UNIVERSITÀ DI PISA

Ph.D. Course in Mathematics

XXXII cycle

Asymptotic behaviour
of rescaled nonlocal functionals
and evolutions

Candidate:
Valerio Pagliari

Supervisor:
Matteo Novaga

«Quello dell'articolo da citare è un grande classico»

Abstract

Taking up a variational viewpoint, we present some nonlocal-to-local asymptotic results for various kinds of integral functionals. The content of the thesis comprises the contributions first appeared in the research papers [BP19, CP18, Pag19, CNP19] in collaboration with J. Berendsen, A. Cesaroni, A. Chambolle, and M. Novaga.

After an initial summary of the basic technical tools, the first original result is discussed in Chapter 2. It is motivated by the work [BBM01] by J. Bourgain, H. Brezis, and P. Mironescu, who proved that the L^p-norm of the gradient of a Sobolev function can be recovered as a suitable limit of iterated integrals involving the difference quotients of the function. A. Ponce later showed that the relation also holds in the sense of Γ-convergence [Pon04]. Loosely speaking, we take into account the rate of this convergence and we establish the Γ-converge of the rate functionals to a second order limit w.r.t. the $H^1_{\mathrm{loc}}(\mathbb{R}^d)$-convergence.

Next, from Chapter 3 on, we move to a geometric context and we consider the nonlocal perimeters associated with a positive kernel K, which we allow to be singular in the origin. Qualitatively, these functionals express a weighted interaction between a given set and its complement. More broadly, we study a total-variation-type nonlocal functional $J_K(\,\cdot\,; \Omega)$, where $\Omega \subset \mathbb{R}^d$ is a measurable set. We establish existence of minimisers of such energy under prescribed boundary conditions, and we prove a criterion for minimality based on calibrations. Due to the nonlocal nature of the problem at stake, the definition of calibration has to be properly chosen. As an application of the criterion, we prove that halfspaces are the unique minimisers of J_K in a ball subject to their own boundary conditions.

A second nonlocal-to-local Γ-convergence result is discussed in Chapter 4. We rescale the kernel K so that, when the scaling parameter approaches 0, the family of rescaled functions tends to the Dirac delta in 0. If K has small tails at infinity, we manage to show that the nonlocal total variations associated with the rescaled kernels Γ-converge w.r.t. the $L^1_{\mathrm{loc}}(\mathbb{R}^d)$-convergence to a local, anisotropic total variation.

Lastly, we consider the nonlocal curvature functional associated with K, which is the geometric L^2-first variation of the nonlocal perimeter. In the same asymptotic regime as above, we retrieve a local, anisotropic mean curvature functional as the limit of rescaled nonlocal curvatures. In particular, the limit is uniform for sets whose boundary is compact and smooth. As a consequence, we establish the locally uniform convergence of the viscosity solutions of the rescaled nonlocal geometric flows to the viscosity solution of the anisotropic mean curvature motion. This is obtained by combining a compactness argument and a set-theoretic approach that relies on the theory of De Giorgi's barriers for evolution equations.

Contents

iii

Introduction

We present some results that describe localisation effects for certain classes of nonlocal functionals which are defined by means of integral kernels. The underlying intuition is that, if we rescale the latter in such a way that they approximate the Dirac delta in the origin as the scaling parameter tends to 0, then in the limit we retrieve some *local* functionals. The precise sense in which the nonlocal-to-local convergence holds will be specified case by case.

We devote these introductory pages to the overview of our main results, for which we provide some motivation and background. At the same time, we outline the structure of the thesis. Our main concern here is conveying general ideas, rather than stating results with full precision. To give a wider perspective to the possibly interested readers, we hint also at some of the literature which, for the sake of brevity, though being related to the subjects we treat, is not mentioned in the main body of the work.

The first chapter includes the basics about functions of bounded variations, finite perimeter sets, and Γ-convergence needed in the sequel. Then, in Chapter 2, we report the results of [CNP19].

We start from the analysis carried out by J. Bourgain, H. Brezis, and P. Mironescu in [BBM01], which has attracted much attention in the years: as an extremely brief selection of follow-ups, we mention [MS02, Dáv02, Pon04, AK09, LS11]. The main result of [BBM01] consists in the approximation of the L^p-norm of the gradient of a Sobolev function by means of iterated integrals involving difference quotients. In [Pon04], A. Ponce extended the result to the functionals

$$\mathscr{F}_\varepsilon(u) := \int_\Omega \int_\Omega \rho_\varepsilon(y-x) f\left(\frac{|u(y)-u(x)|}{|y-x|}\right) \mathrm{d}y\mathrm{d}x,$$

where $u \in L^p(\Omega)$ for some $p \geq 1$, $\{\rho_\varepsilon\}$ is a suitable family of functions in $L^1(\mathbb{R}^d)$, and $f\colon [0,+\infty) \to [0,+\infty)$ is convex. As $\varepsilon \to 0^+$, he proved both pointwise and Γ-convergence of $\{\mathscr{F}_\varepsilon\}$ to a first order limit functional $\mathscr{F}_0$. When $\rho_\varepsilon(x) := \varepsilon^{-d}\rho(\varepsilon^{-1}x)$ for some $\rho \in L^1(\mathbb{R}^d)$ and $u \in W^{1,p}(\Omega)$ is a Sobolev function, we find

$$\mathscr{F}_0(u) = \int_\Omega \int_{\mathbb{R}^d} \rho(z) f(|\nabla u(x) \cdot \hat{z}|)\mathrm{d}z\mathrm{d}x,$$

where, for $z \in \mathbb{R}^d \setminus \{0\}$, $\hat{z} := z/|z|$. We address the asymptotic behaviour of the differences $\mathscr{F}_0 - \mathscr{F}_\varepsilon$, which express the rate of convergence of the ε-functionals to their limit. We prove that, if f is strongly convex, the rescaled rate functionals

$$\mathscr{E}_\varepsilon(u) := \frac{\mathscr{F}_0(u) - \mathscr{F}_\varepsilon(u)}{\varepsilon^2}$$

Γ-converge w.r.t. to the $H^1_{\mathrm{loc}}(\mathbb{R}^d)$-convergence to the second order functional

$$\mathscr{E}_0(u) := \frac{1}{24} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho(z)\, |z|^2\, f''(|\nabla u(x) \cdot \hat{z}|)\, \left|\nabla^2 u(x)\hat{z} \cdot \hat{z}\right|^2 \mathrm{d}z\mathrm{d}x.$$

The assumption of strong convexity on f rules out several interesting scenarios, notably the one of linear growth at infinity. However, the analysis of the case at stake is already nontrivial, and it leads to a conclusion which is analogous to the one of M. Peletier, R. Planqué, and M. Röger in [PPR07] for convolution-type functionals.

Similarly to the works of M. Gobbino and M. Mora [Gob98, GM01], we perform a proof based on a slicing procedure, which reduces the problem to the study of some energies defined on functions of one variable, see Section 2.1. Then, in Section 2.3, we establish the desired result relying on the Γ-convergence of the 1-dimensional functionals. This is not immediate, though, and is achieved by resorting to a compactness result, whose proof, in turn, requires some effort because of the structure of $\mathscr{E}_\varepsilon$.

From Chapter 3 on, we leave Sobolev spaces to move closer to geometric measure theory, and we introduce generalised notions of perimeter, total variation, and mean curvature.

Let $\Omega \subset \mathbb{R}^d$ be measurable and consider an even function $K \colon \mathbb{R}^d \to [0, +\infty)$. Heuristically, K plays the role of a weight that encodes long range interactions between the points in the space. For a measurable function $u \colon \Omega \to \mathbb{R}$ we define its *nonlocal total variation* in Ω as

$$\begin{aligned}
J_K(u;\Omega) = {} & \frac{1}{2} \int_\Omega \int_\Omega K(y-x)\, |u(y) - u(x)|\, \mathrm{d}y\mathrm{d}x \\
& + \int_\Omega \int_{\Omega^c} K(y-x)\, |u(y) - u(x)|\, \mathrm{d}y\mathrm{d}x.
\end{aligned} \tag{0.1}$$

We recall that, for a set $E \subset \mathbb{R}^d$, the perimeter in the sense of E. De Giorgi [DG55, AFP00, Mag12] is defined as the total variation of the distributional gradient of its characteristic function. Accordingly, we define the *nonlocal perimeter* in Ω of E as

$$\mathrm{Per}_K(E;\Omega) := J_K(\chi_E;\Omega),$$

χ_E being the characteristic function.

The first instance of nonlocal perimeter appeared in the celebrated paper [CRS10] by L. Caffarelli, J. Roquejoffre, and O. Savin, who focused on the case $K(x) = |x|^{-d-s}$ for some $s \in (0,1)$, i.e. on *fractional kernels*. Over the years, their work has influenced vastly the research, and in the sequel we suggest a brief list of developments.

Before going into the details of Chapter 3, we present some examples by E. Cinti, J. Serra, and E. Valdinoci [CSV19] with the purpose of motivating the interest in the functionals under consideration,

The first example concerns digital imaging. In order to be displayed on the screen of an electronic device, images are sampled by means of 2-dimensional grids whose elementary cells correspond to the smallest units that the device can handle. These cells are called *pixels*, a short for picture element, and their whole provides the screen representation of the given image. As a general principle, the more the pixels, the more accurate is the digital reproduction; in other words, pixels whose side lengths are small result in high resolution and fidelity. The following discussion shows how nonlocal perimeters are sensitive to changes of resolution, differently from De Giorgi's one.
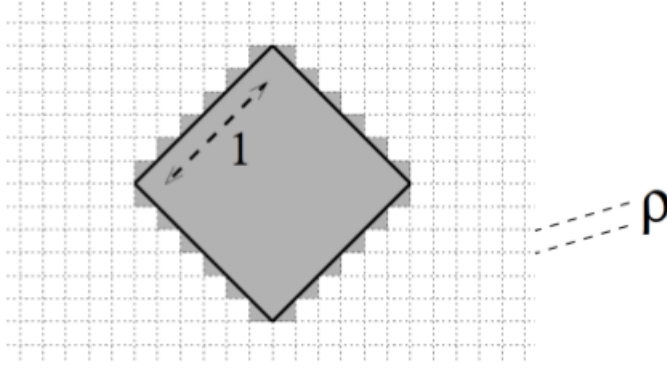
Figure 1: Representation of a unit square by pixels of side length equal to ρ. Source: [CSV19].

Suppose that we want to create a pixel representation of a unit square whose sides are rotated by $45°$ w.r.t. to the reference grid (see Figure 1) . An elementary calculation gets that the standard perimeter of the reproduction, which is displayed in grey in the figure, equals $4\sqrt{2}$, no matter how we choose the side length of the pixels. Hence, we see that the discrepancy between the perimeter of the source image and the one of its pixel approximation is left unchanged under increase of resolution.

In this respect, nonlocal perimeters are far more accurate. For instance, let $s \in (0,1)$ and consider the fractional perimeter

$$\mathrm{Per}_s(E) := \mathrm{Per}_s(E; \mathbb{R}^d) = \int_{E^c} \int_E \frac{\mathrm{d}x\mathrm{d}y}{|y-x|^{2+s}}.$$

Denoting by Q and $\tilde{Q}$ respectively the given square and its pixel representation, we have that

$$\mathrm{Per}_s(\tilde{Q}) = \int_{\tilde{Q}^c} \int_{\tilde{Q} \cap Q^c} \frac{\mathrm{d}x\mathrm{d}y}{|y-x|^{2+s}} + \int_{\tilde{Q}^c} \int_Q \frac{\mathrm{d}x\mathrm{d}y}{|y-x|^{2+s}}$$
$$= \int_{\tilde{Q}^c} \int_{\tilde{Q} \cap Q^c} \frac{\mathrm{d}x\mathrm{d}y}{|y-x|^{2+s}} + \mathrm{Per}_s(Q) - \int_{\tilde{Q} \cap Q^c} \int_Q \frac{\mathrm{d}x\mathrm{d}y}{|y-x|^{2+s}}$$
$$\le \int_{\tilde{Q}^c} \int_{\tilde{Q} \cap Q^c} \frac{\mathrm{d}x\mathrm{d}y}{|y-x|^{2+s}} + \mathrm{Per}_s(Q).$$

The set $\tilde{Q} \cap Q^c$ is the "frame" of pixels that surrounds Q, whence

$$\mathrm{Per}_s(\tilde{Q}) \le \mathrm{Per}_s(Q) + \sum_i \mathrm{Per}_s(P_i),$$

where P_i is a boundary pixel. The number of such pixels is of the order ρ^{-1} and, by a change of variables, it is easy to see that $\mathrm{Per}_s(P_i) = \rho^{2-s} \mathrm{Per}_s(Q)$. On the whole, we find

$$\mathrm{Per}_s(\tilde{Q}) \lesssim (1 + \rho^{1-s}) \mathrm{Per}_s(Q),$$

so that the discrepancy is bounded above by a quantity that vanishes as ρ tends to 0, that is, it vanishes in the high resolution limit.

It is worth mentioning that nonlocal functionals akin to (0.1) have also been studied in connection with image denoising. In a nutshell, this amounts to recover a image of higher quality starting from a distorted datum $u_0 \colon \Omega \to \mathbb{R}$. The result of the process should be a function that is more regular than u_0 and that is, at the same time, not too different from it. To achieve the goal, the idea is to look for solutions to the problem

$$\min_u \left\{ J(u) + \alpha \int_\Omega |u - u_0|^2 \right\},$$

where J acts as a *filter* and α is the so-called *fidelity parameter*, to be suitably tuned. The differences between the models proposed in the literature lie in the choice of the filter J. The classical one is $J(u) = \int_\Omega |\mathrm{D}u|$ [ORF92], but, more recently, several nonlocal filters have been introduced [GO07, AK09, LS11, BN16, BN18].

The second motivational example comes from physics. In [CH58], J. Cahn and J. Hilliard introduced a model for phase transitions in certain binary metallic alloys, and they derived the following expression for the free energy:

$$\mathscr{F}_\varepsilon^{\mathrm{CH}}(u; \Omega) \coloneqq \int_\Omega \left[\frac{\varepsilon^2}{2} |\nabla u(x)|^2 + W(u(x)) \right] \mathrm{d}x.$$

Here, Ω is the container where the system sits, $u \colon \Omega \to [-1, 1]$ represents the concentration of a reference phase, $\varepsilon > 0$ is a (small) scale parameter, and $W \colon \mathbb{R} \to [0, +\infty)$ is a double-well potential which vanishes only in -1 and 1.

When the phases are mixed, that is, u takes also values in $(-1, 1)$, and the system is in suitable physical conditions, experiments show the evolution is steered by the minimisation of $J_\varepsilon^{\mathrm{CH}}$: thus, owing to the pontential W, the two phases separate, and the transition occurs in a thin interface, because the gradient term penalises wide interfaces.

In the description of some systems, it is reasonable to assume that particles interact also if they are far apart; for instance, this is observed in continuum limits of discrete Ising's spins. Such circumstances suggest to modify the free energy of Cahn and Hilliard as to include nonlocal functionals. The option considered by G. Alberti and G. Bellettini in [AB98a, AB98b] is

$$\mathscr{F}_\varepsilon^{\mathrm{AB}}(u; \Omega) \coloneqq \frac{1}{4} \int_\Omega \int_\Omega K_\varepsilon(y - x) |u(y) - u(x)|^2 \, \mathrm{d}y \mathrm{d}x + \int_\Omega W(u(x)) \mathrm{d}x,$$

where

$$K_\varepsilon(x) \coloneqq \frac{1}{\varepsilon^d} K\left(\frac{x}{\varepsilon} \right) \tag{0.2}$$

and K is a positive, long range interaction kernel; see also the paper [BBP01] by G. Bellettini, P. Buttà, and E. Presutti. Analogous energies of fractional type have been investigated as well; the reader interested in this topic may consult the survey [Val13] by E. Valdinoci and references therein.

Observe that, when the phases are separated, the functional $\mathscr{F}_\varepsilon^{\mathrm{AB}}$ equals the first summand in (0.1): indeed, if u takes only the values -1 and 1, then $|u(y) - u(x)|^2 = 2 |u(y) - u(x)|$. Therefore, we see that J_K is related to the free energy of the system at rest.

Let us now overview our analysis of nonlocal perimeters and total variations. Relying on [BP19], we explore some basic properties of the nonlocal energies in Sections 3.1 and

3.2. We allow for a wide family of kernels, which encompasses, for instance, the fractional ones, and, as a first result, we show that the nonlocal total variation of a function u is finite if u is of bounded variation.

Secondly, we show that there exist local minimisers of J_K under prescribed boundary conditions. This corresponds to the existence of solutions to the well-studied Plateau's problem, which amounts to find a hypersurface of least area among the ones spanned by a given boundary.

A simple application of Fatou's Lemma yields $L^1_{\mathrm{loc}}(\mathbb{R}^d)$-lower semicontinuity of J_K, but, in striking contrast with the local case, sequences that have equibounded nonlocal variations are in general not precompact in L^1 (see the lines preceding Remark 3.7 for a counterexample). This hinders a straightforward transliteration of the geometric measure theory approach to the classical Plateau's problem. Nonetheless, it is possible to deduce existence of minimisers *via* the direct method of calculus of variations. The key ingredients in this regard are the convexity of J_K and the validity of a *generalised Coarea Formula*:

$$ J_K(u; \Omega) = \int_0^1 \mathrm{Per}_K(\{\, u > t \,\}; \Omega) \mathrm{d}t. $$

The importance of the latter was firstly pointed out by A. Visintin [Vis91, Vis90], who was concerned with energies for multiphase systems as the one we mentioned above.

The existence of solutions to the nonlocal Plateau's problem raises at least a couple of questions:

- Can we give a *necessary* condition for optimality in terms of the first variation of the nonlocal perimeter?

- Can we devise some *sufficient* condition for optimality?

As in the local case, tackling the first issue leads to the notion of curvature. The geometric first variation of Per_K was computed as soon as nonlocal perimeters were introduced. Indeed, in [CRS10], Caffarelli, Roquejoffre, and Savin identified

$$ H_K(E, x) := -\mathrm{p.v.} \int_{\mathbb{R}^d} K(y - x)\big(\chi_E(y) - \chi_{E^c}(y)\big) \mathrm{d}y \tag{0.3} $$

as the first variation of Per_K, K being a fractional kernel. The relationship holds true in wider generality, and, due to the possible singularity of K, it is not evident in which sense a solution E to Plateau's problem satisfies the Euler-Lagrange equation $H_K(E, x) = 0$ for $x \in \partial E$. It is not even clear, at first sight, for which sets the integral defining the nonlocal curvature is convergent.

In Section 3.4, we give sufficient conditions on K so that $H_K(E, x)$ is finite for all sets E with $C^{1,1}$ boundary and for all $x \in \partial E$. This is not a novelty [CMP15, MRT19], but we also establish an upper bound on H_K that is to come in handy in Chapter 5.

As for the Euler-Lagrange equation $H_K(E, x) = 0$, J. Mazón, J. Rossi, and J. Toledo proved that when $K \in L^1(\mathbb{R}^d)$ and x belongs to the reduced boundary of E, the equality holds pointwise. More broadly, it has to be interpreted in a viscosity sense, see [CRS10]. In a very recent paper [Cab20], X. Cabré has re-proved this fact by utilising an argument that is completely different from the original one (see later on).

On the side of sufficient conditions for optimality, fewer literature concerning our problem is available. *Vice versa*, for De Giorgi's perimeter the well developed theory of

calibrations is at our disposal [Mor08, HL82, Mor90, LM94, CCP12]. In Section 3.3, which grounds on [Pag19], we propose a nonlocal notion of calibration that allows for a minimality criterion that parallels the classical one. A similar task was indepentently accomplished by Cabré [Cab20], who also exploited calibrations to retrieve that nonlocal minimal surface satisfy the equation $H_K = 0$ is the viscosity sense. Another possible application will be discussed in Chapter 4.

We mould nonlocal calibrations on the classical ones. Yet, while the latter are vector fields, following the approach to nonlocal Cheeger's energies in [MRT19], our calibrations are scalar functions on the product space $\mathbb{R}^d \times \mathbb{R}^d$. Then, we prescribe that they satisfy a bound on the norm, a vanishing divergence condition, and agreement with the normal of the calibrated set. Of course, these requirements need to be formulated appropriately as to fit in the nonlocal framework. In particular, inspired again by [MRT19], we resort to the notion of nonlocal divergence introduced by G. Gilboa and S. Osher in [GO08].

While, in general, it is not effortless to construct a calibration, it is simple to exhibit one for graphs [Cab20]. In the very special case of halfspaces, alongside with optimality, we are also able to prove uniqueness [Pag19]. This property had been established previously only for fractional perimeters [CRS10, ADPM11].

A common feature of the phase field models that we sketched above is that, under suitable rescalings, the energies approach asymptotically a surface functional. In the local setting, a classical result by L. Modica and S. Mortola [MM77] states that, when $\varepsilon \to 0^+$, the family $\left\{ \varepsilon^{-1} J_\varepsilon^{\mathrm{CH}} \right\}$ Γ-converges to De Giorgi's perimeter, up to a constant encoding surface tension. In [AB98b] a local, anisotropic surface functional was retrieved as the Γ-limit of $\left\{ \varepsilon^{-1} J_\varepsilon^{\mathrm{AB}} \right\}$, and O. Savin and E. Valdinoci drew similar conclusions for fractional free energies in [SV12]. In a more abstract context, also the limiting behaviour of the functional J_K in (0.1) was studied: for $K(x) = |x|^{-d-s}$ with $s \in (0, 1)$, L. Ambrosio, G. De Philippis, and L. Martinazzi proved in [ADPM11] that, when $s \to 1^-$, De Giorgi's perimeter is obtained in the sense of Γ-convergence.

In Chapter 4, which is based on [BP19, Pag19], we discuss a nonlocal-to-local convergence result which is reminiscent of the previous ones. Indeed, provided that

$$\int_{\mathbb{R}^d} K(x) |x| \, \mathrm{d}x < +\infty,$$

we recover a local, anisotropic total variation J_0 as the Γ-limit of the family $\left\{ \varepsilon^{-1} J_\varepsilon \right\}$, where $J_\varepsilon(u; \Omega) := J_{K_\varepsilon}(u; \Omega)$ and K_ε is as in (0.2). Under the current assumption on the kernel, it is non difficult to show that, when u is a function of bounded variation, $J_\varepsilon(u; \Omega)$ is of order ε. Consequently, to get a nontrivial limit, we put the factor ε^{-1} in front of the rescaled energies.

Our convergence result is closely related to the one in [Pon04], and we rely on that work for what concerns the pointwise limit. Nevertheless, the interactions between the reference set Ω and its complement are not taken into consideration there, so our statement does not follow immediately from the ones of Ponce. In any case, aside from pointwise convergence, our proof is self-contained.

We remark that, paralleling the analysis of Chapter 2, it would be interesting to investigate whether the rate functionals $\varepsilon^{-\alpha}(J_0 - J_\varepsilon)$ Γ-converge for some $\alpha > 0$ to a geometric, second order limit functional.

The first step to prove the Γ-convergence of the rescaled functionals J_ε is reducing to nonlocal perimeters. This is made possible by a result of A. Chambolle, A. Giacomini,

and L. Lussardi [CGL10], which, in turn, grounds on the validity of Coarea Formulas in the sense of Visintin. Then, in Section 4.3, we can easily establish the upper limit inequality, exploiting the density of polyhedra in the family of finite perimeter sets.

The proof of the Γ-lower limit inequality, which is addressed in Section 4.5, is more elaborated. As [AB98b, ADPM11], we adopt the blow-up technique introduced by I. Fonseca and S. Müller [FM93], and we deduce a lower bound where a functional $\tilde{J}_0$, *a priori* distinct from J_0, appears. By construction, the anisotropy of $\tilde{J}_0$ is given only implicitly: indeed, it is defined as the value of a certain cell problem which involves sequences of sets converging in L^1 to a given halfspace. To achieve the conclusion, in Section 4.4 we identify the anisotropy occurring in $\tilde{J}_0$ as the one of J_0. Much in the spirit of [ADPM11], the minimality of halfspaces proves to be crucial at this stage, and this provides an interesting application of the calibration criterion in Chapter 3.

We recalled that nonlocal curvatures can be derived as first variations of nonlocal perimeters. With the result of Chapter 4 on the table, it is spontaneous to wonder if an analogue asymptotic results holds for curvatures. In the affirmative case, it is also natural to ask whether the convergence is inherited by the associated geometric motions, that is, by the L^2-gradient flows of the perimeters. We deal with these questions in the concluding chapter, which includes the contributions in [CP18], and we provide a positive answer to both.

When K is a fractional kernel, N. Abatangelo and E. Valdinoci [AV14] proved that suitable rescalings of the nonlocal curvatures pointwise converge to a multiple of the standard mean curvature when the fractional exponent approaches 1. For radial, L^1 kernels, the same conclusion was achieved in [MRT19]. We establish a congruent theorem in Section 5.2 for a larger class of kernels. We show that, when E is a set with C^2 boundary, in the limit $\varepsilon \to 0^+$, the functionals $\varepsilon^{-1} H_\varepsilon(E, x) := \varepsilon^{-1} H_{K_\varepsilon}(E, x)$, with K_ε as above, tend for all $x \in \partial E$ to an anisotropic mean curvature functional $H_0(\partial E, x)$. If the boundary of E is also compact, we are able to conclude that the convergence is uniform.

The next thing we do is checking that the limit curvature agrees with the limit perimeter found in Chapter 4, namely we notice that H_0 is the first variation of the restriction of J_0 to sets. So, in a sense, the first variation commutes with the limiting procedures.

The last result that we present in this thesis concerns the motions by curvature

$$\partial_t x(t) \cdot \hat{n}(t) = -\frac{1}{\varepsilon} H_\varepsilon(E(t), x(t)) \quad \text{and} \quad \partial_t x(t) \cdot \hat{n}(t) = -H_0(\Sigma(t), x(t)),$$

where $t \mapsto E(t) \subset \mathbb{R}^d$ is an evolution of sets, $\Sigma(t) := \partial E(t)$, $x(t) \in \Sigma(t)$, and $\hat{n}(t)$ is the outer unit normal to $\Sigma(t)$ at $x(t)$. We study them utilizing the level-set method, which is to say that we suppose that the evolving set $E(t)$ and its boundary $\Sigma(t)$ are respectively the 0 superlevel set and the 0 level set of some function $\varphi(t, \cdot)$. Rewriting the previous equations in terms of φ gets formally the following nonlocal and local parabolic partial differential equations:

$$\partial_t \varphi(t, x) + \frac{|\nabla \varphi(t, x)|}{\varepsilon} H_\varepsilon(\{y : \varphi(t, y) \geq \varphi(t, x)\}, x) = 0, \tag{0.4}$$

$$\partial_t \varphi(t, x) + |\nabla \varphi(t, x)| H_0(\{y : \varphi(t, y) = \varphi(t, x)\}, x) = 0. \tag{0.5}$$

Already in the local case, it is well know that a weak notion of solution is needed to cope with the onset of singularities observed in this sort of evolutions. We employ the viscosity theory by L. Evans and J. Spruck [ES92] and by Y. Chen, Y. Giga, and S. Goto [CGG91], who,

in turn, relied on results for Hamilton-Jacobi equations (see for instance [CL83]). In this context, existence and uniqueness of a global-in-time solution are ensured. On the nonlocal side, well posedness for the Cauchy's problem for the motion by fractional curvature was settled by C. Imbert in [Imb09] (see also the paper [LMV19] by D. La Manna and J. Vesa for short time existence of the classical solutions). Lately, existence and uniqueness for the level-set formulation of a broad range of local and nonlocal curvature flows have been established in a unified framework by A. Chambolle, M. Morini, and M. Ponsiglione [CMP15].

Several aspects of motions by nonlocal curvatures have recently been subjects of investigation, mostly in the fractional case: for instance, conservation of convexity [CNR17], formation of neckpinch singularities [CSV18], fattening phenomena [CDNV19], self-shrinkers [CN19]. Instead, going back in time, we see that nonlocal geometric evolutions emerged from physical models that strive to explain plastic behaviour of metallic crystals. According to these theories, the macroscopic phenomenon of plasticity arises as a consequence of microscopic irregularities in the crystalline lattice, known as defects. Among them, we find dislocations, which correspond to linear misalignments in the microscopic structure. Each dislocation produces an elastic field, which in turn exerts the so called Peach-Koehler force on all other defects in the alloy. As a consequence, dislocation lines evolve, and each point on them moves with normal velocity determined by the Peach-Koehler force. In [AHLBM06], O. Alvarez, P. Hoch, Y. Le Bouar, and R. Monneau proposed a mathematical description of dislocation dynamics in terms of a nonlocal eikonal equation, for which they were able to prove short time existence and uniqueness. In this model, the Peach-Koelher force is encoded by a convolution kernel c_0, typically singular, whence the nonlocal nature of the problem. Imbert highlights in [Imb09] that the mathematical expression of the Peach-Koehler force may be understood as a nonlocal curvature of the dislocation line, thus, dislocation dynamics turns out to be a motion by nonlocal curvature.

The explicit expression of the kernel c_0 might be complicated, because it has to capture the physical features of the system. Further, c_0 might change sign and this poses serious technical difficulties, because it opposes the validity of comparison principles. The original model has by then been simplified by various authors, who obtained well-posedness for the problem [ACM05, BL06, IMRM08, NF08, DLFM08].

After the excursus on the connection between nonlocal curvatures and dislocation dynamics, we discuss now the main result of Chapter 5, which shows that the convergence of the curvatures is sufficiently robust to entail convergence of the solutions to the corresponding geometric flows. Precisely, we fix $u_0 \colon \mathbb{R}^d \to \mathbb{R}$ and we let u_ε and u be the viscosity solutions with initial datum u_0 to (0.4) and (0.5), respectively. We prove that the family $\{\, u_\varepsilon \,\}$ locally uniformly converges to u as $\varepsilon \to 0^+$.

A similar result already appeared in the paper [DLFM08] by F. Da Lio, N. Forcadel, and R. Monneau, who also proved convergence of the nonlocal curvature functionals. However, they made assumptions on the kernel that are different from ours, because they focused on a precise dislocation dynamics model. Accordingly, their scaling does not match ours. We also stress that our proof and theirs are based on substantially distinct arguments. Indeed, in [DLFM08], the authors exploit viscosity semilimits and the perturbed test function method, whilst we propose an approach by *geometric barriers*. These were introduced by De Giorgi in [DG94] as weak solutions to a wide class of evolution problems. G. Bellettini, M. Novaga, and M. Paolini considered solution in this class for geometric parabolic PDEs such as (0.5), and they found out that viscosity theory and barriers can be compared

[BGP95, Bel97, BN98, BN00]. In the same spirit, in the recent paper [CDNV19], A. Cesaroni, S. Dipierro, M. Novaga, and E. Valdinoci put in relation barriers and viscosity solutions for nonlocal curvature motions. These comparison results are the cornerstone of the analysis in Section 5.4.

As a concluding bibliographical note, we point out that approximation results for mean curvature motions, either by local or nonlocal operators, have been intensively studied. One of the most renowned is due to J. Bence, B. Merriman, and S. Osher. In [BMO92], they devised a formal strategy to construct fronts evolving by mean curvature, the so called threshold dynamics algorithm. It amounts to a time-discretization, which, at each step, takes the characteristic functions of a suitable set as datum, and yields its evolution according to the heat equation. The convergence of this scheme was rigorously settled in [BG95, Eva93] and subsequently more general diffusion operators were considered [Ish95, IPS99, CN06]. In [CS10] L. Caffarelli and P. Souganidis applied a threshold dynamics scheme to motions by fractional mean curvature. A generalisation encompassing anisotropies and driving forces was proved in [CNR17].

Notation

$a \wedge b, a \vee b$	resp., minimum and maximum between $a, b \in \mathbb{R}$
d	dimension of the ambient space $\mathbb{R}^d$
id	identity map in $\mathbb{R}^d$
E^{c}	complement of the set E in $\mathbb{R}^d$, i.e. $\mathbb{R}^d \setminus E$
$E \triangle F$	symmetric difference between the sets E and F
$E = F$	the sets E and F coincide up to a Lebesgue negligible set
$E \subset F$	the sets E is contained in the set F, up to a Lebesgue negligible set
χ_E	characteristic function of the set E
$\tilde{\chi}_E$	signed characteristic function of the set E, i.e. $\tilde{\chi}_E = \chi_E - \chi_{E^{\mathsf{c}}}$
$x \cdot y$	Euclidean inner product between $x, y \in \mathbb{R}^d$
$|x|$	Euclidean norm of $x \in \mathbb{R}^d$
$\hat{x}$	unit vector parallel to $x \in \mathbb{R}^d \setminus \{0\}$, i.e. $x/|x|$
$x^{\perp}$	subspace of the vectors that are orthogonal to $x \in \mathbb{R}^d \setminus \{0\}$
$\pi_{\hat{x}^{\perp}}$	the matrix that represents the orthogonal projection on $\hat{x}^{\perp}$, i.e id $- \hat{x} \otimes \hat{x}$
e_i	i-th element of the canonical orthonormal basis of $\mathbb{R}^d$
$B(x, r)$	open ball of centre x and radius $r > 0$ in $\mathbb{R}^d$
$\mathbb{S}^{d-1}$	topological boundary of $B(0, 1)$
$\mathrm{dist}(x, E)$	Euclidean distance between the point x and the set E
$\mathrm{dist}(E, F)$	Euclidean distance between the sets E and F
$|\mu|$	total variation measure of the measure μ
$\|\mu\|$	total variation norm of the measure μ
$\mu \llcorner \Omega$	restriction of the measure μ to the set Ω
$\mathscr{L}^d$	d-dimensional Lebesgue measure
$\mathscr{H}^{d-1}$	$(d-1)$-dimensional Hausdorff measure
ω_{d-1}	$(d-1)$-dimensional Lebesgue measure of the unit ball in $\mathbb{R}^{d-1}$
a.e.	$\mathscr{L}^d$-almost everywhere in $\mathbb{R}^d$ or $\mathscr{L}^d \otimes \mathscr{L}^d$-almost everywhere in $\mathbb{R}^d \times \mathbb{R}^d$
$u = v$	the functions $u, v \colon \mathbb{R}^d \to \mathbb{R}$ coincide $\mathscr{L}^d$-a.e.
$\mathrm{D}u$	distributional gradient of the function u
$\mathrm{BV}(\Omega), \mathrm{BV}_{\mathrm{loc}}(\Omega)$	functions of (locally) bounded variation in Ω
M^{t}	transpose of the matrix M
$\mathrm{tr}\,(M)$	trace of the matrix M

Chapter 1

Basic ingredients

In this preliminary chapter we collect some standard notions and results that make up the background of the sequel. We include here just the topics that are of general relevance for the thesis, while we shall provide reminders that pertain to specific aspects when they are needed. Notably, the theories of viscosity solutions and of barriers for geometric evolution equations are presented in Chapter 5. Moreover, we omit proofs, which may be found in the suggested references.

We start with the essentials of geometric measure theory. We recall some facts about measures on the Euclidean space, then we introduce the space of functions of bounded variation and the class of finite perimeter sets.

One of the leitmotifs of our analysis is recovering local functionals as limits of their rescaled nonlocal counterparts. In Chapters 2 and 4, the asymptotics relationships are expressed in terms of Γ-convergence, whose definition we recall in Section 1.3. Typically, the limit functionals that we obtain are not isotropic. For this reason, we sum up in Section 1.2 some properties of anisotropic surface energies.

Before getting to the heart of the matter, let us premise some notations concerning the ambient space of our analysis.

For us, d is always a natural number greater than 1. We work in the d-dimensional Euclidean space, that is, the vector space $\mathbb{R}^d$ endowed with the Euclidean inner product $\cdot$ and the associated norm $|\cdot|$. We let $e_1, \ldots, e_d$ be the elements of the canonical orthonormal basis, and, when $x \in \mathbb{R}^d \setminus \{0\}$, we set

$$\hat{x} := \frac{x}{|x|} \quad \text{and} \quad x^\perp := \left\{ y \in \mathbb{R}^d : x \cdot y = 0 \right\}.$$

We denote the open ball of centre x and radius r in $\mathbb{R}^d$ by $B(x, r)$ and we let $\mathbb{S}^{d-1}$ be the topological boundary of $B(0, 1)$. If $E, F \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$, we put

$$\operatorname{dist}(x, E) := \inf \left\{ |y - x| : y \in E \right\},$$
$$\operatorname{dist}(E, F) := \inf \left\{ |y - x| : x \in E, y \in F \right\}.$$

When $E \subset \mathbb{R}^d$ is a set, we write E^c for its complement in $\mathbb{R}^d$ and χ_E for its *characteristic function*, i.e.

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \in E^\mathsf{c}. \end{cases}$$

If F is another subset of $\mathbb{R}^d$, we let $E \triangle F$ be the symmetric difference between E and F, that is $E \triangle F := (E \cap F^c) \cup (E^c \cap F)$.

1.1 Measures and BV functions

Following the monographs [AFP00] by L. Ambrosio, N. Fusco, and D. Pallara and [Mag12] by F. Maggi, we introduce the by now classical tools to tackle geometric variational problems.

Let X be a set and let $\mathscr{A}$ be a σ-algebra on X. A map $\mu \colon \mathscr{A} \to [-\infty, +\infty]$ is a (real) *measure* on $(X, \mathscr{A})$ if it is σ-additive and $\mu(\emptyset) = 0$. We say that a measure μ is *positive* when $\mu(E) \in [0, +\infty]$ for all $E \in \mathscr{A}$, and that it is *finite* if it is positive and $\mu(E) < +\infty$ for all $E \in \mathscr{A}$. Similarly, we may consider *vector measures*, that is, σ-additive maps $\mu \colon X \to \mathbb{R}^n$ such that $\mu(\emptyset) = 0$ with $n \in \mathbb{N}$ larger than 1.

When μ is either a real- or vector-valued measure on $(X, \mathscr{A})$, we define its *total variation* as

$$|\mu|(E) := \sup \left\{ \sum_{\ell=1}^{+\infty} |\mu(E_\ell)| : E_\ell \in \mathscr{A}, E_\ell \cap E_m = \emptyset \text{ if } \ell \neq m, \text{ and } E = \bigcup_{\ell=1}^{+\infty} E_\ell \right\},$$

which is in turn a finite measure. We dub *norm* of the measure μ the quantity $\|\mu\| := |\mu|(X)$.

In this work, we only consider measures on the Euclidean space $\mathbb{R}^d$. Let $\mathscr{B}$ the Borel σ-algebra, i.e. the σ-algebra generated by open sets. Measures that are defined on each set in $\mathscr{B}$ are called Borel measures. The *support* of a positive measure is the set

$$\operatorname{supp}\mu := \left\{ \bigcap C : C \text{ is closed and } \mu(C^c) = 0 \right\}.$$

We say that a positive Borel measure μ on $\mathbb{R}^d$ is a *Radon measure* if it is

- locally finite: $\mu(C) < +\infty$ for all compact sets $C \subset \mathbb{R}^d$;

- regular: for all $F \subset \mathbb{R}^d$ there exists $E \in \mathscr{B}$ such that $F \subset E$ and $\mu(F) = \mu(E)$.

This concept may also be extended to the vectorial case: a Borel measure μ on $\mathbb{R}^d$ with values in $\mathbb{R}^n$ is a vector-valued Radon measure if there exist a positive measure ν on $(\mathbb{R}^d, \mathscr{B})$ which is Radon and a $L^1(\nu)$ function $f \colon \mathbb{R}^d \to \mathbb{R}^n$ such that $|f| = 1$ ν-a.e. and $\mu = f\nu$. *A posteriori*, one finds $\nu = |\mu|$.

Radon measures satistisfy a bunch of useful properties. Firstly, when equipped with the total variation norm, they form a normed space which coincides with the dual of compactly supported continuous functions:

THEOREM 1.1 (Riesz)
Let $C_c(\mathbb{R}^d; \mathbb{R}^n)$ be the space of $\mathbb{R}^n$-valued, compactly supported, continuous functions on $\mathbb{R}^d$. Then, for all bounded linear functionals $\Lambda \colon C_c(\mathbb{R}^d; \mathbb{R}^n) \to \mathbb{R}$, there exists a unique $\mathbb{R}^n$-valued Radon measure μ on $\mathbb{R}^d$ such that

$$\Lambda(\varphi) = \int_{\mathbb{R}^d} \varphi \cdot \mathrm{d}\mu \quad \text{for all } \varphi \in C_c(\mathbb{R}^d; \mathbb{R}^n).$$

Moreover, for such measure and for all open $V \subset \mathbb{R}^d$, it holds

$$|\mu|(U) = \sup \left\{ \Lambda(\varphi) : \varphi \in C_c(V; \mathbb{R}^n), |\varphi| \leq 1 \right\}$$

In view of the previous representation result, we say that the sequence of Radon measures $\{\,\mu_\ell\,\}$ *weakly-∗ converges to the Radon measure* μ, and we write $\mu_\ell \overset{*}{\rightharpoonup} \mu$, if

$$\lim_{\ell \to +\infty} \int_{\mathbb{R}^d} \varphi \cdot \mathrm{d}\mu_\ell = \int_{\mathbb{R}^d} \varphi \cdot \mathrm{d}\mu \quad \text{for all } \varphi \in C_{\mathrm{c}}(\mathbb{R}^d; \mathbb{R}^n).$$

We stated above that if μ is a vector valued Radon measure, then $\mu = f\,|\mu|$ for some $L^1(|\mu|)$. Relations of this kind hold in wider generality:

THEOREM 1.2 (Besicovitch's Differentiation)
Let μ and ν be respectively a positive and an $\mathbb{R}^n$-valued Radon measure on some open $\Omega \subset \mathbb{R}^d$. If for all Borel sets $E \subset \Omega$ $\mu(E) = 0$ implies $|\nu|\,(E) = 0$, then for μ-a.e. $x \in \mathrm{supp}\mu$ the following limit exists:
$$f(x) := \lim_{r \to 0^+} \frac{\nu(B(x,r))}{\mu(B(x,r))}.$$

Moreover, $f \in L^1(\Omega, \mu; \mathbb{R}^n)$ and $\nu = f\mu$.

We refer to the function f above as the *Radon-Nikodym derivative* of ν w.r.t. to μ and we denote it by $\mathrm{d}\nu/\mathrm{d}\mu$.

Another useful property of Radon measures concerns the maximum numbers of leaves in a foliation which can have positive mass for a Radon measure:

PROPOSITION 1.3
Let μ be a Radon measure on $\mathbb{R}^d$ and let $\{E_\alpha\}_{\alpha \in A}$ be a family of disjoint Borel sets. Then, there exist at most countable $\alpha \in A$ such that $\mu(E_\alpha) > 0$.

Finally, we mention a simple criterion to prove that the restriction of a Borel measure is a Radon measure. We recall that the *restriction* of the measure μ to the set Ω is the measure given by $\mu{\llcorner}\Omega(E) := \mu(E \cap \Omega)$.

PROPOSITION 1.4
If μ is a Borel regular measure on $\mathbb{R}^d$ and E is a μ-measurable set such that $\mu{\llcorner}E$ is locally finite, then $\mu{\llcorner}E$ is a Radon measure.

The measures that occur most frequently in the sequel are the the d-dimensional Lebesgue and the $(d-1)$-dimensional Hausdorff measure on $\mathbb{R}^d$, which we denote respectively by $\mathscr{L}^d$ and $\mathscr{H}^{d-1}$. The Lebesgue measure is an example of Radon measure, while $\mathscr{H}^{d-1}$ is Borel regular, but not locally finite; however, by Proposition 1.4, the restriction $\mathscr{H}^{d-1}{\llcorner}E$ is Radon whenever $\mathscr{H}^{d-1}(E)$ is finite.

We shall henceforth neglect to specify the measure w.r.t. which a set or a function is measurable, when the measure is $\mathscr{L}^d$ or the product $\mathscr{L}^d \times \mathscr{L}^d$ on $\mathbb{R}^d \times \mathbb{R}^d$; similarly, we shall adopt the expression "a.e." in place of "$\mathscr{L}^d$-a.e." and of "$\mathscr{L}^d \times \mathscr{L}^d$-a.e.". If u and v are measurable functions, we shall use "$u = v$" as a shorthand for "$u(x) = v(x)$ for a.e. $x \in \mathbb{R}^d$"; in the same spirit, if E and F are measurable sets, "$E = F$" means that E and F coincide up to a negligible set and "$E \subset F$" that $\mathscr{L}^d(E \cap F^{\mathrm{c}}) = 0$.

We now introduce the class of functions whose distributional derivative is a Radon measure. If u is a function, we denote by $\mathrm{D}u$ its distributional gradient, while we use ∇u for the pointwise one.

Let $\Omega \subset \mathbb{R}^d$ be an open set. A function $u \in L^1(\Omega)$ is *of bounded variation* in Ω if $\mathrm{D}u$ is a $\mathbb{R}^d$-valued Radon measure on Ω such that $\|\mathrm{D}u\| < +\infty$. In this case, we write $u \in \mathrm{BV}(\Omega)$. Since

$$\|\mathrm{D}u\| = \sup\left\{ \int_\Omega u \, \mathrm{div}\,\zeta : \zeta \in C_{\mathrm{c}}^1(\Omega;\mathbb{R}^d), |\zeta| \le 1 \right\}, \tag{1.1}$$

$\mathrm{BV}(\Omega)$ may be equivalently defined as the subspace of all $u \in L^1(\Omega)$ such that the right-hand side in (1.1) is finite. Observe that, in the light of (1.1), the map $u \mapsto \|\mathrm{D}u\|$ is lower semicontinuous w.r.t. the $L_{\mathrm{loc}}^1(\Omega)$ convergence.

The total variation of a BV functions bounds the L^1-norm of its difference quotients. Actually, this property provides a characterisation that we shall exploit several times in our analysis:

PROPOSITION 1.5 (Characterisation *via* difference quotients)
Let $\Omega \subset \mathbb{R}^d$ be open. Then, $u\colon \Omega \to \mathbb{R}$ is a function of bounded variation in Ω if and only if there exists a constant $c \ge 0$ such that for any open set V that is compactly contained in Ω and for any $z \in \mathbb{R}^d$ with $|z| < \mathrm{dist}(V, \Omega^{\mathrm{c}})$ it holds

$$\|\tau_z u - u\|_{L^1(V)} \le c\,|z|,$$

where $\tau_z u(x) \coloneqq u(x + z)$. In particular, it is possible to choose $c = |\mathrm{D}u|\,(\Omega)$.

Finally, we remind a compactness criterion for functions of bounded variation.

THEOREM 1.6
Let $\Omega \subset \mathbb{R}^d$ be an open set and let $\{\,u_\ell\,\}$ be a sequence in $\mathrm{BV}_{\mathrm{loc}}(\Omega)$. If for all open sets V that are compactly contained in Ω it holds $\|u_\ell\|_{L^1(V)} + |\mathrm{D}u_\ell|\,(V) \le c$ for some $c \ge 0$, then there exist a subsequence $\{\,u_{\ell_m}\,\}$ and a function $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ such that $\{\,u_{\ell_m}\,\}$ converges to u in $L_{\mathrm{loc}}^1(\Omega)$.

1.2   Finite perimeter sets and anisotropic surface energies

The material in this section lays the groundwork for Chapters 3 and 4. There, we deal with a class of functionals that we name nonlocal perimeters; here, instead, we introduce the concept of distributional perimeter, which we may consider as the classical one.

The definition of perimeter dates back to the seminal works of R. Caccioppoli and E. De Giorgi. In a nutshell, we identify a set with its characteristic function, we consider the distributional gradient of the latter, and we define its total variation as perimeter of the given set. So, if $E \subset \mathbb{R}^d$ is measurable and $\Omega \subset \mathbb{R}^d$ is an open set, we define the *perimeter* of E in Ω the quantity

$$\mathrm{Per}(E;\Omega) \coloneqq \sup\left\{ \int_E \mathrm{div}\,\zeta : \zeta \in C_{\mathrm{c}}^1(\Omega;\mathbb{R}^d), |\zeta| \le 1 \right\}. \tag{1.2}$$

We say that E is a *set of finite perimeter* in Ω when $\mathrm{Per}(E;\Omega) < +\infty$. Sometimes, we shall use the expression *Caccioppoli set* as a synonym, and we shall refer to Per as De Giorgi's perimeter, when we want to distinguish it from the nonlocal counterparts that occur in the next chapters.

In what follows, we shall frequently identify sets with their characteristic functions. In particular, by saying that a sequence $\{\, E_\ell \,\}$ converges in $L^1_{\mathrm{loc}}(\Omega)$ to E we mean that

$$\lim_{\ell \to +\infty} \mathscr{L}^d\big((E_\ell \triangle E) \cap C\big) = 0 \quad \text{for all compact sets } C \text{ such that } C \subset \Omega,$$

As a consequence of its definition, the perimeter is $L^1_{\mathrm{loc}}(\Omega)$-lower semicontinuous.

Using the language of the previous section, we may say that E has finite perimeter in Ω if its characteristic function χ_E belongs to $\mathrm{BV}_{\mathrm{loc}}(\Omega)$. Accordingly, if E has finite perimeter in Ω, then $\mathrm{D}\chi_E$ is an $\mathbb{R}^d$-valued Radon measure in Ω and

$$\mathrm{Per}(E; \Omega) := |\mathrm{D}\chi_E|\,(\Omega).$$

Moreover, if we denote by $\hat{n}$ the Radon-Nikodym derivative of $\mathrm{D}\chi_E$ w.r.t. $|\mathrm{D}\chi_E|$, it holds

$$\int_E \mathrm{div}\,\zeta\,\mathrm{d}x = -\int_\Omega \zeta \cdot \hat{n}\,\mathrm{d}\,|\mathrm{D}\chi_E| \quad \text{for all } \zeta \in C^1_{\mathrm{c}}(\Omega; \mathbb{R}^d). \tag{1.3}$$

The unit vector field

$$\hat{n}(x) := \lim_{r \to 0^+} \frac{\mathrm{D}\chi_E(B(x,r))}{|\mathrm{D}\chi_E|\,(B(x,r))}, \tag{1.4}$$

which is well defined $|\mathrm{D}\chi_E|$-a.e. $x \in \mathrm{supp}\,|\mathrm{D}\chi_E|$, may be understood as the *inner normal* of E.

Further, let

$$\partial^* E := \left\{\, x \in \mathbb{R}^d : \hat{n}(x) \text{ exists and } |\hat{n}(x)| = 1 \,\right\}.$$

Fundamental results by De Giorgi and H. Federer [DG55, Fed59] show that $\partial^* E$ is $(d-1)$-rectifiable for all measurable E and that $\mathrm{D}\chi_E = \hat{n}\mathscr{H}^{d-1}\llcorner\partial^* E$, whence

$$\mathrm{Per}(E; \Omega) = \mathscr{H}^{d-1}(\partial^* E \cap \Omega),$$

We call the set $\partial^* E$ *reduced boundary* of E. For any $x \in \partial^* E$, it holds

$$\frac{E - x}{r} \to \left\{\, y \in \mathbb{R}^d : y \cdot \hat{n}(x) > 0 \,\right\} \quad \text{as } r \to 0^+ \text{ in } L^1_{\mathrm{loc}}(\mathbb{R}^d). \tag{1.5}$$

Sets of finite perimeter may be exploited to describe various physical models. We are interested in anisotropic surface energies, but we also introduce some more general concepts.

Let $\sigma \colon \mathbb{R}^n \to [0, +\infty]$ be a positevely 1-homogeneous measurable function and let $\Omega \subset \mathbb{R}^d$ be measurable. For any $\mathbb{R}^n$-valued Radon measure on $\mathbb{R}^d$, we define the *anisotropic total variation* of μ as

$$J_\sigma(\mu; \Omega) := \int_\Omega \sigma\left(\frac{\mathrm{d}\mu}{\mathrm{d}\,|\mu|}\right)\mathrm{d}\,|\mu|. \tag{1.6}$$

The anisotropy is encoded by the function σ; when it is constant, the usual total variation is recovered.

The following results sums up the continuity properties of J_σ:

THEOREM 1.7 (Reshetnyak)
Let J_σ be the functional in (1.6).

(i)   *If $\sigma\colon \mathbb{R}^n \to [0, +\infty]$ is 1-homogeneous, lower semicontinuous, and convex, then for all open $\Omega \subset \mathbb{R}^d$ it holds*

$$J_\sigma(\mu; \Omega) \leq \liminf_{k\to+\infty} J_\sigma(\mu_k; \Omega)$$

whenever μ and μ_k, for all $k \in \mathbb{N}$, are $\mathbb{R}^n$-valued Radon measures on $\mathbb{R}^d$ such that $\mu_k \overset{*}{\rightharpoonup} \mu$.

(ii)   *If $\sigma\colon \mathbb{S}^{n-1} \to [0, +\infty)$ is bounded and continuous, then for all open $\Omega \subset \mathbb{R}^d$ it holds*

$$J_\sigma(\mu; \Omega) = \lim_{k\to+\infty} J_\sigma(\mu_k; \Omega)$$

whenever μ and μ_k, for all $k \in \mathbb{N}$, are $\mathbb{R}^n$-valued Radon measures on Ω such that $\mu_k \overset{*}{\rightharpoonup} \mu$, $\|\mu_k\| \to \|\mu\|$, and $\|\mu\| < +\infty$.

For $E \subset \mathbb{R}^d$, we may introduce an *anisotropic perimeter* functional by setting

$$\mathrm{Per}_\sigma(E; \Omega) := J_\sigma(\mathrm{D}\chi_E; \Omega) = \int_{\partial^* E \cap \Omega} \sigma(\hat{n}(x))\mathrm{d}\mathscr{H}^{d-1}(x). \tag{1.7}$$

It is well-known that the total variation of a BV function equals the integral of the perimeter of its level set. We recall an anisotropic variant of such identity:

PROPOSITION 1.8 (Anisotropic Coarea Formula,[CCC$^+$10, Gra09])
Suppose that $\sigma\colon \mathbb{R}^d \to [0, +\infty]$ is a norm and that Ω is an open, bounded set with Lipschitz boundary. Then, for all $u \in \mathrm{BV}(\Omega)$,

$$J_\sigma(\mathrm{D}u; \Omega) = \int_{-\infty}^{+\infty} \mathrm{Per}_\sigma(\{\, u > t \,\}; \Omega)\mathrm{d}t$$

Minimising a surface energy under prescribed boundary conditions is a classical problem. In precise terms, given the reference set Ω and the boundary datum E_0, we look for a set E such that $\mathrm{Per}_\sigma(E; \Omega)$ attains

$$\inf \left\{\, J_\sigma(\mathrm{D}u; \Omega) : u\colon \mathbb{R}^d \to [0, 1] \text{ is measurable and } u = \chi_{E_0} \text{ in } \Omega^{\mathsf{c}}. \right\}$$

When σ is a norm, we can show that the infimum is achieved: indeed, in this case, $J_\sigma(\mathrm{D}u; \Omega) \geq c |\mathrm{D}u|(\Omega)$ for some constant $c > 0$, and standard arguments yield the conclusion.

1.3   Γ-convergence

We include in this section the definition of a celebrated notion of convergence proposed by De Giorgi in the '70s [DGF75]. For a thorough treatment of the subject, we refer to the monographs [DM93] by G. Dal Maso and [Bra02] by A. Braides.

DEFINITION 1.9
Let (X, d) be a metric space and, for all $\varepsilon > 0$, let $f_\varepsilon\colon X \to [-\infty, +\infty]$ be a function. We say that the family $\{\, f_\varepsilon \,\}_{\varepsilon>0}$ Γ-converges as $\varepsilon \to 0^+$ w.r.t. the metric d to the function $f_0\colon X \to [-\infty, +\infty]$ if for any $x \in X$ and for any sequence $\{\, \varepsilon_\ell \,\}_{\ell \in \mathbb{N}}$ such that $\varepsilon_\ell \to 0^+$ the following hold:

(i)   *for any sequence* $\{\, x_{\varepsilon_\ell} \,\}_{\ell\in\mathbb{N}} \subset X$ *such that* $x_{\varepsilon_\ell} \to x$, *we have*

$$f_0(x) \leq \liminf_{\ell\to+\infty} f_{\varepsilon_\ell}(x_{\varepsilon_\ell}); \qquad (1.8)$$

(ii)   *there exists a sequence* $\{\, x_{\varepsilon_\ell} \,\}_{\ell\in\mathbb{N}} \subset X$ *such that* $x_{\varepsilon_\ell} \to x$ *and*

$$\limsup_{\ell\to+\infty} f_{\varepsilon_\ell}(x_{\varepsilon_\ell}) \leq f_0(x). \qquad (1.9)$$

If $\{\, f_\varepsilon \,\}$ Γ*-converges to* f_0, *the we say that* f_0 *is the* Γ*-limit of* $\{\, f_\varepsilon \,\}$.

Formula (1.8) is referred to as Γ-*lower limit inequality*, while (1.9) is known as Γ-*upper limit inequality*. A sequence such that the latter holds is called *recovery sequence*.

We highlight in a separate statement a couple of properties that we shall use in the sequel.

LEMMA 1.10
Let (X, d) *and* $\{\, f_\varepsilon \,\}$ *be as in the previous definition. Let also* $f_0\colon X \to [-\infty, +\infty]$ *be a function.*

(i)   *If* f_0 *is the* Γ-*limit of* $\{\, f_\varepsilon \,\}$, *then it is lower semicontinuous w.r.t. the metric* d.

(ii)   *Let* $Y \subset X$ *be a subset such that for any* $x \in X$ *there exists a sequence* $\{\, y_\ell \,\} \subset Y$ *such that* $y_\ell \to x$ *and* $f_0(y_\ell) \to f_0(x)$. *If for all* $y \in Y$ *there exists a recovery sequence contained in* Y, *then* (1.9) *holds for all* $x \in X$.

A subset Y that fulfils the hypotheses of Lemma 1.10(ii) is said to be *dense in energy* for f_0.

Chapter 2

Rate of convergence of the Bourgain-Brezis-Mironescu formula

Let $\Omega \subset \mathbb{R}^d$ be a smooth, bounded domain. In the renown paper [BBM01], J. Bourgain, H. Brezis, and P. Mironescu considered iterated integrals of the form

$$\int_\Omega \int_\Omega \rho(y-x) \frac{|u(y)-u(x)|^p}{|y-x|^p} \mathrm{d}y \mathrm{d}x, \tag{2.1}$$

where $\rho \in L^1(\mathbb{R}^d)$ is a radial mollifier and $u \in L^p(\Omega)$, with $p \geq 1$. They proved that, if u belongs to the Sobolev space $W^{1,p}(\Omega)$, then the L^p-norm of the weak gradient is achieved as the limit of the quantity above when ρ approaches the Dirac delta in 0. The archetypical case is the study as $s \to 1^-$ of the *Gagliardo seminorm*, which, for $s \in (0,1)$ and $u \in L^p(\Omega)$, is defined as

$$|u|_{W^{s,p}(\Omega)} := \left[\iint_\Omega \int_\Omega \frac{|u(y)-u(x)|^p}{|y-x|^{d+sp}} \mathrm{d}y \mathrm{d}x \right]^{\frac{1}{p}}.$$

In this respect, the analysis in [BBM01] yields

$$\lim_{s \to 1^-} (1-s)^{\frac{1}{p}} |u|_{W^{s,p}(\Omega)} = c \, \|\nabla u\|_{L^p(\Omega)} \qquad \text{if } u \in W^{1,p}(\Omega),$$

with $c := c(d,p) > 0$.

In the same paper, when $p = 1$, the authors also managed to characterise the space of functions of bounded variations $\mathrm{BV}(\Omega)$ w.r.t. the asymptotic behaviour of (2.1). However, whether the total variation of the distributional gradient is attained in the limit was an issue that they left unsolved. Positive answers were soon provided independently by J. Dávila in [Dáv02] and by A. Ponce in [Pon04]. Notably, the latter extended the analysis to more general integral functionals and studied their limits also in sense of Γ-convergence. This is the starting point of the current chapter, which is based on [CNP19] and is devoted to the study of the rate of convergence of a class of functionals appeared in the paper by Ponce. In Section 2.1, we recall a result of his (see also Theorem 4.2 in the next chapter), we state the main result of this chapter, Theorem 2.1 below, and we compare it to the existing literature. The proof is presented in the last section. It grounds on a slicing argument, which reduces the problem to the study of suitable functionals defined on functions of one real variable. The analysis of these auxiliary 1-dimensional functionals is developed in Section 2.2.

2.1 Main result and slicing

Let Ω be an open set with compact Lipschitz boundary, let $G \colon \mathbb{R}^d \to [0, +\infty)$ be an $L^1(\mathbb{R}^d)$-function, and let $f \colon [0, +\infty) \to [0, +\infty)$ be a convex function such that $f(0) = 0$. For $\varepsilon > 0$, we consider the L^1-norm preserving rescalings of G

$$
G_\varepsilon(x) := \frac{1}{\varepsilon^d} G\left(\frac{x}{\varepsilon}\right)
$$

and, for $u \in L^1(\Omega)$, the nonlocal functionals

$$
\mathscr{F}_\varepsilon(u) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G_\varepsilon(z) f\left(\frac{|u(x+z) - u(x)|}{|z|}\right) \mathrm{d}z\mathrm{d}x.
$$

By [Pon04], we know that, as $\varepsilon \to 0^+$, the family $\{\,\mathscr{F}_\varepsilon\,\}$ converges to the limit functional

$$
\mathscr{F}_0(u) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(z) f(|\nabla u(x) \cdot \hat{z}|) \mathrm{d}z\mathrm{d}x,
$$

where $\hat{z} = z/|z|$ for all $z \in \mathbb{R}^d \setminus \{0\}$. Approximations by finite difference functionals have also been studied in relation to free discontinuity problems by M. Gobbino and M. Mora [Gob98, GM01]. We shall see that their method, the so called slicing, can be adapted to our problem as well.

In this chapter, we study the asymptotics of the functionals that express the rate of convergence of $\mathscr{F}_\varepsilon$ to the limit $\mathscr{F}_0$, that is, we let

$$
\begin{aligned}
\mathscr{E}_\varepsilon(u) &:= \frac{\mathscr{F}_0(u) - \mathscr{F}_\varepsilon(u)}{\varepsilon^2} \\
&= \frac{1}{\varepsilon^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[G(z) f(|\nabla u(x) \cdot \hat{z}|) - G_\varepsilon(z) f\left(\frac{|u(x+z) - u(x)|}{|z|}\right) \right] \mathrm{d}z\mathrm{d}x
\end{aligned}
\tag{2.2}
$$

and we consider the limit of $\{\,\mathscr{E}_\varepsilon\,\}$ as ε tends to 0. We prove that the limit is a *second order nonlocal functional* (see (2.6)), under the following set of assumptions:

A1: $G(x) = G(-x)$ for a.e. $x \in \mathbb{R}^d$ and

$$
\int_{\mathbb{R}^d} G(x) \left(1 + |x|^2\right) \mathrm{d}x < +\infty.
\tag{2.3}
$$

A2: there exist $r_0 \geq 0$ and $r_1 > 0$ such that $r_0 < \beta_d\, r_1$, where

$$
\beta_d := \begin{cases} 1 & \text{when } d = 2, \\ \dfrac{d-2}{d-1} & \text{when } d > 2, \end{cases}
\tag{2.4}
$$

and

$$
\operatorname{ess\,inf} \{\, G(x) : x \in B(0, r_1) \setminus B(0, r_0) \,\} > 0.
\tag{2.5}
$$

A3: f is of class C^2, $f(0) = f'(0) = 0$, and

there exists $\alpha > 0$ such that $2f(t) - \alpha t^2$ is convex.

We establish the following:

THEOREM 2.1 (Compactness and Γ-convergence of the rate functionals)
*Let Ω be a bounded open set with Lipschitz boundary and let G and f fulfil **A1**, **A2**, and **A3**. Let also*

$$X := \left\{ u \in H^1_{\mathrm{loc}}(\mathbb{R}^d) : \nabla u = 0 \text{ a.e. in } \Omega^{\mathsf{c}} \right\}$$

and

$$\mathscr{E}_0(u) := \begin{cases} \dfrac{1}{24} \displaystyle\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(z) \, |z|^2 \, f''(|\nabla u(x) \cdot \hat{z}|) \, \left|\nabla^2 u(x)\hat{z} \cdot \hat{z}\right|^2 \mathrm{d}z\mathrm{d}x \\ \hfill \text{if } u \in X \cap H^2_{\mathrm{loc}}(\mathbb{R}^d), \quad (2.6) \\ +\infty \hfill \text{otherwise}. \end{cases}$$

Then, there hold:

(i) *For any family $\{ u_\varepsilon \} \subset X$ such that $\mathscr{E}_\varepsilon(u_\varepsilon) \leq M$ for some $M > 0$, there exists a sequence $\{ \varepsilon_\ell \}$ and a function $u \in H^2(\mathbb{R}^d)$ vanishing outside Ω such that $\nabla u_{\varepsilon_\ell} \to \nabla u$ in $L^2(\mathbb{R}^d)$ as $\ell \to +\infty$.*

(ii) *For any family $\{ u_\varepsilon \} \subset X$ that converges to $u \in X$ in $H^1_{\mathrm{loc}}(\mathbb{R}^d)$,*

$$\mathscr{E}_0(u) \leq \liminf_{\varepsilon \to 0^+} \mathscr{E}_\varepsilon(u_\varepsilon).$$

(iii.a) *For any $u \in X$ such that $\nabla u \in L^\infty(\Omega)$ there exists a family $\{ u_\varepsilon \} \subset X$ that approaches u in $H^1_{\mathrm{loc}}(\mathbb{R}^d)$ and satisfies*

$$\limsup_{\varepsilon \to 0^+} \mathscr{E}_\varepsilon(u_\varepsilon) \leq \mathscr{E}_0(u).$$

(iii.b) *If f'' is bounded, for any $u \in X$ there exists a family $\{ u_\varepsilon \} \subset X$ that converges to u in $H^1_{\mathrm{loc}}(\mathbb{R}^d)$ with the property that*

$$\limsup_{\varepsilon \to 0^+} \mathscr{E}_\varepsilon(u_\varepsilon) \leq \mathscr{E}_0(u).$$

REMARK 2.2
If we replace X with $H^1_{\mathrm{loc}}(\mathbb{R}^d)$, it can be seen that for $u \in H^2_{\mathrm{loc}}(\mathbb{R}^d)$ statements (ii), (iii.a), and (iii.b) in Theorem 2.1 still hold true, the proof remaining essentially unchanged. Conversely, the passage to the unbounded setting invalidates statement (i).

The problem we tackle is not a higher order Γ-limit of $\mathscr{F}_\varepsilon$, which corresponds instead to studying the asymptotics of

$$\frac{\mathscr{F}_\varepsilon - \min \mathscr{F}_0}{\varepsilon^\beta} \qquad \text{for some } \beta > 0.$$

We refer to [AB93] for more details about developments by Γ-convergence. On the other hand, our conclusion is reminiscent of the one obtained in [PPR07] by M. Peletier, P. Planqué and M. Röger, who dealt with a model for bilayer membranes. There, the authors considered the convolution functionals

$$\tilde{\mathscr{F}}_\varepsilon(u) := \int_{\mathbb{R}^d} f\left(G_\varepsilon * u\right) \mathrm{d}x,$$

which converge, as $\varepsilon \to 0^+$, to

$$\tilde{\mathscr{F}}_0(u) = c \int_{\mathbb{R}^d} f(u) \mathrm{d}x, \quad \text{with } c := c(d, G) > 0.$$

They showed that the rate functionals $\tilde{E}_\varepsilon := \varepsilon^{-2}(\tilde{\mathscr{F}}_0 - \tilde{\mathscr{F}}_\varepsilon)$ converge pointwise for $u \in H^1(\mathbb{R}^d)$ to the functional

$$\frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(z)|z|^2 f''(u(x))|\nabla u(x) \cdot \hat{z}|^2 \mathrm{d}z\mathrm{d}x,$$

and they proved as well that the family $\left\{ \tilde{E}_\varepsilon(u) \right\}$ is uniformly bounded if and only if $u \in H^1(\mathbb{R}^d)$. We note that our analysis yields a similar regularity criterion: indeed, if $\mathscr{F}_\varepsilon(u)$ tends to $\mathscr{F}_0(u)$ sufficiently fast, i.e. if $\{ \mathscr{E}_\varepsilon(u) \}$ is uniformly bounded for all small ε, then $u \in H^2(\mathbb{R}^d)$, see Remark 2.19 below.

When the kernel G is radially symmetric and f is the modulus, our study is related to a geometric problem considered in [MS19] to treat a physical model for liquid drops with dipolar repulsion. However, the uniform convexity assumption on f, which we utilise to prove the Γ-inferior limit inequality, excludes this scenario from our analysis, as well as the choice $f(t) = |t|^p$ with $p \geq 1$, $p \neq 2$. Extensions of our result aimed at including these cases are a possible subject of investigation.

As for the assumptions on the kernel, by **A2** we require that the support of G contains a sufficiently large annulus centred at the origin. This might be regarded as a very weak isotropy hypothesis on G, in a sense that will become clearer in the proof of Lemma 2.18 The simplest case for which (2.5) holds is when there exists $\gamma > 0$ such that $G(x) \geq \gamma$ for all $x \in B(0, r_1)$.

REMARK 2.3 (Radial case)
When G is radial, that is, $G(z) = \bar{G}(|z|)$ for some $\bar{G}\colon [0, +\infty) \to [0, +\infty)$, we find

$$\mathscr{F}_0(u) = \|G\|_{L^1(\mathbb{R}^d)} \int_{\mathbb{R}^d} \fint_{\mathbb{S}^{d-1}} f(|\nabla u(x) \cdot e|) \mathrm{d}\mathscr{H}^{d-1}(e) \mathrm{d}x,$$

$$\mathscr{E}_0(u) = \frac{1}{24} \left(\int_{\mathbb{R}^d} G(z)\,|z|^2\,\mathrm{d}z \right) \int_{\mathbb{R}^d} \fint_{\mathbb{S}^{d-1}} f''(|\nabla u(x) \cdot e|)\,\big|\nabla^2 u(x)e \cdot e\big|^2\,\mathrm{d}\mathscr{H}^{d-1}(e)\mathrm{d}x.$$

We prove Theorem 2.1 by a *slicing* procedure, which amounts to express the d-dimensional energies $\mathscr{E}_\varepsilon$ as superpositions of suitable 1-dimensional energies E_ε, regarded as functionals on each line of $\mathbb{R}^d$. Then, we establish the convergence of the functionals E_ε (see Section 2.2), and we recover from it the result concerning $\mathscr{E}_\varepsilon$ (see Section 2.3).

The slicing approach was introduced in [Gob98, GM01]. In our case, the next lemma shows that, for $u \in L^2(\mathbb{R})$ vanishing outside a given bounded interval and $\varepsilon > 0$, the 1-dimensional functionals retrieved by slicing are

$$E_\varepsilon(u) := \frac{1}{\varepsilon^2} \int_{\mathbb{R}} \left[f(u(x)) - f\left(\fint_x^{x+\varepsilon} u(y)dy \right) \right] \mathrm{d}x,$$

$$E_0(u) := \begin{cases} \dfrac{1}{24} \displaystyle\int_{\mathbb{R}} f''(u(x))\left|u'(x)\right|^2 \mathrm{d}x & \text{if } u \in H^1(\mathbb{R}) \cap C_c(\mathbb{R}), \\ +\infty & \text{otherwise.} \end{cases}$$

We recall that, for $z \in \mathbb{R}^d \setminus \{ 0 \}$, we set $\hat{z} := z/|z|$ and

$$\hat{z}^\perp := \left\{ \xi \in \mathbb{R}^d : \xi \cdot z = 0 \right\}.$$

LEMMA 2.4 (Slicing)
For $u \in X$, $z \in \mathbb{R}^d \setminus \{ 0 \}$, and $\xi \in \hat{z}^\perp$, let

$$
w_{\hat{z},\xi}: \quad \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ t & \longmapsto & u(\xi + t\hat{z}). \end{array}
$$

Then, $w'_{\hat{z},\xi}(t) = \nabla u(\xi + t\hat{z}) \cdot \hat{z}$ and

$$
\mathscr{E}_\varepsilon(u) = \int_{\mathbb{R}^d} \int_{z^\perp} G(z) \, |z|^2 \, E_{\varepsilon|z|}(w'_{\hat{z},\xi}) \mathrm{d}\mathcal{H}^{d-1}(\xi) \mathrm{d}z, \tag{2.7}
$$

$$
\mathscr{E}_0(u) = \int_{\mathbb{R}^d} \int_{z^\perp} G(z) \, |z|^2 \, E_0(w'_{\hat{z},\xi}) \mathrm{d}\mathcal{H}^{d-1}(\xi) \mathrm{d}z.
$$

Proof. We prove only (2.7), the case of $\mathscr{E}_0$ being identical.

The formula is a simple application of Fubini's Theorem. Indeed, given the direction $\hat{z} \in \mathbb{S}^{d-1}$, for all $x \in \mathbb{R}^d$, we have $x = \xi + t\hat{z}$ for some $\xi \in \mathbb{R}^d$ such that $\xi \cdot z = 0$ and $t \in \mathbb{R}$. Using this decomposition, we write

$$
\begin{aligned}
\mathscr{F}_\varepsilon(u) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(z) f\left(\frac{|u(x + \varepsilon z) - u(x)|}{\varepsilon \, |z|} \right) \mathrm{d}z \mathrm{d}x \\
&= \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} G(z) f\left(\frac{|w_{\hat{z},\xi}(t + \varepsilon \, |z|) - w_{\hat{z},\xi}(t)|}{\varepsilon \, |z|} \right) dt \mathrm{d}\mathcal{H}^{d-1}(\xi) \mathrm{d}z,
\end{aligned}
$$

whence $\mathscr{E}_\varepsilon(u)$ equals

$$
\frac{1}{\varepsilon^2} \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} G(z) \left[f\big(|w'_{\hat{z},\xi}(t)|\big) - f\left(\frac{|w_{\hat{z},\xi}(t + \varepsilon \, |z|) - w_{\hat{z},\xi}(t)|}{\varepsilon \, |z|} \right) \right] dt \mathrm{d}\mathcal{H}^{d-1}(\xi) \mathrm{d}z.
$$

We obtain (2.7) by multiplying and dividing the integrands by $|z|^2$. $\qquad \square$

2.2 Γ-convergence of the auxiliary functionals

In the current section we analyse the limiting properties of the family of auxiliary functionals E_ε obtained by slicing the d-dimensional energy $\mathscr{E}_\varepsilon$. So, for $u \in L^2(\mathbb{R})$ and $\varepsilon > 0$, we let

$$
U(x) := \int_0^x u(y) \mathrm{d}y, \qquad \mathrm{D}_\varepsilon U(x) := \frac{U(x + \varepsilon) - U(x)}{\varepsilon} = \fint_x^{x+\varepsilon} u(y) \mathrm{d}y, \tag{2.8}
$$

and, as above, we define

$$
E_\varepsilon(u) := \frac{1}{\varepsilon^2} \int_{\mathbb{R}} [f(u(x)) - f(\mathrm{D}_\varepsilon U(x))] \, \mathrm{d}x. \tag{2.9}
$$

We fix an open interval $I := (a, b) \subset \mathbb{R}$ and we consider the closed subspace of $L^2(\mathbb{R})$

$$
Y := \left\{ u \in L^2(\mathbb{R}) : u = 0 \text{ a.e. in } I^c \right\}. \tag{2.10}
$$

By positivity and convexity of f, one can check that $E_\varepsilon(u)$ belongs to $[0, +\infty]$ when $u \in Y$. We want to compute the Γ-limit of $\{ E_\varepsilon \}$ thought as a collection of functionals on Y endowed with the L^2-topology. The candidate limit is

$$
E_0(u) := \begin{cases} \dfrac{1}{24} \displaystyle\int_{\mathbb{R}} f''(u(x)) \left| u'(x) \right|^2 \mathrm{d}x & \text{if } u \in Y \cap H^1(\mathbb{R}), \\ +\infty & \text{otherwise.} \end{cases} \tag{2.11}
$$

Our result reads as follows:

THEOREM 2.5 (Γ-convergence of the auxiliary functionals)
Assume that f is as in **A3**. *Then, as $\varepsilon \to 0^+$, the restriction to Y of the family $\{\, E_\varepsilon \,\}$ Γ-converges w.r.t. the $L^2(\mathbb{R})$-topology to E_0, that is, for any $u \in Y$ the following hold:*

(i) *Whenever $\{\, u_\varepsilon \,\} \subset Y$ is a family that converges to u in $L^2(\mathbb{R})$, we have*

$$E_0(u) \leq \liminf_{\varepsilon \to 0^+} E_\varepsilon(u_\varepsilon).$$

(ii) *There exists $\{\, u_\varepsilon \,\} \subset Y$ converging to u in $L^2(\mathbb{R})$ such that*

$$\limsup_{\varepsilon \to 0^+} E_\varepsilon(u_\varepsilon) \leq E_0(u). \tag{2.12}$$

REMARK 2.6 (Lower order limits)
Taking into account Proposition 2.7 below, *a posteriori* $\{\, \varepsilon^\beta E_\varepsilon \,\}$ Γ-converges to 0 for all $\beta > 0$. However, when $\beta = 1, 2$, this can be proved straightforwardly. Indeed, if for $\varepsilon > 0$ we let

$$F_0(u) := \int_{\mathbb{R}} f(u(x))\mathrm{d}x \quad \text{and} \quad F_\varepsilon(u) := \int_{\mathbb{R}} f(\mathrm{D}_\varepsilon U(x))\mathrm{d}x$$

(recall position (2.8)), then it is easy to see that $\lim_{\varepsilon \to 0^+} F_\varepsilon(u) = F_0(u)$, and, when $u \in Y \cap C^2(\mathbb{R})$, we also find

$$\lim_{\varepsilon \to 0^+} \frac{F_0(u) - F_\varepsilon(u)}{\varepsilon} = \frac{1}{2} \left[f(u(a)) + f(u(b)) \right] = 0.$$

We separate the proofs of the two statements in Theorem 2.5. We focus on the lower limit in Proposition 2.11, which is based on a estimate from below of the energy and on a compactness criterion, established respectively in Lemma 2.8 and in Lemma 2.10. We deal with the Γ-upper limit in the next proposition, where we compute the pointwise limit, as $\varepsilon \to 0^+$, of $E_\varepsilon(u)$ for u smooth.

PROPOSITION 2.7
For all $u \in Y \cap C^2(\mathbb{R})$,
$$\lim_{\varepsilon \to 0^+} E_\varepsilon(u) = E_0(u).$$

More precisely, there exists a continuous, bounded, and increasing function $m\colon [0, +\infty) \to [0, +\infty)$ such that $m(0) = 0$ and

$$|E_\varepsilon(u) - E_0(u)| \leq c\, m(\varepsilon), \tag{2.13}$$

where $c := c(b - a, \|u\|_{C^2(\mathbb{R})}, \|f\|_{C^2([-\|u\|_{C^2(\mathbb{R})}, \|u\|_{C^2(\mathbb{R})}])}) > 0$ is a constant.

Furthermore, for every $u \in Y$, there exists a family $\{\, u_\varepsilon \,\} \subset Y$ that converges to u in $L^2(\mathbb{R})$ and that satisfies (2.12).

Proof. Since $u \in Y \cap C^2(\mathbb{R})$ and $f \in C^2(\mathbb{R})$, $\varepsilon^2 E_\varepsilon(u)$ and $E_0(u)$ are uniformly bounded in ε. Consequently,
$$|E_\varepsilon(u) - E_0(u)| \leq c_\infty \qquad \text{for } \varepsilon > 1 \tag{2.14}$$
for some constant $c_\infty > 0$.

Next, we assume $\varepsilon \in (0, 1]$. If $x \notin (a - \varepsilon, b)$, then $\mathrm{D}_\varepsilon U(x) = 0$, and hence

$$\varepsilon^2 E_\varepsilon(u) = \int_a^b \left[f(u(x)) - f\left(\mathrm{D}_\varepsilon U(x)\right) \right] \mathrm{d}x - \int_{a-\varepsilon}^a f(\mathrm{D}_\varepsilon U(x))\mathrm{d}x.$$

By the regularity of u, for any $x \in (a - \varepsilon, b)$ we have the Taylor's expansion

$$\mathrm{D}_\varepsilon U(x) = u(x) + \frac{\varepsilon}{2} u'(x) + \frac{\varepsilon^2}{6} u''(x_\varepsilon), \qquad \text{with } x_\varepsilon \in (x, x + \varepsilon),$$

which we rewrite as

$$\mathrm{D}_\varepsilon U(x) = u(x) + \varepsilon v_\varepsilon(x), \qquad \text{with } v_\varepsilon(x) := \frac{u'(x)}{2} + \frac{\varepsilon}{6} u''(x_\varepsilon); \qquad (2.15)$$

note that v_ε converges uniformly to $u'/2$ as $\varepsilon \to 0^+$.

Plugging (2.15) into the definition of E_ε, we find

$$\varepsilon^2 E_\varepsilon(u) = -\int_a^b \left[f\big(u(x) + \varepsilon v_\varepsilon(x)\big) - f(u(x)) \right] \mathrm{d}x - \int_{a-\varepsilon}^a f\left(\frac{\varepsilon^2}{6} u''(x_\varepsilon) \right) \mathrm{d}x$$

$$= -\varepsilon \int_a^b f'(u(x)) v_\varepsilon(x) \mathrm{d}x - \frac{\varepsilon^2}{2} \int_a^b f''(w_\varepsilon(x)) v_\varepsilon(x)^2 \mathrm{d}x$$

$$- \int_{a-\varepsilon}^a f\left(\frac{\varepsilon^2}{6} u''(x_\varepsilon) \right) \mathrm{d}x,$$

where w_ε fulfils $w_\varepsilon(x) \in (u(x), u(x) + \varepsilon v_\varepsilon(x_\varepsilon))$ for all $x \in (a, b)$.

It is not difficult to see that

$$\left| \int_{a-\varepsilon}^a f\left(\frac{\varepsilon^2}{6} u''(x_\varepsilon) \right) \mathrm{d}x \right| \le c_1 \varepsilon^5,$$

for a constant $c_1 > 0$ that depends only on $N := \|u\|_{C^2(\mathbb{R})}$ and on $\|f''\|_{L^\infty([-N,N])}$. Also, recalling the definition of v_ε, we have

$$\int_a^b f'(u(x)) v_\varepsilon(x) \mathrm{d}x = \frac{\varepsilon}{6} \int_a^b f'(u(x)) u''(x_\varepsilon) \mathrm{d}x,$$

and, therefore,

$$|E_\varepsilon(u) - E_0(u)| \le \frac{1}{6} \left| -\int_a^b f'(u(x)) u''(x_\varepsilon) \mathrm{d}x - \int_a^b f''(u(x)) u'(x)^2 \mathrm{d}x \right|$$

$$+ \frac{1}{2} \left| \frac{1}{4} \int_a^b f''(u(x)) u'(x)^2 \mathrm{d}x - \int_a^b f''(w_\varepsilon(x)) v_\varepsilon(x)^2 \mathrm{d}x \right| + c_1 \varepsilon^3. \qquad (2.16)$$

Since $u \in Y \cap C^2(\mathbb{R})$, u'' admits a uniform modulus of continuity $m_{u''} : [0, +\infty) \to [0, \infty)$. An integration by parts yields

$$\left| -\int_a^b f'(u(x)) \, u''(x_\varepsilon) \mathrm{d}x - \int_a^b f''(u(x)) u'(x)^2 \mathrm{d}x \right|$$

$$\le \int_a^b \left| f'(u(x)) \right| \left| u''(x) - u''(x_\varepsilon) \right| \mathrm{d}x$$

$$\le c_2 m_{u''}(\varepsilon),$$

where $c_2 := (b - a) \|f'\|_{L^\infty([-N,N])}$.

Analogously, letting $m_{f''}$ be the modulus of continuity of the restriction of f'' to the interval $[-N, N]$, we find as well

$$\left| \frac{1}{4} \int_a^b f''(u(x))u'(x)^2 \mathrm{d}x - \int_a^b f''(w_\varepsilon(x))v_\varepsilon(x)^2 \mathrm{d}x \right|$$

$$\leq \int_a^b |f''(u(x))| \left| \frac{1}{4}u'(x)^2 - v_\varepsilon(x)^2 \right| \mathrm{d}x + \int_a^b |f''(u(x)) - f''(w_\varepsilon(x))| \, v_\varepsilon(x)^2 \mathrm{d}x$$

$$\leq c_3(\varepsilon + m_{f''}(\varepsilon)),$$

with c_3 depending on $b - a$, N, and $\|f''\|_{L^\infty([-N,N])}$.

By combining (2.16) with the inequalities above, we obtain

$$|E_\varepsilon(u) - E_0(u)| \leq c_0\big(m_{u''}(\varepsilon) + m_{f''}(\varepsilon) + \varepsilon + \varepsilon^3\big) \qquad \text{for } \varepsilon \in (0, 1], \qquad (2.17)$$

for a suitable constant $c_0 > 0$. Now, (2.14) and (2.17) get (2.13).

For what concerns (2.12), we construct a subset of Y that is dense in energy, and we invoke Lemma 1.10. The argument is standard, so we just sketch it. If $u \in Y \setminus H^1(\mathbb{R})$, (2.12) holds trivially. Else, by rescaling the domain and mollifying, we can produce a sequence of smooth functions $\{u_\ell\} \subset Y$ that approximate u both uniformly and in $H^1(\mathbb{R})$. Thanks to the continuity of f'', we see that $\lim_{\ell \to +\infty} E_0(u_\ell) = E_0(u)$. Also, $\lim_{\varepsilon \to 0^+} E_\varepsilon(u_\ell) = E_0(u_\ell)$ for any $\ell \in \mathbb{N}$, because u_ℓ is smooth. Then, we are in position to apply Lemma 1.10. $\qquad \square$

With Proposition 2.7 on hand, to achieve the proof of Theorem 2.5, it suffices to validate statement (i), namely, for each $u \in Y$ and for each family $\{u_\varepsilon\} \subset Y$ converging to u in $L^2(\mathbb{R})$ it holds

$$E_0(u) \leq \liminf_{\varepsilon \to 0^+} E_\varepsilon(u_\varepsilon).$$

It is in the proof of this inequality that the assumption of strong convexity of f comes into play: it enables us to provide a lower bound on the energy E_ε, which, in turn, yields that sequences with equibounded energy are precompact w.r.t. the L^2-topology.

LEMMA 2.8 (Lower bound on the energy)
*Assume that **A3** is satisfied. Then, for any $u \in Y$, there holds*

$$E_\varepsilon(u) \geq \sup_{\varphi \in C_c^\infty(\mathbb{R}^2)} \left\{ \int_\mathbb{R} \fint_x^{x+\varepsilon} \left(\frac{u(y) - \mathrm{D}_\varepsilon U(x)}{\varepsilon} \varphi(x, y) - \frac{\varphi(x, y)^2}{4\lambda_\varepsilon(x, y)} \right) \mathrm{d}y\mathrm{d}x \right\}, \quad (2.18)$$

with

$$\lambda_\varepsilon(x, y) := \int_0^1 (1 - \vartheta)f''\big((1 - \vartheta)\mathrm{D}_\varepsilon U(x) + \vartheta u(y)\big)d\vartheta. \qquad (2.19)$$

Moreover,

$$E_\varepsilon(u) \geq \frac{\alpha}{4} \int_\mathbb{R} \int_{-\varepsilon}^\varepsilon H_\varepsilon(r) \left(\frac{u(y + r) - u(y)}{\varepsilon} \right)^2 \mathrm{d}r\mathrm{d}y, \qquad (2.20)$$

where

$$H(r) := (1 - |r|)^+ \quad \text{and} \quad H_\varepsilon(r) := \frac{1}{\varepsilon}H\left(\frac{r}{\varepsilon}\right).$$

Proof. Given $\varepsilon > 0$, let $u \in Y$ be such that $E_\varepsilon(u)$ is finite. We write

$$E_\varepsilon(u) = \frac{1}{\varepsilon^2} \int_{\mathbb{R}} e_\varepsilon(x)\mathrm{d}x, \quad \text{where } e_\varepsilon(x) := \fint_x^{x+\varepsilon} [f(u(y)) - f(\mathrm{D}_\varepsilon U(x))]\mathrm{d}y.$$

The identity

$$f(s) - f(t) = f'(t)(s - t) + (s - t)^2 \int_0^1 (1 - \vartheta) f''((1 - \vartheta)t + \vartheta s)) d\vartheta,$$

gets

$$e_\varepsilon(x) = \fint_x^{x+\varepsilon} \lambda_\varepsilon(x, y) \left(u(y) - \mathrm{D}_\varepsilon U(x)\right)^2 \mathrm{d}y, \tag{2.21}$$

with $\lambda_\varepsilon(x, y)$ as in (2.19). We now make use of the identity $a^2 \geq 2ab - b^2$ for $a, b \in \mathbb{R}$ to recover the bound

$$e_\varepsilon(x) \geq \fint_x^{x+\varepsilon} \left(\frac{u(y) - \mathrm{D}_\varepsilon U(x)}{\varepsilon} \varphi(x, y) - \frac{\varphi(x, y)^2}{4\lambda_\varepsilon(x, y)} \right) \mathrm{d}y,$$

for any $\varphi \in C_c^\infty\left(\mathbb{R}^2\right)$. Hence, (2.18) is proved.

Thanks to the strong convexity of f, $\lambda_\varepsilon(x, y) \geq \alpha/2$ for all $(x, y) \in \mathbb{R}^2$ and $\varepsilon > 0$. Thus, from (2.21) we infer

$$e_h(x) \geq \frac{\alpha}{2} \fint_x^{x+\varepsilon} \left(u(y) - \mathrm{D}_\varepsilon U(x)\right)^2,$$

and, therefore,

$$E_\varepsilon(u) \geq \frac{\alpha}{2} \int_{\mathbb{R}} \fint_x^{x+\varepsilon} \left(\frac{u(y) - \mathrm{D}_\varepsilon U(x)}{\varepsilon} \right)^2 \mathrm{d}y\mathrm{d}x$$

$$\geq \frac{\alpha}{4} \int_{\mathbb{R}} \fint_x^{x+\varepsilon} \fint_x^{x+\varepsilon} \left(\frac{u(z) - u(y)}{\varepsilon} \right)^2 \mathrm{d}z\mathrm{d}y\mathrm{d}x.$$

The last inequality is a consequence of the identity

$$\int |\varphi(y)|^2 \mathrm{d}\mu(y) = \left| \int \varphi(y) d\mu(y) \right|^2 + \frac{1}{2} \int \int |\varphi(z) - \varphi(y)|^2 \mathrm{d}\mu(z)\mathrm{d}\mu(y),$$

which holds whenever μ is a probability measure and $\varphi \in L^2(\mu)$. By Fubini's Theorem and neglecting contributions near the boundary, we find:

$$E_\varepsilon(u) \geq \frac{\alpha}{4} \int_{\mathbb{R}} \fint_{y-\varepsilon}^y \fint_x^{x+\varepsilon} \left(\frac{u(z) - u(y)}{\varepsilon} \right)^2 \mathrm{d}z\mathrm{d}x\mathrm{d}y$$

$$= \frac{\alpha}{4\varepsilon} \int_{\mathbb{R}} \int_{y-\varepsilon}^{y+\varepsilon} \left(1 - \frac{|z - y|}{\varepsilon} \right) \left(\frac{u(z) - u(y)}{\varepsilon} \right)^2 \mathrm{d}z\mathrm{d}y.$$

The change of variables $r = z - y$ yields (2.20). $\qquad \square$

REMARK 2.9

Let $u \in Y \cap H^1(\mathbb{R})$. Then, the family $\{ E_\varepsilon(u) \}$ is bounded above if there exists $c > 0$ such that $f'' \leq c$. To see this, we observe that (2.21) and the definition of λ_ε yield

$$E_\varepsilon(u) \leq \frac{c}{2\varepsilon^2} \int_\mathbb{R} \fint_x^{x+\varepsilon} (u(y) - \mathrm{D}_\varepsilon U(x))^2 \, \mathrm{d}y\mathrm{d}x.$$

Being u in $H^1(\mathbb{R})$, for $y \in (x, x+\varepsilon)$ we have

$$|u(y) - \mathrm{D}_\varepsilon U(x)|^2 \leq \varepsilon \int_x^{x+\varepsilon} |u'(z)|^2 \, \mathrm{d}z = \varepsilon^2 \int_0^1 |u'(x + \varepsilon z)|^2 \, \mathrm{d}z,$$

and we conclude

$$E_\varepsilon(u) \leq \frac{c}{2} \int_\mathbb{R} |u'(x)|^2 \, \mathrm{d}x. \tag{2.22}$$

The current remark will come in handy to prove a characterisation of $H^2_{\mathrm{loc}}(\mathbb{R}^d)$, see Remark 2.19 below.

LEMMA 2.10 (Compactness)

*Assume that f fulfils **A3**. If $\{ u_\varepsilon \} \subset Y$ is a family such that $E_\varepsilon(u_\varepsilon) \leq M$ for some $M \geq 0$, then, there exist a sequence $\{ \varepsilon_\ell \}$ and a function $u \in Y \cap H^1(\mathbb{R})$ such that $\{ u_{\varepsilon_\ell} \}$ converges to u in $L^2(\mathbb{R})$, as $\ell \to +\infty$.*

Proof. We adapt the proof of [AB98b, Theorem 3.1].

By Lemma 2.8, we infer that

$$\frac{\alpha}{4} \int_\mathbb{R} \int_{-\varepsilon}^\varepsilon H_\varepsilon(r) \left(\frac{u_\varepsilon(y+r) - u_\varepsilon(y)}{h} \right)^2 \, \mathrm{d}r\mathrm{d}y \leq M. \tag{2.23}$$

Observe that $H_\varepsilon(r)\mathrm{d}r$ is a probability measure on $[-\varepsilon, \varepsilon]$.

Let $\rho \in C_c^\infty(\mathbb{R})$ be a mollifier such that its support is contained in $[-1, 1]$, $0 \leq \rho \leq H$, and $|\rho'| \leq H$. For all $\varepsilon > 0$, we define

$$\rho_\varepsilon(r) := \frac{1}{c\varepsilon} \rho\left(\frac{r}{\varepsilon} \right), \qquad \text{with } c := \int_\mathbb{R} \rho(r)\mathrm{d}r,$$

and we introduce the regularised functions $v_\varepsilon := \rho_\varepsilon * u_\varepsilon$, Each $v_\varepsilon \colon \mathbb{R} \to \mathbb{R}$ is a smooth function whose support is a subset of $(a-\varepsilon, b+\varepsilon)$, and the family of derivatives $\{ v_\varepsilon' \}_{\varepsilon \in (0,1)}$ is uniformly bounded in $L^2(\mathbb{R})$. Indeed, since $\int_\mathbb{R} \rho'(r)\mathrm{d}r = 0$, we have

$$\int_\mathbb{R} |v_\varepsilon'(y)|^2 \, \mathrm{d}y = \int_\mathbb{R} \left| \int_{-\varepsilon}^\varepsilon \rho_\varepsilon'(r)[u_\varepsilon(y+r) - u_\varepsilon(y)]\mathrm{d}r \right|^2 \mathrm{d}y$$

$$\leq \int_\mathbb{R} \left(\int_{-\varepsilon}^\varepsilon |\rho_\varepsilon'(r)| \, |u_\varepsilon(y+r) - u_\varepsilon(y)| \, \mathrm{d}r \right)^2 \mathrm{d}y$$

$$\leq \frac{1}{c^2} \int_\mathbb{R} \left(\int_{-\varepsilon}^\varepsilon H_\varepsilon(r) \left| \frac{u_\varepsilon(y+r) - u_\varepsilon(y)}{\varepsilon} \right| \mathrm{d}r \right)^2 \mathrm{d}y$$

$$\leq \frac{1}{c^2} \int_\mathbb{R} \int_{-\varepsilon}^\varepsilon H_\varepsilon(r) \left| \frac{u_\varepsilon(y+r) - u_\varepsilon(y)}{\varepsilon} \right|^2 \mathrm{d}r\mathrm{d}y,$$

whence

$$\int_\mathbb{R} |v_\varepsilon'(y)|^2 \, \mathrm{d}y \leq \frac{4M}{c^2\alpha}. \tag{2.24}$$

For all $\varepsilon \in (0, 1)$, let $\tilde{v}_\varepsilon$ be the restriction of v_ε to the interval $(a - 1, b + 1)$. By virtue of Poincaré's inequality, (2.24) entails boundedness in $H_0^1((a - 1, b + 1))$ of the family $\{\tilde{v}_\varepsilon\}_{\varepsilon \in (0,1)}$, and, by Sobolev's Embedding Theorem, there exist a sequence $\{\varepsilon_\ell\}$ and a function $\tilde{u} \in H_0^1([a - 1, b + 1])$ such that $\{\tilde{v}_{\varepsilon_\ell}\}$ uniformly converges to $\tilde{u}$. Since each $\tilde{v}_{\varepsilon_\ell}$ is supported in $(a - \varepsilon_\ell, b + \varepsilon_\ell)$, we see that $\tilde{u} \in H_0^1(\bar{I})$. It follows that $\{v_{\varepsilon_\ell}\}$ converges uniformly to $u \in Y \cap H^1(\mathbb{R})$ if we set

$$u(x) := \begin{cases} \tilde{u}(x) & \text{if } x \in \bar{I}, \\ 0 & \text{otherwise.} \end{cases}$$

Eventually, we focus on the L^2-distance between u_ε and v_ε. As above, we have

$$\int_{\mathbb{R}} |v_\varepsilon(y) - u_\varepsilon(y)|^2 \, \mathrm{d}y = \int_{\mathbb{R}} \left| \int_{-\varepsilon}^{\varepsilon} \rho_\varepsilon(r)[u_\varepsilon(y + r) - u_\varepsilon(y)] \mathrm{d}r \right|^2 \mathrm{d}y$$

$$\leq \int_{\mathbb{R}} \int_{-\varepsilon}^{\varepsilon} \rho_\varepsilon(r) |u_\varepsilon(y + r) - u_\varepsilon(y)|^2 \, \mathrm{d}r\mathrm{d}y$$

$$\leq \frac{1}{c} \int_{\mathbb{R}} \int_{-\varepsilon}^{\varepsilon} H_\varepsilon(r) |u_\varepsilon(y + r) - u_\varepsilon(y)|^2 \, \mathrm{d}r\mathrm{d}y,$$

and, by (2.23), we get

$$\int_{\mathbb{R}} |v_\varepsilon(y) - u_\varepsilon(y)|^2 \, \mathrm{d}y \leq \frac{4M}{c\alpha} \varepsilon^2.$$

The thesis is now proved, because there exists a subsequence $\{v_{\varepsilon_\ell}\}$ that converges uniformly to $u \in Y \cap H^1(\mathbb{R})$. $\qquad\square$

At this stage, we are in position to establish statement (i) in Theorem 2.5, thus concluding the proof of the latter.

PROPOSITION 2.11
Let f satisfy A3. Then, for any $u \in Y$ and for any family $\{u_\varepsilon\} \subset Y$ that converges to u in $L^2(\mathbb{R})$, it holds

$$E_0(u) \leq \liminf_{\varepsilon \to 0^+} E_\varepsilon(u_\varepsilon). \tag{2.25}$$

Proof. Let us pick $u, u_\varepsilon \in Y$ in such a way that $u_\varepsilon \to u$ in $L^2(\mathbb{R})$. We may suppose that the right-hand side in (2.25) is finite, otherwise the conclusion holds trivially. Then, up to extracting a subsequence, which we do not relabel, there exists $\lim_{\varepsilon \to 0^+} E_\varepsilon(u_\varepsilon)$ and it is finite. In particular, for all $\varepsilon > 0$, $E_\varepsilon(u_\varepsilon) \leq M$ for some $M \geq 0$, so that, by Lemma 2.10, $u \in Y \cap H^1(\mathbb{R})$.

We apply formula (2.18) for each u_ε, choosing, for $(x, y) \in \mathbb{R}^2$,

$$\varphi(x, y) = \psi\left(x, \frac{y - x}{\varepsilon}\right), \qquad \text{with } \psi \in C_c^\infty(\mathbb{R}^2).$$

We get

$$E_\varepsilon(u_\varepsilon) \geq \int_{\mathbb{R}} \fint_x^{x+\varepsilon} \frac{u_\varepsilon(y) - \fint_x^{x+\varepsilon} u_\varepsilon}{\varepsilon} \psi\left(x, \frac{y - x}{\varepsilon}\right) \mathrm{d}y\mathrm{d}x$$

$$- \frac{1}{4} \int_{\mathbb{R}} \fint_x^{x+\varepsilon} \frac{\psi\left(x, \frac{y-x}{\varepsilon}\right)^2}{\lambda_\varepsilon(x, y)} \mathrm{d}y\mathrm{d}x, \tag{2.26}$$

where, in agreement with (2.19),

$$\lambda_\varepsilon(x,y) := \int_0^1 (1-\vartheta) f'' \left((1-\vartheta) \fint_x^{x+\varepsilon} u_\varepsilon(z)\mathrm{d}z + \vartheta u_\varepsilon(y) \right) d\vartheta \geq \frac{\alpha}{2}.$$

Let us focus on the right-hand side of (2.26). We have

$$\frac{1}{\varepsilon} \int_\mathbb{R} \fint_x^{x+\varepsilon} \left(\fint_x^{x+\varepsilon} u_\varepsilon(z)\mathrm{d}x \right) \psi\left(x, \frac{y-x}{\varepsilon}\right) \mathrm{d}y\mathrm{d}x$$
$$= \frac{1}{\varepsilon^3} \int_\mathbb{R} \int_x^{x+\varepsilon} \int_x^{x+\varepsilon} u_\varepsilon(z)\psi\left(x, \frac{y-x}{\varepsilon}\right) \mathrm{d}y\mathrm{d}z\mathrm{d}x$$
$$= \frac{1}{\varepsilon^3} \int_\mathbb{R} \int_{z-\varepsilon}^z \int_x^{x+\varepsilon} u_\varepsilon(z)\psi\left(x, \frac{y-x}{\varepsilon}\right) \mathrm{d}y\mathrm{d}x\mathrm{d}z,$$

and, similarly,

$$\int_\mathbb{R} \fint_x^{x+\varepsilon} \frac{u_\varepsilon(y) - \fint_x^{x+\varepsilon} u_\varepsilon}{\varepsilon} \psi\left(x, \frac{y-x}{\varepsilon}\right) \mathrm{d}y\mathrm{d}x$$
$$= \frac{1}{\varepsilon} \int_\mathbb{R} \fint_{y-\varepsilon}^y \fint_x^{x+\varepsilon} u_\varepsilon(y) \left[\psi\left(x, \frac{y-x}{\varepsilon}\right) - \psi\left(x, \frac{z-x}{\varepsilon}\right) \right] \mathrm{d}z\mathrm{d}x\mathrm{d}y.$$

$$(2.27)$$

By changing variables, we get

$$\fint_{y-\varepsilon}^y \fint_x^{x+\varepsilon} \psi\left(x, \frac{y-x}{\varepsilon}\right) \mathrm{d}z\mathrm{d}x = \int_0^1 \int_0^1 \psi(y - \varepsilon r, r)\mathrm{d}q\mathrm{d}r,$$
$$\fint_{y-\varepsilon}^y \fint_x^{x+\varepsilon} \psi\left(x, \frac{z-x}{\varepsilon}\right) \mathrm{d}z\mathrm{d}x = \fint_{y-\varepsilon}^y \int_0^1 \psi(x, r)\mathrm{d}r\mathrm{d}x$$
$$= \int_0^1 \int_0^1 \psi(y - \varepsilon q, r)\mathrm{d}q\mathrm{d}r,$$

hence

$$\frac{1}{\varepsilon} \fint_{y-\varepsilon}^y \fint_x^{x+\varepsilon} \left[\psi\left(x, \frac{y-x}{\varepsilon}\right) - \psi\left(x, \frac{z-x}{\varepsilon}\right) \right] \mathrm{d}z\mathrm{d}x$$
$$= \int_0^1 \int_0^1 \frac{\psi(y - \varepsilon r, r) - \psi(y - \varepsilon q, r)}{h}\mathrm{d}q\mathrm{d}r$$
$$= -\int_0^1 \int_0^1 \int_q^r \partial_1\psi(y - \varepsilon s, r)\mathrm{d}s\mathrm{d}q\mathrm{d}r$$
$$= -\int_0^1 \int_0^1 (r-q) \fint_q^r \partial_1\psi(y - \varepsilon s, r)\mathrm{d}s\mathrm{d}q\mathrm{d}r.$$

Since ψ is smooth, we have that $\partial_1\psi(y - \varepsilon s, r) = \partial_1\psi(y, r) + O(\varepsilon)$ as $\varepsilon \to 0^+$, uniformly for $s \in [0,1]$. Consequently,

$$\frac{1}{\varepsilon} \fint_{y-\varepsilon}^y \fint_x^{x+\varepsilon} \left[\psi\left(x, \frac{y-x}{\varepsilon}\right) - \psi\left(x, \frac{z-x}{\varepsilon}\right) \right] \mathrm{d}z\mathrm{d}x$$
$$= -\int_0^1 \left(r - \frac{1}{2}\right) \partial_1\psi(y, r)\mathrm{d}r + O(\varepsilon).$$

Plugging this equality in (2.27) yields

$$\int_{\mathbb{R}} \fint_x^{x+\varepsilon} \frac{u_\varepsilon(y) - \fint_x^{x+\varepsilon} u_\varepsilon}{h} \psi\left(x, \frac{y-x}{\varepsilon}\right) dy dx$$
$$= -\int_{\mathbb{R}} u_\varepsilon(y) \int_0^1 \left(r - \frac{1}{2}\right) \partial_1 \psi(y, r) dr + O(\varepsilon).$$

Since $u_\varepsilon \to u$ in $L^2(\mathbb{R})$, it is possible to take the limit $\varepsilon \to 0^+$ in the previous formula, getting

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} \fint_x^{x+\varepsilon} \frac{u_\varepsilon(y) - \fint_x^{x+\varepsilon} u_\varepsilon}{h} \psi\left(x, \frac{y-x}{\varepsilon}\right) dy dx$$
$$= -\int_{\mathbb{R}} \int_0^1 u(y)(r - \tfrac{1}{2}) \partial_1 \psi(y, r) dr dy. \quad (2.28)$$

Next, we focus on the second addendum in the right-hand side of (2.26). By Fubini's Theorem and a change of variables, we have

$$\int_{\mathbb{R}} \fint_x^{x+\varepsilon} \frac{\psi(x, \frac{y-x}{\varepsilon})^2}{\lambda_\varepsilon(x, y)} dy dx = \int_{\mathbb{R}} \int_0^1 \frac{\psi(y - \varepsilon r, r)^2}{\lambda_\varepsilon(y - \varepsilon r, y)} dr dy.$$

The function ψ has compact support and $\lambda_\varepsilon \geq \alpha/2$ for all $\varepsilon > 0$, therefore we can apply Lebesgue's Convergence Theorem to let ε vanish in the equality above. We find

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} \int_0^1 \frac{\psi(y - \varepsilon r, r)^2}{\int_0^1 (1 - \vartheta) f''\left((1 - \vartheta) \fint_{y - \varepsilon r}^{y + (1-r)\varepsilon} u_\varepsilon(z) dz + \vartheta u_\varepsilon(y)\right) d\vartheta} dr dy$$
$$= \int_{\mathbb{R}} \int_0^1 \frac{\psi(y, r)^2}{\int_0^1 (1 - \vartheta) f''(u(y)) d\vartheta} dr dy,$$

thus

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} \fint_x^{x+\varepsilon} \frac{\psi(x, \frac{y-x}{\varepsilon})^2}{\lambda_\varepsilon(x, y)} dy dx = 2 \int_{\mathbb{R}} \int_0^1 \frac{\psi(y, r)^2}{f''(u(y))} dr dy. \quad (2.29)$$

Summing up, by (2.28) and (2.29), we deduce

$$\liminf_{\varepsilon \to 0^+} E_\varepsilon(u_\varepsilon) \geq -\int_{\mathbb{R}} \int_0^1 \left[u(y) \left(r - \frac{1}{2}\right) \partial_1 \psi(y, r) dr dy + \frac{1}{2} \int_{\mathbb{R}} \int_0^1 \frac{\psi(y, r)^2}{f''(u(y))} \right] dr dy,$$

for all $\psi \in C_c^\infty(\mathbb{R}^2)$.

Let $\eta \in C_c^\infty(\mathbb{R})$. By a standard approximation argument, one can prove that

$$\psi(x, y) := \eta(x) \left(y - \frac{1}{2}\right)$$

is an admissible test function in the previous estimate. This choice of ψ gets

$$\liminf_{\varepsilon \to 0^+} E_\varepsilon(u_\varepsilon) \geq -\int_0^1 \left(r - \frac{1}{2}\right)^2 dr \int_{\mathbb{R}} u(y) \eta'(y) dy$$
$$- \frac{1}{2} \int_0^1 \left(r - \frac{1}{2}\right)^2 dr \int_{\mathbb{R}} \frac{\eta(y)^2}{f''(u(y))} dy$$
$$= \frac{1}{12} \left[\int_{\mathbb{R}} u'(y) \eta(y) dy - \frac{1}{2} \int_{\mathbb{R}} \frac{\eta(y)^2}{f''(u(y))} dy \right],$$

where we used the identity $\int_0^1 (r - 1/2)^2 \mathrm{d}r = 1/12$ and $u' \in L^2(\mathbb{R})$ is the distributional derivative of u (recall that $u \in H^1(\mathbb{R})$). Since the test function η is arbitrary, we recover (2.25) by taking the supremum w.r.t. $\eta \in C_c^\infty(\mathbb{R})$. □

2.3 Γ-convergence of the rate functionals

We devote this section to the proof of Theorem 2.1, which relies on the Γ-convergence result of the previous section.

We firstly remark that, when the dimension is 1, the Γ-convergence of the functionals $\mathscr{E}_\varepsilon$ follows easily from Theorem 2.5.

COROLLARY 2.12
*Let $\Omega \subset \mathbb{R}$ be bounded and open, and let $X := \{ u \in H_{\mathrm{loc}}^1(\mathbb{R}) : u' = 0 \text{ in } \Omega^c \}$. Suppose that $G \colon \mathbb{R} \to [0, +\infty)$ and $f \colon [0, +\infty) \to [0, +\infty)$ satisfy **A1**, **A2**, and **A3**. If for $\varepsilon > 0$ and $u \in H_{\mathrm{loc}}^1(\mathbb{R})$ we define*

$$\mathscr{E}_\varepsilon(u) := \frac{1}{\varepsilon^2} \int_\mathbb{R} \int_\mathbb{R} G_\varepsilon(z) \left[f\big(|u'(x)|\big) - f\left(\left| \frac{u(x+z) - u(x)}{z} \right| \right) \right] \mathrm{d}z \mathrm{d}x,$$

then the restrictions of the functionals $\mathscr{E}_\varepsilon$ to X Γ-converge w.r.t. the $H_{\mathrm{loc}}^1(\mathbb{R})$-convergence to

$$\mathscr{E}_0(u) := \begin{cases} \dfrac{1}{24} \left(\int_\mathbb{R} G(z) z^2 \mathrm{d}z \right) \int_\mathbb{R} f''(u'(x)) \left| u''(x) \right|^2 \mathrm{d}x & \text{if } u \in X \cap H_{\mathrm{loc}}^2(\mathbb{R}), \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. A change of variables gets

$$\mathscr{E}_\varepsilon(u) = \int_\mathbb{R} G(z) z^2 \left[\frac{1}{(\varepsilon z)^2} \int_\mathbb{R} f\big(|u'(x)|\big) - f\left(\left| \frac{u(x + \varepsilon z) - u(x)}{\varepsilon z} \right| \right) \mathrm{d}x \right] \mathrm{d}z.$$

By (2.9), the quantity between square brackets is $E_{\varepsilon z}(u')$, therefore the conclusion follows by adapting the proof of Theorem 2.5 (see also Proposition 2.13 below). □

We may henceforth assume that $d \geq 2$. In this case, the proof of Theorem 2.1 is more elaborated, because, to be able to apply the results of Section 2.2, we need the functions $w_{\hat{z},\xi}$ in (2.7) to admit a second order weak derivative for a.e. z and ξ. This is no real issue as long as the upper limit inequality is concerned, since we may reason on regular functions; to cope with the lower limit one, instead, we shall make use of the compactness criterion provided by Lemma 2.18.

For the moment being, we can establish a provisional result:

PROPOSITION 2.13
Let $u \in X$. Then:

(i) *If $u \in X \cap H_{\mathrm{loc}}^2(\mathbb{R}^d)$, for any family $\{ u_\varepsilon \} \subset X$ that converges to u in $H_{\mathrm{loc}}^1(\mathbb{R}^d)$, there holds*

$$\mathscr{E}_0(u) \leq \liminf_{\varepsilon \to 0^+} \mathscr{E}_\varepsilon(u_\varepsilon).$$

(ii) *If $u \in X \cap C^3(\mathbb{R}^d)$, then*

$$\mathscr{E}_0(u) = \lim_{\varepsilon \to 0^+} \mathscr{E}_\varepsilon(u). \tag{2.30}$$

Proof. We prove both the assertions by using the slicing formula (2.7).

(i)　For all $\varepsilon > 0$, $z \in \mathbb{R}^d \setminus \{0\}$, and $\xi \in \hat{z}^\perp$, we let $w_{\varepsilon;\hat{z},\xi} \colon \mathbb{R} \to \mathbb{R}$ be defined as $w_{\varepsilon;\hat{z},\xi}(t) := u_\varepsilon(\xi + t\hat{z})$. Then,

$$\mathscr{E}_\varepsilon(u_\varepsilon) = \int_{\mathbb{R}^d} \int_{z^\perp} G(z)\,|z|^2\, E_{\varepsilon|z|}(w'_{\varepsilon;\hat{z},\xi}) \mathrm{d}\mathscr{H}^{d-1}(\xi) \mathrm{d}z,$$

and, by Fatou's Lemma,

$$\liminf_{\varepsilon \to 0^+} \mathscr{E}_\varepsilon(u_\varepsilon) \geq \int_{\mathbb{R}^d} \int_{z^\perp} G(z)\,|z|^2 \left[\liminf_{\varepsilon \to 0^+} E_{\varepsilon|z|}(w'_{\varepsilon;\hat{z},\xi}) \right] \mathrm{d}\mathscr{H}^{d-1}(\xi) \mathrm{d}z.$$
(2.31)

Let $w_{\hat{z},\xi}$ be as in Lemma 2.4. We remark that, for any kernel $\rho \colon \mathbb{R}^d \to [0, +\infty)$ such that $\|\rho\|_{L^1(\mathbb{R}^d)} = 1$, we have

$$\int_{\mathbb{R}^d} |\nabla u_\varepsilon - \nabla u|^2$$
$$\geq \int_{\mathbb{R}^d} \rho(z) \int_{\hat{z}^\perp} \int_{\mathbb{R}} \left| \left(\nabla u_\varepsilon(\xi + t\hat{z}) - \nabla u(\xi + t\hat{z}) \right) \cdot \hat{z} \right|^2 \mathrm{d}t \mathrm{d}\mathscr{H}^{d-1}(\xi) \mathrm{d}z$$
$$= \int_{\mathbb{R}^d} \rho(z) \int_{\hat{z}^\perp} \int_{\mathbb{R}} \left| w'_{\varepsilon;\hat{z},\xi}(t) - w'_{\hat{z},\xi}(t) \right|^2 \mathrm{d}t \mathrm{d}\mathscr{H}^{d-1}(\xi) \mathrm{d}z.$$

The left-hand side vanishes as $\varepsilon \to 0^+$, and, therefore, there exists a subsequence of $\left\{ w'_{\varepsilon;\hat{z},\xi} \right\}$, which we do not relabel, that converges to $w'_{\hat{z},\xi}$ in $L^2(\mathbb{R})$ for a.e. $z \in \mathbb{R}^d$ and $\mathscr{H}^{d-1}$-a.e. $\xi \in \hat{z}^\perp$.

We remind that, by assumption, $w'_{\hat{z},\xi} \in H^1(\mathbb{R})$ for a.e. (z, ξ) and it equals 0 on the complement of some open interval $I_{\hat{z},\xi}$. It follows that we can appeal to Proposition 2.11, obtaining

$$\liminf_{\varepsilon \to 0^+} \mathscr{E}_\varepsilon(u_\varepsilon) \geq \int_{\mathbb{R}^d} \int_{z^\perp} G(z)\,|z|^2\, E_0(w'_{\hat{z},\xi}) \mathrm{d}\mathscr{H}^{d-1}(\xi) \mathrm{d}z = \mathscr{E}_0(u).$$

(ii)　As above, for any fixed $z \in \mathbb{R}^d \setminus \{0\}$ and $\xi \in \hat{z}^\perp$, we define the function $w_{\hat{z},\xi} \in C^3(\mathbb{R})$ setting $w(t) := u(\xi + t\hat{z})$. Since Ω is bounded, there exists $r > 0$ such that, for any choice of z, $w'_{\hat{z},\xi}(t) = \nabla u(\xi + t\hat{z}) \cdot \hat{z} = 0$ whenever $\xi \in z^\perp$ satisfies $|\xi| \geq r$, while $w'_{\hat{z},\xi}$ is supported in an open interval $I_{\hat{z},\xi}$ if $|\xi| < r$.

We apply the slicing formula (2.7), which yields

$$|\mathscr{E}_\varepsilon(u) - \mathscr{E}_0(u)| \leq \int_{\mathbb{R}^d} \int_{z^\perp} G(z)\,|z|^2 \left| E_{\varepsilon|z|}(w'_{\hat{z},\xi}) - E_0(w'_{\hat{z},\xi}) \right| \mathrm{d}\mathscr{H}^{d-1}(\xi) \mathrm{d}z$$

According to Proposition 2.7. there are a constant $c > 0$ and a continuous, bounded, and increasing function $m \colon [0, +\infty) \to [0, +\infty)$ such that $m(0) = 0$ and that

$$|\mathscr{E}_\varepsilon(u) - \mathscr{E}_0(u)| \leq c \int_{\mathbb{R}^d} \int_{z^\perp} G(z)\,|z|^2\, m(\varepsilon\,|z|) \mathrm{d}\mathscr{H}^{d-1}(\xi) \mathrm{d}z.$$

We highlight that here m can be chosen as to depend on ∇u only, and not on $\hat{z}$ and ξ. Thus, by assumption **A1**, we can apply Lebesgue's Convergence Theorem, and (2.30) is proved. □

REMARK 2.14
Proposition 2.13 entails that the Γ-limit of the family $\{\,\varepsilon\mathscr{E}_\varepsilon\,\}$ is 0; the same limit being also found w.r.t. the $L^2_{\mathrm{loc}}(\mathbb{R}^d)$-topology on X. This is the analogue in arbitrary dimension of what we observed in Remark 2.6.

We now deal with the proof of the compactness result, that is, (i) in Theorem 2.1. As in Section 2.2, we start with a lower bound on the energy functionals. Lemma 2.15 below shows that, when f fulfils **A3**, $\mathscr{E}_\varepsilon(u)$ is greater than a double integral which takes into account, for each $z \in \mathbb{R}^d \setminus \{0\}$, the squared projection of the difference quotients of ∇u in the direction of z. Thanks to the slicing formula, the inequality follows with no effort by applying Lemma 2.8 on each line of $\mathbb{R}^d$.

Our approach results in the appearance of an effective kernel $\tilde{G}$ in front of the difference quotients. This function stands as a multidimensional counterpart of the kernel H in Lemma 2.8, and, actually, $\tilde{G}$ depends both on G and on H (see (2.32) for the precise definition). Some properties of the effective kernel are collected in Lemma 2.16.

LEMMA 2.15 (Lower bound on the energy)
Let Ω, G, and f be as above, and set

$$\tilde{G}(z) := \int_{-1}^{1} H(r) G_{|r|}(z)\mathrm{d}r \qquad \text{for a.e. } z \in \mathbb{R}^d, \tag{2.32}$$

with H as in Lemma 2.8. Then, it holds

$$\mathscr{E}_\varepsilon(u) \geq \frac{\alpha}{4} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{G}(z) \left[\frac{(\nabla u(x+\varepsilon z) - \nabla u(x))\cdot\hat{z}}{\varepsilon}\right]^2 \mathrm{d}x\mathrm{d}z. \tag{2.33}$$

Proof. In the following lines, we use d' as a shorthand for $\mathrm{d}\mathscr{H}^{d-1}$. We reduce to the 1-dimensional case by slicing, and then we take advantage of the lower bound provided by Lemma 2.8. Keeping the notation of Lemma 2.4, we find

$$\mathscr{E}_\varepsilon(u)$$

$$\geq \frac{\alpha}{4} \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} \int_{-\varepsilon|z|}^{\varepsilon|z|} H_{\varepsilon|z|}(r) G(z) |z|^2 \left(\frac{w'_{\hat{z},\xi}(t+r) - w'_{\hat{z},\xi}(t)}{\varepsilon|z|}\right)^2 \mathrm{d}r\mathrm{d}t\mathrm{d}'\xi\mathrm{d}z$$

$$= \frac{\alpha}{4} \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} \int_{-\varepsilon|z|}^{\varepsilon|z|} H_{\varepsilon|z|}(r) G(z) \left(\frac{w'_{\hat{z},\xi}(t+r) - w'_{\hat{z},\xi}(t)}{\varepsilon}\right)^2 \mathrm{d}r\mathrm{d}t\mathrm{d}'\xi\mathrm{d}z.$$

To retrieve (2.33) from this bound, it suffices to change variables and use Fubini's Theorem:

$$I := \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} \int_{-\varepsilon|z|}^{\varepsilon|z|} H_{\varepsilon|z|}(r) G(z) \left(\frac{w'_{\hat{z},\xi}(t+r) - w'_{\hat{z},\xi}(t)}{\varepsilon}\right)^2 \mathrm{d}r\mathrm{d}t\mathrm{d}'\xi\mathrm{d}z$$

$$= \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} \int_{-1}^{1} H(r) G(z) \left(\frac{w'_{\hat{z},\xi}(t+\varepsilon|z|r) - w'_{\hat{z},\xi}(t)}{\varepsilon}\right)^2 \mathrm{d}r\mathrm{d}t\mathrm{d}'\xi\mathrm{d}z$$

$$= \int_{-1}^{0} \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} H(r) G_{-r}(z) \left(\frac{w'_{\hat{z},\xi}(t-\varepsilon|z|) - w'_{\hat{z},\xi}(t)}{\varepsilon}\right)^2 \mathrm{d}t\mathrm{d}'\xi\mathrm{d}z\mathrm{d}r$$

$$+ \int_{0}^{1} \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} H(r) G_r(z) \left(\frac{w'_{\hat{z},\xi}(t+\varepsilon|z|) - w'_{\hat{z},\xi}(t)}{\varepsilon}\right)^2 \mathrm{d}t\mathrm{d}'\xi\mathrm{d}z\mathrm{d}r$$

Since $w'_{-\hat{z},\xi}(-s) = -w'_{\hat{z},\xi}(s)$ for all $s \in \mathbb{R}$, we have

$$
\int_{-1}^0 \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} H(r)G_{-r}(z) \left(\frac{w'_{\hat{z},\xi}(t + \varepsilon\,|z|\,) - w'_{\hat{z},\xi}(t)}{\varepsilon} \right)^2 \mathrm{d}t\mathrm{d}'\xi\mathrm{d}z\mathrm{d}r
$$

$$
= \int_{-1}^0 \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} H(r)G_{-r}(z) \left(\frac{w'_{-\hat{z},\xi}\big(-(t + \varepsilon\,|z|)\big) - w'_{-\hat{z},\xi}(-t)}{\varepsilon} \right)^2 \mathrm{d}t\mathrm{d}'\xi\mathrm{d}z\mathrm{d}r
$$

$$
= \int_{-1}^0 \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} H(r)G_{-r}(z) \left(\frac{w'_{\hat{z},\xi}(t + \varepsilon\,|z|\,) - w'_{\hat{z},\xi}(t)}{\varepsilon} \right)^2 \mathrm{d}t\mathrm{d}'\xi\mathrm{d}z\mathrm{d}r.
$$

Thus, we conclude that

$$
I = \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} \left(\int_{-1}^1 H(r)G_{|r|}(z)\mathrm{d}r \right) \left(\frac{w'_{\hat{z},\xi}(t + \varepsilon\,|z|\,) - w'_{\hat{z},\xi}(t)}{\varepsilon} \right)^2 \mathrm{d}t\mathrm{d}'\xi\mathrm{d}z
$$

$$
= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{G}(z) \left[\frac{(\nabla u(x + \varepsilon z) - \nabla u(x)) \cdot \hat{z}}{\varepsilon} \right]^2 \mathrm{d}x\mathrm{d}z.
$$

$\square$

By assumption **A2**, the kernel G is bounded away from 0 in a suitable annulus. The next lemma shows that the effective kernel $\tilde{G}$ in (2.33) inherits a similar property.

LEMMA 2.16
Let $\tilde{G}\colon \mathbb{R}^d \to [0, +\infty)$ be as in (2.32). Then,

$$
\int_{\mathbb{R}^d} \tilde{G}(z) \left(1 + |z|^2\right) \mathrm{d}z < +\infty, \tag{2.34}
$$

and, if β_d and r_1 are the constants in (2.4) and (2.5), then,

$$
\operatorname{ess\,inf} \left\{ \tilde{G}(z) : z \in B(0, \beta_d r_1) \right\} > 0. \tag{2.35}
$$

Proof. The convergence of the integral in (2.34) follows easily from (2.3). Indeed, by the definition of $\tilde{G}$, there holds

$$
\int_{\mathbb{R}^d} \tilde{G}(z)\mathrm{d}z = \int_{-1}^1 \int_{\mathbb{R}^d} H(r)G_{|r|}(z)\mathrm{d}z\mathrm{d}r = \int_{\mathbb{R}^d} G(z)\mathrm{d}z.
$$

Analogously,

$$
\int_{\mathbb{R}^d} \tilde{G}(z)\,|z|^2\,\mathrm{d}z = c \int_{\mathbb{R}^d} G(z)\,|z|^2\,\mathrm{d}z,
$$

for some $c > 0$.

For what concerns (2.35), let us set $\gamma := \operatorname{ess\,inf} \{\, G(z) : z \in B(0, r_1) \setminus B(0, r_0) \,\}$. In view of (2.5), $\gamma > 0$.

We distinguish between the case $z \in B(0, r_0)$ and the case $z \in B(0, r_1) \setminus B(0, r_0)$. In the first situation, for a.e. $z \in \mathbb{R}^d$,

$$\tilde{G}(z) \geq 2 \int_{\frac{|z|}{r_1}}^{\frac{|z|}{r_0}} H(r) G_r(z) \mathrm{d}r \geq 2\gamma \int_{\frac{|z|}{r_1}}^{\frac{|z|}{r_0}} \frac{1}{r^d} H(r) \mathrm{d}r$$

$$= \frac{2\gamma}{|z|^{d-1}} \int_{r_0}^{r_1} s^{d-2} \left(1 - \frac{|z|}{s}\right) \mathrm{d}s.$$

When $z \in B(0, r_1) \setminus B(0, r_0)$, instead, similar computations get

$$\tilde{G}(z) \geq 2 \int_{\frac{|z|}{r_1}}^{1} H(r) G_r(z) \mathrm{d}r = \frac{2\gamma}{|z|^{d-1}} \int_{|z|}^{r_1} s^{d-2} \left(1 - \frac{|z|}{s}\right) \mathrm{d}s \qquad \text{for a.e. } z \in \mathbb{R}^d,$$

so that we obtain

$$\tilde{G}(z) \geq \frac{2\gamma}{|z|^{d-1}} \int_{\max(r_0, |z|)}^{r_1} s^{d-2} \left(1 - \frac{|z|}{s}\right) \mathrm{d}s \qquad \text{for a.e. } z \in \mathbb{R}^d. \qquad (2.36)$$

When $d = 2$, the previous estimate becomes

$$\tilde{G}(z) \geq 2\gamma \left[\frac{r_1 - \max(r_0, |z|)}{|z|} - \log\left(\frac{r_1}{\max(r_0, |z|)}\right)\right] \qquad \text{for a.e. } z \in \mathbb{R}^d.$$

By the concavity of the logarithm, we see that the lower bound is strictly positive if $|z| < r_1 = \beta_2 r_1$.

In the case $d \geq 3$, instead, the right-hand side in (2.36) equals

$$\frac{2\gamma}{(d-1)(d-2)|z|^{d-1}} \left[(d-2)\left(r_1^{d-1} - M^{d-1}\right) - (d-1)|z|\left(r_1^{d-2} - M^{d-2}\right)\right],$$

where we put $M := \max(r_0, |z|)$ for shortness. Hence,

$$\tilde{G}(z) \geq \frac{2\gamma M^{d-2}}{(d-1)(d-2)|z|^{d-1}}$$
$$\cdot \left\{\left(\frac{r_1}{M}\right)^{d-2} [(d-2)r_1 - (d-1)|z|] - [(d-2)M - (d-1)|z|]\right\}$$

for a.e. $z \in \mathbb{R}^d$. When $|z| < (d-2)r_1/(d-1) = \beta_d r_1$, the quantity between braces is strictly positive if

$$\frac{(M - |z|)d - (2M - |z|)}{(r_1 - |z|)d - (2r_1 - |z|)} < \left(\frac{r_1}{M}\right)^{d-2}.$$

Observe that both sides are strictly increasing in d, and that the left-hand one is bounded above by $(M - |z|)/(r_1 - |z|)$. In conclusion, the last inequality holds if

$$\frac{M - |z|}{r_1 - |z|} < \frac{r_1}{M},$$

which, in turn, is true for all $z \in B(0, r_1)$. $\qquad\square$

REMARK 2.17
For our analysis, it is not relevant to identify the largest ball in which $\operatorname{ess\,inf} \tilde{G}$ is strictly positive. For this reason, we are not interested in the optimal β_d in (2.35) for $d \geq 3$.

We can finally prove compactness for families with equibouded energies.

LEMMA 2.18 (Compactness)
Assume that Ω, G, and f are as above. Then, if $\{\, u_\varepsilon \,\} \subset X$ satisfies $\mathscr{E}_\varepsilon(u_\varepsilon) \leq M$ for some $M \geq 0$, there exist a subsequence $\{\, \varepsilon_\ell \,\}$ and a function $u \in H^2(\mathbb{R}^d)$ vanishing outside Ω such that $\nabla u_{\varepsilon_\ell} \to \nabla u$ in $L^2(\mathbb{R}^d)$.

Proof. Let $\tilde{\gamma} := \operatorname{ess\,inf}\{\, \tilde{G}(z) : z \in B(0, \beta_d r_1) \,\}$. Thanks to Lemma 2.16, we know that $\tilde{\gamma} > 0$, and therefore there exists a function $\rho \in C_c^\infty([0, +\infty))$ such that

$$\rho(r) = 0 \qquad \text{if } r \in \left[\frac{\beta_d r_1}{\sqrt{2}}, +\infty \right)$$

and that

$$0 \leq \rho(r) \leq \tilde{\gamma} \quad \text{and} \quad \left| \rho'(r) \right| \leq \tilde{\gamma}.$$

For $\varepsilon > 0$ and $y \in \mathbb{R}^d$, we set

$$\rho_\varepsilon(y) := \frac{1}{c\varepsilon^d} \rho\left(\frac{|y|}{\varepsilon} \right), \qquad \text{with } c := \int_{\mathbb{R}^d} \rho(|y|) \mathrm{d}y,$$

and we introduce the functions

$$v_\varepsilon(x) := \left(\rho_\varepsilon * \left(u_\varepsilon - \sum_{i \in I} a_{\varepsilon,i} \chi_{C_i} \right) \right)(x),$$

where $\{\, C_i \,\}_{i \in I}$ is the family of connected components of Ω^c and each $a_{\varepsilon,i} \in \mathbb{R}$ satisfies $u_\varepsilon(x) = a_{\varepsilon,i}$ in C_i for all $i \in I$ and $\varepsilon > 0$.

Each function v_ε is smooth and, for all $\delta \in (0,1)$, its support is contained in $\Omega_\delta := \{\, x : \operatorname{dist}(x, \Omega) \leq 2^{-1/2} \delta \beta_d r_1 \,\}$ if $\varepsilon \in (0, \delta)$.

Let us fix δ small enough, so that $\partial \Omega_\delta$ is Lipschitz. For such δ, we now prove that the family $\{\, v_\varepsilon \,\}_{\varepsilon \in (0,\delta)}$ is relatively compact in $H_0^1(\Omega_\delta)$. To this aim, we first remark that

$$\int_{\Omega_\delta} \left| \nabla^2 v_\varepsilon \right|^2 = \int_{\Omega_\delta} \left| \Delta v_\varepsilon \right|^2, \tag{2.37}$$

and next we show that the right-hand side is uniformly bounded.

We observe that $\int_{\mathbb{R}^d} \nabla \rho_\varepsilon(y) \mathrm{d}y = 0$ for all $\varepsilon > 0$, because ρ is compactly supported. Hence,

$$\begin{aligned}
\|\Delta v_\varepsilon\|_{L^2(\Omega_\delta)}^2 &= \int_{\mathbb{R}^d} |\Delta v_\varepsilon|^2 \\
&= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \nabla \rho_\varepsilon(y) \cdot \left(\nabla u_\varepsilon(x+y) - \nabla u_\varepsilon(x) \right) \mathrm{d}y \right|^2 \mathrm{d}x \\
&\leq \int_{\mathbb{R}^d} \left[\frac{1}{c\varepsilon^{d+1}} \int_{\mathbb{R}^d} \left| \rho'\left(\frac{|y|}{\varepsilon} \right) \right| \left| \left(\nabla u_\varepsilon(x+y) - \nabla u_\varepsilon(x) \right) \cdot \hat{y} \right| \mathrm{d}y \right]^2 \mathrm{d}x.
\end{aligned}$$

By our choice of ρ and (2.35), we find

$$\|\Delta v_\varepsilon\|_{L^2(\Omega_\delta)}^2 \leq \int_{\mathbb{R}^d} \left[\frac{1}{c\varepsilon} \int_{\mathbb{R}^d} \tilde{G}_\varepsilon(y) \left| \left(\nabla u_\varepsilon(x+y) - \nabla u_\varepsilon(x) \right) \cdot \hat{y} \right| \mathrm{d}y \right]^2 \mathrm{d}x$$

$$\leq \int_{\mathbb{R}^d} \left[\frac{1}{c\varepsilon} \int_{\mathbb{R}^d} \tilde{G}(z) \left| \left(\nabla u_\varepsilon(x+\varepsilon z) - \nabla u_\varepsilon(x) \right) \cdot \hat{z} \right| \mathrm{d}z \right]^2 \mathrm{d}x$$

Further, since $\tilde{G} \in L^1(\mathbb{R}^d)$, Jensen's Inequality and Fubini's Theorem yield

$$\|\Delta v_\varepsilon\|_{L^2(\Omega_\delta)}^2 \leq \frac{\|\tilde{G}\|_{L^1(\mathbb{R}^d)}}{c^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{G}(z) \left[\frac{\left(\nabla u_\varepsilon(x+\varepsilon z) - \nabla u_\varepsilon(x) \right) \cdot \hat{z}}{\varepsilon} \right]^2 \mathrm{d}x\mathrm{d}z.$$

The lower bound (2.33) entails

$$\|\Delta v_\varepsilon\|_{L^2(\Omega_\delta)}^2 \leq \frac{4}{c^2\alpha} \|\tilde{G}\|_{L^1(\mathbb{R}^d)} \mathscr{E}_\varepsilon(u_\varepsilon),$$

so that, in view of the assumption $\mathscr{E}_\varepsilon(u_\varepsilon) \leq M$ and of (2.37), we get

$$\|\nabla^2 v_\varepsilon\|_{L^2(\Omega_\delta)}^2 \leq \frac{4M}{c^2\alpha} \|\tilde{G}\|_{L^1(\mathbb{R}^d)}. \tag{2.38}$$

We argue as in the proof of Lemma 2.10. We recall that, for $\varepsilon \in (0,\delta)$, each v_ε vanishes on the complement of Ω_δ, and thus, by Poincaré's Inequality, (2.38) implies a uniform bound on the norms $\|v_\varepsilon\|_{H_0^2(\Omega_\delta)}$. As a consequence, by Rellich-Kondrachov's Theorem, the family $\{\,\tilde{v}_\varepsilon\,\}_{\varepsilon \in (0,\delta)}$ of the restrictions of the functions v_ε to Ω_δ admits a subsequence $\{\,\tilde{v}_{\varepsilon_\ell}\,\}$ that converges in $H_0^1(\Omega_\delta)$ to a function $\tilde{u} \in H_0^2(\Omega_\delta)$. Actually, the support of $\tilde{u}$ is contained in $\bar{\Omega}$, and, if we put,

$$u(x) := \begin{cases} \tilde{u}(x) & \text{if } x \in \bar{\Omega}, \\ 0 & \text{otherwise,} \end{cases}$$

we infer that $\{\,v_{\varepsilon_\ell}\,\}$ converges in $H^1(\mathbb{R}^d)$ to $u \in \{\,v \in H^2(\mathbb{R}^d) : v = 0 \text{ a.e. in } \Omega^\complement\,\} \subset X$.

To accomplish the proof, we need to show that the $L^2(\mathbb{R}^d)$-distance between ∇u_ε and ∇v_ε vanishes when $\varepsilon \to 0^+$. Since ρ_ε has unit $L^1(\mathbb{R}^d)$-norm and is radial, we have

$$\int_{\mathbb{R}^d} |\nabla v_\varepsilon(x) - \nabla u_\varepsilon(x)|^2 \mathrm{d}x = \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \rho_\varepsilon(y) \left(\nabla u_\varepsilon(x+y) - \nabla u_\varepsilon(x) \right) \mathrm{d}y \right|^2 \mathrm{d}x$$

$$\leq \frac{1}{4} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \rho_\varepsilon(y) \left(\nabla u_\varepsilon(x+y) + \nabla u_\varepsilon(x-y) - 2\nabla u_\varepsilon(x) \right) \mathrm{d}y \right|^2 \mathrm{d}x$$

$$\leq \frac{1}{4} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho_\varepsilon(y) \left| \nabla u_\varepsilon(x+y) + \nabla u_\varepsilon(x-y) - 2\nabla u_\varepsilon(x) \right|^2 \mathrm{d}y\mathrm{d}x.$$

Let id be the identity matrix. For any fixed $y \in \mathbb{R}^d \setminus \{0\}$ and for all $p \in \mathbb{R}^d$, we can rewrite the equality $|p|^2 = |p \cdot y|^2 + |(\text{id} - y \otimes y)p|^2$ as

$$|p|^2 = |p \cdot y|^2 + \int_{\hat{y}^\perp} \pi(|\eta|) |p \cdot \eta|^2 \, \mathrm{d}'\eta$$

$$= |p \cdot y|^2 + \frac{1}{\varepsilon^2} \int_{\hat{y}^\perp} \pi_\varepsilon(\eta) |p \cdot \eta|^2 \, \mathrm{d}'\eta, \tag{2.39}$$

where $\pi\colon [0, +\infty) \to [0, +\infty)$ is a continuous function such that

$$\int_{e_d^\perp} \pi(|\eta|)\, |\eta|^2\, \mathrm{d}'\eta = 1,$$

and $\pi_\varepsilon(\eta) := \varepsilon^{-d+1}\pi(|\eta|/\varepsilon)$. We further prescribe that

$$\pi(r) = 0 \qquad \text{if } r \in \left[\frac{\beta_d r_1}{\sqrt{2}}, +\infty\right)$$

and that the limit $\lim_{r\to 0^+} \pi(r)/r$ is finite.

We apply formula (2.39) to $p_\varepsilon(x,y) := \nabla u_\varepsilon(x+y) + \nabla u_\varepsilon(x-y) - 2\nabla u_\varepsilon(x)$ and we find that

$$\int_{\mathbb{R}^d} |\nabla v_\varepsilon(x) - \nabla u_\varepsilon(x)|^2\, \mathrm{d}x \le \frac{1}{4}\left(I_1 + I_2\right), \tag{2.40}$$

where

$$I_1 := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \rho_\varepsilon(y)\, |y|^2\, |p_\varepsilon(x,y)\cdot \hat{y}|^2\, \mathrm{d}y\mathrm{d}x,$$

$$I_2 := \frac{1}{\varepsilon^2} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp} \rho_\varepsilon(y)\pi_\varepsilon(\eta)\, |p_\varepsilon(x,y)\cdot \eta|^2\, \mathrm{d}'\eta\mathrm{d}y\mathrm{d}x.$$

We first consider I_1. Keeping in mind that ρ is compactly supported and $\rho(|y|) \le \tilde{\gamma} \le \tilde{G}(y)$ for a.e. $y \in B(0, 2^{-1/2}\beta_d r_1)$, we get

$$I_1 \le \frac{(\beta_d r_1)^2}{c}\left[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde{G}_\varepsilon(y)\left|\left(\nabla u_\varepsilon(x+y) - \nabla u_\varepsilon(x)\right)\cdot \hat{y}\right|^2\, \mathrm{d}y\mathrm{d}x \right.$$
$$\left. + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde{G}_\varepsilon(y)\left|\left(\nabla u_\varepsilon(x-y) - \nabla u_\varepsilon(x)\right)\cdot \hat{y}\right|^2\, \mathrm{d}y\mathrm{d}x\right],$$

and, by (2.33),

$$I_1 \le \frac{8(\beta_d r_1)^2 M}{c\alpha}\varepsilon^2. \tag{2.41}$$

As for I_2, we claim that there exist a constant $L > 0$, depending on d, β_d, r_1, $\tilde{\gamma}$, and c, such that

$$I_2 \le \frac{LM}{\alpha}\varepsilon^2. \tag{2.42}$$

To prove this, we write the integrand appearing in I_2 as follows:

$$\begin{aligned}
p_\varepsilon(x,y)\cdot \eta &= \left(\nabla u_\varepsilon(x+y) + \nabla u_\varepsilon(x-y) - 2\nabla u_\varepsilon(x)\right)\cdot \eta \\
&= \left(\nabla u_\varepsilon(x+y) + \nabla u_\varepsilon(x-y) - 2\nabla u_\varepsilon(x-\eta)\right)\cdot \eta \\
&\quad + 2\left(\nabla u_\varepsilon(x-\eta) - \nabla u_\varepsilon(x)\right)\cdot \eta \\
&= \left(\nabla u_\varepsilon(x+y) - \nabla u_\varepsilon(x-\eta)\right)\cdot (\eta + y) \\
&\quad + \left(\nabla u_\varepsilon(x-y) - \nabla u_\varepsilon(x-\eta)\right)\cdot (\eta - y) \\
&\quad - \left(\nabla u_\varepsilon(x+y) - \nabla u_\varepsilon(x-y)\right)\cdot y + 2\left(\nabla u_\varepsilon(x-\eta) - \nabla u_\varepsilon(x)\right)\cdot \eta.
\end{aligned}$$

We plug this expression in the definition of I_2. Using again the symbol d' as a short for $\mathrm{d}\mathscr{H}^{d-1}$, we find

$$I_2 \le \frac{4}{c} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\hat{y}^\perp} \rho(|y|)\pi(|\eta|) \left|\left(\nabla u_\varepsilon(x+\varepsilon y) - \nabla u_\varepsilon(x-\varepsilon\eta)\right) \cdot (\eta+y)\right|^2 \mathrm{d}'\eta\mathrm{d}y\mathrm{d}x$$

$$+ \frac{4}{c} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\hat{y}^\perp} \rho(|y|)\pi(|\eta|) \left|\left(\nabla u_\varepsilon(x-\varepsilon y) - \nabla u_\varepsilon(x-\varepsilon\eta)\right) \cdot (\eta-y)\right|^2 \mathrm{d}'\eta\mathrm{d}y\mathrm{d}x$$

$$+ \frac{8}{c} \|\pi\|_{L^1(e_d^\perp)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho(|y|)\,|y|^2 \left|\left(\nabla u_\varepsilon(x+\varepsilon y) - \nabla u_\varepsilon(x)\right) \cdot \hat{y}\right|^2 \mathrm{d}y\mathrm{d}x$$

$$+ \frac{16}{c} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\hat{y}^\perp} \rho(|y|)\pi(|\eta|) \left|\left(\nabla u_\varepsilon(x+\varepsilon\eta) - \nabla u_\varepsilon(x)\right) \cdot \eta\right|^2 \mathrm{d}'\eta\mathrm{d}y\mathrm{d}x.$$

We estimate separately each of the contributions on the right-hand side.

Let us set $\mathbb{S}_+^{d-1} := \left\{\, e \in \mathbb{S}^{d-1} : e \cdot e_d > 0 \,\right\}$ and $\mathbb{S}_-^{d-1} := \left\{\, e \in \mathbb{S}^{d-1} : e \cdot e_d < 0 \,\right\}$. Hereafter, we denote by L any strictly positive constant depending only on d, β_d, r_1, and on the norms of ρ and π, possibly changing from line to line.

By Coarea Formula, we can rewrite the first addendum as follows:

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\hat{y}^\perp} \rho(|y|)\pi(|\eta|) \left|\left(\nabla u_\varepsilon(x+\varepsilon y) - \nabla u_\varepsilon(x-\varepsilon\eta)\right) \cdot (\eta+y)\right|^2 \mathrm{d}'\eta\mathrm{d}y\mathrm{d}x$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\hat{y}^\perp} \rho(|y|)\pi(|\eta|) \left|\left(\nabla u_\varepsilon(x+\varepsilon(\eta+y)) - \nabla u_\varepsilon(x)\right) \cdot (\eta+y)\right|^2 \mathrm{d}'\eta\mathrm{d}x\mathrm{d}y$$

$$= \int_{\mathbb{S}_+^{d-1}} \int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_{e^\perp} r^{d-1}\rho(r)\pi(|\eta|)$$

$$\cdot \left|\left(\nabla u_\varepsilon(x+\varepsilon(\eta+re)) - \nabla u_\varepsilon(x)\right) \cdot (\eta+re)\right|^2 \mathrm{d}'\eta\mathrm{d}r\mathrm{d}x\mathrm{d}'e$$

$$= \int_{\mathbb{S}_+^{d-1}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |y|^2\, |y \cdot e|^{d-1}\, \rho(|y \cdot e|)\pi\big(|(\mathrm{id} - e \otimes e)y|\big)$$

$$\cdot \left|\left(\nabla u_\varepsilon(x+\varepsilon y) - \nabla u_\varepsilon(x)\right) \cdot \hat{y}\right|^2 \mathrm{d}y\mathrm{d}x\mathrm{d}'e.$$

Similarly, we have

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\hat{y}^\perp} \rho(|y|)\pi(|\eta|) \left|\left(\nabla u_\varepsilon(x-\varepsilon y) - \nabla u_\varepsilon(x-\varepsilon\eta)\right) \cdot (\eta-y)\right|^2 \mathrm{d}'\eta\mathrm{d}y\mathrm{d}x$$

$$= \int_{\mathbb{S}_-^{d-1}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |y|^2\, |y \cdot e|^{d-1}\, \rho(|y \cdot e|)\pi\big(|(\mathrm{id} - e \otimes e)y|\big)$$

$$\cdot \left|\left(\nabla u_\varepsilon(x+\varepsilon y) - \nabla u_\varepsilon(x)\right) \cdot \hat{y}\right|^2 \mathrm{d}y\mathrm{d}x\mathrm{d}'e,$$

and thus

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\hat{y}^\perp} \rho(|y|)\pi(|\eta|) \left|\left(\nabla u_\varepsilon(x+\varepsilon y) - \nabla u_\varepsilon(x-\varepsilon\eta)\right) \cdot (\eta+y)\right|^2 \mathrm{d}'\eta\mathrm{d}y\mathrm{d}x$$

$$+ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\hat{y}^\perp} \rho(|y|)\pi(|\eta|) \left|\left(\nabla u_\varepsilon(x-\varepsilon y) - \nabla u_\varepsilon(x-\varepsilon\eta)\right) \cdot (\eta-y)\right|^2 \mathrm{d}'\eta\mathrm{d}y\mathrm{d}x$$

$$= \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |y \cdot e|^{d-1}\, |y|^2\, \rho(|y \cdot e|)\pi\big(|(\mathrm{id} - e \otimes e)y|\big)$$

$$\cdot \left|\left(\nabla u_\varepsilon(x+\varepsilon y) - \nabla u_\varepsilon(x)\right) \cdot \hat{y}\right|^2 \mathrm{d}y\mathrm{d}x\mathrm{d}'e.$$

Let us recall that $\rho(r) = \pi(r) = 0$ if $r \notin [0, 2^{-1/2}\beta_d r_1)$, whence, for any $e \in \mathbb{S}^{d-1}$, the product $\rho(|y \cdot e|)\pi\big(|(\mathrm{id} - e \otimes e)y|\big)$ vanishes outside the cylinder

$$C_e := \Big\{ y \in \mathbb{R}^d : |y \cdot e|, |(\mathrm{id} - e \otimes e)y| \in [0, 2^{-1/2}\beta_d r_1) \Big\} \subset B(0, \beta_d r_1).$$

Consequently, the last multiple integral equals

$$\int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}^d} \int_{C_e} |y \cdot e|^{d-1} |y|^2 \rho(|y \cdot e|)\pi\big(|(\mathrm{id} - e \otimes e)y|\big)$$
$$\cdot \big|(\nabla u_\varepsilon(x + \varepsilon y) - \nabla u_\varepsilon(x)) \cdot \hat{y}\big|^2 \,\mathrm{d}y\mathrm{d}x\mathrm{d}'e,$$

which, in turn, is bounded above by

$$L \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}^d} \int_{C_e} \tilde{G}(y) \big|(\nabla u_\varepsilon(x + \varepsilon y) - \nabla u_\varepsilon(x)) \cdot \hat{y}\big|^2 \,\mathrm{d}y\mathrm{d}x\mathrm{d}'e \leq \frac{LM}{\alpha}\varepsilon^2.$$

We then obtain

$$\frac{4}{c} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\hat{y}^\perp} \rho(|y|)\pi(|\eta|) \big|(\nabla u_\varepsilon(x + \varepsilon y) - \nabla u_\varepsilon(x - \varepsilon \eta)) \cdot (\eta + y)\big|^2 \,\mathrm{d}'\eta\mathrm{d}y\mathrm{d}x$$

$$+ \frac{4}{c} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\hat{y}^\perp} \rho(|y|)\pi(|\eta|) \big|(\nabla u_\varepsilon(x - \varepsilon y) - \nabla u_\varepsilon(x - \varepsilon \eta)) \cdot (\eta - y)\big|^2 \,\mathrm{d}'\eta\mathrm{d}y\mathrm{d}x$$

$$\leq \frac{LM}{\alpha}\varepsilon^2. \quad (2.43)$$

Next, we have

$$\frac{8}{c}\|\pi\|_{L^1(e_d^\perp)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho(|y|) |y|^2 \big|(\nabla u_\varepsilon(x + \varepsilon y) - \nabla u_\varepsilon(x)) \cdot \hat{y}\big|^2 \,\mathrm{d}y\mathrm{d}x \leq \frac{LM}{\alpha}\varepsilon^2, \quad (2.44)$$

$$\frac{16}{c} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\hat{y}^\perp} \rho(|y|)\pi(|\eta|) \big|(\nabla u_\varepsilon(x + \varepsilon \eta) - \nabla u_\varepsilon(x)) \cdot \eta\big|^2 \,\mathrm{d}'\eta\mathrm{d}y\mathrm{d}x \leq \frac{LM}{\alpha}\varepsilon^2. \quad (2.45)$$

The bound in (2.44) may be deduced as the one in (2.41), so, to establish (2.42), we are only left to prove (2.45). To this purpose, let $\psi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ be a test function. By a standard argument and Fubini's Theorem we have that

$$\int_{\mathbb{R}^d} \int_{\hat{y}^\perp} \rho(|y|)\pi(|\eta|)\psi(y, \eta)\mathrm{d}'\eta\mathrm{d}y$$

$$= \lim_{\delta \to 0^+} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|y|}{2\delta} \chi_{\{t < \delta\}}(|\eta \cdot y|)\rho(|y|)\pi(|\eta|)\psi(y, \eta)d\eta dy$$

$$= \lim_{\delta \to 0^+} \int_{\mathbb{R}^d} \frac{\pi(|\eta|)}{|\eta|} \left(\int_{\mathbb{R}^d} \frac{|\eta|}{2\delta} \chi_{\{t < \delta\}}(|\eta \cdot y|)\rho(|y|) |y| \psi(y, \eta)\mathrm{d}y \right) d\eta$$

$$= \int_{\mathbb{R}^d} \int_{\hat{\eta}^\perp} \frac{\pi(|\eta|)}{|\eta|}\rho(|y|) |y| \psi(y, \eta)\mathrm{d}'yd\eta.$$

It follows that

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\hat{y}^\perp} \rho(|y|)\pi(|\eta|) \big|(\nabla u_\varepsilon(x + \varepsilon \eta) - \nabla u_\varepsilon(x)) \cdot \eta\big|^2 \,\mathrm{d}'\eta\mathrm{d}y\mathrm{d}x$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\hat{\eta}^\perp} \frac{\pi(|\eta|)}{|\eta|}\rho(|y|) |y| \big|(\nabla u_\varepsilon(x + \varepsilon \eta) - \nabla u_\varepsilon(x)) \cdot \eta\big|^2 \,\mathrm{d}'yd\eta\mathrm{d}x$$

$$\leq L \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{G}(\eta) \big|(\nabla u_\varepsilon(x + \varepsilon \eta) - \nabla u_\varepsilon(x)) \cdot \eta\big|^2 \,d\eta\mathrm{d}x$$

(recall that we assume $\lim_{r \to 0^+} \pi(r)/r$ to be finite).

In conclusion, combining (2.40), (2.41), and (2.42), we infer

$$\int_{\mathbb{R}^d} |\nabla v_\varepsilon(x) - \nabla u_\varepsilon(x)|^2 \, \mathrm{d}x \leq \frac{LM}{\alpha} \varepsilon^2.$$

This concludes the proof. □

Eventually, we observe that Theorem 2.1 follows from the results of the current section.

Proof of Theorem 2.1. Lemma 2.18 shows that statement (i) holds.

As for the lower limit inequality, for any $u \in X$ and for any $\{u_\varepsilon\} \subset X$ that converges to u in $H^1_{\mathrm{loc}}(\mathbb{R}^d)$, we can assume that $\mathscr{E}_\varepsilon(u_\varepsilon) \leq M$ for all $\varepsilon > 0$ and some $M \geq 0$; otherwise, the lower limit inequality holds trivially. Then, by Lemma 2.18, u belongs to $H^2_{\mathrm{loc}}(\mathbb{R}^d)$ and we can invoke Proposition 2.13, which yields statement (ii).

We prove the upper limit inequality in the same fashion as the 1-dimensional case (see the proof of Proposition 2.7). Let us suppose that $u \in X \cap H^2_{\mathrm{loc}}(\mathbb{R}^d)$. By mollification we can obtain a sequence $\{u_\ell\} \subset X$ of smooth functions that converges to u in $H^2_{\mathrm{loc}}(\mathbb{R}^d)$. More precisely, $\{\nabla u_\ell\}$ and $\{\nabla^2 u_\ell\}$ converge in $L^2(\mathbb{R}^d)$ respectively to ∇u and $\nabla^2 u$. If the gradient of u is bounded in Ω or f'' is bounded, $f''(|\nabla u_\ell(x) \cdot \hat{z}|) \leq M$ for a.e. x, all z and some $M > 0$. It follows that we can exploit the Dominated Convergence Theorem to get $\lim_{\ell \to +\infty} \mathscr{E}_0(u_\ell) = \mathscr{E}_0(u)$. So, statements (iii.a) and (iii.b) are achieved through the approximation by smooth functions, Proposition 2.13, and Lemma 1.10. □

REMARK 2.19 (A regularity criterion)
When Ω, K, and f are as in Theorem 2.1 and f'' is bounded, as a consequence of our results, we see that a function $u \in X$ belongs to $H^2_{\mathrm{loc}}(\mathbb{R}^d)$ if and only if $\mathscr{E}_\varepsilon(u) \leq M$ for some $M > 0$ and for all sufficiently small ε. Indeed, on one hand, if $\mathscr{E}_\varepsilon(u) \leq M$ we can invoke Lemma 2.18 with the choice $u_\varepsilon = u$. On the other, by slicing and Remark 2.9, when $f'' \leq c$ we get

$$\mathscr{E}_\varepsilon(u) \leq \frac{c}{2} \left(\int_{\mathbb{R}^d} K(z) |z|^2 \, dz \right) \int_{\mathbb{R}^d} |\nabla^2 u(x)|^2 \, \mathrm{d}x.$$

Chapter 3

Nonlocal perimeters
and nonlocal curvatures

In the first chapter, we revised the key points of the well-established theory of finite perimeter sets in $\mathbb{R}^d$. Now, we go beyond it and we focus on its nonlocal extension. The main sources of our exposition are the papers [BP19, Pag19].

Given a reference set $\Omega \subset \mathbb{R}^d$ and a measurable function $K \colon \mathbb{R}^d \to [0, +\infty)$, for us the *nonlocal perimeter* associated with K of a set $E \subset \mathbb{R}^d$ in Ω is

$$
\begin{aligned}
\mathrm{Per}_K(E, \Omega) := & \int_{E \cap \Omega} \int_{E^c \cap \Omega} K(y - x) \mathrm{d}y \mathrm{d}x \\
& + \int_{E \cap \Omega} \int_{E^c \cap \Omega^c} K(y - x) \mathrm{d}y \mathrm{d}x + \int_{E \cap \Omega^c} \int_{E^c \cap \Omega} K(y - x) \mathrm{d}y \mathrm{d}x
\end{aligned}
$$

(recall that, when F is a set, $F^c := \mathbb{R}^d \setminus F$). Functionals of this form were introduced in [CRS10] by L. Caffarelli, J. Roquejoffre, and O. Savin, who focused on the fractional case, that is, $K(x) = |x|^{-d-s}$ with $s \in (0, 1)$. Their analysis was motivated by phase field models where long range interactions occur. We refer to the Introduction for a brief account of the reasons to study this sort of functionals, which, by now, have been intensively investigated. We shall suggest the works that are more closely related to ours among the ones in the vast available literature in the body of the chapter.

In analogy with the classical case, it is natural to consider the nonlocal counterparts of two other functionals, namely the total variation and the mean curvature. We define them in Section 3.2 and Section 3.4, respectively. Ahead of this, to get acquainted with nonlocal perimeters, Section 3.1 provides an overview of their basic properties: semicontinuity, submodularity, relation with the De Giorgi's perimeter. In Section 3.2 we also deal with existence of solutions to Plateau's problem. In this, an important role is played by a generalised Coarea Formula, whose simple proof we recall. Then, in Section 3.3, we propose a notion of calibration that is tailored for the nonlocal Plateau's problem, and we prove that calibrated functions are optimal w.r.t. their own boundary condition (see the recent paper [Cab20] by X. Cabré for similar results, too). A case in which we are able to produce an explicit calibration is the one of halfspaces, which turn out to be also the unique minimisers, see Theorem 3.14. Finally, moving from sufficient to necessary conditions for optimality, in Section 3.4 we establish an upper bound on the nonlocal curvature of sets with $C^{1,1}$ boundary.

3.1 Elementary properties of nonlocal perimeters

In a naïve manner, we might say that a functional named perimeter should measure the extension of the locus that separates a set from its complement. In the case of nonlocal perimeters, this is achieved by means of a measurable function $K \colon \mathbb{R}^d \to [0, +\infty)$, which we imagine to express some interaction between the points in $\mathbb{R}^d$. Then, we define the *nonlocal perimeter* associated with K of a set $E \subset \mathbb{R}^d$ as

$$\mathrm{Per}_K(E) := \int_E \int_{E^c} K(y - x) \mathrm{d}y \mathrm{d}x. \tag{3.1}$$

The intuitive idea behind such a position is that the interaction between the points $x \in E$ and $y \in E^c$ must "cross" the boundary of E, so the size of the latter can by quantified by the iterated integral at stake.

We would also like to give a notion of nonlocal perimeter restricted to a reference set Ω, in analogy to (1.2). To this aim, we firstly introduce for any measurable E, F the coupling

$$L_K(E; F) := \int_E \int_F K(y - x) \mathrm{d}y \mathrm{d}x,$$

and we discuss some of its properties.

By Tonelli's Theorem,

$$L_K(E; F) = L_K(F; E) = \int_{E \times F} K(y - x) \mathrm{d}y \mathrm{d}x,$$

because K is positive. It follows that we may assume without loss of generality that K is even, i.e.

$$K(x) = K(-x) \quad \text{for a.e. } x \in \mathbb{R}^d. \tag{3.2}$$

We also observe that when E_1, E_2, F are measurable

$$L_K(E_1; F) = L_K(E_2; F) \quad \text{if } \mathscr{L}^d(E_1 \triangle E_2) = 0,$$
$$L_K(E_1 \cup E_2; F) = L_K(E_1; F) + L_K(E_2; F) \quad \text{if } \mathscr{L}^d(E_1 \cap E_2) = 0,$$

and

$$\mathrm{Per}_K(E) = L_K(E; E^c).$$

Whether the interaction $L_K(E; F)$ is finite, it depends on the "regularity" of K, E, and F. We consider various possibilities:

(i) If K is in $L^1(\mathbb{R}^d)$ and its support is contained in $B(0, r)$ for some $r > 0$, then the interaction is determined by the points that are at a distance smaller than r from E and F:

$$L_K(E; F) = \int_{\{x \in E : \mathrm{dist}(x, F) < r\}} \int_{\{y \in F : \mathrm{dist}(y, E) < r\}} K(y - x) \mathrm{d}y \mathrm{d}x.$$

In [MRT19], J. Mazón, J. Rossi, and J. Toledo studied nonlocal perimeters and curvatures defined by kernels in this class.

(ii) When $K \in L^1(\mathbb{R}^d)$, $L_K(E; F)$ is finite as soon as one between E and F has finite measure. Indeed,

$$L_K(E; F) = \int_{\mathbb{R}^d} K(z) \, |E \cap (F - z)| \, \mathrm{d}z \leq \|K\|_{L^1(\mathbb{R}^d)} \left(\mathscr{L}^d(E) \wedge \mathscr{L}^d(F) \right).$$

(iii) As a third case, we allow K to be singular in the origin. Precisely, let us assume that

$$\int_{\mathbb{R}^d} K(x)(1 \wedge |x|) \mathrm{d}x < +\infty. \tag{3.3}$$

Then, we can bound the coupling L_K from above, provided some information about the mutual position of the sets is available. Indeed, on one hand, because of the singularity of K, L_K might blow up if the sets overlap on a region of full measure. On the other hand, if $\mathscr{L}^d(E \cap F) = 0$, then $F \subset E^{\mathsf{c}}$ up to negligible sets, so that

$$L_K(E; F) \leq L_K(E; E^{\mathsf{c}}) = \mathrm{Per}_K(E).$$

We shall prove in Lemma 3.2 that $\mathrm{Per}_K(E)$ is bounded above by the BV-norm of χ_E, up to a constant that depends on K.

Observe that, by (3.3), $K \in L^1(B(0, r)^{\mathsf{c}})$ for all balls $B(0, r)$ with centre in the origin and radius $r > 0$.

We gather here some examples of kernels such that (3.3) holds.

EXAMPLE 3.1
Functions in L^1 satisfy (3.3) trivially. Instead, a relevant example of kernels that exhibit a singularity in 0 is given by the fractional ones ([CRS10, Lud14]). We say that K is of *fractional* type if

$$K(x) = \frac{a(x)}{|x|^{d+s}},$$

with $s \in (0, 1)$ and $a \colon \mathbb{R}^d \to \mathbb{R}$ a measurable even function such that $0 < m \leq a(x) \leq M$ for any $x \in \mathbb{R}^d$ for some positive m and M.

A third class is formed by the kernels K such that $\int_{\mathbb{R}^d} K(x) \, |x| \, \mathrm{d}x < +\infty$. This requirement allows for a fractional-type behaviour near the origin, but it implies faster-than-L^1 decay at infinity at the same time. We shall deal with functions of this type in the next chapter, see Theorem 4.1.

Now, we use the coupling L_K to define the nonlocal perimeter restricted to a reference measurable set Ω, which we always assume to have strictly positive measure.

Again, we consider the interaction between the given set E and its complement. An initial attempt might be setting $\mathrm{Per}_K(E; \Omega)$ equal to $L_K(E \cap \Omega; E^{\mathsf{c}} \cap \Omega)$, but in this way we would be neglecting contributions arising from the portions of boundary that E and Ω share. For this reason, we take into consideration also terms where Ω^{c} appears, and we define the *nonlocal perimeter* of a measurable set E in Ω as

$$\begin{aligned}
\mathrm{Per}_K(E; \Omega) := \; & L_K(E \cap \Omega; E^{\mathsf{c}} \cap \Omega) \\
& + L_K(E \cap \Omega; E^{\mathsf{c}} \cap \Omega^{\mathsf{c}}) + L_K(E \cap \Omega^{\mathsf{c}}; E^{\mathsf{c}} \cap \Omega).
\end{aligned} \tag{3.4}$$

An illustration of what the different terms encode is provided by Figure 3.1.

Figure 3.1: The three contributions that compound Per_K. The thick lines represent the portions of boundary crossed by the interaction.

(a) $L_K(E \cap \Omega; E^c \cap \Omega)$

(b) $L_K(E \cap \Omega; E^c \cap \Omega^c)$

(c) $L_K(E \cap \Omega^c; E^c \cap \Omega)$
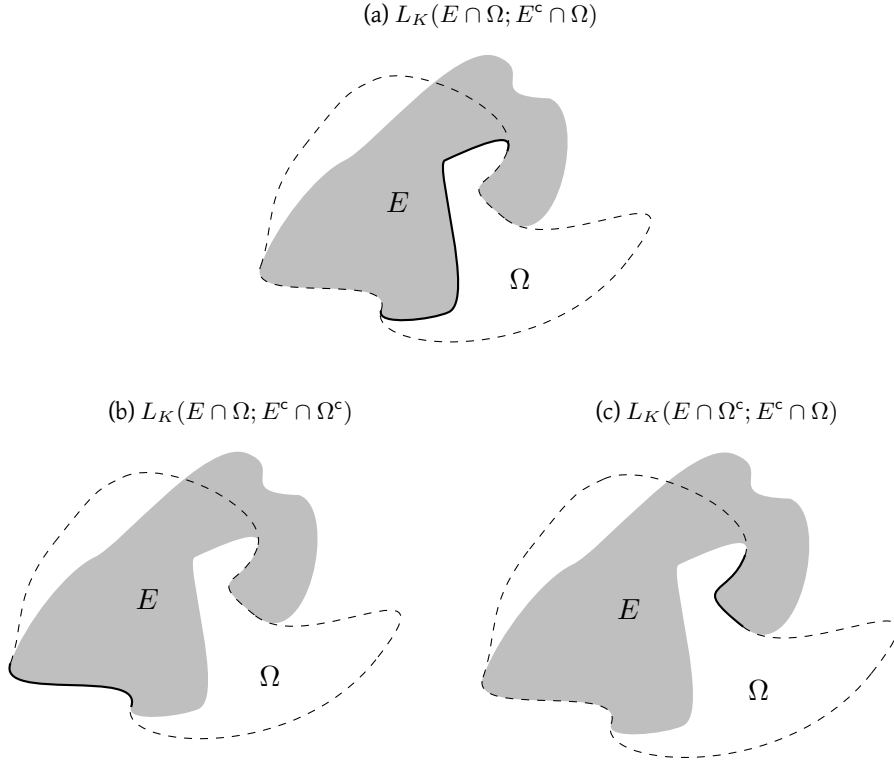
A simple but useful fact is that $\mathrm{Per}_K(E; \Omega)$ may be rewritten in terms of the characteristic function of E:

$$\mathrm{Per}_K(E; \Omega) = \frac{1}{2} \int_\Omega \int_\Omega K(y - x) \left| \chi_E(y) - \chi_E(x) \right| \mathrm{d}y \mathrm{d}x$$
$$+ \int_\Omega \int_{\Omega^c} K(y - x) \left| \chi_E(y) - \chi_E(x) \right| \mathrm{d}y \mathrm{d}x. \tag{3.5}$$

We also note that the nonlocal perimeter in (3.4) coincides with $\mathrm{Per}_K(\,\cdot\,; \mathbb{R}^d)$ and that $\mathrm{Per}_K(E; \Omega) = \mathrm{Per}_K(E)$ when $\mathscr{L}^d(E \cap \Omega^c) = 0$.

We collect some basic properties of the nonlocal perimeter in the following lemma:

LEMMA 3.2
Let $\Omega \subset \mathbb{R}^d$ be an open set. If $K \colon \mathbb{R}^d \to [0, +\infty)$ fulfils (3.2) and (3.3), then the functional $\mathrm{Per}_K(\,\cdot\,; \Omega)$ defined by (3.4) satisfies the following:

(i) $\mathrm{Per}_K(\emptyset; \Omega) = 0$, $\mathrm{Per}_K(E; \Omega) = \mathrm{Per}_K(F; \Omega)$ *if* $\mathscr{L}^d(E \triangle F) = 0$, *and* $\mathrm{Per}_K(E + z; \Omega + z) = \mathrm{Per}_K(E; \Omega)$ *for all* $z \in \mathbb{R}^d$.

(ii)   *Let E be a set such that $\mathscr{L}^d(E) < +\infty$. If $\Omega \neq \mathbb{R}^d$ and there exists $r > 0$ such that E*

is a finite perimeter set in $\Omega_r := \left\{ x \in \mathbb{R}^d : \operatorname{dist}(x, \Omega) < r \right\}$, *then*

$$
\begin{aligned}
\operatorname{Per}_K(E; \Omega) \leq {}& \left(\mathscr{L}^d(E) \vee \frac{\operatorname{Per}(E; \Omega_r)}{2} \right) \int_{\mathbb{R}^d} K(x)(1 \wedge |x|) \mathrm{d}x \\
& + \frac{\mathscr{L}^d(E)}{2r} \int_{B(0,1)} K(x) |x| \, \mathrm{d}x;
\end{aligned}
\tag{3.6}
$$

in particular, E has finite nonlocal K-perimeter in Ω. Also, if E is a finite perimeter set in $\mathbb{R}^d$, then

$$
\operatorname{Per}_K(E) \leq \left(\mathscr{L}^d(E) \vee \frac{\operatorname{Per}(E)}{2} \right) \int_{\mathbb{R}^d} K(x)(1 \wedge |x|) \mathrm{d}x.
$$

(iii) $\operatorname{Per}_K(\,\cdot\,; \Omega)$ *is lower semicontinuous w.r.t. the $L^1_{\mathrm{loc}}(\mathbb{R}^d)$-convergence.*

(iv) $\operatorname{Per}_K(\,\cdot\,; \Omega)$ *is submodular, i.e. for all $E, F \in \mathscr{M}$ it holds*

$$
\operatorname{Per}_K(E \cap F; \Omega) + \operatorname{Per}_K(E \cup F; \Omega) \leq \operatorname{Per}_K(E; \Omega) + \operatorname{Per}_K(F; \Omega).
$$

Proof. Statement (i) follows easily from the definition of perimeter thanks to the properties of Lebesgue's integral.

To prove (ii), we take advantage of formula (3.5):

$$
\begin{aligned}
L_K(E \cap \Omega; E^{\mathsf{c}} \cap \Omega) &= \frac{1}{2} \int_\Omega \int_\Omega K(y - x) |\chi_E(y) - \chi_E(x)| \, \mathrm{d}y \mathrm{d}x \\
&= \frac{1}{2} \int_\Omega \int_{\mathbb{R}^d} K(z) |\chi_E(x + z) - \chi_E(x)| \chi_\Omega(x + z) \mathrm{d}z \mathrm{d}x \\
&= \frac{1}{2} \int_{\mathbb{R}^d} \int_{\Omega \cap (\Omega - z)} K(z) |\chi_E(x + z) - \chi_E(x)| \, \mathrm{d}x \mathrm{d}z \\
&= \frac{1}{2} \int_{B(0,1)} \int_{\Omega \cap (\Omega - z)} K(z) |\chi_E(x + z) - \chi_E(x)| \, \mathrm{d}x \mathrm{d}z \\
&\quad + \frac{1}{2} \int_{B(0,1)^{\mathsf{c}}} \int_{\Omega \cap (\Omega - z)} K(z) |\chi_E(x + z) - \chi_E(x)| \, \mathrm{d}x \mathrm{d}z
\end{aligned}
$$

The last double integral may be easily bounded by the triangular inequality:

$$
\frac{1}{2} \int_{B(0,1)^{\mathsf{c}}} \int_{\Omega \cap (\Omega - z)} K(z) |\chi_E(x + z) - \chi_E(x)| \, \mathrm{d}x \mathrm{d}z \leq \mathscr{L}^d(E \cap \Omega) \int_{B(0,1)^{\mathsf{c}}} K(z) \mathrm{d}z.
$$

Moreover, for any $z \in B(0, 1)$ and $x \in \Omega \cap (\Omega - z)$, we may appeal to Proposition 1.5, which gets

$$
\frac{1}{2} \int_{B(0,1)} \int_{\Omega \cap (\Omega - z)} K(z) |\chi_E(x + z) - \chi_E(x)| \, \mathrm{d}x \mathrm{d}z \leq \frac{\operatorname{Per}(E; \Omega)}{2} \int_{B(0,1)} K(z) |z| \, \mathrm{d}z.
$$

On the whole,

$$
L_K(E \cap \Omega; E^{\mathsf{c}} \cap \Omega) \leq \left(\mathscr{L}^d(E \cap \Omega) \vee \frac{\operatorname{Per}(E; \Omega)}{2} \right) \int_{\mathbb{R}^d} K(z)(1 \wedge |z|) \mathrm{d}z; \tag{3.7}
$$

this concludes the proof when $\Omega = \mathbb{R}^d$.

We focus now on the case when Ω is a proper subset of $\mathbb{R}^d$. Without loss of generality, we may assume that E is a finite perimeter set in Ω_r for $r \in (0, 1]$. By reasoning as above, we obtain

$$L_K(E \cap \Omega; E^{\mathsf{c}} \cap \Omega^{\mathsf{c}}) + L_K(E \cap \Omega^{\mathsf{c}}; E^{\mathsf{c}} \cap \Omega)$$
$$= \int_\Omega \int_{\Omega^{\mathsf{c}}} K(y - x) \, |\chi_E(y) - \chi_E(x)| \, \mathrm{d}y \mathrm{d}x$$
$$= \frac{1}{2} \int_{B(0,r)} \int_{\Omega \cap (\Omega^{\mathsf{c}} - z)} K(z) \, |\chi_E(x + z) - \chi_E(x)| \, \mathrm{d}x \mathrm{d}z$$
$$+ \frac{1}{2} \int_{B(0,r)^{\mathsf{c}}} \int_{\Omega \cap (\Omega^{\mathsf{c}} - z)} K(z) \, |\chi_E(x + z) - \chi_E(x)| \, \mathrm{d}x \mathrm{d}z.$$

Again, we are in position to apply Proposition 1.5 and the triangular inequality. We get

$$L_K(E \cap \Omega; E^{\mathsf{c}} \cap \Omega^{\mathsf{c}}) + L_K(E \cap \Omega^{\mathsf{c}}; E^{\mathsf{c}} \cap \Omega)$$
$$\leq \frac{\mathrm{Per}(E; \Omega_r)}{2} \int_{B(0,r)} K(z) \, |z| \, \mathrm{d}z + \frac{\mathscr{L}^d(E)}{2} \int_{B(0,r)^{\mathsf{c}}} K(z) \mathrm{d}z$$
$$\leq \frac{\mathrm{Per}(E; \Omega_r)}{2} \int_{B(0,1)} K(z) \, |z| \, \mathrm{d}z$$
$$+ \frac{\mathscr{L}^d(E)}{2} \left(\frac{1}{r} \int_{B(0,1)} K(z) \, |z| \, \mathrm{d}z + \int_{B(0,1)^{\mathsf{c}}} K(z) \mathrm{d}z \right).$$

We now recover (3.6) by combing (3.7) and the last estimate.

For what concerns assertion (iii), the lower semicontinuity is a consequence of formula (3.5) and of Fatou's Lemma.

Eventually, we prove statement (iv). It suffices to rewrite suitably each contribution: for instance, we have

$$L_K((E \cup F) \cap \Omega; E^{\mathsf{c}} \cap F^{\mathsf{c}} \cap \Omega)$$
$$= L_K(E \cap \Omega; E^{\mathsf{c}} \cap \Omega) + L_K(F \cap \Omega; F^{\mathsf{c}} \cap \Omega)$$
$$- L_K(E \cap \Omega; E^{\mathsf{c}} \cap F \cap \Omega) - L_K(F \cap \Omega; E \cap F^{\mathsf{c}} \cap \Omega)$$
$$- L_K(E \cap F \cap \Omega; E^{\mathsf{c}} \cap F^{\mathsf{c}} \cap \Omega)$$

and

$$L_K(E \cap F \cap \Omega; (E^{\mathsf{c}} \cup F^{\mathsf{c}}) \cap \Omega)$$
$$= L_K(E \cap F \cap \Omega; E^{\mathsf{c}} \cap F^{\mathsf{c}} \cap \Omega)$$
$$+ L_K(E \cap F \cap \Omega; E^{\mathsf{c}} \cap F \cap \Omega) + L_K(E \cap F \cap \Omega; E \cap F^{\mathsf{c}} \cap \Omega).$$

In the end, one finds

$$\mathrm{Per}_K(E; \Omega) + \mathrm{Per}_K(F; \Omega) = \mathrm{Per}_K(E \cap F; \Omega) + \mathrm{Per}_K(E \cup F; \Omega)$$
$$+ 2L_K(E \cap F^{\mathsf{c}} \cap \Omega; E^{\mathsf{c}} \cap F \cap \Omega)$$
$$+ 2L_K(E \cap F^{\mathsf{c}} \cap \Omega; E^{\mathsf{c}} \cap F \cap \Omega^{\mathsf{c}})$$
$$+ 2L_K(E \cap F^{\mathsf{c}} \cap \Omega^{\mathsf{c}}; E^{\mathsf{c}} \cap F \cap \Omega).$$

$\square$

REMARK 3.3 (Generalised perimeters)
Definition (3.4) might seem very different from the one of De Giorgi's perimeter. Nevertheless, Lemma 3.2 shows that the classical and the nonlocal perimeter have some useful properties in common. Actually, in [CMP15] A. Chambolle, M. Morini, and M. Ponsiglione proposed that a set functional p is a *generalised perimeter* if

(i) $p(\emptyset) = 0, p(E) = p(F)$ whenever $\mathscr{L}^d(E \triangle F) = 0$, and p is invariant under translations;

(ii) it is finite on the closures of open sets with compact C^2 boundary;

(iii) it is $L^1_{\mathrm{loc}}(\mathbb{R}^d)$-lower semicontinuous;

(iv) it is submodular.

In [CMP15], the functional in (3.1) appears as an instance of generalised perimeter. In that work, the authors are concerned with geometric evolutions driven by generalised curvatures, and Per_K is listed as an example of functional whose first variation is a curvature in that sense. We shall come back to this point later on, see Section 3.4 and Chapter 5.

3.2 Coarea Formula and Plateau's problem

We recalled in Section 1.2 that the theory of De Giorgi's perimeter may be formulated in terms of functions of bounded variation. In the same spirit, we introduce a nonlocal functional that might be regarded as a *nonlocal total variation*. Standing the previous assumptions on Ω and K, for any measurable $u \colon \mathbb{R}^d \to \mathbb{R}$ we set

$$J_K^1(u; \Omega) \coloneqq \frac{1}{2} \int_\Omega \int_\Omega K(y-x) \left| u(y) - u(x) \right| \mathrm{d}x \mathrm{d}y,$$

$$J_K^2(u; \Omega) \coloneqq \int_\Omega \int_{\Omega^c} K(y-x) \left| u(y) - u(x) \right| \mathrm{d}x \mathrm{d}y, \tag{3.8}$$

$$J_K(u; \Omega) \coloneqq J_K^1(u; \Omega) + J_K^2(u; \Omega).$$

By a small abuse of notation, we shall write $J_K^i(E; \Omega) \coloneqq J_K^i(\chi_E; \Omega)$ for $i = 1, 2$ and $J_K(E; \Omega) \coloneqq J_K(\chi_E; \Omega)$, so that (3.5) reads

$$\mathrm{Per}_K(E; \Omega) = J_K^1(E; \Omega) + J_K^2(E; \Omega) = J_K(E; \Omega).$$

By imitating the proof of statement (ii) in Lemma 3.2, the reader may verify that the BV-norm bounds the nonlocal total variation:

PROPOSITION 3.4
Let Ω be an open set and let $K \colon \mathbb{R}^d \to [0, +\infty)$ satisfy (3.2) and (3.3). Suppose also that $u \in L^1(\mathbb{R}^d)$. Then, if $\Omega \neq \mathbb{R}^d$ and there exists $r > 0$ such that $u \in \mathrm{BV}(\Omega_r)$, with $\Omega_r \coloneqq \left\{ x \in \mathbb{R}^d : \mathrm{dist}(x, \Omega) < r \right\}$,

$$J_K(u; \Omega) \le \left(\|u\|_{L_1(\mathbb{R}^d)} \vee \frac{|\mathrm{D}u|(\Omega_r)}{2} \right) \int_{\mathbb{R}^d} K(x)(1 \wedge |x|) \mathrm{d}x$$

$$+ \frac{\|u\|_{L_1(\mathbb{R}^d)}}{2r} \int_{B(0,1)} K(x) \left| x \right| \mathrm{d}x.$$

Moreover, if $u \in \mathrm{BV}(\mathbb{R}^d)$, then

$$J_K(u) \le \left(\|u\|_{L_1(\mathbb{R}^d)} \vee \frac{\|\mathrm{D}u\|}{2} \right) \int_{\mathbb{R}^d} K(x)(1 \wedge |x|) \mathrm{d}x.$$

In the remainder of this Section, we investigate some of the main features of the nonlocal total variation. We firstly state two fundamental properties of $J_K(\,\cdot\,;\Omega)$ whose validation is straightforward.

PROPOSITION 3.5
The nonlocal total variation $J_K(\,\cdot\,;\Omega)$ is convex and lower semicontinuous w.r.t. the $L^1_{\mathrm{loc}}(\mathbb{R}^d)$-convergence.

Another tool that will be crucial for the forthcoming analysis is a generalised version of the Coarea Formula. Following A. Visintin [Vis91, Vis90], we say that a functional J defined on measurable scalar functions fulfils the *generalised Coarea Formula* if

$$J(u) = \int_{-\infty}^{+\infty} J(\chi_{\{\,u>t\,\}})\mathrm{d}t. \tag{3.9}$$

Hereafter, if $u\colon \mathbb{R}^d \to \mathbb{R}$ is a function and $t \in \mathbb{R}$, we set

$$\{\,u>t\,\} := \{\,x \in \Omega : u(x) > t\,\}.$$

It is not difficult to show that the nonlocal total variation J_K and the K-perimeter satisfy a coarea-type equality.

PROPOSITION 3.6 (Nonlocal Coarea Formula)
Let $K\colon \mathbb{R}^d \to [0,+\infty)$ and $u\colon \mathbb{R}^d \to \mathbb{R}$ be measurable. It holds

$$J_K^i(u;\Omega) = \int_{-\infty}^{+\infty} J_K^i(\{\,u>t\,\};\Omega)\mathrm{d}t \quad \text{for } i=1,2,$$

thus

$$J_K(u;\Omega) = \int_{-\infty}^{+\infty} \mathrm{Per}_K(\{\,u>t\,\};\Omega)\mathrm{d}t.$$

The proof has already appeared several times in the literature (see [CSV19] or [Vis91, CN18, MRT19, ADPM11]).

Proof. Given $x,y \in \Omega$, we may suppose that $u(x) \le u(y)$. We observe that the function $t \mapsto \chi_{\{\,u>t\,\}}(x) - \chi_{\{\,u>t\,\}}(y)$ vanishes for $t \in (-\infty, u(x)) \cup (u(y), +\infty)$, therefore

$$\int_{\mathbb{R}} \left|\chi_{\{\,u>t\,\}}(y) - \chi_{\{\,u>t\,\}}(x)\right|\mathrm{d}t = \int_{u(x)}^{u(y)} \left|\chi_{\{\,u>t\,\}}(y) - \chi_{\{\,u>t\,\}}(x)\right|\mathrm{d}t$$
$$= |u(y) - u(x)|.$$

By Tonelli's Theorem,

$$J_K^1(u;\Omega) := \frac{1}{2}\int_{\Omega}\int_{\Omega} K(x-y)\,|u(x)-u(y)|\,\mathrm{d}x\mathrm{d}y$$
$$= \frac{1}{2}\int_{\mathbb{R}}\int_{\Omega}\int_{\Omega} K(x-y)\left|\chi_{\{\,u>t\,\}}(x) - \chi_{\{\,u>t\,\}}(y)\right|\mathrm{d}x\mathrm{d}y\mathrm{d}t$$
$$= \int_{\mathbb{R}} J_K^1(\{\,u>t\,\};\Omega)\mathrm{d}t$$

The equality involving $J_K^2(u;\Omega)$ may be proved in the same manner. $\square$

The validity of the generalised Coarea Formula has interesting consequences from a variational perspective. For instance, in [CGL10], A. Chambolle, A. Giacomini, and L. Lussardi proved that a proper L^1-lower semicontinuous functional that satisfies the generalised Coarea Formula and whose restriction to characteristic functions is submodular is necessarily convex. Thus, we might see the convexity of $J_K(\,\cdot\,;\Omega)$ as an example of this general principle. Another useful contribution in [CGL10] that involves the Coarea Formula will be invoked in Section 4.2, when we deal with the asymptotics of rescaled energies.

Proposition 3.6 comes in handy also when addressing a quite natural variational problem concerning J_K. Its classical analogue is the well-known Plateau's problem, which amounts to minimise the perimeter under prescribed boundary conditions. If Ω is a given references set and E_0 is a measurable set playing the role of the boundary datum, one usually considers the class of competitors $\mathscr{F} := \{\, E : E \cap \Omega^{\mathsf{c}} = E_0 \cap \Omega^{\mathsf{c}} \,\}$. Remember that, in our notation, equality holds up to negligible sets. Existence of minimisers may be easily achieved thanks to the compactness criterion Theorem 1.6, and the L^1_{loc}-semicontinuity of De Giorgi's perimeter.

The approach does not require modifications if the classical perimeter is replaced by an anisotropic surface energy such as Per_σ in (1.7), provided the anisotropy σ is a norm. Conversely, the very same strategy is not effective in the nonlocal framework, because sequences that have uniformly bounded nonlocal perimeter are not necessarily precompact in L^1. As an example of this phenomenon, we may consider a sequence $\{\, E_\ell \,\}$ whose elements are subsets of some Ω with finite measure, and we observe that, for any $K \in L^1(\Omega)$, it holds $\mathrm{Per}_K(E_\ell;\Omega) \leq 3\,\|K\|_{L^1(\mathbb{R}^d)}\,\mathscr{L}^d(\Omega)$.

REMARK 3.7

It would be natural to investigate under which general conditions on K coercivity of $\mathrm{Per}_K(\,\cdot\,,\Omega)$ is ensured. In a recent paper [CS19], J. Chaker and L. Silvestre provided an answer for a functionals of the form

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(x,y)\,|u(y) - u(x)|^2\,\mathrm{d}x\mathrm{d}y.$$

Even though, in general, a uniform bound on the nonlocal perimeter is not informative, the direct method of calculus of variations is still a viable option.

THEOREM 3.8 (Existence of solutions to Plateau's problem)
Let $K\colon \mathbb{R}^d \to [0,+\infty)$ be measurable and let $\Omega \subset \mathbb{R}^d$ be an open set with finite measure. We suppose that E_0 is a measurable set such that $\mathrm{Per}_K(E_0;\Omega) < +\infty$, and we define the family

$$\mathscr{F} := \Big\{\, v\colon \mathbb{R}^d \to [0,1] : v \text{ is measurable and } v = \chi_{E_0} \text{ in } \Omega^{\mathsf{c}} \,\Big\}. \tag{3.10}$$

Then, there exists $E \subset \mathbb{R}^d$ such that $\chi_E \in \mathscr{F}$ and

$$\mathrm{Per}_K(E;\Omega) \leq J_K(v;\Omega) \quad \textit{for any } v \in \mathscr{F}.$$

We premise some comments about the statement before dealing with its proof.

REMARK 3.9 (Truncation)
For $s \in \mathbb{R}$, let us set $T(s) := 0 \vee (s \wedge 1)$. Observe that $T \circ \chi_{E_0} = \chi_{E_0}$ and $J_K(T \circ u;\Omega) \leq J_K(u;\Omega)$. Therefore,

$$\inf\{\, J_K(v;\Omega) : v \in \mathscr{F} \,\} = \inf\{\, J_K(v;\Omega) : v\colon \mathbb{R}^d \to \mathbb{R} \text{ is measurable and } v = \chi_{E_0} \text{ in } \Omega^{\mathsf{c}}, \,\}.$$

and we see that choice of $\mathscr{F}$ as the class of competitors is not restrictive.

REMARK 3.10 (The class of competitors is nonempty)
Let us suppose that E_0 has finite perimeter in some open set Ω_0 that strictly contains the closure of Ω and that K fulfils (3.3). Then, by Lemma 3.2–(ii), $\mathrm{Per}_K(E_0; \Omega) < +\infty$.

Proof of Theorem 3.8. Let $\{ u_\ell \} \subset \mathscr{F}$ be a minimising sequence. Since Ω has finite measure, for any choice of $p \in (1, +\infty)$, $\{ u_\ell \}$ is bounded in $L^p(\Omega; [0, 1])$. It follows that there exist a subsequence $\{ u_{\ell_m} \}$ and some $u \in L^p(\Omega; [0, 1])$ such that $\{ u_{\ell_m}|_\Omega \}$ weakly converges in L^p to u. Let us extend u to the whole $\mathbb{R}^d$ setting $u|_{\Omega^c} = \chi_{E_0}$. By Proposition 3.5, we know that J_K is convex and lower semicontinuous w.r.t. strong convergence in $L^p(\mathbb{R}^d; [0, 1])$ for any $p \in [1, +\infty)$, therefore it is lower semicontinuous w.r.t. the weak $L^p(\Omega; [0, 1])$-convergence. We deduce

$$\lim_{k \to +\infty} J_K(u_{\ell_m}; \Omega) \geq J_K(u; \Omega)$$

(recall that $u_{\ell_m} = u = \chi_{E_0}$ in Ω^c), that is, u is a minimiser for $J_K(\,\cdot\,, \Omega)$.

To conclude the proof, it suffices to show that we can recover a set E such that $\chi_E \in \mathscr{F}$ and $\mathrm{Per}_K(E; \Omega) \leq J_K(u; \Omega)$. We do this by the Coarea Formula: since

$$J_K(u; \Omega) = \int_0^1 \mathrm{Per}_K(\{ u > t \}; \Omega)\mathrm{d}t,$$

for some $t^* \in (0, 1)$ it must hold $J_K(u; \Omega) \geq \mathrm{Per}_K(\{ u > t^* \}; \Omega)$. Then, the choice $E := \chi_{\{ u > t^* \}}$ gets the conclusion. $\qquad\square$

3.3 A nonlocal notion of calibration

In the previous Section we proved existence of solutions to the nonlocal Plateau's problem. Now, relying on the paper [Pag19], we want to validate the nonlocal version of the classical sufficiency principle for minimality based on the concept of calibration.

The notion of calibration may be expressed in very general terms (see [Mor08, HL82] and references therein). As far as least area surfaces are concerned, we say that a (classical) *calibration* for the finite perimeter set E is a divergence-free vector field $\zeta \colon \mathbb{R}^d \to \mathbb{R}^d$ such that $|\zeta(x)| \leq 1$ a.e. and $\zeta(x) = \hat{n}(x)$ for $\mathscr{H}^{d-1}$-a.e. $x \in \partial^* E$, $\hat{n}$ being the measure-theoretic inner normal to E defined in (1.4).

It can be shown that if the set E admits a calibration, then its perimeter equals the infimum in the classical, isotropic Plateau's problem. To transpose this minimality criterion in the nonlocal setting, a suitable notion of calibration has to be introduced.

We remind that we assume that $\mathbb{R}^d \times \mathbb{R}^d$ is equipped with the product measure $\mathscr{L}^d \otimes \mathscr{L}^d$.
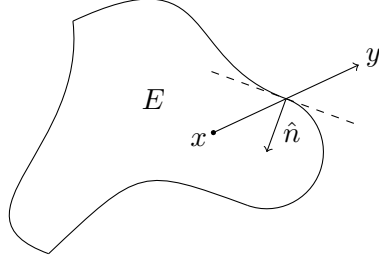
DEFINITION 3.11
Let $u \colon \mathbb{R}^d \to [0, 1]$ and $\zeta \colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be measurable functions. We say that ζ is a nonlocal calibration *for u if the following hold:*

(i) $|\zeta(x, y)| \leq 1$ *for a.e.* $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$;

(ii)   *for a.e.* $x \in \mathbb{R}^d$,

$$\lim_{r \to 0^+} \int_{B(x,r)^c} K(y - x)\left(\zeta(y, x) - \zeta(x, y)\right) \mathrm{d}y = 0; \qquad (3.11)$$

Figure 3.2: If ζ is a calibration for the set E (i.e. for χ_E) and x, y are as in the picture, then $\zeta(x, y) = -1$.

(iii)   *for a.e.* $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ *such that* $u(x) \neq u(y)$,

$$\zeta(x, y)(u(y) - u(x)) = |u(y) - u(x)|. \tag{3.12}$$

Some comments about the definition are in order. Suppose that $\zeta \colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a calibration for $u \colon \mathbb{R}^d \to [0, 1]$.

(i)   Up to replacing ζ with $\tilde{\zeta}(x, y) := (\zeta(x, y) - \zeta(y, x))/2$, we may always assume that $\zeta(x, y) = -\zeta(y, x)$.

(ii)   In view of (3.3), the integral in (3.11) is convergent for each $r > 0$. We may interpret (3.11) as a nonlocal reformulation of the classical vanishing divergence condition. Nonlocal notions of gradient and divergence operators where introduced by G. Gilboa and S. Osher in [GO08], and they have already been exploited to study nonlocal perimeters by Mazón, Rossi, and Toledo in [MRT19], where the authors propose a notion of K-calibrable set in relation to a nonlocal Cheeger energy.

(iii)   Suppose that $u = \chi_E$ for some measurable $E \subset \mathbb{R}^d$. By (3.12), ζ must satisfy

$$\zeta(x, y) = \begin{cases} -1 & \text{if } x \in E, y \in E^{\mathsf{c}} \\ 1 & \text{if } x \in E^{\mathsf{c}}, y \in E. \end{cases}$$

Heuristically, this means that the calibration gives the sign of the inner product between the vector $y - x$ and the inner normal to E at the "crossing point", provided the boundary of E is sufficiently regular (see Figure 3.2). Indeed, if we imagine to displace a particle from to x and y, ζ equals -1 when the particle exits E, and it equals 1 if the particles enters E.

REMARK 3.12

In a very recent, independent work [Cab20], Cabré studied essentially the same concept of nonlocal calibration: given an open bounded set $\Omega \subset \mathbb{R}^d$ and a measurable $E \subset \mathbb{R}^d$ such that $E = \{ f_E > 0 \}$ for some measurable $f_E \colon \mathbb{R}^d \to \mathbb{R}$, he introduced the set functional

$$\mathscr{C}_\Omega(F) := \int \int_{(\Omega^{\mathsf{c}} \times \Omega^{\mathsf{c}})^{\mathsf{c}}} K(y - x)(\chi_F(y) - \chi_F(x)) \operatorname{sign}(f_E(y) - f_E(x)) \mathrm{d}y \mathrm{d}x,$$

where $F \subset \mathbb{R}^d$ satisfies $F \cap \Omega^c = E \cap \Omega^c$. In [Cab20, Theorem 2.4] the author gives sufficient conditions for the set E to be a minimiser for Plateau's problem as well as conditions to grant uniqueness. As applications, he establishes the local minimality of graphs with 0 nonlocal curvature and, very interestingly, re-proves a result in [CRS10] stating that minimisers have null nonlocal curvature in a viscosity sense.

The existence of a calibration is a sufficient condition for a function u to minimise the energy J_K w.r.t compact perturbations, as the following statement shows:

THEOREM 3.13
Let $E_0 \subset \mathbb{R}^d$ be a measurable set such that $J_K(\chi_{E_0}; \Omega) < +\infty$, and let $\mathscr{F}$ be the family in (3.10). If for some $u \in \mathscr{F}$ there exists a calibration ζ, then

$$J_K(u; \Omega) \leq J_K(v; \Omega) \quad \text{for all } v \in \mathscr{F}.$$

Also, when $K > 0$, if $\tilde{u} \in \mathscr{F}$ is another minimiser, then ζ is a calibration for $\tilde{u}$ as well.

Proof. For any $v \in \mathscr{F}$, let

$$a(v) := \frac{1}{2} \int_\Omega \int_\Omega K(y-x)\zeta(x,y)(v(y)-v(x))\mathrm{d}y\mathrm{d}x,$$

$$b_1(v) := -\int_\Omega \int_{\Omega^c} K(y-x)\zeta(x,y)v(x)\mathrm{d}y\mathrm{d}x,$$

$$b_0 := \int_\Omega \int_{\Omega^c} K(y-x)\zeta(x,y)\chi_{E_0}(y)\mathrm{d}y\mathrm{d}x.$$

Then, by the definitions of $J_K(\,\cdot\,; \Omega)$, ζ, and $\mathscr{F}$,

$$J_K(v; \Omega) \geq a(v) + b_1(v) + b_0. \tag{3.13}$$

Since it is not restrictive to assume that $J_K(v; \Omega)$ is finite, we can suppose that $a(v)$, $b_1(v)$, and b_0 are finite as well.

First of all, we show that $a(v) = -b_1(v)$ for all $v \in \mathscr{F}$. Since we may always suppose that ζ is antisymmetric, we have

$$a(v) = -\int_\Omega \int_\Omega K(y-x)\zeta(x,y)v(x)\mathrm{d}y\mathrm{d}x.$$

Besides, (3.11) gets

$$0 = -2 \lim_{r \to 0^+} \int_{B(x,r)^c} K(y-x)\zeta(x,y)\mathrm{d}y$$

$$= -2 \lim_{r \to 0^+} \int_{B(x,r)^c \cap \Omega} K(y-x)\zeta(x,y)\mathrm{d}y - 2\int_{\Omega^c} K(y-x)\zeta(x,y)\mathrm{d}y,$$

whence

$$a(v) = -\lim_{r \to 0^+} \int_\Omega \int_{B(x,r)^c \cap \Omega} K(y-x)\zeta(x,y)v(x)\mathrm{d}y\mathrm{d}x = -b_1(v).$$

We plug the previous equality in (3.13) and obtain

$$J_K(v; \Omega) \geq b_0 \quad \text{for all } v \in \mathscr{F}. \tag{3.14}$$

To infer the optimality of u, observe that u attains the lower bounds, because, being ζ a calibration for u, equality holds in (3.13) when $v = u$.

Now, let K be strictly positive and assume that $\tilde{u} \in \mathscr{F}$ is another minimiser of $J_K(\,\cdot\,; \Omega)$, or, in other words, assume that $J_K(\tilde{u}; \Omega) = b_0$. We claim that for a.e. $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ such that $\tilde{u}(x) \neq \tilde{u}(y)$ it holds

$$\zeta(x, y)\,(\tilde{u}(y) - \tilde{u}(x)) = |\tilde{u}(y) - \tilde{u}(x)|\,. \tag{3.15}$$

The equality holds trivially for a.e. $(x, y) \in \Omega^{\mathsf{c}} \times \Omega^{\mathsf{c}}$, because $u = \tilde{u}$ in Ω^{c}. Furthermore, from (3.13) we have

$$b_0 = J_K(\tilde{u}; \Omega) \geq a(\tilde{u}) + b_1(\tilde{u}) + b_0 = b_0,$$

thus

$$\frac{1}{2} \int_\Omega \int_\Omega K(y - x)\left[|\tilde{u}(y) - \tilde{u}(x)| - \zeta(x, y)(\tilde{u}(y) - \tilde{u}(x))\right] \mathrm{d}y\mathrm{d}x$$
$$+ \int_\Omega \int_{\Omega^{\mathsf{c}}} K(y - x)\left[|\tilde{u}(y) - \tilde{u}(x)| - \zeta(x, y)(\tilde{u}(y) - \tilde{u}(x))\right] \mathrm{d}y\mathrm{d}x = 0.$$

Since the integrand is positive and $K > 0$, (3.15) is satisfied for a.e. $(x, y) \in \Omega \times \mathbb{R}^d$. Eventually, in the case $x \in \Omega^{\mathsf{c}}$ and $y \in \Omega$, we can derive the conclusion thanks to the antisymmetry of ζ. □

We put the previous theorem in action to prove that halfspaces are the unique local minimisers of $J_K(\,\cdot\,; B(0, 1))$ w.r.t. their own boundary condition. This is in line with the case of local perimeters like Per_σ in (1.7) associated with strictly convex anisotropies σ (see the monograph by F. Maggi [Mag12]). An analogue in the nonlocal setting is already available for fractional perimeters [CRS10, ADPM11] and, in wider generality, when the kernel is radial and strictly decreasing [BP19]. The use of calibrations enables us to extend the result to all positive, even kernels.

THEOREM 3.14
For all $\hat{n} \in \mathbb{S}^{d-1}$, set $H_{\hat{n}} := \left\{\, x \in \mathbb{R}^d : x \cdot \hat{n} > 0 \,\right\}$ and

$$\zeta_{\hat{n}}(x, y) := \mathrm{sign}((y - x) \cdot \hat{n}). \tag{3.16}$$

Then, $\zeta_{\hat{n}}$ is a calibration for $\chi_{H_{\hat{n}}}$ in the sense of Definition 3.11, and

$$J_K(\chi_{H_{\hat{n}}}; B(0, 1)) \leq J_K(v; B(0, 1))$$

for all measurable $v \colon \mathbb{R}^d \to [0, 1]$ such that $v(x) = \chi_{H_{\hat{n}}}(x)$ in $B(0, 1)^{\mathsf{c}}$.

Moreover, if $K > 0$, for any other minimiser u satisfying the same constraint, it holds $u(x) = \chi_{H_{\hat{n}}}(x)$.

Proof. It is readily shown that ζ is a calibration for $\chi_{H_{\hat{n}}}$, so that, in the light of Theorem 3.13, $\chi_{H_{\hat{n}}}$ is a minimiser of the nonlocal perimeter subject to its own boundary conditions.

To prove uniqueness, we suppose that $u \colon \mathbb{R}^d \to [0, 1]$ is another minimiser. Again by Theorem 3.13, we infer that $\zeta_{\hat{n}}$ is a calibration for u as well. Explicitly,

$$\mathrm{sign}((y - x) \cdot \hat{n})(u(y) - u(x)) = |u(y) - u(x)| \qquad \text{a.e. } (x, y) \in \mathbb{R}^d \times \mathbb{R}^d,$$

whence

$$u(x) \le u(y) \qquad \text{for a.e. } x, y \in \mathbb{R}^d \text{ such that } x \cdot \hat{n} < y \cdot \hat{n}. \tag{3.17}$$

We now focus on the superlevel sets of u: for $t \in (0, 1)$, we define

$$E_t := \{ u > t \},$$

and we observe that if $(x, y) \in E_t \times E_t^{\mathsf{c}}$, it must be $x \cdot \hat{n} \ge y \cdot \hat{n}$, otherwise, by (3.17) we would have $u(x) \le u(y)$. Therefore, there exists $\lambda_t \in \mathbb{R}$ such that $E_t \subset \{ x : x \cdot \hat{n} \ge \lambda_t \}$ and $E_t^{\mathsf{c}} \subset \{ y : y \cdot \hat{n} \le \lambda_t \}$, whence $\mathscr{L}^d(E_t \triangle \{ x : x \cdot \hat{n} \ge \lambda_t \}) = 0$ for all $t \in (0, 1)$. Recalling that it holds $u = \chi_H$ in $B(0, 1)^{\mathsf{c}}$, we infer that $\lambda_t = 0$ and this gets

$$\mathscr{L}^d(E_t \triangle H_{\hat{n}}) = 0 \qquad \text{for all } t \in (0, 1).$$

Summarising, we found that $u \colon \mathbb{R}^d \to [0, 1]$ is a function such that, for all $t \in (0, 1)$, the superlevel set E_t coincides with the halfspace $H_{\hat{n}}$, up to a negligible set. To conclude, we let $\{ t_\ell \}_{\ell \in \mathbb{N}} \subset (0, 1)$ be a sequence that converges to 0 when $\ell \to +\infty$. Because it holds

$$\{ x : u(x) = 0 \} = \bigcap_{\ell \in \mathbb{N}} E_{t_\ell}^{\mathsf{c}} \quad \text{and} \quad \{ x : u(x) = 1 \} = \bigcap_{\ell \in \mathbb{N}} E_{1-t_\ell},$$

we see that $\mathscr{L}^d(\{ x : u(x) = 0 \} \triangle H_{\hat{n}}^{\mathsf{c}}) = 0$ and $\mathscr{L}^d(\{ x : u(x) = 1 \} \triangle H_{\hat{n}}) = 0$. Thus, $u = \chi_{H_{\hat{n}}}$ in $\mathbb{R}^d$ $\hfill\square$

REMARK 3.15
Observe that $\zeta_{\hat{n}}$ in (3.16) is a calibration for all $u \colon \mathbb{R}^d \to \mathbb{R}$ such that $u(x) = f(x \cdot \hat{n})$ for some increasing $f \colon \mathbb{R} \to \mathbb{R}$. Hence, functions of this form are local minimisers.

We shall make use of Theorem 3.14 in the next chapter, see Section 4.4.

3.4 Nonlocal curvatures

Let $E \subset \mathbb{R}^d$ be a set with smooth boundary and let $E_t := \Phi(t, E)$, where $\Phi \colon \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is a smooth function such that $\Phi(0, x) = x$ and $\Phi(t, \cdot)$ is a diffeomorphism for all $t \in \mathbb{R}$. Then, it can be proved [CMP15] that

$$\frac{\mathrm{d}\operatorname{Per}_K(E_t)}{\mathrm{d}t}\Big|_{t=0} = \int_{\partial E} H_K(E, x) \zeta(x) \cdot \hat{n}(x) \mathrm{d}\mathscr{H}^{d-1}(x),$$

where $\zeta(x) := \partial_t \Phi(0, x)$, $\hat{n}(x)$ is the outer unit normal to ∂E at x, and

$$H_K(E, x) := -\lim_{r \to 0^+} \int_{B(x,r)^{\mathsf{c}}} K(y - x) \left(\chi_E(y) - \chi_{E^{\mathsf{c}}}(y) \right) \mathrm{d}y. \tag{3.18}$$

Thus, we see that H_K is the geometric first variation of the nonlocal perimeter, and, consistently with the classical case, we dub it *nonlocal curvature*.

Once again, in the fractional case this kind of nonlocal functional was introduced in the seminal work [CRS10]. For a discussion of the properties of fractional curvatures, we refer to the paper [AV14] by N. Abatangelo and E. Valdinoci. A wider study of general, translation-invariant curvature functionals was carried out in [CMP15] (see Remark 3.19 and Chapter 5 for more details).

Figure 3.3: $Q_\lambda(e)$ is the region comprised between an upward and a downward quadric.
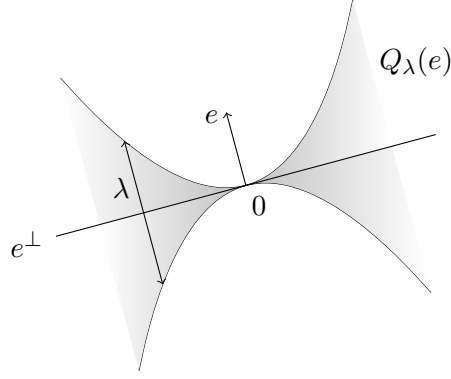
When $K \in L^1(\mathbb{R}^d)$, the nonlocal curvature can be written as a convolution:

$$H_K(E, x) = -(K * \tilde{\chi}_E)(x) := - \int_{\mathbb{R}^d} K(y - x)\tilde{\chi}_E(y)\mathrm{d}y.$$

Hereafter, if E is a set,

$$\tilde{\chi}_E(x) := \begin{cases} 1 & \text{if } x \in E, \\ -1 & \text{if } x \in E^{\mathsf{c}}. \end{cases}$$

In general, H_K has to be defined by means of the principal value because we include singular kernels in our analysis. Precisely, we make the following assumptions: $K \colon \mathbb{R}^d \to [0, +\infty)$ is a measurable, even function such that

$$K \in L^1(B(0, r)^{\mathsf{c}}) \qquad \text{for all } r > 0 \tag{3.19}$$

and

$$K \in L^1(Q_\lambda(e)) \qquad \text{for all } e \in \mathbb{S}^{d-1} \text{ and } \lambda > 0, \tag{3.20}$$

where, for any $e \in \mathbb{S}^{d-1}$ and $\lambda > 0$, $e^\perp := \left\{ y \in \mathbb{R}^d : y \cdot e = 0 \right\}$, $\pi_{e^\perp} := \mathrm{id} - e \otimes e$ is the orthogonal projection on $e^\perp$, and

$$Q_\lambda(e) := \left\{ y \in \mathbb{R}^d : |y \cdot e| \leq \frac{\lambda}{2} |\pi_{e^\perp}(y)|^2 \right\}$$

(see Figure 3.3). The two requirements grant that sets with $C^{1,1}$ boundary have finite nonlocal curvature, see Proposition 3.17 below.

We collect some basic properties of the nonlocal curvature.

LEMMA 3.16
Let $E \subset \mathbb{R}^d$ be an open set such that $H_K(E, x)$ is finite for some $x \in \partial E$.

(i)   *For any $z \in \mathbb{R}^d$ and any orthogonal matrix R, if $T(y) := Ry + z$, then*

$$H_K(E, x) = H_{\tilde{K}}(T(E), T(x)), \tag{3.21}$$

where $\tilde{K} := K \circ R^{\mathsf{t}}$, R^{t} being the transpose of R. In particular, H_K is invariant under translation.

(ii)   If $F \subset E$ and $x \in \partial E \cap \partial F$, then $H_K(E, x) \le H_K(F, x)$.

(iii) If $\{\, E_\ell \,\}$ is a sequence of sets that converges to E in C^2 and the points $x_\ell \in \partial E_\ell$ satisfy $x_\ell \to x$, then $H_K(E_\ell, x_\ell)$ converges to $H_K(E, x)$.

(iv)   If E is convex, then $H_K(E, x) \ge 0$.

Proof. The first two statements follow readily from the definition of H_K, while the third one can be established similarly to Proposition 3.17 below. As for (iv), let $T(y) := 2x - y$. Notice that, for any $r > 0$,

$$\int_{B(0,r)^c} K(y - x)\chi_E(y)\mathrm{d}y = \int_{B(0,r)^c} K(y - x)\chi_{T(E)}(y)\mathrm{d}y$$

and that, by convexity, $T(E) \subset E^c$. Then, the right-hand side of 3.18 is positive. $\square$

Finally, we show that the nonlocal curvature is finite on sets with $C^{1,1}$ boundaries [Imb09, CMP15]. We include the proof for the sake of completeness, and also to recover (3.22), which we shall utilise in Chapter 5. We use the following notation: for $e \in \mathbb{S}^{d-1}$, $x \in \mathbb{R}^d$ and $\delta > 0$, we denote the cylinder of centre x and axis e as

$$C_e(x, \delta) := \Big\{\, y \in \mathbb{R}^d : y = x + z + te,\ \text{with } z \in e^\perp \cap B(0, \delta), t \in (-\delta, \delta) \,\Big\}.$$

PROPOSITION 3.17
Let $E \subset \mathbb{R}^d$ be an open set such that ∂E is a $C^{1,1}$-hypersurface. Then, for all $x \in \partial E$, there exist $\bar\delta, \lambda > 0$ such that

$$|H_K(E, x)| \le \int_{Q_{\lambda, \bar\delta}(\hat n)} K(y)\mathrm{d}y + \int_{B(0,\bar\delta)^c} K(y)\mathrm{d}y, \tag{3.22}$$

where $\hat n$ is the outer unit normal in x and $Q_{\lambda, \bar\delta}(\hat n) := \big\{\, y \in Q_\lambda(\hat n) : |\pi_{\hat n^\perp}(y)| < \bar\delta \,\big\}$. In particular, $H_K(E, x)$ is finite.

Proof. Since $\Sigma := \partial E$ is regular, there exist $\bar\delta := \bar\delta(x)$ and a function $f \colon \hat n^\perp \cap B(0, \bar\delta) \to (-\bar\delta, \bar\delta)$ of class $C^{1,1}$ such that

$$\Sigma \cap C_{\hat n}(x, \bar\delta) = \Big\{\, y = x + z - f(z)\hat n : z \in \hat n^\perp \cap B(0, \bar\delta) \,\Big\},$$

$$E \cap C_{\hat n}(x, \bar\delta) = \Big\{\, y = x + z - t\hat n : z \in \hat n^\perp \cap B(0, \bar\delta), t \in (f(z), \bar\delta) \,\Big\},$$

$$|f(z)| \le \frac{\lambda}{2}|z|^2 \quad \text{for some } \lambda > 0. \tag{3.23}$$

We may always suppose that $r < \bar\delta$ and split the integral in (3.18) into the sum

$$\int_C K(y - x)\tilde\chi_E(y)\chi_{B(x,r)^c}(y)\mathrm{d}y + \int_{C^c} K(y - x)\tilde\chi_E(y)\mathrm{d}y,$$

where $C := C_{\hat n}(x, \bar\delta)$ for short. Thanks to (3.19), the second term is finite: indeed, since $B(x, \bar\delta) \subset C$, we have that

$$\left| \int_{C^c} K(y - x)\tilde\chi_E(y)\mathrm{d}y \right| \le \int_{B(0,\bar\delta)^c} K(y)\mathrm{d}y. \tag{3.24}$$

So, it remains to show that the integral

$$I_r := \int_C K(y - x)\tilde{\chi}_E(y)\chi_{B(x,r)^c}(y)\mathrm{d}y$$

is bounded by a constant that does not depend on r. In view of (5.20), and since K belongs to $L^1(B(0,r)^c)$ for any $r > 0$, we can write

$$I_r = \int_{\hat{n}^\perp \cap B(0,\bar{\delta})} \left[\int_{f(z)}^{\bar{\delta}} K(z - t\hat{n})b_r(z,t)\mathrm{d}t - \int_{-\bar{\delta}}^{f(z)} K(z - t\hat{n})b_r(z,t)\mathrm{d}t \right] \mathrm{d}\mathscr{H}^{d-1}(z),$$

where, for $(z,t) \in [\hat{n}^\perp \cap B(0,\bar{\delta})] \times (-\bar{\delta},\bar{\delta})$,

$$b_r(z,t) := \begin{cases} 0 & \text{if } |z| < r \text{ and } |t| < \sqrt{r^2 - |z|^2} \\ 1 & \text{otherwise.} \end{cases} \tag{3.25}$$

Recalling that K is even, we get

$$I_r = \int_{\hat{n}^\perp \cap B(0,\bar{\delta})} \left[\int_{f(z)}^{\bar{\delta}} K(z - t\hat{n})b_r(z,t)\mathrm{d}t - \int_{-f(-z)}^{\bar{\delta}} K(z - t\hat{n})b_r(z,t)\mathrm{d}t \right] \mathrm{d}\mathscr{H}^{d-1}(z)$$

$$= -\int_{\hat{n}^\perp \cap B(0,\bar{\delta})} \int_{-f(-z)}^{f(z)} K(z - t\hat{n})b_r(z,t)\mathrm{d}t\mathrm{d}\mathscr{H}^{d-1}(z)$$

By (3.23), we infer

$$|I_r| \le \int_{\hat{n}^\perp \cap B(0,\bar{\delta})} \int_{-\frac{\lambda}{2}|z|^2}^{\frac{\lambda}{2}|z|^2} K(z - t\hat{n})b_r(z,t)\mathrm{d}t\mathrm{d}\mathscr{H}^{d-1}(z)$$

$$= \int_{Q_{\lambda,\bar{\delta}}(\hat{n})} K(y)\chi_{B(0,r)^c}(y)\mathrm{d}y.$$

Assumption (3.20) allows to take the limit in the last inequality, and we obtain (3.22). $\quad\square$

REMARK 3.18
The second integral in (3.22) takes into account the "tails" of K, while the first one is related to the mean curvature of Σ. Indeed, λ in (3.23) measures how much the hypersurface bends in a neighbourhood of x.

REMARK 3.19 (Generalised curvatures)
According to [CMP15], a function H defined on the couples (E,x), where $E \subset \mathbb{R}^d$ is an open set with compact C^2 boundary and $x \in \partial E$, is a *generalised curvature* if

(i)   it is invariant under translation;

(ii)   it is monotone, that is, if $E \subset F$ and $x \in \partial E \cap \partial F$, then $H(E,x) \ge H(F,x)$;

(iii)   it is continuous, that is, if $E_\ell \to E$ in C^2 and $x_\ell \to x$, with $x_\ell \in \partial E_\ell$ and $x \in \partial E$, then $H(E_\ell, x_\ell)$ converges to $H(E,x)$.

Well-posedness for the level set flows associated with motions by generalised curvature (be it local or nonlocal) was established in the same paper. Lemma 3.16 and Proposition 3.17 ensure that H_K lies in framework of [CMP15]. We shall gain profit of this in Chapter 5.

Chapter 4

Γ-convergence of rescaled nonlocal total variations

In Chapter 3 we discussed some general properties of the nonlocal functionals Per_K and J_K, and we drew some comparisons between them and, respectively, De Giorgi's perimeter and the total variation. A further link, of asymptotic nature, is the subject of the present chapter, whose content firstly appeared in [BP19].

Let us describe the problem we deal with. When $u \colon \mathbb{R}^d \to [0, 1]$ is a measurable function, recalling (3.2), we define

$$J_\varepsilon^1(u; \Omega) := \frac{1}{2} \int_\Omega \int_\Omega K_\varepsilon(y - x) \, |u(y) - u(x)| \, \mathrm{d}y \mathrm{d}x,$$

$$J_\varepsilon^2(u; \Omega) := \int_\Omega \int_{\Omega^\mathrm{c}} K_\varepsilon(y - x) \, |u(y) - u(x)| \, \mathrm{d}y \mathrm{d}x,$$

$$J_\varepsilon(u; \Omega) := J_{K_\varepsilon}(u; \Omega) = J_\varepsilon^1(u; \Omega) + J_\varepsilon^2(u; \Omega),$$

where, for $\varepsilon > 0$, we let

$$K_\varepsilon(x) := \frac{1}{\varepsilon^d} K\left(\frac{x}{\varepsilon}\right).$$

We are interested in the limit of the family of functionals $\{ J_\varepsilon(\,\cdot\,; \Omega) \}$ with respect to the $L^1_{\mathrm{loc}}(\mathbb{R}^d)$-convergence as $\varepsilon \to 0^+$. This is somewhat reminiscent of the Bourgain-Brezis-Mironescu formula that we mentioned in Chapter 2. We shall be more precise on this connection in a while, when we comment the statement of the main result of the chapter, Theorem 4.1 below.

Note that the rescaled kernel K_ε has the same mass as K, i.e. formally $\|K_\varepsilon\|_{L^1(\mathbb{R}^d)} = \|K\|_{L^1(\mathbb{R}^d)}$, but, as ε approaches 0, it gets more and more concentrated around the origin. So, qualitatively, we guess that the limiting procedure has a sort of localisation effect, and, in particular, we expect to recover a total-variation-type functional. We shall see in Section 4.1 that Γ-convergence is the tool to describe the phenomenon rigorously. For the definition of this kind of convergence and its fundamental properties, we refer to Section 1.3.

Besides Γ-convergence, we also present a compactness property of families that have equibounded energies. We establish it in Section 4.1, while in the following one we outline our approach to the Γ-convergence result. Detailed proofs are expounded in the remaining sections.

4.1 Main result and proof of compactness

The current chapter is devoted to the proof of the following:

THEOREM 4.1 (Compactness and Γ-convergence of the rescaled energy)
Let $\Omega \subset \mathbb{R}^d$ be an open bounded set with Lipschitz boundary. Let also $K \colon \mathbb{R}^d \to (0, +\infty)$ be an even function such that

$$c_K := \frac{1}{2} \int_{\mathbb{R}^d} K(x) \, |x| \, dx < +\infty, \tag{4.1}$$

and define for any measurable $u \colon \mathbb{R}^d \to [0, 1]$

$$J_0(u; \Omega) := \begin{cases} \dfrac{1}{2} \displaystyle\int_{\mathbb{R}^d} K(z) \left(\int_\Omega |z \cdot \mathrm{D}u| \right) \mathrm{d}z & \text{if } u \in \mathrm{BV}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Then, the following hold:

(i) *For all $\ell \in \mathbb{N}$, let $\varepsilon_\ell > 0$ and let $u_\ell \colon \Omega \to [0, 1]$ be a measurable function. If $\varepsilon_\ell \to 0^+$ and*

$$\frac{1}{\varepsilon_\ell} J^1_{\varepsilon_\ell}(u_\ell; \Omega) \quad \text{is uniformly bounded,}$$

then $\{\, u_\ell \,\}$ is precompact in $L^1(\Omega)$ and its cluster points belong to $\mathrm{BV}(\Omega; [0, 1])$.

(ii) *As $\varepsilon \to 0^+$, the family $\{\, \varepsilon^{-1} J_\varepsilon(\,\cdot\,; \Omega)\,\}$ Γ-converges in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ to $J_0(\,\cdot\,; \Omega)$.*

Assertion (i) provides a compactness criterion for families that are equibounded in energy and it grants some extra regularity for the limits point of such families. The second aspect will be useful for the proof of the lower limit inequality. The criterion has been already established in [AB98b] by G. Alberti and G. Bellettini in a slightly different framework. For the reader's convenience, in this Section we shall include its proof.

Our theorem is very close to a result by Ponce:

THEOREM 4.2 (Corollary 2 and Theorem 8 in [Pon04])
Let Ω an opens set with compact, Lipschitz boundary and let K and K_ε be as above. If $u \in \mathrm{BV}(\Omega)$, then

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_\Omega \int_\Omega K_\varepsilon(y - x) \, |u(y) - u(x)| \, \mathrm{d}y \mathrm{d}x = J_0(u; \Omega) \tag{4.2}$$

In addition, if Ω is also bounded, the right-hand side is the Γ-limit as $\varepsilon \to 0^+$ of the left-hand one w.r.t. the $L^1(\Omega)$-distance.

We remark that, in [Pon04], contributions to the energy functional of the form of J^2_ε are not considered, so, to prove our result, some explanation concerning the Γ-upper limit would be still required, even if one took the results by Ponce for granted. Besides, there, the approach to the Γ-lower limit inequality relies on representation formulas for the relaxations of a certain class of integral functionals. In fact, following [BP19], we propose an independent argument, which combines the pointwise limit (4.2) and Theorem 3.14 about the minimality of halfspaces.

We notice that in (4.1) we require a condition that is stronger than (3.3). Heuristically, the faster-than-L^1 decay at infinity entails that the "local" contribution to the energy,

i.e. J_ε^1, prevails in the large scale limit; for this reason, the same limit of Theorem 4.2 is recovered. We shall motivate the choice of the scaling factor below in this Section, see Remark 4.5.

We conclude this section by proving the first statement of Theorem 4.1, that is, the compactness criterion. We recall that an analogous result appeared in [AB98b]. We premise a couple of lemmas.

LEMMA 4.3
Let $G \in L^1(\mathbb{R}^d)$ be a positive function. Then, for any $u \in L^\infty(\mathbb{R}^d)$ it holds

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} (G * G)(z) \, |u(x+z) - u(x)| \, \mathrm{d}z\mathrm{d}x \leq 4 \, \|G\|_{L^1(\mathbb{R}^d)} \, J_G(u; \mathbb{R}^d).$$

Proof. The proof is elementary:

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} (G * G)(z) \, |u(x+z) - u(x)| \, \mathrm{d}z\mathrm{d}x$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(y)G(z-y) \, |u(x+z) - u(x)| \, \mathrm{d}y\mathrm{d}z\mathrm{d}x$$

$$\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(y)G(z-y) \, |u(x+z) - u(x+y)| \, \mathrm{d}y\mathrm{d}z\mathrm{d}x$$

$$+ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(y)G(z-y) \, |u(x+y) - u(x)| \, \mathrm{d}y\mathrm{d}z\mathrm{d}x$$

$$= 2 \, \|G\|_{L^1(\mathbb{R}^d)} \int_{\mathbb{R}^d \times \mathbb{R}^d} G(y) \, |u(x+y) - u(x)| \, \mathrm{d}y\mathrm{d}x.$$

$\square$

The second lemma is more closely related to the nonlocal nature of the problem we treat. It shows that the rescaled nonlocal energy is bounded, or even vanishing for $\varepsilon \to 0^+$, provided that the sets appearing in the functional are separated by a regular "frame", or that the function under consideration is sufficiently regular.

LEMMA 4.4
Let E_1 and E_2 be measurable sets in $\mathbb{R}^d$ and let $u\colon \mathbb{R}^d \to [0,1]$ be measurable.

(i) If either there exists a set F with finite perimeter in $\mathbb{R}^d$ such that, up to negligible sets, $E_1 \subset F$ and $E_2 \subset F^c$, or if $u \in \mathrm{BV}(\mathbb{R}^d)$, then

$$\frac{1}{\varepsilon} \int_{E_1} \int_{E_2} K_\varepsilon(y-x) \, |u(y) - u(x)| \, \mathrm{d}y\mathrm{d}x \leq c,$$

where c is a constant such that either $c = c(K, \mathrm{Per}(F))$ or $c = c(K, \|\mathrm{D}u\|)$.

(ii) If $\mathrm{dist}(E_1, E_2) > 0$, then

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_{E_1} \int_{E_2} K_\varepsilon(y-x) \, |u(y) - u(x)| \, \mathrm{d}y\mathrm{d}x = 0$$

Proof. The argument is the same as Lemma 3.2-(ii).

For what concerns (i), suppose firstly that $E_1 \subset F$ and $E_2 \subset F^c$. We bound the double integral over E_1 and E_2 by the nonlocal K_ε-perimeter of F in $\mathbb{R}^d$, which in turn is bounded by the classical perimeter of F:

$$\int_{E_1} \int_{E_2} K_\varepsilon(y-x) |u(y)-u(x)| \, \mathrm{d}y\mathrm{d}x$$
$$\leq 2 \int_F \int_{F^c} K_\varepsilon(y-x) \mathrm{d}y\mathrm{d}x$$
$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(z) |\chi_F(x+\varepsilon z) - \chi_F(x)| \, \mathrm{d}z\mathrm{d}x$$
$$\leq 2\varepsilon c_K \operatorname{Per}(F).$$

On the other hand, if $u \in \mathrm{BV}(\mathbb{R}^d)$, a similar reasoning shows

$$\int_{E_1} \int_{E_2} K_\varepsilon(y-x) |u(y)-u(x)| \, \mathrm{d}y\mathrm{d}x \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K_\varepsilon(y-x) |u(y)-u(x)| \, \mathrm{d}y\mathrm{d}x$$
$$\leq 2\varepsilon c_K \|\mathrm{D}u\|.$$

Turning to (ii), we have

$$\frac{1}{\varepsilon} \int_{E_1} \int_{E_2} K_\varepsilon(y-x) |u(y)-u(x)| \, \mathrm{d}y\mathrm{d}x \leq \frac{2}{\varepsilon r} \int_{E_1} \int_{E_2} K_\varepsilon(y-x) |y-x| \, \mathrm{d}y\mathrm{d}x$$
$$= \frac{2}{r} \int_{E_1} \int_{\mathbb{R}^d} K(z) |z| \, \chi_{E_2}(x+\varepsilon z) \mathrm{d}z\mathrm{d}x,$$

where $r := \operatorname{dist}(E_1, E_2) > 0$. Statement (ii) follows by Lebesgue's Theorem. $\qquad\square$

REMARK 4.5 (Choice of the scaling factor)
The previous Lemma shows that $J_\varepsilon(u;\Omega) = O(\varepsilon)$ when u is a function of bounded variation in $\mathbb{R}^d$. Accordingly, we multiply J_ε by ε^{-1} to recover a nontrivial limit in our Γ-convergence theorem.

Now we are ready to prove the compactness criterion.

Proof of statement (i) in Theorem 4.1. For notational convenience, we write respectively ε and u_ε in place of ε_ℓ and of u_ℓ.

We are going to exhibit a sequence $\{ v_\varepsilon \}$ that is asymptotically equivalent in $L^1(\mathbb{R}^d)$ to $\{ u_\varepsilon \}$, and that is precompact as well. To this aim, we extend each u_ε outside Ω setting $u_\varepsilon = 0$ (in wider generality, any constant extension would fit). Let $\rho \in C_c^\infty(\mathbb{R}^d)$ be positive and let

$$v_\varepsilon(x) := (\rho_\varepsilon * u_\varepsilon)(x),$$

where

$$\rho_\varepsilon(x) := \frac{1}{c\varepsilon^d} \rho\left(\frac{x}{\varepsilon}\right), \quad \text{with } c := \int_{\mathbb{R}^d} \rho(x)\mathrm{d}x.$$

Notice that any v_ε is supported in some ball B containing Ω. By properties of convolutions,

$$\int_{\mathbb{R}^d} |v_\varepsilon(x) - u_\varepsilon| \, (x)\mathrm{d}x \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\rho_\varepsilon(z)| \, |u_\varepsilon(x+z) - u_\varepsilon(x)| \, \mathrm{d}z\mathrm{d}x \qquad (4.3)$$

and

$$\int_B |\nabla v_\varepsilon(x)| \, \mathrm{d}x = \int_{\mathbb{R}^d} |\nabla v_\varepsilon(x)| \, \mathrm{d}x$$
$$\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\nabla \rho_\varepsilon(z)| \, |u_\varepsilon(x+z) - u_\varepsilon(x)| \, \mathrm{d}z\mathrm{d}x. \tag{4.4}$$

We claim that, if we choose the mollifier ρ suitably, then the asymptotic equivalence between $\{\, u_\varepsilon \,\}$ and $\{\, v_\varepsilon \,\}$ and a uniform bound on the BV-norm of $\{\, v_\varepsilon \,\}$ follow respectively from (4.3) and (4.4). In turn, the claim entails the conclusion: by virtue of Theorem 1.6, up to extraction of subsequences, $\{\, v_\varepsilon \,\}$ converges to some $u \in \mathrm{BV}(B)$ in $L^1(B)$, and hence $\{\, u_\varepsilon \,\}$ tends in $L^1(B)$ to the same function.

Let us now validate the claim. We define the function $T(s) := 0 \vee (s \wedge 1)$ and the truncated kernel $G := T \circ K \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. We observe that the convolution $G * G$ is positive and continuous, therefore we can pick $\rho \in C_c^\infty(\mathbb{R}^d)$ such that

$$0 \leq \rho \leq G * G \quad \text{and} \quad |\nabla \rho| \leq G * G. \tag{4.5}$$

Setting

$$G_\varepsilon(x) := \frac{1}{\varepsilon^d} G\left(\frac{x}{\varepsilon}\right),$$

we infer from (4.3) and (4.4)

$$\int_{\mathbb{R}^d} |v_\varepsilon(x) - u_\varepsilon(x)| \, \mathrm{d}x \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |G_\varepsilon * G_\varepsilon(z)| \, |u_\varepsilon(x+z) - u_\varepsilon(x)| \, \mathrm{d}z\mathrm{d}x$$

and

$$\int_{\mathbb{R}^d} |\nabla v_\varepsilon(x)| \, \mathrm{d}x \leq \frac{1}{\varepsilon} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |G_\varepsilon * G_\varepsilon(z)| \, |u_\varepsilon(x+z) - u_\varepsilon(x)| \, \mathrm{d}z\mathrm{d}x.$$

Both the right-hand sides of the inequalities can be bounded above by Lemma 4.3. Indeed, we have

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |G_\varepsilon * G_\varepsilon(z)| \, |u_\varepsilon(x+z) - u_\varepsilon(x)| \, \mathrm{d}z\mathrm{d}x$$
$$\leq 4 \, \|G\|_{L^1(\mathbb{R}^d)} \, J_{G_\varepsilon}(u_\varepsilon; \mathbb{R}^d)$$
$$\leq 4 \, \|G\|_{L^1(\mathbb{R}^d)} \, J_\varepsilon(u_\varepsilon; \mathbb{R}^d)$$
$$= 4 \, \|G\|_{L^1(\mathbb{R}^d)} \left(J_\varepsilon^1(u_\varepsilon; \Omega) + J_\varepsilon^2(u_\varepsilon; \Omega) + J_\varepsilon^1(u_\varepsilon; \Omega^c) \right)$$

Note that $J_\varepsilon^1(u_\varepsilon; \Omega^c) = 0$, because we set $u_\varepsilon = 0$ outside of Ω. Moreover, by assumption, there exists $M \geq 0$ such that

$$J_\varepsilon^1(u_\varepsilon; \Omega) \leq \varepsilon M.$$

In third place, we can invoke Lemma 4.4 on $J_\varepsilon^2(u_\varepsilon; \Omega)$, because the boundary of Ω is Lipschitz. On the whole, we deduce

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |G_\varepsilon * G_\varepsilon(z)| \, |\chi_{E_\varepsilon}(x+z) - \chi_{E_\varepsilon}(x)| \, \mathrm{d}z\mathrm{d}x = O(\varepsilon),$$

as desired. $\qquad\square$

REMARK 4.6 (Locality defect)
In the previous proof, we utilised the identity

$$J_\varepsilon(u; \mathbb{R}^d) = J_\varepsilon^1(u; \Omega) + J_\varepsilon^2(u; \Omega) + J_\varepsilon^1(u; \Omega^c).$$

More broadly, let F be an arbitrary measurable set. If we split a given reference domain Ω in the disjoint regions $\Omega \cap F$ and $\Omega \cap F^c$, we have

$$J_K^1(u; \Omega) = J_K^1(u; \Omega \cap F) + J_K^1(u; \Omega \cap F^c)$$
$$+ \int_{\Omega \cap F} \int_{\Omega \cap F^c} K(y - x) \, |u(y) - u(x)| \, \mathrm{d}x \mathrm{d}y.$$

We remark that the sum of the energies stored in each of the sets of the partition is smaller than the energy of Ω. This can be seen as a feature of nonlocality. The difference is given by the mutual interaction between $\Omega \cap F$ and $\Omega \cap F^c$, which, borrowing the terminology suggested in [AB98b], we may dub *locality defect*.

4.2 Overview of the proving strategy

Before discussing in depth the proof of the Γ-convergence result in Theorem 4.1, we summarise in this Section the strategy that we adopt.

The key point of our approach is the possibility of reasoning in terms of sets rather than functions. Indeed, thanks to an abstract result in [CGL10] by A. Chambolle, A. Giacomini, and L. Lussardi, the Γ-convergence of $\{ \varepsilon^{-1} J_\varepsilon \}$ regarded as a family of functionals on measurable sets to the restriction of J_0 is sufficient to yield the more general result of Theorem 4.1. We include the precise statement in the next lines; we recall that we say that the collection of measurable sets $\{ E_\varepsilon \}$ $L^1_{\mathrm{loc}}(\mathbb{R}^d)$-converges to E if $\{ \chi_{E_\varepsilon} \}$ converges to χ_E w.r.t. the $L^1_{\mathrm{loc}}(\mathbb{R}^d)$-convergence.

PROPOSITION 4.7 (Proposition 3.5 in [CGL10])
Let $U \subset \mathbb{R}^d$ be open and bounded. Suppose that $\{ J_\ell \}$ is a sequence of convex functionals on $L^1(U)$ such that the generalised Coarea Formula (3.9) holds for all $\ell \in \mathbb{N}$ and let us set for all measurable $E \subset U$

$$\tilde{J}_\ell(E) := J_\ell(\chi_E)$$

Suppose that there exists a functional $\tilde{J}$ defined on measurable sets of U such that the sequence $\{ \tilde{J}_\ell \}$ Γ-converges to $\tilde{J}$ w.r.t. the L^1-convergence, and put

$$J(u) := \int_{-\infty}^{+\infty} \tilde{J}(\chi_{\{ u > t \}}) \mathrm{d}t.$$

Then, the sequence $\{ J_\ell \}$ Γ-converges to J w.r.t. L^1-convergence.

Recalling Propositions 1.8, 3.5, and 3.6, we see that the Γ-convergence result in Theorem 4.1 holds as soon as we prove the following:

THEOREM 4.8 (Γ-convergence of rescaled nonlocal perimeters)
Let Ω and K be as in Theorem 4.1. For $E \subset \mathbb{R}^d$ measurable, consider

$$J_\varepsilon^i(E; \Omega) := J_\varepsilon^i(\chi_E; \Omega) \quad \text{for } i = 1, 2, \quad J_\varepsilon(E; \Omega) := J_\varepsilon(\chi_E; \Omega),$$

and

$$J_0(E; \Omega) := J_0(\chi_E; \Omega).$$

Then, for any given measurable E, we have that

(i)   *for any family $\{ E_\varepsilon \}$ that converges to E in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ as $\varepsilon \to 0^+$, it holds*

$$J_0(E; \Omega) \leq \liminf_{\varepsilon \to 0^+} \frac{1}{\varepsilon} J^1_\varepsilon(E_\varepsilon; \Omega). \tag{4.6}$$

(ii)   *there exists a family $\{ E_\varepsilon \}$ that converges to E in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ as $\varepsilon \to 0^+$ and that satisfies*

$$\limsup_{\varepsilon \to 0^+} \frac{1}{\varepsilon} J_\varepsilon(E_\varepsilon; \Omega) \leq J_0(E; \Omega).$$

The two assertions in Theorem 4.8 clearly entail the Γ-convergence of $\{ \varepsilon^{-1} J_\varepsilon(\cdot ; \Omega) \}$ to $J_0(\cdot ; \Omega)$ w.r.t. the $L^1_{\mathrm{loc}}(\mathbb{R}^d)$-convergence, because $J^2_\varepsilon(\cdot ; \Omega)$ is positive.

Our result is akin to several others already appeared in the literature. In addition to [BBM01, Dáv02, Pon04], we refer to the aforementioned [AB98b], where the asymptotics of a nonlocal model for phase transitions is studied. The authors consider the rescaled energy

$$\mathscr{F}_\varepsilon(u; \Omega) := \frac{1}{4} \int_\Omega \int_\Omega K_\varepsilon(y - x) \big(u(y) - u(x) \big)^2 \mathrm{d}y \mathrm{d}x + \int_\Omega W(u(x)) \mathrm{d}x,$$

where $W : \mathbb{R} \to [0, +\infty)$ is a double-well potential. Under the same summability assumptions on K as ours, they establish the Γ-convergence of the family $\{ \varepsilon^{-1} \mathscr{F}_\varepsilon(\cdot ; \Omega) \}$ to a limit anisotropic surface energy. Interactions between Ω and its complement are not relevant in this framework. The reader interested in this sort of energies may consult also [BBP01].

Another result that is linked to Theorem 4.8 was proved by L. Ambrosio, G. De Philippis, and L. Martinazzi in [ADPM11]. There, they showed that fractional perimeters Γ-converge, as the fractional parameter approaches 1, to De Giorgi's perimeter. The scaling is substantially different from ours, but some of the techniques exploited in that work still fit in our setting. For the asymptotics of fractional energies, we refer to the survey [Val13].

Let us now outline how we deal with the proof of Theorem 4.8. We start with a qualitative picture.

The contributions J^1_ε and J^2_ε that compound the rescaled nonlocal perimeter functional have different asymptotic behaviours: when ε is small, the former is concentrated near the portions of the boundary of E inside Ω, the latter instead captures the parts that are close to $\partial \Omega$. Proposition 4.11 in Section 4.3 makes the heuristics precise, showing that the pointwise limit and the Γ-limit do not agree. However, once the pointwise limit has been computed, it is possible to recover the Γ-upper limit by a density result, see Lemma 4.12.

As for the Γ-lower limit, it is convenient to regard J_0 as an anisotropic perimeter functional. We observe that we can write

$$J_0(E; \Omega) = \begin{cases} \displaystyle\int_{\partial^* E \cap \Omega} \sigma_K(\hat{n}(x)) \mathrm{d}\mathscr{H}^{d-1}(x) & \text{if } E \text{ is a finite perimeter set in } \Omega, \\ +\infty & \text{otherwise,} \end{cases} \tag{4.7}$$

where $\hat{n}\colon \partial^* E \to \mathbb{S}^{d-1}$ is the measure-theoretic inner normal of E defined by (1.4) and $\sigma_K\colon \mathbb{R}^d \to [0, +\infty)$ is the anisotropy

$$\sigma_K(p) \coloneqq \frac{1}{2}\int_{\mathbb{R}^d} K(z)\,|z\cdot p|\,\mathrm{d}z, \qquad \text{for } p \in \mathbb{R}^d. \tag{4.8}$$

This function is evidently a norm on $\mathbb{R}^d$.

REMARK 4.9 (The radial case [BP19])
When K is radial, J_0 equals De Giorgi's perimeter times a constant that depends on the dimension of the space and on K. To see this, admit that $K(x) = \bar{K}(|x|)$ for some $\bar{K}\colon [0, +\infty) \to [0, +\infty)$. Then, for any $\hat{p} \in \mathbb{S}^{d-1}$, we find

$$\begin{aligned}
\sigma_K(\hat{p}) &= \frac{1}{2}\left(\int_0^{+\infty} \bar{K}(r)r^d\mathrm{d}r\right)\int_{\mathbb{S}^{d-1}} |e\cdot\hat{p}|\,\mathrm{d}\mathscr{H}^{d-1}(e) \\
&= c_K \fint_{\mathbb{S}^{d-1}} |e\cdot e_d|\,\mathrm{d}\mathscr{H}^{d-1}(e),
\end{aligned}$$

where $e_d \coloneqq (0, \ldots, 0, 1)$ is the last element of the canonical basis.

Theorem 4.8 was proved for radial and strictly decreasing kernels in [BP19]. The content of this chapter shows that the same arguments may be conveniently adapted to a more general setting.

Following [ADPM11, AB98b], we prove the lower limit inequality *via* the strategy introduced by I. Fonseca and S. Müller in [FM93], which amounts to turn the proof of (4.6) into an inequality of Radon-Nikodym derivatives. To accomplish the task, the compactness criterion that we proved in the previous section comes in handy.

REMARK 4.10 (Semicontinuity of the Γ-limit)
If Theorem 4.1 holds, then we obtain as a by-product that J_0 is $L^1_{\mathrm{loc}}(\mathbb{R}^d)$-lower semicontinuos. Indeed, Γ-limits are always lower semicontinuous [Bra02]. More straightforwardly, the semicontinuity may be seen as a consequence of the following equality, which holds for all $u \in \mathrm{BV}(\Omega)$ [Gra09]:

$$J_0(u;\Omega) = \sup\left\{ -\int_\Omega u\operatorname{div}\zeta : \zeta \in C^1_c(\Omega;\mathbb{R}^d), \sigma_K^\circ(\zeta) \le 1 \right\}.$$

Here, σ_K° is the dual norm of σ_K, that is

$$\sigma_K^\circ(q) \coloneqq \sup\left\{ q\cdot p : p \in \mathbb{R}^d \text{ and } \sigma_K(p) \le 1 \right\} \qquad \text{for all } q \in \mathbb{R}^d.$$

The previous formula may be seen as a more general version of (1.1).

Note further that, being σ_K a norm $\mathbb{R}^d$, (semi)continuity w.r.t. other notions of convergence may be derived for the Γ-limit as a consequence of Reshetnyak's Theorems 1.7.

4.3 Pontwise limit and Γ-upper limit

In the current section, we establish the Γ-upper limit inequality: we show that, for any given measurable $E \subset \mathbb{R}^d$, there exists a family $\{ E_\varepsilon \}$ that converges to E in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ as $\varepsilon \to 0^+$ and that has the property that

$$\limsup_{\varepsilon\to 0^+} \frac{1}{\varepsilon} J_\varepsilon(E_\varepsilon;\Omega) \le J_0(E;\Omega). \tag{4.9}$$

We obtain the inequality above in two steps. We firstly compute the pointwise limit of $\{ \varepsilon^{-1} J_\varepsilon \}$ by taking advantage of (4.2). We find that, as $\varepsilon \to 0^+$, $\{ J_\varepsilon \}$ approaches

the presumed Γ-limit J_0 plus a boundary contribution. Next, we prove that the latter is negligible in the Γ-limit by reasoning on a suitable class of sets that is dense in energy for J_0 (recall Lemma 1.10(ii)).

We now study the pointwise convergence of $\{\,\varepsilon^{-1}J_\varepsilon\,\}$. An application of the Bourgain-Brezis-Mironescu formula to the study of nonlocal perimeters already appeared in [MRT19], where J. Mazón, J. Rossi, and J. Toledo exploited the approximation established by J. Dávila in [Dáv02]. In [MRT19] the authors focus on compactly supported, L^1 kernels. Here, we include a slightly more general statement:

PROPOSITION 4.11 (Pointwise limit of the rescaled nonocal perimeter)
Let $\Omega \subset \mathbb{R}^d$ be an open set whose boundary is compact and Lipschitz. Let also (4.1) hold. Then, if E is a finite perimeter set in $\mathbb{R}^d$,

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} J_\varepsilon(E; \Omega) = J_0(E; \Omega) + \int_{\partial^* E \cap \partial\Omega} \sigma_K(\hat{n}(x)) \mathrm{d}\mathcal{H}^{d-1}(x). \qquad (4.10)$$

Proof. By virtue of Theorem 4.2,

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} J_\varepsilon^1(E; \Omega) = J_0(E; \Omega),$$

so, we only have to take care of J_ε^2. We start from the K_ε-perimeter of E in the whole space and we rewrite it as the sum of three contributions:

$$\int_E \int_{E^c} K_\varepsilon(y - x) \mathrm{d}x\mathrm{d}y = J_\varepsilon^1(E; \Omega) + J_\varepsilon^2(E; \Omega) + J_\varepsilon^1(E; \Omega^c).$$

Since the topological boundary of Ω is negligible, we may appeal again to Theorem 4.2 and take the limit $\varepsilon \to 0^+$. Let $\operatorname{int} \Omega^c$ be the interior of Ω^c. We get

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} J_\varepsilon^2(E; \Omega) = J_0(E) - J_0(E; \Omega) - J_0(E; \operatorname{int} \Omega^c)$$

$$= \int_{\partial^* E \cap \partial\Omega} \sigma_K(\hat{n}(x)) \mathrm{d}\mathcal{H}^{d-1}(x),$$

as desired.

$\square$

Let us now go back to (4.9). The inequality holds trivially if the right-hand side is not finite, therefore we may assume that E is a Caccioppoli set in Ω. Note as well that if E has finite perimeter in the whole space $\mathbb{R}^d$ and if

$$\mathcal{H}^{d-1}(\partial^* E \cap \partial\Omega) = 0, \qquad (4.11)$$

then, in view of Proposition 4.11, the choice $E_\varepsilon \coloneqq E$ for all $\varepsilon > 0$ defines a recovery family. When (4.11) hold, we say that E is *transversal* to Ω.

At this stage, by Lemma 1.10(ii), the proof of the upper limit inequality is accomplished once we show that the class of finite perimeter sets in $\mathbb{R}^d$ that are transversal to Ω is dense in energy; in particular, we consider approximations by polyhedra. We say that a set $P \subset \mathbb{R}^d$ is a d-dimensional *polyhedron* if it is an open set whose boundary is a Lipschitz hypersurface contained in the union of a finite number of affine hyperplanes.

LEMMA 4.12 (Density of polyhedra)
Let E be a finite perimeter set in Ω. Then, there exists a family $\{\, P_\varepsilon \,\}$ of polyhedra with the following properties:

(i) $\mathscr{H}^{d-1}(\partial P_\varepsilon \cap \partial \Omega) = 0$;

(ii) $P_\varepsilon \to E$ in $L^1(\Omega)$ and $\operatorname{Per}(P_\varepsilon; \Omega) \to \operatorname{Per}(E; \Omega)$;

(iii) $J_0(P_\varepsilon; \Omega) \to J_0(E; \Omega)$.

Proof. Assertion (ii) is a standard one, see for instance [Mag12]. Also, a family that satisfies (ii) may be refined so that (i) hold as well, see for instance [ADPM11, Proposition 15]. Lastly, (iii) is a consequence of Reshetnyak's Continuity Theorem and (ii). □

4.4   Characterisation of the anisotropy

This section is devoted to the proof of a characterisation of the anisotropic norm σ_K that appears in the definition of the limit functional J_0. A key tool in the proof is the local minimality of halfspaces. The characterisation proves to be useful to establish the Γ-inferior limit inequality in Theorem 4.8, see the next section.

Recall that, for all $p \in \mathbb{R}^d$,

$$\sigma_K(p) := \frac{1}{2} \int_{\mathbb{R}^d} K(z) \, |z \cdot p| \, \mathrm{d}z.$$

By homogeneity, without loss of generality, we can think of σ_K as function defined on $\mathbb{S}^{d-1}$. For $p \in \mathbb{R}^d \setminus \{0\}$, let us recall the notations

$$\hat{p} := \frac{p}{|p|} \quad \text{and} \quad H_{\hat{p}} := \left\{\, x \in \mathbb{R}^d : x \cdot \hat{p} > 0 \,\right\}.$$

Also, in this section we use the symbol B_r as a shorthand for $B(0, r)$, with $r > 0$, and we set $B := B_1 = B(0, 1)$.

Our goal is validating the following:

LEMMA 4.13
For any $\hat{p} \in \mathbb{S}^{d-1}$,

$$\sigma_K(\hat{p}) = \inf \left\{ \liminf_{\varepsilon \to 0^+} \frac{1}{\omega_{d-1}\varepsilon} J_\varepsilon^1(E_\varepsilon; B) : E_\varepsilon \to H_{\hat{p}} \text{ in } L^1(B) \right\}, \qquad (4.12)$$

where ω_{d-1} is the $(d-1)$-dimensional Lebesgue measure of the unit ball in $\mathbb{R}^{d-1}$.

For $\hat{p} \in \mathbb{S}^{d-1}$, we label the quantity in the right-hand side of (4.12):

$$\sigma'_K(\hat{p}) := \inf \left\{ \liminf_{\varepsilon \to 0^+} \frac{1}{\omega_{d-1}\varepsilon} J_\varepsilon^1(E_\varepsilon; B) : E_\varepsilon \to H_{\hat{p}} \text{ in } L^1(B) \right\}. \qquad (4.13)$$

We highlight that the previous equality corresponds to the definition of the surface tension given in the phase-field model in [AB98b].

Observe that, by virtue of Theorem 4.2, we can compute σ_K as a pointwise limit:

$$\sigma_K(\hat{p}) = \lim_{\varepsilon \to 0^+} \frac{1}{\omega_{d-1}\varepsilon} J_\varepsilon^1(H_{\hat{p}}; B). \tag{4.14}$$

It follows that $\sigma_K(\hat{p}) \geq \sigma_K'(\hat{p})$.

To the purpose of proving the reverse inequality, we introduce a third function σ_K'': for $\hat{p} \in \mathbb{S}^{d-1}$ and $\delta \in (0,1)$, we let

$$\sigma_K''(\hat{p}) := \inf\left\{ \liminf_{\varepsilon \to 0^+} \frac{1}{\omega_{d-1}\varepsilon} J_\varepsilon^1(E_\varepsilon; B) : E_\varepsilon \to H_{\hat{p}} \text{ in } L^1(B) \text{ and } E_\varepsilon \triangle H_{\hat{p}} \subset B_{1-\delta} \right\}.$$

Our notation does not explicitly show the dependence of σ_K'' on the parameter δ since *a posteriori* the values of σ_K'' are not influenced by it.

The thesis of Lemma 4.13 holds as soon as we prove that $\sigma_K(\hat{p}) \leq \sigma_K''(\hat{p}) \leq \sigma_K'(\hat{p})$ for all $\hat{p} \in \mathbb{S}^{d-1}$. We establish the former inequality by taking advantage of the minimality of halfspaces, i.e. Theorem 3.14, while for the latter we need the following technical result, which parallels a similar one proved in [ADPM11] for fractional perimeters:

LEMMA 4.14 (Gluing)
Suppose that $\delta_1, \delta_2 \in \mathbb{R}$ satisfy $\delta_2 > \delta_1 > 0$, and that E_1, E_2 are measurable sets such that $J_K^1(E_i; \Omega) < +\infty$ for both $i = 1, 2$. Let us define

$$\Omega_1 := \{ x \in \Omega : \operatorname{dist}(x, \Omega^{\mathsf{c}}) \leq \delta_1 \} \quad \text{and} \quad \Omega_2 := \{ x \in \Omega : \operatorname{dist}(x, \Omega^{\mathsf{c}}) > \delta_2 \}.$$

Then, there exists a measurable set F with the following properties:

(i) $F \cap \Omega_1 = E_1 \cap \Omega_1$ and $F \cap \Omega_2 = E_2 \cap \Omega_2$;

(ii) $\mathscr{L}^d\big((F \triangle E_2) \cap \Omega\big) \leq \mathscr{L}^d\big((E_1 \triangle E_2) \cap \Omega\big)$;

(iii) *for all $\eta > 0$*

$$\begin{aligned}
J_\varepsilon^1(F; \Omega) \leq\ & J_\varepsilon^1(E_1; \Omega_\eta) + J_\varepsilon^1(E_2; \Omega) \\
& + \frac{\varepsilon}{\delta_2 - \delta_1} \int_{\mathbb{R}^d} K(z)\,|z|\,\mathrm{d}z\,\mathscr{L}^d\big((E_1 \triangle E_2) \cap \Omega\big) \\
& + \frac{\varepsilon}{\eta} \int_{\Omega \cap \Omega_2^{\mathsf{c}}} \int_{\mathbb{R}^d} K(z)\,|z|\,\chi_{\Omega \cap \Omega_2^{\mathsf{c}}}(x + \varepsilon z)\mathrm{d}z\mathrm{d}x.
\end{aligned} \tag{4.15}$$

where $\Omega_\eta := \{ x \in \Omega : \operatorname{dist}(x, \Omega^{\mathsf{c}}) \leq \delta_2 + \eta \}$.

Proof. We consider a function $w \colon \Omega \to [0,1]$, which is, loosely speaking, a convex combination of the data $u := \chi_{E_1}$ and $v := \chi_{E_2}$. Precisely, we set $w := (1 - \varphi)u + \varphi v$, where $\varphi \in C_{\mathrm{c}}^\infty(\mathbb{R}^d)$ is such that

$$0 \leq \varphi \leq 1 \text{ in } \Omega, \quad \varphi = 0 \text{ in } \Omega_1, \quad \varphi = 1 \text{ in } \Omega_2, \quad \text{and} \quad |\nabla \varphi| \leq \frac{2}{\delta_2 - \delta_1}.$$

We claim that, for all $\eta > 0$, w satisfies the following inequality:

$$\begin{aligned}
J_\varepsilon^1(w; \Omega) \leq\ & J_\varepsilon^1(E_1; \Omega_\eta) + J_\varepsilon^1(E_2; \Omega) \\
& + \frac{\varepsilon}{\delta_2 - \delta_1} \int_{\mathbb{R}^d} K(z)\,|z|\,\mathrm{d}z\,\mathscr{L}^d\big((E_1 \triangle E_2) \cap \Omega\big) \\
& + \frac{\varepsilon}{\eta} \int_{\Omega \cap \Omega_2^{\mathsf{c}}} \int_{\mathbb{R}^d} K(z)\,|z|\,\chi_{\Omega \cap \Omega_2^{\mathsf{c}}}(x + \varepsilon z)\mathrm{d}z\mathrm{d}x.
\end{aligned} \tag{4.16}$$

Let us assume provisionally that the claim holds. Then, by virtue of the Coarea Formula, there exists $t^* \in (0, 1)$ such that (4.15) holds for the superlevel $F := \{ w > t^* \}$. Furthermore, we see that such set fulfils (i) and (ii) as well. Indeed, φ is supported in $\Omega \cap \Omega_1^{\mathsf{c}}$ and it equals 1 in Ω_2, whence $F \cap \Omega_1 = E_1 \cap \Omega_1$ and $F \cap \Omega_2 = E_2 \cap \Omega_2$. As for (ii), we remark that $x \in E_2 \cap F^{\mathsf{c}} \cap \Omega$ and $x \in E_2^{\mathsf{c}} \cap F \cap \Omega$ respectively get the equalities $w(x) = (1 - \varphi(x))\chi_{E_1}(x) + \varphi(x) \le t^* < 1$ and $w(x) = (1 - \varphi(x))\chi_{E_1}(x) > t^* > 0$, which in turn entail $x \in E_1^{\mathsf{c}} \cap E_2 \cap \Omega$ and $x \in E_1 \cap E_2^{\mathsf{c}} \cap \Omega$.

So, to achieve the conclusion, we only need to validate (4.16). To this aim, we explicit the integrand appearing in $J_K^1(w; \Omega)$: when $x, y \in \Omega$, we have

$$
\begin{aligned}
|w(y) - w(x)| &\le (1 - \varphi(y)) \, |u(y) - u(x)| + \varphi(y) \, |v(y) - v(x)| \\
&\quad + |\varphi(y) - \varphi(x)| \, |v(x) - u(x)| \\
&\le \chi_{\{ \varphi \ne 1 \}}(y) \, |u(y) - u(x)| + \chi_{\{ \varphi \ne 0 \}}(y) \, |v(y) - v(x)| \\
&\quad + |\varphi(y) - \varphi(x)| \, |v(x) - u(x)| \, .
\end{aligned}
$$

Since $\{ \varphi \ne 0 \} \subset \Omega \cap \Omega_1^{\mathsf{c}}$ and $\{ \varphi \ne 1 \} \subset \Omega \cap \Omega_2^{\mathsf{c}}$, we see that

$$
\begin{aligned}
2J_\varepsilon^1(w; \Omega) &\le \int_\Omega \int_{\Omega \cap \Omega_2^{\mathsf{c}}} K_\varepsilon(y - x) \, |u(y) - u(x)| \, \mathrm{d}y \mathrm{d}x \\
&\quad + \int_\Omega \int_{\Omega \cap \Omega_1^{\mathsf{c}}} K_\varepsilon(y - x) \, |v(y) - v(x)| \, \mathrm{d}y \mathrm{d}x \\
&\quad + \int_\Omega \int_\Omega K_\varepsilon(y - x) \, |\varphi(y) - \varphi(x)| \, |v(x) - u(x)| \, \mathrm{d}y \mathrm{d}x .
\end{aligned}
$$

We bound each of three double integrals above separately. Firstly, we divide Ω in the disjoint regions Ω_η and $\Omega \cap \Omega_\eta^{\mathsf{c}}$. We obtain

$$
\begin{aligned}
\int_\Omega &\int_{\Omega \cap \Omega_2^{\mathsf{c}}} K_\varepsilon(y - x) \, |u(y) - u(x)| \, \mathrm{d}y \mathrm{d}x \\
&= \int_{\Omega_\eta} \int_{\Omega \cap \Omega_2^{\mathsf{c}}} K_\varepsilon(y - x) \, |u(y) - u(x)| \, \mathrm{d}y \mathrm{d}x \\
&\quad + \int_{\Omega \cap \Omega_\eta^{\mathsf{c}}} \int_{\Omega \cap \Omega_2^{\mathsf{c}}} K_\varepsilon(y - x) \, |u(y) - u(x)| \, \mathrm{d}y \mathrm{d}x .
\end{aligned}
$$

Note that

$$
\int_{\Omega_\eta} \int_{\Omega \cap \Omega_2^{\mathsf{c}}} K_\varepsilon(y - x) \, |u(y) - u(x)| \, \mathrm{d}y \mathrm{d}x \le 2J_\varepsilon^1(u; \Omega_\eta)
$$

and that

$$
\begin{aligned}
\int_{\Omega \cap \Omega_\eta^{\mathsf{c}}} &\int_{\Omega \cap \Omega_2^{\mathsf{c}}} K_\varepsilon(y - x) \, |u(y) - u(x)| \, \mathrm{d}y \mathrm{d}x \\
&\le \frac{2\varepsilon}{\eta} \int_{\Omega \cap \Omega_\eta^{\mathsf{c}}} \int_{\Omega \cap \Omega_2^{\mathsf{c}}} K_\varepsilon(y - x) \frac{|y - x|}{\varepsilon} \mathrm{d}y \mathrm{d}x \\
&= \frac{2\varepsilon}{\eta} \int_{\Omega \cap \Omega_\eta^{\mathsf{c}}} \int_{\mathbb{R}^d} K(z) \, |z| \, \chi_{\Omega \cap \Omega_2^{\mathsf{c}}}(x + \varepsilon z) \mathrm{d}z \mathrm{d}x ,
\end{aligned}
$$

therefore,

$$\int_\Omega \int_{\Omega \cap \Omega_2^c} K_\varepsilon(y-x) \, |u(y) - u(x)| \, \mathrm{d}y \mathrm{d}x$$

$$\leq 2J_\varepsilon^1(u; \Omega_\eta) + \frac{2\varepsilon}{\eta} \int_{\Omega \cap \Omega_\eta^c} \int_{\mathbb{R}^d} K(z) \, |z| \, \chi_{\Omega \cap \Omega_2^c}(x + \varepsilon z) \mathrm{d}z \mathrm{d}x. \quad (4.17)$$

Next, as for the second integral, we have

$$\int_\Omega \int_{\Omega \cap \Omega_1^c} K_\varepsilon(y-x) \, |v(y) - v(x)| \, \mathrm{d}y \mathrm{d}x \leq 2J_\varepsilon^1(v; \Omega). \quad (4.18)$$

Finally, we observe that

$$|\varphi(y) - \varphi(x)| \leq \frac{2}{\delta_2 - \delta_1} \, |y - x|$$

and hence

$$\int_\Omega \int_\Omega K_\varepsilon(y-x) \, |\varphi(y) - \varphi(x)| \, |v(x) - u(x)| \, \mathrm{d}y \mathrm{d}x$$

$$\leq \frac{2\varepsilon}{\delta_2 - \delta_1} \int_{\mathbb{R}^d} K(z) \, |z| \, \mathrm{d}z \int_\Omega |v(x) - u(x)| \, \mathrm{d}x \quad (4.19)$$

Combining (4.17), (4.18), and (4.19) we retrieve (4.16). □

Now, we are in position to prove a chain of inequalities which yields Lemma 4.13 as a corollary:

LEMMA 4.15
For all $\hat{p} \in \mathbb{S}^{d-1}, \sigma_K(\hat{p}) \leq \sigma_K''(\hat{p}) \leq \sigma_K'(\hat{p})$.

Proof. We begin with the inequality $\sigma_K \leq \sigma_K''$. Let E_ε be measurable subsets of $\mathbb{R}^d$ such that $E_\varepsilon \cap B^c = H_{\hat{p}} \cap B^c$ for all $\varepsilon > 0$, and suppose that $E_\varepsilon \to H_{\hat{p}}$ in $L^1(B)$. By Theorem 3.14, we have that

$$0 \leq J_\varepsilon(E_\varepsilon; B) - J_\varepsilon(H_{\hat{p}}; B)$$
$$= J_\varepsilon^1(E_\varepsilon; B) - J_\varepsilon^1(H_{\hat{p}}; B) + J_\varepsilon^2(E_\varepsilon; B) - J_\varepsilon^2(H_{\hat{p}}; B).$$

If in addition we require that $E_\varepsilon \triangle H_{\hat{p}} \subset B_{1-\delta}$, a direct computation shows

$$J_\varepsilon^2(E_\varepsilon; B) - J_\varepsilon^2(H_{\hat{p}}; B)$$
$$= L_\varepsilon(E_\varepsilon \cap H_{\hat{p}}^c; H_{\hat{p}}^c \cap B^c) + L_\varepsilon(E_\varepsilon^c \cap H_{\hat{p}}; H_{\hat{p}} \cap B^c)$$
$$- L_\varepsilon(E_\varepsilon^c \cap H_{\hat{p}}; H_{\hat{p}}^c \cap B^c) - L_\varepsilon(E_\varepsilon \cap H_{\hat{p}}^c; H_{\hat{p}} \cap B^c),$$

and so

$$\left| J_\varepsilon^2(E_\varepsilon; B) - J_\varepsilon^2(H_{\hat{p}}; B) \right| \leq L_\varepsilon(E_\varepsilon \triangle H_{\hat{p}}; B^c).$$

Since $E_\varepsilon \triangle H_{\hat{p}} \subset B_{1-\delta}$, we notice that $|y - x| \geq \delta$ if $y \in B^{\mathsf{c}}$ and $x \in E_\varepsilon \triangle H_{\hat{p}}$, and thus

$$\frac{1}{\varepsilon} \left| J_\varepsilon^2(E_\varepsilon; B) - J_\varepsilon^2(H_{\hat{p}}; B) \right| \leq \frac{1}{\delta} \int_{E_\varepsilon \triangle H_{\hat{p}}} \int_{B^{\mathsf{c}}} K_\varepsilon(y - x) \frac{|y - x|}{\varepsilon} \mathrm{d}y \mathrm{d}x$$

$$\leq \frac{1}{\delta} \mathscr{L}^d(E_\varepsilon \triangle H_{\hat{p}}) \int_{\mathbb{R}^d} K(z) |z| \, \mathrm{d}z.$$

We deduce that

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \left| J_\varepsilon^2(E_\varepsilon; B) - J_\varepsilon^2(H_{\hat{p}}; B) \right| = 0,$$

whence

$$0 \leq \liminf_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \left[J_\varepsilon(E_\varepsilon; B) - J_\varepsilon(H_{\hat{p}}; B) \right]$$

$$= \liminf_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \left[J_\varepsilon^1(E_\varepsilon; B) - J_\varepsilon^1(H_{\hat{p}}; B) \right].$$

Recalling (4.14) and the definition of σ_K'', we conclude that $\sigma_K(\hat{p}) \leq \sigma_K''(\hat{p})$.

We now focus on the inequality $\sigma_K'' \leq \sigma_K'$. We pick a family $\{ E_\varepsilon \}$ that converges to $H_{\hat{p}}$ in $L^1(B)$, and we assume, without loss of generality, that $J_\varepsilon^1(E_\varepsilon; B)$ is finite. We exploit Lemma 4.14 taking as data, for any ε, E_ε and $H_{\hat{p}}$. We obtain a family $\{ F_\varepsilon \}$ that converges to $H_{\hat{p}}$ in $L^1(B)$ and that fulfils $F_\varepsilon \triangle H_{\hat{p}} \subset B_{1-\delta}$. Moreover, for any $\eta \in \left(0, (1 - \delta)/2 \right)$ and for $\delta_1 = \delta$ and $\delta_2 = \delta + \eta$, formula (4.15) yields

$$\frac{1}{\varepsilon} J_\varepsilon^1(F_\varepsilon; B) \leq \frac{1}{\varepsilon} J_\varepsilon^1(H_{\hat{p}}; B \cap B_{1-\delta-2\eta}^{\mathsf{c}}) + \frac{1}{\varepsilon} J_\varepsilon^1(E_\varepsilon; B)$$

$$+ \frac{1}{\eta} \int_{\mathbb{R}^d} K(z) |z| \, \mathrm{d}z \, \mathscr{L}^d\left((H_{\hat{p}} \triangle E_\varepsilon) \cap B \right)$$

$$+ \frac{1}{\eta} \int_{B_{1-\delta-2\eta}} \int_{\mathbb{R}^d} K(z) |z| \, \chi_{B \cap B_{1-\delta-\eta}^{\mathsf{c}}}(x + \varepsilon z) \mathrm{d}z \mathrm{d}x.$$

By utilizing (4.2) we find

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} J_\varepsilon^1(H_{\hat{p}}; B \cap B_{1-\delta-2\eta}^{\mathsf{c}}) = J_0(H_{\hat{p}}; B \cap B_{1-\delta-2\eta}^{\mathsf{c}}),$$

and thus, in the limit $\varepsilon \to 0^+$, we infer

$$\sigma_K''(\hat{p}) \leq \liminf_{\varepsilon \to 0^+} \frac{1}{\varepsilon \omega_{d-1}} J_\varepsilon^1(F_\varepsilon; B)$$

$$\leq \frac{1}{\omega_{d-1}} J_0(H_{\hat{p}}; B \cap B_{1-\delta-2\eta}^{\mathsf{c}}) + \liminf_{\varepsilon \to 0^+} \frac{1}{\varepsilon \omega_{d-1}} J_\varepsilon^1(E_\varepsilon; B),$$

In view of the arbitrariness of the family $\{ E_\varepsilon \}$, we achieve the conclusion by letting η and δ vanish. □

4.5 Γ-lower limit

We conclude the proof of Theorem 4.8 by validating the Γ-lower limit inequality. We remind that the method we apply was introduced in [FM93] and that it was applied in [ADPM11, AB98b] to establish Γ-convergence results that are close to ours.

Our task is showing that, for all measurable E and for all families $\{ E_\varepsilon \}$ such $E_\varepsilon \to E$ in $L^1_{\text{loc}}(\mathbb{R}^d)$ as $\varepsilon \to 0^+$, we have

$$J_0(E; \Omega) \leq \liminf_{\varepsilon \to 0^+} \frac{1}{\varepsilon} J^1_\varepsilon(E_\varepsilon; \Omega). \tag{4.20}$$

Firstly, we notice that (4.20) holds trivially when the right-hand side is not finite, thus we may suppose that

$$\liminf_{\varepsilon \to 0^+} \frac{1}{\varepsilon} J^1_\varepsilon(E_\varepsilon; \Omega) < +\infty.$$

Besides, up to extraction of a subsequence, we may assume that the the lower limit is a limit, and hence the ratios $\{ \varepsilon^{-1} J^1_\varepsilon(E_\varepsilon; \Omega) \}$ are uniformly bounded in ε. In view of the compactness criterion on page 52, the bound yields regularity for the limit set E, which turns out to be a finite perimeter set in Ω.

We introduce the densities

$$f_\varepsilon(x) := \begin{cases} \dfrac{1}{2\varepsilon} \displaystyle\int_{E^c_\varepsilon \cap \Omega} K_\varepsilon(y - x) \mathrm{d}y & \text{if } x \in \Omega \cap E_\varepsilon, \\[3mm] \dfrac{1}{2\varepsilon} \displaystyle\int_{E_\varepsilon \cap \Omega} K_\varepsilon(y - x) \mathrm{d}y & \text{if } x \in \Omega \cap E^c_\varepsilon, \end{cases}$$

and the measures

$$\mu_\varepsilon := f_\varepsilon \mathscr{L}^d \llcorner \Omega.$$

For any $\varepsilon > 0$, the total variation of μ_ε in Ω equals $\varepsilon^{-1} J^1_\varepsilon(E_\varepsilon; \Omega)$, thus $\{ \mu_\varepsilon \}$ is a family of positive measures with uniformly bounded total variation in Ω. It follows from Theorem 1.1 and Banach-Alaoglu's Theorem that there exists a finite positive measure μ on Ω such that (up to subsequences) $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$ as $\varepsilon \to 0^+$, and, by the lower semicontinuity of the total variation,

$$\|\mu\| \leq \liminf_{\varepsilon \to 0} \frac{1}{\varepsilon} J^1_\varepsilon(E_\varepsilon; \Omega).$$

We now see that (4.20) holds true if

$$\|\mu\| \geq J_0(E; \Omega). \tag{4.21}$$

In order to establish the previous inequality, we recall that we know that E is a finite perimeter set in Ω, so for a point $x \in \partial^* E \cap \Omega$ we can consider the Radon-Nikodym derivative

$$\frac{\mathrm{d}\mu}{\mathrm{d}\nu}(x) := \lim_{r \to 0} \frac{\mu(B(x, r))}{\omega_{d-1} r^{d-1}},$$

where $\nu := \mathscr{H}^{d-1} \llcorner \partial^* E$. Since the sequence μ_ε weakly-$*$ converges to μ, by Proposition 1.3, for all $r > 0$ but at most a countable set N, we have $\nu(B(x, r)) = \lim_{\varepsilon \to 0} \nu_\varepsilon(B(x, r))$. Thus,

$$\begin{aligned} \frac{\mathrm{d}\mu}{\mathrm{d}\nu}(x) &= \lim_{r \to 0, r \notin N} \left[\lim_{\varepsilon \to 0} \frac{\mu_\varepsilon(B(x, r))}{\omega_{d-1} r^{d-1}} \right] \\ &= \lim_{\ell \to +\infty} \frac{\mu_{\varepsilon_\ell}(B(x, r_\ell))}{\omega_{d-1} r_\ell^{d-1}} \end{aligned}$$

for suitable subsequences $\{\,\varepsilon_\ell\,\}$ and $\{\,r_\ell\,\}$ satisfying

$$\lim_{\ell\to+\infty} r_\ell = \lim_{\ell\to+\infty} \frac{\varepsilon_\ell}{r_\ell} = 0.$$

Rewriting the last equality explicitly, we have

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x) = \lim_{\ell\to+\infty} \frac{1}{2\omega_{d-1}\varepsilon_\ell r_\ell^{d-1}} \left[\int_{E_{\varepsilon_\ell}\cap B(x,r_\ell)\cap\Omega} \int_{E_{\varepsilon_\ell}^{\mathsf{c}}\cap\Omega} K_{\varepsilon_\ell}(y-x)\mathrm{d}y\mathrm{d}x \right.$$
$$\left. + \int_{E_{\varepsilon_\ell}^{\mathsf{c}}\cap B(x,r_\ell)\cap\Omega} \int_{E_{\varepsilon_\ell}\cap\Omega} K_{\varepsilon_\ell}(y-x)\mathrm{d}y\mathrm{d}x \right],$$

and hence

$$\frac{\mathrm{d}\mu}{\mathrm{d}\nu}(x) \geq \limsup_{\ell\to+\infty} \frac{1}{\omega_{d-1}\varepsilon_\ell r_\ell^{d-1}} J_{\varepsilon_\ell}^1(E_{\varepsilon_\ell}; B(x,r_\ell)\cap\Omega)$$
$$= \limsup_{\ell\to+\infty} \frac{1}{\omega_{d-1}\varepsilon_\ell r_\ell^{d-1}} J_{\varepsilon_\ell}^1(E_{\varepsilon_\ell}; B(x,r_\ell)),$$

because $B(x,r_\ell)\subset\Omega$ when ℓ is large enough. A change of variables yields

$$\frac{\mathrm{d}\mu}{\mathrm{d}\nu}(x) \geq \limsup_{\ell\to+\infty} \frac{r_\ell}{\omega_{d-1}\varepsilon_\ell} J_{\frac{\varepsilon_\ell}{r_\ell}}^1 \left(\frac{E_{\varepsilon_\ell}-x}{r_\ell}; B \right),$$

In view of position (4.13), it is clear that

$$\frac{\mathrm{d}\mu}{\mathrm{d}\nu}(x) \geq \sigma_K'(\hat{n}(x)),$$

which gets

$$\|\mu\| \geq \int_{\partial^* E\cap\Omega} \sigma_K'(\hat{n}(x))\mathrm{d}\mathscr{H}^{d-1}(x).$$

The characterisation of σ_K provided by Lemma 4.13 yields the conclusion.

Chapter 5

Convergence of evolutions by rescaled nonlocal curvatures

In Chapter 4 we proved that, by a suitable rescaling, we can recover a local perimeter functional as the Γ-limit of nonlocal perimeters. We also saw in Section 3.4 that the functional

$$H_K(E, x) = -\lim_{r \to 0^+} \int_{B(x,r)^{\mathsf{c}}} K(y-x)\left(\chi_E(y) - \chi_{E^{\mathsf{c}}}(y)\right) \mathrm{d}y$$

is the geometric first variation of the nonlocal perimeter Per_K. Relying on the paper [CP18], in the current chapter we show that we can produce a localisation effect for the nonlocal curvature by means of the same procedure that we applied to the nonlocal perimeter: for any $\varepsilon > 0$ we set

$$K_\varepsilon(x) := \frac{1}{\varepsilon^d} K\left(\frac{x}{\varepsilon}\right),$$

we define

$$H_\varepsilon(E, x) := H_{K_\varepsilon}(E, x) \tag{5.1}$$

for a set $E \subset \mathbb{R}^d$ and $x \in \partial E$, and then we let ε tend to 0. As we discuss in Section 5.1, the limit of the family $\left\{\, \varepsilon^{-1} H_\varepsilon \,\right\}$ agrees with the analysis in Chapter 4, in the sense that the limit curvature, which is local, is the first variation of the surface energy

$$J_0(E) = \int_{\partial^* E} \sigma_K(\hat{n}(x)) \mathrm{d}\mathscr{H}^{d-1}(x)$$

obtained as the Γ-limit of the rescaled nonlocal perimeters.

The study of the asymptotics of rescaled nonlocal curvatures is expounded in Section 5.2. Then, we tackle a second issue: is the relationship between curvatures carried over to the solutions of the related geometric motions? More precisely, we investigate whether the solutions of

$$\partial_t x(t) \cdot \hat{n} = -\frac{1}{\varepsilon} H_\varepsilon(E(t), x(t)),$$

converge, as ε vanishes, to the solution of

$$\partial_t x(t) \cdot \hat{n} = -H_0(E(t), x(t)),$$

where $t \mapsto E(t)$ is an evolution of sets in $\mathbb{R}^d$, $\hat{n}$ is the outer unit normal to $\partial E(t)$ at the point $x(t)$, and H_0 is the first variation of J_0.

To answer, we firstly explain in Section 5.3 what kind of solutions we take into consideration, and we recall the well-posedness results for local and nonlocal curvature flows that are available in the framework of level set formulations. By this method, the geometric evolutions are cast into degenerate parabolic equations, to which viscosity theory provides an adequate definition of solution. Then, we prove that solutions in this class to the motions by rescaled nonlocal curvatures locally uniformly converge to the viscosity solution of the limit curvature flow. This is the main result of the chapter, which we establish in Section 5.4 by resorting to the set-theoretic approach of geometric barriers, formerly introduced by De Giorgi [DG94]. We use barriers to transfer the geometric information given by the convergence result of Section 5.2 to viscosity solutions. In this respect, comparison results between barriers and level set flows [BN98, CDNV19] are crucial.

5.1 Preliminaries

Since we want to mimic the approach of Chapter 4, as an initial step we compute the candidate limit curvature, that is, the first variation of the functional (4.7) on page 57. We premise the assumptions on the kernel K that we use throughout the chapter.

As before, $K \colon \mathbb{R}^d \to [0, +\infty)$ is a measurable function such that

$$K(y) = K(-y) \quad \text{for a.e. } y \tag{5.2}$$

and

$$K \in W^{1,1}(B(0,r)^{\mathsf{c}}) \quad \text{for all } r > 0. \tag{5.3}$$

This allows both K and ∇K to be singular in 0 and, at the same time, implies convergence of their integrals at infinity. Moreover, we assume that

$$K(y), \ |y|\,|\nabla K(y)| \in L^1(Q_\lambda(e)) \quad \text{for all } e \in \mathbb{S}^{d-1} \text{ and } \lambda > 0, \tag{5.4}$$

where, for any $e \in \mathbb{S}^{d-1}$ and $\lambda > 0$,

$$Q_\lambda(e) := \left\{ y \in \mathbb{R}^d : |y \cdot e| \leq \frac{\lambda}{2} \left| \pi_{e^\perp}(y) \right|^2 \right\}.$$

(see again Figure 3.3 on page 47). These requirements are in line with the ones in Section 3.4, and, in particular, they ensure that sets with $C^{1,1}$ boundary have finite nonlocal curvature.

To the purpose of proving a nonlocal-to-local converge result that is uniform w.r.t. the point in which the curvatures are evaluated, we need to describe the behaviour of K in quantitative terms. So, we suppose that

$$\lim_{r \to 0^+} r \int_{B(0,r)^{\mathsf{c}}} K(y)\mathrm{d}y = 0. \tag{5.5}$$

Besides, we want our analysis to encompass anisotropic kernels. Still, we have to admit some control on the mass of K in $Q_\lambda(e)$, uniformly in e. We therefore assume that for all $\lambda > 0$ there exists $a_\lambda > 0$ such that for all $e \in \mathbb{S}^{d-1}$

$$\int_{Q_\lambda(e)} K(y)\mathrm{d}y \leq a_\lambda. \tag{5.6}$$

In addition, we require that there exist $a_0, b_0 > 0$ such that for all $e \in \mathbb{S}^{d-1}$

$$\limsup_{\lambda \to 0^+} \frac{1}{\lambda} \int_{Q_\lambda(e)} K(y) \mathrm{d}y \leq a_0, \tag{5.7}$$

$$\limsup_{\lambda \to 0^+} \frac{1}{\lambda} \int_{Q_\lambda(e)} |\nabla K(y)| \, |y| \, \mathrm{d}y \leq b_0. \tag{5.8}$$

We assume as well that for all $e \in \mathbb{S}^{d-1}$

$$\lim_{\lambda \to +\infty} \frac{1}{\lambda} \int_{Q_\lambda(e)} K(y) \mathrm{d}y = 0. \tag{5.9}$$

Lastly, we let K be bounded above by a fractional kernel away from the origin, that is,

$$K(y) \leq \frac{m}{|y|^{d+1+s}} \quad \text{if } y \in B(0,1)^{\mathsf{c}} \tag{5.10}$$

for some $m > 0$ and $s \in (0,1)$. This hypothesis entails that, for all $\alpha < s$,

$$\lim_{r \to +\infty} r^{1+\alpha} \int_{B(0,r)^{\mathsf{c}}} K(y) \mathrm{d}y = 0. \tag{5.11}$$

Actually, most of the results in the sequel hold even if the weaker assumption (5.11) replaces (5.10).

Let us describe two classes of singular kernels that satisfy (5.2) – (5.10).

EXAMPLE 5.1
If $K \colon \mathbb{R}^d \to [0, +\infty)$ is even and there exist $m, \mu > 0$ and $s, \sigma \in (0,1)$ such that

$$K(y), |y||\nabla K(y)| \leq \frac{\mu}{|y|^{d+\sigma}} \quad \text{for all } y \in B(0,1)$$

and that

$$K(y), |y||\nabla K(y)| \leq \frac{m}{|y|^{d+1+s}} \quad \text{for all } y \in B(0,1)^{\mathsf{c}},$$

then, K fulfils all the previous requirements.

Fractional kernels with exponential decay at infinity, i.e. even functions $K \colon \mathbb{R}^d \to [0, +\infty)$ for which there exist $m, \mu > 0$ and $s \in (0,1)$ such that

$$K(y), |y||\nabla K(y)| \leq \frac{\mu e^{-m|y|}}{|y|^{d+s}} \quad \forall y \in \mathbb{R}^d,$$

also fit in our setting.

We now derive the candidate limit curvature starting from the anisotropic surface energy J_0. In this case, we suppose that K fits in the framework of the Γ-convergence statement concerning perimeters, i.e. Theorem 4.1. As above, $p^\perp$ is the hyperplane of vectors orthogonal to $p \in \mathbb{R}^d \setminus \{0\}$ and $H_{\hat{p}} := \{ x \in \mathbb{R}^d : x \cdot \hat{p} > 0 \}$.

LEMMA 5.2 (Derivatives of the anisotropy)
Let $K \colon \mathbb{R}^d \to [0, +\infty)$ be a function such the assumptions (5.2) - (5.7) hold, and suppose that

$$c_K := \frac{1}{2} \int_{\mathbb{R}^d} K(z) |z| \, \mathrm{d}y < +\infty.$$

Let σ_K be the norm in (4.8), i.e.

$$\sigma_K(p) := \frac{1}{2} \int_{\mathbb{R}^d} K(z) \, |z \cdot p| \, \mathrm{d}z, \qquad \text{for } p \in \mathbb{R}^d.$$

Then, $\sigma_K \in C^2(\mathbb{R}^d \setminus \{0\})$ and

$$\nabla \sigma_K(p) = \int_{H_{\hat{p}}} K(z) z \, \mathrm{d}z, \qquad \nabla^2 \sigma_K(p) = \frac{1}{|p|} \int_{\hat{p}^\perp} K(z) z \otimes z \, \mathrm{d}\mathscr{H}^{d-1}(z).$$

The calculation of the gradient is immediate; as for the Hessian, we preliminarily recall a result about absolute continuity on lines of Sobolev functions. We refer to the monograph [HKST15] for the details.

THEOREM 5.3
Let $\Omega \subset \mathbb{R}^d$ be an open set. For any $p \in [1, +\infty)$, $u : \Omega \to \mathbb{R}$ belongs to the Sobolev space $W^{1,p}(\Omega)$ if and only if it belongs to $L^p(\Omega)$ and the following property holds: for any $e \in \mathbb{S}^{d-1}$ there exists $N_e \subset e^\perp$ such that $\mathscr{H}^{d-1}(N_e) = 0$ and for all $z \in e^\perp \cap N_e^{\mathsf{c}}$ the function $I \ni t \mapsto u(z + te)$ is absolutely continuous on any compact interval I such that $z + te \in \Omega$ when $t \in I$.

Further, for any $e \in \mathbb{S}^{d-1}$ and a.e. $y \in \Omega$, the classical directional derivative $\partial_e u$ exists and it coincides with $\nabla u(y) \cdot e$.

In the light of the previous characterisation, when (5.3) holds, the kernel K is absolutely continuous on lines in $B(0, r)^{\mathsf{c}}$ for all $r > 0$. We use this property to establish the following:

LEMMA 5.4
For all $e \in \mathbb{S}^{d-1}$,

$$\int_{e^\perp} K(z) \, |z|^2 \, \mathrm{d}\mathscr{H}^{d-1}(z) \leq a_0, \tag{5.12}$$

a_0 being as in (5.7), and there holds

$$\lim_{r \to +\infty} r^\beta \int_{e^\perp \cap B(0,r)^{\mathsf{c}}} K(z) \, |z|^2 \, \mathrm{d}\mathscr{H}^{d-1}(z) = 0 \quad \text{for all } \beta < s. \tag{5.13}$$

Proof. By Theorem 5.3, for any $e \in \mathbb{S}^{d-1}$ and any $\ell \in \mathbb{N}$, there exists a $\mathscr{H}^{d-1}$-negligible $N_\ell \subset \{ z \in e^\perp : \ell \, |z| \geq 1 \}$ such that, for all $z \in e^\perp \cap N_\ell^{\mathsf{c}}$ with $\ell \, |z| \geq 1$, the function $t \mapsto K(z + te)$ is absolutely continuous when t belongs to closed, bounded intervals. By the arbitrariness of $\ell \in \mathbb{N}$, we conclude that for $\mathscr{H}^{d-1}$-a.e. $z \in e^\perp$, $[a, b] \ni t \mapsto K(z + te)$ is absolutely continuous for any $a, b \in \mathbb{R}$.

Hence, by the Mean Value Theorem, for $\mathscr{H}^{d-1}$-almost every $z \in e^\perp$ we find

$$\lim_{\lambda \to 0^+} \frac{1}{\lambda} \int_{-\frac{\lambda}{2}|z|^2}^{\frac{\lambda}{2}|z|^2} K(z + te) \mathrm{d}t = K(z) \, |z|^2 \,. \tag{5.14}$$

By definition of $Q_\lambda(e)$,

$$\frac{1}{\lambda} \int_{e^\perp} \int_{-\frac{\lambda}{2}|z|^2}^{\frac{\lambda}{2}|z|^2} K(z + te) \mathrm{d}t \mathrm{d}\mathscr{H}^{d-1}(z) = \frac{1}{\lambda} \int_{Q_\lambda(e)} K(y) \mathrm{d}y,$$

and the right hand side is finite for any $\lambda > 0$, thanks to (5.4). By appealing to (5.7) and (5.14) , we can take the limit $\lambda \to 0^+$ on both sides of the equality, and this yields (5.12).

Estimate (5.13) is an easy consequence of assumption (5.10). $\square$

Now we are ready to prove Lemma 5.2.

Proof. By Lebesgue's Theorem, we see that $\sigma_K \in C^1(\mathbb{R}^d \setminus \{0\})$, with

$$\nabla \sigma_K(p) = \frac{1}{2} \int_{\mathbb{R}^d} K(z) z \frac{z \cdot p}{|z \cdot p|} \mathrm{d}z = \int_{H_{\hat{p}}} K(z) z \mathrm{d}z.$$

Let now $i, j \in \{1, \ldots, d\}$ and fix $p \in \mathbb{R}^d \setminus \{0\}$. We have

$$\partial_{i,j}^2 \sigma_K(p) := \lim_{h \to 0^+} \frac{1}{h} \left[\int_{H_{\widehat{p+he_j}}} K(z) z_i \mathrm{d}z - \int_{H_{\hat{p}}} K(z) z_i \mathrm{d}z \right],$$

where $z_i := z \cdot e_i$ for all $z \in \mathbb{R}^d$ and all i. We rewrite the term between square brackets as

$$\int_{\hat{p}^\perp} \int_{\frac{h z_j}{|p| + h \hat{p}_j}}^0 K(z + t\hat{p})(z_i + t\hat{p}_i) \mathrm{d}t \mathrm{d}\mathscr{H}^{d-1}(z),$$

and, reasoning as in the proof of the previous Lemma, we apply the Mean Value Theorem to get

$$\partial_{i,j}^2 \sigma_K(p) := \lim_{h \to 0^+} \frac{1}{|p| + h\hat{p}_j} \int_{\hat{p}^\perp} K(z + t_h \hat{p})(z_i + t_h \hat{p}_i) z_j \mathrm{d}\mathscr{H}^{d-1}(z),$$

with t_h between 0 and $h z_j / (|p| + h\hat{p}_j)$. Thanks to (5.12), we can take the limit $h \to 0^+$ and we conclude the computation of the Hessian. Next, we show that the latter is continuous.

For all $e \in \mathbb{S}^{d-1}$, set

$$M_K(e) := \int_{e^\perp} K(z) z \otimes z \mathrm{d}\mathscr{H}^{d-1}(z).$$

Fix a unit vector e and consider a sequence of rotations $\{R_\ell\}$ such that $R_\ell \to \mathrm{id}$. We have

$$|M_K(R_\ell e) - M_K(e)| = \left| \int_{e^\perp} K(R_\ell z) R_\ell z \otimes R_\ell z \mathrm{d}\mathscr{H}^{d-1} - \int_{e^\perp} K(z) z \otimes z \mathrm{d}\mathscr{H}^{d-1} \right|$$

$$\leq \left| \int_{e^\perp} K(R_\ell z) \left[R_\ell z \otimes R_\ell z - z \otimes z \right] \mathrm{d}\mathscr{H}^{d-1} \right|$$

$$+ \left| \int_{e^\perp} [K(R_\ell z) - K(z)] z \otimes z \mathrm{d}\mathscr{H}^{d-1} \right|.$$

Since $K \in L^1(B(0,r)^c)$ for all $r > 0$, it holds

$$\lim_{\ell \to +\infty} \| K \circ R_\ell - K \|_{L^1(B(0,r)^c)} = 0,$$

whence $K(R_\ell z) \to K(z)$ for $\mathscr{H}^{d-1}$-a.e. $z \in e^\perp$. This, together with (5.12), gets that the upper bound on $|M_K(R_\ell e) - M_K(e)|$ vanishes as $\ell \to +\infty$. Therefore, $\nabla^2 \sigma_K(p) = |p|^{-1} M_K(\hat{p})$ is continuous on $\mathbb{R}^d \setminus \{0\}$. $\qquad\square$

Let Σ a C^2 hypersurface in $\mathbb{R}^d$. For $x \in \mathbb{R}^d$, we define the following *anisotropic mean curvature* functional:

$$H_0(\Sigma, x) := -\operatorname{div}\left(\nabla\sigma_K(\hat{n}(x))\right),$$

with σ_K as above and $\hat{n}(x)$ is the outer unit normal to Σ at x. Note that this is the geometric first variation of the surface energy J_0 in (4.7). Since Σ is smooth, we can find an open neighbourhood of x and $\varphi \in C^2(\mathbb{R}^d)$ such that $\Sigma \cap U = \{\, y \in U : \varphi(y) = 0 \,\}, \nabla\varphi(x) \neq 0$. Let us admit that $\nabla\varphi(x)/|\nabla\varphi(x)|$ is the outer normal at x. Then, we find

$$H_0(\Sigma, x) = -\frac{1}{|\nabla\varphi(x)|}\operatorname{tr}\left(M_K(\widehat{\nabla\varphi(x)})\nabla^2\varphi(x)\right), \tag{5.15}$$

where tr is the trace operator and

$$M_K(e) := \int_{e^\perp} K(z) z \otimes z \, \mathrm{d}\mathscr{H}^{d-1}(z) \qquad \text{for all } e \in \mathbb{S}^{d-1} \tag{5.16}$$

because, if $p \in \mathbb{R}^d \setminus \{0\}$, by Lemma 5.2, $M_K(\hat{p}) = \nabla^2\sigma_K(\hat{p})$.

A third, useful expression of the local curvature functional is

$$H_0(\Sigma, x) = -\frac{1}{|\nabla\varphi(x)|}\int_{\nabla\varphi(x)^\perp} K(z)\nabla^2\varphi(x)z \cdot z \, \mathrm{d}\mathscr{H}^{d-1}(z). \tag{5.17}$$

REMARK 5.5
Let Σ be a smooth hypersurface whose outer unit normal at a given point x is $\hat{n}$, and consider the map $T(y) := Ry + z$, where R is an orthogonal matrix and $z \in \mathbb{R}^d$. Then, using (5.17) it is easy to check that it holds

$$H_0(\Sigma, x) = \tilde{H}_0(T(\Sigma), T(x)), \tag{5.18}$$

where $\tilde{H}_0$ is the anisotropic mean curvature functional associated with the kernel $\tilde{K} := K \circ R^{\mathrm{t}}$.

To prove our claim, we observe that if $\Sigma = \{\, y \in \mathbb{R}^d : \varphi(y) = 0 \,\}$ for some smooth $\varphi \colon \mathbb{R}^d \to \mathbb{R}$, then $T(\Sigma) = \{\, y \in \mathbb{R}^d : \psi(y) = 0 \,\}$ with $\psi(y) := \varphi(R^{\mathrm{t}}(y - x))$. We have

$$\nabla\psi(T(y)) = R\nabla\varphi(y) \quad \text{and} \quad \nabla^2\psi(T(y)) = R\nabla^2\varphi(y)R^{\mathrm{t}},$$

and, therefore,

$$\tilde{H}_0(T(\Sigma), T(x)) = -\frac{1}{|R\nabla\varphi(x)|}\int_{R(\hat{n}^\perp)} \tilde{K}(z)\left(R\nabla^2\varphi(x)R^{\mathrm{t}}\right)z \cdot z \, \mathrm{d}\mathscr{H}^{d-1}(z)$$

$$= -\frac{1}{|\nabla\varphi(x)|}\int_{\hat{n}^\perp} K(z)\nabla^2\varphi(x)z \cdot z \, \mathrm{d}\mathscr{H}^{d-1}(z).$$

REMARK 5.6 (Connection with standard mean curvature)
When K is radial, that is $K(x) = \bar{K}(|x|)$ for some $\bar{K} \colon [0, +\infty) \to [0, +\infty)$, then H_0 coincides with a multiple of the standard mean curvature.

To see this, let Σ be a C^2 hypersurface such that $0 \in \Sigma$ and $\Sigma \cap U = \{\, y \in U : \varphi(y) = 0 \,\}$ for some neighbourhood U of 0 and some smooth function $\varphi \colon U \to \mathbb{R}$. We suppose also that $\nabla\varphi(0) \neq 0$ and that the outer unit normal to Σ at 0 is e_d. We recall the expression of the mean curvature H of Σ at 0:

$$H(\Sigma, 0) = -\frac{1}{\omega_{d-1}|\nabla\varphi(0)|}\int_{\mathbb{S}^{d-2}} \nabla^2\varphi(0)e \cdot e \, \mathrm{d}\mathscr{H}^{d-2}(e),$$

with ω_{d-1} the $(d-1)$-dimensional Lebesgue measure of the unit ball in $\mathbb{R}^{d-1}$.

If $K(x) = \bar{K}(|x|)$, formula (5.12) gets

$$c_K = \int_0^{+\infty} r^d \bar{K}(r) \mathrm{d}r < +\infty,$$

and, consequently, we have

$$H_0(\Sigma, 0) = -\frac{c_K}{|\nabla\varphi(0)|} \int_{\mathbb{S}^{d-2}} \nabla\varphi(0)e \cdot e \, \mathrm{d}\mathscr{H}^{d-2}(e) = \omega_{d-1}\, c_K H(\Sigma, 0).$$

5.2 Convergence of the rescaled nonlocal curvatures

This section is devoted to the proof of the uniform convergence of the rescaled nonlocal curvatures in (5.1) to the local functional H_0. This is accomplished in two steps: first, in Proposition 5.7, we establish pointwise convergence of the curvatures, with a precise estimate of the error, and then, in Theorem 5.9, we show that it is possible to make the error estimate uniform when the hypersurfaces at stake are compact.

Let us introduce some notation. For $e \in \mathbb{S}^{d-1}$, $x \in \mathbb{R}^d$ and $\delta > 0$, we let

$$C_e(x, \delta) := \left\{ y \in \mathbb{R}^d : y = x + z + te, \text{ with } z \in e^\perp \cap B(0, \delta), t \in (-\delta, \delta) \right\}.$$

When $E \subset \mathbb{R}^d$ is a set with boundary $\Sigma := \partial E$ of class C^2, then for all $x \in \Sigma$, there exist a open neighbourhood U of x and $\varphi \in C^2(\mathbb{R}^d)$ such that

$$\Sigma \cap U = \{y \in U : \varphi(y) = 0\}, \quad E \cap U = \{y \in U : \varphi(y) > 0\},$$

and $\nabla\varphi(y) \neq 0$ for all $y \in \Sigma \cap U$. We use the symbol $\hat{n} := \hat{n}(x)$ for the *outer* unit normal to Σ at x. By the Implicit Function Theorem, there exist $\bar{\delta} := \bar{\delta}(x) > 0$ and $f \colon \hat{n}^\perp \cap B(0, \bar{\delta}) \to (-\bar{\delta}, \bar{\delta})$ such that

$$\Sigma \cap C_{\hat{n}}(x, \bar{\delta}) = \left\{ y = x + z - f(z)\hat{n} : z \in \hat{n}^\perp \cap B(0, \bar{\delta}) \right\}, \tag{5.19}$$

$$E \cap C_{\hat{n}}(x, \bar{\delta}) = \left\{ y = x + z - t\hat{n} : z \in \hat{n}^\perp \cap B(0, \bar{\delta}), t \in (f(z), \bar{\delta}) \right\}, \tag{5.20}$$

and $\inf_{y \in C_{\hat{n}}(x, \bar{\delta})} |\nabla\varphi(y)| > 0$.

PROPOSITION 5.7 (Rate of convergence and pointwise limit)
Let $E \subset \mathbb{R}^d$ be a set whose topological boundary Σ is of class C^2. Let us fix $x \in \Sigma$ and let $\bar{\delta}$ and f be as above. Let also $s \in (0, 1)$ be the exponent in (5.10). Then, for all $\alpha, \beta \in (0, s)$, there exist $q > 1$ and $\bar{\varepsilon} \in (0, 1)$ such that $q\bar{\varepsilon} \leq \bar{\delta}$ and that, for all $\varepsilon \in (0, \bar{\varepsilon})$ and all $\delta \in (q\varepsilon, \bar{\delta})$, it holds

$$\left| \frac{1}{\varepsilon} H_\varepsilon(E, x) - H_0(\Sigma, x) \right| \leq \mathrm{Err}(\varepsilon, \delta), \tag{5.21}$$

where

$$\mathrm{Err}(\varepsilon, \delta) := \frac{1}{\delta}\left(\frac{\varepsilon}{\delta}\right)^\alpha + (b_0 + 1)\left\|\nabla^2 f\right\|^2_{L^\infty(D)}\delta + a_0\, O_f(\delta) + \left|\nabla^2 f(0)\right|\left(\frac{\varepsilon}{\delta}\right)^\beta, \tag{5.22}$$

$D := \hat{n}^\perp \cap B(0, \bar{\delta})$, *and*

$$O_f(\delta) := \sup_{z \in B(0, \delta)} \left|\nabla^2 f(z) - \nabla^2 f(0)\right|. \tag{5.23}$$

Furthermore, for all $\gamma \in \left(0, \alpha/(\alpha + 1)\right)$, it is possible to choose $\delta = q\varepsilon^\gamma$ in (5.21), so that

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} H_\varepsilon(E, x) = H_0(\Sigma, x).$$

Proof. We firstly focus on the bound on the rate of convergence. We observe that to validate (5.21) it suffices to reason on the case when $x = 0$ and $\hat{n} = e_d$. Indeed, if $x \neq 0$ or $\hat{n} \neq e_d$, by formulas (3.21) and (5.18), we have

$$H_\varepsilon(E, x) = \tilde{H}_\varepsilon(T(E), 0) \quad \text{and} \quad H_0(\Sigma, x) = \tilde{H}_0(T(\Sigma), 0),$$

where $T(y) := R(y - x)$, R is the matrix associated with the canonical basis of $\mathbb{R}^d$ that represents a rotation such that $R\hat{n} = e_d$, and $\tilde{H}_\varepsilon$ and $\tilde{H}_0$ are defined as in (3.18) and (5.17), the kernels K_ε and K being replaced respectively by $\tilde{K}_\varepsilon := K_\varepsilon \circ R^{\mathsf{t}}$ and $\tilde{K} := K \circ R^{\mathsf{t}}$.

We argue as in the proof of estimate (3.22): we consider $f \colon D \to (-\bar{\delta}, \bar{\delta})$ of class C^2 such that (5.19) and (5.20) hold. Moreover, $f(0) = 0$, $\nabla f(0) = 0$, and

$$\partial_i f = \frac{\partial_i \varphi}{\partial_d \varphi} \tag{5.24}$$

$$\partial_{i,j}^2 f = \frac{1}{\partial_d \varphi} \left(\partial_{i,j}^2 \varphi + \partial_i f \, \partial_{j,d}^2 \varphi + \partial_j f \, \partial_{i,d}^2 \varphi + \partial_i f \, \partial_j f \, \partial_{d,d}^2 \varphi \right), \tag{5.25}$$

where, for $i, j = 1, \ldots, d - 1$, ∂_i and $\partial_{i,j}^2$ denote respectively the i-th partial derivative and the second order partial derivative w.r.t. the i-th and j-th variable.

Let us now fix ε and δ such that $0 < \varepsilon < \delta < \bar{\delta}$. We write $\varepsilon^{-1} H_\varepsilon$ as

$$\frac{1}{\varepsilon} H_\varepsilon(E, 0) = I_\varepsilon^0 + I_\varepsilon^\infty,$$

where

$$I_\varepsilon^0 := -\frac{1}{\varepsilon} \int_C K_\varepsilon(y) \tilde{\chi}_E(y) \mathrm{d}y, \qquad I_\varepsilon^\infty := -\frac{1}{\varepsilon} \int_{C^c} K_\varepsilon(y) \tilde{\chi}_E(y) \mathrm{d}y,$$

and $C := C_{e_d}(0, \delta)$ for short. The former of the two integrals takes into account the interactions with points that are close to 0, and, when ε is small, we expect it to approximate the anisotropic mean curvature. The second terms, instead, encodes the contribution given by a region far away from 0. Note that the integral defining I_ε^0 has to be understood in the principal value sense. As suggested by this heuristic, we write

$$\left| \frac{1}{\varepsilon} H_\varepsilon(E, 0) - H_0(\Sigma, 0) \right| \leq \left| I_\varepsilon^0 - H_0(\Sigma, 0) \right| + \left| I_\varepsilon^\infty \right|. \tag{5.26}$$

We may easily bound the second summand on the right-hand side by means of (5.11): for all $\alpha < s$ there exists $q_\infty > 1$ such that

$$|I_\varepsilon^\infty| \leq \frac{1}{\varepsilon} \int_{B\left(0, \frac{\delta}{\varepsilon}\right)^c} K(y) \mathrm{d}y \leq \frac{1}{\delta} \left(\frac{\varepsilon}{\delta} \right)^\alpha \quad \text{whenever } q_\infty \varepsilon < \delta. \tag{5.27}$$

To deal with the difference between I_ε^0 and $H_0(\Sigma, 0)$, we rewrite both the terms in a more convenient way. We introduce the function $f_\varepsilon(z) := f(\varepsilon z)/\varepsilon$, and we notice that

$$I_\varepsilon^0 = \frac{1}{\varepsilon} \int_{e_d^\perp \cap B(0, \delta)} \left[\int_{-\delta}^{f(z)} K_\varepsilon(z + t e_d) \mathrm{d}t - \int_{f(z)}^{\delta} K_\varepsilon(z + t e_d) \mathrm{d}t \right] \mathrm{d}\mathscr{H}^{d-1}(z)$$

$$= \frac{1}{\varepsilon} \int_{e_d^\perp \cap B\left(0, \frac{\delta}{\varepsilon}\right)} \int_{-f_\varepsilon(-z)}^{f_\varepsilon(z)} K(z + t e_d) \mathrm{d}t \mathrm{d}\mathscr{H}^{d-1}(z).$$

Thanks to the regularity of f, for all $z \in D$ we have

$$f_\varepsilon(z) = \frac{\varepsilon}{2} \nabla^2 f(\varepsilon \vartheta z) z \cdot z \quad \text{for some } \vartheta \in (0,1) \, ,$$

so that, when t ranges between $-f_\varepsilon(-z)$ and $f_\varepsilon(z)$, it holds

$$|t| \leq \frac{\varepsilon}{2} \left\| \nabla^2 f \right\|_{L^\infty(D)} |z|^2 . \tag{5.28}$$

Further, we remark that

$$H_0(\Sigma, 0) = \int_{e_d^\perp} K(z) \nabla^2 f(0) z \cdot z \mathrm{d}\mathscr{H}^{d-1}(z).$$

Indeed, $H_0(\Sigma, 0)$ may be cast as in (5.17), and by (5.24), (5.25) we have

$$-\frac{1}{|\partial_d \varphi(0)|} \nabla^2 \varphi(0) z \cdot z = \nabla^2 f(0) z \cdot z \quad \text{when } z \in e_d^\perp .$$

We introduce the quantity

$$I_\varepsilon^1 := \frac{1}{\varepsilon} \int_{e_d^\perp \cap B\left(0, \frac{\delta}{\varepsilon}\right)} K(z) \left[f_\varepsilon(z) + f_\varepsilon(-z) \right] \mathrm{d}\mathscr{H}^{d-1}(z)$$

to help us quantify the difference between I_ε^0 and $H_0(\Sigma, 0)$. By the triangular inequality,

$$\left| I_\varepsilon^0 - H_0(\Sigma, 0) \right| \leq \left| I_\varepsilon^0 - I_\varepsilon^1 \right| + \left| I_\varepsilon^1 - H_0(\Sigma, 0) \right| , \tag{5.29}$$

and we estimate the summands in the right-hand side separately.

We observe that

$$\left| I_\varepsilon^0 - I_\varepsilon^1 \right| \leq \frac{1}{\varepsilon} \int_{e_d^\perp \cap B\left(0, \frac{\delta}{\varepsilon}\right)} \left| \int_{-f_\varepsilon(-z)}^{f_\varepsilon(z)} \left[K(z + t e_d) - K(z) \right] \mathrm{d}t \right| \mathrm{d}\mathscr{H}^{d-1}(z). \tag{5.30}$$

Since $K \in W^{1,1}(B(0,r)^c)$ for all $r > 0$, by Theorem 5.3, for $\mathscr{H}^{d-1}$-a.e. $z \in e_d^\perp$ it holds

$$K(z + t e_d) - K(z) = \int_0^t \partial_d K(z + s e_d) \mathrm{d}s,$$

and this, combined with (5.28), implies that

$$|K(z + t e_d) - K(z)| \leq \int_{-\frac{\varepsilon}{2} \|\nabla^2 f\|_{L^\infty(D)} |z|^2}^{\frac{\varepsilon}{2} \|\nabla^2 f\|_{L^\infty(D)} |z|^2} |\nabla K(z + s e_d)| \mathrm{d}s.$$

By plugging this inequality in (5.30), we infer

$$|I_\varepsilon^0 - I_\varepsilon^1|$$

$$\leq \left\| \nabla^2 f \right\|_{L^\infty(D)} \int_{e_d^\perp \cap B\left(0, \frac{\delta}{\varepsilon}\right)} |z|^2 \int_{-\frac{\varepsilon}{2} \|\nabla^2 f\|_{L^\infty(D)} |z|^2}^{\frac{\varepsilon}{2} \|\nabla^2 f\|_{L^\infty(D)} |z|^2} |\nabla K(z + s e_d)| \mathrm{d}s \mathrm{d}\mathscr{H}^{d-1}(z)$$

$$\leq \frac{\delta}{\varepsilon} \left\| \nabla^2 f \right\|_{L^\infty(D)} \int_{Q(\varepsilon)} |y| \, |\nabla K(y)| \, \mathrm{d}y,$$

where $Q(\varepsilon) := Q_{\varepsilon \|\nabla^2 f\|_{L^\infty(D)}}(e_d)$. In view of (5.8), there exists $\eta \in (0, \bar{\delta})$ such that

$$\left| I_\varepsilon^0 - I_\varepsilon^1 \right| \leq (b_0 + 1) \left\| \nabla^2 f \right\|_{L^\infty(D)}^2 \delta \quad \text{whenever } \varepsilon < \eta. \tag{5.31}$$

Then, we have

$$\left| I_\varepsilon^1 - H_0(\Sigma, 0) \right| \leq O_f(\delta) \int_{e_d^\perp \cap B(0, \frac{\delta}{\varepsilon})} K(z) |z|^2 \, d\mathscr{H}^{d-1}(z)$$
$$+ \left| \nabla^2 f(0) \right| \int_{e_d^\perp \cap B(0, \frac{\delta}{\varepsilon})^c} K(z) |z|^2 \, d\mathscr{H}^{d-1}(z),$$

with O_f defined in (5.23). As a consequence of (5.13), for all $\beta < s$, there exists $q_1 > 0$ such that, if $q_1 \varepsilon < \delta$, then

$$\int_{e_d^\perp \cap B(0, \frac{\delta}{\varepsilon})^c} K(z) |z|^2 \, d\mathscr{H}^{d-1}(z) \leq \left(\frac{\varepsilon}{\delta} \right)^\beta;$$

Hence, bearing in mind (5.12), we infer

$$\left| I_\varepsilon^1 - H_0(\Sigma, 0) \right| \leq a_0 \, O_f(\delta) + \left| \nabla^2 f(0) \right| \left(\frac{\varepsilon}{\delta} \right)^\beta \quad \text{whenever } q_1 \varepsilon < \delta. \tag{5.32}$$

Summing up, we can now accomplish the proof of (5.21) by choosing the parameters q and $\bar{\varepsilon}$ in such a way that the estimates above are fulfilled at once. Inequalities (5.26) and (5.29) get

$$\left| \frac{1}{\varepsilon} H_\varepsilon(E, 0) - H_0(\Sigma, 0) \right| \leq \left| I_\varepsilon^0 - I_\varepsilon^1 \right| + \left| I_\varepsilon^1 - H_0(\Sigma, 0) \right| + \left| I_\varepsilon^\infty \right|,$$

therefore, if we set $q := \max\{q_\infty, q_1\} > 1$ with q_∞ and q_1 as above, both (5.27) and (5.32) hold for all $\varepsilon, \delta > 0$ such that $q\varepsilon < \delta < \bar{\delta}$. Besides, if we pick $\bar{\varepsilon} := \min\left\{ \eta, \bar{\delta}/q \right\}$, (5.31) is satisfied too whenever $\varepsilon < \bar{\varepsilon}$.

Once rate estimate (5.21) is on hand, the pointwise convergence follows straightforwardly. Indeed, for $\gamma \in \left(0, \alpha/(\alpha + 1) \right)$, $q\varepsilon^\gamma > q\varepsilon$ and $\mathrm{Err}(\varepsilon, q\varepsilon^\gamma)$ vanishes as ε tends to 0. $\qquad \square$

REMARK 5.8
Assumptions (5.5) and (5.9) are not exploited in the proof of Lemma 5.7. We shall utilize them to establish Proposition 5.20.

When Σ is a smooth, compact hypersurface, uniform convergence of the rescaled nonlocal curvatures stems from a refinement of the previous error estimate.

THEOREM 5.9
Let K satisfy all the assumptions in Section 5.1 and let $E \subset \mathbb{R}^d$ be a set whose topological boundary Σ is compact and of class C^2. Then,

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} H_\varepsilon(E, x) = H_0(\Sigma, x) \quad \text{uniformly in } x \in \Sigma.$$

Proof. By definition of C^2 hypersurface, for any $x \in \Sigma$ there exist an open set $U_x \subset \mathbb{R}^d$ and a C^2 function $\varphi_x \colon U_x \to \mathbb{R}$ such that

$$\Sigma \cap U_x = \{\, y \in U_x : \varphi_x(y) = 0 \,\}.$$

For all $x \in \Sigma$, we also choose another open set V_x, in such a way that it is compactly contained in U_x and that $\inf_{y \in V_x} |\nabla \varphi(y)| > 0$. Observe that, by compactness, Σ admits a finite cover $\{\, V_1, \dots, V_L \,\}$ extracted from the collection $\{\, V_x \,\}_{x \in \Sigma}$.

If $x \in \Sigma$ belongs to V_ℓ for some ℓ, we choose $\bar{\delta}_x$ so small that the closure of the cylinder $C_{\hat{n}_x}(x, \bar{\delta}_x)$ is contained in V_ℓ, where $\hat{n}_x$ is the outer unit normal to Σ at x. We now apply Proposition 5.7 at the point x selecting $\bar{\delta} = \bar{\delta}_x$. We get, for all $\alpha, \beta \in (0, s)$, the existence of $q > 1$ and $\bar{\varepsilon} \in (0,1)$ such that $q\bar{\varepsilon} \leq \bar{\delta}_x$ and that for all $\gamma \in \big(0, \alpha/(\alpha+1)\big)$ and all $\varepsilon \in (0, \bar{\varepsilon})$ it holds

$$\left| \frac{1}{\varepsilon} H_\varepsilon(E, x) - H_0(\Sigma, x) \right| \leq \mathrm{Err}(\varepsilon, q\varepsilon^\gamma)$$

$$\leq c \Big(\varepsilon^{\alpha - \gamma(1+\alpha)} + \big\| \nabla^2 f_x \big\|_{L^\infty(\hat{n}_x^\perp \cap B(0, \bar{\delta}_x))} \varepsilon^\gamma + O_{f_x}(q\varepsilon^\gamma) + \big| \nabla^2 f_x(0) \big| \varepsilon^{(1-\gamma)\beta} \Big),$$

where $c := c(\alpha, \beta, a_0, b_0) > 0$ and f_x is the implicit function such that (5.19) holds in $C_{\hat{n}_x}(x, \bar{\delta}_x)$.

Recall that $x \in V_\ell$ and let us denote by φ_ℓ a C^2 function φ such that $\Sigma \cap V_\ell = \{\, y \in V_\ell : \varphi(y) = 0 \,\}$. By formulas (5.24) and (5.25), one may show that $\big| \nabla^2 f_x(0) \big|$ and $\big\| \nabla^2 f_x \big\|_{L^\infty(\hat{n}_x^\perp \cap B(0, \bar{\delta}_x))}$ are bounded above by quantities that depend only on the infimum of $|\nabla \varphi|$ on V_ℓ, which is strictly positive, and on the values of $\nabla \varphi_\ell$ and $\nabla^2 \varphi_\ell$ in V_ℓ. In turn, the latter are bounded in V_ℓ, because φ_ℓ is C^2. Similarly, $O_{f_x}(\delta)$ is smaller than some $O_\ell(\delta)$ that is defined in terms of φ_ℓ and of its derivatives, and that vanishes as $\delta \to 0^+$.

In conclusion, since there is a finite number of φ_ℓ, it suffices to maximise w.r.t. the parameter ℓ to obtain a bound on $\big| \varepsilon^{-1} H_\varepsilon(E, x) - H_0(\Sigma, x) \big|$ that is uniform in $x \in \Sigma$ and that vanishes when $\varepsilon \to 0^+$. $\qquad\square$

5.3 Level set formulations of motions by curvature

We now revise the approach to mean curvature motions *via* level set formulations. We include the existence and uniqueness results for the related viscosity solutions and recall their connections with the notion of geometric barriers.

We consider the geometric evolutions driven by the curvatures $\varepsilon^{-1} H_\varepsilon$ and H_0 defined respectively in (5.1) and (5.17): if $t \mapsto E(t)$ is an evolution of sets, we formally prescribe that

$$\partial_t x(t) \cdot \hat{n} = -\frac{1}{\varepsilon} H_\varepsilon(E(t), x(t)), \qquad \partial_t x(t) \cdot \hat{n} = -H_0(E(t), x(t)), \tag{5.33}$$

where $\hat{n}$ is the outer unit normal to $\partial E(t)$ at the point $x(t)$. We accompany these equations with an initial datum E_0, which we assume to be a bounded set.

To cast the geometric laws (5.33) in the level set form, we interpret the initial datum E_0 as the 0 superlevel set of a certain function $u_0 \colon \mathbb{R}^d \to \mathbb{R}$, that is, $E_0 = \{\, x : u_0(x) \geq 0 \,\}$ and $\partial E_0 = \{\, x : u_0(x) = 0 \,\}$. Hereafter, we assume that

$$u_0 \colon \mathbb{R}^d \to \mathbb{R} \text{ is Lipschitz continuous and constant outside a compact set } C \subset \mathbb{R}^d. \tag{5.34}$$

Then, we consider the Cauchy's problems:

$$\begin{cases} \partial_t u(t,x) + \frac{|\nabla u(t,x)|}{\varepsilon} H_\varepsilon(\{\, y : u(t,y) \geq u(t,x) \,\}, x) = 0 & (t,x) \in [0,+\infty) \times \mathbb{R}^d \\ u(0,x) = u_0(x) & x \in \mathbb{R}^d \end{cases} \quad (5.35)$$

$$\begin{cases} \partial_t u(t,x) - \mathrm{tr}\left(M_K\left(\widehat{\nabla u(t,x)} \right) \nabla^2 u(t,x) \right) = 0 & (t,x) \in [0,\infty) \times \mathbb{R}^d \\ u(0,x) = u_0(x) & x \in \mathbb{R}^d \end{cases} \quad (5.36)$$

Observe that

$$|\nabla u(x)|\, H_0(\{\, y : u(y) = u(x) \,\}, x) = -\mathrm{tr}\left(M_K\left(\widehat{\nabla u(x)} \right) \nabla^2 u(x) \right).$$

Parabolic equations defined by operators of this form are well studied. We shall consider solutions in the following viscosity sense (see [DLFM08] and references therein):

DEFINITION 5.10 (Solution to the limit problem)
A function $u \colon [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ is a viscosity subsolution*(resp.* supersolution*) to Cauchy's problem* (5.36) *if it is locally bounded, upper semicontinuous (resp. lower semicontinuous function), and satisfies the following:*

(i) $u(0,x) \leq u_0(x)$ *for all $x \in \mathbb{R}^d$, (resp. $u(0,x) \geq u_0(x)$);*

(ii)   *for all $(t,x) \in (0,+\infty) \times \mathbb{R}^d$ and for all $\varphi \in C^2([0,+\infty) \times \mathbb{R}^d)$ such that $u - \varphi$ has a maximum at (t,x) (resp. a minimum at (t,x)) it holds*

$$\partial_t \varphi(t,x) \leq 0 \quad (\textit{resp. } \partial_t \varphi(t,x) \geq 0) \qquad \textit{when } \nabla \varphi(t,x) = 0 \textit{ and } \nabla^2 \varphi(t,x) = 0$$

or

$$\partial_t \varphi(t,x) - \mathrm{tr}\left(M_K\left(\widehat{\nabla \varphi(t,x)} \right) \nabla^2 \varphi(t,x) \right) \leq 0 \,(\textit{resp. } \geq 0) \quad \textit{otherwise.}$$

A continuous function $u \colon [0,+\infty) \times \mathbb{R}^d \to \mathbb{R}$ is a viscosity solution *to* (5.36) *if it is both a viscosity sub- and supersolution.*

As for existence, note that the function

$$F_0 \colon \quad \begin{array}{ccc} \mathbb{R}^d \setminus \{0\} \times \mathrm{Sym}(d) & \longrightarrow & \mathbb{R} \\ (p,X) & \longmapsto & -\mathrm{tr}\left(M_K\left(\hat{p} \right) X \right) \end{array}$$

that defines the problem (5.36) satisfies the following:

(i)   it is continuous;

(ii)   it is geometric, that is for all $\lambda > 0$, $\sigma \in \mathbb{R}$, $p \in \mathbb{R}^d \setminus \{\, 0 \,\}$ and $X \in \mathrm{Sym}(d)$ it holds $F_0(\lambda p, \lambda X + \sigma p \otimes p) = \lambda F_0(p, X)$.

(iii) it is degenerate elliptic, that is $F_0(p, X) \geq F_0(p, Y)$ for all $p \in \mathbb{R}^d \setminus \{0\}$ and $X, Y \in \mathrm{Sym}(d)$ such that $X \leq Y$.

Since the classic by Y. Chen, Y. Giga, and S. Goto [CGG91], it is well known that that these three conditions grant existence and uniqueness of a viscosity solution:

THEOREM 5.11

Assume that (5.34) holds. Then, Cauchy's problem (5.36) admits a unique bounded Lipschitz viscosity solution in $[0, +\infty) \times \mathbb{R}^d$, which is constant in $\mathbb{R}^d \setminus C$, for some compact set $C \subset \mathbb{R}^d$.

In second place, we recall the notion of viscosity solution for the nonlocal problems. which goes back to the work [Sle03], see also [Imb09, DLFM08, CMP15].

DEFINITION 5.12 (Solution to the rescaled problems)

A locally bounded, upper semicontinuous function (resp. lower semicontinuous) $u_\varepsilon \colon [0, +\infty) \times \mathbb{R}^d \to \mathbb{R}$ is a viscosity subsolution (resp. supersolution) to the problem (5.35) if

(i) $u_\varepsilon(0, x) \leq u_0(x)$ *for all* $x \in \mathbb{R}^d$ *(resp.* $u_\varepsilon(0, x) \geq u_0(x)$*);*

(ii)   *for all* $(t, x) \in (0, +\infty) \times \mathbb{R}^d$ *and for all* $\varphi \in C^2([0, +\infty) \times \mathbb{R}^d)$ *such that* $u_\varepsilon - \varphi$ *has a maximum at* (t, x) *(resp. has a minimum at* (t, x)*), it holds*

$$\partial_t \varphi(t, x) \leq 0 \quad (\text{resp. } \partial_t \varphi(t, x) \geq 0) \qquad \text{when } \nabla \varphi(t, x) = 0,$$

or

$$\partial_t \varphi(t, x) + \frac{|\varphi(t, x)|}{\varepsilon} H_\varepsilon(\{ y : \varphi(t, y) \geq \varphi(t, x) \}, x) \leq 0$$

$$\left(\text{resp. } \partial_t \varphi(t, x) + \frac{|\varphi(t, x)|}{\varepsilon} H_\varepsilon(\{ y : \varphi(t, y) > \varphi(t, x) \}, x) \geq 0\right) \quad \text{otherwise.}$$

A continuous function $u_\varepsilon \colon [0, +\infty) \times \mathbb{R}^d \to \mathbb{R}$ *is a viscosity solution to (5.35) if it is both a viscosity sub- and supersolution.*

Using this concept of solution, comparison results and well posedness of (5.35) are available, as proved in a very general setting by A. Chambolle, M. Morini, and M. Ponsiglione in [CMP15] (see also the paper by C. Imbert [Imb09]).

THEOREM 5.13 (Comparison principle and existence of solutions to the nonlocal problem)

If the kernel K is as in Section 5.1 and (5.34) holds, then, for all $\varepsilon > 0$, if $v_\varepsilon, w_\varepsilon \colon [0, +\infty) \times \mathbb{R}^d \to \mathbb{R}$ are respectively a sub- and a supersolution to (5.35), then $v_\varepsilon(t, x) \leq w_\varepsilon(t, x)$ for all $(t, x) \in [0, +\infty) \times \mathbb{R}^d$.

Moreover, (5.35) admits a unique bounded, Lipschitz viscosity solution in $[0, +\infty) \times \mathbb{R}^d$, which is constant in $\mathbb{R}^d \setminus C$, for some compact set $C \subset \mathbb{R}^d$.

Summing up, we know that, for every initial datum u_0 fulfilling (5.34), there exists a unique viscosity solution u_ε to (5.36) and a unique viscosity solution u to (5.36). We define the related *level set flows* as follows: for every $\lambda \in \mathbb{R}$, we put

$$E^>_{\varepsilon, \lambda}(t) = \left\{ x \in \mathbb{R}^d : u_\varepsilon(t, x) > \lambda \right\}, \quad E^\geq_{\varepsilon, \lambda}(t) = \left\{ x \in \mathbb{R}^d : u_\varepsilon(t, x) \geq \lambda \right\}, \quad (5.37)$$

$$E^>_\lambda(t) = \left\{ x \in \mathbb{R}^d : u(t, x) > \lambda \right\}, \quad E^\geq_\lambda(t) = \left\{ x \in \mathbb{R}^d : u(t, x) \geq \lambda \right\}. \quad (5.38)$$

It is well known that, as long as they are smooth, these families are solutions to the geometric flows (5.33) with initial data respectively $\{ u_0 > \lambda \}$ and $\{ u_0 \geq \lambda \}$.

Geometric evolutions can be formulated as PDEs involving the distance function from the moving front, see for instance the survey [Amb00] by L. Ambrosio. In the following

definitions, we use it to express a regularity property both in time and space for sets (see (ii) below) evolving according to a generic geometric law.

DEFINITION 5.14
Let $0 \leq t_0 < t_1 < +\infty$. We say that the evolution of sets $[t_0, t_1] \ni t \mapsto D(t)$ is a geometric subsolution (resp. supersolution) to the flow associated with the curvature functional H if

(i) $D(t)$ *is closed and $\partial D(t)$ is compact for all $t \in [t_0, t_1]$;*

(ii) *there exists an open set $U \subset \mathbb{R}^d$ such that the distance function $(t, x) \mapsto \mathrm{d}_{D(t)}(x)$ is of class C^∞ in $[t_0, t_1] \times U$ and $\partial D(t) \subset U$ for all $t \in [t_0, t_1]$;*

(iii) *for all $t \in (t_0, t_1)$ and $x(t) \in \partial D(t)$, it holds*

$$\partial_t x(t) \cdot \hat{n} \leq -H(D(t), x(t)) \qquad (\text{resp. } \partial_t x(t) \cdot \hat{n} \geq -H(D(t), x(t)), \quad (5.39)$$

where $\hat{n}$ is the outer unit normal to $D(t)$ at x.

When strict inequalities hold, $D(t)$ is called strict geometric subsolution (resp. supersolution)*.*

REMARK 5.15
For any $p \in \mathbb{R}^d \setminus \{0\}$ and $X \in \mathrm{Sym}(d)$, by (5.12) we have

$$|\mathrm{tr}(M_K(\hat{p})X)| = \frac{1}{2} \left| \int_{\hat{p}^\perp} K(z) \, Xz \cdot z \mathrm{d}\mathscr{H}^{d-1}(z) \right| \leq \frac{a_0}{2} |X| ,$$

This ensures that geometric sub- and supersolution for the flow driven by H_0 exist.

Next, we remind the notion of geometric barriers w.r.t. these smooth evolutions:

DEFINITION 5.16
Let $T > 0$ and $\mathcal{F}^-$ and $\mathcal{F}^+$ be, respectively, the families of strict geometric sub- and supersolution to the flow associated with some curvature functional H, as in Definition 5.14.

(i) *We say that the evolution of sets $[0, T] \ni t \mapsto E(t)$ is an* outer barrier *w.r.t. $\mathcal{F}^-$ (resp. $\mathcal{F}^+$) if whenever $[t_0, t_1] \subset [0, T]$ and $[t_0, t_1] \ni t \mapsto D(t)$ is a smooth strict subsolution (resp. $F(t)$ is a smooth strict supersolution) such that $D(t_0) \subset E(t_0)$, then we get $D(t_1) \subset E(t_1)$ (resp. such that $F(t_0) \subset E(t_0)$, then we get $F(t_1) \subset E(t_1)$).*

(ii) *Analogously, $[0, T] \ni t \mapsto E(t)$ is an* inner barrier *w.r.t. the family $\mathcal{F}^-$ (resp. $\mathcal{F}^+$) if whenever $[t_0, t_1] \subset [0, T]$ and $[t_0, t_1] \ni t \mapsto D(t)$ is a smooth strict subsolution (resp. supersolution) such that $E(t_0) \subset \mathrm{int}(D(t_0))$, then $E(t_1) \subset \mathrm{int}(D(t_1))$ (resp. $E(t_0) \subset \mathrm{int}(F(t_0))$, then $E(t_1) \subset \mathrm{int}(F(t_1))$).*

We are interested in barriers is motivated by the next results, which show that they are comparable with level sets flows. The case of the local curvatures was studied in [BN98, Theorem 3.2] by G. Bellettini and M. Novaga. For further reading about barriers for more general local motions, we refer also to [BN00]. In the nonlocal setting, recently an analogous comparison principle has been established in [CDNV19, Proposition A.10] by A. Cesaroni, S. Dipierro, M. Novaga, and E. Valdinoci.

THEOREM 5.17

Let u be the unique solution to (5.36) *with initial datum u_0 as in* (5.34). *Let $E_\lambda^>$, $E_\lambda^\geq$ the sets defined in* (5.38).

(i) *The map $[0, T] \ni t \mapsto E_\lambda^>(t)$ is the minimal outer barrier for the family of strict geometric subsolutions associated with H_0, that is $E_\lambda^>(t)$ is an outer barrier and $E_\lambda^>(t) \subset E(t)$ for any other outer barrier $E(t)$.*

(ii) *The map $[0, T] \ni t \mapsto E_\lambda^\geq(t)$ is the maximal inner barrier for the family of geometric strict supersolutions associated with H_0, that is $E_\lambda^\geq(t)$ is an inner barrier and $E(t) \subset E_\lambda^\geq(t)$ for any other inner barrier $E(t)$.*

PROPOSITION 5.18

Let $u_\varepsilon \colon [0, +\infty) \times \mathbb{R}^d \to \mathbb{R}$ be the viscosity solution to (5.35) *with initial datum u_0 as in* (5.34). *Let $E_{\varepsilon,\lambda}^>$, $E_{\varepsilon,\lambda}^\geq$ be as in* (5.37). *Then, the evolutions $t \mapsto E_{\varepsilon,\lambda}^>(t)$ and $t \mapsto E_{\varepsilon,\lambda}^\geq(t)$ are, respectively, an outer barrier w.r.t geometric strict subsolutions to* (5.35) *and an inner barrier w.r.t geometric strict supersolutions to* (5.35).

5.4 Convergence of the solutions to the nonlocal flows

Now that we clarified what class of solutions for geometric motions we address to, we can formulate our main result:

THEOREM 5.19

Let K satisfy all the assumptions in Section 5.1. Let $u_0 \colon \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz continuous function that is constant outside a compact set. Let $u_\varepsilon, u \colon [0, +\infty) \times \mathbb{R}^d \to \mathbb{R}$ be respectively the unique continuous viscosity solution to (5.35) *and* (5.36) *with initial datum u_0. Then*

$$\lim_{\varepsilon \to 0^+} u_\varepsilon(t, x) = u(t, x) \quad \text{locally uniformly in } [0, +\infty) \times \mathbb{R}^d.$$

An analogous result was proved by F. Da Lio, N. Forcadel, and R. Monneau [DLFM08, Theorem 1.4]. Though the authors were concerned with a problem which resembles much ours, their assumptions on the interaction kernel were dictated by a model of dislocation dynamics and are different from the ones we consider. Indeed, they assumed K to be bounded in a neighbourhood of 0, hence nonsingular, and to decay as $|x|^{-d-1}$ at infinity. Perforce, their scaling of the curvature,

$$\frac{1}{\varepsilon \log \varepsilon} H_{K_\varepsilon},$$

does not coincide with ours. However, the proofs are partly similar. Indeed, as [DLFM08], we firstly prove a compactness result for the family of solutions to Cauchy's problems (5.35), but, in contrast with the work of Da Lio, Forcadel, and Monneau, we show that the cluster point must be unique and coincide with the viscosity solution to (5.36) by means of De Giorgi's barriers.

Let us first prove the compactness result, which is not completely novel. We include the proof because it has not been stated explicitly yet for our setting.

PROPOSITION 5.20

Assume that $u_0 \colon \mathbb{R}^d \to \mathbb{R}$ is as in (5.34) and let u_ε be the unique continuous viscosity solution to (5.35). Then,

$$|u_\varepsilon(t,x) - u_\varepsilon(t,y)| \le \|\nabla u_0\|_{L^\infty(\mathbb{R}^d)} |x - y| \quad \text{for all } t \in [0,T] \text{ and } x, y \in \mathbb{R}^d, \quad (5.40)$$

and there exists a constant $c > 0$ independent of ε such that

$$|u_\varepsilon(t,x) - u_\varepsilon(s,x)| \le \|\nabla u_0\|_{L^\infty(\mathbb{R}^d)} \sqrt{c\,|t - s|} \quad \text{for all } t, s \in [0,T] \text{ and } x \in \mathbb{R}^d. \quad (5.41)$$

Proof. By standard arguments [DLFM08, CDNV19], one can see that the Lipschitz continuity of the datum and the comparison principle yield the equi-Lipschitz estimate (5.40).

As for the equi-Hölder continuity, we adapt the strategy of Section 5 in [DLFM08]. Some care is needed because of the possible singularity of our interaction kernel.

For $\eta > 0$ and $x \in \mathbb{R}^d$ fixed, we consider

$$\varphi(t,y) = Lt + A\sqrt{|y - x|^2 + \eta^2} + u_0(x), \quad (5.42)$$

where we set $A \coloneqq \|\nabla u_0\|_{L^\infty(\mathbb{R}^d)}$. We claim that, for $L > 0$ sufficiently large, φ is a supersolution to (5.35) for any $\varepsilon \in (0,1)$.

We firstly note that $\varphi(0,y) \ge u_0(y)$ thanks to the Lipschitz continuity of u_0. Also, we observe that, for any $y \in \mathbb{R}^d$,

$$\left\{ z \in \mathbb{R}^d : \varphi(t,z) \ge \varphi(t,y) \right\} = B(x, |y - x|)^{\mathsf{c}},$$

whence, to show that φ is a supersolution, it suffices to find L so large that

$$\frac{L}{A} \ge \frac{|y - x|}{\varepsilon\sqrt{|y - x|^2 + \eta^2}} H_\varepsilon(B(x, |y - x|), y) \quad \text{for all } y \in \mathbb{R}^d \text{ and } \varepsilon \in (0,1).$$

Recalling that the nonlocal curvature is invariant under translations, if we set $e \coloneqq \widehat{y - x}$ and $r \coloneqq |y - x|$, we have that the last inequality holds if and only if

$$\frac{L}{A} \ge \frac{r}{\varepsilon\sqrt{r^2 + \eta^2}} H_\varepsilon(B(-re, r), 0) \quad \text{for all } r > 0, e \in \mathbb{S}^{d-1}, \text{ and } \varepsilon \in (0,1). \quad (5.43)$$

Hence, we want to prove that there exists $L_0 \coloneqq L_0(\eta) > 0$ such that

$$\sup_{r > 0,\, e \in \mathbb{S}^{d-1}} \sup_{\varepsilon \in (0,1)} \frac{r}{\varepsilon\sqrt{r^2 + \eta^2}} H_\varepsilon(B(-re, r), 0) \le L_0. \quad (5.44)$$

This clearly gets (5.43) for $L = AL_0$.

To recover (5.44), we use inequality (3.22). Retracing its proof, we see that, when $E = B(-re, r)$ and $x = 0$, we can choose $\lambda = \lambda_0/r$ for some $\lambda_0 > 0$ and $\bar\delta = r/2$, so that we get

$$0 \le H_\varepsilon(B(-re, r), 0) \le \int_{Q_{\frac{\lambda_0 \varepsilon}{r}}(e)} K(y)\mathrm{d}y + \int_{B\left(0, \frac{r}{2\varepsilon}\right)^{\mathsf{c}}} K(y)\mathrm{d}y.$$

It follows that

$$\frac{r}{\varepsilon\sqrt{r^2+\eta^2}}H_\varepsilon(B(-re,r),0) \leq \frac{r}{\varepsilon\eta}\left[\int_{Q_{\frac{\lambda_0\varepsilon}{r}}(e)} K(y)\mathrm{d}y + \int_{B\left(0,\frac{r}{2\varepsilon}\right)^{\mathrm{c}}} K(y)\mathrm{d}y\right].$$

We exploit the hypotheses on K: by (5.5), (5.9), (5.7), and (5.11), there exist $\lambda, \Lambda > 0$ with the following properties:

(i) $\lambda < \Lambda$;

(ii)   if $r < \lambda\varepsilon$, then

$$\frac{r}{\varepsilon}\int_{Q_{\frac{\lambda_0\varepsilon}{r}}(e)} K(y)\mathrm{d}y \leq \frac{1}{2} \quad\text{and}\quad \frac{r}{\varepsilon}\int_{B\left(0,\frac{r}{2\varepsilon}\right)^{\mathrm{c}}} K(y)\mathrm{d}y \leq \frac{1}{2}$$

and, consequently,

$$\frac{r}{\varepsilon\sqrt{r^2+\eta^2}}H_\varepsilon(B(-re,r),0) \leq \frac{1}{\eta}; \tag{5.45}$$

(iii)   if $r > \Lambda\varepsilon$, then

$$\frac{r}{\varepsilon}\int_{Q_{\frac{\lambda_0\varepsilon}{r}}(e)} K(y)\mathrm{d}y \leq a_0 + \frac{1}{2} \quad\text{and}\quad \frac{r}{\varepsilon}\int_{B\left(0,\frac{r}{2\varepsilon}\right)^{\mathrm{c}}} K(y)\mathrm{d}y \leq \frac{1}{2}$$

and, accordingly,

$$\frac{r}{\varepsilon\sqrt{r^2+\eta^2}}H_\varepsilon(B(-re,r),0) \leq \frac{a_0+1}{\eta}. \tag{5.46}$$

Now, only the case $\lambda\varepsilon \leq r \leq \Lambda\varepsilon$ is left to discuss. In this intermediate regime, recalling (5.6), we easily obtain

$$\frac{r}{\varepsilon\sqrt{r^2+\eta^2}}H_\varepsilon(B(-re,r),0) \leq \frac{\Lambda}{\eta}\left(c + \int_{B\left(0,\frac{\lambda}{2}\right)^{\mathrm{c}}} K(y)\mathrm{d}y\right), \tag{5.47}$$

with $c > 0$ depending only on λ.

By the whole of (5.45), (5.46), and (5.47), we infer that there exists a constant $c := c(a_0, \lambda, \Lambda) > 0$ such that

$$\sup_{r>0,\,e\in\mathbb{S}^{d-1}}\sup_{\varepsilon\in(0,1)} \frac{r}{\varepsilon\sqrt{r^2+\eta^2}}H_\varepsilon(B(-re,r),0) \leq \frac{c}{\eta}$$

and, thus, (5.43) holds if we pick $L = cA/\eta$.

Summing up, we proved that, for any fixed $x \in \mathbb{R}^d$, the function

$$\varphi(t,y) = A\left(\frac{c}{\eta}t + \sqrt{|y-x|^2+\eta^2}\right) + u_0(x)$$

is a supersolution to (5.35) for any $\varepsilon > 0$. A similar reasoning shows that there exists $c = c(a_0, \lambda, \Lambda)$ such that the function

$$\psi(t, y) := -A\left(\frac{c}{\eta}t + \sqrt{|y - x|^2 + \eta^2}\right) + u_0(x),$$

is a subsolution to (5.35) for all $x \in \mathbb{R}^d$ and $\varepsilon > 0$.

We apply the comparison principle in Theorem 5.13 to u_ε, φ, and ψ. This yields that, for all $(t, x) \in [0, T] \times \mathbb{R}^d$ and all $\eta > 0$,

$$|u_\varepsilon(t, x) - u_0(x)| \leq \|\nabla u_0\|_{L^\infty(\mathbb{R}^d)}\left(\frac{c}{\eta}t + \eta\right).$$

Being η arbitrary, we may set $\eta = \sqrt{ct}$ and obtain

$$|u_\varepsilon(t, x) - u_0(x)| \leq 2\|\nabla u_0\|_{L^\infty(\mathbb{R}^d)}\sqrt{ct}.$$

In turn, this entails (5.41), because the problem (5.35) is invariant w.r.t. translations in time, it admits a unique solution, and $\|\nabla u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R}^d)} \leq \|\nabla u_0\|_{L^\infty(\mathbb{R}^d)}$ for all $t \in [0, T]$. $\quad\square$

Now, we deal with the proof of Theorem 5.19. Thanks to the previous proposition, we know that the family $\{u_\varepsilon\}$ of the solutions to (5.35) is precompact in $C([0, T] \times \mathbb{R}^d)$. It follows that, for some sequence $\{\varepsilon_\ell\}$ and some $v \in C([0, T] \times \mathbb{R}^d)$, u_{ε_ℓ} converges to v locally uniformly as $\ell \to +\infty$ The desired conclusion is achieved as soon as we prove $v = u$, u being the solution to (5.36). In view of the following lemma, we see that the equality may be inferred by certain inclusions of the level sets of u and v:

LEMMA 5.21
Let $f, g\colon \mathbb{R}^d \to \mathbb{R}$ be two continuous functions such that for all $\lambda \in \mathbb{R}$ there hold

$$\left\{x \in \mathbb{R}^d : f(x) > \lambda\right\} \subseteq \left\{x \in \mathbb{R}^d : g(x) \geq \lambda\right\},$$

$$\left\{x \in \mathbb{R}^d : g(x) > \lambda\right\} \subseteq \left\{x \in \mathbb{R}^d : f(x) \geq \lambda\right\}.$$

Then, $f(x) = g(x)$ for all $x \in \mathbb{R}^d$.

Proof. Let $\bar{x} \in \mathbb{R}^d$ and assume that $g(\bar{x}) = \lambda$. Then, for all $\mu > 0$, we have

$$\bar{x} \in \left\{x : g(x) > \lambda - \mu\right\} \subseteq \left\{x : f(x) \geq \lambda - \mu\right\},$$

which yields $f(\bar{x}) \geq \lambda$. If $f(\bar{x}) > \lambda$, then

$$\bar{x} \in \left\{x : f(x) > \lambda + \mu_0\right\} \subseteq \left\{x : g(x) \geq \lambda + \mu_0 > \lambda\right\}$$

some $\mu_0 > 0$, which contradicts the equality $g(\bar{x}) = \lambda$. Thus, it necessarily holds $f(\bar{x}) = \lambda$. $\quad\square$

So, to show that any cluster point v of the family $\{u_\varepsilon\}$ coincides with u, we compare its superlevel sets with the ones of u. To this purpose, we introduce the *set-theoretic upper limits* of the level set flows $t \mapsto E^>_{\varepsilon,\lambda}(t)$, $t \mapsto E^\geq_{\varepsilon,\lambda}(t)$ associated with u_ε (recall (5.37)). We define

$$\tilde{E}^>_\lambda(t) := \bigcap_{\varepsilon < 1} \bigcup_{\eta < \varepsilon} E^>_{\eta,\lambda}(t) \quad \text{and} \quad \tilde{E}^\geq_\lambda(t) := \bigcap_{\varepsilon < 1} \bigcup_{\eta < \varepsilon} E^\geq_{\eta,\lambda}(t). \tag{5.48}$$

REMARK 5.22

It is readily shown that, for any $\bar{\varepsilon} < 1$,

$$\tilde{E}_\lambda^>(t) = \bigcap_{\varepsilon < \bar{\varepsilon}} \bigcup_{\eta < \varepsilon} E_{\eta,\lambda}^>(t) \quad \text{and} \quad \tilde{E}_\lambda^\geq(t) = \bigcap_{\varepsilon < \bar{\varepsilon}} \bigcup_{\eta < \varepsilon} E_{\eta,\lambda}^\geq(t).$$

The proof of the main result of this chapter follows. In our argument, the uniform convergence of curvatures established in Theorem 5.9 plays a major role.

Proof of Theorem 5.19. Let $v \in C([0,T] \times \mathbb{R}^d)$ be the limit point w.r.t. the locally uniform convergence of some subsequence of $\{ u_\varepsilon \}$, which we do not relabel. Such v exists as a consequence of Proposition 5.20.

We now show that for every $\lambda \in \mathbb{R}$,

$$\left\{ x \in \mathbb{R}^d : v(t,x) > \lambda \right\} \subseteq \tilde{E}_\lambda^>(t) \subseteq \tilde{E}_\lambda^\geq(t) \subseteq \left\{ x \in \mathbb{R}^d : v(t,x) \geq \lambda \right\}, \qquad (5.49)$$

with $\tilde{E}_\lambda^>$ and $\tilde{E}_\lambda^\geq$ as in (5.48). Without loss of generality, we may discuss just the case $\lambda = 0$. We only utilise the pointwise convergence of u_ε to v: if $\bar{x} \in \mathbb{R}^d$ is such that $v(t,\bar{x}) = \mu$ for some $\mu > 0$, then there exists $\bar{\varepsilon} > 0$ such that

$$u_\varepsilon(t,\bar{x}) \geq \frac{\mu}{2} > 0 \quad \text{for all } \varepsilon < \bar{\varepsilon},$$

and, hence, $\bar{x} \in \tilde{E}_0^>(t)$. Thus, we deduce $\left\{ x \in \mathbb{R}^d : v(t,x) > 0 \right\} \subseteq \tilde{E}_0^>(t)$.

On the other hand, if $\bar{x} \in \tilde{E}_0^\geq(t)$, then for all $\varepsilon < 1$ there exists $\eta_\varepsilon < \varepsilon$ such that $u_{\eta_\varepsilon}(t,\bar{x}) \geq 0$. Taking the limit $\varepsilon \to 0^+$, we get

$$v(t,\bar{x}) = \lim_{\varepsilon \to 0^+} u_{\eta_\varepsilon}(t,\bar{x}) \geq 0.$$

Secondly, we prove that, for all $\lambda \in \mathbb{R}$,

$$\left\{ x \in \mathbb{R}^d : u(t,x) > \lambda \right\} \subseteq \tilde{E}_\lambda^>(t) \subseteq \tilde{E}_\lambda^\geq(t) \subseteq \left\{ x \in \mathbb{R}^d : u(t,x) \geq \lambda \right\}, \qquad (5.50)$$

where u is the viscosity solution to (5.36).

We begin by showing that $\tilde{E}_\lambda^>$ and $\tilde{E}_\lambda^\geq$ are an outer and an inner barrier, respectively, for the family of strict geometric subsolutions and supersolutions, respectively, associated with the flow driven by H_0. If this assertion hold, then Theorem 5.17 immediately gets (5.50), because it states that $\left\{ x \in \mathbb{R}^d : u(t,x) > \lambda \right\}$ is the minimal outer barrier for the family of strict geometric subsolutions, and $\left\{ x \in \mathbb{R}^d : u(t,x) \geq \lambda \right\}$ is the maximal inner barrier for the family of strict geometric supersolutions.

We consider only the case of $\tilde{E}_0^>(t)$, since the proofs for $\lambda \neq 0$ and for $\tilde{E}_0^\geq$ are the same.

For some $0 \leq t_0 < t_1 \leq T$, let $t \mapsto D(t)$ be an evolution of sets which is a strict geometric subsolution to the anisotropic mean curvature motion when $t \in [t_0, t_1]$. Explicitly, we assume that there exists $\gamma > 0$ such that

$$\partial_t x(t) \cdot \hat{n}_D(t, x(t)) \leq -H_0(\partial D(t), x(t)) - \gamma \quad \text{for all } t \in (t_0, t_1] \text{ and } x(t) \in \partial D(t), \tag{5.51}$$

where $\hat{n}_D$ is the outer unit normal to $D(t)$. We suppose as well that

$$D(t_0) \subset \tilde{E}_0^>(t_0). \tag{5.52}$$

Our goal is verifying that it holds $D(t_1) \subset \tilde{E}_0^{>}(t_1)$.

Bearing in mind definition (5.48), we get from (5.52) that for all $\varepsilon < 1$ there exists $\eta_\varepsilon \leq \varepsilon$ such that

$$D(t_0) \subseteq E_{\eta_\varepsilon,0}^{-}(t_0). \tag{5.53}$$

Since the second fundamental forms of $\partial D(t)$ are uniformly bounded for $t \in [t_0, t_1]$, we may appeal to Theorem 5.9, which yields

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} H_\varepsilon(D(t), x) = H_0(D(t), x) \quad \text{uniformly in } t \in [t_0, t_1] \text{ and } x \in \partial D(t).$$

Accordingly, there exists $\bar{\varepsilon} := \bar{\varepsilon}(\gamma)$ such that, for all $\varepsilon < \bar{\varepsilon}$,

$$\partial_t x(t) \cdot \hat{n}_D(t, x(t)) \leq -\frac{1}{\varepsilon} H_\varepsilon(D(t), x(t)) - \frac{\gamma}{2} \quad \text{for all } t \in (t_0, t_1] \text{ and } x(t) \in \partial D(t),$$

or, in other words, $t \mapsto D(t)$ is a strict geometric subsolution to all the rescaled problems of parameter $\varepsilon \in (0, \bar{\varepsilon})$. By (5.53) and Proposition 5.18, we infer that for all $\varepsilon < \bar{\varepsilon}$ there exists $\eta_\varepsilon \leq \varepsilon$ such that

$$D(t) \subset E_{\eta_\varepsilon,0}^{-}(t) \quad \text{for all } t \in [t_0, t_1].$$

In view of Remark 5.22, it holds as well that

$$D(t) \subseteq \tilde{E}_0^{>}(t) \quad \text{for all } t \in [t_0, t_1],$$

and we get that $D(t_1) \subseteq \tilde{E}_0^{>}(t_1)$, as desired.

We can finally draw the conclusion. Indeed, by (5.49) and (5.50), for every $\lambda \in \mathbb{R}$ and $t \in [0, T]$, we have

$$\left\{ x \in \mathbb{R}^d : v(t, x) > \lambda \right\} \subseteq \left\{ x \in \mathbb{R}^d : u(t, x) \geq \lambda \right\},$$
$$\left\{ x \in \mathbb{R}^d : u(t, x) > \lambda \right\} \subseteq \left\{ x \in \mathbb{R}^d : v(t, x) \geq \lambda \right\},$$

and, from this, Lemma 5.21 yields $u = v$. $\qquad\qquad\square$

Bibliography

[AV14] N. Abatangelo and E. Valdinoci, *A notion of nonlocal curvature*, Numerical Functional Analysis and Optimization **35** (2014), 793–815.

[AB98a] G. Alberti and G. Bellettini, *A nonlocal anisotropic model for phase transitions. Part I: the optimal profile problem*, Mathematische Annalen **310** (1998), no. 3, 527–560.

[AB98b] ——, *A non-local anisotropic model for phase transitions. Part II: asymptotic behaviour of rescaled energies*, European Journal of Applied Mathematics **9** (1998), no. 3, 261–284.

[ACM05] O. Alvarez, P. Cardaliaguet, and R. Monneau, *Existence and uniqueness for dislocation dynamics with nonnegative velocity*, Interfaces Free Bound. **7** (2005), no. 4, 415–434.

[AHLBM06] O. Alvarez, P. Hoch, Y. Le Bouar, and R. Monneau, *Dislocation dynamics: Short-time existence and uniqueness of the solution*, Arch. Ration. Mech. Anal. **181** (2006), no. 3, 449–504.

[Amb00] L. Ambrosio, *Geometric evolution problems, distance function and viscosity solutions*, Calculus of Variations and Partial Differential Equations: Topics on Geometrical Evolution Problems and Degree Theory, 2000, pp. 5–93.

[ADPM11] L. Ambrosio, G. De Philippis, and L. Martinazzi, *Gamma-convergence of nonlocal perimeter functionals*, Manuscripta Mathematica **134** (2011), no. 3, 377–403.

[AFP00] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Science Publications, Clarendon Press, 2000.

[AB93] G. Anzellotti and S. Baldo, *Asymptotic development by Γ-convergence*, Appl. Math. Optim. **27** (1993), no. 2, 105–123.

[AK09] G. Aubert and P. Kornprobst, *Can the nonlocal characterization of Sobolev spaces by Bourgain et al. be useful for solving variational problems?*, SIAM J. Numer. Anal. **47** (2009), 844—860.

[BG95] G. Barles and C. Georgelin, *A Simple Proof of Convergence for an Approximation Scheme for Computing Motions by Mean Curvature*, SIAM J. Numer. Anal. **32** (1995), no. 2, 484–500.

[BL06] G. Barles and O. Ley, *Nonlocal first-order Hamilton-Jacobi equations modelling dislocations dynamics*, Commun. Partial Differ. Equations **31** (2006), no. 8, 1191–1208.

[BP88] G. Barles and B. Perthame, *Exit time problems in optimal control and vanishing viscosity method*, SIAM J. Control Optim. **26** (1988), no. 5, 1133–1148.

[Bel97] G. Bellettini, *Alcuni risultati sulle minime barriere per movimenti geometrici di insiemi*, Bollettino UMI **7** (1997), 485-512.

[BBP01] G. Bellettini, P. Buttà, and E. Presutti, *Sharp Interface Limits for Non-Local Anisotropic Interactions*, Archive for Rational Mechanics and Analysis **159** (2001), no. 2, 109–135.

[BN98] G. Bellettini and M. Novaga, *Comparison Results between Minimal Barriers and Viscosity Solutions for Geometric Evolutions*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **26** (1998), no. 1, 97–131.

[BN00] ——, *Some aspects of De Giorgi's barriers for geometric evolutions*, Calculus of Variations and Partial Differential Equations: Topics on Geometrical Evolution Problems and Degree Theory, 2000, pp. 115–151.

[BGP95] Bellettini G. and M. Paolini, *Some results on minimal barriers in the sense of De Giorgi applied to driven motion by mean curvature*, Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5) **19** (1995), 43–67.

[BMO92] J. Bence, B. Merriman, and S. Osher, *Diffusion generated motion by mean curvature*, Computational Crystal Growers Workshop, 1992, pp. 73–83.

[BP19] J. Berendsen and V. Pagliari, *On the asymptotic behaviour of nonlocal perimeters*, ESAIM: COCV **25** (2019).

[BBM01] J. Bourgain, H. Brezis, and P. Mironescu, *Another look at Sobolev spaces* (J. L. Menaldi, E. Rofman, and A. Sulem, eds.), IOS Press, 2001.

[Bra02] A. Braides, *Γ-convergence for Beginners*, Oxford Lecture Series in Mathematics, Oxford University Press, 2002.

[BN18] H. Brezis and H. Nguyen, *Non-local Functionals Related to the Total Variation and Connections with Image Processing*, Annals of PDE **4** (2018), no. 1, 9.

[BN16] _____, *Two subtle convex nonlocal approximations of the BV-norm*, Nonlinear Analysis **137** (2016), 222–245.

[Cab20] X. Cabré, *Calibrations and null-Lagrangians for nonlocal perimeters and an application to the viscosity theory*, Annali di Matematica Pura ed Applicata (1923-) (2020).

[CRS10] L. Caffarelli, J. Roquejoffre, and O. Savin, *Nonlocal minimal surfaces*, Comm. Pure Appl. Math. **63** (2010), no. 9, 1111–1144.

[CS10] L. Caffarelli and P. Souganidis, *Convergence of Nonlocal Threshold Dynamics Approximations to Front Propagation*, Arch. Ration. Mech. Anal. **195** (2010), no. 1, 1–23.

[CV11] L. Caffarelli and E. Valdinoci, *Uniform estimates and limiting arguments for nonlocal minimal surfaces*, Calculus of Variations and Partial Differential Equations **41** (2011), no. 1, 203–240.

[CH58] J. Cahn and J. Hilliard, *Free energy of a nonuniform system I: interfacial free energy*, J. Chem. Phys. **28** (1958), 258–267.

[CDNV19] A. Cesaroni, S. Dipierro, M. Novaga, and E. Valdinoci, *Fattening and nonfattening phenomena for planar nonlocal curvature flows*, Mathematische Annalen **375** (2019), no. 1-2, 687–736.

[CN18] A. Cesaroni and M. Novaga, *The isoperimetric problem for nonlocal perimeters*, Discrete Contin. Dyn. Syst., Ser. S **11** (2018), no. 3, 425–440.

[CN19] _____, *Symmetric self-shrinkers for the fractional mean curvature flow*, The Journal of Geometric Analysis (2019), 1–18.

[CP18] A. Cesaroni and V. Pagliari, *Convergence of nonlocal geoemetric flows to anisotropic mean curvature motions*, preprint: arXiv:1811.01732 (2018).

[CS19] J. Chaker and L. Silvestre, *Coercivity estimates for integro-differential operators*, preprint: arXiv:1904.13014 (2019).

[CCC⁺10] A. Chambolle, V. Caselles, D. Cremers, M. Novaga, and T. Pock, *An introduction to Total Variation for Image Analysis*, Theoretical Foundations and Numerical Methods for Sparse Recovery, 2010, pp. 263–340.

[CCP12] A. Chambolle, D. Cremers, and T. Pock, *A convex approach to minimal partitions*, SIAM Journal on Imaging Sciences **5** (2012), no. 4, 1113–1158.

[CGL10] A. Chambolle, A. Giacomini, and L. Lussardi, *Continuous limits of discrete perimeters*, ESAIM: M2AN **44** (2010), no. 2, 207–230.

[CMP15] A. Chambolle, M. Morini, and M. Ponsiglione, *Nonlocal curvature flows*, Arch. Ration. Mech. Anal. **218** (2015), no. 3, 1263–1329.

[CN06] A. Chambolle and M. Novaga, *Convergence of an algorithm for the anisotropic and crystalline mean curvature flow*, SIAM J. Math. Anal. **37** (2006), no. 6, 1978–1987.

[CNP19] A. Chambolle, M. Novaga, and V. Pagliari, *On the convergence rate of some nonlocal energies*, preprint: arXiv:1907.06030 (2019).

[CNR17] A. Chambolle, M. Novaga, and B. Ruffini, *Some results on anisotropic fractional mean curvature flows*, Interfaces Free Bound. **19** (2017), no. 3, 393–415.

[CGG91] Y. Chen, Y. Giga, and S. Goto, *Uniqueness and Existence of Viscosity Solutions of Generalized Mean Curvature Flow Equations*, J. Differential Geom. **33** (1991), 749–786.

[CSV19] E. Cinti, J. Serra, and E. Valdinoci, *Quantitative flatness results and BV-estimates for stable nonlocal minimal surfaces*, Journal of Differential Geometry **112** (2019), no. 3, 447–504.

[CSV18] E. Cinti, C. Sinestrari, and E. Valdinoci, *Neckpinch singularities in fractional mean curvature flows*, Proc. Amer. Math. Soc. **146** (2018), no. 6, 2637–2646.

[CL83] M. Crandall and P. Lions, *Viscosity Solutions of Hamilton-Jacobi Equations*, Trans. Amer. Math. Soc. **277** (1983), 1-42.

[DLFM08] F. Da Lio, N. Forcadel, and R. Monneau, *Convergence of a non-local eikonal equation to anisotropic mean curvature motion. Application to dislocations dynamics*, J. Eur. Math. Soc. (JEMS) **10** (2008), no. 4, 1105–1119.

[DM93] G. Dal Maso, *An Introduction to Γ-convergence*, Progress in nonlinear differential equations and their applications, Birkhäuser, 1993.

[Dáv02] J. Dávila, *On an open question about functions of bounded variation*, Calculus of Variations and Partial Differential Equations **15** (2002), no. 4, 519–527.

[DG94] E. De Giorgi, *Barriers, boundaries, motion of manifolds*, 1994. Conference held at Dipartimento di Matematica, Univ. of Pavia.

[DG55] ———, *Nuovi teoremi relativi alle misure $(r-1)$-dimensionali in uno spazio a r dimensioni*, Ricerche Mat. **4** (1955), 95-113.

[DGF75] E. De Giorgi and T. Franzoni, *Su un tipo di convergenza variazionale*, Atti Accad. Naz. Lincei, VIII. Ser., Rend., Cl. Sci. Fis. Mat. Nat. **58** (1975), 842–850.

[Eva93] L. Evans, *Convergence of an algorithm for mean curvature motion*, Indiana Univ. Math. J. **42** (1993), no. 2, 533–557.

[ES92] L. Evans and J. Spruck, *Motion of level sets by mean curvature. II*, Transactions of the american mathematical society **330** (1992), no. 1, 321–332.

[Fed59] H. Federer, *A note on the Gauss-Green theorem*, Proc. Amer. Math. Soc. **9** (1959), 447–451.

[FM93] I. Fonseca and S. Müller, *Relaxation of quasiconvex functionals in $BV(\Omega; \mathbb{R}^p)$ for integrands $f(x, u, \nabla u)$*, Arch. Ration. Mech. Anal. **123** (1993), no. 1, 1–49.

[NF08] C. Imbert N. Forcadel and R. Monneau, *Homogenization of some particle systems with two-body interactions and of the dislocation dynamics*, DCDS-A **23** (2008), no. 3, 785–826.

[GO07] G. Gilboa and S. Osher, *Nonlocal linear image regularization and supervised segmentation*, Multiscale Model. Simul. **6** (2007), 595–630.

[GO08] ———, *Nonlocal operators with applications to image processing*, Multiscale Model. Simul. **7** (2008), no. 3, 1005–1028.

[Gob98] M. Gobbino, *Finite difference approximation of the Mumford-Shah functional*, Comm. Pure Appl. Math. **51** (1998), no. 2, 197–228.

[GM01] M. Gobbino and M. Mora, *Finite-difference approximation of free-discontinuity problems*, Proceedings of the Royal Society of Edinburgh Section A: Mathematics **131** (2001), no. 3, 567–595.

[Gra09] M. Grasmair, *A coarea formula for anisotropic total variation regularisation*, 2009. unpublished.

[HL82] R. Harvey and H. Lawson Jr., *Calibrated geometries*, Acta Math. **148** (1982), 47–157.

[HKST15] J. Heinonen, P. Koskela, N. Shanmugalingam, and J. T. Tyson, *Sobolev Spaces on Metric Measure Spaces* (2015).

[Imb09] C. Imbert, *Level set approach for fractional mean curvature flows*, Interfaces Free Bound. **11** (2009), no. 1, 153–176.

[IMRM08] C. Imbert, R. Monneau, and E. Rouy-Mironescu, *Homogenization of first order equations with u/ε-periodic Hamiltonians. Part II: application to dislocations dynamics*, Comm. in PDEs **33** (2008), no. 1, 479–516.

[Ish95] H. Ishii, *A generalization of the Bence, Merriman and Osher algorithm for motion by mean curvature*, Proceedings of the international conference on curvature flows and related topics held in Levico, Italy, June 27-July 2nd, 1994 (1995), 111–127.

[IPS99] H. Ishii, G. Pires, and P. Souganidis, *Threshold dynamics type approximation schemes for propagating fronts*, J. Math. Soc. Japan **51** (1999), no. 2, 267–308.

[LMV19] D. La Manna and J. Vesa, *Short time existence of the classical solution to the fractional Mean curvature flow*, preprint: arXiv:1906.10990 (2019).

[LM94] G. Lawlor and F. Morgan, *Paired calibrations applied to soap films, immiscible fluids, and surfaces or networks minimizing other norms*, Pacific journal of Mathematics **166** (1994), no. 1, 55–83.

[LS11] G. Leoni and D. Spector, *Characterization of Sobolev and BV spaces*, J. Funct. Anal. **261** (2011), 2926 – 2958.

[Lud14] M. Ludwig, *Anisotropic fractional perimeters*, J. Diff. Geom. **96** (2014), no. 1, 77–93.

[Mag12] F. Maggi, *Sets of Finite Perimeter and Geometric Variational Problems: An Introduction to Geometric Measure Theory*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2012.

[MRT19] J. M. Mazón, J. D. Rossi, and J. Toledo, *Nonlocal perimeter, curvature and minimal surfaces for measurable sets*, Journal d'Analyse Mathématique (2019).

[MS02] V. Maz'ya and T. Shaposhnikova, *On the Bourgain, Brezis, and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces*, J. Funct. Anal. **195** (2002), 230–238.

[MM77] L. Modica and S. Mortola, *Un esempio di Γ^--convergenza*, Boll. Un. Mat. Ital. **5** (1977), 285 – 299.

[Mor08] F. Morgan, *Geometric Measure Theory: A Beginner's Guide*, Elsevier Science, 2008.

[Mor90] ———, *Calibrations and new singularities in area-minimizing surfaces: a survey*, Variational methods, 1990, pp. 329–342.

[MS19] C. Muratov and T. Simon, *A nonlocal isoperimetric problem with dipolar repulsion*, Communications in Mathematical Physics **372** (2019), no. 3, 1059–1115.

[ORF92] S. Osher, L. Rudin, and E. Fatemi, *Nonlinear total variation based noise removal algorithms*, Physica D **60** (1992), no. 1-4, 259–268.

[Pag19] V. Pagliari, *Halfspaces minimise nonlocal perimeter: a proof* via *calibrations*, Annali di Matematica Pura ed Applicata (1923-) (2019), 1–12.

[PPR07] M. Peletier, R. Planqué, and M. Röger, *Sobolev regularity via the convergence rate of convolutions and Jensen's inequality*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **6** (2007), no. 4, 499–510.

[Pon04] A. Ponce, *A new approach to Sobolev spaces and connections to Γ-convergence*, Calculus of Variations and Partial Differential Equations **19** (2004), no. 3, 229–255.

[SV12] O. Savin and E. Valdinoci, *Γ-convergence for nonlocal phase transitions*, Ann. Inst. H. Poincaré Anal. Non Linéaire **29** (2012), no. 4, 479–500.

[Sle03] D. Slepčev, *Approximation schemes for propagation of fronts with nonlocal velocities and Neumann boundary conditions*, Nonlinear Anal. **52** (2003), 79–115.

[Val13] E. Valdinoci, *A fractional framework for perimeters and phase transitions*, Milan J. Math. **81** (2013), no. 1, 1–23.

[Vis91] A. Visintin, *Generalized coarea formula and fractal sets*, Japan J. Indust. Appl. Math. **81** (1991), no. 1, 175-201.

[Vis90] ———, *Nonconvex functionals related to multiphase systems*, SIAM journal on mathematical analysis **21** (1990), no. 5, 1281–1304.

Index