STRICHARTZ ESTIMATES AND FOURIER RESTRICTION THEOREMS ON THE HEISENBERG GROUP

HAJER BAHOURI, DAVIDE BARILARI, AND ISABELLE GALLAGHER

Abstract. This paper is dedicated to the proof of Strichartz estimates on the Heisenberg group $\mathbb{H}^d$ for the linear Schrödinger and wave equations involving the sublaplacian. The Schrödinger equation on $\mathbb{H}^d$ is an example of a totally non-dispersive evolution equation: for this reason the classical approach that permits to obtain Strichartz estimates from dispersive estimates is not available. Our approach, inspired by the Fourier transform restriction method initiated in [41], is based on Fourier restriction theorems on $\mathbb{H}^d$, using the non-commutative Fourier transform on the Heisenberg group. It enables us to obtain also an anisotropic Strichartz estimate for the wave equation, for a larger range of indices than was previously known.

1. Introduction

1.1. **Strichartz estimates.** In the past decades, Strichartz estimates for linear evolution equations such as the Schrödinger and wave equations, have been a central tool in the study of semilinear and quasilinear equations, which appear in numerous physical applications. In many cases and particularly in $\mathbb{R}^n$, the proof of those inequalities, which involve space-time Lebesgue norms, is a combination of an abstract functional analysis argument known as the TT^*-argument (see [28]) and of a dispersive estimate. Concerning the Schrödinger equation on $\mathbb{R}^n$

$$(S) \qquad \begin{cases} i\partial_t u - \Delta u = 0 \\ u_{|t=0} = u_0 \,, \end{cases}$$

the dispersive estimate writes (for $t \neq 0$)

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{(4\pi|t|)^{\frac{n}{2}}} \|u_0\|_{L^1(\mathbb{R}^n)} \,, \tag{1.1}$$

and can be easily derived from the explicit expression of the solution, which is based on Fourier analysis:

$$u(t,\cdot) = \frac{\mathrm{e}^{i\frac{|\cdot|^2}{4t}}}{(4\pi i t)^{\frac{n}{2}}} \star u_0 \,.$$

The dispersive inequality (1.1) which expresses that waves with different frequencies move at different velocities, gives rise when u_0 is in $L^2(\mathbb{R}^n)$ to the following Strichartz estimate (see for instance [13, 14, 15, 20, 34]) for the solution to the free Schrödinger equation

$$\|u\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))} \leq C(p,q)\|u_0\|_{L^2(\mathbb{R}^n)} \,, \tag{1.2}$$

where (p,q) satisfies the scaling admissibility condition

$$\frac{2}{q} + \frac{n}{p} = \frac{n}{2} \quad \text{with} \quad q \geq 2 \quad \text{and} \quad (n,q,p) \neq (2,2,\infty)\,. \tag{1.3}$$

It is worth noticing that the dispersive inequality (1.1) also yields the following Strichartz inequalities for the inhomogeneous Schrödinger equation, which have proven to be of paramount importance in the study of semilinear and quasilinear Schrödinger equations

Date: November 8, 2019.

(one can for instance consult the monograph [4] and the references therein): if (p, q) and (p_1, q_1) satisfy the admissibility condition (1.3), then

$$\|u\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))} \leq C(p,q,p_1,q_1)\Big(\|u_0\|_{L^2(\mathbb{R}^n)} + \|i\partial_t u - \Delta u\|_{L^{q_1'}(\mathbb{R},L^{p_1'}(\mathbb{R}^n))}\Big),$$

denoting by a' the dual exponent of any $a \in [1, \infty]$.

In the case of the wave equation on $\mathbb{R}^n$

$$(W) \qquad \begin{cases} \partial_t^2 u - \Delta u = 0 \\ (u, \partial_t u)_{|t=0} = (u_0, u_1), \end{cases}$$

the solution of which may be written by means of the Fourier transform $\mathcal{F}$ (in the case when $\mathcal{F}u_0$ and $\mathcal{F}u_1$ are supported in a ring) as

$$u(t) = \sum_{\pm} \mathcal{F}^{-1}(e^{\pm it|\xi|}\gamma_\pm(\xi)), \quad \gamma_\pm(\xi) \stackrel{\text{def}}{=} \frac{1}{2}\big(\mathcal{F}u_0(\xi) \pm \frac{1}{i|\xi|}\mathcal{F}u_1(\xi)\big), \tag{1.4}$$

the dispersive estimate writes (for $t \neq 0$)

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C}{|t|^{\frac{n-1}{2}}}\big(\|u_0\|_{L^1(\mathbb{R}^n)} + \|u_1\|_{L^1(\mathbb{R}^n)}\big). \tag{1.5}$$

Its proof requires more elaborate techniques involving oscillatory integrals and the application of a stationary phase theorem. This dispersive estimate leads to the following Strichartz estimate (see for instance [4, 28, 34] and the references therein)

$$\|u\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))} \leq C(p,q)\big(\|\nabla u_0\|_{L^2(\mathbb{R}^n)} + \|u_1\|_{L^2(\mathbb{R}^n)}\big), \tag{1.6}$$

where (p, q) satisfies the scaling admissibility condition

$$\frac{1}{q} + \frac{n}{p} = \frac{n}{2} - 1 \quad \text{with} \quad p, q \geq 2 \quad \text{and} \quad q < \infty. \tag{1.7}$$

If (p, q) and (p_1, q_1) satisfy (1.7), one can also infer

$$\|u\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))} \leq C(p,q,p_1,q_1)\Big(\|\nabla u_0\|_{L^2(\mathbb{R}^n)} + \|u_1\|_{L^2(\mathbb{R}^n)} + \|\partial_t^2 u - \Delta u\|_{L^{q_1'}(\mathbb{R},L^{p_1'}(\mathbb{R}^n))}\Big).$$

When some loss of dispersion occurs, as for instance in the case of compact Riemannian manifolds and of some bounded domains, or as was highlighted by Bahouri, Gérard and Xu in [10] in the case of the Schrödinger operator on $\mathbb{H}^d$ (where it is shown that there is no dispersion at all), the Euclidean strategy referred to above fails and the problem of obtaining Strichartz estimates is considered as very difficult. Strichartz estimates in the setting of compact Riemannian manifolds and bounded domains (with a possible loss of derivatives) have been obtained in a number of works (see for instance Bourgain [16], Burq, Gérard and Tzvetkov [17], Ivanovici, Lebeau and Planchon [31] and the references therein). The case of the hyperbolic space (noncompact and negatively curved) is also considered in [2].

Even if the study of PDEs associated with sublaplacians on nilpotent groups is nowadays classical (see for instance the pioneering works [27, 30, 37]), obtaining Strichartz estimates for the Schrödinger operator on the Heisenberg group is still open and has to our knowledge never been tackled. Note that the Heisenberg group is one of the simplest examples of a noncommutative Lie group, whence our interest in proving those estimates in this setting. We are confident that our methods should apply to more general nilpotent Lie groups, provided some harmonic and Fourier analysis tools (that will be introduced in the setting of $\mathbb{H}^d$ in Section 3) are extended from the Heisenberg framework to the context of these groups. This is for instance the case of H-type groups or more generally of step 2 stratified Lie groups: in [22] and [7] the lack of dispersion for the associated Schrödinger operators is indeed proved. We also refer to [11, 12] for a discussion about the link between dispersion, restriction estimates and the heat semigroup and to [26] for a study of the cubic Schrödinger equation on the Heisenberg group.

In this paper our main goal is thus to establish Strichartz estimates for the solutions to the linear Schrödinger equation on the Heisenberg group $\mathbb{H}^d$, involving the sublaplacian, as well as for the wave equation. As already mentioned, in [10] the authors show the absence of dispersion – they actually prove that the Schrödinger equation on $\mathbb{H}^d$ behaves as a transport equation with respect to one direction, known as the vertical direction (i.e., along the orbits of the Reeb vector field). But as will be clear later, a salutary fact is that the Schrödinger operator on $\mathbb{H}^d$ behaves rather well in the complement to that vertical direction. This enables us to derive anisotropic Strichartz estimates for the Schrödinger operator on $\mathbb{H}^d$, by adapting the Fourier transform restriction analysis initiated in [39] and [41] in the Euclidean case (see also [25]); this also leads to new, anisotropic Strichartz estimates for the wave equation, at least for the radial case. The approach we set up here is somewhat more challenging than in the Euclidean case because the Fourier analysis on the Heisenberg group is an intricate tool.

1.2. **Basic facts about the Heisenberg group.** Let us start by recalling that the d-dimensional Heisenberg group $\mathbb{H}^d$ can be defined as $T^\star\mathbb{R}^d\times\mathbb{R}$ where $T^\star\mathbb{R}^d$ is the cotangent bundle, endowed with the noncommutative product law

$$(Y,s)\cdot(Y',s')\stackrel{\text{def}}{=}\big(Y+Y',s+s'+2\sigma(Y,Y')\big)\,, \tag{1.8}$$

where[1] $w=(Y,s)=(y,\eta,s)$ and $w'=(Y',s')=(y',\eta',s')$ are elements of $\mathbb{H}^d$, while σ denotes the canonical symplectic form on $T^\star\mathbb{R}^d$ defined by

$$\sigma(Y,Y')\stackrel{\text{def}}{=}\langle\eta,y'\rangle-\langle\eta',y\rangle\quad\text{for all}\quad(Y,Y')\in T^\star\mathbb{R}^d\times T^\star\mathbb{R}^d\,, \tag{1.9}$$

with $\langle\eta,y\rangle$ the value of the one-form η at y.

With this point of view, the Haar measure on $\mathbb{H}^d$ is simply the Lebesgue measure on the space $T^\star\mathbb{R}^d\times\mathbb{R}$. In particular, one can define the following (noncommutative) convolution product for any two integrable functions f and g:

$$f\star g(w)\stackrel{\text{def}}{=}\int_{\mathbb{H}^d}f(w\cdot v^{-1})g(v)\,dv=\int_{\mathbb{H}^d}f(v)g(v^{-1}\cdot w)\,dv\,. \tag{1.10}$$

Even though the convolution on the Heisenberg group is noncommutative, if one defines the Lebesgue spaces $L^p(\mathbb{H}^d)$ to be simply $L^p(T^\star\mathbb{R}^d\times\mathbb{R})$, then one still obtains Hölder and Young inequalities, in their classical and weak versions. In order to distinguish the vertical coordinate from the others, we shall also be using, for any two real numbers $1\le p,r\le\infty$, the anistropic Lebesgue spaces $L^{p,r}(\mathbb{H}^d)$ endowed with the mixed norms

$$\|f\|_{L^{p,r}(\mathbb{H}^d)}\stackrel{\text{def}}{=}\|f\|_{L^p(T^\star\mathbb{R}^d,L^r(\mathbb{R}))}=\bigg(\int\bigg(\int|f(Y,s)|^r\,ds\bigg)^{\frac{p}{r}}\,dY\bigg)^{\frac1p}\,.$$

In the framework of the Heisenberg group, the scale invariance is investigated through the family of dilation operators $(\delta_a)_{a>0}$ (which are compatible with the product law (1.8)) defined by

$$\delta_a(Y,s)\stackrel{\text{def}}{=}(aY,a^2s)\,. \tag{1.11}$$

As the determinant of δ_a is a^{2d+2}, it is natural to define the homogeneous dimension of $\mathbb{H}^d$ to be $Q\stackrel{\text{def}}{=}2d+2$.

[1]The variable Y is called the horizontal variable, while the variable s is known as the vertical variable.

The Schwartz class $\mathcal{S}(\mathbb{H}^d)$ coincides with $\mathcal{S}(\mathbb{R}^{2d+1})$, and can be characterized by the action of the sublaplacian

$$\Delta_{\mathbb{H}} u \stackrel{\text{def}}{=} \sum_{j=1}^{d} (\mathcal{X}_j^2 u + \Xi_j^2 u)\,,$$

where the horizontal vector fields $\mathcal{X}_j$ and Ξ_j are defined for $j \in \{1, \dots, d\}$ by

$$\mathcal{X}_j \stackrel{\text{def}}{=} \partial_{y_j} + 2\eta_j \partial_s \quad \text{and} \quad \Xi_j \stackrel{\text{def}}{=} \partial_{\eta_j} - 2y_j \partial_s\,. \tag{1.12}$$

We also define the horizontal gradient

$$\nabla_{\mathbb{H}} u \stackrel{\text{def}}{=} (\mathcal{X}_1 u, \dots, \mathcal{X}_d u, \Xi_1 u, \dots, \Xi_d u)\,.$$

The purpose of this paper is to establish Strichartz estimates for the linear Schrödinger and wave equations on $\mathbb{H}^d$ associated with the sublaplacian

$$(S_{\mathbb{H}}) \quad \begin{cases} i\partial_t u - \Delta_{\mathbb{H}} u = f \\ u_{|t=0} = u_0\,, \end{cases} \qquad (W_{\mathbb{H}}) \quad \begin{cases} \partial_t^2 u - \Delta_{\mathbb{H}} u = f \\ (u, \partial_t u)_{|t=0} = (u_0, u_1)\,. \end{cases}$$

As in the Euclidean case, among the most notable achievements of Fourier analysis on the Heisenberg group that we review in Section 3, one can mention that one can explicitly solve those equations by means of the Fourier transform. However as shown by the following proposition established in [10], $(S_{\mathbb{H}})$ is a model for totally non-dispersive evolution equations.

Proposition 1.1 ([10])**.** *There exists a function u_0 in the Schwartz class $\mathcal{S}(\mathbb{H}^d)$ such that the solution to the free Schrödinger equation $(S_{\mathbb{H}})$ (with $f \equiv 0$) satisfies*

$$u(t, Y, s) = u_0(Y, s + 4td)\,.$$

Remark 1.2. *Since the translation $(Y, s) \mapsto (Y, s + s_0)$ leaves the Lebesgue measure invariant for all $s_0 \in \mathbb{R}$, the solution constructed in Proposition 1.1 satisfies*

$$\forall p \in [1, \infty]\,,\ \|u(t, \cdot)\|_{L^p(\mathbb{H}^d)} = \|u_0\|_{L^p(\mathbb{H}^d)}$$

which shows that one cannot hope for a dispersion phenomenon of the type (1.5).

Proof. In order to establish Proposition 1.1, let us introduce a family of functions on $\mathbb{H}^d$ which are the analogues of the solutions associated with plane waves in the classical Euclidean case, namely

$$(t, x) \in \mathbb{R} \times \mathbb{R}^n \longmapsto e^{i|\xi|^2 t + i\langle \xi, x\rangle} \in \mathbb{S}^1$$

which of course satisfy

$$(i\partial_t - \Delta) e^{i|\xi|^2 t + i\langle \xi, x\rangle} = 0\,.$$

Similarly, consider the family of functions

$$\Theta_\lambda : (Y, s) \in \mathbb{H}^d \longmapsto e^{is\lambda} e^{-\lambda |Y|^2} \in \mathbb{C}\,. \tag{1.13}$$

One can readily check that

$$-\Delta_{\mathbb{H}} \Theta_\lambda = 4\lambda d \Theta_\lambda\,, \tag{1.14}$$

therefore the functions

$$(t, Y, s) \in \mathbb{R} \times \mathbb{H}^d \longmapsto \Theta_\lambda(Y, s + 4td) \in \mathbb{C}$$

satisfy

$$(i\partial_t - \Delta_{\mathbb{H}})(\Theta_\lambda(Y, s + 4td)) = 0\,.$$

Now let g be a function in $\mathcal{D}(]0, \infty[)$, and define

$$u(t, Y, s) \stackrel{\text{def}}{=} \int_{\mathbb{R}} \Theta_\lambda(Y, s + 4td) g(\lambda)\, |\lambda|^d d\lambda\,.$$

It stems from the Lebesgue derivation theorem that u solves the Cauchy problem $(S_{\mathbb{H}})$ with $f \equiv 0$ and initial data

$$u_0(Y,s) = \int_{\mathbb{R}} \Theta_\lambda(Y,s) g(\lambda)\, |\lambda|^d d\lambda\,, \tag{1.15}$$

which easily ends the proof of the proposition. □

Actually, as we shall see in Section 3 page 12, there is a family of functions $(\Theta_\lambda^{(\ell)})_{\ell\in\mathbb{N}}$ on $\mathbb{H}^d$ such that[2]

$$(i\partial_t - \Delta_{\mathbb{H}})(\Theta_\lambda^{(\ell)}(Y, s + 4t(2\ell+d))) = 0\,. \tag{1.16}$$

This readily ensures that the solution to the free Schrödinger equation $(S_{\mathbb{H}})$ associated to the Cauchy data

$$u_0^{(\ell)}(Y,s) \stackrel{\text{def}}{=} \int_{\mathbb{R}} \Theta_\lambda^{(\ell)}(Y,s) g(\lambda)\, |\lambda|^d d\lambda\,,$$

with $g \in \mathcal{D}(]0,\infty[)$, behaves as a transport equation, with velocity depending on ℓ. More precisely, we have

$$(i\partial_t - \Delta_{\mathbb{H}}) u_0^{(\ell)} = i(\partial_t - 4(2\ell+d)\partial_s) u_0^{(\ell)}\,, \tag{1.17}$$

which again highlights the fact that one cannot hope for a dispersion phenomenon of the type (1.5).

Remark 1.3. *In [10] the authors also prove that every solution to the wave equation on $\mathbb{H}^d$ satisfies the dimension-independent dispersive estimate*

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{H}^d)} \leq \frac{C}{|t|^{\frac{1}{2}}} (\|u_0\|_{L^1(\mathbb{H}^d)} + \|u_1\|_{L^1(\mathbb{H}^d)})\,, \tag{1.18}$$

and show by an example similar to the ones above that this estimate is optimal. The rate of decay in (1.18) regardless to the dimension is due to the fact that only the center is involved in the dispersive effect. Note also that compared with the Euclidean framework, there is an exchange in the rates of decay between the wave and the Schrödinger equations on $\mathbb{H}^d$. It is also proved in [10] that the dispersive estimate (1.18) gives rise to a Strichartz estimate

$$\|u\|_{L^q_t(\mathbb{R},L^p(\mathbb{H}^d))} \leq C_{p,q,p_1,q_1} \Big(\|\nabla_{\mathbb{H}^d} u_0\|_{L^2(\mathbb{H}^d)} + \|u_1\|_{L^2(\mathbb{H}^d)} + \|f\|_{L_t^{q_1'}(\mathbb{R},L^{p_1'}(\mathbb{H}^d))} \Big)$$

with $\dfrac{1}{q} + \dfrac{Q}{p} = \dfrac{Q}{2} - 1$ *and* $q \geq 2Q-1$.

1.3. **Statements of the results.** Our first goal in this paper is to establish the following Strichartz estimates for the Schrödinger equation on $\mathbb{H}^d$ for radial data — note that the Fourier transform in the radial setting is much easier to handle, and the geometry of sets on the Fourier side is also much easier to describe in the radial case (see for example (4.1) in Section 4.1 for the sphere), so we restrict our attention to that framework in this article. A function f on $\mathbb{H}^d$ is said to be *radial* if it is invariant under the action of the unitary group $U(d)$ of $T^\star\mathbb{R}^d$, which implies that f can be written under the form $f(Y,s) = f(|Y|,s)$.

Theorem 1. *Given (p,q) and (p_1,q_1) belonging to the admissible set*

$$\mathcal{A}^S \stackrel{\text{def}}{=} \Big\{ (p,q) \in [2,\infty]^2 \,/\, q \leq p \quad \text{and} \quad \frac{2}{q} + \frac{2d}{p} = \frac{Q}{2} \Big\}\,,$$

[2] The function $\Theta_\lambda^{(0)}$ corresponds to the function Θ_λ given by (1.13).

there is a constant C_{p,q,p_1,q_1} *such that the solution to the Schrödinger equation* $(S_{\mathbb{H}})$ *associated with radial data satisfies the following Strichartz estimate (denoting by* a' *the dual exponent of any* $a \in [1,\infty]$*)*

$$\|u\|_{L^\infty_s\left(\mathbb{R},L^q_t(\mathbb{R},L^p(T^*\mathbb{R}^d))\right)} \leq C_{p,q,p_1,q_1}\Big(\|u_0\|_{L^2(\mathbb{H}^d)} + \|f\|_{L^1_s\left(\mathbb{R},L^{q_1'}_t(\mathbb{R},L^{p_1'}(T^*\mathbb{R}^d))\right)}\Big)\,. \tag{1.19}$$

In the case of the wave equation we obtain the following Strichartz estimate.

Theorem 2. *With the above notation, given* (p,q) *and* (p_1,q_1) *belonging to the admissible set*

$$\mathcal{A}^W \stackrel{\text{def}}{=} \Big\{(p,q) \in [2,\infty]^2 \,/\, q \leq p \quad \textit{and} \quad \frac{1}{q} + \frac{2d}{p} = \frac{Q}{2} - 1\Big\}\,,$$

there is a constant C_{p,q,p_1,q_1} *such that the solution to the wave equation* $(W_{\mathbb{H}})$ *associated with radial data satisfies the following Strichartz estimate:*

$$\|u\|_{L^\infty_s\left(\mathbb{R},L^q_t(\mathbb{R},L^p(T^*\mathbb{R}^d))\right)} \leq C_{p,q,p_1,q_1}\Big(\|\nabla_{\mathbb{H}^d}u_0\|_{L^2(\mathbb{H}^d)} + \|u_1\|_{L^2(\mathbb{H}^d)} + \|f\|_{L^1_s(\mathbb{R},L^{q_1'}_t(\mathbb{R},L^{p_1'}(T^*\mathbb{R}^d)))}\Big)\,.$$

Our strategy of proof of the estimates is closely related to the method developed in [39] (the reader may consult [40] and the references therein for an overview on this subject in the Euclidean framework, as well as Section 2 below) consisting in reducing the problem to the study of the restriction operator on a manifold in Fourier space — with additional non negligible technicalities owing to the complexity of the Fourier transform on the Heisenberg group. That is actually the main achievement of this paper. At this stage, one should mention the Fourier restriction theorem on $\mathbb{H}^d$ due to Müller ([35]), where the author investigated the restriction of the Heisenberg Fourier transform on the unit sphere and emphasized the separate roles of the horizontal and vertical variables of $\mathbb{H}^d$.

Other results extending the restriction theorem of Müller to more general nilpotent groups through spectral analysis have been considered in [18, 19] and [32, 33]. Finally, let us mention that applications of non commutative Fourier analysis have been also used to study the heat equation associated to sublaplacians on groups, see for instance [1]. For our purposes, we need Fourier restriction estimates in a direct product of the Heisenberg group and the real line, which will be obtained by combining the methods of Müller [35] and Tomas-Stein [41].

Remark 1.4. *Notice that the relations between* p *and* q *given in the admissibility sets* $\mathcal{A}^S$ *and* $\mathcal{A}^W$ *are dictated by scaling, and are the same as in the Euclidean case* $\mathbb{R}^n$*, where* Q *is replaced by* n *on the right-hand side (but not on the left one, due to the anisotropy).*

Remark 1.5. *Switching* s *and* t *in the proofs of Theorems 1 and 2, one readily gets the following Strichartz estimates with a reduced range of indices due to the presence of a velocity coefficient with respect to the variable* s *underlined above in* (1.17)*. More precisely regarding the Schrödinger equation* $(S_{\mathbb{H}})$ *associated with radial data, there holds*

$$\|u\|_{L^\infty_t(\mathbb{R},L^q_s(\mathbb{R},L^p(T^*\mathbb{R}^d)))} \leq C_{p,q,p_1,q_1}\Big(\|u_0\|_{L^2(\mathbb{H}^d)} + \|f\|_{L^1_t(\mathbb{R},L^{q_1'}_s(\mathbb{R},L^{p_1'}(T^*\mathbb{R}^d)))}\Big)\,,$$

for (p,q) *and* (p_1,q_1) *belonging to the admissible set*

$$\widetilde{\mathcal{A}}^S \stackrel{\text{def}}{=} \Big\{(p,q) \in [2,\infty]^2 \,/\, q \leq p \quad \textit{with} \quad \frac{1}{p} + \frac{1}{q} < \frac{1}{2} \quad \textit{and} \quad \frac{2}{q} + \frac{2d}{p} = \frac{Q}{2}\Big\}\,.$$

Concerning the wave equation ($W_{\mathbb{H}}$)*, the estimate reads*

$$\|u\|_{L^\infty_t(\mathbb{R},L^q_s(\mathbb{R},L^p(T^*\mathbb{R}^d)))} \leq C_{p,q,p_1,q_1}\Big(\|\nabla_{\mathbb{H}^d}u_0\|_{L^2(\mathbb{H}^d)} + \|u_1\|_{L^2(\mathbb{H}^d)} + \|f\|_{L^1_t(\mathbb{R},L^{q_1'}_s(\mathbb{R},L^{p_1'}(T^*\mathbb{R}^d)))}\Big)\,,$$

for (p,q) *and* (p_1,q_1) *belonging to the admissible set*

$$\widetilde{\mathcal{A}}^W \stackrel{\text{def}}{=} \Big\{(p,q)\in[2,\infty]^2\,/\,q\leq p \quad \textit{with} \quad \frac{1}{p}+\frac{1}{q}<\frac{1}{2} \quad \textit{and} \quad \frac{1}{q}+\frac{2d}{p}=\frac{Q}{2}-1\Big\}\,.$$

1.4. **Layout.** The proof of Theorems 1 and 2 is addressed in Section 5. A short illustration of the proof in the (well-known) Euclidean case is provided in Section 2 for the convenience of the reader. The Fourier transform on $\mathbb{H}^d$ and the space of frequencies $\widehat{\mathbb{H}}^d$ are defined and described in Section 3, while Section 4 is dedicated to the study of the restriction of the Heisenberg Fourier transform to the unit sphere of the frequency space $\widehat{\mathbb{H}}^d$: this is not strictly necessary to the proof of our main results but will be a way of introducing our methods, by recovering the results of Müller [35] in a slightly simpler setting. Finally in the Appendix we recall some properties of λ-twisted convolutions which are needed in the proof.

To avoid heaviness, all along this article C will denote a positive constant which may vary from line to line. We also use $f\lesssim g$ to denote an estimate of the form $f\leq Cg$.

Acknowledgments. The authors wish to thank warmly Jean-Yves Chemin for numerous enlightening discussions.

The second author was supported by the Grant ANR-15-CE40-0018 “Sub-Riemannian Geometry and Interactions” of the French ANR.

2. Fourier restriction theorem and its applications in the Euclidean space

In this section we recall some classical results on the Fourier restriction problem and its application to PDEs in the classical, Euclidean setting for the convenience of the reader, since we shall follow a similar approach in our framework. To keep the notation consistent with the case of the Heisenberg group that follows, we distinguish $\mathbb{R}^n$ and its dual $\widehat{\mathbb{R}}^n$, which is of course isomorphic to $\mathbb{R}^n$ itself.

2.1. **Restriction theorems.** The Fourier transform $\mathcal{F}(f)$ of a function f in $L^1(\mathbb{R}^n)$ is continuous, thus it makes sense to restrict $\mathcal{F}(f)$ to any subset of $\widehat{\mathbb{R}}^n$. However, the Fourier transform of a function in $L^2(\mathbb{R}^n)$ is, in general, only in $L^2(\widehat{\mathbb{R}}^n)$, hence completely arbitrary on a set $\widehat{S}$ of $\widehat{\mathbb{R}}^n$ of measure zero.

Indeed, in general, the Fourier transform of a function in L^p for $p>1$ cannot be restricted to an hyperplane. As one can easily check, the function $f:\mathbb{R}^n\to\mathbb{R}$ defined by

$$f(x)=\frac{e^{-|x'|^2}}{1+|x_1|}\,,\qquad x=(x_1,x')\in\mathbb{R}^n, \tag{2.1}$$

belongs to $L^p(\mathbb{R}^n)$, for all $p>1$, but its Fourier transform does not admit a restriction on the hyperplane $\widehat{S}$ of $\widehat{\mathbb{R}}^n$ defined by $\widehat{S}=\{\xi\in\widehat{\mathbb{R}}^n\,/\,\xi_1=0\}$.

Tomas and Stein made the surprising discovery that one can restrict the Fourier transform of $L^p(\mathbb{R}^n)$ functions, for $p>1$ (and close to 1), to hypersurfaces $\widehat{S}$ that are “sufficiently curved”, as for instance the sphere. More generally, given a hypersurface $\widehat{S}\subset\widehat{\mathbb{R}}^n$ endowed with a smooth measure $d\sigma$, the restriction problem asks for which pairs (p,q) an inequality of the form

$$\|\mathcal{F}(f)|_{\widehat{S}}\|_{L^q(\widehat{S},d\sigma)}\leq C\|f\|_{L^p(\mathbb{R}^n)} \tag{2.2}$$

holds for all f in $\mathcal{S}(\mathbb{R}^n)$. Despite all the recent progresses in this field, this question is not completely settled in its general form and remains a topical issue. For a general survey on these questions we refer to the book of Stein [38] and the text of Tao [40]. In what follows, we focus on the case $q = 2$. By a duality argument, the above question for $q = 2$ is equivalent to asking whether the adjoint operator R_S^* defined by

$$R_S^* g \stackrel{\text{def}}{=} \mathcal{F}^{-1}(g d\sigma)$$

is continuous from $L^2(\widehat{S}, d\sigma)$ to $L^{p'}(\mathbb{R}^n)$, where p' is the dual exponent of p. A basic counterexample shows that the range of p for which the estimate holds cannot be the entire interval $1 \leq p \leq 2$; for details we refer to [39].

Example 1 (Knapp)**.** *Let $\widehat{S}$ be the $(n-1)$-dimensional sphere in $\widehat{\mathbb{R}}^n$ endowed with the standard measure $d\mu$. Let g_δ be the characteristic function of a spherical cap*

$$\widehat{C}_\delta \stackrel{\text{def}}{=} \{x \in \widehat{S} : |x \cdot e_n| < \delta\}\,.$$

With some computation one can prove that as $\delta \to 0$,

$$\|g_\delta\|_{L^2(\widehat{S}, d\mu)} \sim \delta^{(n-1)/2}, \qquad \|\mathcal{F}^{-1}(g_\delta)\|_{L^{p'}(\mathbb{R}^n)} \geq C\delta^{n-1}\delta^{-(n+1)/p'}\,,$$

hence the estimate can hold only if $p' \geq (2n+2)/(n-1)$, i.e., if $p \leq (2n+2)/(n+3)$.

The above range is indeed the correct one in the case of a surface with non vanishing curvature. This is the statement of the so-called Tomas-Stein theorem.

Theorem 3 ([41])**.** *Let $\widehat{S}$ be a smooth compact hypersurface in $\widehat{\mathbb{R}}^n$ with non vanishing Gaussian curvature at every point, and let $d\sigma$ be a smooth measure on $\widehat{S}$. Then there holds for every $f \in \mathcal{S}(\mathbb{R}^n)$ and every $p \leq (2n+2)/(n+3)$,*

$$\|\mathcal{F}(f)|_{\widehat{S}}\|_{L^2(\widehat{S}, d\sigma)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}\,.$$

A similar result is possible for surfaces with vanishing Gaussian curvature (that are not flat). In this case the range of p is smaller depending on the order of tangency of the surface to its tangent space. The assumption about compactness of $\widehat{S}$ can be removed by replacing $d\sigma$ with a compactly supported smooth measure.

2.2. **Application of restriction theorems to some PDEs.** Restriction estimates have several applications, from spectral theory to number theory. Here we recall some of these to PDEs: indeed, the restriction theorem can be efficiently applied to obtain Strichartz estimates on the solutions to some PDEs. Here we focus on the Schrödinger and wave equations, for which these estimates were first discovered by Strichartz in his seminal work [39].

Let us first consider the classical Schrödinger equation (S) in $\mathbb{R}^n$, recalled in the introduction page 1. Given a solution $u(t,x)$ of this equation, the Fourier transform $\widehat{u}(t,\xi)$ with respect to the spatial variable x satisfies

$$i\partial_t \widehat{u}(t,\xi) = -|\xi|^2 \widehat{u}(t,\xi), \qquad \widehat{u}(0,\xi) = \widehat{u}_0(\xi). \tag{2.3}$$

Solving the corresponding ODE and taking the inverse Fourier transform one has

$$u(t,x) = \int_{\widehat{\mathbb{R}}^n} e^{i(x\cdot\xi + t|\xi|^2)} \widehat{u}_0(\xi) d\xi\,. \tag{2.4}$$

Formula (2.4) can be interpreted as the restriction of the Fourier transform on the paraboloid $\widehat{S}$ in the space of frequencies $\widehat{\mathbb{R}}^{n+1} = \widehat{\mathbb{R}} \times \widehat{\mathbb{R}}^n$, defined as

$$\widehat{S} \stackrel{\text{def}}{=} \Big\{(\alpha,\xi) \in \widehat{\mathbb{R}} \times \widehat{\mathbb{R}}^n \mid \alpha = |\xi|^2\Big\}\,.$$

Let us endow $\widehat{S}$ with the measure $d\sigma = d\xi$ induced by the projection $\pi : \widehat{\mathbb{R}} \times \widehat{\mathbb{R}}^n \to \widehat{\mathbb{R}}^n$ onto the second factor. More formally one should write[3] $d\sigma = (\pi|_{\widehat{S}}^{-1})_\sharp d\xi$. Notice that $\pi|_{\widehat{S}}$ is invertible and $d\sigma$ is not the intrinsic surface measure of $\widehat{S}$, which is written in coordinates as $d\mu = \sqrt{1+2|\xi|}d\xi$.

Given $\widehat{u}_0 : \widehat{\mathbb{R}}^n \to \mathbb{C}$ define $g : \widehat{S} \to \mathbb{C}$ as $g = \widehat{u}_0 \circ \pi|_{\widehat{S}}$. In other words $g(|\xi|^2, \xi) = \widehat{u}_0(\xi)$. By construction, for $\widehat{u}_0 \in L^2(\widehat{\mathbb{R}}^n)$ one has $g \in L^2(\widehat{S}, d\mu)$ and $\|u_0\|_{L^2(\widehat{\mathbb{R}}^n)} = \|g\|_{L^2(\widehat{S},d\mu)}$. Then

$$u(t,x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi + t|\xi|^2)}\widehat{u}_0(\xi)d\xi = \int_{\widehat{S}} e^{iy\cdot z} g(z)d\sigma(z)$$

where $y = (t,x)$ and $z = (\alpha, \xi)$. Theorem 3, in dual form, tells us that

$$\|\mathcal{F}^{-1}(gd\sigma)\|_{L^{p'}(\widehat{\mathbb{R}}^{n+1})} \leq C_p \|g\|_{L^2(\widehat{S},d\mu)}\,, \tag{2.5}$$

for all $g \in L^2(\widehat{S}, d\mu)$ and all $p' \geq 2(n+2)/n$ (we stress that we apply the result in dimension $n+1$, i.e., in $\mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1}$).

Hence applying the statement to g related to a initial data u_0 such that $\widehat{u}_0$ is supported on a unit ball (which can be translated in a compact support for $d\sigma$) one has by the Plancherel formula

$$\|u\|_{L^{p'}(\mathbb{R}^{n+1})} \leq C\|u_0\|_{L^2(\mathbb{R}^n)}\,, \tag{2.6}$$

for all $p' \geq 2(n+2)/n$.

A scaling argument and the density of spectrally localized functions in $L^2(\mathbb{R}^n)$, give for $p' = 2 + \frac{4}{n}$ and all $u_0 \in L^2(\mathbb{R}^n)$

$$\|u\|_{L^{\frac{2n+4}{n}}(\mathbb{R}, L^{\frac{2n+4}{n}}(\mathbb{R}^n))} \leq C\|u_0\|_{L^2(\mathbb{R}^n)}\,. \tag{2.7}$$

One can similarly prove a Strichartz estimate for the wave equation (W) in the Euclidean space recalled on page 2, by using the representation formula (1.4). The solution can be seen as a sum of two parts, each of which is the restriction of the Fourier transform on one of the two halves of the cone

$$\widehat{S}_\pm \stackrel{\text{def}}{=} \Big\{(\alpha,\xi) \in \widehat{\mathbb{R}} \times \widehat{\mathbb{R}}^n \mid \alpha^2 = |\xi|^2, \pm\alpha > 0\Big\}\,,$$

each of which endowed with the measure defined by the projection $\pi : \widehat{\mathbb{R}} \times \widehat{\mathbb{R}}^n \to \widehat{\mathbb{R}}^n$ onto the second factor (cf. the discussion above).

Now let us first assume that $\gamma_\pm$ is frequency localized in a unit ring $\mathcal{C}_1$ centered at zero. Then for any $p' \geq 2(n+2)/n$ we have

$$\|u\|_{L^{p'}(\mathbb{R}^{n+1})} \leq C\|\mathcal{F}_{\mathbb{H}}^{-1}\gamma_\pm\|_{L^2(\mathbb{R}^n)}\,.$$

As above, for $p' = (2n+2)/(n-2)$, we conclude by scaling arguments and the density in $L^2(\mathbb{H}^d)$ of functions whose Fourier transform is compactly supported in rings centered at zero.

Remark 2.1. *Notice that to apply the Fourier restriction to evolution PDEs and obtain Strichartz estimates, one applies the result to a surface in the space* $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$, *namely the paraboloid and the cone for the Schrödinger and wave equation, respectively.*

When dealing with equations defined on the Heisenberg group $\mathbb{H}^d$, *one is naturally lead to consider surfaces in the space* $\mathbb{R} \times \mathbb{H}^d$, *which is not equal to* $\mathbb{H}^{d'}$ *for some* d'. *Hence it is not enough to know restriction theorems in* $\widehat{\mathbb{H}}^d$ *(cf. Section 4) but one needs to adapt these results to surfaces in* $\widehat{\mathbb{R}} \times \widehat{\mathbb{H}}^d$ *(cf. Section 5).*

[3]Given $T : M \to N$ and μ measure on M we can define a measure $T_\sharp\mu$ on N as $T_\sharp\mu(A) = \mu(T^{-1}(A))$.

3. Fourier analysis on $\mathbb{H}^d$

3.1. **The Fourier transform on $\mathbb{H}^d$.** As the Heisenberg group is noncommutative, defining the Fourier transform of integrable functions on $\mathbb{H}^d$ by means of characters is not relevant. The standard way consists in using irreducible representations of $\mathbb{H}^d$, and in that case the Heisenberg Fourier transform $\mathfrak{F}_{\mathbb{H}}f(\lambda)$ is not a complex valued function on some "frequency space" as in the Euclidean case, but a family of bounded operators on $L^2(\mathbb{R}^d)$ (see Corwin and Greenleaf [21] for instance for more details). Starting from the so-called Schrödinger representation, in [6] and [5] the authors introduce an alternative definition of the Fourier transform on $\mathbb{H}^d$ in terms of functions acting on some frequency set $\widetilde{\mathbb{H}}^d$. This point of view (which turns out to be equivalent to the classical definition) consists in defining the Fourier transform of an integrable function f on $\mathbb{H}^d$ by projecting $\mathfrak{F}_{\mathbb{H}}(\lambda)$ onto the orthonormal basis of $L^2(\mathbb{R}^d)$ given by Hermite functions. This enables to see the Fourier transform of a function f in $L^1(\mathbb{H}^d)$ as the mean value of f modulated by some oscillatory functions in the following way:

$$\mathcal{F}_{\mathbb{H}}f(\widehat{w}) \stackrel{\text{def}}{=} \int_{\mathbb{H}^d} \overline{e^{is\lambda}\mathcal{W}(\widehat{w},Y)}\, f(Y,s)\, dY\, ds\,, \tag{3.1}$$

for any $\widehat{w} \stackrel{\text{def}}{=} (n,m,\lambda)$ in $\widetilde{\mathbb{H}}^d \stackrel{\text{def}}{=} \mathbb{N}^{2d} \times \mathbb{R}\setminus\{0\}$, with $\mathcal{W}$ the Wigner transform of the (renormalized) Hermite functions

$$\mathcal{W}(\widehat{w},Y) \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} e^{2i\lambda\langle \eta,z\rangle} H_{n,\lambda}(y+z) H_{m,\lambda}(-y+z)\, dz\,. \tag{3.2}$$

Here $H_{m,\lambda}$ stands for the renormalized Hermite function on $\mathbb{R}^d$, namely $H_{m,\lambda}(x) \stackrel{\text{def}}{=} |\lambda|^{\frac{d}{4}} H_m(|\lambda|^{\frac12}x)$, with $(H_m)_{m\in\mathbb{N}^d}$ the Hermite orthonormal basis of $L^2(\mathbb{R}^d)$ given by the eigenfunctions of the harmonic oscillator:

$$-(\Delta - |x|^2)H_m = (2|m|+d)H_m\,,$$

specifically

$$H_m \stackrel{\text{def}}{=} \Big(\frac{1}{2^{|m|}m!}\Big)^{\frac12} \prod_{j=1}^{d} \big(-\partial_j H_0 + x_j H_0\big)^{m_j}\,, \tag{3.3}$$

with $H_0(x) \stackrel{\text{def}}{=} \pi^{-\frac{d}{4}} e^{-\frac{|x|^2}{2}}$, $m! \stackrel{\text{def}}{=} m_1!\cdots m_d!$ and $|m| \stackrel{\text{def}}{=} m_1+\cdots+m_d$.

In this setting, the classical statements of Fourier analysis hold in a similar way to the Euclidean case, namely the inversion and Fourier-Plancherel formulae read

$$f(w) = \frac{2^{d-1}}{\pi^{d+1}} \int_{\widetilde{\mathbb{H}}^d} e^{is\lambda}\mathcal{W}(\widehat{w},Y)\mathcal{F}_{\mathbb{H}}f(\widehat{w})\, d\widehat{w} \tag{3.4}$$

and

$$\big(\mathcal{F}_{\mathbb{H}}f|\mathcal{F}_{\mathbb{H}}g\big)_{L^2(\widetilde{\mathbb{H}}^d)} = \frac{\pi^{d+1}}{2^{d-1}} (f|g)_{L^2(\mathbb{H}^d)}\,, \tag{3.5}$$

with the notation

$$\int_{\widetilde{\mathbb{H}}^d} \theta(\widehat{w})\, d\widehat{w} \stackrel{\text{def}}{=} \int_{\mathbb{R}} \sum_{(n,m)\in\mathbb{N}^{2d}} \theta(n,m,\lambda)|\lambda|^d\, d\lambda\,. \tag{3.6}$$

By straightforward computations we find that

$$-\Delta_{\mathbb{H}}(e^{is\lambda}\mathcal{W}(\widehat{w},Y)) = 4|\lambda|(2|m|+d)e^{is\lambda}\mathcal{W}(\widehat{w},Y)\,, \tag{3.7}$$

for any $\widehat{w} = (n,m,\lambda)$ in $\widetilde{\mathbb{H}}^d$, which readily implies that

$$\mathcal{F}_{\mathbb{H}}(\Delta_{\mathbb{H}}f)(\widehat{w}) = -4|\lambda|(2|m|+d)\mathcal{F}_{\mathbb{H}}(f)(\widehat{w})\,.$$

This formula allows to give a definition of a function whose Fourier transform is compactly supported, in the following way.

Definition 3.1. *We say that a function f on $\mathbb{H}^d$ is frequency localized in a ball $\mathcal{B}_\Lambda$ centered at 0 of radius Λ if there exists an even function ψ in $\mathcal{D}(\mathbb{R})$ supported in $\mathcal{B}_1$ and equal to 1 near 0 such that*[4]

$$f = \psi(-\Lambda^{-2}\Delta_{\mathbb{H}})\, f\,,$$

which is equivalent to stating that for any $\widehat{w} = (m, n, \lambda)$ in $\widetilde{\mathbb{H}}^d$,

$$\mathcal{F}_{\mathbb{H}}(f)(n, m, \lambda) = \psi(\Lambda^{-2}4|\lambda|(2|m| + d))\, \mathcal{F}_{\mathbb{H}}(f)(n, m, \lambda)\,.$$

Similarly we say that a function f on $\mathbb{H}^d$ is frequency localized in a ring $\mathcal{C}_\Lambda$ centered at 0 of small radius $\Lambda/2$ and large radius Λ if there exists an even function ϕ in $\mathcal{D}(\mathbb{R})$ supported in $\mathcal{C}_1$ and equal to 1 in a ring $\mathcal{C}'$ contained in $\mathcal{C}_1$ such that

$$f = \phi(-\Lambda^{-2}\Delta_{\mathbb{H}})\, f\,,$$

which is equivalent to stating that for any $\widehat{w} = (m, n, \lambda)$ in $\widetilde{\mathbb{H}}^d$,

$$\mathcal{F}_{\mathbb{H}}(f)(n, m, \lambda) = \phi(\Lambda^{-2}4|\lambda|(2|m| + d))\, \mathcal{F}_{\mathbb{H}}(f)(n, m, \lambda)\,.$$

One of the interests of this definition lies in the following proposition, whose proof may be found in [9] and [10].

Lemma 3.2. *With the above notation,*

- *if f is frequency localized in $\mathcal{B}_\Lambda$, then for all $1 \leq p \leq q \leq \infty$, $k \in \mathbb{N}$ and $\beta \in \mathbb{N}^{2d}$ with $|\beta| = k$, there is a constant C_k depending only on k such that*

$$\|\mathcal{X}^\beta f\|_{L^q(\mathbb{H}^d)} \leq C_k \Lambda^{k+Q(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\mathbb{H}^d)}\,, \tag{3.8}$$

  where $\mathcal{X}^\beta$ denotes a product of $|\beta|$ vectors fields of type (1.12);
- *if f is frequency localized in $\mathcal{C}_\Lambda$, then for all $p \geq 1$ and $s \in \mathbb{R}$, there is a constant C_s depending only on s such that*

$$\frac{1}{C_s}\Lambda^s \|f\|_{L^p(\mathbb{H}^d)} \leq \|(-\Delta_{\mathbb{H}})^{\frac{s}{2}} f\|_{L^p(\mathbb{H}^d)} \leq C_s \Lambda^s \|f\|_{L^p(\mathbb{H}^d)}\,. \tag{3.9}$$

It will be useful later on to observe that for any function f in $L^1(\mathbb{H}^d)$ and any positive real number a, there holds

$$\forall \widehat{w} = (n, m, \lambda) \in \widetilde{\mathbb{H}}^d\,,\ \mathcal{F}_{\mathbb{H}}(f \circ \delta_a)(\widehat{w}) = a^{-Q}\mathcal{F}_{\mathbb{H}}(f)(n, m, a^{-2}\lambda)\,. \tag{3.10}$$

Let us also emphasize that if f and g are two functions of $L^1(\mathbb{H}^d)$ then for any $\widehat{w} = (n, m, \lambda)$ in $\widetilde{\mathbb{H}}^d$,

$$\mathcal{F}_{\mathbb{H}}(f \star g)(\widehat{w}) = (\mathcal{F}_{\mathbb{H}} f \cdot \mathcal{F}_{\mathbb{H}} g)(\widehat{w}) \stackrel{\text{def}}{=} \sum_{p \in \mathbb{N}^d} \mathcal{F}_{\mathbb{H}} f(n, p, \lambda) \mathcal{F}_{\mathbb{H}} g(p, m, \lambda)\,. \tag{3.11}$$

In the radial framework (recall that f is radial if it is invariant under the action of the unitary group $U(d)$ of $T^\star\mathbb{R}^d$), which is our concern in this paper, it turns out that for any function f in $L^1_{\text{rad}}(\mathbb{H}^d)$ there holds

$$\mathcal{F}_{\mathbb{H}}(f)(n, m, \lambda) = \mathcal{F}_{\mathbb{H}}(f)(n, m, \lambda)\delta_{n,m} = \mathcal{F}_{\mathbb{H}}(f)(|n|, |n|, \lambda)\delta_{n,m}\,. \tag{3.12}$$

The interested reader can consult for instance [8, 24, 36]. Actually the Fourier transform $\mathcal{F}_{\mathbb{H}}$ acts in the following way on radial functions:

$$\mathcal{F}_{\mathbb{H}}(f)(\ell, \ell, \lambda) = \begin{pmatrix} \ell + d - 1 \\ \ell \end{pmatrix}^{-1} \int_{\mathbb{H}^d} \overline{e^{is\lambda}\widetilde{\mathcal{W}}(\ell, \lambda, Y)} f(Y, s)\, dY\, ds\,,$$

[4] where $\psi(-\Delta_{\mathbb{H}})$ is defined by the functional calculus of the self-adjoint operator $-\Delta_{\mathbb{H}}$.

with (see for example [8, 23, 36] for further details)

$$\widetilde{\mathcal{W}}(\ell,\lambda,Y)\stackrel{\rm def}{=}\sum_{\substack{n\in\mathbb{N}^d\\|n|=\ell}}\mathcal{W}(n,n,\lambda,Y)=e^{-|\lambda||Y|^2}L_\ell^{(d-1)}(2|\lambda||Y|^2)\,, \tag{3.13}$$

where $L_\ell^{(d-1)}$ stands for the Laguerre polynomial of order ℓ and type $d-1$ given for $x\geq 0$ by

$$L_\ell^{(d-1)}(x)\stackrel{\rm def}{=}\sum_{k=0}^{\ell}(-1)^k\binom{\ell+d-1}{\ell-k}\frac{x^k}{k!}\,\cdot$$

Note that the family of functions $(\Theta_\lambda^{(\ell)})_{\ell\in\mathbb{N}}$ mentioned in the introduction of this paper, satisfying the transport equation (1.16), is defined by the formula

$$\Theta_\lambda^{(\ell)}(Y,s)\stackrel{\rm def}{=}e^{is\lambda}\widetilde{\mathcal{W}}(\ell,\lambda,Y)\,.$$

Equation (1.16) then follows simply from the fact that

$$-\Delta_{\mathbb{H}}\Theta_\lambda^{(\ell)}=4|\lambda|(2\ell+d)\Theta_\lambda^{(\ell)}\,.$$

Obviously the inversion and Fourier-Plancherel formulae write in that case

$$f(w)=\frac{2^{d-1}}{\pi^{d+1}}\sum_{\ell\in\mathbb{N}}\int_{\mathbb{R}}e^{is\lambda}\widetilde{\mathcal{W}}(\ell,\lambda,Y)\mathcal{F}_{\mathbb{H}}(f)(\ell,\ell,\lambda)\,|\lambda|^d\,d\lambda$$

and

$$(f|g)_{L^2(\mathbb{H}^d)}=\frac{2^{d-1}}{\pi^{d+1}}\sum_{\ell\in\mathbb{N}}\binom{\ell+d-1}{\ell}\int_{\mathbb{R}}\mathcal{F}_{\mathbb{H}}(f)(\ell,\ell,\lambda)\,\overline{\mathcal{F}_{\mathbb{H}}(g)(\ell,\ell,\lambda)}\,|\lambda|^d\,d\lambda\,.$$

Moreover since for any element R of $U(d)$, the automorphism θ_R of $\mathbb{H}^d$ defined by

$$\theta_R(Y,s)\stackrel{\rm def}{=}(R(Y),s)$$

preserves the Haar measure of $\mathbb{H}^d$, we have

$$(f\star g)\circ\theta_R=(f\circ\theta_R)\star(g\circ\theta_R)\,,$$

which implies that the space $L^1_{\rm rad}(\mathbb{H}^d)$ equipped with its standard structure of linear space and with the convolution product is a commutative sub-algebra of $L^1(\mathbb{H}^d)$. We deduce that in this framework, (3.11) reduces to

$$\mathcal{F}_{\mathbb{H}}(f\star g)(\ell,\ell,\lambda)=\mathcal{F}_{\mathbb{H}}f(\ell,\ell,\lambda)\mathcal{F}_{\mathbb{H}}g(\ell,\ell,\lambda)\,. \tag{3.14}$$

Finally it will be important to observe that there holds for all $w=(Y,s)$ in $\mathbb{H}^d$, in the radial setting,

$$\sum_{\substack{n\in\mathbb{N}^d\\|n|=\ell}}\mathcal{F}_{\mathbb{H}}(f\circ\tau_w)(n,n,\lambda)=\mathcal{F}_{\mathbb{H}}(f)(\ell,\ell,\lambda)e^{-is\lambda}e^{-|\lambda||Y|^2}L_\ell^{(d-1)}(2|\lambda||Y|^2)\,, \tag{3.15}$$

where τ_w denotes the left translate defined by $\tau_w(w')\stackrel{\rm def}{=}w\cdot w'$.

3.2. **Frequency space for the Heisenberg group.** In [6], the authors show that the following distance $\widehat{d}$ on $\widetilde{\mathbb{H}}^d=\mathbb{N}^{2d}\times\mathbb{R}\setminus\{0\}$

$$\widehat{d}(\widehat{w},\widehat{w}')\stackrel{\rm def}{=}\big|\lambda(n+m)-\lambda'(n'+m')\big|_1+|(n-m)-(n'-m')|_1+d|\lambda-\lambda'|\,, \tag{3.16}$$

where $|\cdot|_1$ denotes the ℓ^1 norm on $\mathbb{R}^d$, is appropriate and that the completion of the set $\widetilde{\mathbb{H}}^d$ for this distance is the set

$$\widehat{\mathbb{H}}^d\stackrel{\rm def}{=}(\mathbb{N}^{2d}\times\mathbb{R}\setminus\{0\})\cup\widehat{\mathbb{H}}^d_0\quad\text{with}\quad\widehat{\mathbb{H}}^d_0\stackrel{\rm def}{=}\{(\dot{x},k)\in\mathbb{R}^d_{\mp}\times\mathbb{Z}^d\}\quad\text{and}\quad\mathbb{R}^d_{\mp}\stackrel{\rm def}{=}(\mathbb{R}_-)^d\cup(\mathbb{R}_+)^d\,.$$

It readily stems from (3.1) that the following continuous embedding holds:

$$\mathcal{F}_{\mathbb{H}} : L^1(\mathbb{H}^d) \hookrightarrow L^\infty(\widehat{\mathbb{H}}^d)\,. \tag{3.17}$$

Combining the Fourier-Plancherel formula (3.5) together with interpolation theory, we deduce that, for all $1 \leq p \leq 2$, the Hausdorff-Young inequality holds

$$\|\mathcal{F}_{\mathbb{H}} f\|_{L^{p'}(\widehat{\mathbb{H}}^d)} \leq \|f\|_{L^p(\mathbb{H}^d)}\,,$$

where p' is the dual exponent of p.

This new approach enabled the authors in [5] to extend $\mathcal{F}_{\mathbb{H}}$ to $\mathcal{S}'(\mathbb{H}^d)$, the set of tempered distributions: note that since the Schwartz class $\mathcal{S}(\mathbb{H}^d)$ coincides with $\mathcal{S}(\mathbb{R}^{2d+1})$ then similarly $\mathcal{S}'(\mathbb{H}^d)$ is noting else than $\mathcal{S}'(\mathbb{R}^{2d+1})$. As in the Euclidean case, this extension is done by duality and the starting point is the characterization of $\mathcal{S}(\widehat{\mathbb{H}}^d)$ as the range[5] of $\mathcal{S}(\mathbb{H}^d)$ by $\mathcal{F}_{\mathbb{H}}$. Actually in [5], the authors prove that the Fourier transform $\mathcal{F}_{\mathbb{H}}$ is a bicontinuous isomorphism between the spaces $\mathcal{S}(\mathbb{H}^d)$ and $\mathcal{S}(\widehat{\mathbb{H}}^d)$, and that the map $\mathcal{F}_{\mathbb{H}}$ can be continuously extended from $\mathcal{S}'(\mathbb{H}^d)$ into $\mathcal{S}'(\widehat{\mathbb{H}}^d)$ in the following way:

$$\mathcal{F}_{\mathbb{H}} : \left\{ \begin{array}{ccc} \mathcal{S}'(\mathbb{H}^d) & \longrightarrow & \mathcal{S}'(\widehat{\mathbb{H}}^d) \\ T & \longmapsto & \Big[\theta \mapsto \langle T, {}^t\mathcal{F}_{\mathbb{H}}\theta\rangle_{\mathcal{S}'(\mathbb{H}^d)\times\mathcal{S}(\mathbb{H}^d)}\Big]\,, \end{array}\right. \tag{3.18}$$

where

$${}^t\mathcal{F}_{\mathbb{H}}\theta(y,\eta,s) \stackrel{\text{def}}{=} \frac{\pi^{d+1}}{2^{d-1}}(\mathcal{F}_{\mathbb{H}}^{-1}\theta)(y,-\eta,-s)\,. \tag{3.19}$$

Let us also emphasize that if T is a tempered distribution on $\mathbb{H}^d$, then for all f in $\mathcal{S}(\mathbb{H}^d)$ and all w in $\mathbb{H}^d$,

$$(T \star f)(w) = \langle T, \check{f} \circ \tau_{w^{-1}}\rangle_{\mathcal{S}'(\mathbb{H}^d)\times\mathcal{S}(\mathbb{H}^d)}\,, \tag{3.20}$$

where $\check{f}(w) \stackrel{\text{def}}{=} f(w^{-1})$.

4. A restriction theorem on the sphere in the frequency space $\widehat{\mathbb{H}}^d$

Our purpose here is to recover a Fourier restriction result on the sphere of $\widehat{\mathbb{H}}^d$ due to Müller [35]. Our approach is rather different to [35] as we use the Fourier transform as a key tool in obtaining a representation of the Fourier restriction operator, whereas Müller uses a spectral representation. As will be seen in Paragraph 5.1 in the proof of Strichartz estimates for the wave and Schrödinger operators on $\mathbb{H}^d$, the interest of our approach is that it can easily be applied to more general frameworks.

4.1. **Study of the surface measure on the sphere of the frequency space.** The aim of this section is to recover the Heisenberg Fourier transform restriction result of Müller in [35]. To this end, let us start by introducing $\mathbb{S}_{\widehat{\mathbb{H}}^d}$ the unit sphere on $\widehat{\mathbb{H}}^d$: denoting by $\widehat{0}$ the origin of $\widehat{\mathbb{H}}^d$ (that is the point of $\widehat{\mathbb{H}}^d$ corresponding to $(\dot{x},k) = (0,0)$, with the notation of Paragraph 3.2), the sphere of $\widehat{\mathbb{H}}^d$ centered at the origin with radius 1 is defined by

$$\mathbb{S}_{\widehat{\mathbb{H}}^d} \stackrel{\text{def}}{=} \Big\{(n,n,\lambda) \in \widehat{\mathbb{H}}^d \,/\, (2|n|+d)|\lambda| = 1\Big\} \bigcup \Big\{(\dot{x},0) \in \widehat{\mathbb{H}}^d_0 \,/\, |\dot{x}|_1 = 1\Big\}\,, \tag{4.1}$$

[5] We refer to [5] for the definition of $\mathcal{S}(\widehat{\mathbb{H}}^d)$.

and the surface measure $d\sigma_{\mathbb{S}_{\widehat{\mathbb{H}}^d}}$ is given for all θ in $\mathcal{S}(\widehat{\mathbb{H}}^d)$ by the following formula:

$$\langle d\sigma_{\mathbb{S}_{\widehat{\mathbb{H}}^d}}, \theta\rangle_{\mathcal{S}'(\widehat{\mathbb{H}}^d)\times\mathcal{S}(\widehat{\mathbb{H}}^d)} = \sum_{n\in\mathbb{N}^d} \frac{1}{(2|n|+d)^{d+1}} \Big(\theta(n,n,\frac{1}{2|n|+d}) + \theta\Big(n,n,\frac{-1}{2|n|+d}\Big)\Big)\,. \tag{4.2}$$

We observe that one can show that the measure of $\widehat{\mathbb{H}}^d_0$ with respect to $d\widehat{w}$ is zero, and thus in all that follows, we shall agree that the measure $d\widehat{w}$ has been extended by 0 to the whole of $\widehat{\mathbb{H}}^d_0$, and we shall keep the same notation $d\widehat{w}$ for the measure on the whole of $\widehat{\mathbb{H}}^d$.

More generally, if $\mathbb{S}_{\widehat{\mathbb{H}}^d}(\sqrt{R})$ denotes the sphere of $\widehat{\mathbb{H}}^d$ centered at the origin $\widehat{0}$ of radius $\sqrt{R}$, let us prove that for all θ in $\mathcal{F}_{\mathbb{H}}(\mathcal{S}_{\rm rad}(\mathbb{H}^d))$, there holds[6]

$$\begin{aligned}\langle d\sigma_{\mathbb{S}_{\widehat{\mathbb{H}}^d}(\sqrt{R})}, \theta\rangle_{\mathcal{S}'(\widehat{\mathbb{H}}^d)\times\mathcal{S}(\widehat{\mathbb{H}}^d)} \\ = \sum_{n\in\mathbb{N}^d} \frac{R^d}{(2|n|+d)^{d+1}} \Big(\theta(n,n,\frac{R}{2|n|+d}) + \theta(n,n,\frac{-R}{2|n|+d})\Big)\,.\end{aligned} \tag{4.3}$$

We start indeed with the general formula

$$\int_{\widehat{\mathbb{H}}^d} \theta(\widehat{w})\, d\widehat{w} = \int_0^\infty \Big(\int_{\mathbb{S}_{\widehat{\mathbb{H}}^d}(\sqrt{R})} \theta(\widehat{w})\, d\sigma_{\mathbb{S}_{\widehat{\mathbb{H}}^d}}(\sqrt{R})\Big)\, dR\,,$$

for all θ in $\mathcal{F}_{\mathbb{H}}(\mathcal{S}_{\rm rad}(\mathbb{H}^d))$. In view of (3.6) and (3.12), we have

$$\int_{\widehat{\mathbb{H}}^d} \theta(\widehat{w})\, d\widehat{w} = \sum_{n\in\mathbb{N}^d} \int_{\mathbb{R}} \theta(n,n,\lambda)\, |\lambda|^d d\lambda\,, \tag{4.4}$$

which thanks to the Fubini theorem and the change of variable $R = (2|n|+d)|\lambda|$ yields

$$\begin{aligned}&\sum_{n\in\mathbb{N}^d} \int_{\mathbb{R}} \theta(n,n,\lambda)\, |\lambda|^d d\lambda \\ &= \sum_{n\in\mathbb{N}^d} \int_0^\infty \frac{R^d}{(2|n|+d)^{d+1}} \Big(\theta\Big(n,n,\frac{R}{2|n|+d}\Big) + \theta\Big(n,n,\frac{-R}{2|n|+d}\Big)\Big) dR\,,\end{aligned}$$

which proves (4.3).

In order to investigate boundedness properties of the restriction of $\mathcal{F}_{\mathbb{H}}$ to $\mathbb{S}_{\widehat{\mathbb{H}}^d}$, we shall adapt the Euclidean proof due to Tomas-Stein (see [41], and Section 2 of this paper). To this end, let us first compute $\mathcal{F}_{\mathbb{H}}^{-1}(d\sigma_{\mathbb{S}_{\widehat{\mathbb{H}}^d}})$. By definition, the tempered distribution

$$G \stackrel{\rm def}{=} \mathcal{F}_{\mathbb{H}}^{-1}(d\sigma_{\mathbb{S}_{\widehat{\mathbb{H}}^d}})$$

satisfies for all θ in $\mathcal{S}(\widehat{\mathbb{H}}^d)$

$$\langle d\sigma_{\mathbb{S}_{\widehat{\mathbb{H}}^d}}, \theta\rangle_{\mathcal{S}'(\widehat{\mathbb{H}}^d)\times\mathcal{S}(\widehat{\mathbb{H}}^d)} = \langle G, {}^t\mathcal{F}_{\mathbb{H}}\theta\rangle_{\mathcal{S}'(\mathbb{H}^d)\times\mathcal{S}(\mathbb{H}^d)}\,.$$

Let us prove the following proposition.

Proposition 4.1. *With the above notation, G is the bounded function on $\mathbb{H}^d$ defined by*

$$G(Y,s) = \frac{2^d}{\pi^{d+1}} \sum_{n\in\mathbb{N}^d} \frac{1}{(2|n|+d)^{d+1}} \cos\Big(\frac{s}{2|n|+d}\Big) \mathcal{W}\Big(n,n,1,\frac{Y}{\sqrt{2|n|+d}}\Big)\,. \tag{4.5}$$

[6] According to the definition of $\widehat{d}$ introduced in (3.16), if $A > 1$ then the sphere $\mathbb{S}_{\widehat{\mathbb{H}}^d}(A)$ is a much more complex set than the unit sphere, and that is why we focus here on the radial framework.

Proof. According to (3.19) and to the fact that the Fourier transform $\mathcal{F}_{\mathbb{H}}$ is a bicontinuous isomorphism between the spaces $\mathcal{S}(\mathbb{H}^d)$ and $\mathcal{S}(\widehat{\mathbb{H}}^d)$, we have (with $\theta \stackrel{\text{def}}{=} \mathcal{F}_{\mathbb{H}} f$)

$$\langle d\sigma_{\mathbb{S}_{\widehat{\mathbb{H}}^d}}, \theta\rangle_{\mathcal{S}'(\widehat{\mathbb{H}}^d)\times\mathcal{S}(\widehat{\mathbb{H}}^d)} = \frac{\pi^{d+1}}{2^{d-1}}\langle G, \widetilde{f}\rangle_{\mathcal{S}'(\mathbb{H}^d)\times\mathcal{S}(\mathbb{H}^d)}\,, \tag{4.6}$$

with $\widetilde{f}(y,\eta,s) \stackrel{\text{def}}{=} f(y,-\eta,-s)$.

Now observe that

$$\theta\Big(n,n,\frac{1}{2|n|+d}\Big) = \int_{\mathbb{H}^d} e^{-i\frac{s}{2|n|+d}}\,\mathcal{W}\Big(n,n,1,\frac{Y}{\sqrt{2|n|+d}}\Big)\, f(Y,s)\,dY\,ds\,. \tag{4.7}$$

Indeed, we have by easy computations

$$\mathcal{W}\Big(n,n,\frac{1}{2|n|+d},Y\Big) = \mathcal{W}\Big(n,n,1,\frac{Y}{\sqrt{2|n|+d}}\Big)\,,$$

which gives rise to (4.7) according to the definition of $\mathcal{F}_{\mathbb{H}}$ and to the fact that for all λ in $\mathbb{R}\setminus\{0\}$, the function $\mathcal{W}(n,n,\lambda,Y)$ is real valued.

Besides by an obvious change of variable, one also has

$$\mathcal{W}\Big(n,n,-1,\frac{Y}{\sqrt{2|n|+d}}\Big) = \mathcal{W}\Big(n,n,1,\frac{Y}{\sqrt{2|n|+d}}\Big)\,,$$

which implies that

$$\theta\Big(n,n,-\frac{1}{2|n|+d}\Big) = \int_{\mathbb{H}^d} e^{i\frac{s}{2|n|+d}}\,\mathcal{W}\Big(n,n,1,\frac{Y}{\sqrt{2|n|+d}}\Big)\, f(Y,s)\,dY\,ds\,. \tag{4.8}$$

Invoking (4.2), this gives rise to

$$\begin{aligned}\langle d\sigma_{\mathbb{S}_{\widehat{\mathbb{H}}^d}}, \theta\rangle_{\mathcal{S}'(\widehat{\mathbb{H}}^d)\times\mathcal{S}(\widehat{\mathbb{H}}^d)} &= \sum_{n\in\mathbb{N}^d}\frac{2}{(2|n|+d)^{d+1}}\\ &\qquad\times\int_{\mathbb{H}^d}\cos\Big(\frac{s}{2|n|+d}\Big)\,\mathcal{W}\Big(n,n,1,\frac{Y}{\sqrt{2|n|+d}}\Big)\, f(Y,s)\,dY\,ds\,,\end{aligned}$$

which by an obvious change of variable ends the proof of Formula (4.5).

Furthermore, we have the following classical combinatorial identity

$$\#\{n\in\mathbb{N}^d, |n|=\ell\} = \binom{\ell+d-1}{\ell}\,, \tag{4.9}$$

and since the modulus of the Wigner transform of the (renormalized) Hermite functions defined by (3.2) is bounded by one, Stirling's formula implies that G belongs to $L^\infty(\mathbb{H}^d)$. The result follows. □

Remark 4.2. *Arguing as for the unit sphere, we readily gather that for all θ belonging to $\mathcal{F}_{\mathbb{H}}(\mathcal{S}_{\rm rad}(\mathbb{H}^d))$, there holds*

$$\langle d\sigma_{\mathbb{S}_{\widehat{\mathbb{H}}^d}(\sqrt{R})}, \theta\rangle_{\mathcal{S}'(\widehat{\mathbb{H}}^d)\times\mathcal{S}(\widehat{\mathbb{H}}^d)} = \langle G_R, {}^t\mathcal{F}_{\mathbb{H}}\theta\rangle_{\mathcal{S}'(\mathbb{H}^d)\times\mathcal{S}(\mathbb{H}^d)}\,,$$

where, in view of (3.10), G_R is given by

$$G_R(Y,s) \stackrel{\text{def}}{=} R^d(G\circ\delta_{\sqrt{R}})(Y,s)\,. \tag{4.10}$$

We thus recover Formula (19) derived by Müller in [35].

4.2. **The restriction theorem.** Now let us state the restriction theorem, due to Müller in [35], and sketch its proof for the convenience of the reader.

Theorem 4 ([35]). *If $1 \leq p \leq 2$, then*

$$\|\mathcal{F}_{\mathbb{H}}(f)_{|\mathbb{S}_{\widetilde{\mathbb{H}}^d}}\|_{L^2(\mathbb{S}_{\widetilde{\mathbb{H}}^d})} \leq C_p \|f\|_{L^{p,1}(\mathbb{H}^d)}\,, \tag{4.11}$$

for all radial functions f in $\mathcal{S}_{\rm rad}(\mathbb{H}^d)$.

Remark 4.3. *In light of (4.2), Theorem 4 writes*

$$\Big[\sum_{n\in\mathbb{N}^d} \frac{1}{(2|n|+d)^{d+1}}\Big(\Big|\mathcal{F}_{\mathbb{H}}(f)(n,n,\frac{1}{2|n|+d})\Big|^2 + \Big|\mathcal{F}_{\mathbb{H}}(f)(n,n,\frac{-1}{2|n|+d})\Big|^2\Big)\Big]^{\frac12} \leq C_p\|f\|_{L^{p,1}(\mathbb{H}^d)}\,,$$

for f in $\mathcal{S}_{\rm rad}(\mathbb{H}^d)$. Besides, since $\mathcal{S}(\mathbb{H}^d)$ is dense in the space $L^{p,1}(\mathbb{H}^d)$, when the estimate of Theorem 4 holds we can define $\mathcal{F}_{\mathbb{H}} f$ on $\mathbb{S}_{\widetilde{\mathbb{H}}^d}$ (a.e. with respect to $d\sigma_{\mathbb{S}_{\widetilde{\mathbb{H}}^d}}$), for each function f in $L^{p,1}(\mathbb{H}^d)$. Let us also emphasize that the gain we get here with respect to the horizontal variable Y is better than the one obtained in Euclidean case, since the index p ranges from 1 to 2 with no further restriction, contrary to the Euclidean case (recall Theorem 3). Note that a counterexample for large values of p (namely $p > 4n/(2n-1)$) is provided in [35].

Remark 4.4. *By duality, Inequality (4.11) is equivalent to the following estimate*

$$\|\mathcal{F}_{\mathbb{H}}^{-1}(\theta_{|\mathbb{S}_{\widetilde{\mathbb{H}}^d}})\|_{L^{p'}(\mathbb{H}^d)} \leq C_p\, \|\theta_{|\mathbb{S}_{\widetilde{\mathbb{H}}^d}}\|_{L^2(\mathbb{S}_{\widetilde{\mathbb{H}}^d})}\,, \tag{4.12}$$

for all $2 \leq p' \leq \infty$ and all θ in $\mathcal{F}_{\mathbb{H}}(\mathcal{S}_{\rm rad})(\mathbb{H}^d)$.

Proof of Theorem 4. First note that the case when $p = 1$ is a straightforward consequence of (3.17). Then by interpolation, it suffices to investigate the case when $p = 2$. For that purpose, we shall proceed using purely Fourier analysis arguments, and sketch the proof due to Müller (see [35] for further details). Our goal here is to establish the following estimate

$$\sum_{\ell\in\mathbb{N}}\Big(\Big|\widetilde{\Theta}\Big(\ell,\ell,\frac{1}{2\ell+d}\Big)\Big|^2 + \Big|\widetilde{\Theta}\Big(\ell,\ell,\frac{-1}{2\ell+d}\Big)\Big|^2\Big) \leq C_2\|f\|^2_{L^{2,1}(\mathbb{H}^d)}\,,$$

where

$$\begin{aligned}\widetilde{\Theta}(\ell,\ell,\lambda) &\stackrel{\rm def}{=} \Big[\begin{pmatrix}\ell+d-1\\ \ell\end{pmatrix}^{-1}\frac{1}{(2\ell+d)^{d+1}}\Big]^{\frac12}\sum_{\substack{n\in\mathbb{N}^d\\|n|=\ell}}\mathcal{F}_{\mathbb{H}} f(n,n,\lambda)\\ &= \Big[\begin{pmatrix}\ell+d-1\\ \ell\end{pmatrix}^{-1}\frac{1}{(2\ell+d)^{d+1}}\Big]^{\frac12}\int_{\mathbb{H}^d} e^{-is\lambda}\widetilde{\mathcal{W}}(\ell,\lambda,Y)\, f(Y,s)\,dY\,ds\,.\end{aligned} \tag{4.13}$$

This amounts to proving that the operator T defined by

$$Tf \stackrel{\rm def}{=} \Big(\widetilde{\Theta}\Big(\ell,\ell,\frac{1}{2\ell+d}\Big)\Big)_{\ell\in\mathbb{N}}\,,$$

with $\widetilde{\Theta}$ obtained from f through (4.13), is bounded from $L^{2,1}(\mathbb{H}^d)$ into $\ell^2(\mathbb{N})$, or equivalently that its adjoint T^* is bounded from $\ell^2(\mathbb{N})$ into $L^{2,\infty}(\mathbb{H}^d)$.

Now for any sequence $\underline{a} = (a_\ell)_{\ell\in\mathbb{N}}$ in $\ell^2(\mathbb{N})$, the operator T^* is given by

$$T^*(\underline{a})(Y,s) = \sum_{\ell=0}^{\infty} a_\ell\, e^{-is\frac{1}{2\ell+d}}\, K\Big(\ell,\frac{1}{2\ell+d},Y\Big)\,,$$

with

$$K\Big(\ell,\frac{1}{2\ell+d},Y\Big) \stackrel{\text{def}}{=} \Big[\binom{\ell+d-1}{\ell}^{-1}\frac{1}{(2\ell+d)^{d+1}}\Big]^{\frac12}\widetilde{\mathcal{W}}(\ell,\lambda,Y)\,.$$

But by Lemma 4.2 in [35], we know that for all ℓ, m in $\mathbb{N}$

$$\int_{T^\star\mathbb{R}^d}\Big|K\Big(\ell,\frac{1}{2\ell+d},Y\Big)\Big|\,\Big|K\Big(m,\frac{1}{2m+d},Y\Big)\Big|dY=\mathcal{O}\Big(\frac{1}{\max(\ell,m)}\Big)\,. \tag{4.14}$$

We deduce that

$$\|T^*(\underline{a})\|_{L^{2,\infty}(\mathbb{H}^d)}\lesssim\sum_{\ell\leq m}\frac{|a_\ell||a_m|}{m}=\sum_m|a_m|b_m\,,$$

with $b_m \stackrel{\text{def}}{=} \dfrac{1}{m}\displaystyle\sum_{\ell\leq m}|a_\ell|$. This ensures the result thanks to the following Hardy inequality (see [3]):

$$\|\underline{b}\|_{\ell^p(\mathbb{N})}\leq C_p\|\underline{a}\|_{\ell^p(\mathbb{N})}\,, \tag{4.15}$$

available for all $1<p<\infty$, which achieves the proof of Theorem 4. □

5. Proof of Strichartz estimates

5.1. **Restriction theorem on $\widehat{\mathbb{R}}\times\widehat{\mathbb{H}}^d$.** As we explained in Section 2 (cf. in particular Remark 2.1), to apply efficiently restriction estimates to PDEs to get Strichartz estimates, one has to investigate the Tomas-Stein restriction theorem on submanifolds of the product $\widehat{\mathbb{R}}\times\widehat{\mathbb{H}}^d$, where $\widehat{\mathbb{R}}$ stands for the dual group of $\mathbb{R}$. In the following we set

$$\mathbb{D}\stackrel{\text{def}}{=}\mathbb{R}\times\mathbb{H}^d\quad\text{and}\quad\widehat{\mathbb{D}}\stackrel{\text{def}}{=}\widehat{\mathbb{R}}\times\widehat{\mathbb{H}}^d\,.$$

5.1.1. *The Fourier transform on $\mathbb{R}\times\mathbb{H}^d$.* A combination of Fourier analysis on the real line and on the Heisenberg group leads directly to an efficient Fourier theory on $\mathbb{D}$ whose Haar measure is obviously nothing else than the Lebesgue measure. We define the Fourier transform of f in $L^1(\mathbb{D})$ as follows:

$$\mathcal{F}_{\mathbb{D}}f(\alpha,\widehat{w})\stackrel{\text{def}}{=}\int_{\mathbb{D}}\overline{e^{it\alpha}e^{is\lambda}\mathcal{W}(\widehat{w},Y)}\,f(t,Y,s)\,dt\,dY\,ds\,, \tag{5.1}$$

for any $(\alpha,\widehat{w})\in\widehat{\mathbb{D}}$. The Fourier transform $\mathcal{F}_{\mathbb{D}}$ inherits all the properties of $\mathcal{F}_{\mathbb{H}}$ and of the Fourier transform on the real line $\mathcal{F}$. In particular, the inversion and Fourier-Plancherel formulae take the following forms:

$$f(t,w)=\frac{2^{d-2}}{\pi^{d+2}}\int_{\widehat{\mathbb{D}}}e^{it\alpha}e^{is\lambda}\mathcal{W}(\widehat{w},Y)\mathcal{F}_{\mathbb{D}}f(\alpha,\widehat{w})\,d\alpha\,d\widehat{w} \tag{5.2}$$

and

$$(\mathcal{F}_{\mathbb{D}}f|\mathcal{F}_{\mathbb{D}}g)_{L^2(\widehat{\mathbb{D}})}=\frac{\pi^{d+2}}{2^{d-2}}(f|g)_{L^2(\mathbb{D})}\,. \tag{5.3}$$

In the sequel, we shall say that a function f on $\mathbb{D}$ is radial if it is invariant under the action of $U(d)$, in the sense that for any R of $U(d)$ and any (t,Y,s) of $\mathbb{D}$, we have

$$f(t,R(Y),s)=f(t,Y,s)\,.$$

It readily stems from Relation (3.12) that if f belongs to $L^1_{\rm rad}(\mathbb{D})$, then for all $(\alpha,\widehat{w})\in\widehat{\mathbb{D}}$,

$$\mathcal{F}_{\mathbb{D}}(f)(\alpha,n,m,\lambda)=\mathcal{F}_{\mathbb{D}}(f)(\alpha,|n|,|n|,\lambda)\delta_{n,m}\,. \tag{5.4}$$

To avoid any confusion, we shall denote in what follows by $\star_{\mathbb{D}}$ the noncommutative convolution product on $\mathbb{D}$, namely

$$f\star_{\mathbb{D}}g(t,w)\stackrel{\text{def}}{=}\int_{\mathbb{D}}f(t-t',w\cdot v^{-1})g(t',v)\,dt'\,dv\,, \tag{5.5}$$

which of course enjoys Young's inequalities and satisfies

$$\mathcal{F}_{\mathbb{D}}(f \star_{\mathbb{D}} g)(\alpha, \widehat{w}) = \sum_{p\in\mathbb{N}^d} \mathcal{F}_{\mathbb{D}} f(\alpha, n, p, \lambda)\mathcal{F}_{\mathbb{D}} g(\alpha, p, m, \lambda)\,. \tag{5.6}$$

Patching Fourier analysis on the real line and on the Heisenberg group, one can easily check that $L^1_{\rm rad}(\mathbb{D})$ is a commutative sub-algebra of $L^1(\mathbb{D})$ where (5.6) reduces to

$$\mathcal{F}_{\mathbb{D}}(f \star_{\mathbb{D}} g)(\alpha, |n|, |n|, \lambda) = \mathcal{F}_{\mathbb{D}} f(\alpha, |n|, |n|, \lambda)\mathcal{F}_{\mathbb{D}} g(\alpha, |n|, |n|, \lambda)\,. \tag{5.7}$$

Besides, it is worth noticing that $\mathcal{F}_{\mathbb{D}}$ is a bicontinuous isomorphism between the space $\mathcal{S}(\mathbb{D})$ (which coincides with $\mathcal{S}(\mathbb{R}^{2d+2})$) and $\mathcal{S}(\widehat{\mathbb{D}})$ — which can be defined naturally from the definition of $\mathcal{S}(\widehat{\mathbb{H}}^d)$. The map $\mathcal{F}_{\mathbb{D}}$ can then be continuously extended from $\mathcal{S}'(\mathbb{D})$ into $\mathcal{S}'(\widehat{\mathbb{D}})$ by duality according to the following formula:

$$\mathcal{F}_{\mathbb{D}} : \begin{cases} \mathcal{S}'(\mathbb{D}) & \longrightarrow \quad \mathcal{S}'(\widehat{\mathbb{D}}) \\ \quad T & \longmapsto \quad \Big[\theta \mapsto \langle T, {}^t\mathcal{F}_{\mathbb{D}}\theta\rangle_{\mathcal{S}'(\mathbb{D})\times\mathcal{S}(\mathbb{D})}\Big]\,, \end{cases}$$

with

$$ {}^t\mathcal{F}_{\mathbb{D}}\theta(t, y, \eta, s) \stackrel{\rm def}{=} \frac{\pi^{d+2}}{2^{d-2}} (\mathcal{F}_{\mathbb{D}}^{-1}\theta)(-t, y, -\eta, -s)\,. \tag{5.8}$$

5.1.2. *A surface measure.* Let us define the set

$$\Sigma \stackrel{\rm def}{=} \Big\{(\alpha, \widehat{w}) = (\alpha, (n, n, \lambda)) \in \widehat{\mathbb{D}} \,/\, \alpha = 4|\lambda|(2|n| + d)\Big\}\,. \tag{5.9}$$

We endow Σ with the measure $d\Sigma$ induced by the projection $\pi : \widehat{\mathbb{R}} \times \widehat{\mathbb{H}}^d \to \widehat{\mathbb{H}}^d$ onto the second factor. Following the notation of Section 2, $d\Sigma = (\pi|_\Sigma)^{-1}_\sharp d\widehat{w}$. More explicitly, recalling (4.4), for all Θ in $\mathcal{S}(\widehat{\mathbb{D}})$, we have

$$\begin{aligned}\langle d\Sigma, \Theta\rangle_{\mathcal{S}'(\widehat{\mathbb{D}})\times\mathcal{S}(\widehat{\mathbb{D}})} &= \int_\Sigma \Theta(\alpha, \widehat{w})\, d\Sigma(\alpha, \widehat{w}) \\ &\stackrel{\rm def}{=} \sum_{n\in\mathbb{N}^d} c_n^{d+1} \int_0^\infty \Big(\Theta(\alpha, n, n, \alpha c_n) + \Theta(\alpha, n, n, -\alpha c_n)\Big)\, \alpha^d\, d\alpha\,,\end{aligned}$$

where to simplify notation we have set

$$c_n \stackrel{\rm def}{=} (4(2|n| + d))^{-1}\,.$$

Notice that if $\Theta : \Sigma \subset \widehat{\mathbb{D}} \to \mathbb{C}$ is defined as $\Theta = \theta \circ \pi|_\Sigma$, where $\theta : \widehat{\mathbb{H}}^d \to \mathbb{C}$, then by construction for all $1 \leq p \leq \infty$

$$\|\Theta\|_{L^p(\Sigma, d\Sigma)} = \|\theta\|_{L^p(\widehat{\mathbb{H}}^d)}\,. \tag{5.10}$$

Our purpose here is to show that every (appropriate) function f has a Fourier transform $\mathcal{F}_{\mathbb{D}} f$ that restricts to Σ. Actually as in the classical case, this restriction property is best dealt with in compact subsets of Σ. Thus, we shall consider Σ endowed with the surface measure $d\Sigma_{\rm loc} \stackrel{\rm def}{=} \psi(\alpha) d\Sigma$ defined by

$$\begin{aligned}\int_\Sigma \Theta(\alpha, \widehat{w})\, d\Sigma_{\rm loc}(\alpha, \widehat{w}) \stackrel{\rm def}{=} &\sum_{n\in\mathbb{N}^d} c_n^{d+1} \\ &\times \int_0^\infty \Big(\Theta(\alpha, n, n, \alpha c_n) + \Theta(\alpha, n, n, -\alpha c_n)\Big)\, \alpha^d\, \psi(\alpha)\, d\alpha\,,\end{aligned} \tag{5.11}$$

with ψ any smooth, nonnegative, even function, compactly supported in $\mathbb{R}$ with an L^∞ norm at most 1.

Proceeding as for the restriction theorem on the sphere of $\widehat{\mathbb{H}}^d$, let us first compute

$$G_{\Sigma_{\rm loc}} \stackrel{\rm def}{=} \mathcal{F}_{\mathbb{D}}^{-1}(d\Sigma_{\rm loc})\,.$$

Proposition 5.1. *With the above notation, $G_{\Sigma_{\rm loc}}$ is the bounded function on $\mathbb{D}$ defined by*

$$G_{\Sigma_{\rm loc}}(t,w) = 2\pi \int_0^\infty G_\alpha(w)\, e^{-it\,\alpha}\psi(\alpha)\, d\alpha\,, \tag{5.12}$$

where G_R is given by (4.10).

Proof. Arguing as in the proof of Proposition 4.1 and using the fact that the Fourier transform $\mathcal{F}_{\mathbb{D}}$ is a bicontinuous isomorphism between the spaces $\mathcal{S}(\mathbb{D})$ and $\mathcal{S}(\widehat{\mathbb{D}})$, there holds (with $\Theta \stackrel{\rm def}{=} \mathcal{F}_{\mathbb{D}} f$)

$$\langle d\Sigma_{\rm loc}, \Theta\rangle_{\mathcal{S}'(\widehat{\mathbb{D}})\times\mathcal{S}(\widehat{\mathbb{D}})} = \Bigl(\frac{\pi}{2}\Bigr)^d \langle G_{\Sigma_{\rm loc}}, \widetilde f\rangle_{\mathcal{S}'(\mathbb{D})\times\mathcal{S}(\mathbb{D})}\,, \tag{5.13}$$

with $\widetilde f(t,y,\eta,s) \stackrel{\rm def}{=} f(-t,y,-\eta,-s)$. By definition, we have for any non negative real number α

$$\Theta(\alpha,n,n,\pm\alpha c_n) = \int_{\mathbb{D}} e^{\mp is\alpha c_n}\, \mathcal{W}\Bigl(n,n,1,\alpha^{\frac12}\sqrt{c_n}\,Y\Bigr)\, e^{-it\,\alpha} f(t,Y,s)\, dt\, dY\, ds\,,$$

which implies in view of (5.11) that

$$\begin{aligned}\langle d\Sigma_{\rm loc}, \Theta\rangle_{\mathcal{S}'(\widehat{\mathbb{D}})\times\mathcal{S}(\widehat{\mathbb{D}})} &= \sum_{n\in\mathbb{N}^d} 2c_n^{d+1}\\ &\quad\times \int_0^\infty\!\!\int_{\mathbb{D}} \cos\Bigl(s\alpha c_n\Bigr)\, \mathcal{W}\Bigl(n,n,1,\alpha^{\frac12}\sqrt{c_n}\,Y\Bigr)\, e^{-it\,\alpha} f(t,Y,s)\, dt\, dY\, ds\, \alpha^d\psi(\alpha)\, d\alpha\,.\end{aligned}$$

This achieves the proof of the result thanks to the Fubini theorem and Formula (4.10), after an obvious change of variables. □

5.1.3. *Restriction theorem.* Our aim now is to establish the following restriction result for the "dual set" $\Sigma_{\rm loc}$ of $\widehat{\mathbb{D}}$ defined by (5.9) and endowed with the measure $d\Sigma_{\rm loc}$.

Theorem 5. *If $1\leq q\leq p\leq 2$, then*

$$\|\mathcal{F}_{\mathbb{D}}(f)_{|\Sigma_{\rm loc}}\|_{L^2(d\Sigma_{\rm loc})} \leq C_{p,q}\|f\|_{L^1_s(\mathbb{R},L^q_t(\mathbb{R},L^p(T^*\mathbb{R}^d))}\,, \tag{5.14}$$

for all radial functions f in $\mathcal{S}_{\rm rad}(\mathbb{R}\times\mathbb{H}^d)$.

Remark 5.2. *By duality, Theorem 5 may be rephrased as follows : for any $2\leq p'\leq q'\leq \infty$, there holds*

$$\|\mathcal{F}_{\mathbb{D}}^{-1}(\Theta_{|\Sigma_{\rm loc}})\|_{L^\infty_s(\mathbb{R},L^{q'}_t(\mathbb{R},L^{p'}(T^*\mathbb{R}^d))} \leq C_{p,q}\|\Theta_{|\Sigma_{\rm loc}}\|_{L^2(d\Sigma_{\rm loc})}\,, \tag{5.15}$$

for all $\Theta\in\mathcal{F}_{\mathbb{D}}(\mathcal{S}_{\rm rad}(\mathbb{R}\times\mathbb{H}^d))$.

Proof of Theorem 5. We handle differently the cases $1\leq p<2$ and $p=2$. To undertake the case $1\leq p<2$, we shall follow the Euclidean strategy outlined in Section 2. To this end, let us introduce $R_{\Sigma_{\rm loc}}$ the restriction operator on $\Sigma_{\rm loc}$ defined for any function f in $\mathcal{S}(\mathbb{D})$ by

$$R_{\Sigma_{\rm loc}} f \stackrel{\rm def}{=} \mathcal{F}_{\mathbb{D}}(f)_{|\Sigma_{\rm loc}} = \Bigl(\int_{\mathbb{D}} \overline{e^{is\lambda+it\alpha}\mathcal{W}(\widehat w, Y)}\, f(t,Y,s)\, dY\, ds\, dt\Bigr)_{|\Sigma_{\rm loc}}\,,$$

and by $R^*_{\Sigma_{\rm loc}}$ its adjoint. By definition

$$\langle R_{\Sigma_{\rm loc}} f, R_{\Sigma_{\rm loc}} f\rangle_{L^2(\Sigma_{\rm loc})} = \sum_{n\in\mathbb{N}^d} c_n^{d+1}\int_0^\infty \Big(\big|\Theta(\alpha,n,n,\alpha c_n)\big|^2 + \big|\Theta(\alpha,n,n,-\alpha c_n)\big|^2\Big)\psi(\alpha)\alpha^d\,d\alpha\,,$$

with $\Theta \stackrel{\rm def}{=} \mathcal{F}_{\mathbb{D}}(f)$, which readily implies that

$$\begin{aligned}(5.16)\quad \langle R_{\Sigma_{\rm loc}} f, R_{\Sigma_{\rm loc}} f\rangle_{L^2(\Sigma_{\rm loc})} &= \sum_{n\in\mathbb{N}^d} c_n^{d+1}\\ \times \int_0^\infty \Big[\Theta(\alpha,n,n,\alpha c_n)&\int_{\mathbb{D}} e^{is\alpha c_n+it\alpha}\mathcal{W}\Big(n,n,\alpha c_n,Y\Big)\,\overline{f(t,Y,s)}\,dt\,dY\,ds\\ +\Theta(\alpha,n,n,-\alpha c_n)&\int_{\mathbb{D}} e^{-is\alpha c_n+it\alpha}\mathcal{W}(n,n,-\alpha c_n,Y)\,\overline{f(t,Y,s)}\,dt\,dY\,ds\Big]\psi(\alpha)\alpha^d\,d\alpha\,.\end{aligned}$$

In view of (5.11), this leads to

$$(5.17)\qquad \langle R_{\Sigma_{\rm loc}} f, R_{\Sigma_{\rm loc}} f\rangle_{L^2(\Sigma_{\rm loc})} = \langle R^*_{\Sigma_{\rm loc}} R_{\Sigma_{\rm loc}} f, f\rangle_{L^2(\mathbb{D})}\,,$$

with (for $w=(Y,s)$)

$$(5.18)\qquad (R^*_{\Sigma_{\rm loc}} R_{\Sigma_{\rm loc}} f)(t,w) = \int_\Sigma e^{is\lambda+it\alpha}\mathcal{W}(\widehat w,Y)\mathcal{F}_{\mathbb{D}}(f)(\alpha,\widehat w)d\Sigma_{\rm loc}(\alpha,\widehat w)\,.$$

Since f belongs to $\mathcal{S}_{\rm rad}(\mathbb{D})$, combining (3.12) together with (3.13) and (3.15) we infer that the operator $R^*_{\Sigma_{\rm loc}} R_{\Sigma_{\rm loc}}$ writes in the radial setting[7] :

$$(5.19)\qquad (R^*_{\Sigma_{\rm loc}} R_{\Sigma_{\rm loc}} f)(t,Y,s) = \int_\Sigma \mathcal{F}_{\mathbb{D}}(f\circ\tau_{(-t,Y,-s)})(\alpha,\widehat w)\;d\Sigma_{\rm loc}(\alpha,\widehat w)\,.$$

By (3.20) and (5.13), this gives rise to

$$(5.20)\qquad (R^*_{\Sigma_{\rm loc}} R_{\Sigma_{\rm loc}} f)(t,Y,s) = \Big(\frac{\pi}{2}\Big)^d (G_{\Sigma_{\rm loc}}\star_{\mathbb{D}}\check f)(-t,-Y,s)\,,$$

for all f in $\mathcal{S}_{\rm rad}(\mathbb{D})$.

Now applying the Hölder inequality to (5.17), we deduce that

$$\begin{aligned}\|R_{\Sigma_{\rm loc}} f\|^2_{L^2(\Sigma_{\rm loc})} &\leq \|R^*_{\Sigma_{\rm loc}} R_{\Sigma_{\rm loc}} f\|_{L^\infty_s(\mathbb{R},L^{q'}_t(\mathbb{R},L^{p'}(T^*\mathbb{R}^d)))}\|f\|_{L^1_s(\mathbb{R},L^q_t(\mathbb{R},L^p(T^*\mathbb{R}^d)))}\\ &\leq C_d\|\check f\star_{\mathbb{D}} G_{\Sigma_{\rm loc}}\|_{L^\infty_s(\mathbb{R},L^{q'}_t(\mathbb{R},L^{p'}(T^*\mathbb{R}^d)))}\|f\|_{L^1_s(\mathbb{R},L^q_t(\mathbb{R},L^p(T^*\mathbb{R}^d)))}\,,\end{aligned}$$

for some irrelevant constant C_d which may change from line to line. Then as in the Euclidean case, to complete the proof of Estimate (5.14), we are reduced to proving that $R^*_{\Sigma_{\rm loc}} R_{\Sigma_{\rm loc}}$ is bounded from $L^1_s(\mathbb{R},L^q_t(\mathbb{R},L^p(T^*\mathbb{R}^d)))$ into $L^\infty_s(\mathbb{R},L^{q'}_t(\mathbb{R},L^{p'}(T^*\mathbb{R}^d)))$. For that purpose, let us start by observing that in light of (4.10) and (5.12), we have

$$f\star_{\mathbb{D}} G_{\Sigma_{\rm loc}}(t,Y,s) = C_d\sum_\ell \frac{1}{(2\ell+d)^{d+1}} \times\sum_\pm\int_0^\infty f\star_{\mathbb{D}}\Big(e^{\pm i\alpha_\ell-it\alpha}\widetilde{\mathcal{W}}\Big(\ell,\alpha_\ell,Y\Big)\Big)\alpha^d\,\psi(\alpha)d\alpha\,,$$

where $\widetilde{\mathcal{W}}$ is given by (3.13) and where we have defined

$$\alpha_\ell \stackrel{\rm def}{=} \frac{\alpha}{4(2\ell+d)}\,.$$

[7]where of course $(f\circ\tau_{(\tau,w)})(t,v) = f(t+\tau,w\cdot v)$.

An easy computation shows that for any real number λ,

$$f \star_{\mathbb{D}} \left(e^{\pm i\lambda s - it\alpha}\widetilde{\mathcal{W}}(\ell,\lambda,Y)\right) = e^{\pm i\lambda s - it\alpha}\left(f^{\alpha,\pm\lambda} \star_{\pm\lambda} \widetilde{\mathcal{W}}(\ell,\lambda,Y)\right),$$

where $\star_\lambda$ denotes the twisted convolution defined in (A.1), and with

$$f^{\alpha,\beta}(Y) \stackrel{\text{def}}{=} \int e^{is\beta+it\alpha} f(t,Y,s)\,dtds\,. \tag{5.21}$$

Defining $\psi_+(\alpha) \stackrel{\text{def}}{=} \psi(\alpha)\mathbf{1}_{\alpha>0}$, it follows that

$$\begin{aligned} f \star_{\mathbb{D}} G_{\Sigma_{\rm loc}}(t,s,Y) &= C_d \sum_\ell \frac{1}{(2\ell+d)^{d+1}} \\ &\qquad \times \sum_\pm \int_0^\infty e^{\pm is\alpha_\ell - it\alpha}\Big(f^{\alpha,\pm\alpha_\ell} \star_{\pm\alpha_\ell} \widetilde{\mathcal{W}}(\ell,\alpha_\ell,Y)\Big)\alpha^d\,\psi(\alpha)\,d\alpha \\ &= C_d \sum_\ell\sum_\pm \frac{1}{(2\ell+d)^{d+1}}\mathcal{F}_{\alpha\to t}\Big(e^{\pm is\alpha_\ell}\alpha^d\psi_+(\alpha)\Big(f^{\alpha,\pm\alpha_\ell} \star_{\pm\alpha_\ell} \widetilde{\mathcal{W}}(\ell,\alpha_\ell,Y)\Big)\Big)\,. \end{aligned}$$

Now let us fix $s \in \mathbb{R}$ and $Y \in T^*\mathbb{R}^d$, and compute the $L_t^{q'}(\mathbb{R})$ norm of the function of t appearing on the right-hand side of the equality, for $q' \geq 2$: by the Hausdorff-Young inequality we find

$$\|f \star_{\mathbb{D}} G_{\Sigma_{\rm loc}}\|_{L_t^{q'}(\mathbb{R})} \lesssim \sum_\ell\sum_\pm \frac{1}{(2\ell+d)^{d+1}}\Big\|e^{\pm is\alpha_\ell}\alpha^d\psi(\alpha)\Big(f^{\alpha,\pm\alpha_\ell} \star_{\pm\alpha_\ell} \widetilde{\mathcal{W}}(\ell,\alpha_\ell,Y)\Big)\Big\|_{L^q_\alpha(\mathbb{R}^+)}\,.$$

Now noticing that as soon as $q' \geq p' \geq 2$, thanks to Minkowski's inequality we have for any function g defined on $\mathbb{R} \times T^*\mathbb{R}^d$

$$\begin{aligned} \|\mathcal{F}_{\alpha\to t} g\|_{L^{q'}(\mathbb{R},L^{p'}(T^*\mathbb{R}^d))} &\leq \|\mathcal{F}_{\alpha\to t} g\|_{L^{p'}(T^*\mathbb{R}^d,L^{q'}(\mathbb{R}))} \\ &\lesssim \|g\|_{L^{p'}(T^*\mathbb{R}^d,L^{q}(\mathbb{R}))} \\ &\lesssim \|g\|_{L^{q}(\mathbb{R},L^{p'}(T^*\mathbb{R}^d))}\,, \end{aligned} \tag{5.22}$$

we deduce that for $q' \geq p' > 2$ (since we assumed here that $1 \leq p < 2$)

$$\begin{aligned} \|f \star_{\mathbb{D}} G_{\Sigma_{\rm loc}}\|_{L_s^\infty(\mathbb{R},L_t^{q'}(\mathbb{R},L^{p'}(T^*\mathbb{R}^d)))} &\lesssim \sum_\ell\sum_\pm \frac{1}{(2\ell+d)^{d+1}} \\ &\qquad \times \Big\|\alpha^d\psi(\alpha)\Big(f^{\alpha,\pm\alpha_\ell} \star_{\pm\alpha_\ell} \widetilde{\mathcal{W}}(\ell,\alpha_\ell,Y)\Big)\Big\|_{L^q_\alpha(\mathbb{R},L^{p'}(T^*\mathbb{R}^d))}\,. \end{aligned}$$

But by (A.2), there holds

$$\Big\|\Big(f^{\alpha,\pm\frac{\alpha}{4(2\ell+d)}} \star_{\pm\frac{\alpha}{4(2\ell+d)}} \widetilde{\mathcal{W}}\Big(\ell,\frac{\alpha}{4(2\ell+d)},Y\Big)\Big)\Big\|_{L^{p'}(T^*\mathbb{R}^d)} \lesssim \ell^{d-1+\frac{2}{p'}}\,\alpha^{-\frac{2d}{p'}}\|\mathcal{F}_{\mathbb{R}}f(\alpha,\cdot)\|_{L^{p,1}(\mathbb{H}^d)}\,,$$

which (since $p' > 2$) implies that

$$\|f \star_{\mathbb{D}} G_{\Sigma_{\rm loc}}\|_{L_s^\infty(\mathbb{R},L_t^{q'}(\mathbb{R},L^{p'}(T^*\mathbb{R}^d)))} \lesssim \Big\|\|\mathcal{F}_{\mathbb{R}}(f)(\alpha,\cdot)\|_{L^{p,1}(\mathbb{H}^d)}\,\alpha^{d(1-\frac{2}{p'})}\,\psi(\alpha)\Big\|_{L^q_\alpha(\mathbb{R})}\,.$$

Then, applying a Hölder estimate in α followed by the Hausdorff-Young inequality, we get for any $a \geq 2$

$$\begin{aligned} \|f \star_{\mathbb{D}} G_{\Sigma_{\rm loc}}\|_{L_s^\infty(\mathbb{R},L_t^{q'}(\mathbb{R},L^{p'}(T^*\mathbb{R}^d)))} &\lesssim \|\mathcal{F}_{\mathbb{R}}(f)\|_{L^a_\alpha(\mathbb{R},L^{p,1}(\mathbb{H}^d))}\|\alpha^{d(1-\frac{2}{p'})}\,\psi(\alpha)\|_{L^b_\alpha(\mathbb{R})} \\ &\lesssim \|f\|_{L_t^{a'}(\mathbb{R},L^{p,1}(\mathbb{H}^d))}\|\alpha^{d(1-\frac{2}{p'})}\,\psi(\alpha)\|_{L^b_\alpha(\mathbb{R})}\,, \end{aligned}$$

where of course a' is the conjugate exponent of a and $\dfrac{1}{a}+\dfrac{1}{b}=\dfrac{1}{q}\cdot$

Finally selecting $a' = q$ and thanks again to Minkowski's inequality, we get for all real numbers $q' \geq p' > 2$

$$\|f \star_{\mathbb{D}} G_{\Sigma_{\rm loc}}\|_{L^\infty_s(\mathbb{R}, L^{q'}_t(\mathbb{R}, L^{p'}(T^*\mathbb{R}^d))} \lesssim \|f\|_{L^1_s(\mathbb{R}, L^q_t(\mathbb{R}, L^p(T^*\mathbb{R}^d))}$$

which completes the proof of the result in the case when $1 \leq q \leq p < 2$.

Finally the proof in the case when $p = 2$ is in the same spirit than the one concerning the unit dual sphere of the Heisenberg group already outlined in the proof of Theorem 4. By definition, our aim here is to show that (for $q \leq 2$)

$$\sum_{\ell \in \mathbb{N}} \int_0^\infty \Big(\Big| \widetilde\Theta\Big(\alpha, \ell, \ell, \frac{\alpha}{2\ell + d}\Big) \Big|^2 + \Big| \widetilde\Theta\Big(\alpha, \ell, \ell, \frac{-\alpha}{2\ell + d}\Big) \Big|^2 \Big) \psi(\alpha) \alpha^d \, d\alpha \lesssim \|f\|^2_{L^1_s(\mathbb{R}, L^q_t(\mathbb{R}, L^2(T^*\mathbb{R}^d)))} ,$$

where with the notation introduced in (4.13) and on page 17

$$\begin{aligned} \widetilde\Theta\Big(\alpha, \ell, \ell, \lambda\Big) &\stackrel{\rm def}{=} \Big[\binom{\ell + d - 1}{\ell}^{-1} \frac{1}{(2\ell + d)^{d+1}} \Big]^{\frac12} \sum_{\substack{n \in \mathbb{N}^d \\ |n| = \ell}} \mathcal{F}_{\mathbb{D}} f(\alpha, n, n, \lambda) \\ &= \int_{\mathbb{D}} e^{-it\alpha} e^{-is\lambda} K(\ell, \lambda, Y) \, f(t, Y, s) dt \, dY \, ds \, . \end{aligned} \tag{5.23}$$

For that purpose, let us establish that the operator $T_{\mathbb{D}}$ defined on $\mathcal{S}_{\rm rad}(\mathbb{D})$ by

$$T_{\mathbb{D}} f \stackrel{\rm def}{=} \Big(\widetilde\Theta(\alpha, \ell, \ell, \alpha_\ell) \Big)_{\ell \in \mathbb{N}} ,$$

where f is related to Θ through (5.23), is bounded from $L^1_s(\mathbb{R}, L^q_t(\mathbb{R}, L^2(T^*\mathbb{R}^d)))$ into the space $L^2(\mathbb{N} \times \mathbb{R})$ endowed with the measure $\ell^2(\mathbb{N}) \otimes L^2(\mathbb{R}^+, \psi(\alpha)\alpha^d \, d\alpha)$, or equivalently that its adjoint $T^*_{\mathbb{D}}$ is bounded from $L^2(\mathbb{N} \times \mathbb{R})$ into $L^\infty_s(\mathbb{R}, L^{q'}_t(\mathbb{R}, L^2(T^*\mathbb{R}^d)))$.

For $\underline{a}(\alpha) = (a_\ell(\alpha))_{\ell \in \mathbb{N}}$ in $L^2(\mathbb{N} \times \mathbb{R})$, the operator $T^*_{\mathbb{D}}$ is given by

$$T^*_{\mathbb{D}}(\underline{a})(t, Y, s) = \sum_{\ell = 0}^\infty \int_0^\infty a_\ell(\alpha) \, e^{-it\alpha} \, e^{-is\alpha_\ell} \, K(\ell, \alpha_\ell, Y) \psi(\alpha) \alpha^d \, d\alpha \, .$$

Combining (5.22) together with the Hausdorff-Young inequality, we find that for any fixed s in $\mathbb{R}$, there holds

$$\|T^*_{\mathbb{D}}(\underline{a})(\cdot, \cdot, s)\|_{L^{q'}_t(\mathbb{R}, L^2(T^*\mathbb{R}^d))} \lesssim \|g\|_{L^q(\mathbb{R}, L^2(T^*\mathbb{R}^d))} ,$$

where

$$g(\alpha, Y) \stackrel{\rm def}{=} \alpha^d \psi(\alpha) \sum_{\ell = 0}^\infty a_\ell(\alpha) \, K(\ell, \alpha_\ell, Y) \, .$$

Now taking advantage of (4.14) and performing an obvious change of variable, we get

$$\Big\| \sum_{\ell = 0}^\infty a_\ell(\alpha) \, K(\ell, \ell, \alpha_\ell, Y) \Big\|^2_{L^2(T^*\mathbb{R}^d)} \lesssim \alpha^d \sum_m |a_m(\alpha)| b_m(\alpha) ,$$

where of course $b_m(\alpha) \stackrel{\rm def}{=} \dfrac{1}{m} \displaystyle\sum_{\ell \leq m} |a_\ell(\alpha)|$. Thanks to the Hardy inequality (4.15), this ensures that

$$\|g(\alpha, \cdot)\|_{L^2(T^*\mathbb{R}^d)} \lesssim \alpha^{\frac{3d}{2}} \psi(\alpha) \|\underline{a}(\alpha)\|_{\ell^2(\mathbb{N})} ,$$

which by Hölder's inequality gives rise to

$$\|g\|_{L^q(\mathbb{R}, L^2(T^*\mathbb{R}^d))} \lesssim \|\alpha^d \psi^{\frac12}(\alpha)\|_{L^{\frac{2-q}{2q}}(\mathbb{R})} \|\underline{a}\|_{L^2(\mathbb{N} \times \mathbb{R})} \, .$$

This achieves the proof of the Fourier restriction estimate (5.14). □

Remark 5.3. *In the case of the wave equation, we consider the sets*

$$\Sigma_\pm \stackrel{\text{def}}{=} \Big\{(\alpha,\widehat{w}) = (\alpha,(n,n,\lambda)) \in \widehat{\mathbb{D}} \,/\, \alpha^2 = 4|\lambda|(2|n|+d)\,,\ \pm\alpha>0\Big\}\,. \tag{5.24}$$

Each of those is endowed with the measure induced by the projection $\pi : \widehat{\mathbb{R}} \times \widehat{\mathbb{H}}^d \to \widehat{\mathbb{H}}^d$ *onto the second factor. Explicitly, for all* Θ *in* $\mathcal{S}(\widehat{\mathbb{D}})$*, we have*

$$\begin{aligned}\langle d\Sigma_\pm, \Theta\rangle_{\mathcal{S}'(\widehat{\mathbb{D}})\times\mathcal{S}(\widehat{\mathbb{D}})} &= \int_\Sigma \Theta(\alpha,\widehat{w})\, d\Sigma(\alpha,\widehat{w}) \\ &\stackrel{\text{def}}{=} \sum_{n\in\mathbb{N}^d} c_n^{d+1}\int_0^\infty \Big(\Theta(\alpha,n,n,\alpha^2 c_n) + \Theta(\alpha,n,n,-\alpha^2 c_n)\Big)\,\alpha^d\, d\alpha\,.\end{aligned}$$

Following the same argument as above, one proves (5.15) *for the corresponding localized measures.*

5.2. **Strichartz estimates for the Schrödinger operator on $\mathbb{H}^d$.** The aim of this section is to prove the Strichartz estimate stated in Theorem 1. By duality arguments, one can reduce to the free Schrödinger equation. Let u_0 be a function in $\mathcal{S}_{\text{rad}}(\mathbb{H}^d)$ and consider the Cauchy problem

$$(S_{\mathbb{H}}) \qquad \begin{cases} i\partial_t u - \Delta_{\mathbb{H}} u = 0 \\ u_{|t=0} = u_0\,. \end{cases}$$

As in the Euclidean case, Fourier analysis allows us to explicitly solve $(S_{\mathbb{H}})$. More precisely, taking the partial Fourier transform with respect to the variable w in the Heisenberg group, we obtain for all $(t,\widehat{w})$ in $\mathbb{R}\times\widetilde{\mathbb{H}}^d$

$$\begin{cases} i\dfrac{d}{dt}\mathcal{F}_{\mathbb{H}}(u)(t,n,m,\lambda) &= -4|\lambda|(2|m|+d)\mathcal{F}_{\mathbb{H}}(u)(t,n,m,\lambda) \\ \mathcal{F}_{\mathbb{H}}(u)_{|t=0} &= \mathcal{F}_{\mathbb{H}} u_0\,. \end{cases}$$

By integration, this leads to

$$\mathcal{F}_{\mathbb{H}}(u)(t,n,m,\lambda) = e^{4it|\lambda|(2|m|+d)}\mathcal{F}_{\mathbb{H}}(u_0)(|n|,|n|,\lambda)\delta_{n,m}\,.$$

Then applying the inverse Fourier formula (3.4), we infer that the solution of the Cauchy problem $(S_{\mathbb{H}})$ can be expressed as follows:

$$u(t,Y,s) = \frac{2^{d-1}}{\pi^{d+1}}\int_{\widehat{\mathbb{H}}^d} e^{is\lambda}\,\mathcal{W}(\widehat{w},Y)\,e^{4it|\lambda|(2|m|+d)}\,\mathcal{F}_{\mathbb{H}}(u_0)(|n|,|n|,\lambda)\delta_{n,m}\,d\widehat{w}\,. \tag{5.25}$$

That explicit representation of the solution to $(S_{\mathbb{H}})$ can be re-expressed as the inverse Fourier transform in $\widehat{\mathbb{D}}$ of $\mathcal{F}_{\mathbb{H}}(u_0)\,d\Sigma$, where as defined in Paragraph 5.1.2, $d\Sigma$ is the measure in $\widehat{\mathbb{D}}$ given for any regular function Φ on $\widehat{\mathbb{D}}$:

$$\int_{\widehat{\mathbb{D}}} \Phi(\alpha,\widehat{w})\,d\Sigma(\alpha,\widehat{w}) = \int_{\widehat{\mathbb{H}}^d}\Phi(4|\lambda|(2|m|+d),\widehat{w})\,d\widehat{w}\,, \tag{5.26}$$

and which is determined by Formula (5.11).

Now in order to establish Estimate (1.19), let us first discuss the case of initial data frequency localized, in the sense of Definition 3.1, in the unit ball $\mathcal{B}_1$; then we shall generalize this to any ball of radius Λ by a scaling argument, and finally the result will follow by density.

So let us start by assuming that u_0 is frequency localized in the unit ball. Then by the restriction inequality (5.15), and (5.10) we have for any $2\le p\le q\le\infty$

$$\|u\|_{L^\infty_s(\mathbb{R},L^q_t(\mathbb{R},L^p(T^*\mathbb{R}^d))} \le C\|\mathcal{F}_{\mathbb{H}}u_0\|_{L^2(\widehat{\mathbb{H}}^d)} = C\|u_0\|_{L^2(\mathbb{H}^d)}\,,$$

where we used Plancherel formula. Now if u_0 is frequency localized in the ball $\mathcal{B}_\Lambda$, then by virtue of (3.10) the function

$$u_{0,\Lambda} \stackrel{\text{def}}{=} u_0 \circ \delta_{\Lambda^{-1}}$$

is frequency localized in $\mathcal{B}_1$, and it gives rise to the solution

$$u_\Lambda(t,Y,s) \stackrel{\text{def}}{=} u(\Lambda^{-2}t, \Lambda^{-1}Y, \Lambda^{-2}s)$$

of $(S_{\mathbb{H}})$. Since

$$\|u_\Lambda\|_{L^\infty_s(\mathbb{R}, L^q_t(\mathbb{R}, L^p(T^*\mathbb{R}^d)))} = \Lambda^{\frac{2}{q}+\frac{2d}{p}} \|u\|_{L^\infty_s(\mathbb{R}, L^q_t(\mathbb{R}, L^p(T^*\mathbb{R}^d)))}$$

and

$$\|u_{0,\Lambda}\|_{L^2(\mathbb{H}^d)} = \Lambda^{d+1}\|u_0\|_{L^2(\mathbb{H}^d)}\,,$$

we infer that for u_0 frequency localized in the ball $\mathcal{B}_\Lambda$, there holds

$$\|u\|_{L^\infty_s(\mathbb{R}, L^q(\mathbb{R}_t, L^p(T^*\mathbb{R}^d)))} \leq C\Lambda^{d+1-\frac{2}{q}-\frac{2d}{p}} \|u_0\|_{L^2(\mathbb{H}^d)}\,.$$

It suffices now to choose p and q satisfying

$$\frac{2}{q} + \frac{2d}{p} = d+1 = \frac{Q}{2}$$

to conclude the proof of the estimate by density of spectrally localized functions in $L^2(\mathbb{H}^d)$. This ends the proof of Theorem 1. □

5.3. **Strichartz estimates for the wave operator on $\mathbb{H}^d$.** The aim of this section is to prove the Strichartz estimate stated in Theorem 2. The method is identical to the previous section: again we reduce to the free wave equation and consider for (u_0, u_1) in $\mathcal{S}_{\rm rad}(\mathbb{H}^d)$ frequency localized in a unit ring in the sense of Definition 3.1, the Cauchy problem

$$(W_{\mathbb{H}}) \qquad \begin{cases} \partial_t^2 u - \Delta_{\mathbb{H}} u = 0 \\ (u, \partial_t u)_{|t=0} = (u_0, u_1)\,. \end{cases}$$

Taking the Fourier transform we find that for all $(t, \widehat{w})$ in $\mathbb{R} \times \widetilde{\mathbb{H}}^d$,

$$\begin{cases} \dfrac{d^2}{dt^2}\mathcal{F}_{\mathbb{H}}(u)(t,n,m,\lambda) &=& -4|\lambda|(2|m|+d)\mathcal{F}_{\mathbb{H}}(u)(t,n,m,\lambda) \\ (\mathcal{F}_{\mathbb{H}}u, \mathcal{F}_{\mathbb{H}}\partial_t u)_{|t=0} &=& (\mathcal{F}_{\mathbb{H}}u_0, \mathcal{F}_{\mathbb{H}}u_1)\,. \end{cases}$$

By integration, this leads to

$$\mathcal{F}_{\mathbb{H}}(u)(t,n,m,\lambda) = \sum_{\pm} e^{\pm 2it\sqrt{|\lambda|(2|m|+d)}} \mathcal{F}_{\mathbb{H}}(\gamma_\pm)(|n|,|n|,\lambda)\delta_{n,m}\,,$$

where similarly to (1.4) we have defined

$$\mathcal{F}_{\mathbb{H}}(\gamma_\pm)(n,n,\lambda) \stackrel{\text{def}}{=} \frac{1}{2}\Big(\mathcal{F}_{\mathbb{H}}(u_0) \pm \frac{1}{i\sqrt{4|\lambda|(2n+d)}}\mathcal{F}_{\mathbb{H}}(u_1)\Big)\,.$$

Then applying the inverse Fourier formula (3.4), we infer that the solution of the Cauchy problem $(W_{\mathbb{H}})$ can be expressed as follows:

$$u(t,Y,s) = \sum_{\pm} \frac{2^{d-1}}{\pi^{d+1}} \int_{\widehat{\mathbb{H}}^d} e^{is\lambda}\, \mathcal{W}(\widehat{w}, Y)\, e^{\pm 2it\sqrt{|\lambda|(2|m|+d)}}\, \mathcal{F}_{\mathbb{H}}(\gamma_\pm)(|n|,|n|,\lambda)\delta_{n,m}\, d\widehat{w}\,. \tag{5.27}$$

This can be written as the inverse Fourier transform in $\widehat{\mathbb{D}}$ of $\mathcal{F}_{\mathbb{H}}(\gamma_\pm)\, d\Sigma_\pm$, where we recall that by Remark 5.3, for any regular function Φ on $\widehat{\mathbb{D}}$

$$\int_{\widehat{\mathbb{D}}} \Phi(\alpha, \widehat{w})\, d\Sigma_\pm(\alpha, \widehat{w}) = \int_{\widehat{\mathbb{H}}^d} \Phi(\sqrt{4|\lambda|(2|m|+d)}, \widehat{w})\, d\widehat{w}\,. \tag{5.28}$$

By hypothesis $\gamma_\pm$ is frequency localized in a unit ring. Then for any $2 \leq p \leq q \leq \infty$, we have by virtue of the restriction inequality (5.15)

$$\|u\|_{L^\infty_s(\mathbb{R}, L^q_t(\mathbb{R}, L^p(T^\star\mathbb{R}^d)))} \leq C \|\mathcal{F}_{\mathbb{H}}^{-1}\gamma_\pm\|_{L^2(\mathbb{H}^d)}\,.$$

Next if $\gamma_\pm$ is frequency localized in the ring $\mathcal{C}_\Lambda$, then by (3.10) the function

$$u_{0,\Lambda} \stackrel{\text{def}}{=} u_0 \circ \delta_{\Lambda^{-1}}$$

is frequency localized in the ring $\mathcal{C}_1$ and gives rise to the solution

$$u_\Lambda(t, Y, s) \stackrel{\text{def}}{=} u(\Lambda^{-1}t, \Lambda^{-1}Y, \Lambda^{-2}s)$$

of $(W_{\mathbb{H}})$. Since

$$\|u_\Lambda\|_{L^\infty_s(\mathbb{R}, L^q_t(\mathbb{R}, L^p(T^\star\mathbb{R}^d)))} = \Lambda^{\frac{1}{q}+\frac{2d}{p}} \|u\|_{L^\infty_s(\mathbb{R}, L^q_t(\mathbb{R}, L^p(T^\star\mathbb{R}^d))}$$

and

$$\|u_{0,\Lambda}\|_{L^2(\mathbb{H}^d)} = \Lambda^{d+1} \|\gamma_\pm\|_{L^2(\mathbb{H}^d)}\,,$$

we infer that for u_0 frequency localized in the ring $\mathcal{C}_\Lambda$, there holds

$$\|u\|_{L^\infty_s(\mathbb{R}, L^q(\mathbb{R}_t, L^p(T^\star\mathbb{R}^d)))} \leq C\Lambda^{d+1-\frac{1}{q}-\frac{2d}{p}} \|\gamma_\pm\|_{L^2(\mathbb{H}^d)}\,.$$

Then by Bernstein's Lemma 3.2, and in particular estimate (3.9), followed by Plancherel's inequality we infer that

$$\|u\|_{L^\infty_s(\mathbb{R}, L^q(\mathbb{R}_t, L^p(T^\star\mathbb{R}^d)))} \leq C\Lambda^{d-\frac{1}{q}-\frac{2d}{p}} \Big(\|\nabla_{\mathbb{H}^d} u_0\|_{L^2(\mathbb{H}^d)} + \|u_1\|_{L^2(\mathbb{H}^d)}\Big)\,.$$

To conclude the proof of the estimate, it suffices now to choose p and q satisfying

$$\frac{1}{q} + \frac{2d}{p} = d = \frac{Q}{2} - 1$$

and to use the density in $L^2(\mathbb{H}^d)$ of functions whose Fourier transform is compactly supported away from zero. This proves Theorem 2. □

Appendix A. Proof of a technical result

We have used the following lemma due to Müller (see [35]), whose proof we sketch below.

Lemma A.1. *Defining for f, g in $\mathcal{S}(T^\star\mathbb{R}^d)$ and λ in $\mathbb{R}\setminus\{0\}$ the λ-twisted convolution*

$$(f \star_\lambda g)(Y) \stackrel{\text{def}}{=} \int_{T^\star\mathbb{R}^d} f(Y-w)g(w)e^{2i\lambda\sigma(Y,w)}dw\,, \tag{A.1}$$

with σ defined in (1.9), the following estimate holds: there exists a positive constant C_d such that

$$\|f \star_\lambda \widetilde{\mathcal{W}}(\ell,\lambda,\cdot)\|_{L^{p'}(T^\star\mathbb{R}^d)} \leq C_d\,|\lambda|^{-\frac{2d}{p'}}\,\ell^{(d-1)(1-\frac{2}{p'})}\,\|f\|_{L^p(T^\star\mathbb{R}^d)}\,, \tag{A.2}$$

for all $1 \leq p \leq 2$ and all integers $\ell \geq 1$, where the function $\widetilde{\mathcal{W}}(\ell,\lambda,Y)$ is defined by (3.13).

Proof of Lemma A.1. Let us start by establishing that for f, g in $\mathcal{S}(T^\star\mathbb{R}^d)$ and $1 \leq p \leq 2$, the following estimate holds:

$$\|f \star_\lambda g\|_{L^{p'}(T^\star\mathbb{R}^d)} \leq C_{p,d}\,|\lambda|^{-\frac{d}{p'}}\,\|f\|_{L^p(T^\star\mathbb{R}^d)} \|g\|^{\frac{2}{p'}}_{L^2(T^\star\mathbb{R}^d)} \|g\|^{1-\frac{2}{p'}}_{L^\infty(T^\star\mathbb{R}^d)}\,.$$

By definition,

$$(f \star_\lambda g)(Y) = \int_{T^\star\mathbb{R}^d} f(Y-w)g(w)e^{2i\lambda\sigma(Y,w)}dw\,,$$

which easily implies by Young's inequalities that

$$\|f \star_\lambda g\|_{L^\infty(T^\star\mathbb{R}^d)} \leq \|f\|_{L^1(T^\star\mathbb{R}^d)} \|g\|_{L^\infty(T^\star\mathbb{R}^d)}\,.$$

Therefore invoking the method of real interpolation, we are reduced to showing that

$$\|f\star_\lambda g\|_{L^2(T^\star\mathbb{R}^d)}\leq C_d\,|\lambda|^{-\frac{d}{2}}\,\|f\|_{L^2(T^\star\mathbb{R}^d)}\|g\|_{L^2(T^\star\mathbb{R}^d)}\,,$$

and this follows by an easy dilation argument from the well-known fact that $(L^2(T^\star\mathbb{R}^d),\star_1)$ is a Hilbert algebra (see for instance [29]).

Let us now focus on Estimate (A.2). We first recall that

$$\widetilde{\mathcal{W}}(\ell,\lambda,Y)=\sum_{|n|=\ell}\mathcal{L}_n(|\lambda|^{\frac{1}{2}}\,Y)\,,$$

where for $Z=(Z_1,\cdots,Z_d)$ in $T^\star\mathbb{R}^d$, we denote $\mathcal{L}_n(Z)=e^{-|Z|^2}\Pi_{j=1}^d L_{n_j}(2|Z_j|)$ with L_{n_j} the Laguerre polynomial of order n_j and type 0.

Define now for $n\in\mathbb{N}^d$ the operator T_n on $L^2(T^\star\mathbb{R}^d)$ by

$$T_n f\stackrel{\rm def}{=}f\star_\lambda\mathcal{L}_n(|\lambda|^{\frac{1}{2}}\cdot)\,,\tag{A.3}$$

so that

$$Tf=f\star_\lambda\widetilde{\mathcal{W}}(\ell,\lambda,\cdot)=\sum_{|n|=\ell}T_nf\,.\tag{A.4}$$

Then using the fact that the Laguerre polynomials $(L_k)_{k\in\mathbb{N}}$ are pairwise orthogonal on $[0,\infty[$ with respect to the measure $e^{-x}dx$, we infer that the family of operators $(T_n)_{|n|=\ell}$ is also pairwise orthogonal, and thus denoting by $\|T\|$ the norm of T defined by (A.4) as an operator on $L^2(T^\star\mathbb{R}^d)$, we can conclude that

$$\|T\|_{\mathcal{L}(L^2(T^\star\mathbb{R}^d))}=\max_{|n|=\ell}\|T_n\|_{\mathcal{L}(L^2(T^\star\mathbb{R}^d))}\,.\tag{A.5}$$

Since $\|\mathcal{L}_n(|\lambda|^{\frac{1}{2}}\cdot)\|_{L^2(T^\star\mathbb{R}^d)}=|\lambda|^{-\frac{d}{2}}\,\|\mathcal{L}_n\|_{L^2(T^\star\mathbb{R}^d)}$, one obtains

$$\|T_n\|_{\mathcal{L}(L^2(T^\star\mathbb{R}^d))}\lesssim|\lambda|^{-d}\,.$$

In view of (A.5), this implies that

$$\|f\star_\lambda\widetilde{\mathcal{W}}(\ell,\lambda,\cdot)\|_{L^2(T^\star\mathbb{R}^d)}\lesssim|\lambda|^{-d}\,\|f\|_{L^2(T^\star\mathbb{R}^d)}\,.$$

Finally, by definition

$$\|\widetilde{\mathcal{W}}(\ell,\lambda,\cdot)\|_{L^\infty(T^\star\mathbb{R}^d)}\lesssim\ell^{d-1}\,,$$

which ensures that

$$\|f\star_\lambda\widetilde{\mathcal{W}}(\ell,\lambda,\cdot)\|_{L^\infty(T^\star\mathbb{R}^d)}\lesssim\ell^{d-1}\,\|f\|_{L^1(T^\star\mathbb{R}^d)}\,,$$

and completes the proof of Estimate (A.2) by interpolation. □

References

[1] A. Agrachev, U. Boscain, J.P. Gauthier and F. Rossi, The intrinsic hypoelliptic Laplacian and its heat kernel on unimodular Lie groups, *J. Funct. Anal.*, **256**, pages 2621–2655, 2009

[2] J.-Ph. Anker and V. Pierfelice, Nonlinear Schrödinger equation on real hyperbolic spaces, *Ann. Inst. H. Poincaré (C) Non Linear Analysis*, **26**, pages 1853–1869, 2009.

[3] R. Askey and and S. Waigner, Mean convergence of expansions in Laguerre and Hermite series, *American Journal of Mathematics*, **87**, pages 695–708, 1965.

[4] H. Bahouri, J.-Y. Chemin et R. Danchin, Fourier analysis and applications to nonlinear partial differential equations, *Grundlehren der Mathematischen Wisserchaften, Springer Verlag*, **343**, 2011.

[5] H. Bahouri, J.-Y. Chemin and R. Danchin, Fourier transform of tempered distributions on the Heisenberg group, *Annales Henri Lebesgue*, **1**, pages 1–45, 2018.

[6] H. Bahouri, J.-Y. Chemin and R. Danchin, A frequency space for the Heisenberg group, *Annales de l'Institut de Fourier*, **69**, pages 365–407, 2019.

[7] H. Bahouri, C. Fermanian-Kammerer and I. Gallagher, Dispersive estimates for the Schrödinger operator on step 2 stratified Lie groups, *Analysis and PDE*, **9**, pages 545–574, 2016.

[8] H. Bahouri, C. Fermanian-Kammerer and I. Gallagher, Phase-space analysis and pseudo-differential calculus on the Heisenberg group, *Astérisque, Bulletin de la Société Mathématique de France*, **340**, 2012.
[9] H. Bahouri and I. Gallagher: Paraproduit sur le groupe de Heisenberg et applications, *Revista Matemática Iberoamericana*, **17**(1), pages 69–105, 2001.
[10] H. Bahouri, P. Gérard, and C.-J. Xu, Espaces de Besov et estimations de Strichartz généralisées sur le groupe de Heisenberg, *Journal d'Analyse Mathématique*, **82**, pages 93–118, 2000.
[11] F. Bernicot, and El M. Ouhabaz, Restriction estimates via the derivatives of the heat semigroup and connection with dispersive estimates, *Math. Res. Lett.*, **20**, pages 1047–1058, 2013
[12] F. Bernicot, and V. Samoyeau, Dispersive estimates with loss of derivatives via the heat semigroup and the wave operator, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, **17**, pages 969–1029, 2017
[13] J. Bourgain, A remark on Schrödinger operators, *Israel Journal of Mathematics*, **77**, pages 1–16, 1992.
[14] J. Bourgain, Refinements of Strichartz' inequality and applications to 2D-NLS with critical nonlinearity, *International Mathematical Research Notices*, **5** pages 253–283, 1998.
[15] J. Bourgain, Some new estimates on oscillatoryon integrals, *Essays on Fourier Analysis in Honor of Elias M. Stein, Princeton Math*, **42**, pages 83–112, 1995.
[16] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and application to nonlinear evolution equations I, *Geom. and Funct. Anal.*, **3**, pages 107–156, 1993.
[17] N. Burq, P. Gérard and N. Tzvetkov, Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds, *American Journal of Mathematics*, **126**, pages 569–605, 2004.
[18] V. Casarino and P. Ciatti, A restriction theorem for Métivier groups, *Advances in Mathematics*, **245**, pages 52–77, 2013.
[19] V. Casarino and P. Ciatti, Restriction estimates for the free two step nilpotent group on three generators, *arXiv e-prints*, arXiv:1704.07876, 2017.
[20] T. Cazenave, Equations de Schrödinger non linéaires en dimension deux, *Proceedings of the Royal Society of Edinburgh. Section A*, **84**, pages 327–346, 1979.
[21] L.-J. Corwin and F.-P. Greenleaf, Representations of nilpotent Lie groups and their applications, Part 1: Basic theory and examples, *Cambridge studies in advanced Mathematics*, **18**, Cambridge university Press, 1990.
[22] M. Del Hierro, Dispersive and Strichartz estimates on H-type groups, *Studia Math*, **169**, pages 1–20, 2005.
[23] A. Erdélyi, W. Magnus, F. Oberhettinger, and F.-G. Tricomi, Higher Transcendental Functions, *New York, McGraw-Hill*, **2**, 1953.
[24] J. Faraut and K. Harzallah: *Deux cours d'analyse harmonique*, École d'Été d'analyse harmonique de Tunis. Progress in Mathematics, Birkhäuser, 1984.
[25] C. Fefferman, Inequalities for strongly singular convolution operators, *Acta Math*, **124**, pages 9–36, 1970.
[26] L. Gassot, On the radially symmetric traveling waves for the Schrödinger equation on the Heisenberg group, *arXiv:1904.07010.*
[27] B. Gaveau, Principe de moindre action, propagation de la chaleur et estimées sous-elliptiques sur certains groupes nilpotents, *Acta Mathematica*, **139**, pages 95–153, 1977.
[28] J. Ginibre and G. Velo, Generalized Strichartz inequalities for the wave equations, *Journal of Functional Analysis*, **133**, pages 50–68, 1995.
[29] R. Howe, Quantum mechanics and partial differential equations, *Journal of Functional Analysis*, **38**, pages 188–254, 1980.
[30] A. Hulanicki, A functional calculus for Rockland operators on nilpotent Lie groups, *Studia Mathematica*, **78**, pages 253–266, 1984.
[31] O. Ivanovici, G. Lebeau and F. Planchon, Dispersion for the wave equation inside strictly convex domains I: the Friedlander model case, *Annals of Mathematics*, **180**, pages 323–380, 2014.
[32] H. Liu and M. Song, A restriction theorem for the H-type groups, *Proc. Amer. Math. Soc.*, **139**, pages 2713–2720, 2011.
[33] H. Liu and M. Song, A restriction theorem for Grushin operators, *Front. Math. China*, **11**, pages 365–375, 2016.
[34] M. Keel and T. Tao, Endpoint Strichartz estimates, *American Journal of Mathematics*, **120**, pages 955–980, 1998.
[35] D. Müller, A restriction theorem for the Heisenberg group, *Annals of Mathematics*, **131**, pages 567–587, 1990.
[36] A. I. Nachman, The Wave Equation on the Heisenberg Group, *Communications in Partial Differential Equations*, **7**, pages 675–714, 1982.
[37] L.P. Rothschild and E.M. Stein, Hypoelliptic differential operators and nilpotent groups. *Acta Math.*, **137**, pages 247–320, 1976.

[38] E.M. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality, Oscillatory integrals, Princeton University Press, 1993.

[39] R. Strichartz, Restriction Fourier transform of quadratic surfaces and decay of solutions of the wave equations, *Duke Mathematical Journal*, **44**, pages 705-714, 1977.

[40] T. Tao, Some recent progress on the restriction conjecture, *Fourier Analysis and Convexity*, Appl. Numer. Harmon. Anal., Birkhauser Boston, pages 217-243, 2004.

[41] P. A. Tomas, A restriction theorem for the Fourier transform, *Bull. Amer. Math. Soc.*, **81**, pages 477-478, 1975.

(H. Bahouri) LAMA UMR 8050, Université Paris-Est Créteil, 61, avenue du Général de Gaulle, 94010 Créteil Cedex, France

E-mail address: hbahouri@math.cnrs.fr

(D. Barilari) IMJ-PRG UMR 7586, Université Paris Diderot (Paris 7), Bâtiment Sophie Germain, Case 7012, 75205 PARIS Cedex 13, France

E-mail address: davide.barilari@imj-prg.fr

(I. Gallagher) DMA UMR 8553, École normale supérieure, CNRS, PSL Research University, 75005 Paris, and UFR de mathématiques, Université Paris-Diderot, Sorbonne Paris-Cité, 75013 Paris, France.

E-mail address: gallagher@math.ens.fr