda(\mathcal{S}(\beta))$ the first Dirichlet eigenvalue of the Laplace-Beltrami operator on $\mathcal{S}(\beta)$. Then we define $\alpha(\beta)$ to be the positive root of the equation

$$\alpha(\beta)\,(N-2+\alpha(\beta)) = \lambda(\mathcal{S}(\beta)),$$

i.e.

$$\alpha(\beta) = \frac{\sqrt{(N-2)^2 + 4\,\lambda(\mathcal{S}(\beta))} - (N-2)}{2} > 0. \tag{2.2}$$

Note that the map $\beta \to \lambda(\mathcal{S}(\beta))$ is increasing and thus so is the map $\beta \to \alpha(\beta)$. Furthermore, we have $\lambda(\mathcal{S}(0)) = N - 1$ and an associated eigenfunction is given by $\varphi(x) = x_1$. Correspondingly, we obtain

$$\alpha(0) = 1.$$

More generally, we have

$$\begin{aligned} &\lambda(\mathcal{S}(\beta)) \searrow N-1 \quad &&\text{and} \quad \alpha(\beta) \searrow 1, \quad &&\text{as } \beta \searrow 0, \\ &\lambda(\mathcal{S}(\beta)) \nearrow +\infty \quad &&\text{and} \quad \alpha(\beta) \nearrow +\infty, \quad &&\text{as } \beta \nearrow 1. \end{aligned} \tag{2.3}$$

In the case $N = 2$, it is easily seen that $\lambda(\mathcal{S}(\beta))$ is the first Dirichlet eigenvalue of the operator $\varphi \mapsto -\varphi''$ on the interval

$$(-\arccos\beta, \arccos\beta).$$

Thus, in this case we have

$$\lambda(\mathcal{S}(\beta)) = \left(\frac{\pi}{2\,\arccos\beta}\right)^2 \qquad \text{and} \qquad \alpha(\beta) = \frac{\pi}{2\,\arccos\beta} \tag{2.4}$$

Let us now precisely state the inner cone condition that will be used throughout the paper.

Definition 2.4. Let $\Omega \subset \mathbb{R}^N$ be an open bounded set. For a given $R > 0$, we indicate

$$\Omega_R = \{x \in \Omega \,:\, d_\Omega(x) > R\}.$$

We say that Ω satisfies an *inner cone condition of index* $\beta \in (0, 1)$ if there exists $0 < R < \operatorname{diam}(\Omega)$ such that

$$\forall x \in \overline{\Omega} \setminus \Omega_R \ \ \exists \text{ an isometry } \mathcal{O} \text{ of } \mathbb{R}^N \text{ such that } \mathcal{O}(x) = 0 \text{ and } \ x + \mathcal{O}\,(\Gamma(\beta, R)) \subset \Omega. \tag{2.5}$$

We also define the homogeneity index of a set Ω as follows

Definition 2.5. Let $\Omega \subset \mathbb{R}^N$ be an open bounded set. We define its *cone index* as

$$\beta_\Omega = \inf\big\{\beta \in (0, 1) \,:\, \Omega \text{ satisfies an inner cone condition of index } \beta \in (0, 1)\big\},$$

and its *homogeneity index* as

$$\alpha_\Omega = \inf\big\{\alpha(\beta) \,:\, \beta \geq \beta_\Omega\big\} = \alpha(\beta_\Omega).$$

Remark 2.6. It is a classical fact that if Ω is an open bounded Lipschitz set, then it satisfies an inner cone condition with index β, for all β such that

$$\frac{\kappa_\Omega}{\sqrt{1 + \kappa_\Omega^2}} < \beta < 1.$$

Here κ_Ω is the Lipschitz constant defined in (2.1). Hence

$$\beta_\Omega \leq \frac{\kappa_\Omega}{\sqrt{1 + \kappa_\Omega^2}} \qquad \text{and} \qquad \alpha_\Omega < +\infty. \tag{2.6}$$

In particular, by Remark 2.2 and (2.3), one has that

$$\Omega \text{ of class } C^1 \qquad \Longrightarrow \qquad \beta_\Omega = 0 \quad \text{and} \quad \alpha_\Omega = 1. \tag{2.7}$$

Note however that for a Lipschitz set the inequality in (2.6) can be strict. This is the case, for example, when the set has "concave" corners, see Figure 1.

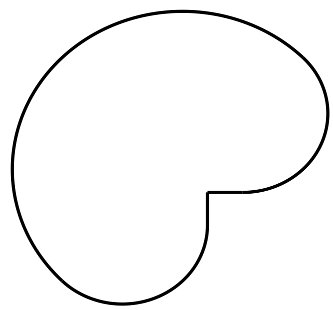

FIGURE 1. A set with Lipschitz constant $\kappa_\Omega = 1$, which satisfies $\beta_\Omega = 0$.

3. LEAST ENERGY SOLUTIONS

Definition 3.1. Let $\Omega \subset \mathbb{R}^N$ be an open bounded set. For $1 < q < 2$, we define $w_{\Omega,q}$ to be the unique solution of

$$\min_{\varphi \in \mathcal{D}^{1,2}_0(\Omega)} \left\{ \frac{1}{2} \int_\Omega |\nabla \varphi|^2\, dx - \frac{1}{q} \int_\Omega \varphi^q\, dx \,:\, \varphi \geq 0 \text{ a.e. in } \Omega \right\}.$$

Existence follows by using the Direct Methods in the Calculus of Variations, while for uniqueness we refer for example to [7, Lemma 2.2].

Remark 3.2. The function $w_{\Omega,q}$ is the unique non-negative solution of

$$\min_{\varphi \in \mathcal{D}^{1,2}_0(\Omega)} \left\{ \frac{1}{2} \int_\Omega |\nabla \varphi|^2\, dx - \frac{1}{q} \int_\Omega |\varphi|^q\, dx \right\},$$

as well. Consequently, when $\Omega \subset \mathbb{R}^N$ is connected, it is the unique non-negative solution of the Lane-Emden equation (1.4). Then, it is not difficult to see that

$$\lambda_1(\Omega; q) = \|w_{\Omega,q}\|^{q-2}_{L^q(\Omega)},$$

and the rescaled function

$$u = \frac{w_{\Omega,q}}{\|w_{\Omega,q}\|_{L^q(\Omega)}},$$

is the (unique) first positive $q-$eigenfunction of Ω, with unit L^q norm.

On the other hand, when Ω is disconnected, equation (1.4) has many non-negative solutions. More precisely, if Ω has k connected components $\Omega_1, \dots, \Omega_k$, then (1.4) has exactly $2^k - 1$ positive solutions, given by

$$\delta_1\, w_{\Omega_1,q} + \dots + \delta_k\, w_{\Omega_k,q},$$

where each $\delta_i \in \{0,1\}$ and they are not all equal to 0. In this case, the function $w_{\Omega,q}$ is given by

$$w_{\Omega,q} = \sum_{i=1}^{k} w_{\Omega_i,q}, \tag{3.1}$$

and it gives the positive least energy solution of (1.4). Accordingly, in this case the first $q-$eigenvalue of Ω is given by (see [6, Corollary 2.6])

$$\lambda_1(\Omega;q) = \left[\sum_{i=1}^{k} \left(\frac{1}{\lambda_1(\Omega_i;q)}\right)^{\frac{q}{2-q}}\right]^{\frac{q-2}{q}},$$

and it is not simple.

The following Hardy-type inequality is a particular case of a general result proved in [7]. This simpler result is enough for our purposes and it is obtained by taking $\delta = 1$ in [7, Theorem 3.1 & Corollary 3.4].

Proposition 3.3 (Hardy-Lane-Emden inequality)**.** *Let $1 < q < 2$ and let $\Omega \subset \mathbb{R}^N$ be an open bounded set. Then for every $\varphi \in \mathcal{D}^{1,2}_0(\Omega)$, we have*

$$\int_\Omega w_{\Omega,q}^{q-2}\, |\varphi|^2\, dx \le \int_\Omega |\nabla \varphi|^2\, dx.$$

In particular, the embedding

$$\mathcal{D}^{1,2}_0(\Omega) \hookrightarrow L^2(\Omega; w_{\Omega,q}^{q-2}),$$

is continuous.

Given an open bounded set $\Omega \subset \mathbb{R}^N$, we denote by $\mathcal{G}_\Omega$ the Green function for the Dirichlet-Laplacian on Ω. We also recall the notation

$$\Omega_R = \{x \in \Omega\, :\, d_\Omega(x) > R\}.$$

The following result is contained in [9, Proposition 3.6]. A similar result, under slightly stronger assumptions and with a worse control on the constants, was previously obtained in [19].

Proposition 3.4. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded Lipschitz set which satisfies an inner cone condition of index $\beta \in (0,1)$. Let $R > 0$ be such that (2.5) hold true. Then, for $\alpha = \alpha(\beta)$ defined in (2.2), we have*

$$\mathcal{G}_\Omega(x,y) \ge \frac{1}{C}\, d_\Omega(x)^\alpha, \qquad \text{for every } x \in \Omega \text{ and } y \in \Omega_R, \tag{3.2}$$

where the constant $C > 0$ depends on N, Ω and R only.

From the previous result, we immediately get the following one. This is an essential ingredient for the proof of our main result.

Corollary 3.5. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded Lipschitz set, with homogeneity index α_Ω. For $1 < q < 2$, let $w_{\Omega,q}$ be the function in Definition 3.1. Then, for all $\alpha > \alpha_\Omega$ we have*

$$w_{\Omega,q}(x) \ge \frac{1}{\mathcal{C}}\, d_\Omega(x)^\alpha, \qquad \text{for every } x \in \Omega, \tag{3.3}$$

where the constant $\mathcal{C} > 0$ depends on N, q, Ω and α (and it might degenerate as α converges to α_Ω).

Proof. The function $w_{\Omega,q}$ solves

$$-\Delta w_{q,\Omega} = w_{\Omega,q}^{q-1}, \qquad \text{in } \Omega,$$

with homogeneous Dirichlet conditions. Thus, by the representation formula for Poisson's equation, we have

$$w_{\Omega,q}(x) = \int_\Omega \mathcal{G}_\Omega(x,y)\, w_{\Omega,q}(y)^{q-1}\, dy, \qquad \text{for } x \in \Omega.$$

We fix $\alpha > \alpha_\Omega$ and we let $\beta > \beta_\Omega$ be such that $\alpha = \alpha(\beta)$. By definition of cone index β_Ω, we have that Ω satisfies an inner cone condition of index β. We can thus use (3.2) to infer

$$\begin{aligned} w_{\Omega,q}(x) \geq \int_{\Omega_R} \mathcal{G}_\Omega(x,y)\, w_{\Omega,q}(y)^{q-1}\, dy &\geq \frac{1}{C} \int_{\Omega_R} d_\Omega(x)^\alpha\, w_{\Omega,q}(y)^{q-1}\, dy \\ &\geq \frac{1}{C} \left(\inf_{\Omega_R} w_{\Omega,q}\right)^{q-1} |\Omega_R|\, d_\Omega(x)^\alpha. \end{aligned} \tag{3.4}$$

Observe that $\inf_{\Omega_R} w_{\Omega,q} > 0$, by the minimum principle. We can now use the lack of homogeneity of the equation to improve the previous estimate: by passing to the minimum over Ω_R in (3.4), we get

$$\inf_{\Omega_R} w_{\Omega,q} \geq \frac{1}{C} \left(\inf_{\Omega_R} w_{\Omega,q}\right)^{q-1} |\Omega_R|\, R^\alpha,$$

that is

$$\inf_{\Omega_R} w_{\Omega,q} \geq \left(\frac{1}{C}\, |\Omega_R|\, R^\alpha\right)^{\frac{1}{2-q}}.$$

By spending this information in (3.4), we finally get

$$w_{\Omega,q}(x) \geq \frac{1}{C} \left(\frac{1}{C}\, |\Omega_R|\, R^\alpha\right)^{\frac{q-1}{2-q}} |\Omega_R|\, d_\Omega(x)^\alpha,$$

as desired. □

4. An abstract result

The following result is the key ingredient for the proof of our main result.

Proposition 4.1. *Let $1 < q < 2$ and let $\Omega \subset \mathbb{R}^N$ be an open bounded set. Let us assume that*

$$\text{the embedding } \mathcal{D}_0^{1,2}(\Omega) \hookrightarrow L^2(\Omega; w_{\Omega,q}^{q-2}) \text{ is compact.} \tag{4.1}$$

Then there exists $\delta > 0$ such that the neighborhood

$$\mathcal{I}_\delta(w_{\Omega,q}) = \left\{\varphi \in \mathcal{D}_0^{1,2}(\Omega)\, :\, \|\varphi - w_{\Omega,q}\|_{L^1(\Omega)} < \delta\right\},$$

does not contain any solution of the Lane-Emden equation (1.4).

Proof. We argue by contradiction. We assume that there exists a sequence $\{U_n\}_{n\in\mathbb{N}} \subset \mathcal{D}_0^{1,2}(\Omega)$ of solutions of the equation (1.4), such that

$$\lim_{n\to\infty} \|U_n - w_{\Omega,q}\|_{L^1(\Omega)} = 0.$$

We first observe that by Corollary B.3, we automatically get

$$\lim_{n\to\infty} \|\nabla w_{\Omega,q} - \nabla U_n\|_{L^2(\Omega)} = 0,$$

as well.

By subtracting the equations satisfied by $w_{\Omega,q}$ and U_n, we get

$$\int_\Omega \langle \nabla(w_{\Omega,q} - U_n), \nabla\varphi\rangle\, dx = \int_\Omega (w_{\Omega,q}^{q-1} - |U_n|^{q-2}\, U_n)\, \varphi\, dx, \tag{4.2}$$

for every $\varphi \in \mathcal{D}^{1,2}_0(\Omega)$. We introduce the function

$$W_n(x) = \begin{cases} \dfrac{w_{\Omega,q}(x)^{q-1} - |U_n(x)|^{q-2}\, U_n(x)}{w_{\Omega,q}(x) - U_n(x)}, & \text{if } w_{\Omega,q}(x) \neq U_n(x),\\ 0, & \text{otherwise.}\end{cases}$$

Then equation (4.2) can be rewritten as

$$\int_\Omega \langle \nabla(w_{\Omega,q} - U_n), \nabla\varphi\rangle\, dx = \int_\Omega W_n\, (w_{\Omega,q} - U_n)\, \varphi\, dx.$$

Thus, if we define the rescaled sequence

$$\phi_n = \frac{w_{\Omega,q} - U_n}{\|w_{\Omega,q} - U_n\|_{L^2(\Omega; w_{\Omega,q}^{q-2})}}, \qquad \text{for } n \in \mathbb{N},$$

we get that $\phi_n \in \mathcal{D}^{1,2}_0(\Omega)$ is a weak solution of

$$-\Delta \phi_n = W_n\, \phi_n. \tag{4.3}$$

We observe that

$$0 \le W_n(x) \le 2^{2-q}\, w_{\Omega,q}^{q-2}(x), \qquad \text{for every } x \in \Omega, \tag{4.4}$$

thanks to Lemma A.1. Then, by testing the equation (4.3) with ϕ_n itself and recalling the normalization taken, we get

$$\int_\Omega |\nabla \phi_n|^2\, dx \le 2^{2-q} \int_\Omega w_{\Omega,q}^{q-2}\, |\phi_n|^2\, dx = 2^{2-q}, \qquad \text{for every } n \in \mathbb{N}.$$

Thus, the assumption on the compactness of the embedding $\mathcal{D}^{1,2}_0(\Omega) \hookrightarrow L^2(\Omega; w_\Omega^{q-2})$ implies that $\{\phi_n\}_{n\in\mathbb{N}}$ converges (up to a subsequence) weakly in $\mathcal{D}^{1,2}_0(\Omega)$ and strongly in $L^2(\Omega; w_{\Omega,q}^{q-2})$ to $\phi \in \mathcal{D}^{1,2}_0(\Omega)$. In particular, we still have

$$\|\phi\|_{L^2(\Omega; w_{\Omega,q}^{q-2})} = 1, \qquad \text{so that } \ \phi \not\equiv 0. \tag{4.5}$$

Observe that we have

$$\limsup_{n\to\infty} W_n(x) \le (q-1)\, w_{\Omega,q}^{q-2}(x), \qquad \text{for a. e. } x \in \Omega. \tag{4.6}$$

This can be seen as follows: take $\Omega' \Subset \Omega$, thus by the minimum principle $w_{\Omega,q} \ge 1/C_{\Omega'}$ on Ω', for an appropriate $C_{\Omega'} > 0$. By using the equation and standard Elliptic Regularity, we have that U_n converges uniformly on $\overline{\Omega'}$ to $w_{\Omega,q}$. Thus, for n large enough (depeding on Ω' only), the function

$$t \mapsto \Big(|w_{\Omega,q}(x) + t\, (U_n(x) - w_{\Omega,q}(x))|^{q-2}\, (w_{\Omega,q}(x) + t\, (U_n(x) - w_{\Omega,q}(x)))\Big)$$

is differentiable, for every $x \in \Omega'$. Then we write

$$\begin{aligned} w_{\Omega,q}^{q-1} - |U_n|^{q-2}\, U_n &= -\int_0^1 \frac{d}{dt}\Big(|w_{\Omega,q} + t\, (U_n - w_{\Omega,q})|^{q-2}\, (w_{\Omega,q} + t\, (U_n - w_{\Omega,q}))\Big)\, dt \\ &= (q-1)\, \left(\int_0^1 |w_{\Omega,q} + t\, (U_n - w_{\Omega,q})|^{q-2}\, dt\right)\, (w_{\Omega,q} - U_n). \end{aligned}$$

From the previous equation we get

$$W_n \le (q-1)\int_0^1 |w_{\Omega,q}+t\,(U_n-w_{\Omega,q})|^{q-2}\,dt, \qquad \text{on } \Omega',$$

for n large enough. By passing to the limit as n goes to ∞, we get the desired conclusion for every $x\in\Omega'$. By arbitrariness of $\Omega'\Subset\Omega$, we finally get the claim (4.6).

By testing equation (4.3) with ϕ_n itself, we get

$$\int_\Omega |\nabla\phi_n|^2\,dx=\int_\Omega W_n\,|\phi_n|^2\,dx\le\int_\Omega W_n\,\Big||\phi_n|^2-|\phi|^2\Big|\,dx+\int_\Omega W_n\,|\phi|^2\,dx.$$

We now use the lower semicontinuity of the L^2 norm on the left-hand side and the Fatou Lemma in the second term on the right-hand side. Indeed, note that by (4.4) and the Hardy-Lane-Emden inequality (i.e. Proposition 3.3), $W_n\,|\phi|^2$ is bounded by the L^1 function $2^{2-q}\,w_{\Omega,q}^{q-2}\,|\phi|^2$.

In view of (4.6) and (4.5), we get

$$\int_\Omega |\nabla\phi|^2\,dx\le\limsup_{n\to\infty}\int_\Omega W_n\,\Big||\phi_n|^2-|\phi|^2\Big|\,dx+(q-1). \tag{4.7}$$

We are only left with handling the limit

$$\limsup_{n\to\infty}\int_\Omega W_n\,\Big||\phi_n|^2-|\phi|^2\Big|\,dx.$$

This is done as follows: by elementary manipulations and (4.4)

$$\begin{aligned}\int_\Omega W_n\,\Big||\phi_n|^2-|\phi|^2\Big|\,dx&=\int_\Omega W_n\,\Big|\phi_n-\phi\Big|\,\Big|\phi_n+\phi\Big|\,dx\\&\le 2^{2-q}\int_\Omega w_{\Omega,q}^{q-2}\,\Big|\phi_n-\phi\Big|\,\Big|\phi_n+\phi\Big|\,dx\\&\le 2^{2-q}\left(\int_\Omega w_{\Omega,q}^{q-2}\,|\phi_n-\phi|^2\,dx\right)^{\frac12}\left(\int_\Omega w_{\Omega,q}^{q-2}\,|\phi_n+\phi|^2\,dx\right)^{\frac12}.\end{aligned}$$

By using the strong convergence in $L^2(\Omega;w_{\Omega,q}^{q-2})$ for the first integral and the Hardy-Lane-Emden inequality to bound the second integral, we get

$$\limsup_{n\to\infty}\int_\Omega W_n\,\Big||\phi_n|^2-|\phi|^2\Big|\,dx=0.$$

By using this in (4.7), we finally end up with

$$\int_\Omega |\nabla\phi|^2\,dx\le(q-1).$$

On the other hand, again by Proposition 3.3 we must have

$$1=\int_\Omega w_\Omega^{q-2}\,|\phi^2|\,dx\le\int_\Omega|\nabla\phi|^2\,dx.$$

Since $q<2$, the last two inequalities give the desired contradiction. □

We now proceed to analyze the situation for the first $q-$eigenvalue. Before doing this, we need the following result.

Lemma 4.2. *Let $1<q<2$ and let $\Omega\subset\mathbb{R}^N$ be an open bounded set, such that the embedding $\mathcal{D}_0^{1,2}(\Omega)\hookrightarrow L^2(\Omega;w_{\Omega,q}^{q-2})$ is compact. Then Ω must have a finite number of connected components.*

Proof. We argue by contradiction and assume that Ω has countably many connected components $\{\Omega_n\}_{n\in\mathbb{N}}$. We take the sequence

$$\varphi_n = \frac{w_{\Omega_n,q}}{\|\nabla w_{\Omega_n,q}\|_{L^2(\Omega_n)}}, \qquad n \in \mathbb{N},$$

which is bounded in $\mathcal{D}^{1,2}_0(\Omega)$. By using Poincaré inequality, we have

$$\int_\Omega |\varphi_n|^2\,dx = \int_{\Omega_n} |\varphi_n|^2\,dx \le \frac{1}{\lambda_1(\Omega_n)} \int_{\Omega_n} |\nabla\varphi_n|^2\,dx = \frac{1}{\lambda_1(\Omega_n)}, \tag{4.8}$$

where we recall that λ_1 stands for the first eigenvalue of the Dirichlet-Laplacian. We now observe that by the *Faber-Krahn inequality*, we have

$$\lambda_1(\Omega_n) \ge |\Omega_n|^{-\frac{2}{N}} \left(|B|^{\frac{2}{N}}\,\lambda_1(B)\right),$$

where B is any $N-$dimensional ball. Moreover, since Ω is bounded, we have

$$\lim_{n\to\infty} |\Omega_n| = 0.$$

The last two formulas show that $\lambda_1(\Omega_n)$ diverges to $+\infty$, as n goes to ∞. By (4.8) we thus get that φ_n converges strongly to 0 in $L^2(\Omega)$.

On the other hand, by computing the weighted L^2 norm of φ_n and recalling (3.1), we have

$$\int_\Omega \frac{|\varphi_n|^2}{w_{\Omega,q}^{2-q}}\,dx = \int_{\Omega_n} \frac{|\varphi_n|^2}{w_{\Omega_n,q}^{2-q}}\,dx = \frac{\displaystyle\int_{\Omega_n} w_{\Omega_n,q}^q\,dx}{\displaystyle\int_{\Omega_n} |\nabla w_{\Omega_n,q}|^2\,dx} = 1,$$

where in the last identity we used that $w_{\Omega_n,q}$ solves the Lane-Emden equation on Ω_n. This contradicts the compactness of the embedding $\mathcal{D}^{1,2}_0(\Omega) \hookrightarrow L^2(\Omega; w_{\Omega,q}^{q-2})$. $\square$

By using Proposition 4.1, we can now get the following result for the first $q-$eigenvalue.

Proposition 4.3. *Let $1 < q < 2$ and let $\Omega \subset \mathbb{R}^N$ be an open bounded set. Let us assume that condition (4.1) holds. Then the first $q-$eigenvalue*

$$\lambda_1(\Omega; q) = \min_{u\in\mathcal{D}^{1,2}_0(\Omega)} \left\{\int_\Omega |\nabla u|^2\,dx \,:\, \int_\Omega |u|^q\,dx = 1\right\},$$

is isolated in $\mathrm{Spec}(\Omega; q)$.

Proof. We argue by contradiction and assume that there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathrm{Spec}(\Omega; q)$ such that

$$\lim_{n\to\infty} \lambda_n = \lambda_1(\Omega; q). \tag{4.9}$$

We have to consider two different cases: either Ω is connected or not.

Case 1: Ω is connected. There exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{D}^{1,2}_0(\Omega)$ of normalized $q-$eigenfunctions for Ω, i. e.

$$-\Delta u_n = \lambda_n\,|u_n|^{q-2}\,u_n, \text{ in } \Omega, \qquad \text{with } \|u_n\|_{L^q(\Omega)} = 1.$$

By using the equation, it is not difficult to see that

$$\lim_{n\to\infty} \|\nabla u_n - \nabla u\|_{L^2(\Omega)} = 0, \tag{4.10}$$

where $u \in \mathcal{D}^{1,2}_0(\Omega)$ is a first $q-$eigenfunction, with unit L^q norm. Since we are assuming Ω to be connected, the first $q-$eigenvalue is simple (see [5, Theorem 3.1]). Thus, upon replacing u_n with $-u_n$, we can suppose that u is the first positive $q-$eigenfunction, with unit L^q norm.

We note that the rescaled functions

$$U = \lambda_1(\Omega; q)^{\frac{1}{q-2}}\, u \qquad \text{and} \qquad U_n = \lambda_n^{\frac{1}{q-2}}\, u_n,$$

solve the Lane-Emden equation (1.4) in Ω. Moreover, by uniqueness of the positive solution, U coincides with $w_{\Omega,q}$. From (4.9) and (4.10), we can thus infer that

$$\lim_{n\to\infty} \|\nabla U_n - \nabla w_{\Omega,q}\|_{L^2(\Omega)} = 0.$$

However, this contradicts Proposition 4.1.

Case 2: Ω is not connected. By Lemma 4.2, we know that assumption (4.1) guarantees that Ω has a finite number $\Omega_1, \dots, \Omega_k$ of connected components. By the "spin formula" of [6, Proposition 2.1 & Corollary 2.2], we know that

$$\lambda_n = \left[\sum_{i=1}^{k} \left(\frac{\delta_{n,i}}{\lambda_{n,i}}\right)^{\frac{q}{2-q}}\right]^{\frac{q-2}{q}},$$

for some $\lambda_{n,i} \in \mathrm{Spec}(\Omega_i; q)$ and $\delta_{n,i} \in \{0,1\}$ such that

$$\sum_{i=1}^{k} \delta_{n,i} \neq 0, \qquad \text{for every } n \in \mathbb{N}.$$

Moreover, by [6, Corollary 2.6], we must have

$$\lambda_1(\Omega; q) = \left[\sum_{i=1}^{k} \left(\frac{1}{\lambda_1(\Omega_i; q)}\right)^{\frac{q}{2-q}}\right]^{\frac{q-2}{q}}.$$

In light of (4.9), we thus get that[2]

$$\lambda_n = \left[\sum_{i=1}^{k} \left(\frac{1}{\lambda_{n,i}}\right)^{\frac{q}{2-q}}\right]^{\frac{q-2}{q}},$$

for n large enough, with $\lambda_{n,i}$ converging to $\lambda_1(\Omega_i; q)$, for every $i = 1, \dots, k$. However, this contradicts the fact that each $\lambda_1(\Omega_i; q)$ is isolated, by the first part of the proof. $\square$

[2]Observe that the function

$$f(t_1, \dots, t_k) = \left[\sum_{i=1}^{k} t_i^{\frac{q}{2-q}}\right]^{\frac{q-2}{q}}, \qquad \text{for } t_i \le \frac{1}{\lambda_1(\Omega_i; q)},$$

is strictly decreasing in each argument and it uniquely attains its minimum when

$$t_i = \frac{1}{\lambda_1(\Omega_i; q)}, \qquad \text{for every } i = 1, \dots, k.$$

Thus (4.9) entails that

$$\lim_{n\to\infty} \frac{\delta_{n,i}}{\lambda_{n,i}} = \frac{1}{\lambda_1(\Omega_i; q)}, \qquad \text{for every } i = 1, \dots, k.$$

5. A weighted embedding

The results of the previous section lead us to study conditions on Ω under which

$$\text{the embedding } \mathcal{D}^{1,2}_0(\Omega) \hookrightarrow L^2(\Omega; w_{\Omega,q}^{q-2}) \text{ is compact.}$$

We have seen in Lemma 4.2 that a necessary condition is that Ω has a finite number of connected components. We now provide a sufficient condition, as well. The sharpness of the assumptions is discussed in Example 5.3 below.

Proposition 5.1. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded Lipschitz set, with homogeneity index α_Ω. Then:*

- *if $1 \le \alpha_\Omega \le 2$, the embedding $\mathcal{D}^{1,2}_0(\Omega) \hookrightarrow L^2(\Omega; w_{\Omega,q}^{q-2})$ is compact for every $1 < q < 2$;*
- *if $\alpha_\Omega > 2$, the embedding $\mathcal{D}^{1,2}_0(\Omega) \hookrightarrow L^2(\Omega; w_{\Omega,q}^{q-2})$ is compact for every*
$$2 - \frac{2}{\alpha_\Omega} < q < 2.$$

In particular, by (2.7), the embedding $\mathcal{D}^{1,2}_0(\Omega) \hookrightarrow L^2(\Omega; w_{\Omega,q}^{q-2})$ is compact for every $1 < q < 2$ if Ω is of class C^1.

Proof. Since $\Omega \subset \mathbb{R}^N$ is an open bounded Lipschitz set, there exists a constant $C_\Omega > 0$ such that the classical *Hardy inequality* holds

$$\int_\Omega \frac{|\varphi|^2}{d_\Omega^2}\,dx \le C_\Omega \int_\Omega |\nabla\varphi|^2\,dx, \qquad \text{for every } \varphi \in \mathcal{D}^{1,2}_0(\Omega), \tag{5.1}$$

see [22, Théorème 1.6] or also [17, Theorem 8.4]. We now discuss separately the two cases:

Case $\alpha_\Omega \le 2$. In this case, we have

$$\alpha_\Omega \le 2 < \frac{2}{2-q}, \qquad \text{for every } 1 < q < 2.$$

Thus, we can fix $\alpha_\Omega < \alpha < 2/(2-q)$ such that (3.3) holds. By (5.1) and Hölder's inequality with exponents

$$\frac{2}{(2-q)\,\alpha} \qquad \text{and} \qquad \frac{2}{2-(2-q)\,\alpha},$$

we get for every $\varphi \in \mathcal{D}^{1,2}_0(\Omega)$

$$\begin{aligned}\int_\Omega \frac{|\varphi|^2}{d_\Omega^{(2-q)\,\alpha}}\,dx &\le \left(\int_\Omega \frac{|\varphi|^2}{d_\Omega^2}\right)^{\frac{(2-q)\,\alpha}{2}} \left(\int_\Omega |\varphi|^2\,dx\right)^{1-\frac{2-q}{2}\,\alpha}\\ &\le C_\Omega^{\frac{(2-q)\,\alpha}{2}} \left(\int_\Omega |\nabla\varphi|^2\,dx\right)^{\frac{(2-q)\,\alpha}{2}} \left(\int_\Omega |\varphi|^2\,dx\right)^{1-\frac{2-q}{2}\,\alpha}.\end{aligned}$$

Moreover, by (3.3) we have $d_\Omega^\alpha \le \mathcal{C}\, w_{\Omega,q}$. Thus we get the following interpolation inequality

$$\int_\Omega \frac{|\varphi|^2}{w_{\Omega,q}^{2-q}}\,dx \le \mathcal{C}^{2-q}\, C_\Omega^{\frac{(2-q)\,\alpha}{2}} \left(\int_\Omega |\nabla\varphi|^2\,dx\right)^{\frac{(2-q)\,\alpha}{2}} \left(\int_\Omega |\varphi|^2\,dx\right)^{1-\frac{2-q}{2}\,\alpha}.$$

By using this and recalling that the embedding $\mathcal{D}^{1,2}_0(\Omega) \hookrightarrow L^2(\Omega)$ is compact for an open bounded set, we get the conclusion.

Case $\alpha_\Omega > 2$. In this case, we have

$$\alpha_\Omega < \frac{2}{2-q}, \qquad \text{for every } q \text{ such that } 2 - \frac{2}{\alpha_\Omega} < q < 2.$$

We can repeat the previous argument and get again the desired conclusion. □

By the very definition of α_Ω, the assumption $\alpha_\Omega \leq 2$ means that the "corners" of Ω should not be "too narrow". On the other hand, when $\alpha_\Omega > 2$, we have that the higher the value of α_Ω is, the smaller is the set of exponents q for which the relevant embedding is compact.

Once again, the two-dimensional case is easier to understand. For $N = 2$, we can reformulate the previous result as follows

Corollary 5.2 (Two-dimensional case)**.** *Let $\Omega \subset \mathbb{R}^2$ be an open bounded Lipschitz set, with cone index $\beta_\Omega \in [0, 1)$. Then:*

- *if $\beta_\Omega \leq \cos(\pi/4)$, the embedding $\mathcal{D}^{1,2}_0(\Omega) \hookrightarrow L^2(\Omega; w^{q-2}_{\Omega,q})$ is compact for every $1 < q < 2$;*
- *if $\beta_\Omega > \cos(\pi/4)$, the embedding $\mathcal{D}^{1,2}_0(\Omega) \hookrightarrow L^2(\Omega; w^{q-2}_{\Omega,q})$ is compact for every*
$$2 - \frac{4}{\pi}\arccos(\beta_\Omega) < q < 2.$$

Proof. By (2.4) we know that
$$\alpha_\Omega = \frac{\pi}{2\,\arccos(\beta_\Omega)}.$$
From this we get
$$\alpha_\Omega \leq 2 \qquad \Longleftrightarrow \qquad \arccos\beta_\Omega \geq \frac{\pi}{4} \qquad \Longleftrightarrow \qquad \beta_\Omega \leq \cos\left(\frac{\pi}{4}\right),$$
and thus the conclusion follows from Proposition 5.1. □

The assumptions in Proposition 5.1 are sharp. In fact, when these are not in force, the compactness of the embedding can badly fail, even among convex sets. Indeed, we can produce an open bounded convex set $\Omega \subset \mathbb{R}^N$ such that

- $\alpha_\Omega > 2$;
- for every $1 < q < 2 - 2/\alpha_\Omega$, the embedding $\mathcal{D}^{1,2}_0(\Omega) \hookrightarrow L^2(\Omega; w^{q-2}_{\Omega,q})$ is not compact.

This is shown in the following

Example 5.3. We use the same notations of Definition 2.3. Let $\beta > 0$ be such that

$$2\,N < \lambda(\mathcal{S}(\beta)). \tag{5.2}$$

For every $R > 0$, we consider the convex cone $\Gamma(\beta, R)$. We observe that the function
$$\Phi(t) = t\,(N - 2 + t), \qquad \text{for } t \geq 0.$$
is monotone increasing and $\Phi(2) = 2\,N$. By recalling the definition of $\alpha_{\Gamma(\beta,R)}$, this implies that
$$\text{condition (5.2)} \qquad \Longleftrightarrow \qquad \alpha_{\Gamma(\beta,R)} > 2.$$
We show that for every

$$1 < q < 2 - \frac{2}{\alpha_{\Gamma(\beta,R)}}, \tag{5.3}$$

the embedding
$$\mathcal{D}^{1,2}_0(\Gamma(\beta, R)) \hookrightarrow L^2(\Gamma(\beta, R); w^{q-2}_{\Gamma(\beta,R),q}),$$

is not compact. In order to prove this, it is sufficient to observe that with our choice (5.3) we have

$$\frac{2}{2-q} < \alpha_{\Gamma(\beta,R)} \qquad \text{which implies} \qquad \Phi\left(\frac{2}{2-q}\right) < \Phi(\alpha_{\Gamma(\beta,R)}) = \lambda(\mathcal{S}(\beta)).$$

Thus the claimed assertion follows from Proposition C.1 below.

6. Proofs of the main results

6.1. **Proofs.** By combining the compactness result of Proposition 5.1 with Proposition 4.1, we now get the following more general version of Theorem A. We recall that α_Ω is the homogeneity index of Ω, defined in Definition 2.4.

Theorem 6.1. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded Lipschitz set, with homogeneity index α_Ω. Let us suppose that Ω has a finite number of connected components. If we set*

$$q_\Omega := \max\left\{2 - \frac{2}{\alpha_\Omega}, 1\right\}, \tag{6.1}$$

then, for every $q_\Omega < q < 2$, the positive least energy solution $w_{\Omega,q} \in \mathcal{D}^{1,2}_0(\Omega)$ of equation (1.4) is isolated in the $L^1(\Omega)$ norm topology, i.e. there exists $\delta > 0$ such that the neighborhood

$$\mathcal{I}_\delta(w_{\Omega,q}) = \left\{\varphi \in \mathcal{D}^{1,2}_0(\Omega) \,:\, \|\varphi - w_{\Omega,q}\|_{L^1(\Omega)} < \delta\right\},$$

does not contain any other solution of the Lane-Emden equation.

Remark 6.2. We recall that for a C^1 set, we have $\alpha_\Omega = 1$. Thus in this case from (6.1) we get

$$q_\Omega = 1,$$

and we recover the statement of Theorem A. More generally, observe that $q_\Omega = 1$ whenever $\alpha_\Omega \leq 2$.

As for the claimed isolation result of Theorem B, this follows by combining Proposition 5.1 with Proposition 4.3. The final outcome is again slightly more general, as we can admit Lipschitz sets. We still indicate by q_Ω the exponent defined in (6.1).

Theorem 6.3. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded Lipschitz set, with homogeneity index α_Ω. Let us suppose that Ω has a finite number of connected components. Then, for every $q_\Omega < q < 2$, the first $q-$eigenvalue*

$$\lambda_1(\Omega; q) = \min_{u \in \mathcal{D}^{1,2}_0(\Omega)} \left\{\int_\Omega |\nabla u|^2\,dx \,:\, \int_\Omega |u|^q\,dx = 1\right\},$$

is isolated in $\mathrm{Spec}(\Omega; q)$.

Finally, for ease of exposition, we find it useful to state the previous results for $N = 2$. Here, the interplay between the cone index β_Ω and the exponent q_Ω is cleaner. Indeed, by recalling (2.4), we have

$$\alpha_\Omega = \frac{\pi}{2\,\arccos(\beta_\Omega)},$$

thus we get the following

Corollary 6.4 (Two dimensional case)**.** *Let $\Omega \subset \mathbb{R}^2$ be an open bounded Lipschitz set with cone index β_Ω. The conclusions of Theorems 6.1 and 6.3 hold for every*

$$\max\left\{2 - \frac{4}{\pi}\,\arccos(\beta_\Omega),\, 1\right\} < q < 2.$$

6.2. **Non-negative solutions with higher energy.** In the next example we show that when Ω is not connected, Theorem A *cannot* be extended to non-negative solutions not having least energy (recall Remark 3.2). This is similar to the example of [6, Theorem 3.1].

Example 6.5. Let $1 < q < 2$ and let $\Omega = \Omega_1 \cup \Omega_2$, with $\Omega_1, \Omega_2 \subset \mathbb{R}^N$ open bounded sets with smooth boundary, such that $\Omega_1 \cap \Omega_2 = \emptyset$. Then there exists a sequence $\{U_n\}_{n\in\mathbb{N}} \subset \mathcal{D}^{1,2}_0(\Omega)$ of distinct solutions of the Lane-Emden equation (1.4) and a positive solution U of the same equation, such that

$$\lim_{n\to\infty} \|\nabla U_n - \nabla U\|_{L^2(\Omega)} = 0.$$

We start by taking the positive least energy solution $w_{\Omega_1,q}$ of Ω_1. We consider it to be extended by 0 on the whole Ω. We then take $u_n \in \mathcal{D}^{1,2}_0(\Omega_2)$ to be a $q-$eigenfunction of Ω_2 with unit L^q norm, associated to the $n-$th variational eigenvalue $\lambda_{n,LS}(\Omega_2; q)$ of Ω_2 (recall the definition (1.3)). Again, we consider it to be extended by 0 on the whole Ω. We then set

$$U_n = w_{\Omega_1,q} + \lambda_{n,LS}(\Omega_2; q)^{\frac{1}{q-2}}\, u_n,$$

which solves, by construction

$$-\Delta U_n = |U_n|^{q-2}\, U_n, \qquad \text{in } \Omega.$$

By recalling that $\lambda_{n,LS}(\Omega_2; q)$ diverges to $+\infty$ as n goes to ∞ and using that $2 - q < 2$, we then obtain

$$\begin{aligned}\lim_{n\to\infty} \|\nabla U_n - \nabla w_{\Omega_1,q}\|_{L^2(\Omega)} &= \lim_{n\to\infty} \lambda_{n,LS}(\Omega_2; q)^{\frac{1}{q-2}}\, \|\nabla u_n\|_{L^2(\Omega)}\\ &= \lim_{n\to\infty} \lambda_{n,LS}(\Omega_2; q)^{\frac{1}{q-2}+\frac{1}{2}} = 0,\end{aligned}$$

which is the desired conclusion.

Appendix A. A pointwise inequality

The following simple inequality has been useful in order to prove our main result.

Lemma A.1. *Let $0 < \alpha < 1$, then for every $a > 0$ and $b \in \mathbb{R}$ we have*

$$|a^\alpha - |b|^{\alpha-1}\, b| \le 2^{1-\alpha}\, a^{\alpha-1}\, |a - b|.$$

Proof. We first suppose that $a \ge b \ge 0$, then we write (recall that $a > 0$)

$$\begin{aligned}|a^\alpha - |b|^{\alpha-1}\, b| = a^\alpha - b^\alpha &= a^\alpha\, \left(1 - \left(\frac{b}{a}\right)^\alpha\right)\\ &\le a^\alpha\, \left(1 - \frac{b}{a}\right) = a^{\alpha-1}\,(a - b),\end{aligned}$$

where we used that $t \le t^\alpha$ for every $0 \le t \le 1$.

We now suppose that $b > a > 0$, then by proceeding as before, we find

$$|a^\alpha - |b|^{\alpha-1}\, b| = b^\alpha - a^\alpha \le b^{\alpha-1}\,(b - a).$$

By observing that the power $\alpha - 1$ is negative and using the hypothesis $b > a > 0$, we prove the inequality in this case, as well.

Finally, we suppose that $a > 0 \ge b$. In this case, by using the concavity of the map $t \mapsto t^\alpha$, we obtain

$$|a^\alpha - |b|^{\alpha-1}\, b| = a^\alpha + (-b)^\alpha \le 2^{1-\alpha}\,(a - b)^\alpha = 2^{1-\alpha}\,(a - b)^{\alpha-1}\,(a - b).$$

Since b is negative and $\alpha - 1 < 0$, we can further use that $(a-b)^{\alpha-1} \le a^{\alpha-1}$ and get the desired conclusion. $\square$

Appendix B. A uniform L^∞ estimate

For $1 < q < 2$, solutions of the Lane-Emden equation enjoys the following *universal* global L^∞ estimate. This fact should be well-known, it is mentioned for example in [14, page 149]. We provide for completeness a precise estimate, under optimal assumptions on the set. In this section, we drop the requirement that the open set Ω should be bounded.

Proposition B.1. *Let $1 < q < 2$ and let $\Omega \subset \mathbb{R}^N$ be an open set, such that*

$$\lambda_1(\Omega; q) := \min_{u \in \mathcal{D}^{1,2}_0(\Omega)} \left\{ \int_\Omega |\nabla u|^2\, dx \,:\, \int_\Omega |u|^q\, dx = 1 \right\} > 0.$$

For every solution $u \in \mathcal{D}^{1,2}_0(\Omega)$ of the Lane-Emden equation (1.4), we have $u \in L^\infty(\Omega)$, with the universal estimate

$$\|u\|_{L^\infty(\Omega)} \le \begin{cases} C_{N,q}\, \lambda_1(\Omega; q)^{\frac{q}{2\,(q-2)}\,\frac{2^*-2}{2^*-q}}, & \text{if } N \ge 3, \\[2mm] C_q\, \lambda_1(\Omega; q)^{\frac{q}{2\,(q-2)}}, & \text{if } N = 2. \end{cases}$$

Proof. By setting $\lambda = \|u\|^{q-2}_{L^q(\Omega)}$, we see that u solves

$$-\Delta u = \lambda\, \|u\|^{2-q}_{L^q(\Omega)}\, |u|^{q-2}\, u.$$

We can then apply the estimate of [5, Proposition 2.5] and obtain

$$\|u\|_{L^\infty(\Omega)} \le \begin{cases} C_{N,q} \left(\sqrt{\lambda}\right)^{\frac{2^*}{2^*-q}} \|u\|_{L^q(\Omega)}, & \text{if } N \ge 3, \\[2mm] C_q \sqrt{\dfrac{\lambda}{\lambda_1(\Omega; q)}}\, \sqrt{\lambda}\, \|u\|_{L^q(\Omega)}, & \text{if } N = 2. \end{cases}$$

By recalling the definition of λ, we obtain the $L^\infty - L^q$ estimate

$$\|u\|_{L^\infty(\Omega)} \le \begin{cases} C_{N,q}\, \|u\|^{\frac{q}{2}\,\frac{2^*-2}{2^*-q}}_{L^q(\Omega)}, & \text{if } N \ge 3, \\[2mm] C_q \sqrt{\dfrac{1}{\lambda_1(\Omega; q)}}\, \|u\|^{q-1}_{L^q(\Omega)}, & \text{if } N = 2. \end{cases}$$

We only need to show that the L^q norm admits a universal estimate. For this, from the equation we have the energy identity

$$\int_\Omega |\nabla u|^2\, dx = \int_\Omega |u|^q\, dx.$$

By using the definition of $\lambda_1(\Omega; q)$, this entails that

$$\lambda_1(\Omega; q) \left(\int_\Omega |u|^q\, dx \right)^{\frac{2}{q}} \le \int_\Omega |u|^q\, dx.$$

By using that $2/q > 1$, we obtain the desired conclusion. $\square$

Remark B.2. The previous estimate guarantees that the L^∞ norm of a solution of the Lane-Emden equation can be controlled from above in terms of a (negative) power of the sharp Poincaré-Sobolev constant $\lambda_1(\Omega; q)$. This estimate can not be reversed. Indeed, by taking the "slab–type" sequence

$$\Omega_n = (-n, n)^{N-1} \times (-1, 1),$$

we know that the positive least energy solution $w_{\Omega_n,q}$ can be bounded uniformly in $L^\infty(\Omega)$, see [7, Proposition 4.3]. On the other hand, we have

$$\lim_{n\to\infty} \lambda_1(\Omega_n; q) = 0.$$

The previous result permits to infer that on the space of the solutions of the Lane-Emden equation in Ω, the $L^1(\Omega)$ strong topology and the $\mathcal{D}^{1,2}_0(\Omega)$ strong topology are actually equivalent.

Corollary B.3. *Let $1 < q < 2$ and let $\Omega \subset \mathbb{R}^N$ be an open set with finite measure. There exists a constant $C > 0$ depending on N, q and $\lambda_1(\Omega; q)$ only, such that for every pair u, v of solutions of the Lane-Emden equation (1.4), we have*

$$\|\nabla u - \nabla v\|_{L^2(\Omega)} \le C \sqrt{\|u - v\|_{L^1(\Omega)}}.$$

Proof. By subtracting the equations satisfied by u and v, we get

$$\int_\Omega \langle \nabla(u - v), \nabla\varphi\rangle \, dx = \int_\Omega (|u|^{q-2}\, u - |v|^{q-2}\, v)\, \varphi \, dx,$$

for every $\varphi \in \mathcal{D}^{1,2}_0(\Omega)$. We use this identity with $\varphi = u - v$, so to get

$$\begin{aligned}\int_\Omega |\nabla u - \nabla v|^2 \, dx &= \int_\Omega (|u|^{q-2}\, u - |v|^{q-2}\, v)\,(u - v)\, dx \\ &\le \int_\Omega (|u|^{q-1} + |v|^{q-1})\, |u - v| \, dx \\ &\le \left(\|u\|^{q-1}_{L^\infty(\Omega)} + \|v\|^{q-1}_{L^\infty(\Omega)}\right) \|u - v\|_{L^1(\Omega)}.\end{aligned}$$

If we now use the uniform L^∞ estimate of Proposition B.1, we get the desired conclusion. □

Appendix C. Defect of compactness in convex cones

In this section, we show that the embedding

$$\mathcal{D}^{1,2}_0(\Omega) \hookrightarrow L^2(\Omega; w^{q-2}_{\Omega,q}),$$

fails to be compact in a narrow convex cone. For completeness, we will make a more refined analysis, aiming at identifying the energy levels at which the loss of compactness occurs. We will see that this is linked to the exact determination of a Hardy-type sharp constant.

Throughout this section, we still use the notation of Definition 2.3 and set

$$\Phi(t) = t\,(N - 2 + t), \qquad \text{for } t \ge 0.$$

Recall that this defines a monotone increasing function.

In order to state the main outcome of this appendix, we introduce a definition: given $1 < q < 2$, $0 \le \beta < 1$, and $R > 0$, we call *"concentration energy at the tip" of the convex cone* $\Gamma(\beta, R)$ the quantity

$$\inf_{\varphi\in C_0^\infty(\Gamma(\beta,R))}\left[\lim_{n\to\infty}\frac{\displaystyle\int_{\Gamma(\beta,R)}|\nabla\varphi_n|^2\,dx}{\displaystyle\int_{\Gamma(\beta,R)}|\varphi_n|^2\,w_{\Gamma(\beta,R),q}^{q-2}\,dx}\right], \tag{C.1}$$

where for all $\varphi\in C_0^\infty(\Gamma(\beta,R))$ and for all $n\in\mathbb{N}$, we set $\varphi_n(x)=n^{\frac{N-2}{2}}\,\varphi(n\,x)$. Then we have the following.

Proposition C.1. *Let $1<q<2$ and let $0\le\beta<1$ be such that*

$$\Phi\left(\frac{2}{2-q}\right)<\lambda(\mathcal{S}(\beta)).$$

Then for every $R>0$ we have:

1) *the value (C.1) does not depend on R, we indicate it by $\mathcal{C}_q(\beta)$. Moreover, $\mathcal{C}_q(\beta)>1$ and*

$$\lim_{\beta\to1^-}\mathcal{C}_q(\beta)=1;$$

2) *for every $t>\mathcal{C}_q(\beta)$, the set*

$$\mathcal{E}_{\Gamma(\beta,R),q}(t)=\left\{\varphi\in\mathcal{D}_0^{1,2}(\Gamma(\beta,R))\,:\,\int_{\Gamma(\beta,R)}w_{\Gamma(\beta,R),q}^{q-2}\,|\varphi|^2\,dx=1,\ \int_{\Gamma(\beta,R)}|\nabla\varphi|^2\,dx\le t\right\},$$

is not precompact in $L^2(\Gamma(\beta,R);w_{\Gamma(\beta,R),q}^{q-2})$.

Before giving the proof of Proposition C.1, we need some intermediate expedient results.

Lemma C.2 (A special solution)**.** *Let $1<q<2$ and let β be such that*

$$\Phi\left(\frac{2}{2-q}\right)<\lambda(\mathcal{S}(\beta)). \tag{C.2}$$

Then there exists a positive function $\psi\in\mathcal{D}_0^{1,2}(\mathcal{S}(\beta))\cap L^\infty(\mathcal{S}(\beta))$ such that

$$V(x)=|x|^{\frac{2}{2-q}}\,\psi\left(\frac{x}{|x|}\right),$$

is a positive solution of

$$-\Delta V=V^{q-1},\quad \text{in }\Gamma(\beta,+\infty),\qquad V=0,\quad\text{on }\partial\Gamma(\beta,+\infty).$$

Moreover, for every $R>0$ we have

$$w_{\Gamma(\beta,R),q}(x)\le V(x),\qquad \text{for } x\in\Gamma(\beta,R). \tag{C.3}$$

Finally, we have

$$\lim_{R\to+\infty}w_{\Gamma(\beta,R),q}=V,\qquad\text{uniformly on every }\Gamma(\beta,r). \tag{C.4}$$

Proof. We start by considering the variational problem

$$\mu_q(\beta) = \min_{\varphi\in\mathcal{D}^{1,2}_0(\mathcal{S}(\beta))\setminus\{0\}} \frac{\displaystyle\int_{\mathcal{S}(\beta)} |\nabla_\tau\varphi|^2\, d\mathcal{H}^{N-1} - \Phi\left(\frac{2}{2-q}\right)\int_{\mathcal{S}(\beta)} |\varphi|^2\, d\mathcal{H}^{N-1}}{\left(\displaystyle\int_{\mathcal{S}(\beta)} |\varphi|^q\, d\mathcal{H}^{N-1}\right)^{\frac{2}{q}}},$$

where ∇_τ denotes the tangential gradient. By definition of $\lambda(\mathcal{S}(\beta))$, we have

$$\int_{\mathcal{S}(\beta)} |\nabla_\tau\varphi|^2\, d\mathcal{H}^{N-1} \geq \lambda(\mathcal{S}(\beta))\int_{\mathcal{S}(\beta)} |\varphi|^2\, d\mathcal{H}^{N-1}, \qquad \text{for } \varphi\in\mathcal{D}^{1,2}_0(\mathcal{S}(\beta)).$$

Then, keeping in mind the choice (C.2) of β, by applying the Direct Methods in the Calculus of Variations we easily get that the value $\mu_q(\beta)$ is attained by a function $\widetilde{\psi}$, which can be taken to be positive and normalized by the condition

$$\int_{\mathcal{S}(\beta)} |\widetilde{\psi}|^q\, d\mathcal{H}^{N-1} = 1. \tag{C.5}$$

Moreover, still thanks to (C.2), we can assure that $\mu_q(\beta) > 0$. We now observe that $\widetilde{\psi}$ weakly solves

$$-\Delta_g\widetilde{\psi} - \Phi\left(\frac{2}{2-q}\right)\widetilde{\psi} = \mu_q(\beta)\,\widetilde{\psi}^{q-1}, \qquad \text{in } \mathcal{S}(\beta),$$

where Δ_g is the Laplace-Beltrami operator on $\mathbb{S}^{N-1}$. If we now set

$$\psi = \mu_q(\beta)^{-\frac{1}{2-q}}\,\widetilde{\psi}, \tag{C.6}$$

this function solves

$$-\Delta_g\psi - \Phi\left(\frac{2}{2-q}\right)\psi = \psi^{q-1}. \tag{C.7}$$

By writing the Laplacian in spherical coordinates, it is easily seen that the function

$$V(x) = |x|^{\frac{2}{2-q}}\,\psi\left(\frac{x}{|x|}\right),$$

has the claimed properties.

We now prove the property (C.3). For this, we use a comparison principle similar to that of [7, Lemma 2.7]. We observe that the restriction of V to $\Gamma(\beta,R)$ is the unique solution of the variational problem

$$\min_{\varphi\in W^{1,2}(\Gamma(\beta,R))}\left\{\frac{1}{2}\int_{\Gamma(\beta,R)} |\nabla\varphi|^2\, dx - \frac{1}{q}\int_{\Gamma(\beta,R)} \varphi^q\, dx \,:\, \varphi\geq 0,\ \varphi = V \text{ on } \partial\Gamma(\beta,R)\right\}. \tag{C.8}$$

We test the minimality of V in (C.8) comparing with the value corresponding to the function $\varphi = \max\{w_{\Gamma(\beta,R),q},\, V\}$. In this way, we get

$$\begin{aligned}\frac{1}{2}\int_{\{w_{\Gamma(\beta,R),q}>V\}} |\nabla w_{\Gamma(\beta,R),q}|^2\, dx &- \frac{1}{q}\int_{\{w_{\Gamma(\beta,R),q}>V\}} w^q_{\Gamma(\beta,R),q}\, dx\\ &\geq \frac{1}{2}\int_{\{w_{\Gamma(\beta,R),q}>V\}} |\nabla V|^2\, dx - \frac{1}{q}\int_{\{w_{\Gamma(\beta,R),q}>V\}} V^q\, dx.\end{aligned} \tag{C.9}$$

If we now introduce

$$\widetilde{\varphi}=\min\big\{w_{\Gamma(\beta,R),q},V\big\},$$

and add on both sides of (C.9) the term

$$\frac{1}{2}\int_{\{w_{\Gamma(\beta,R),q}\le V\}}|\nabla w_{\Gamma(\beta,R),q}|^2\,dx-\frac{1}{q}\int_{\{w_{\Gamma(\beta,R),q}\le V\}}w_{\Gamma(\beta,R),q}^q\,dx,$$

we get

$$\frac{1}{2}\int_{\Gamma(\beta,R)}|\nabla w_{\Gamma(\beta,R),q}|^2\,dx-\frac{1}{q}\int_{\Gamma(\beta,R)}w_{\Gamma(\beta,R),q}^q\,dx\ge\frac{1}{2}\int_{\Gamma(\beta,R)}|\nabla\widetilde{\varphi}|^2\,dx-\frac{1}{q}\int_{\Gamma(\beta,R)}\widetilde{\varphi}^q\,dx. \tag{C.10}$$

On the other hand, by Definition 3.1 the function $w_{\Gamma(\beta,R),q}$ is the unique solution of

$$\min_{\varphi\in\mathcal{D}_0^{1,2}(\Gamma(\beta,R))}\left\{\frac{1}{2}\int_{\Gamma(\beta,R)}|\nabla\varphi|^2\,dx-\frac{1}{q}\int_{\Gamma(\beta,R)}\varphi^q\,dx\,:\,\varphi\ge 0\right\}.$$

Hence, since $\widetilde{\varphi}$ is admissible for this problem, equation (C.10) shows that

$$\widetilde{\varphi}=\min\big\{w_{\Gamma(\beta,R),q},V\big\}=w_{\Gamma(\beta,R),q},$$

which is the desired estimate (C.3).

Finally, we prove (C.4). We first observe that by [7, Lemma 2.7], we get that

$$w_{\Gamma(\beta,R_1),q}\le w_{\Gamma(\beta,R_2),q},\qquad\text{for every } R_1\le R_2.$$

Thus the pointwise limit

$$W(x):=\lim_{R\to+\infty}w_{\Gamma(\beta,R),q}(x),$$

exists by monotonicity and it is finite, thanks to (C.3). Moreover, by proceeding as in the proof of [7, Proposition 5.1], it is not difficult to see that W solves the Lane-Emden equation in the infinite cone $\Gamma(\beta,+\infty)$. By using the scaling properties of the Lane-Emden equation and the uniqueness of the positive least energy solution in $\Gamma(\beta,R)$, we observe that

$$w_{\Gamma(\beta,R),q}(x)=R^{\frac{2}{2-q}}\,w_{\Gamma(\beta,1),q}\left(\frac{x}{R}\right),\qquad\text{for } x\in\Gamma(\beta,R). \tag{C.11}$$

Thus, for every $\lambda>0$, we have

$$\begin{aligned}W(\lambda\,x)=\lim_{R\to+\infty}w_{\Gamma(\beta,R),q}(\lambda\,x)&=\lim_{R\to+\infty}R^{\frac{2}{2-q}}\,w_{\Gamma(\beta,1),q}\left(\frac{\lambda\,x}{R}\right)\\&=\lambda^{\frac{2}{2-q}}\lim_{R\to+\infty}\left(\frac{R}{\lambda}\right)^{\frac{2}{2-q}}w_{\Gamma(\beta,1),q}\left(\frac{\lambda\,x}{R}\right)\\&=\lambda^{\frac{2}{2-q}}\lim_{R\to+\infty}w_{\Gamma(\beta,R/\lambda)}(x)\\&=\lambda^{\frac{2}{2-q}}\,W(x).\end{aligned}$$

This shows that W is $2/(2-q)-$homogeneous, so that it can be written as

$$W(x)=|x|^{\frac{2}{2-q}}\,W\left(\frac{x}{|x|}\right),\qquad\text{for every } x\in\Gamma(\beta,+\infty).$$

In order to conclude, we just need to show that

$$\psi(\omega)=W(\omega),\qquad\text{for every }\omega\in\mathcal{S}(\beta),$$

where ψ is still the function defined in (C.6).

Since W solves the Lane-Emden equation, by writing the Laplacian in spherical coordinates we get that $\omega \to W(\omega)$ must be a positive solution of equation (C.7), as well. We now adapt the trick by Brezis and Oswald (see [4] and also [7, Lemma 2.2]), based on Picone's inequality, in order to show uniqueness for (C.7). We take the weak formulations

$$\int_{\mathcal{S}(\beta)} \langle \nabla_\tau \psi, \nabla_\tau \varphi \rangle \, d\mathcal{H}^{N-1} - \Phi\left(\frac{2}{2-q}\right) \int_{\mathcal{S}(\beta)} \psi \, \varphi \, d\mathcal{H}^{N-1} = \int_{\mathcal{S}(\beta)} \psi^{q-1} \, \varphi \, d\mathcal{H}^{N-1},$$

and

$$\int_{\mathcal{S}(\beta)} \langle \nabla_\tau W, \nabla_\tau \varphi \rangle \, d\mathcal{H}^{N-1} - \Phi\left(\frac{2}{2-q}\right) \int_{\mathcal{S}(\beta)} W \, \varphi \, d\mathcal{H}^{N-1} = \int_{\mathcal{S}(\beta)} W^{q-1} \, \varphi \, d\mathcal{H}^{N-1}.$$

Then, for every $\varepsilon > 0$, we insert the test function $\varphi = (W^2/(\varepsilon + \psi) - \psi)$ in the first equation and the test function $\varphi = (\psi^2/(\varepsilon + W) - W)$ in the second one. By summing up the resulting identities, we get

$$\begin{aligned}
&\int_{\mathcal{S}(\beta)} \left\langle \nabla_\tau \psi, \nabla_\tau \left(\frac{W^2}{\varepsilon + \psi} - \psi \right) \right\rangle d\mathcal{H}^{N-1} + \int_{\mathcal{S}(\beta)} \left\langle \nabla_\tau W, \nabla_\tau \left(\frac{\psi^2}{\varepsilon + W} - W \right) \right\rangle d\mathcal{H}^{N-1} \\
&\quad - \Phi\left(\frac{2}{2-q}\right) \int_{\mathcal{S}(\beta)} \psi \left(\frac{W^2}{\varepsilon + \psi} - \psi \right) d\mathcal{H}^{N-1} - \Phi\left(\frac{2}{2-q}\right) \int_{\mathcal{S}(\beta)} W \left(\frac{\psi^2}{\varepsilon + W} - W \right) d\mathcal{H}^{N-1} \\
&\quad = \int_{\mathcal{S}(\beta)} \psi^{q-1} \left(\frac{W^2}{\varepsilon + \psi} - \psi \right) d\mathcal{H}^{N-1} + \int_{\mathcal{S}(\beta)} W^{q-1} \left(\frac{\psi^2}{\varepsilon + W} - W \right) d\mathcal{H}^{N-1}.
\end{aligned}$$

We now use *Picone's inequality*, so that

$$\int_{\mathcal{S}(\beta)} \left\langle \nabla_\tau \psi, \nabla_\tau \frac{W^2}{\varepsilon + \psi} \right\rangle d\mathcal{H}^{N-1} = \int_{\mathcal{S}(\beta)} \left\langle \nabla_\tau (\psi + \varepsilon), \nabla_\tau \frac{W^2}{\varepsilon + \psi} \right\rangle d\mathcal{H}^{N-1} \le \int_{\mathcal{S}(\beta)} |\nabla_\tau W|^2 \, d\mathcal{H}^{N-1},$$

and

$$\int_{\mathcal{S}(\beta)} \left\langle \nabla_\tau W, \nabla_\tau \frac{\psi^2}{\varepsilon + W} \right\rangle d\mathcal{H}^{N-1} = \int_{\mathcal{S}(\beta)} \left\langle \nabla_\tau (W + \varepsilon), \nabla_\tau \frac{\psi^2}{\varepsilon + W} \right\rangle d\mathcal{H}^{N-1} \le \int_{\mathcal{S}(\beta)} |\nabla_\tau \psi|^2 \, d\mathcal{H}^{N-1}.$$

A further passage to the limit as ε goes to 0 leads to

$$\int_{\mathcal{S}(\beta)} \left(\psi^{q-2} \, W^2 - \psi^q \right) d\mathcal{H}^{N-1} + \int_{\mathcal{S}(\beta)} \left(W^{q-2} \, \psi^2 - W^q \right) d\mathcal{H}^{N-1} \le 0.$$

The last two integrals can be rearranged as follows

$$\int_{\mathcal{S}(\beta)} (\psi^{q-2} - W^{q-2}) \, (W^2 - \psi^2) \, d\mathcal{H}^{N-1} \le 0.$$

On the other hand, by virtue of the fact that $q < 2$, we have

$$(a^{q-2} - b^{q-2}) \, (b^2 - a^2) > 0, \qquad \text{for every } a, b > 0 \text{ such that } a \neq b.$$

The last two displays shows that we must have $\psi = W$ on $\mathcal{S}(\beta)$. The proof is now complete. □

Remark C.3. We observe that from the equation (C.7), we have

$$\begin{aligned}
\int_{\mathcal{S}(\beta)} \psi^q \, d\mathcal{H}^{N-1} &= \int_{\mathcal{S}(\beta)} |\nabla_\tau \psi|^2 \, d\mathcal{H}^{N-1} - \Phi\left(\frac{2}{2-q}\right) \int_{\mathcal{S}(\beta)} \psi^2 \, d\mathcal{H}^{N-1} \\
&\ge \left(\lambda(\mathcal{S}(\beta)) - \Phi\left(\frac{2}{2-q}\right) \right) \int_{\mathcal{S}(\beta)} \psi^2 \, d\mathcal{H}^{N-1},
\end{aligned}$$

where we also used Poincaré's inequality on $\mathcal{S}(\beta)$. By recalling (C.5) and (C.6), we also have

$$\int_{\mathcal{S}(\beta)} \psi^q \, d\mathcal{H}^{N-1} = \mu_q(\beta)^{-\frac{1}{2-q}}.$$

Lemma C.4. *Let $1 < q < 2$ and let $0 \le \beta < 1$ be such that*

$$\Phi\left(\frac{2}{2-q}\right) < \lambda(\mathcal{S}(\beta)).$$

For every $0 < R \le +\infty$, we define the Hardy-type constant

$$\sigma(\beta, R) = \inf_{\varphi \in C_0^\infty(\Gamma(\beta,R))\setminus\{0\}} \frac{\displaystyle\int_{\Gamma(\beta,R)} |\nabla\varphi|^2 \, dx}{\displaystyle\int_{\Gamma(\beta,R)} |\varphi|^2 \, V^{q-2} \, dx},$$

where V is the function of Lemma C.2. Then we have

$$\sigma(\beta, R) = \sigma(\beta, +\infty) =: \sigma(\beta) > 1,$$

and

$$\lim_{\beta\to 1^-} \sigma(\beta) = 1.$$

Proof. We first prove that $\sigma(\beta, R) = \sigma(\beta, +\infty)$, then we show that $\sigma(\beta, +\infty) > 1$.

Since $C_0^\infty(\Gamma(\beta, R)) \subset C_0^\infty(\Gamma(\beta, +\infty))$, we immediately have that

$$\sigma(\beta, R) \ge \sigma(\beta, +\infty).$$

In order to prove the reverse inequality, for every $\varepsilon > 0$ we take $\varphi_\varepsilon \in C_0^\infty(\Gamma(\beta, +\infty))$ such that

$$\frac{\displaystyle\int_{\Gamma(\beta,+\infty)} |\nabla\varphi_\varepsilon|^2 \, dx}{\displaystyle\int_{\Gamma(\beta,+\infty)} |\varphi_\varepsilon|^2 \, V^{q-2} \, dx} < \sigma(\beta, +\infty) + \varepsilon.$$

We now take the rescaled function

$$\varphi_{\varepsilon,n}(x) = n^{\frac{N-2}{2}} \, \varphi_\varepsilon(n\, x), \qquad \text{for } n \in \mathbb{N},$$

and observe that

$$\frac{\displaystyle\int_{\Gamma(\beta,+\infty)} |\nabla\varphi_{\varepsilon,n}|^2 \, dx}{\displaystyle\int_{\Gamma(\beta,+\infty)} |\varphi_{\varepsilon,n}|^2 \, V^{q-2} \, dx} = \frac{\displaystyle\int_{\Gamma(\beta,+\infty)} |\nabla\varphi_\varepsilon|^2 \, dx}{\displaystyle\int_{\Gamma(\beta,+\infty)} |\varphi_\varepsilon|^2 \, V^{q-2} \, dx},$$

thanks to the $2-$homogeneity of V^{2-q}. Moreover, for n large enough we also have $\varphi_{\varepsilon,n} \in C_0^\infty(\Gamma(\beta, R))$. This in turn permits us to infer that

$$\sigma(\beta, R) < \sigma(\beta, +\infty) + \varepsilon.$$

By arbitrariness of $\varepsilon > 0$, we obtain that $\sigma(\beta, R) = \sigma(\beta, +\infty)$.

We are now left with estimating the Hardy-type constant $\sigma(\beta) := \sigma(\beta, +\infty)$. We first prove that $\sigma(\beta) > 1$. For this, we recall that the function V satisfies

$$\int_{\Gamma(\beta,+\infty)} V^{q-1} \, \varphi \, dx = \int_{\Gamma(\beta,+\infty)} \langle \nabla V, \nabla\varphi \rangle \, dx, \qquad \text{for every } \varphi \in C_0^\infty(\Gamma(\beta, +\infty)).$$

Given $\eta \in C_0^\infty(\Gamma(\beta,+\infty))$, we use the previous identity with the choice $\varphi = \eta^2/V$. An application of *Picone's identity* leads to

$$\int_{\Gamma(\beta,+\infty)} \eta^2\, V^{q-2}\, dx = \int_{\Gamma(\beta,+\infty)} |\nabla \eta|^2\, dx - \int_{\Gamma(\beta,+\infty)} \left|\eta\, \frac{\nabla V}{V} - \nabla \eta\right|^2\, dx.$$

By dividing both sides by the weighted L^2 norm of η, we get

$$\sigma(\beta) = \inf_{\eta\in C_0^\infty(\Gamma(\beta,R))\setminus\{0\}} \frac{\displaystyle\int_{\Gamma(\beta,R)} |\nabla \eta|^2\, dx}{\displaystyle\int_{\Gamma(\beta,R)} |\eta|^2\, V^{q-2}\, dx} = 1 + \inf_{\eta\in C_0^\infty(\Gamma(\beta,R))\setminus\{0\}} \frac{\displaystyle\int_{\Gamma(\beta,R)} \left|\eta\, \frac{\nabla V}{V} - \nabla \eta\right|^2\, dx}{\displaystyle\int_{\Gamma(\beta,R)} |\eta|^2\, V^{q-2}\, dx}.$$

We perform the change of variable $\eta = \varphi\, V$, so that the last minimization problem can be transformed into

$$\Theta(\beta) := \inf_{\varphi\in C_0^\infty(\Gamma(\beta,R))\setminus\{0\}} \frac{\displaystyle\int_{\Gamma(\beta,R)} |\nabla \varphi|^2\, V^2\, dx}{\displaystyle\int_{\Gamma(\beta,R)} |\varphi|^2\, V^q\, dx}.$$

In order to show that $\sigma(\beta) > 1$, it is sufficient to prove that $\Theta(\beta) > 0$. For this, we use spherical coordinates, the specific form of V and the one-dimensional Hardy's inequality (see [20, equation (1.3.1)]), i.e.

$$\int_0^R |f'(t)|^2\, t^{N-1+\frac{4}{2-q}}\, dt \geq C_{N,q} \int_0^R |f(t)|^2\, t^{N-1+\frac{2\,q}{2-q}}\, dt.$$

This is valid for every smooth function f, such that $f(R) = 0$. We denote by $C_{N,q} > 0$ the sharp constant, whose precise value has no bearing in what follows. By proceeding as explained above, we get

$$\begin{aligned}
\Theta(\beta) &= \inf_{\varphi\in C_0^\infty(\Gamma(\beta,R))\setminus\{0\}} \frac{\displaystyle\int_{\Gamma(\beta,R)} |\nabla \varphi|^2\, |x|^{\frac{4}{2-q}}\, \psi^2\, dx}{\displaystyle\int_{\Gamma(\beta,R)} |\varphi|^2\, |x|^{\frac{2\,q}{2-q}}\, \psi^q\, dx} \\
&= \inf_{\varphi\in C_0^\infty(\Gamma(\beta,R))\setminus\{0\}} \frac{\displaystyle\int_{\mathcal{S}(\beta)} \left(\int_0^R \left[|\partial_\varrho \varphi|^2 + \varrho^{-2}\, |\nabla_\tau \varphi|^2\right] \varrho^{N-1+\frac{4}{2-q}}\, d\varrho\right) \psi^2\, d\mathcal{H}^{N-1}}{\displaystyle\int_0^R \left(\int_{\mathcal{S}(\beta)} |\varphi|^2\, \psi^q\, d\mathcal{H}^{N-1}\right) \varrho^{N-1+\frac{2\,q}{2-q}}\, d\varrho} \qquad \text{(C.12)} \\
&\geq \inf_{\varphi\in C_0^\infty(\Gamma(\beta,R))\setminus\{0\}} \frac{\displaystyle\int_0^R \left(\int_{\mathcal{S}(\beta)} \left[C_{N,q}\, |\varphi|^2 + |\nabla_\tau \varphi|^2\right] \psi^2\, d\mathcal{H}^{N-1}\right) \varrho^{N-1+\frac{2\,q}{2-q}}\, d\varrho}{\displaystyle\int_0^R \left(\int_{\mathcal{S}(\beta)} |\varphi|^2\, \psi^q\, d\mathcal{H}^{N-1}\right) \varrho^{N-1+\frac{2\,q}{2-q}}\, d\varrho}.
\end{aligned}$$

We now observe that there exists a constant $C > 0$ such that

$$\frac{1}{C}\, \mathrm{dist}_g(\omega, \partial\mathcal{S}(\beta)) \leq \psi(\omega) \leq C\, \mathrm{dist}_g(\omega, \partial\mathcal{S}(\beta)), \qquad \text{for } \omega \in \mathcal{S}(\beta). \tag{C.13}$$

Here we denote by $\mathrm{dist}_g(\cdot, \partial\mathcal{S}(\beta))$ the geodesic distance on $\mathcal{S}(\beta)$ from the boundary. Thus, for every $\varrho \in [0, R]$, we can apply the weighted Poincaré inequality of [17, Theorem 8.2] to the compactly supported function $\omega \mapsto \varphi(\varrho\,\omega)$. This gives

$$\begin{aligned}\int_{\mathcal{S}(\beta)} \Big[C_{N,q}\,|\varphi|^2 + |\nabla_\tau \varphi|^2\Big]\,\psi^2\,d\mathcal{H}^{N-1} &\geq \gamma \int_{\mathcal{S}(\beta)} |\varphi|^2\,d\mathcal{H}^{N-1} \\ &\geq \frac{\gamma}{\left(\displaystyle\max_{\omega\in\mathcal{S}(\beta)} \mathrm{dist}_g(\omega, \partial\mathcal{S}(\beta))\right)^q} \int_{\mathcal{S}(\beta)} |\varphi|^2\,\psi^q\,d\mathcal{H}^{N-1},\end{aligned}$$

for a suitable constant $\gamma > 0$. By inserting this estimate in (C.12), we get $\Theta(\beta) > 0$ as desired.

Finally, we show that $\sigma(\beta) \to 1$ as β goes to 1, i.e.

$$\lim_{\beta\to 1^-} \Theta(\beta) = 0. \tag{C.14}$$

For every $\varepsilon > 0$, by definition of sharp constant we know that there exists f_ε such that

$$\int_0^R |f_\varepsilon'(t)|^2\,t^{N-1+\frac{4}{2-q}}\,dt < C_{N,q}\,(1+\varepsilon) \int_0^R |f_\varepsilon(t)|^2\,t^{N-1+\frac{2\,q}{2-q}}\,dt. \tag{C.15}$$

We then take $g \in C_0^\infty(\mathcal{S}(\beta))$ and insert the test function $\varphi(x) = f_\varepsilon(|x|)\,g(x/|x|)$ in the minimization problem which defines $\Theta(\beta)$. By using spherical coordinates, recalling the definition of V and using (C.15), we get

$$\Theta(\beta) \leq C_{N,q}\,(1+\varepsilon)\,\frac{\displaystyle\int_{\mathcal{S}(\beta)} |g|^2\,\psi^2\,d\mathcal{H}^{N-1}}{\displaystyle\int_{\mathcal{S}(\beta)} |g|^2\,\psi^q\,d\mathcal{H}^{N-1}} + \frac{\displaystyle\int_{\mathcal{S}(\beta)} |\nabla_\tau g|^2\,\psi^2\,d\mathcal{H}^{N-1}}{\displaystyle\int_{\mathcal{S}(\beta)} |g|^2\,\psi^q\,d\mathcal{H}^{N-1}}.$$

By taking the limit as ε goes to 0, this gives

$$\Theta(\beta) \leq \frac{\displaystyle\int_{\mathcal{S}(\beta)} \left[C_{N,q}\,|g|^2 + |\nabla_\tau g|^2\right]\,\psi^2\,d\mathcal{H}^{N-1}}{\displaystyle\int_{\mathcal{S}(\beta)} |g|^2\,\psi^q\,d\mathcal{H}^{N-1}},$$

for every $g \in C_0^\infty(\mathcal{S}(\beta))$. In particular, for every $\varepsilon > 0$ we take a compactly supported approximation of the unit g_ε, i.e. $g_\varepsilon \in C_0^\infty(\mathcal{S}(\beta))$ with

$$0 \leq g_\varepsilon \leq 1, \qquad g_\varepsilon \equiv 1 \text{ on } \mathcal{S}(\beta+\varepsilon), \qquad |\nabla_\tau g_\varepsilon| \leq \frac{C}{\varepsilon}.$$

We also use that

$$|\psi| \leq C'\,\varepsilon, \qquad \text{on } \mathcal{S}(\beta)\setminus\mathcal{S}(\beta+\varepsilon),$$

which follows from (C.13). Then we obtain

$$\Theta(\beta) \leq \frac{C_{N,q}\displaystyle\int_{\mathcal{S}(\beta)} |g_\varepsilon|^2\,\psi^2\,d\mathcal{H}^{N-1} + (C')^2\,C\,\mathcal{H}^{N-1}(\mathcal{S}(\beta)\setminus\mathcal{S}(\beta+\varepsilon))}{\displaystyle\int_{\mathcal{S}(\beta)} |g_\varepsilon|^2\,\psi^q\,d\mathcal{H}^{N-1}}.$$

By taking the limit as ε goes to 0, we obtain the estimate

$$\Theta(\beta) \le C_{N,q} \, \frac{\displaystyle\int_{\mathcal{S}(\beta)} \psi^2 \, d\mathcal{H}^{N-1}}{\displaystyle\int_{\mathcal{S}(\beta)} \psi^q \, d\mathcal{H}^{N-1}}.$$

In particular, by recalling Remark C.3, we end up with the upper bound

$$\Theta(\beta) \le C_{N,q} \left(\lambda(\mathcal{S}(\beta)) - \Phi\left(\frac{2}{2-q}\right)\right)^{-1}.$$

Thus the claimed asymptotic behavior (C.14) of $\Theta(\beta)$ is proved, by recalling that $\lambda(\mathcal{S}(\beta)) \to +\infty$, as $\beta \to 1$. $\square$

We can now prove the main result of this section.

Proof of Proposition C.1. Let us fix a nontrivial function $\varphi \in C^\infty_0(\Gamma(\beta, R))$ and take the sequence

$$\varphi_n(x) = n^{\frac{N-2}{2}} \, \varphi(n\,x), \qquad n \in \mathbb{N}.$$

It is not difficult to see that $\{\varphi_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{D}^{1,2}_0(\Gamma(\beta, R))$ and it converges to 0, strongly in $L^2(\Gamma(\beta, R))$. Indeed, we have

$$\int_{\Gamma(\beta,R)} |\nabla \varphi_n|^2 \, dx = \int_{\Gamma(\beta,R/n)} |\nabla \varphi_n|^2 \, dx = \int_{\Gamma(\beta,R)} |\nabla \varphi|^2 \, dx, \tag{C.16}$$

and

$$\int_{\Gamma(\beta,R)} |\varphi_n|^2 \, dx = \int_{\Gamma(\beta,R/n)} |\varphi_n|^2 \, dx = n^{-2} \int_{\Gamma(\beta,R)} |\varphi|^2 \, dx. \tag{C.17}$$

Let us compute the weighted L^2 norm of φ_n. At this aim, we observe that (C.11) yields

$$n^{\frac{2}{2-q}} \, w_{\Gamma(\beta,R),q}\left(\frac{x}{n}\right) = w_{\Gamma(\beta,n\,R),q}(x),$$

thus by using a change of variables we get

$$\begin{aligned} \int_{\Gamma(\beta,R)} |\varphi_n|^2 \, w^{q-2}_{\Gamma(\beta,R),q} \, dx &= \int_{\Gamma(\beta,R/n)} n^{N-2} \, |\varphi(n\,x)|^2 \, w_{\Gamma(\beta,R),q}(x)^{q-2} \, dx \\ &= \int_{\Gamma(\beta,R)} |\varphi|^2 \, w^{q-2}_{\Gamma(\beta,n\,R),q} \, dx. \end{aligned} \tag{C.18}$$

This gives

$$\lim_{n\to\infty} \frac{\displaystyle\int_{\Gamma(\beta,R)} |\nabla \varphi_n|^2 \, dx}{\displaystyle\int_{\Gamma(\beta,R)} |\varphi_n|^2 \, w^{q-2}_{\Gamma(\beta,R),q} \, dx} = \lim_{n\to\infty} \frac{\displaystyle\int_{\Gamma(\beta,R)} |\nabla \varphi|^2 \, dx}{\displaystyle\int_{\Gamma(\beta,R)} |\varphi|^2 \, w^{q-2}_{\Gamma(\beta,n\,R),q} \, dx}.$$

We now use that

$$\lim_{n\to\infty} w_{\Gamma(\beta,n\,R),q} = V, \qquad \text{uniformly on } \Gamma(\beta, R),$$

thanks to Lemma C.2. This yields

$$\lim_{n\to\infty} \frac{\displaystyle\int_{\Gamma(\beta,R)} |\nabla \varphi_n|^2\,dx}{\displaystyle\int_{\Gamma(\beta,R)} |\varphi_n|^2\,w_{\Gamma(\beta,R),q}^{q-2}\,dx} = \frac{\displaystyle\int_{\Gamma(\beta,R)} |\nabla \varphi|^2\,dx}{\displaystyle\int_{\Gamma(\beta,R)} |\varphi|^2\,V^{q-2}\,dx}.$$

By taking the infimum over $\varphi\in C^\infty_0(\Gamma(\beta,R))$ and using Lemma C.4, we get

$$\inf_{\varphi\in C^\infty_0(\Gamma(\beta,R))}\left[\lim_{n\to\infty} \frac{\displaystyle\int_{\Gamma(\beta,R)} |\nabla \varphi_n|^2\,dx}{\displaystyle\int_{\Gamma(\beta,R)} |\varphi_n|^2\,w_{\Gamma(\beta,R),q}^{q-2}\,dx}\right] = \sigma(\beta),$$

and thus the conclusion of point 1).

We now show the loss of compactness. Given $t>\mathcal{C}_q(\beta)=\sigma(\beta)$, by point 1) we know that there exists $\varphi\in C^\infty_0(\Gamma(\beta,R))$ such that

$$\lim_{n\to\infty} \frac{\displaystyle\int_{\Gamma(\beta,R)} |\nabla \varphi_n|^2\,dx}{\displaystyle\int_{\Gamma(\beta,R)} |\varphi_n|^2\,w_{\Gamma(\beta,R),q}^{q-2}\,dx} < t,$$

where as before we set $\varphi_n(x)=n^{(N-2)/2}\,\varphi(n\,x)$. This shows that the rescaled sequence

$$\psi_n=\frac{\varphi_n}{\left(\displaystyle\int_{\Gamma(\beta,R)} |\varphi_n|^2\,w_{\Gamma(\beta,R),q}^{q-2}\,dx\right)^{\frac{1}{2}}},\qquad n\in\mathbb{N},$$

belongs to $\mathcal{E}_{\Gamma(\beta,R),q}(t)$ for n large enough. However, this sequence can not converge strongly in the weighted L^2 space, since by construction we have (recall (C.17) and (C.18))

$$\int_{\Gamma(\beta,R)} w_{\Gamma(\beta,R),q}^{q-2}\,|\psi_n|^2\,dx=1 \quad\text{and}\quad \int_{\Gamma(\beta,R)} |\psi_n|^2\,dx=\frac{1}{n^2}\,\frac{\displaystyle\int_{\Gamma(\beta,R)}|\varphi|^2\,dx}{\displaystyle\int_{\Gamma(\beta,R)}|\varphi|^2\,w_{\Gamma(\beta,n\,R),q}^{q-2}\,dx}\to 0.$$

This concludes the proof. □

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