EQUILIBRIUM CONFIGURATIONS FOR EPITAXIALLY STRAINED FILMS AND MATERIAL VOIDS IN THREE-DIMENSIONAL LINEAR ELASTICITY

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Abstract. We extend the results about existence of minimizers, relaxation, and approximation proven by Chambolle et al. in 2002 and 2007 for an energy related to epitaxially strained crystalline films, and by Braides, Chambolle, and Solci in 2007 for a class of energies defined on pairs of function-set. We study these models in the framework of three-dimensional linear elasticity, where a major obstacle to overcome is the lack of any *a priori* assumption on the integrability properties of displacements. As a key tool for the proofs, we introduce a new notion of convergence for $(d-1)$-rectifiable sets that are jumps of $GSBD^p$ functions, called σ^p_{sym}-convergence.

1. Introduction

The last years have witnessed a remarkable progress in the mathematical and physical literature towards the understanding of stress driven rearrangement instabilities (SDRI), i.e., morphological instabilities of interfaces between elastic phases generated by the competition between elastic and surface energies of (isotropic or anisotropic) perimeter type. Such phenomena are for instance observed in the formation of material voids inside elastically stressed solids. Another example is hetero-epitaxial growth of elastic thin films, when thin layers of highly strained hetero-systems, such as InGaAs/GaAs or SiGe/Si, are deposited onto a substrate: in case of a mismatch between the lattice parameters of the two crystalline solids, the free surface of the film is flat until a critical value of the thickness is reached, after which the free surface becomes corrugated (see e.g. [4, 42, 45, 46, 52, 54] for some physical and numerical literature).

From a mathematical point of view, the common feature of functionals describing SDRI is the presence of both stored elastic bulk and surface energies. In the static setting, problems arise concerning existence, regularity, and stability of equilibrium configurations obtained by energy minimization. The analysis of these issues is by now mostly developed in dimension two only.

Starting with the seminal work by Bonnetier and Chambolle [9] who proved existence of equilibrium configurations, several results have been obtained in [5, 7, 31, 33, 41, 44] for hetero-epitaxially strained elastic thin films in 2d. We also refer to [27, 28, 49] for related energies and to [48] for a unified model for SDRI. In the three dimensional setting, results are limited to the geometrically nonlinear setting or to linear elasticity under antiplane-shear assumption [8, 18]. In a similar fashion, regarding the study of material voids in elastic solids, there are works about existence and regularity in dimension two [12, 30] and a relaxation result in higher dimensions [11] for nonlinearly elastic energies or in linear elasticity under antiplane-shear assumption.

The goal of the present paper is to extend the results about relaxation, existence, and approximation obtained for energies related to material voids [11] and to epitaxial growth

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[9, 18], respectively, to the case of linear elasticity in arbitrary space dimensions. As already observed in [18], the main obstacle for deriving such generalizations lies in the fact that a deep understanding of the function space of *generalized special functions of bounded deformation* ($GSBD$) is necessary. Indeed, our strategy is based extensively on using the theory on $GSBD$ functions which, initiated by Dal Maso [25], was developed over the last years, see e.g. [14, 15, 17, 19, 20, 21, 22, 23, 35, 36, 38, 47]. In fact, as a byproduct of our analysis, we introduce two new notions related to this function space, namely (1) a version of the space with functions attaining also the value infinity and (2) a novel notion for convergence of rectifiable sets, which we call $\sigma^p_{\rm sym}$-convergence. Let us stress that in this work we consider exclusively a static setting. For evolutionary models, we mention the recent works [32, 39, 40, 51].

We now introduce the models under consideration in a slightly simplified way, restricting ourselves to three space dimensions. To describe material voids in elastically stressed solids, we consider the following functional defined on pairs of function-set (see [52])

$$F(u,E) = \int_{\Omega\setminus E} \mathbb{C}\, e(u) : e(u)\,\mathrm{d}x + \int_{\Omega\cap\partial E} \varphi(\nu_E)\,\mathrm{d}\mathcal{H}^2\,, \tag{1.1}$$

where $E \subset \Omega$ represents the (sufficiently smooth) shape of voids within an elastic body with reference configuration $\Omega \subset \mathbb{R}^3$, and u is an elastic displacement field. The first part of the functional represents the elastic energy depending on the linear strain $e(u) := \frac{1}{2}\big((\nabla u)^T + \nabla u\big)$, where $\mathbb{C}$ denotes the fourth-order positive semi-definite tensor of elasticity coefficients. (In fact, we can incorporate more general elastic energies, see (2.2) below.) The surface energy depends on a (possibly anisotropic) density φ evaluated at the outer normal ν_E to E. This setting is usually complemented with a volume constraint on the voids E and nontrivial prescribed Dirichlet boundary conditions for u on a part of $\partial\Omega$. We point out that the boundary conditions are the reason why the solid is elastically stressed.

A variational model for epitaxially strained films can be regarded as a special case of (1.1) and corresponds to the situation where the material domain is the subgraph of an unknown nonnegative function h. More precisely, we assume that the material occupies the region

$$\Omega_h^+ := \{x \in \omega\times\mathbb{R} : 0 < x_3 < h(x_1,x_2)\}$$

for a given bounded function $h\colon \omega \to [0,\infty)$, $\omega \subset \mathbb{R}^2$, whose graph represents the free profile of the film. We consider the energy

$$G(u,h) = \int_{\Omega_h^+} \mathbb{C}\, e(u) : e(u)\,\mathrm{d}x + \int_\omega \sqrt{1+|\nabla h(x_1,x_2)|^2}\,\mathrm{d}(x_1,x_2)\,. \tag{1.2}$$

Here, u satisfies prescribed boundary data on $\omega\times\{0\}$ which corresponds to the interface between film and substrate. This Dirichlet boundary condition models the case of a film growing on an infinitely rigid substrate and is the reason for the film to be strained. We observe that (1.2) corresponds to (1.1) when φ is the Euclidean norm, $\Omega = \omega\times(0,M)$ for some $M > 0$ large enough, and $E = \Omega\setminus\Omega_h^+$.

Variants of the above models (1.1) and (1.2) have been studied by Braides, Chambolle, and Solci [11] and by Chambolle and Solci [18], respectively, where the linearly elastic energy density $\mathbb{C}\,e(u) : e(u)$ is replaced by an elastic energy satisfying a 2-growth (or p-growth, $p > 1$) condition in the full gradient ∇u with quasiconvex integrands. These works are devoted to giving a sound mathematical formulation for determining equilibrium configurations. By means of variational methods and geometric measure theory, they study the relaxation of the functionals in terms of *generalized functions of bounded variation* ($GSBV$) which allows to incorporate the possible roughness of the geometry of voids or films. Existence of minimizers for the relaxed functionals and the approximation of (the counterpart of) G through a phase-field Γ-convergence result are addressed. In fact, the two articles have been written almost simultaneously with many similarities in both the setting and the proof strategy.

Therefore, we prefer to present the extension of both works to the $GSBD$ setting (i.e., to three-dimensional linear elasticity) in a single work to allow for a comprehensive study of different applications. We now briefly discuss our main results.

(a) Relaxation of F: We first note that, for fixed E, $F(\cdot, E)$ is weakly lower semicontinuous in H^1 and, for fixed u, $F(u,\cdot)$ can be regarded as a lower semicontinuous functional on sets of finite perimeter. The energy defined on pairs (u,E), however, is not lower semicontinuous since, in a limiting process, the voids E may collapse into a discontinuity of the displacement u. The relaxation has to take this phenomenon into account, in particular collapsed surfaces need to be counted twice in the relaxed energy. Provided that the surface density φ is a norm in $\mathbb{R}^3$, we show that the relaxation takes the form (see Proposition 2.1)

$$\overline{F}(u,E) = \int_{\Omega\setminus E} \mathbb{C}\, e(u) : e(u)\, \mathrm{d}x + \int_{\Omega\cap\partial^* E} \varphi(\nu_E)\, \mathrm{d}\mathcal{H}^2 + \int_{J_u\cap(\Omega\setminus E)^1} 2\,\varphi(\nu_u)\, \mathrm{d}\mathcal{H}^2\,, \tag{1.3}$$

where E is a set of finite perimeter with essential boundary $\partial^* E$, $(\Omega\setminus E)^1$ denotes the set of points of density 1 of $\Omega\setminus E$, and $u \in GSBD^2(\Omega)$. Here, $e(u)$ denotes the approximate symmetrized gradient of class $L^2(\Omega;\mathbb{R}^{3\times 3})$ and J_u is the jump set with corresponding measure-theoretical normal ν_u. (We refer to Section 3 for the definition and the main properties of this function space. Later, we will also consider more general elastic energies and work with the space $GSBD^p(\Omega)$, $1<p<\infty$, i.e., $e(u)\in L^p(\Omega;\mathbb{R}^{3\times3})$.)

(b) Minimizer for $\overline{F}$: In Theorem 2.2, we show that such a relaxation result can also be proved by imposing additionally a volume constraint on E (which reflects mass conservation) and by prescribing boundary data for u. For this version of the relaxed functional, we prove the existence of minimizers, see Theorem 2.3.

(c) Relaxation of G: For the model (1.2) describing epitaxially strained crystalline films, we show in Theorem 2.4 that the lower semicontinous envelope takes the form

$$\overline{G}(u,h) = \int_{\Omega_h^+} \mathbb{C}\, e(u) : e(u)\, \mathrm{d}x + \mathcal{H}^2(\Gamma_h) + 2\,\mathcal{H}^2(\Sigma)\,, \tag{1.4}$$

where $h \in BV(\omega;[0,\infty))$ and Γ_h denotes the (generalized) graph of h. Here, u is again a $GSBD^2$-function and the set $\Sigma\subset\mathbb{R}^3$ is a "vertical" rectifiable set describing the discontinuity set of u inside the subgraph Ω_h^+. Similar to the last term in (1.3), this contribution has to be counted twice. We remark that in [31] the set Σ is called "vertical cuts". Also here a volume constraint may be imposed.

(d) Minimizer for $\overline{G}$: In Theorem 2.5, we show compactness for sequences with bounded G energy. In particular, this implies existence of minimizers for $\overline{G}$ (under a volume constraint).

(e) Approximation for $\overline{G}$: In Theorem 2.6, we finally prove a phase-field Γ-convergence approximation of $\overline{G}$. We remark that we can generalize the assumptions on the regularity of the Dirichlet datum. Whereas in [18, Theorem 5.1] the class $H^1\cap L^\infty$ was considered, we show that it indeed suffices to assume H^1-regularity.

We now provide some information on the proof strategy highlighting in particular the additional difficulties compared to [11, 18]. Here, we will also explain why two new technical tools related to the space $GSBD$ have to be introduced.

(a) The proof of the lower inequality for the relaxation $\overline{F}$ is closely related to the analog in [11]: we use an approach by slicing, exploit the lower inequality in one dimension, and a localization method. To prove the upper inequality, it is enough to combine the corresponding upper bound from [11] with a density result for $GSBD^p$ ($p > 1$) functions [15], slightly adapted for our purposes, see Lemma 5.6.

(b) We point out that, in [11], the existence of minimizers has not been addressed due to the lack of a compactness result. In this sense, our study also delivers a conceptionally new result without corresponding counterpart in [11]. The main difficulty lies in the fact that, for configurations with finite energy (1.3), small pieces of the body could be disconnected from the bulk part, either by the voids E or by the jump set J_u. Thus, since there are *no a priori bounds* on the displacements, the function u could attain arbitrarily large values on certain components, and this might rule out measure convergence for minimizing sequences. We remark that truncation methods, used to remedy this issue in scalar problems, are not applicable in the vectorial setting. This problem was solved only recently by general compactness results, both in the $GSBV^p$ and the $GSBD^p$ setting. The result [37] in $GSBV^p$ delivers a *selection principle for minimizing sequences* showing that one can always find at least one minimizing sequence converging in measure. With this, existence of minimizers for the energies in [11] is immediate.

Our situation in linear elasticity, however, is more delicate since a comparable strong result is not available in $GSBD$. In [17, Theorem 1.1], a compactness and lower semicontinuity result in $GSBD^p$ is derived relying on the idea that minimizing sequences may "converge to infinity" on a set of finite perimeter. In the present work, we refine this result by introducing a topology which induces this kind of nonstandard convergence. To this end, we need to define the new space $GSBD^p_\infty$ consisting of $GSBD^p$ functions which may also attain the value infinity. With these new techniques at hand, we can prove a general compactness result in $GSBD^p_\infty$ (see Theorem 5.7) which particularly implies the existence of minimizers for (1.3).

(c) Although the functional G in (1.2) is a special case of F, the relaxation result is not an immediate consequence, due to the additional constraint that the domain is the subgraph of a function. Indeed, in the lower inequality, a further crucial step is needed in the description of the (variational) limit of $\partial\Omega_{h_n}$ when $h_n \to h$ in $L^1(\omega)$. In particular, the vertical set Σ has to be identified, see (1.4).

This issue is connected to the problem of detecting all possible limits of jump sets J_{u_n} of converging sequences $(u_n)_n$ of $GSBD^p$ functions. In the $GSBV^p$ setting, the notion of σ^p-convergence of sets is used, which has originally been developed by Dal Maso, Francfort, and Toader [26] to study quasistatic crack evolution in nonlinear elasticity. (We refer also to the variant [43] which is independent of p.) In this work, we introduce an analogous notion in the $GSBD^p$ setting which we call σ^p_{sym}-convergence. The definition is a bit more complicated compared to the $GSBV$ setting since it has to be formulated in the frame of $GSBD^p_\infty$ functions possibly attaining the value infinity. We believe that this notion may be of independent interest and is potentially helpful to study also other problems such as quasistatic crack evolution in linear elasticity [38]. We refer to Section 4 for the definition and properties of σ^p_{sym}-convergence, as well as for a comparison to the corresponding notion in the $GSBV^p$ setting.

Showing the upper bound for the relaxation result is considerably more difficult than the analogous bound for $\overline{F}$. In fact, one has to guarantee that recovery sequences are made up by sets that are still subgraphs. We stress that this cannot be obtained by some general existence results, but is achieved through a very careful construction (pp. 28-36), that follows only partially the analogous one in [18]. We believe that the construction in [18] could indeed be improved by adopting an approach similar to ours, in order to take also some pathological situations into account.

(d) To show the existence of minimizers of G, the delicate step is to prove that minimizing sequences have subsequences which converge (at least) in measure. In the $GSBV^p$ setting, this is simply obtained by applying a Poincaré inequality on vertical slices through the film. The same strategy cannot be pursued in $GSBD^p$ since by slicing in a certain direction not all components can be controlled. As a remedy, we proceed in two steps. We first use the novel compactness result in $GSBD^p_\infty$ to identify a limit which might attain the value infinity on a set of finite perimeter G_∞. Then, *a posteriori*, we show that actually $G_\infty = \emptyset$ by means of a slicing argument in various directions, see Subsection 6.1 for details.

(e) For the phase-field approximation, we combine a variant of the construction in the upper inequality for $\overline{G}$ with the general strategy of the corresponding approximation result in [18]. The latter is slightly modified in order to proceed without L^∞-bound on the displacements.

The paper is organized as follows. In Section 2, we introduce the setting of our two models on material voids in elastic solids and epitaxially strained films. Here, we also present our main relaxation, existence, and approximation results. Section 3 collects definition and main properties of the function space $GSBD^p$. In this section, we also define the space $GSBD^p_\infty$ and show basic properties. In Section 4 we introduce the novel notion of σ^p_{sym}-convergence and prove a compactness result for sequences of rectifiable sets with bounded Hausdorff measure. Section 5 is devoted to the analysis of functionals defined on pairs of function-set. Finally, in Section 6 we investigate the model for epitaxially strained films and prove the relaxation, existence, and approximation results.

2. Setting of the problem and statement of the main results

In this section, we give the precise definitions of the two energy functionals and present the main relaxation, existence, and approximation results. In the following, $f\colon \mathbb{R}^{d\times d} \to [0,\infty)$ denotes a convex function satisfying the growth condition ($|\cdot|$ is the Frobenius norm on $\mathbb{R}^{d\times d}$)

$$c_1|\zeta^T + \zeta|^p - c_2 \le f(\zeta) \le c_2(|\zeta^T + \zeta|^p + 1) \qquad \text{for all } \zeta \in \mathbb{R}^{d\times d} \tag{2.1}$$

for some $1 < p < +\infty$. For an open subset $\Omega \subset \mathbb{R}^d$, we will denote by $L^0(\Omega;\mathbb{R}^d)$ the space of $\mathcal{L}^d$-measurable functions $v\colon \Omega \to \mathbb{R}^d$ endowed with the topology of the convergence in measure. We let $\mathfrak{M}(\Omega)$ be the family of all $\mathcal{L}^d$-measurable subsets of Ω.

2.1. Energies on pairs function-set: material voids in elastically stressed solids.

Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain. We introduce an energy functional defined on pairs function-set. Given a norm φ on $\mathbb{R}^d$ and $f\colon \mathbb{R}^{d\times d} \to [0,\infty)$, we let $F\colon L^0(\Omega;\mathbb{R}^d) \times \mathfrak{M}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ be defined by

$$F(u,E) = \begin{cases} \displaystyle\int_{\Omega\setminus E} f(e(u))\,\mathrm{d}x + \int_{\Omega\cap\partial E} \varphi(\nu_E)\,\mathrm{d}\mathcal{H}^{d-1} & \\ \qquad\qquad\qquad \text{if } \partial E \text{ Lipschitz, } u|_{\Omega\setminus\overline{E}} \in W^{1,p}(\Omega\setminus\overline{E};\mathbb{R}^d),\ u|_E = 0, & \\ +\infty \qquad\qquad\quad \text{otherwise,} & \end{cases} \tag{2.2}$$

where $e(u) := \frac{1}{2}\big((\nabla u)^T + \nabla u\big)$ denotes the symmetrized gradient, and ν_E the outer normal to E. We point out that the energy is determined by E and the values of u on $\Omega\setminus\overline{E}$. The condition $u|_E = 0$ is for definiteness only. We denote by $\overline{F}\colon L^0(\Omega;\mathbb{R}^d) \times \mathfrak{M}(\Omega) \to \mathbb{R}\cup\{+\infty\}$ the lower semicontinuous envelope of the functional F with respect to the convergence in measure for the functions and the $L^1(\Omega)$-convergence of characteristic functions of sets, i.e.,

$$\overline{F}(u,E) = \inf\Big\{ \liminf_{n\to\infty} F(u_n,E_n)\colon\ u_n \to u \text{ in } L^0(\Omega;\mathbb{R}^d) \text{ and } \chi_{E_n} \to \chi_E \text{ in } L^1(\Omega)\Big\}. \tag{2.3}$$

In the following, for any $s \in [0,1]$ and any $E \in \mathfrak{M}(\Omega)$, E^s denotes the set of points with density s for E. By $\partial^* E$ we indicate its essential boundary, see [3, Definition 3.60]. For the definition of the space $GSBD^p(\Omega)$, $p > 1$, we refer to Section 3 below. In particular, by $e(u) = \frac{1}{2}((\nabla u)^T + \nabla u)$ we denote the approximate symmetrized gradient, and by J_u the jump set of u with measure-theoretical normal ν_u. We characterize $\overline{F}$ as follows.

Proposition 2.1 (Characterization of the lower semicontinuous envelope $\overline{F}$)**.** *Suppose that f is convex and satisfies (2.1), and that φ is a norm on $\mathbb{R}^d$. Then there holds*

$$\overline{F}(u,E) = \begin{cases} \displaystyle\int_{\Omega\setminus E} f(e(u))\,\mathrm{d}x + \int_{\Omega\cap\partial^* E} \varphi(\nu_E)\,\mathrm{d}\mathcal{H}^{d-1} + \int_{J_u\cap(\Omega\setminus E)^1} 2\,\varphi(\nu_u)\,\mathrm{d}\mathcal{H}^{d-1} & \\ & \text{if } u = u\,\chi_{E^0} \in GSBD^p(\Omega) \text{ and } \mathcal{H}^{d-1}(\partial^* E) < +\infty\,, \\ +\infty & \text{otherwise.} \end{cases}$$

Moreover, if $\mathcal{L}^d(E) > 0$, then for any $(u,E) \in L^0(\Omega;\mathbb{R}^d)\times\mathfrak{M}(\Omega)$ there exists a recovery sequence $(u_n,E_n)_n \subset L^0(\Omega;\mathbb{R}^d)\times\mathfrak{M}(\Omega)$ such that $\mathcal{L}^d(E_n) = \mathcal{L}^d(E)$ for all $n \in \mathbb{N}$.

The last property shows that it is possible to incorporate a volume constraint on E in the relaxation result. We now move on to consider a Dirichlet minimization problem associated to F. We will impose Dirichlet boundary data $u_0 \in W^{1,p}(\mathbb{R}^d;\mathbb{R}^d)$ on a subset $\partial_D\Omega \subset \partial\Omega$. For technical reasons, we suppose that $\partial\Omega = \partial_D\Omega \cup \partial_N\Omega \cup N$ with $\partial_D\Omega$ and $\partial_N\Omega$ relatively open, $\partial_D\Omega \cap \partial_N\Omega = \emptyset$, $\mathcal{H}^{d-1}(N) = 0$, $\partial_D\Omega \neq \emptyset$, $\partial(\partial_D\Omega) = \partial(\partial_N\Omega)$, and that there exist a small $\overline{\delta} > 0$ and $x_0 \in \mathbb{R}^d$ such that for every $\delta \in (0,\overline{\delta})$ there holds

$$O_{\delta,x_0}(\partial_D\Omega) \subset \Omega\,, \tag{2.4}$$

where $O_{\delta,x_0}(x) := x_0 + (1-\delta)(x-x_0)$. (These assumptions are related to Lemma 5.6 below.) In the following, we denote by $\mathrm{tr}(u)$ the trace of u on $\partial\Omega$ which is well defined for functions in $GSBD^p(\Omega)$, see Section 3. In particular, it is well defined for functions u considered in (2.2) satisfying $u|_{\Omega\setminus\overline{E}} \in W^{1,p}(\Omega\setminus\overline{E};\mathbb{R}^d)$ and $u|_E = 0$. By ν_Ω we denote the outer unit normal to $\partial\Omega$.

We now introduce a version of F taking boundary data into account. Given $u_0 \in W^{1,p}(\mathbb{R}^d;\mathbb{R}^d)$, we set

$$F_{\mathrm{Dir}}(u,E) = \begin{cases} F(u,E) + \int_{\partial_D\Omega\cap\partial E} \varphi(\nu_E)\,\mathrm{d}\mathcal{H}^{d-1} & \text{if} \quad \mathrm{tr}(u) = \mathrm{tr}(u_0) \text{ on } \partial_D\Omega\setminus\overline{E}, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.5}$$

Similar to (2.3), we define the lower semicontinuous envelope $\overline{F}_{\mathrm{Dir}}$ by

$$\overline{F}_{\mathrm{Dir}}(u,E) = \Big\{ \liminf_{n\to\infty} F_{\mathrm{Dir}}(u_n,E_n)\colon\ u_n \to u \text{ in } L^0(\Omega;\mathbb{R}^d) \text{ and } \chi_{E_n} \to \chi_E \text{ in } L^1(\Omega) \Big\}\,. \tag{2.6}$$

We have the following characterization.

Theorem 2.2 (Characterization of the lower semicontinuous envelope $\overline{F}_{\mathrm{Dir}}$)**.** *Suppose that f is convex and satisfies (2.1), that φ is a norm on $\mathbb{R}^d$, and that (2.4) is satisfied. Then there holds*

$$\overline{F}_{\mathrm{Dir}}(u,E) = \overline{F}(u,E) + \int_{\partial_D\Omega\cap\partial^* E} \varphi(\nu_E)\,\mathrm{d}\mathcal{H}^{d-1} + \int_{\{\mathrm{tr}(u)\neq\mathrm{tr}(u_0)\}\cap(\partial_D\Omega\setminus\partial^* E)} 2\,\varphi(\nu_\Omega)\,\mathrm{d}\mathcal{H}^{d-1}\,. \tag{2.7}$$

Moreover, if $\mathcal{L}^d(E) > 0$, then for any $(u,E) \in L^0(\Omega;\mathbb{R}^d)\times\mathfrak{M}(\Omega)$ there exists a recovery sequence $(u_n,E_n)_n \subset L^0(\Omega;\mathbb{R}^d)\times\mathfrak{M}(\Omega)$ such that $\mathcal{L}^d(E_n) = \mathcal{L}^d(E)$ for all $n \in \mathbb{N}$.

The proof of Proposition 2.1 and Theorem 2.2 will be given in Subsection 5.2. We close this subsection with an existence result for $\overline{F}_{\mathrm{Dir}}$, under a volume constraint for the voids.

Theorem 2.3 (Existence of minimizers for $\overline{F}_{\rm Dir}$)**.** *Suppose that f is convex and satisfies* (2.1)*, and that φ is a norm on $\mathbb{R}^d$. Let $m > 0$. Then the minimization problem*

$$\inf\big\{\overline{F}_{\rm Dir}(u,E)\colon (u,E)\in L^0(\Omega;\mathbb{R}^d)\times\mathfrak{M}(\Omega),\ \mathcal{L}^d(E)=m\big\}$$

admits solutions.

For the proof, we refer to Subsection 5.3. It relies on the lower semicontinuity of $\overline{F}_{\rm Dir}$ and a compactness result in the general space $GSBD^p_\infty$ (cf. (3.10)), see Theorem 5.7.

2.2. **Energies on domains with a subgraph constraint: epitaxially strained films.** We now consider the problem of displacement fields in a material domain which is the subgraph of an unknown nonnegative function h. Assuming that h is defined on a Lipschitz domain $\omega\subset\mathbb{R}^{d-1}$, displacement fields u will be defined on the subgraph

$$\Omega_h^+ := \{x\in\omega\times\mathbb{R}\colon 0<x_d<h(x')\},$$

where here and in the following we use the notation $x=(x',x_d)$ for $x\in\mathbb{R}^d$. To model Dirichlet boundary data at the flat surface $\omega\times\{0\}$, we will suppose that functions are extended to the set $\Omega_h := \{x\in\omega\times\mathbb{R}\colon -1<x_d<h(x')\}$ and satisfy $u=u_0$ on $\omega\times(-1,0)$ for a given function $u_0\in W^{1,p}(\omega\times(-1,0);\mathbb{R}^d)$, $p>1$. In the application to epitaxially strained films, u_0 represents the substrate and h represents the profile of the free surface of the film.

For convenience, we introduce the reference domain $\Omega := \omega\times(-1,M+1)$ for $M>0$. We define the energy functional $G\colon L^0(\Omega;\mathbb{R}^d)\times L^1(\omega;[0,M])\to\mathbb{R}\cup\{+\infty\}$ by

$$G(u,h)=\int_{\Omega_h^+} f(e(u(x)))\,\mathrm{d}x+\int_\omega\sqrt{1+|\nabla h(x')|^2}\,\mathrm{d}x' \tag{2.8}$$

if $h\in C^1(\omega;[0,M])$, $u|_{\Omega_h}\in W^{1,p}(\Omega_h;\mathbb{R}^d)$, $u=0$ in $\Omega\setminus\Omega_h$, and $u=u_0$ in $\omega\times(-1,0)$, and $G(u,h)=+\infty$ otherwise. Here, $f\colon\mathbb{R}^{d\times d}\to[0,\infty)$ denotes a convex function satisfying (2.1), and as before we set $e(u):=\frac12\big((\nabla u)^T+\nabla u\big)$. Notice that, in contrast to [9], we suppose that the functions h are equibounded by a value M: this is for technical reasons only and is indeed justified from a mechanical point of view since other effects come into play for very high crystal profiles.

We study the relaxation of G with respect to the $L^0(\Omega;\mathbb{R}^d)\times L^1(\omega;[0,M])$ topology, i.e., its lower semicontinuous envelope $\overline{G}\colon L^0(\Omega;\mathbb{R}^d)\times L^1(\omega;[0,M])\to\mathbb{R}\cup\{+\infty\}$, defined as

$$\overline{G}(u,h)=\inf\big\{\liminf_{n\to\infty}G(u_n,h_n)\colon\ u_n\to u \text{ in } L^0(\Omega;\mathbb{R}^d),\quad h_n\to h\text{ in } L^1(\omega)\big\}\,.$$

We characterize $\overline{G}$ as follows, further assuming that the Lipschitz set $\omega\subset\mathbb{R}^{d-1}$ is uniformly star-shaped with respect to the origin, i.e.,

$$tx\subset\omega\quad\text{for all}\quad t\in(0,1),\ x\in\partial\omega. \tag{2.9}$$

Theorem 2.4 (Characterization of the lower semicontinuous envelope $\overline{G}$)**.** *Suppose that f is convex satisfying* (2.1) *and that* (2.9) *holds. Then we have*

$$\overline{G}(u,h)=\begin{cases}\displaystyle\int_{\Omega_h^+} f(e(u))\,\mathrm{d}x+\mathcal{H}^{d-1}(\partial^*\Omega_h\cap\Omega)+2\mathcal{H}^{d-1}(J_u'\cap\Omega_h^1)\\ \qquad\qquad \text{if } u=u\chi_{\Omega_h}\in GSBD^p(\Omega),\ u=u_0 \text{ in } \omega\times(-1,0),\ h\in BV(\omega;[0,M])\,,\\ +\infty \qquad\quad \text{otherwise,}\end{cases}$$

where

$$J_u' := \{(x',x_d+t)\colon\ x\in J_u,\ t\geq 0\}. \tag{2.10}$$

The assumption (2.9) on ω is more general than the one considered in [18], where ω is assumed to be a torus. We point out, however, that both assumptions are only of technical nature and could be dropped at the expense of more elaborated estimates, see also [18]. The proof of this result will be given in Subsection 6.1.

We note that the functional G could be considered with an additional volume constraint on the film, i.e., $\mathcal{L}^d(\Omega_h^+) = \int_\omega h(x')\,\mathrm{d}x'$ is fixed. An easy adaptation of the proof shows that the relaxed functional $\overline{G}$ is not changed under this constraint, see Remark 6.7 for details.

In Subsection 6.2, we further prove the following general compactness result, from which we deduce the existence of equilibrium configurations for epitaxially strained films.

Theorem 2.5 (Compactness for $\overline{G}$)**.** *Suppose that f is convex and satisfies* (2.1)*. For any $(u_n, h_n)_n$ with $\sup_n G(u_n, h_n) < +\infty$, there exist a subsequence (not relabeled) and functions $u \in GSBD^p(\Omega)$, $h \in BV(\omega; [0, M])$ with $u = u\chi_{\Omega_h}$ and $u = u_0$ on $\omega \times (-1, 0)$ such that*

$$(u_n, h_n) \to (u, h) \quad \text{in} \quad L^0(\Omega; \mathbb{R}^d) {\times} L^1(\omega)\,.$$

In particular, general properties of relaxation (see e.g. [24, Theorem 3.8]) imply that, given $0 < m < M\mathcal{H}^{d-1}(\omega)$, the minimization problem

$$\inf \Big\{ \overline{G}(u, h)\colon (u, E) \in L^0(\Omega; \mathbb{R}^d) \times L^1(\omega),\ \mathcal{L}^d(\Omega_h^+) = m \Big\} \tag{2.11}$$

admits solutions. Moreover, fixed m and the volume constraint $\mathcal{L}^d(\Omega_h^+) = m$ for G and $\overline{G}$, any cluster point for minimizing sequences of G is a minimum point for $\overline{G}$.

Our final issue is a phase-field approximation of $\overline{G}$. The idea is to represent any subgraph Ω_h by a (regular) function v which will be an approximation of the characteristic function χ_{Ω_h} at a scale of order ε. Let $W\colon [0, 1] \to [0, \infty)$ be continuous, with $W(1) = W(0) = 0$, $W > 0$ in $(0, 1)$, and let $(\eta_\varepsilon)_\varepsilon$ with $\eta_\varepsilon > 0$ and $\eta_\varepsilon \varepsilon^{1-p} \to 0$ as $\varepsilon \to 0$. Let $c_W := (\int_0^1 \sqrt{2W(s)}\,\mathrm{d}s)^{-1}$. In the reference domain $\Omega = \omega {\times} (-1, M + 1)$, we introduce the functionals

$$G_\varepsilon(u, v) := \int_\Omega \bigg((v^2 + \eta_\varepsilon) f(e(u)) + c_W \Big(\frac{W(v)}{\varepsilon} + \frac{\varepsilon}{2} |\nabla v|^2 \Big) \bigg) \,\mathrm{d}x\,, \tag{2.12}$$

if

$$u \in W^{1,p}(\Omega; \mathbb{R}^d)\,, \quad u = u_0 \text{ in } \omega {\times} (-1, 0)\,,$$
$$v \in H^1(\Omega; [0, 1])\,, \quad v = 1 \text{ in } \omega {\times} (-1, 0)\,,\, v = 0 \text{ in } \omega {\times} (M, M + 1) \quad \partial_d v \le 0\ \mathcal{L}^d\text{-a.e. in } \Omega\,,$$

and $G_\varepsilon(u, v) := +\infty$ otherwise. The following phase-field approximation is the analog of [18, Theorem 5.1] in the frame of linear elasticity. We remark that here, differently from [18], we assume only $u_0 \in W^{1,p}(\omega \times (-1, 0); \mathbb{R}^d)$, and not necessarily $u_0 \in L^\infty(\omega \times (-1, 0); \mathbb{R}^d)$. For the proof we refer to Subsection 6.3.

Theorem 2.6. *Let $u_0 \in W^{1,p}(\omega \times (-1, 0); \mathbb{R}^d)$. For any decreasing sequence $(\varepsilon_n)_n$ of positive numbers converging to zero, the following hold:*

(i) For any $(u_n, v_n)_n$ with $\sup_n G_{\varepsilon_n}(u_n, v_n) < +\infty$, there exist $u \in L^0(\Omega; \mathbb{R}^d)$ and $h \in BV(\omega; [0, M])$ such that, up to a subsequence, $u_n \to u$ a.e. in Ω, $v_n \to \chi_{\Omega_h}$ in $L^1(\Omega)$, and

$$\overline{G}(u, h) \le \liminf_{n\to+\infty} G_{\varepsilon_n}(u_n, v_n)\,. \tag{2.13}$$

(ii) For any (u, h) with $\overline{G}(u, h) < +\infty$, there exists $(u_n, v_n)_n$ such that $u_n \to u$ a.e. in Ω, $v_n \to \chi_{\Omega_h}$ in $L^1(\Omega)$, and

$$\limsup_{n\to\infty} G_{\varepsilon_n}(u_n, v_n) = \overline{G}(u, h)\,.$$

3. Preliminaries

In this section, we recall the definition and main properties of the function space $GSBD^p$. Moreover, we introduce the space $GSBD^p_\infty$ of functions which may attain the value infinity.

3.1. **Notation.** For every $x \in \mathbb{R}^d$ and $\varrho > 0$, let $B_\varrho(x) \subset \mathbb{R}^d$ be the open ball with center x and radius ϱ. For x, $y \in \mathbb{R}^d$, we use the notation $x \cdot y$ for the scalar product and $|x|$ for the Euclidean norm. By $\mathbb{M}^{d\times d}$ and $\mathbb{M}^{d\times d}_{\rm sym}$ we denote the set of matrices and symmetric matrices, respectively. We write χ_E for the indicator function of any $E \subset \mathbb{R}^n$, which is 1 on E and 0 otherwise. If E is a set of finite perimeter, we denote its essential boundary by $\partial^* E$, and by E^s the set of points with density s for E, see [3, Definition 3.60]. We indicate the minimum and maximum value between $a, b \in \mathbb{R}$ by $a \wedge b$ and $a \vee b$, respectively. The symmetric difference of two sets $A, B \subset \mathbb{R}^d$ is indicated by $A \triangle B$.

We denote by $\mathcal{L}^d$ and $\mathcal{H}^k$ the n-dimensional Lebesgue measure and the k-dimensional Hausdorff measure, respectively. For any locally compact subset $B \subset \mathbb{R}^d$, (i.e. any point in B has a neighborhood contained in a compact subset of B), the space of bounded $\mathbb{R}^m$-valued Radon measures on B [respectively, the space of $\mathbb{R}^m$-valued Radon measures on B] is denoted by $\mathcal{M}_b(B;\mathbb{R}^m)$ [resp., by $\mathcal{M}(B;\mathbb{R}^m)$]. If $m = 1$, we write $\mathcal{M}_b(B)$ for $\mathcal{M}_b(B;\mathbb{R})$, $\mathcal{M}(B)$ for $\mathcal{M}(B;\mathbb{R})$, and $\mathcal{M}^+_b(B)$ for the subspace of positive measures of $\mathcal{M}_b(B)$. For every $\mu \in \mathcal{M}_b(B;\mathbb{R}^m)$, its total variation is denoted by $|\mu|(B)$. Given $\Omega \subset \mathbb{R}^d$ open, we use the notation $L^0(\Omega;\mathbb{R}^d)$ for the space of $\mathcal{L}^d$-measurable functions $v\colon \Omega \to \mathbb{R}^d$.

Definition 3.1. Let $E \subset \mathbb{R}^d$, $v \in L^0(E;\mathbb{R}^m)$, and $x \in \mathbb{R}^d$ such that

$$\limsup_{\varrho \to 0^+} \frac{\mathcal{L}^d(E \cap B_\varrho(x))}{\varrho^d} > 0\,.$$

A vector $a \in \mathbb{R}^d$ is the *approximate limit* of v as y tends to x if for every $\varepsilon > 0$ there holds

$$\lim_{\varrho \to 0^+} \frac{\mathcal{L}^d(E \cap B_\varrho(x) \cap \{|v - a| > \varepsilon\})}{\varrho^d} = 0\,,$$

and then we write

$$\operatorname*{ap\,lim}_{y \to x} v(y) = a\,.$$

Definition 3.2. Let $U \subset \mathbb{R}^d$ be open and $v \in L^0(U;\mathbb{R}^m)$. The *approximate jump set* J_v is the set of points $x \in U$ for which there exist a, $b \in \mathbb{R}^m$, with $a \neq b$, and $\nu \in \mathbb{S}^{d-1}$ such that

$$\operatorname*{ap\,lim}_{(y-x)\cdot\nu>0,\, y\to x} v(y) = a \quad \text{and} \quad \operatorname*{ap\,lim}_{(y-x)\cdot\nu<0,\, y\to x} v(y) = b\,.$$

The triplet (a, b, ν) is uniquely determined up to a permutation of (a, b) and a change of sign of ν, and is denoted by $(v^+(x), v^-(x), \nu_v(x))$. The jump of v is the function defined by $[v](x) := v^+(x) - v^-(x)$ for every $x \in J_v$.

We note that J_v is a Borel set with $\mathcal{L}^d(J_v) = 0$, and that $[v]$ is a Borel function.

3.2. BV **and** BD **functions.** Let $U \subset \mathbb{R}^d$ be open. We say that a function $v \in L^1(U)$ is a *function of bounded variation* on U, and we write $v \in BV(U)$, if $\mathrm{D}_i v \in \mathcal{M}_b(U)$ for $i = 1, \dots, d$, where $\mathrm{D}v = (\mathrm{D}_1 v, \dots, \mathrm{D}_d v)$ is its distributional derivative. A vector-valued function $v\colon U \to \mathbb{R}^m$ is in $BV(U;\mathbb{R}^m)$ if $v_j \in BV(U)$ for every $j = 1, \dots, m$. The space $BV_{\rm loc}(U)$ is the space of $v \in L^1_{\rm loc}(U)$ such that $\mathrm{D}_i v \in \mathcal{M}(U)$ for $i = 1, \dots, d$.

A function $v \in L^1(U;\mathbb{R}^d)$ belongs to the space of *functions of bounded deformation* if the distribution $\mathrm{E}v := \frac{1}{2}((\mathrm{D}v)^T + \mathrm{D}v)$ belongs to $\mathcal{M}_b(U;\mathbb{M}^{d\times d}_{\mathrm{sym}})$. It is well known (see [2, 55]) that for $v \in BD(U)$, J_v is countably $(\mathcal{H}^{d-1}, d-1)$ rectifiable, and that

$$\mathrm{E}v = \mathrm{E}^a v + \mathrm{E}^c v + \mathrm{E}^j v\,,$$

where $\mathrm{E}^a v$ is absolutely continuous with respect to $\mathcal{L}^d$, $\mathrm{E}^c v$ is singular with respect to $\mathcal{L}^d$ and such that $|\mathrm{E}^c v|(B) = 0$ if $\mathcal{H}^{d-1}(B) < \infty$, while $\mathrm{E}^j v$ is concentrated on J_v. The density of $\mathrm{E}^a v$ with respect to $\mathcal{L}^d$ is denoted by $e(v)$.

The space $SBD(U)$ is the subspace of all functions $v \in BD(U)$ such that $\mathrm{E}^c v = 0$. For $p \in (1,\infty)$, we define

$$SBD^p(U) := \{v \in SBD(U)\colon e(v) \in L^p(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}),\ \mathcal{H}^{d-1}(J_v) < \infty\}\,.$$

Analogous properties hold for BV, such as the countable rectifiability of the jump set and the decomposition of $\mathrm{D}v$. The spaces $SBV(U;\mathbb{R}^m)$ and $SBV^p(U;\mathbb{R}^m)$ are defined similarly, with ∇v, the density of $\mathrm{D}^a v$, in place of $e(v)$. For a complete treatment of BV, SBV functions and BD, SBD functions, we refer to [3] and to [2, 6, 55], respectively.

3.3. GBD **functions.** We now recall the definition and the main properties of the space GBD of *generalized functions of bounded deformation*, introduced in [25], referring to that paper for a general treatment and more details. Since the definition of GBD is given by slicing (differently from the definition of GBV, cf. [1, 29]), we first need to introduce some notation. Fixed $\xi \in \mathbb{S}^{d-1} := \{\xi \in \mathbb{R}^d\colon |\xi| = 1\}$, we let

$$\Pi^\xi := \{y \in \mathbb{R}^d\colon y\cdot\xi = 0\}, \qquad B^\xi_y := \{t \in \mathbb{R}\colon y + t\xi \in B\} \quad \text{for any } y \in \mathbb{R}^d \text{ and } B \subset \mathbb{R}^d\,, \tag{3.1}$$

and for every function $v\colon B \to \mathbb{R}^d$ and $t \in B^\xi_y$ let

$$v^\xi_y(t) := v(y + t\xi), \qquad \widehat{v}^\xi_y(t) := v^\xi_y(t)\cdot\xi\,. \tag{3.2}$$

Definition 3.3 ([25])**.** Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, and let $v \in L^0(\Omega;\mathbb{R}^d)$. Then $v \in GBD(\Omega)$ if there exists $\lambda_v \in \mathcal{M}^+_b(\Omega)$ such that one of the following equivalent conditions holds true for every $\xi \in \mathbb{S}^{d-1}$:

(a) for every $\tau \in C^1(\mathbb{R})$ with $-\frac{1}{2} \le \tau \le \frac{1}{2}$ and $0 \le \tau' \le 1$, the partial derivative $\mathrm{D}_\xi\big(\tau(v\cdot\xi)\big) = \mathrm{D}\big(\tau(v\cdot\xi)\big)\cdot\xi$ belongs to $\mathcal{M}_b(\Omega)$, and for every Borel set $B \subset \Omega$

$$\big|\mathrm{D}_\xi\big(\tau(v\cdot\xi)\big)\big|(B) \le \lambda_v(B);$$

(b) $\widehat{v}^\xi_y \in BV_{\mathrm{loc}}(\Omega^\xi_y)$ for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$, and for every Borel set $B \subset \Omega$

$$\int_{\Pi^\xi} \Big(\big|\mathrm{D}\widehat{v}^\xi_y\big|\big(B^\xi_y \setminus J^1_{\widehat{v}^\xi_y}\big) + \mathcal{H}^0\big(B^\xi_y \cap J^1_{\widehat{v}^\xi_y}\big)\Big)\,\mathrm{d}\mathcal{H}^{d-1}(y) \le \lambda_v(B)\,,$$

where $J^1_{\widehat{u}^\xi_y} := \Big\{t \in J_{\widehat{u}^\xi_y} : |[\widehat{u}^\xi_y]|(t) \ge 1\Big\}$.

The function v belongs to $GSBD(\Omega)$ if $v \in GBD(\Omega)$ and $\widehat{v}^\xi_y \in SBV_{\mathrm{loc}}(\Omega^\xi_y)$ for every $\xi \in \mathbb{S}^{d-1}$ and for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$.

$GBD(\Omega)$ and $GSBD(\Omega)$ are vector spaces, as stated in [25, Remark 4.6], and one has the inclusions $BD(\Omega) \subset GBD(\Omega)$, $SBD(\Omega) \subset GSBD(\Omega)$, which are in general strict (see [25, Remark 4.5 and Example 12.3]). Every $v \in GBD(\Omega)$ has an *approximate symmetric gradient* $e(v) \in L^1(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})$ such that for every $\xi \in \mathbb{S}^{d-1}$ and $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$ there holds

$$e(v)(y + t\xi)\xi\cdot\xi = (\widehat{v}^\xi_y)'(t) \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in \Omega^\xi_y\,. \tag{3.3}$$

We recall also that by the area formula (cf. e.g. [53, (12.4)]; see [2, Theorem 4.10] and [25, Theorem 8.1]) it follows that for any $\xi \in \mathbb{S}^{d-1}$

$$(J_v^\xi)_y^\xi = J_{\widehat{v}_y^\xi} \quad \text{for } \mathcal{H}^{d-1}\text{-a.e. } y \in \Pi^\xi, \ \text{ where } \ J_v^\xi := \{x \in J_v \colon [v](x) \cdot \xi \neq 0\}\,, \tag{3.4a}$$

$$\int_{\Pi^\xi} \mathcal{H}^0(J_{\widehat{v}_y^\xi})\,\mathrm{d}\mathcal{H}^{d-1}(y) = \int_{J_v^\xi} |\nu_v \cdot \xi|\,\mathrm{d}\mathcal{H}^{d-1}\,. \tag{3.4b}$$

Moreover, there holds

$$\mathcal{H}^{d-1}(J_v \setminus J_v^\xi) = 0 \qquad \text{for } \mathcal{H}^{d-1}\text{-a.e. } \xi \in \mathbb{S}^{d-1}\,. \tag{3.5}$$

Finally, if Ω has Lipschitz boundary, for each $v \in GBD(\Omega)$ the traces on $\partial\Omega$ are well defined in the sense that for $\mathcal{H}^{d-1}$-a.e. $x \in \partial\Omega$ there exists $\mathrm{tr}(v)(x) \in \mathbb{R}^d$ such that

$$\mathop{\mathrm{ap\,lim}}_{y \to x,\ y \in \Omega} v(y) = \mathrm{tr}(v)(x).$$

For $1 < p < \infty$, the space $GSBD^p(\Omega)$ is defined by

$$GSBD^p(\Omega) := \{u \in GSBD(\Omega) \colon e(u) \in L^p(\Omega; \mathbb{M}^{d\times d}_{\mathrm{sym}}),\ \mathcal{H}^{d-1}(J_u) < \infty\}\,.$$

We recall below two general density and compactness results in $GSBD^p$, from [15] and [17].

Theorem 3.4 (Density in $GSBD^p$)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open, bounded set with finite perimeter and let $\partial\Omega$ be a $(d-1)$-rectifiable, $p > 1$, $\psi(t) = t \wedge 1$, and $u \in GSBD^p(\Omega)$. Then there exist $u_n \in SBV^p(\Omega; \mathbb{R}^d) \cap L^\infty(\Omega; \mathbb{R}^d)$ such that each J_{u_n} is closed in Ω and included in a finite union of closed connected pieces of C^1 hypersurfaces, $u_n \in W^{1,\infty}(\Omega \setminus J_{u_n}; \mathbb{R}^d)$, and:*

$$\int_\Omega \psi(|u_n - u|)\,\mathrm{d}x \to 0\,, \tag{3.6a}$$

$$\|e(u_n) - e(u)\|_{L^p(\Omega)} \to 0\,, \tag{3.6b}$$

$$\mathcal{H}^{d-1}(J_{u_n} \triangle J_u) \to 0\,. \tag{3.6c}$$

We refer to [15, Theorem 1.1]. In contrast to [15], we use here the function $\psi(t) := t \wedge 1$ for simplicity. It is indeed easy to check that [15, (1.1e)] implies (3.6a).

Theorem 3.5 ($GSBD^p$ compactness)**.** *Let $\Omega \subset \mathbb{R}$ be an open, bounded set, and let $(u_n)_n \subset GSBD^p(\Omega)$ be a sequence satisfying*

$$\sup\nolimits_{n\in\mathbb{N}} \big(\|e(u_n)\|_{L^p(\Omega)} + \mathcal{H}^{d-1}(J_{u_n})\big) < +\infty.$$

Then, there exists a subsequence, still denoted by $(u_n)_n$, such that the set $A := \{x \in \Omega \colon |u_n(x)| \to \infty\}$ has finite perimeter, and there exists $u \in GSBD^p(\Omega)$ such that

$$\begin{aligned}
&\text{(i)} \quad u_n \to u \qquad \textit{in } L^0(\Omega \setminus A; \mathbb{R}^d),\\
&\text{(ii)} \quad e(u_n) \rightharpoonup e(u) \qquad \textit{weakly in } L^p(\Omega \setminus A; \mathbb{M}^{d\times d}_{\mathrm{sym}}),\\
&\text{(iii)} \quad \liminf_{n\to\infty} \mathcal{H}^{d-1}(J_{u_n}) \geq \mathcal{H}^{d-1}(J_u \cup (\partial^* A \cap \Omega)).
\end{aligned} \tag{3.7}$$

Moreover, for each $\Gamma \subset \Omega$ with $\mathcal{H}^{d-1}(\Gamma) < +\infty$, there holds

$$\liminf_{n\to\infty} \mathcal{H}^{d-1}(J_{u_n} \setminus \Gamma) \geq \mathcal{H}^{d-1}\big((J_u \cup (\partial^* A \cap \Omega)) \setminus \Gamma\big)\,. \tag{3.8}$$

Proof. We refer to [17]. The additional statement (3.8) is proved, e.g., in [38, Theorem 2.5]. $\square$

Later, as a byproduct of our analysis, we will generalize the lower semicontinuity property (3.7)(iii) to anisotropic surface energies, see Corollary 5.5.

3.4. $GSBD^p_\infty$ **functions.** Inspired by the previous compactness result, we now introduce a space of $GSBD^p$ functions which may also attain a limit value ∞. Define $\bar{\mathbb{R}}^d := \mathbb{R}^d \cup \{\infty\}$. The sum on $\bar{\mathbb{R}}^d$ is given by $a + \infty = \infty$ for any $a \in \bar{\mathbb{R}}^d$. There is a natural bijection between $\bar{\mathbb{R}}^d$ and $\mathbb{S}^d = \{\xi \in \mathbb{R}^{d+1} : |\xi| = 1\}$ given by the stereographic projection of $\mathbb{S}^d$ to $\bar{\mathbb{R}}^d$: for $\xi \neq e_{d+1}$, we define

$$\phi(\xi) = \frac{1}{1-\xi_{d+1}}(\xi_1, \dots, \xi_d),$$

and let $\phi(e_{d+1}) = \infty$. By $\psi : \bar{\mathbb{R}}^d \to \mathbb{S}^d$ we denote the inverse. Note that

$$d_{\bar{\mathbb{R}}^d}(x,y) := |\psi(x) - \psi(y)| \quad \text{for } x, y \in \bar{\mathbb{R}}^d \tag{3.9}$$

induces a bounded metric on $\bar{\mathbb{R}}^d$. We define

$$\begin{aligned} GSBD^p_\infty(\Omega) := \Big\{ & u \in L^0(\Omega; \bar{\mathbb{R}}^d) \colon\ A^\infty_u := \{u = \infty\} \text{ satisfies } \mathcal{H}^{d-1}(\partial^* A^\infty_u) < +\infty, \\ & \tilde{u}_t := u\chi_{\Omega \setminus A^\infty_u} + t\chi_{A^\infty_u} \in GSBD^p(\Omega) \ \text{ for all } t \in \mathbb{R}^d \Big\}. \end{aligned} \tag{3.10}$$

Symbolically, we will also write

$$u = u\chi_{\Omega \setminus A^\infty_u} + \infty\chi_{A^\infty_u}.$$

Moreover, for any $u \in GSBD^p_\infty(\Omega)$, we set $e(u) = 0$ in A^∞_u, and

$$J_u = J_{u\chi_{\Omega \setminus A^\infty_u}} \cup (\partial^* A^\infty_u \cap \Omega). \tag{3.11}$$

In particular, we have

$$e(u) = e(\tilde{u}_t) \ \ \mathcal{L}^d\text{-a.e. on } \Omega \quad \text{ and } \quad J_u = J_{\tilde{u}_t} \ \ \mathcal{H}^{d-1}\text{-a.e.} \quad \text{ for almost all } t \in \mathbb{R}\,, \tag{3.12}$$

where $\tilde{u}_t$ is the function from (3.10). Hereby, we also get a natural definition of a normal ν_u to the jump set J_u, and the slicing properties described in (3.3)–(3.5) still hold. Finally, we point out that all definitions are consistent with the usual ones if $u \in GSBD^p(\Omega)$, i.e., if $A^\infty_u = \emptyset$. Since $GSBD^p(\Omega)$ is a vector space, we observe that the sum of two functions in $GSBD^p_\infty(\Omega)$ lies again in this space.

A metric on $GSBD^p_\infty(\Omega)$ is given by

$$d(u,v) := \int_\Omega d_{\bar{\mathbb{R}}^d}(u(x), v(x))\, \mathrm{d}x\,, \tag{3.13}$$

where $d_{\bar{\mathbb{R}}^d}$ is the distance in (3.9). We now state compactness properties in $GSBD^p_\infty(\Omega)$.

Lemma 3.6 (Compactness in $GSBD^p_\infty$)**.** *For $L > 0$ and $\Gamma \subset \Omega$ with $\mathcal{H}^{d-1}(\Gamma) < +\infty$, we introduce the sets*

$$\begin{aligned} X_L(\Omega) &= \left\{ v \in GSBD^p_\infty(\Omega) \colon \mathcal{H}^{d-1}(J_v) \le L, \ \ \|e(v)\|_{L^p(\Omega)} \le 1 \right\}, \\ X_\Gamma(\Omega) &= \left\{ v \in GSBD^p_\infty(\Omega) \colon \mathcal{H}^{d-1}(J_v \setminus \Gamma) = 0, \ \ \|e(v)\|_{L^p(\Omega)} \le 1 \right\}. \end{aligned} \tag{3.14}$$

Then the sets $X_L(\Omega), X_\Gamma(\Omega) \subset GSBD^p_\infty(\Omega)$ are compact with respect to the metric d.

Proof. For $X_L(\Omega)$, the statement follows from Theorem 3.5 and the definitions (3.10)–(3.11): in fact, given a sequence $(u^n)_n \subset X_L(\Omega)$, we consider a sequence $(\tilde{u}^n_{t_n})_n \subset GSBD^p(\Omega)$ as in (3.10), for suitable $(t_n)_n \subset \mathbb{R}^d$ with $|t_n| \to \infty$. This implies

$$d(u^n, \tilde{u}^n_{t_n}) \to 0 \text{ as } n \to \infty. \tag{3.15}$$

Then, by Theorem 3.5 there exists $v \in GSBD^p(\Omega)$ and $A = \{x \in \Omega \colon |\tilde{u}^n_{t_n}(x)| \to \infty\}$ such that $\tilde{u}^n_{t_n} \to v$ in $L^0(\Omega \setminus A; \mathbb{R}^d)$. We define $u = v\chi_{\Omega \setminus A} + \infty\chi_A \in GSBD^p_\infty(\Omega)$. By (3.7)(ii),(iii) and (3.11) we get that $u \in X_L(\Omega)$. We observe that $d(\tilde{u}^n_{t_n}, u) \to 0$ and then by (3.15) also $d(u^n, u) \to 0$.

The proof for the set $X_\Gamma(\Omega)$ is similar, where we additionally use (3.8) to ensure that $\mathcal{H}^{d-1}(J_u \setminus \Gamma) = 0$. $\square$

In the next sections, we will use the following notation. We say that a sequence $(u_n)_n \subset GSBD^p_\infty(\Omega)$ *converges weakly* to $u \in GSBD^p_\infty(\Omega)$ if

$$\sup\nolimits_{n\in\mathbb{N}} \big(\|e(u_n)\|_{L^p(\Omega)} + \mathcal{H}^{d-1}(J_{u_n})\big) < +\infty \quad \text{and} \quad d(u_n, u) \to 0 \text{ for } n \to \infty\,. \tag{3.16}$$

4. The σ^p_{sym}-convergence of sets

This section is devoted to the introduction of a convergence of sets in the framework of $GSBD^p$ functions analogous to σ^p-convergence defined in [26] for the space $GSBV^p$. This type of convergence of sets will be useful to study the lower limits in the relaxation results in Subsection 6.1 and the compactness properties in Subsection 6.2. We believe that this notion may be of independent interest and is potentially helpful to study also other problems such as quasistatic crack evolution.

We start by recalling briefly the definition of σ^p-convergence in [26]: a sequence of sets $(\Gamma_n)_n$ σ^p-converges to Γ if (i) for any sequence $(u_n)_n$ converging to u in $GSBV^p$ with $J_{u_n} \subset \Gamma_n$, it holds $J_u \subset \Gamma$ and (ii) there exists a $GSBV^p$ function whose jump is Γ, which is approximated by $GSBV^p$ functions with jump included in Γ_n. For sequences of sets $(\Gamma_n)_n$ with $\sup_n \mathcal{H}^{d-1}(\Gamma_n) < +\infty$, a compactness result with respect to σ^p-convergence is obtained by means of Ambrosio's compactness theorem [1], see [26, Theorem 4.7] and [18, Theorem 3.3]. We refer to [26, Section 4.1] for a general motivation to consider such a kind of convergence.

We now introduce the notion of σ^p_{sym}-convergence. In the following, we use the notation $A \tilde{\subset} B$ if $\mathcal{H}^{d-1}(A \setminus B) = 0$ and $A \tilde{=} B$ if $A \tilde{\subset} B$ and $B \tilde{\subset} A$. As before, by $(G)^1$ we denote the set of points with density 1 for $G \subset \mathbb{R}^d$. Recall also the definition and properties of $GSBD^p_\infty$ in Subsection 3.4, in particular (3.16).

Definition 4.1 (σ^p_{sym}-convergence)**.** Let $U \subset \mathbb{R}^d$ be open, let $U' \supset U$ be open with $\mathcal{L}^d(U' \setminus U) > 0$, and let $p \in (1, \infty)$. We say that a sequence $(\Gamma_n)_n \subset \overline{U} \cap U'$ with $\sup_{n\in\mathbb{N}} \mathcal{H}^{d-1}(\Gamma_n) < +\infty$ σ^p_{sym}-converges to a pair (Γ, G_∞) satisfying $\Gamma \subset \overline{U} \cap U'$ together with

$$\mathcal{H}^{d-1}(\Gamma) < +\infty, \quad G_\infty \subset U, \quad \partial^* G_\infty \cap U' \,\tilde{\subset}\, \Gamma, \quad \text{and} \quad \Gamma \cap (G_\infty)^1 = \emptyset \tag{4.1}$$

if there holds:

(i) for any sequence $(v_n)_n \subset GSBD^p_\infty(U')$ with $J_{v_n} \tilde{\subset} \Gamma_n$ and $v_n = 0$ in $U' \setminus U$, if a subsequence $(v_{n_k})_k$ converges weakly in $GSBD^p_\infty(U')$ to $v \in GSBD^p_\infty(U')$, then $J_v \setminus \Gamma \tilde{\subset} (G_\infty)^1$.

(ii) there exists a function $v \in GSBD^p_\infty(U')$ and a sequence $(v_n)_n \subset GSBD^p_\infty(U')$ converging weakly in $GSBD^p_\infty(U')$ to v such that $J_{v_n} \tilde{\subset} \Gamma_n$, $v_n = 0$ on $U' \setminus U$ for all $n \in \mathbb{N}$, $J_v \tilde{=} \Gamma$, and $\{v = \infty\} = G_\infty$.

Our definition deviates from σ^p-convergence in the sense that, besides a limiting $(d-1)$-rectifiable set Γ, there exists also a set of finite perimeter G_∞. Roughly speaking, in view of $\partial^* G_\infty \subset \Gamma \cup \partial U$, this set represents the parts which are completely disconnected by Γ from the rest of the domain. The behavior of functions cannot be controlled there, i.e., a sequence $(v_n)_n$ as in (i) may converge to infinity on this set or exhibit further cracks. In [26], it was possible to work with truncations to avoid such a phenomenon. In $GSBD$, however, this truncation technique is not available and we therefore need a more general definition involving the space $GSBD^p_\infty$ and a set of finite perimeter G_∞.

Moreover, due to the presence of the set G_∞, in contrast to the definition of σ^p-convergence, it is essential to control the functions in a set $U' \setminus U$: the assumptions $\mathcal{L}^d(U' \setminus U) > 0$ and $G_\infty \subset U$ are crucial since otherwise, if $U' = U$, conditions (i) and (ii) would always be trivially satisfied with $G_\infty = U$ and $\Gamma = \emptyset$. We also point out that, given a sequence $(\Gamma_n)_n$, the $\sigma^p_{\rm sym}$-limit is not unique: consider $\Gamma_n = \partial B$ for some small ball $B \subset U$ for all $n \in \mathbb{N}$. Then possible limits (Γ, G_∞) are $(\partial B, B)$ and $(\partial B, \emptyset)$. (To see this, use that $X_{\partial B}(U')$, defined in (3.14), is closed.)

The main goal of this section is to prove the following compactness result for $\sigma^p_{\rm sym}$-convergence.

Theorem 4.2 (Compactness of $\sigma^p_{\rm sym}$-convergence)**.** *Let $U \subset \mathbb{R}^d$ be open, let $U' \supset U$ be open with $\mathcal{L}^d(U' \setminus U) > 0$, and let $p \in (1, \infty)$. Then, every sequence $(\Gamma_n)_n \subset U$ with $\sup_n \mathcal{H}^{d-1}(\Gamma_n) < +\infty$ has a $\sigma^p_{\rm sym}$-convergent subsequence with limit (Γ, G_∞) satisfying $\mathcal{H}^{d-1}(\Gamma) \le \liminf_{n\to\infty} \mathcal{H}^{d-1}(\Gamma_n)$.*

For the proof, we need the following two auxiliary results.

Lemma 4.3. *Let $(v_i)_i \subset GSBD^p(\Omega)$ such that $\|e(v_i)\|_{L^p(\Omega)} \le 1$ for all i and $\Gamma := \bigcup_{i=1}^\infty J_{v_i}$ satisfies $\mathcal{H}^{d-1}(\Gamma) < +\infty$. Then there exist constants $c_i > 0$, $i \in \mathbb{N}$, such that $\sum_{i=1}^\infty c_i \le 1$ and $v := \sum_{i=1}^\infty c_i v_i \in GSBD^p(\Omega)$ satisfies $J_v \tilde{=} \bigcup_{i=1}^\infty J_{v_i}$.*

Lemma 4.4. *Let $V \subset \mathbb{R}^d$ and suppose that two sequences $(u_n)_n, (v_n)_n \in L^0(V; \bar{\mathbb{R}}^d)$ satisfy $|u_n|, |v_n| \to \infty$ on V. Then for $\mathcal{L}^1$-a.e. $\theta \subset (0,1)$ there holds*

$$|(1-\theta)u_n(x) + \theta v_n(x)| \to \infty \quad \textit{for a.e. } x \in V.$$

We postpone the proof of the lemmas and proceed with the proof of Theorem 4.2.

Proof of Theorem 4.2. For $\Gamma \subset U$ with $\mathcal{H}^{d-1}(\Gamma) < +\infty$ we define

$$X(\Gamma) = \big\{ v \in GSBD^p_\infty(U') \colon J_v \tilde{\subset} \Gamma, \ \ \|e(v)\|_{L^p(U')} \le 1, \quad v = 0 \text{ on } U' \setminus U \big\}.$$

The set $X(\Gamma)$ is compact with respect to the metric d introduced in (3.13). This follows from Lemma 3.6 and the fact that $\{v \in L^0(U'; \bar{\mathbb{R}}^d) \colon v = 0 \text{ on } U' \setminus U\}$ is closed with respect to d.

Since we treat any $v \in GSBD^p_\infty(U')$ as a constant function in the exceptional set A^∞_v (namely we have no jump and $e(v) = 0$ therein, see (3.12)), we get that the convex combination of two $v, v' \in X(\Gamma)$ is still in $X(\Gamma)$. (Recall that the sum on $\bar{\mathbb{R}}^d$ is given by $a + \infty = \infty$ for any $a \in \bar{\mathbb{R}}^d$.)

Step 1: Identification of a compact subset. Consider $(\Gamma_n)_n \subset U$ with $\sup_{n\in\mathbb{N}} \mathcal{H}^{d-1}(\Gamma_n) < +\infty$. Fix $\delta > 0$ small and define

$$L := \liminf_{n\to\infty} \mathcal{H}^{d-1}(\Gamma_n) + \delta\,. \tag{4.2}$$

Thus, up to a subsequence (not relabeled), there holds that each $X(\Gamma_n)$ is compact and is contained in $X_L(U')$ defined in (3.14). Hence, a subsequence (not relabeled) converges in the Hausdorff sense (with the Hausdorff distance induced by d) to a compact set $K \subset X_L(U')$.

We first observe that the function identical to zero lies in K. We now show that K is convex. Choose $u, v \in K$ and $\theta \in (0,1)$. We need to check that $w := (1-\theta)u + \theta v \in K$. Observe that $A^\infty_w = A^\infty_u \cup A^\infty_v$, where A^∞_u, A^∞_v, and A^∞_w are the exceptional sets given in (3.10). There exist sequences $(u_n)_n$ and $(v_n)_n$ with $u_n, v_n \in X(\Gamma_n)$ such that $d(u_n, u) \to 0$ and $d(v_n, v) \to 0$. In particular, note that $|u_n| \to \infty$ on A^∞_u and $|v_n| \to \infty$ on A^∞_v. By Lemma 4.4 and a diagonal argument we can choose $(\theta_n)_n \subset (0,1)$ with $\theta_n \to \theta$ such that $w_n := (1-\theta_n)u_n + \theta_n v_n$ satisfies $|w_n| \to \infty$ on $A^\infty_u \cap A^\infty_v$. As clearly $|w_n| \to \infty$ on $A^\infty_u \triangle A^\infty_v$ and $(1-\theta_n)u_n + \theta_n v_n \to (1-\theta)u + \theta v$ in measure on $U' \setminus (A^\infty_u \cup A^\infty_v)$, we get $d(w_n, w) \to 0$. Since $X(\Gamma_n)$ is convex, there holds $w_n \in X(\Gamma_n)$. Then $d(w_n, w) \to 0$ implies $w \in K$, as desired.

Step 2: Definition of Γ and G_∞. Fix $v, v' \in K$ and again denote by A^∞_v, $A^\infty_{v'}$ the sets where the functions attain the value ∞. Since $\{x \in J_v \setminus \partial^* A^\infty_v \colon [v](x) = t\}$ has negligible $\mathcal{H}^{d-1}$-measure up to a countable set of points t, we find some $\theta \in (0,1)$ such that $w := \theta v + (1-\theta)v'$ satisfies

$$J_w \tilde{\subset} J_v \cup J_{v'}, \qquad (J_v \cup J_{v'}) \setminus J_w \tilde{\subset} (A^\infty_v \cup A^\infty_{v'})^1. \tag{4.3}$$

Here, we particularly point out that $\{w = \infty\} = A^\infty_v \cup A^\infty_{v'}$ and that $\partial^*(A^\infty_v \cup A^\infty_{v'}) \cap U' \tilde{\subset} J_w$ by (3.11). Note that $w \in K$ since K is convex. Since $w \in K \subset X_L(U')$, (4.3) implies

$$\mathcal{H}^{d-1}((J_v \cup J_{v'}) \setminus (A^\infty_v \cup A^\infty_{v'})^1) \le \mathcal{H}^{d-1}(J_w) \le L, \qquad \mathcal{H}^{d-1}(\partial^*(A^\infty_v \cup A^\infty_{v'}) \cap U') \le \mathcal{H}^{d-1}(J_w) \le L\,.$$

Since K is compact with respect to the metric d, we can choose a sequence $(y_i)_i \subset GSBD^p_\infty(U')$ with $y_i = 0$ on $U' \setminus U$ which is d-dense in K. By the above convexity argument, we find

$$\mathcal{H}^{d-1}\Big(\bigcup\nolimits_{i=1}^k J_{y_i} \setminus \big(\bigcup\nolimits_{i=1}^k A_i\big)^1\Big) \le L, \qquad \mathcal{H}^{d-1}\Big(\partial^*\big(\bigcup\nolimits_{i=1}^k A_i\big) \cap U'\Big) \le L \tag{4.4}$$

for all $k \in \mathbb{N}$, where

$$A_i := A^\infty_{y_i} = \{y_i = \infty\}\,.$$

We define

$$G_\infty := \bigcup\nolimits_{i=1}^\infty A_i\,. \tag{4.5}$$

By passing to the limit $k \to \infty$ in (4.4), we get $\mathcal{H}^{d-1}(\partial^* G_\infty \cap U') \le L$ and $\mathcal{H}^{d-1}(\bigcup_{i=1}^k J_{y_i} \setminus (G_\infty)^1) \le L$ for all $k \in \mathbb{N}$. Passing again to the limit $k \to \infty$, and setting

$$\Gamma := \bigcup\nolimits_{i=1}^\infty J_{y_i} \setminus (G_\infty)^1 \tag{4.6}$$

we get $\mathcal{H}^{d-1}(\Gamma) \le L$. Notice that $\Gamma \cap (G_\infty)^1 = \emptyset$ by definition. Moreover, the fact that $y_i = 0$ on $U' \setminus U$ for all $i \in \mathbb{N}$ implies both that $G_\infty \subset U$ and that $\Gamma \subset \overline{U} \cap U'$. By (4.2) and the arbitrariness of δ we get $\mathcal{H}^{d-1}(\Gamma) \le \liminf_{n\to\infty} \mathcal{H}^{d-1}(\Gamma_n)$. Since $\partial^* A_i \cap U' \tilde{\subset} J_{y_i}$ for all $i \in \mathbb{N}$ by (3.11), we also get $\Gamma \tilde{\supset} \partial^* G_\infty \cap U'$. Thus, (4.1) is satisfied.

We now claim that for each $v \in K$ there holds

$$J_v \setminus \Gamma \tilde{\subset} (G_\infty)^1. \tag{4.7}$$

Indeed, for any fixed $v \in K$, there is a sequence $(y_k)_k = (y_{i_k})_k$ with $d(y_k, v) \to 0$, by the density of $(y_i)_i$. Consider the functions $\tilde{v}_k := y_k(1 - \chi_{G_\infty})$ that d-converge to $\tilde{v} := v(1-\chi_{G_\infty})$: since $J_{\tilde{v}_k} \tilde{\subset} \Gamma$ for any k (we employ (4.6) and that $\partial^* G_\infty \cap U' \tilde{\subset} \Gamma$), the fact that $X(\Gamma)$ is closed gives that $J_{\tilde{v}} \tilde{\subset} \Gamma$. This implies (4.7).

Step 3: Proof of properties (i) and (ii). We first show (i). Given a sequence $(v_n)_n \subset GSBD^p_\infty(U')$ with $J_{v_n} \tilde{\subset} \Gamma_n$ and $v_n = 0$ on $U' \setminus U$, and a subsequence $(v_{n_k})_k$ that converges weakly in $GSBD^p_\infty(U')$ to v, we clearly get $v \in K$ by Hausdorff convergence of $X(\Gamma_n) \to K$. (More precisely, consider λv_{n_k} and λv for $\lambda > 0$ such that $\|e(\lambda v_{n_k})\|_{L^p(U')} \le 1$ for all k.) By (4.7), this implies $J_v \setminus \Gamma \tilde{\subset} (G_\infty)^1$. This shows (i).

We now address (ii). Recalling the choice of the sequence $(y_i)_i \subset K$, for each $i \in \mathbb{N}$, we choose $\tilde{y}_i = y_i \chi_{U' \setminus G_\infty} + t_i \chi_{G_\infty} \in GSBD^p(U')$ for some $t_i \in \mathbb{R}^d$ such that $J_{\tilde{y}_i} \tilde{=} J_{y_i} \setminus (G_\infty)^1$. (Almost every t_i works. Note that the function indeed lies in $GSBD^p(U)$, see (3.10) and (4.5).) In view of (4.6), we also observe that $\bigcup_i J_{\tilde{y}_i} = \Gamma$.

By Lemma 4.3 (recall $(y_i)_i \subset K \subset X_L(\Omega)$) we get a function $\tilde{v} = \sum_{i=1}^\infty c_i \tilde{y}_i \in GSBD^p(U')$ such that $J_{\tilde{v}} \tilde{=} \Gamma$, where $\sum_{i=1}^\infty c_i \le 1$. We also define $v = \tilde{v} \chi_{U' \setminus G_\infty} + \infty \chi_{G_\infty} \in GSBD^p_\infty(U')$. Note that $\{v = \infty\} = G_\infty$ and $J_v \tilde{=} \Gamma$ since $\Gamma \cap (G_\infty)^1 = \emptyset$ and $\partial^* G_\infty \cap U' \subset \Gamma$. Then by the convexity of K, we find $z_k := \sum_{i=1}^k c_i y_i \in K$. (Here we also use that the function identical to zero lies in K.) As $G_\infty = \bigcup_{i=1}^\infty A_i$, we obtain $d(z_k, v) \to 0$ for $k \to \infty$. Thus, also $v \in K$

since K is compact. As $X(\Gamma_n)$ converges to K in Hausdorff convergence, we find a sequence $(v_n)_n \subset GSBD^p_\infty(U')$ with $J_{v_n} \tilde{\subset} \Gamma_n$, $v_n = 0$ on $U' \setminus U$, and $d(v_n, v) \to 0$. This shows (ii). $\square$

The next corollary will be instrumental in the following.

Corollary 4.5. *Let $U \subset \mathbb{R}^d$ be open, let $U' \supset U$ be open with $\mathcal{L}^d(U' \setminus U) > 0$, and let $p \in (1, \infty)$. Suppose that the sequence $(\Gamma_n)_n \subset U$ with $\sup_n \mathcal{H}^{d-1}(\Gamma_n) < +\infty$ is σ^p_{sym}-convergent with limit (Γ, G_∞). Let $(v_n)_n \subset GSBD^p_\infty(U')$ with $J_{v_n} \tilde{\subset} \Gamma_n$ and $v_n = 0$ in $U' \setminus U$ which converges weakly in $GSBD^p_\infty(U')$ to $v \in GSBD^p_\infty(U')$. Then Γ_n σ^p_{sym}-converges also to the limit $(\Gamma, G_\infty \cup \{v = \infty\})$.*

Proof. This follows immediately from the proof of Theorem 4.2: up to rescaling we may assume that $\|e(v_n)\|_{L^p(U')} \leq 1$. Note that the function v lies in K due to the Hausdorff convergence $X(\Gamma_n) \to K$. We now include v into the countable, dense subset $(y_i)_i \subset K$ introduced in Step 2 of the above proof. Then, we observe that the definition $G_\infty = \bigcup_{i=1}^\infty A_i$ in (4.5) can be replaced by $\bigcup_{i=1}^\infty A_i \cup \{v = \infty\}$. $\square$

Next, we prove Lemma 4.3. To this end, we will need the following measure-theoretical result. (See [34, Lemma 4.1, 4.2] and note that the statement in fact holds in arbitrary space dimensions for measurable functions.)

Lemma 4.6. *Let $\Omega \subset \mathbb{R}^d$ with $\mathcal{L}^d(\Omega) < \infty$, and $N \in \mathbb{N}$. Then for every sequence $(u_n)_n \subset L^0(\Omega; \mathbb{R}^N)$ with*

$$\mathcal{L}^d\Big(\bigcap\nolimits_{n\in\mathbb{N}} \bigcup\nolimits_{m \geq n} \{|u_m - u_n| > 1\}\Big) = 0 \tag{4.8}$$

there exist a subsequence (not relabeled) and an increasing concave function $\psi : [0, \infty) \to [0, \infty)$ with $\lim_{t\to\infty} \psi(t) = +\infty$ such that

$$\sup_{n\geq 1} \int_\Omega \psi(|u_n|)\,\mathrm{d}x < +\infty.$$

Proof of Lemma 4.3. Let $(v_i)_i \subset GSBD^p(\Omega)$ be given satisfying the assumptions of the lemma. First, choose $0 < d_i < 2^{-i}$ such that

$$\mathcal{L}^d\Big(\Big\{|v_i| \geq \frac{1}{2^i d_i}\Big\}\Big) \leq 2^{-i}, \qquad \mathcal{H}^{d-1}\Big(\Big\{x \in J_{v_i} \colon |[v_i](x)| \geq \frac{1}{d_i}\Big\}\Big) \leq 2^{-i}. \tag{4.9}$$

Our goal is to select constants $c_i \in (0, d_i)$ such that the function $v := \sum_{i=1}^\infty c_i v_i$ lies in $GSBD^p(\Omega)$ and satisfies $J_v \tilde{=} \Gamma := \bigcup_{i=1}^\infty J_{v_i}$. We proceed in two steps: we first show that for each choice $c_i \in (0, d_i)$ the function $v = \sum_{i=1}^\infty c_i v_i$ lies indeed in $GSBD^p(\Omega)$ (Step 1). Afterwards, we prove that for a specific choice there holds $J_v \tilde{=} \Gamma$.

Step 1. Given $c_i \in (0, d_i)$, we define $u_k = \sum_{i=1}^k c_i v_i$. Fix $m \geq n$. We observe that

$$\{|u_m - u_n| > 1\} = \Big\{\big|\sum\nolimits_{i=n+1}^m c_i v_i\big| > 1\Big\} \subset \bigcup\nolimits_{i=n+1}^m \Big\{|c_i v_i| \geq 2^{-i}\Big\} \subset \bigcup\nolimits_{i=n+1}^m \Big\{|v_i| \geq \frac{1}{2^i d_i}\Big\}.$$

By passing to the limit $m \to \infty$ and by using (4.9) we get

$$\mathcal{L}^d\Big(\bigcup\nolimits_{m\geq n} \{|u_m - u_n| > 1\}\Big) \leq \sum\nolimits_{i=n+1}^\infty \mathcal{L}^d\Big(\Big\{|v_i| \geq \frac{1}{2^i d_i}\Big\}\Big) \leq \sum\nolimits_{i=n+1}^\infty 2^{-i} = 2^{-n}.$$

This shows that the sequence $(u_k)_k$ satisfies (4.8), and therefore there exist a subsequence (not relabeled) and an increasing continuous function $\psi : [0, \infty) \to [0, \infty)$ with $\lim_{t\to\infty} \psi(t) = +\infty$ such that $\sup_{k\geq 1} \int_\Omega \psi(|u_k|)\,\mathrm{d}x < +\infty$. Recalling also that $\|e(v_i)\|_{L^p(\Omega)} \leq 1$ for all i and $\mathcal{H}^{d-1}(\Gamma) < +\infty$, we are now in the position to apply the $GSBD^p$-compactness result [25, Theorem 11.3] (alternatively, one could apply Theorem 3.5 and observe that the limit v satisfies

$\mathcal{L}^d(\{v = \infty\}) = 0$), to get that the function $v = \sum_{i=1}^\infty c_i v_i$ lies in $GSBD^p(\Omega)$. For later purposes, we note that by (3.8) (which holds also in addition to [25, Theorem 11.3]) we obtain

$$J_v \tilde{\subset} \bigcup_{i=1}^\infty J_{v_i} = \Gamma. \tag{4.10}$$

This concludes Step 1 of the proof.

Step 2. We define the constants $c_i \in (0, d_i)$ iteratively by following the arguments in [26, Lemma 4.5]. Suppose that $(c_i)_{i=1}^k$, and a decreasing sequence $(\varepsilon_i)_{i=1}^k \subset (0,1)$ have been chosen such that the functions $u_j = \sum_{i=1}^j c_i v_i$, $1 \le j \le k$, satisfy

$$\begin{aligned} &\text{(i)} \quad J_{u_j} \tilde{=} \bigcup_{i=1}^j J_{v_i}, \\ &\text{(ii)} \quad \mathcal{H}^{d-1}(\{x \in J_{u_j} \colon |[u_j](x)| \le \varepsilon_j\}) \le 2^{-j}, \end{aligned} \tag{4.11}$$

and, for $2 \le j \le k$, there holds

$$c_j \le \varepsilon_{j-1} d_j 2^{-j-1}. \tag{4.12}$$

(Note that in the first step we can simply set $c_1 = 1/4$ and $0 < \varepsilon_1 < 1$ such that (4.11)(ii) holds.)

We pass to the step $k+1$ as follows. Note that there is a set $N_0 \subset \mathbb{R}$ of negligible measure such that for all $t \in \mathbb{R} \setminus N_0$ there holds $J_{u_k + t v_{k+1}} \tilde{=} J_{u_k} \cup J_{v_{k+1}}$. We choose $c_{k+1} \in \mathbb{R} \setminus N_0$ such that additionally $c_{k+1} \le \varepsilon_k d_{k+1} 2^{-k-2}$. Then (4.11)(i) and (4.12) hold. We can then choose $\varepsilon_{k+1} \le \varepsilon_k$ such that also (4.11)(ii) is satisfied.

We proceed in this way for all $k \in \mathbb{N}$. Let us now introduce the sets

$$E_k = \bigcup_{m \ge k} \{x \in J_{u_m} \colon \ |[u_m](x)| \le \varepsilon_m\}, \quad F_k = \bigcup_{m \ge k} \{x \in J_{v_m} \colon \ |[v_m](x)| > 1/d_m\}\,. \tag{4.13}$$

Note by (4.9) and (4.11)(ii) that

$$\mathcal{H}^{d-1}(E_k \cup F_k) \le 2 \sum_{m \ge k} 2^{-m} = 2^{2-k}\,. \tag{4.14}$$

We now show that for all $k \in \mathbb{N}$ there holds

$$J_{u_k} \tilde{\subset} J_v \cup E_k \cup F_k\,. \tag{4.15}$$

To see this, we first observe that for $\mathcal{H}^{d-1}$-a.e. $x \in \Gamma = \bigcup_{i=1}^\infty J_{v_i}$ there holds

$$[v](x) = [u_k](x) + \sum_{i=k+1}^\infty c_i [v_i](x)\,. \tag{4.16}$$

Moreover, we get that $c_i \le \varepsilon_k d_i 2^{-i-1}$ for all $i \ge k+1$ by (4.12) and the fact that $(\varepsilon_i)_i$ is decreasing. Fix $x \in J_{u_k} \setminus (E_k \cup F_k)$. Then by (4.13) and (4.16) we get

$$|[v](x)| \ge |[u_k](x)| - \sum_{i=k+1}^\infty c_i |[v_i](x)| \ge \varepsilon_k - \sum_{i=k+1}^\infty \frac{c_i}{d_i} \ge \varepsilon_k \Big(1 - \sum_{i=k+1}^\infty 2^{-i-1}\Big) \ge \varepsilon_k/2\,,$$

where we have used that $|[u_k](x)| \ge \varepsilon_k$ and $[v_i](x) \le 1/d_i$, for $i \ge k+1$. Thus, $[v](x) \ne 0$ and therefore $x \in J_v$. Consequently, we have shown that $\mathcal{H}^{d-1}$-a.e. $x \in J_{u_k} \setminus (E_k \cup F_k)$ lies in J_v. This shows (4.15).

We now conclude the proof as follows: by (4.11)(i) and (4.15) we get that

$$\bigcup_{i=1}^l J_{v_i} \tilde{=} J_{u_l} \tilde{\subset} J_v \cup E_l \cup F_l \tilde{\subset} J_v \cup E_k \cup F_k$$

for all $l \ge k$, where we used that the sets $(E_k)_k$ and $(F_k)_k$ are decreasing. Taking the union with respect to l, we get that $\Gamma \tilde{\subset} J_v \cup E_k \cup F_k$ for all $k \in \mathbb{N}$. By (4.14) this implies $\mathcal{H}^{d-1}(\Gamma \setminus J_v) \le 2^{2-k}$. Since $k \in \mathbb{N}$ was arbitrary, we get $\Gamma \tilde{\subset} J_v$. This along with (4.10) shows $J_v \tilde{=} \Gamma$ and concludes the proof. $\square$

We close this section with the proof of Lemma 4.4.

Proof of Lemma 4.4. Let $B = \{x \in V\colon\ \limsup_{n\to\infty} |u_n(x) - v_n(x)| < +\infty\}$. For $\theta \in (0,1)$, define $w_n^\theta = (1-\theta)u_n + \theta v_n$ and observe that $|w_n^\theta| \to \infty$ on B for all θ since $|u_n| \to \infty$ on V. Let $D_\theta = \{x \in V \setminus B\colon\ \limsup_{n\to\infty} |w_n^\theta(x)| < +\infty\}$. As $|u_n - v_n| \to \infty$ on $V \setminus B$ and thus $|w_n^{\theta_1} - w_n^{\theta_2}| = |(\theta_1-\theta_2)(v_n - u_n)| \to \infty$ on $V\setminus B$ for all $\theta_1 \neq \theta_2$, we obtain $D_{\theta_1} \cap D_{\theta_2} = \emptyset$. This implies that $\mathcal{L}^d(D_\theta) > 0$ for an at most countable number of different θ. We note that for all θ with $\mathcal{L}^d(D_\theta) = 0$ there holds $|w_n^\theta| \to \infty$ a.e. on V. This yields the claim. □

5. Functionals defined on pairs of function-set

This section is devoted to the proofs of the results announced in Subsection 2.1. Before proving the relaxation and existence results, we address the lower bound separately since this will be instrumental also for Section 6.

5.1. **The lower bound.**

In this subsection we prove a lower bound for functionals defined on pairs of function-set which will be needed for the proof of Theorem 2.2–Theorem 2.4. We will make use of the definition of $GSBD^p_\infty(\Omega)$ in Subsection 3.4. In particular, we refer to the definition of $e(u)$ and of the jump set J_u with its normal ν_u, see (3.11)–(3.12), as well as to the notion of weak convergence in $GSBD^p_\infty(\Omega)$, see (3.16). We recall also that for any $s \in [0,1]$ and any $E \in \mathfrak{M}(\Omega)$, E^s denotes the set of points with density s for E, see [3, Definition 3.60].

Theorem 5.1 (Lower bound)**.** *Let $\Omega \subset \mathbb{R}^d$ be open and bounded, let $1 < p < \infty$. Consider a sequence of Lipschitz sets $(E_n)_n \subset \Omega$ with $\sup_{n\in\mathbb{N}} \mathcal{H}^{d-1}(\partial E_n) < +\infty$ and a sequence of functions $(u_n)_n \subset GSBD^p(\Omega)$ such that $u_n|_{\Omega\setminus\overline{E_n}} \in W^{1,p}(\Omega \setminus \overline{E_n};\mathbb{R}^d)$ and $u_n = 0$ in E_n. Let $u \in GSBD^p_\infty(\Omega)$ and $E \in \mathfrak{M}(\Omega)$ such that u_n converges weakly in $GSBD^p_\infty(\Omega)$ to u and*

$$\chi_{E_n} \to \chi_E \text{ in } L^1(\Omega)\,. \tag{5.1}$$

Then, for any norm φ on $\mathbb{R}^d$ there holds

$$e(u_n)\chi_{\Omega\setminus(E_n\cup A^\infty_u)} \rightharpoonup e(u)\chi_{\Omega\setminus(E\cup A^\infty_u)} \quad \text{weakly in } L^p(\Omega;\mathbb{M}^{d\times d}_{\rm sym})\,, \tag{5.2a}$$

$$\int_{J_u\cap E^0} 2\varphi(\nu_u)\,\mathrm{d}\mathcal{H}^{d-1} + \int_{\Omega\cap\partial^*E} \varphi(\nu_E)\,\mathrm{d}\mathcal{H}^{d-1} \le \liminf_{n\to+\infty} \int_{\Omega\cap\partial E_n} \varphi(\nu_{E_n})\,\mathrm{d}\mathcal{H}^{d-1}\,, \tag{5.2b}$$

where $A^\infty_u = \{u = \infty\}$.

In the proof, we need the following two auxiliary results, see [11, Proposition 4, Lemma 5].

Proposition 5.2. *Let Ω be an open subset of $\mathbb{R}^d$ and μ be a finite, positive set function defined on $\mathfrak{M}(\Omega)$. Let $\lambda \in \mathcal{M}^+_b(\Omega)$, and $(g_i)_{i\in\mathbb{N}}$ be a family of positive Borel functions on Ω. Assume that $\mu(U) \ge \int_U g_i\,\mathrm{d}\lambda$ for every U and i, and that $\mu(U\cup V) \ge \mu(U)+\mu(V)$ whenever $U, V \subset\subset \Omega$ and $\overline{U}\cap\overline{V} = \emptyset$ (superadditivity). Then $\mu(U) \ge \int_U (\sup_{i\in\mathbb{N}} g_i)\,\mathrm{d}\lambda$ for every $U \in \mathfrak{M}(\Omega)$.*

Lemma 5.3. *Let $\Gamma \subset E^0$ be a $(d-1)$-rectifiable subset, $\xi \in \mathbb{S}^{d-1}$ such that ξ is not orthogonal to the normal ν_Γ to Γ at any point of Γ. Then, for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$, the set E^ξ_y (see (3.1)) has density 0 in t for every $t \in \Gamma^\xi_y$.*

Proof of Theorem 5.1. Since u_n converges weakly in $GSBD^p_\infty(\Omega)$ to u, (3.16) implies

$$\sup\nolimits_{n\in\mathbb{N}} \big(\|e(u_n)\|^p_{L^p(\Omega)} + \mathcal{H}^{d-1}(J_{u_n}) + \mathcal{H}^{d-1}(\partial E_n)\big) =: M < +\infty. \tag{5.3}$$

Consequently, Theorem 3.5 and the fact that $d(u_n, u) \to 0$, see (3.13) and (3.16), imply that $A_u^\infty = \{u = \infty\} = \{x \in \Omega \colon |u_n(x)| \to \infty\}$ and

$$u_n \to u \qquad \mathcal{L}^d\text{-a.e. in } \Omega \setminus A_u^\infty\,, \tag{5.4a}$$

$$e(u_n) \rightharpoonup e(u) \quad \text{weakly in } L^p(\Omega \setminus A_u^\infty; \mathbb{M}^{d\times d}_{\mathrm{sym}})\,. \tag{5.4b}$$

By (5.1), (5.4a), $u_n = 0$ on E_n, and the definition of A_u^∞, we have

$$E \cap A_u^\infty = \emptyset \quad \text{and} \quad u = 0 \ \ \mathcal{L}^d\text{-a.e. in } E. \tag{5.5}$$

Then (5.4b) gives (5.2a).

We now show (5.2b) which is the core of the proof. Let φ^* be the dual norm of φ and observe that (see, e.g. [10, Section 4.1.2])

$$\varphi(\nu) = \max_{\xi\in\mathbb{S}^{d-1}} \frac{\nu\cdot\xi}{\varphi^*(\xi)} = \max_{\xi\in\mathbb{S}^{d-1}} \frac{|\nu\cdot\xi|}{\varphi^*(\xi)}\,, \tag{5.6}$$

where the second equality holds since $\varphi(\nu) = \varphi(-\nu)$.

As a preparatory step, we consider a set $B \subset \Omega$ with Lipschitz boundary and a function v with $v|_{\Omega\setminus\overline{B}} \in W^{1,p}(\Omega\setminus\overline{B};\mathbb{R}^d)$ and $v = 0$ in B (observe that $v \in GSBD^p(\Omega)$). Recall the notation in (3.1)–(3.2). Let $\varepsilon \in (0,1)$ and $U \in \mathfrak{M}(\Omega)$. For each $\xi \in \mathbb{S}^{d-1}$ and $y \in \Pi^\xi$, we define

$$F^\xi_\varepsilon(\widehat{v}^\xi_y, B^\xi_y; U^\xi_y) = \varepsilon \int_{U^\xi_y\setminus B^\xi_y} |(\widehat{v}^\xi_y)'|^p \,\mathrm{d}t + \mathcal{H}^0(\partial B^\xi_y \cap U^\xi_y)\frac{1}{\varphi^*(\xi)}\,. \tag{5.7}$$

By Fubini-Tonelli Theorem, with the slicing properties (3.3), (3.4), (3.5), for a.e. $\xi\in\mathbb{S}^{d-1}$ there holds

$$\int_{\Pi^\xi} F^\xi_\varepsilon(\widehat{v}^\xi_y, B^\xi_y; U^\xi_y)\,\mathrm{d}\mathcal{H}^{d-1}(y) = \varepsilon\int_{U\setminus B} |e(v)\xi\cdot\xi|^p\,\mathrm{d}x + \int_{U\cap\partial B} \frac{|\nu_B\cdot\xi|}{\varphi^*(\xi)}\,\mathrm{d}\mathcal{H}^{d-1}\,.$$

Since $|e(v)| \geq |e(v)\xi\cdot\xi|$, the previous estimate along with (5.6) implies

$$\int_{\Pi^\xi} F^\xi_\varepsilon(\widehat{v}^\xi_y, B^\xi_y; U^\xi_y)\,\mathrm{d}\mathcal{H}^{d-1}(y) \leq \varepsilon\|e(v)\|^p_{L^p(U\setminus B)} + \int_{U\cap\partial B}\varphi(\nu_B)\,\mathrm{d}\mathcal{H}^{d-1}\,.$$

By applying this estimate for the sequence of pairs (u_n, E_n), we get by (5.3)

$$\int_{\Pi^\xi} F^\xi_\varepsilon((\widehat{u}_n)^\xi_y, (E_n)^\xi_y; U^\xi_y)\,\mathrm{d}\mathcal{H}^{d-1}(y) \leq M\varepsilon + \int_{U\cap\partial E_n}\varphi(\nu_{E_n})\,\mathrm{d}\mathcal{H}^{d-1} \leq M(\|\varphi\|_{L^\infty(\mathbb{S}^{d-1})} + \varepsilon) \tag{5.8}$$

for all $U\in\mathfrak{M}(\Omega)$. Since $d(u_n,u)\to 0$, we have that $d((\widehat{u}_n)^\xi_y, \widehat{u}^\xi_y) = \int_{\Omega^\xi_y} d_{\overline{\mathbb{R}}^d}((\widehat{u}_n)^\xi_y, \widehat{u}^\xi_y)\,\mathrm{d}x \to 0$ for $\xi\in\mathbb{S}^{d-1}$ and $\mathcal{H}^{d-1}$-a.e. $y\in\Pi^\xi$. In particular, this implies

$$(\widehat{u}_n)^\xi_y \to \widehat{u}^\xi_y \quad \mathcal{L}^1\text{-a.e. in } (\Omega\setminus A_u^\infty)^\xi_y\,, \tag{5.9a}$$

$$|(\widehat{u}_n)^\xi_y| \to +\infty \quad \mathcal{L}^1\text{-a.e. in } (A_u^\infty)^\xi_y\,. \tag{5.9b}$$

By using (5.8) and Fatou's lemma we obtain

$$\liminf_{n\to\infty} F^\xi_\varepsilon((\widehat{u}_n)^\xi_y, (E_n)^\xi_y; U^\xi_y) < +\infty \tag{5.10}$$

for $\mathcal{H}^{d-1}$-a.e. $y\in\Pi^\xi$ and any $U\in\mathfrak{M}(\Omega)$. Then, we may find a subsequence $(u_m)_m = (u_{n_m})_m$, depending on ε, ξ, and y, such that

$$\lim_{m\to\infty} F^\xi_\varepsilon((\widehat{u}_m)^\xi_y, (E_m)^\xi_y; U^\xi_y) = \liminf_{n\to\infty} F^\xi_\varepsilon((\widehat{u}_n)^\xi_y, (E_n)^\xi_y; U^\xi_y) \tag{5.11}$$

for any $U \in \mathfrak{M}(\Omega)$. At this stage, up to passing to a further subsequence, we have

$$\mathcal{H}^0\big(\partial(E_m)^\xi_y\big) = N^\xi_y \in \mathbb{N}\,,$$

independently of m, so that the points in $\partial(E_m)^\xi_y$ converge, as $m \to \infty$, to $M^\xi_y \le N^\xi_y$ points

$$t_1, \dots, t_{M^\xi_y}\,,$$

which are either in ∂E^ξ_y or in a finite set $S^\xi_y := \{t_1, \dots, t_{M^\xi_y}\} \setminus \partial E^\xi_y \subset (E^\xi_y)^0 \cup (E^\xi_y)^1$, where $(\cdot)^0$ and $(\cdot)^1$ denote the sets with one-dimensional density 0 or 1, respectively. Notice that E^ξ_y is thus the union of $M^\xi_y/2 - \#S^\xi_y$ intervals (up to a finite set of points) on which there holds $\widehat{u}^\xi_y = 0$, see (5.5) and (5.9a). In view of (5.7) and (5.10), $((\widehat{u}_m)^\xi_y)'$ are equibounded (with respect to m) in $L^p_{\mathrm{loc}}(t_j, t_{j+1})$, for any interval

$$(t_j, t_{j+1}) \subset \Omega^\xi_y \setminus (E^\xi_y \cup S^\xi_y)\,.$$

Then, as in the proof of [17, Theorem 1.1], we have two alternative possibilities on (t_j, t_{j+1}): either $(\widehat{u}_m)^\xi_y$ converge locally uniformly in (t_j, t_{j+1}) to $\widehat{u}^\xi_y$, or $|(\widehat{u}_m)^\xi_y| \to +\infty$ $\mathcal{L}^1$-a.e. in (t_j, t_{j+1}). Recalling that $J_{\widehat{u}^\xi_y} = \partial(A^\infty_u)^\xi_y \cup \big((J^\xi_u)^\xi_y \setminus (A^\infty_u)^\xi_y\big)$, see (3.4a) and (3.11), we find

$$J_{\xi,y} := J_{\widehat{u}^\xi_y} \cap (E^\xi_y)^0 \subset S^\xi_y \cap (E^\xi_y)^0\,. \tag{5.12}$$

We notice that any point in S^ξ_y is the limit of two distinct sequences of points $(p^1_m)_m$, $(p^2_m)_m$ with p^1_m, $p^2_m \in \partial(E_m)^\xi_y$. Thus, in view of (5.7) and (5.11), for any $U \in \mathfrak{M}(\Omega)$ we derive

$$\begin{aligned}\varepsilon \int_{U^\xi_y \setminus (E\cup A^\infty_u)^\xi_y} |(\widehat{u}^\xi_y)'|^p \,\mathrm{d}t + \mathcal{H}^0(U^\xi_y \cap \partial E^\xi_y)\frac{1}{\varphi^*(\xi)} + \mathcal{H}^0(U^\xi_y \cap J_{\xi,y})\frac{2}{\varphi^*(\xi)} \\ \le \liminf_{m\to\infty} F^\xi_\varepsilon((\widehat{u}_m)^\xi_y, (E_m)^\xi_y; U^\xi_y) = \liminf_{n\to\infty} F^\xi_\varepsilon((\widehat{u}_n)^\xi_y, (E_n)^\xi_y; U^\xi_y)\,.\end{aligned} \tag{5.13}$$

We apply Lemma 5.3 to the rectifiable set $J_u \cap E^0 \cap \{\xi\cdot\nu_u \neq 0\}$ and get that for $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$

$$y + t\xi \in J_u \cap E^0 \cap \{\xi\cdot\nu_u \neq 0\} \quad \Rightarrow \quad t \in (E^\xi_y)^0\,.$$

This along with (5.12)–(5.13), the slicing properties (3.3)–(3.5) (which also hold for $GSBD^p_\infty(\Omega)$ functions), and Fatou's lemma yields that for all $\xi \in \mathbb{S}^{d-1} \setminus N_0$, for some N_0 with $\mathcal{H}^{d-1}(N_0) = 0$, there holds

$$\begin{aligned}&\varepsilon \int_{U\setminus(E\cup A^\infty_u)} |e(u)\xi\cdot\xi|^p \,\mathrm{d}x + \int_{U\cap\partial^* E} \frac{|\nu_E\cdot\xi|}{\varphi^*(\xi)}\,\mathrm{d}\mathcal{H}^{d-1} + \int_{J_u\cap E^0\cap U} \frac{2|\nu_u\cdot\xi|}{\varphi^*(\xi)}\,\mathrm{d}\mathcal{H}^{d-1} \\ &\le \int_{\Pi^\xi} \liminf_{n\to\infty} F^\xi_\varepsilon((\widehat{u}_n)^\xi_y, (E_n)^\xi_y; U^\xi_y)\ \mathrm{d}\mathcal{H}^{d-1} \le \liminf_{n\to\infty} \int_{\Pi^\xi} F^\xi_\varepsilon((\widehat{u}_n)^\xi_y, (E_n)^\xi_y; U^\xi_y)\ \mathrm{d}\mathcal{H}^{d-1}.\end{aligned}$$

Introducing the set function $\mu : \mathfrak{M}(\Omega) \to [0, +\infty)$

$$\mu(U) := \liminf_{n\to+\infty} \int_{U\cap\partial E_n} \varphi(\nu_{E_n})\,\mathrm{d}\mathcal{H}^{d-1} \quad \text{for } U \in \mathfrak{M}(\Omega)\,, \tag{5.14}$$

and letting $\varepsilon \to 0$ we find by (5.8) for all $\xi \in \mathbb{S}^{d-1} \setminus N_0$ that

$$\int_{U\cap\partial^* E} \frac{|\nu_E\cdot\xi|}{\varphi^*(\xi)}\,\mathrm{d}\mathcal{H}^{d-1} + \int_{J_u\cap E^0\cap U} \frac{2|\nu_u\cdot\xi|}{\varphi^*(\xi)}\,\mathrm{d}\mathcal{H}^{d-1} \le \mu(U)\,. \tag{5.15}$$

The set function μ is clearly superadditive. Let $\lambda = \mathcal{H}^{d-1} \llcorner \big(J_u \cap E^0\big) + \mathcal{H}^{d-1} \llcorner \partial^* E$ and define

$$g_i = \begin{cases} \dfrac{2|\nu_u \cdot \xi_i|}{\varphi^*(\xi_i)} & \text{on } J_u \cap E^0\,, \\ \dfrac{|\nu_E \cdot \xi_i|}{\varphi^*(\xi_i)} & \text{on } \partial^* E\,, \end{cases}$$

where $(\xi_i)_i \subset \mathbb{S}^{d-1} \setminus N_0$ is a dense sequence in $\mathbb{S}^{d-1}$. By (5.15) we have $\mu(U) \geq \int_U g_i \, \mathrm{d}\lambda$ for all $i \in \mathbb{N}$ and all $U \in \mathfrak{M}(\Omega)$. Then, Proposition 5.2 yields $\mu(\Omega) \geq \int_\Omega \sup_i g_i \, \mathrm{d}\lambda$. In view of (5.6) and (5.14), this implies (5.2b) and concludes the proof. □

5.2. **Relaxation for functionals defined on pairs of function-set.** In this subsection we give the proof of Proposition 2.1 and Theorem 2.2. For the upper bound, we recall the following result proved in [11, Proposition 9, Remark 14].

Proposition 5.4. *Let $u \in L^1(\Omega;\mathbb{R}^d)$ and $E \in \mathfrak{M}(\Omega)$ such that $\mathcal{H}^{d-1}(\partial^* E) < +\infty$ and $u\chi_{E^0} \in GSBV^p(\Omega;\mathbb{R}^d)$. Then, there exists a sequence $(u_n)_n \subset W^{1,p}(\Omega;\mathbb{R}^d)$ and $(E_n)_n \subset \mathfrak{M}(\Omega)$ with E_n of class C^∞ such that $u_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$, $\chi_{E_n} \to \chi_E$ in $L^1(\Omega)$, and*

$$\nabla u_n \chi_{\Omega \setminus E_n} \to \nabla u \chi_{\Omega \setminus E} \quad \text{in } L^p(\Omega;\mathbb{M}^{d\times d}),$$

$$\limsup_{n\to\infty} \int_{\partial E_n \cap \Omega} \varphi(\nu_{E_n}) \, \mathrm{d}\mathcal{H}^{d-1} \leq \int_{J_u \cap E^0} 2\varphi(\nu_u) \, \mathrm{d}\mathcal{H}^{d-1} + \int_{\partial^* E \cap \Omega} \varphi(\nu_E) \, \mathrm{d}\mathcal{H}^{d-1}\,.$$

Moreover, if $\mathcal{L}^d(E) > 0$, one can guarantee in addition the condition $\mathcal{L}^d(E_n) = \mathcal{L}^d(E)$ for $n \in \mathbb{N}$.

Proof of Proposition 2.1. We first prove the lower inequality, and then the upper inequality. The lower inequality relies on Theorem 5.1 and the upper inequality on a density argument along with Proposition 5.4.

The lower inequality. Suppose that $u_n \to u$ in $L^0(\Omega;\mathbb{R}^d)$ and $\chi_{E_n} \to \chi_E$ in $L^1(\Omega)$. Without restriction, we can assume that $\sup_n F(u_n, E_n) < +\infty$. In view of (2.2) and $\min_{\mathbb{S}^{d-1}} \varphi > 0$, this implies $\mathcal{H}^{d-1}(\partial E_n) < +\infty$. Moreover, by (2.1) the functions $v_n := u_n \chi_{\Omega \setminus E_n}$ lie in $GSBD^p(\Omega)$ with $J_{v_n} \subset \partial E_n \cap \Omega$ and satisfy $\sup_n \|e(v_n)\|_{L^p(\Omega)} < +\infty$. This along with the fact that $u_n \to u$ in measure shows that v_n converges weakly in $GSBD^p_\infty(\Omega)$ to $u\chi_{E^0}$, see (3.16), where we point out that $A^\infty_u = \{u = \infty\} = \emptyset$. In particular, $u\chi_{E^0} \in GSBD^p_\infty(\Omega)$ and, since $A^\infty_u = \emptyset$, even $u\chi_{E^0} \in GSBD^p(\Omega)$, cf. (3.10). As also (5.1) holds, we can apply Theorem 5.1. The lower inequality now follows from (5.2) and the fact that f is convex.

The upper inequality. We first observe the following: given $u \in L^0(\Omega;\mathbb{R}^d)$ and $E \in \mathfrak{M}(\Omega)$ with $\mathcal{H}^{d-1}(\partial^* E) < \infty$ and $u\chi_{E^0} \in GSBD^p(\Omega)$, we find an approximating sequence $(v_n)_n \subset L^1(\Omega;\mathbb{R}^d)$ with $v_n\chi_{E^0} \in GSBV^p(\Omega;\mathbb{R}^d)$ such that

$$\begin{aligned} &\text{(i)} \quad v_n \to u\chi_{E^0} \quad \text{in } L^0(\Omega;\mathbb{R}^d)\,, \\ &\text{(ii)} \quad e(v_n)\chi_{\Omega\setminus E} \to e(u)\chi_{\Omega\setminus E} \quad \text{in } L^p(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})\,, \\ &\text{(iii)} \quad \mathcal{H}^{d-1}\big((J_{v_n} \triangle J_u) \cap E^0\big) \to 0. \end{aligned}$$

This can be seen by approximating first $u\chi_{E^0}$ by a sequence $(\widetilde{u}_n)_n$ by means of Theorem 3.4, and by setting $v_n := \widetilde{u}_n \chi_{E^0}$ for every n. It is then immediate to verify that the conditions in (3.6) for $(\widetilde{u}_n)_n$ imply the three conditions above.

By this approximation, (2.1), and a diagonal argument, it thus suffices to construct a recovery sequence for $u \in L^1(\Omega;\mathbb{R}^d)$ with $u\chi_{E^0} \in GSBV^p(\Omega;\mathbb{R}^d)$. To this end, we apply Proposition 5.4 to obtain $(u_n, E_n)_n$ and we consider the sequence $u_n\chi_{\Omega\setminus E_n}$. We further observe that, if $\mathcal{L}^d(E) > 0$, this recovery sequences $(u_n, E_n)_n$ can be constructed ensuring $\mathcal{L}^d(E_n) = \mathcal{L}^d(E)$ for $n \in \mathbb{N}$. □

As a consequence, we obtain the following lower semicontinuity result in $GSBD^p_\infty$.

Corollary 5.5 (Lower semicontinuity in $GSBD^p_\infty$)**.** *Let us suppose that a sequence $(u_n)_n \subset GSBD^p_\infty(\Omega)$ converges weakly in $GSBD^p_\infty(\Omega)$ to $u \in GSBD^p_\infty(\Omega)$, see (3.16). Then for each norm ϕ on $\mathbb{R}^d$ there holds*

$$\int_{J_u} \phi(\nu_u)\, \mathrm{d}\mathcal{H}^{d-1} \le \liminf_{n\to\infty} \int_{J_{u_n}} \phi(\nu_{u_n})\, \mathrm{d}\mathcal{H}^{d-1}.$$

Proof. Let $\varepsilon > 0$ and $f(\zeta) = \varepsilon|\zeta^T + \zeta|^p$ for $\zeta \in \mathbb{M}^{d\times d}$. The upper inequality in Proposition 2.1 (for u_n and $E = \emptyset$) shows that for each $u_n \in GSBD^p_\infty(\Omega)$, we can find a Lipschitz set E_n with $\mathcal{L}^d(E_n) \le \frac{1}{n}$ and $v_n \in L^0(\Omega;\mathbb{R}^d)$ with $v_n|_{\Omega\setminus \overline{E}_n} \in W^{1,p}(\Omega \setminus \overline{E}_n;\mathbb{R}^d)$, $v_n|_{E_n} = 0$, and $d(v_n, u_n) \le \frac{1}{n}$ (see (3.13)) such that

$$\int_{\Omega\setminus E_n} \varepsilon|e(v_n)|^p\, \mathrm{d}x + \int_{\Omega\cap\partial E_n} \phi(\nu_{E_n})\, \mathrm{d}\mathcal{H}^{d-1} \le \int_\Omega \varepsilon|e(u_n)|^p\, \mathrm{d}x + \int_{J_{u_n}} 2\phi(\nu_{u_n})\, \mathrm{d}\mathcal{H}^{d-1} + \frac{1}{n}\,. \tag{5.16}$$

(Strictly speaking, if $u_n \notin GSBD^p(\Omega)$, Proposition 2.1 is not applied for u_n, but for some representative given in (3.10).) Observe that $d(v_n, u) \to 0$ as $n \to \infty$, and thus v_n converges weakly to u in $GSBD^p_\infty(\Omega)$. By applying Theorem 5.1 on (v_n, E_n) and using $E = \emptyset$ we get

$$\int_{J_u} 2\phi(\nu_u)\, \mathrm{d}\mathcal{H}^{d-1} \le \liminf_{n\to\infty} \int_{\Omega\cap\partial E_n} \phi(\nu_{E_n})\, \mathrm{d}\mathcal{H}^{d-1}\,.$$

This, along with (5.16), $\sup_{n\in\mathbb{N}} \|e(u_n)\|_{L^p(\Omega)} < +\infty$, and the arbitrariness of ε yields the result. $\square$

We now address the relaxation of F_{Dir}, see (2.5), i.e., a version of F with boundary data.

We take advantage of the following approximation result which is obtained by following the lines of [15, Theorem 5.5], where an analogous approximation is proved for Griffith functionals with Dirichlet boundary conditions. The new feature with respect to [15, Theorem 5.5] is that, besides the construction of approximating functions with the correct boundary data, also approximating sets are constructed. For convenience of the reader, we give a sketch of the proof in Appendix A highlighting the adaptations needed with respect to [15, Theorem 5.5]. In the following, we denote by $\overline{F}'_{\mathrm{Dir}}$ the functional on the right hand side of (2.7).

Lemma 5.6. *Suppose that $\partial_D\Omega \subset \partial\Omega$ satisfies (2.4). Consider $(v, H) \in L^0(\Omega;\mathbb{R}^d) \times \mathfrak{M}(\Omega)$ such that $\overline{F}'_{\mathrm{Dir}}(v, H) < +\infty$. Then there exist $(v_n, H_n) \in (L^p(\Omega;\mathbb{R}^d) \cap SBV^p(\Omega;\mathbb{R}^d)) \times \mathfrak{M}(\Omega)$ such that J_{v_n} is closed in Ω and included in a finite union of closed connected pieces of C^1 hypersurfaces, $v_n \in W^{1,p}(\Omega \setminus J_{v_n};\mathbb{R}^d)$, $v_n = u_0$ in a neighborhood $V_n \subset \Omega$ of $\partial_D\Omega$, H_n is a set of finite perimeter, and*

(i) $v_n \to v$ *in* $L^0(\Omega;\mathbb{R}^d)$,
(ii) $\chi_{H_n} \to \chi_H$ *in* $L^1(\Omega)$,
(iii) $\limsup_{n\to\infty} \overline{F}'_{\mathrm{Dir}}(v_n, H_n) \le \overline{F}'_{\mathrm{Dir}}(v, H)$.

Proof of Theorem 2.2. First, we denote by Ω' a bounded open set with $\Omega \subset \Omega'$ and $\Omega' \cap \partial\Omega = \partial_D\Omega$. By F' and $\overline{F}'$ we denote the analogs of the functionals F and $\overline{F}$, respectively, defined on $L^0(\Omega';\mathbb{R}^d)\times\mathfrak{M}(\Omega')$. Given $u \in L^0(\Omega;\mathbb{R}^d)$, we define the extension $u' \in L^0(\Omega';\mathbb{R}^d)$ by setting

$u' = u_0$ on $\Omega' \setminus \Omega$ for fixed boundary values $u_0 \in W^{1,p}(\mathbb{R}^d;\mathbb{R}^d)$. Then, we observe

$$F'(u',E) = F_{\rm Dir}(u,E) + \int_{\Omega'\setminus\Omega} f(e(u_0))\,\mathrm{d}x\,, \quad \overline{F}'(u',E) = \overline{F}_{\rm Dir}(u,E) + \int_{\Omega'\setminus\Omega} f(e(u_0))\,\mathrm{d}x\,. \tag{5.17}$$

Therefore, the lower inequality follows from Proposition 2.1 applied for $F', \overline{F}'$ instead of $F, \overline{F}$.

We now address the upper inequality. In view of Lemma 5.6 and by a diagonal argument, it is enough to prove the result in the case where, besides the assumptions in the statement, also $u \in L^p(\Omega;\mathbb{R}^d) \cap SBV^p(\Omega;\mathbb{R}^d)$ and $u = u_0$ in a neighborhood $U \subset \Omega$ of $\partial_D\Omega$.

By $(u_n,E_n)_n$ we denote a recovery sequence for (u,E) given by Proposition 5.4. In general, the functions $(u_n)_n$ do not satisfy the boundary conditions required in (2.5). Let $\delta > 0$ and let $V \subset\subset U$ be a smaller neighborhood of $\partial_D\Omega$. In view of (2.1)–(2.2), by a standard diagonal argument in the theory of Γ-convergence, it suffices to find a sequence $(v_n)_n \subset L^1(\Omega;\mathbb{R}^d)$ with $v_n|_{\Omega\setminus\overline{E}_n} \in W^{1,p}(\Omega\setminus\overline{E}_n;\mathbb{R}^d)$, $v_n = 0$ on E_n, and $v_n = u_0$ on $V \setminus \overline{E}_n$ such that

$$\limsup_{n\to\infty} \|v_n - u\chi_{E^0}\|_{L^1(\Omega)} \le \delta, \qquad \limsup_{n\to\infty} \|e(v_n) - e(u_n)\chi_{\Omega\setminus E_n}\|_{L^p(\Omega)} \le \delta. \tag{5.18}$$

To this end, choose $\psi \in C^\infty(\Omega)$ with $\psi = 1$ in $\Omega \setminus U$ and $\psi = 0$ on V. For $M > 0$, we define the truncation u^M by $u^M_i = (-M \vee u_i) \wedge M$, where u_i denotes the i-th component, $i = 1,\dots,d$. In a similar fashion, we define u^M_n. By Proposition 5.4 we then get $\chi_{E_n} \to \chi_E$ in $L^1(\Omega)$ and

$$u^M_n \to u^M \quad \text{in} \quad L^p(\Omega;\mathbb{R}^d), \qquad \nabla u^M_n \chi_{\Omega\setminus E_n} \to \nabla u^M \chi_{E^0} \quad \text{in} \quad L^p(\Omega;\mathbb{M}^{d\times d})\,.$$

We define $v_n := (\psi u^M_n + (1-\psi)u_0)\chi_{\Omega\setminus E_n}$. Clearly, $v_n = u_0$ on $V \setminus \overline{E}_n$. By $\nabla v_n = \psi\nabla u^M_n + (1-\psi)\nabla u_0 + \nabla\psi \otimes (u^M_n - u_0)$ on $\Omega \setminus E_n$, $u = u_0$ on U, and Proposition 5.4 we find

$$\limsup_{n\to\infty} \|v_n - u\|_{L^1(\Omega)} \le \|u - u^M\|_{L^1(\Omega)}\,,$$
$$\limsup_{n\to\infty} \|e(v_n) - e(u_n)\chi_{\Omega\setminus E_n}\|_{L^p(\Omega)} \le \|\nabla u - \nabla u^M\|_{L^p(E^0)} + \|\nabla\psi\|_\infty \|u - u^M\|_{L^p(\Omega)}\,.$$

For M sufficiently large, we obtain (5.18) since $u = u\chi_{E^0}$. This concludes the proof. □

5.3. **Compactness and existence results for the relaxed functional.** We start with the following general compactness result.

Theorem 5.7 (Compactness)**.** *For every $(u_n,E_n)_n$ with $\sup_n F(u_n,E_n) < +\infty$, there exist a subsequence (not relabeled), $u \in GSBD^p_\infty(\Omega)$, and $E \in \mathfrak{M}(\Omega)$ with $\mathcal{H}^{d-1}(\partial^*E) < +\infty$ such that u_n converges weakly in $GSBD^p_\infty(\Omega)$ to u and $\chi_{E_n} \to \chi_E$ in $L^1(\Omega)$.*

Proof. Let $(u_n,E_n)_n$ with $\sup_n F(u_n,E_n) < +\infty$. As by the assumptions on φ there holds $\sup_{n\in\mathbb{N}} \mathcal{H}^{d-1}(\partial E_n) < +\infty$, a compactness result for sets of finite perimeter (see [3, Remark 4.20]) implies that there exists $E \subset \Omega$ with $\mathcal{H}^{d-1}(\partial^*E) < +\infty$ such that $\chi_{E_n} \to \chi_E$ in $L^1(\Omega)$, up to a subsequence (not relabeled).

Since the functions $u_n = u_n\chi_{\Omega\setminus E_n}$ satisfy $J_{u_n} \subset \partial E_n$, we get $\sup_n \mathcal{H}^{d-1}(J_{u_n}) < +\infty$. Moreover, by the growth assumptions on f (see (2.1)) we get that $\|e(u_n)\|_{L^p(\Omega)}$ is uniformly bounded. Thus, by Theorem 3.5, u_n converges (up to a subsequence) weakly in $GSBD^p_\infty(\Omega)$ to some $u \in GSBD^p_\infty(\Omega)$. This concludes the proof. □

Based on the above compactness properties, we now prove the following result on existence and convergence of minimizers. Clearly, Theorem 2.3 is then an immediate consequence. We call any affine function $a\colon \mathbb{R}^d \to \mathbb{R}^d$ with skew-symmetric gradient an *infinitesimal rigid motion*.

Proposition 5.8 (Existence of minimizers)**.** *Let $m > 0$. Let $(u_n, E_n)_n$ be a minimizing sequence for*

$$\inf\left\{F_{\rm Dir}(u,E)\colon (u,E)\in L^0(\Omega;\mathbb{R}^d)\times\mathfrak{M}(\Omega),\ \mathcal{L}^d(E)=m\right\}. \tag{5.19}$$

Then, there exist a set of finite perimeter E and $u \in GSBD^p_\infty(\Omega)$ such that, up to a (not relabeled) subsequence, u_n converges weakly in $GSBD^p_\infty(\Omega)$ to u, for every infinitesimal rigid motion $a\colon \mathbb{R}^d \to \mathbb{R}^d$ the function $v_a := u\chi_{\Omega\setminus A^\infty_u} + a\chi_{A^\infty_u}$ (recall $A^\infty_u = \{u = \infty\}$) is such that (v_a, E) is a minimizing pair of

$$\inf\left\{\overline{F}_{\rm Dir}(u,E)\colon (u,E)\in L^0(\Omega;\mathbb{R}^d)\times\mathfrak{M}(\Omega),\ \mathcal{L}^d(E)=m\right\}, \tag{5.20}$$

and there holds

$$\lim_{n\to+\infty} F_{\rm Dir}(u_n,E_n) = \overline{F}_{\rm Dir}(v_a,E)\,.$$

Proof. In principle, the result follows from general properties of relaxations, see e.g. [24, Theorem 3.8]. The topology of the compactness result in Theorem 5.7, however, is slightly different from the one in (2.6). Therefore, we briefly give some details, in particular on the cluster points of minimizing sequences for problem (5.19).

As in the proof of Theorem 2.2 above, see (5.17), we work with the functionals $F', \overline{F}'$ extended to Ω', where Ω' satisfies $\Omega' \cap \partial\Omega = \partial_D\Omega$. Functions $u \in L^0(\Omega;\mathbb{R}^d)$ are extended to $u' \in L^0(\Omega';\mathbb{R}^d)$ by setting $u' = u_0$ on $\Omega' \setminus \Omega$. Let $(u_n, E_n)_n$ be a minimizing sequence for (5.19). Then $(u'_n, E_n)_n$ is a minimizing sequence for the extended functional F'. We apply Theorem 5.7 to $(u'_n, E_n)_n$ and obtain the existence of a limiting pair (u', E) such that $u' = u_0$ in $\Omega' \setminus \Omega$, $E \subset \Omega$, and $\mathcal{L}^d(E) = m$ such that, up to a (not relabeled) subsequence, u'_n converges weakly in $GSBD^p_\infty(\Omega')$ to u' and $\chi_{E_n} \to \chi_E$ in $L^1(\Omega')$. By Theorem 5.1 we obtain

$$\int_{\Omega'\setminus(E\cup A^\infty_u)} f(e(u'))\,\mathrm{d}x \le \liminf_{n\to+\infty}\int_{\Omega'\setminus E_n} f(e(u'_n))\,\mathrm{d}x\,, \tag{5.21a}$$

$$\int_{\Omega'\cap\partial^* E}\varphi(\nu_E)\,\mathrm{d}\mathcal{H}^{d-1} + \int_{J_{u'}\cap E^0\cap\Omega'} 2\varphi(\nu_{u'})\,\mathrm{d}\mathcal{H}^{d-1} \le \liminf_{n\to+\infty}\int_{\partial E_n}\varphi(\nu_{E_n})\,\mathrm{d}\mathcal{H}^{d-1}\,. \tag{5.21b}$$

For any infinitesimal rigid motion $a\colon \mathbb{R}^d \to \mathbb{R}^d$ we define $v_a := u\chi_{\Omega\setminus A^\infty_u} + a\chi_{A^\infty_u}$ on Ω. By (2.2), Proposition 2.1, (5.17), (5.21), and $u' = u'_n = u_0$ on $\Omega' \setminus \Omega$ we get

$$\begin{aligned}\overline{F}_{\rm Dir}(v_a,E) &\le \int_{\Omega\setminus(E\cup A^\infty_u)} f(e(u'))\,\mathrm{d}x + \int_{\Omega'\cap\partial^* E}\varphi(\nu_E)\,\mathrm{d}\mathcal{H}^{d-1} + \int_{J_{u'}\cap E^0\cap\Omega'} 2\varphi(\nu_{u'})\,\mathrm{d}\mathcal{H}^{d-1}\\ &\le \liminf_{n\to+\infty}\Big(\int_{\Omega\setminus E_n} f(e(u'_n))\,\mathrm{d}x + \int_{\partial E_n}\varphi(\nu_{E_n})\,\mathrm{d}\mathcal{H}^{d-1}\Big) = \liminf_{n\to+\infty} F_{\rm Dir}(u_n,E_n)\,.\end{aligned}$$

This implies $\overline{F}_{\rm Dir}(v_a, E) \le \inf F_{\rm Dir}$. (The infimum is understood with volume constraint, i.e., in the sense of (5.19).) By general properties of relaxation (see e.g. [24, Theorem 3.8]) and the upper bound in Theorem 2.2 (for recovery sequences satisfying the volume constraint) we conclude that (v_a, E) is a minimizer of (5.20) and that $\lim_{n\to\infty} F_{\rm Dir}(u_n, E_n) = \overline{F}_{\rm Dir}(v_a, E)$. $\square$

6. Functionals on domains with a subgraph constraint

In this section we prove the the results announced in Subsection 2.2.

6.1. **Relaxation of the energy G.** This subsection is devoted to the proof of Theorem 2.4. The lower inequality is obtained by exploiting the tool of $\sigma^p_{\rm sym}$-convergence introduced in Section 4. The corresponding analysis will prove to be useful also for the compactness theorem in the next subsection. The proof of the upper inequality is quite delicate, and a careful procedure is needed to guarantee that the approximating displacements are still defined on a domain which is the subgraph of a function. We only follow partially the strategy in [18, Proposition 4.1], and employ also other arguments in order to improve the $GSBV$ proof which might fail in some pathological cases.

Consider a Lipschitz set $\omega \subset \mathbb{R}^{d-1}$ which is uniformly star-shaped with respect to the origin, see (2.9). We recall the notation $\Omega = \omega \times (-1, M+1)$ and

$$\Omega_h = \{x \in \Omega\colon\ -1 < x_d < h(x')\}, \qquad \Omega^+_h = \Omega_h \cap \{x_d > 0\} \tag{6.1}$$

for $h\colon \omega \to [0, M]$ measurable, where we write $x = (x', x_d)$ for $x \in \mathbb{R}^d$. Moreover, we let $\Omega^+ = \Omega \cap \{x_d > 0\}$.

The lower inequality. Consider $(u_n, h_n)_n$ with $\sup_n G(u_n, h_n) < +\infty$. Then, we have that $h_n \in C^1(\omega; [0, M])$, $u_n|_{\Omega_{h_n}} \in W^{1,p}(\Omega_{h_n}; \mathbb{R}^d)$, $u_n|_{\Omega \setminus \Omega_{h_n}} = 0$, and $u_n = u_0$ on $\omega \times (-1, 0)$. Suppose that $(u_n, h_n)_n$ converges in $L^0(\Omega; \mathbb{R}^d) \times L^1(\omega)$ to (u, h). We let

$$\Gamma_n := \partial\Omega_{h_n} \cap \Omega = \{x \in \Omega\colon\ h_n(x') = x_d\} \tag{6.2}$$

be the graph of the function h_n. Note that $\sup_n \mathcal{H}^{d-1}(\Gamma_n) < +\infty$. We take $U = \omega \times (-\frac{1}{2}, M)$ and $U' = \Omega = \omega \times (-1, M+1)$, and apply Theorem 4.2, to deduce that $(\Gamma_n)_n$ $\sigma^p_{\rm sym}$-converges (up to a subsequence) to a pair (Γ, G_∞). A fundamental step in the proof will be to show that

$$G_\infty = \emptyset\,. \tag{6.3}$$

We postpone the proof of this property to Steps 3–4 below. We first characterize the limiting set Γ (Step 1) and prove the lower inequality (Step 2). We point out that in Steps 1–2 we follow partially the lines of [18, Subsection 3.2] whereas in Steps 3–4 we deal with additional technical difficulties arising in the $GSBD$ setting. In the whole proof, for simplicity we omit to write $\tilde{\subset}$ and $\tilde{=}$ to indicate that the inclusions hold up to $\mathcal{H}^{d-1}$-negligible sets.

Step 1: Characterization of the limiting set Γ. Let us prove that the set

$$\Sigma := \Gamma \cap \Omega^1_h \tag{6.4}$$

is vertical, that is

$$(\Sigma + te_d) \cap \Omega^1_h \subset \Sigma \quad \text{ for any } t \geq 0\,. \tag{6.5}$$

This follows as in [18, Subsection 3.2]: in fact, consider $(v_n)_n$ and v as in Definition 4.1(ii). In particular, $v_n = 0$ on $U' \setminus U$, $J_{v_n} \subset \Gamma_n$, and, in view of (6.3), v is $\mathbb{R}^d$-valued with $\Gamma = J_v$. The functions $v'_n(x) := v_n(x', x_d - t)\chi_{\Omega_{h_n}}(x)$ (with $t > 0$, extended by zero in $\omega \times (-1, -1+t)$) converge to $v'(x) := v(x', x_d - t)\chi_{\Omega_h}(x)$ in measure on U'. Since $J_{v'_n} \subset \Gamma_n$, Definition 4.1(i) implies $J_{v'} \setminus \Gamma \subset (G_\infty)^1$. As $G_\infty = \emptyset$ by (6.3), we get $J_{v'} \subset \Gamma$, so that

$$(\Sigma + te_d) \cap \Omega^1_h = (\Gamma + te_d) \cap \Omega^1_h = (J_v + te_d) \cap \Omega^1_h = J_{v'} \cap \Omega^1_h \subset \Gamma \cap \Omega^1_h = \Sigma\,,$$

where we have used $\Gamma = J_v$. This shows (6.5). In particular, $\nu_\Sigma \cdot e_d = 0$ $\mathcal{H}^{d-1}$-a.e. in Σ. Next, we show that

$$\mathcal{H}^{d-1}(\partial^*\Omega_h \cap \Omega) + 2\mathcal{H}^{d-1}(\Sigma) \leq \liminf_{n\to\infty} \int_\omega \sqrt{1 + |\nabla h_n(x')|^2}\, \mathrm{d}x'\,. \tag{6.6}$$

To see this, we again consider functions $(v_n)_n$ and v satisfying Definition 4.1(ii). In particular, we have $J_{v_n} \subset \Gamma_n$ and $J_v = \Gamma$. Since Γ_n is the graph of a C^1 function, we either get $v_n|_{\Omega_{h_n}} \equiv \infty$ or, by Korn's inequality, we have $v_n|_{\Omega_{h_n}} \in W^{1,p}(\Omega_{h_n}; \mathbb{R}^d)$. Since $v_n = 0$ on $U' \setminus U$, we obtain

$v_n|_{\Omega_{h_n}} \in W^{1,p}(\Omega_{h_n};\mathbb{R}^d)$. We apply Theorem 5.1 for $E_n = \Omega \setminus \Omega_{h_n}$, $E = \Omega \setminus \Omega_h$, and the sequence of functions $w_n := v_n \chi_{\Omega\setminus E_n} = v_n \chi_{\Omega_{h_n}}$.

Observe that $\chi_{E_n} \to \chi_E$ in $L^1(\Omega)$. Moreover, w_n converges weakly in $GSBD^p_\infty(\Omega)$ to $w := v\chi_{\Omega_h}$ since v_n converges weakly in $GSBD^p_\infty(\Omega)$ to v and $\sup_n \mathcal{H}^{d-1}(\partial E_n) < +\infty$. By (5.2b) for $\varphi \equiv 1$ on $\mathbb{S}^{d-1}$ there holds

$$\mathcal{H}^{d-1}(\partial^*\Omega_h \cap \Omega) + 2\mathcal{H}^{d-1}(J_w \cap \Omega_h^1) \le \liminf_{n\to\infty} \mathcal{H}^{d-1}(\partial\Omega_{h_n} \cap \Omega)\,,$$

where we used that $E^0 = \Omega_h^1$ and $\partial^*E \cap \Omega = \partial^*\Omega_h \cap \Omega$. Since $J_v = \Gamma$ and $J_w \cap \Omega_h^1 = J_v \cap \Omega_h^1 = \Sigma$, we indeed get (6.6), where for the right hand side we use that $\partial\Omega_{h_n}$ is the graph of the function $h_n \in C^1(\omega;[0,M])$. For later purposes in Step 4, we also note that by Corollary 5.5 for $\phi(\nu) = |\xi\cdot\nu|$, with $\xi \in \mathbb{S}^{d-1}$ fixed, we get

$$\int_\Gamma |\nu_\Gamma \cdot \xi|\,\mathrm{d}\mathcal{H}^{d-1} = \int_{J_v} |\nu_v \cdot \xi|\,\mathrm{d}\mathcal{H}^{d-1} \le \liminf_{n\to\infty} \int_{J_{v_n}} |\nu_{v_n}\cdot\xi|\,\mathrm{d}\mathcal{H}^{d-1} \le \liminf_{n\to\infty} \int_{\Gamma_n} |\nu_{\Gamma_n}\cdot\xi|\,\mathrm{d}\mathcal{H}^{d-1}\,. \tag{6.7}$$

(Strictly speaking, as ϕ is only a seminorm, we apply Corollary 5.5 for $\phi + \varepsilon$ for any $\varepsilon > 0$.)

Step 2: The lower inequality. We now show the lower bound. Recall that $(u_n,h_n)_n$ converges in $L^0(\Omega;\mathbb{R}^d)\times L^1(\omega)$ to (u,h) and that $(G(u_n,h_n))_n$ is bounded. Then, (2.1) and $\min_{\mathbb{S}^{d-1}}\varphi > 0$ along with Theorem 3.5 and the fact that $\mathcal{L}^d(\{x\in\Omega\colon |u_n(x)|\to\infty\}) = 0$ imply that the limit $u = u\chi_{\Omega_h}$ lies in $GSBD^p(\Omega)$. There also holds $u = u_0$ on $\omega\times(-1,0)$ by (3.7)(i) and the fact that $u_n = u_0$ on $\omega\times(-1,0)$ for all $n\in\mathbb{N}$. In particular, we observe that $u_n = u_n\chi_{\Omega_{h_n}}$ converges weakly in $GSBD^p_\infty(\Omega)$ to u, cf. (3.16). The fact that $h \in BV(\omega;[0,M])$ follows from a standard compactness argument. This shows $\overline{G}(u,h) < +\infty$.

To obtain the lower bound for the energy, we again apply Theorem 5.1 for $E_n = \Omega\setminus\Omega_{h_n}$ and $E = \Omega\setminus\Omega_h$. Consider the sequence of functions $v_n := \psi u_n \chi_{\Omega\setminus E_n} = \psi u_n$, where $\psi \in C^\infty(\Omega)$ with $\psi = 1$ in a neighborhood of $\Omega^+ = \Omega\cap\{x_d > 0\}$ and $\psi = 0$ on $\omega\times(-1,-\frac{1}{2})$. We observe that $v_n = 0$ on $U'\setminus U = \omega\times((-1,-\frac{1}{2}]\cup[M,M+1))$ and that v_n converges to $v := \psi u \in GSBD^p(\Omega)$ weakly in $GSBD^p_\infty(\Omega)$. Now we apply Theorem 5.1. First, notice that (5.2a), $\psi = 1$ on Ω^+, and the fact that $A_u^\infty = \emptyset$ imply $e(u_n)\chi_{\Omega^+_{h_n}} \rightharpoonup e(u)\chi_{\Omega^+_h}$ weakly in $L^p(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})$. This along with the convexity of f yields

$$\int_{\Omega_h^+} f(e(u))\,\mathrm{d}x \le \liminf_{n\to\infty} \int_{\Omega^+_{h_n}} f(e(u_n))\,\mathrm{d}x\,. \tag{6.8}$$

Moreover, applying Definition 4.1(i) on the sequence $(v_n)_n$, which satisfies $v_n = 0$ on $U'\setminus U$ and $J_{v_n} \subset \Gamma_n$, we observe $J_u = J_v \subset \Gamma$, where we have also used (6.3). Recalling the definition of $J_u' = \{(x', x_d + t)\colon x \in J_u,\, t \ge 0\}$, see (2.10), and using (6.4)–(6.5) we find $J'_u \cap \Omega_h^1 \subset \Sigma$. Thus, by (6.6), we obtain

$$\mathcal{H}^{d-1}(\partial^*\Omega_h \cap \Omega) + 2\mathcal{H}^{d-1}(J'_u \cap \Omega_h^1) \le \liminf_{n\to\infty} \int_\omega \sqrt{1 + |\nabla h_n(x')|^2}\,\mathrm{d}x'\,. \tag{6.9}$$

Collecting (6.8) and (6.9) we conclude the lower inequality. To conclude the proof, it remains to confirm (6.3).

Step 3: Slicing arguments for rectifiable sets. To show (6.3) we need some auxiliary estimates. Recall the notation introduced in (3.1) and consider $\Lambda \subset \Omega$ with $\mathcal{H}^{d-1}(\Lambda) < +\infty$ satisfying the property $\mathcal{H}^0(\Lambda_y^{e_d}) \le 1$ for all $y \in \Pi^{e_d} \cap (\omega\times\{0\})$. Let $\omega_\Lambda \subset \omega \subset \mathbb{R}^{d-1}$ be the measurable set with

$$\mathcal{H}^0\big(\Lambda_y^{e_d}\big) = 1 \qquad \text{if and only if } y \in \Pi^{e_d} \cap (\omega_\Lambda \times \{0\})\,.$$

In the arguments in Step 4 below, Λ will represent the graph of a function defined on ω. By the area formula (cf. e.g. [53, (12.4) in Section 12]) there holds

$$\int_{\Lambda} |\nu_\Lambda \cdot e_d| \, \mathrm{d}\mathcal{H}^{d-1} = \int_{\omega \times \{0\}} \mathcal{H}^0\big(\Lambda_y^{e_d}\big) \, \mathrm{d}\mathcal{H}^{d-1}(y) = \mathcal{H}^{d-1}(\omega_\Lambda). \tag{6.10}$$

For general $\nu = (\nu_i)_{i=1}^d, \xi = (\xi_i)_{i=1}^d \in \mathbb{S}^{d-1}$ there holds

$$|\nu \cdot \xi - \nu \cdot e_d| = |(\xi - e_d) \cdot \nu| \leq |\nu_d(1-\xi_d)| + |\nu \cdot \xi - \nu_d \xi_d| \leq |\nu_d(1-\xi_d)| + \sum\nolimits_{i=1}^{d-1} |\xi_i| \,.$$

Thus, for $\delta > 0$ small, taking $\xi \in \mathbb{S}^{d-1}$ with $|\xi_d| \geq 1 - \delta$, we get

$$|\nu \cdot e_d| - C_\delta \leq |\nu \cdot \xi| \leq |\nu \cdot e_d| + C_\delta \,, \qquad \text{for some } C_\delta \to 0 \text{ as } \delta \to 0\,. \tag{6.11}$$

In view of (6.10), this implies for all $\xi \in \mathbb{S}^{d-1}$ with $|\xi_d| \geq 1 - \delta$ that

$$\Big| \int_{\Lambda} |\nu_\Lambda \cdot \xi| \, \mathrm{d}\mathcal{H}^{d-1} - \mathcal{H}^{d-1}(\omega_\Lambda) \Big| \leq \int_{\Lambda} \big| |\nu_\Lambda \cdot \xi| - |\nu_\Lambda \cdot e_d| \big| \, \mathrm{d}\mathcal{H}^{d-1} \leq C_\delta \mathcal{H}^{d-1}(\Lambda)\,. \tag{6.12}$$

In a similar fashion, for $\Psi \subset \Omega$ with $\mathcal{H}^{d-1}(\Psi) < +\infty$ satisfying the property

$$\mathcal{H}^0\big(\Psi_y^{e_d}\big) \geq 2 \quad \text{ for all } y \in \omega_\Psi \times \{0\} \text{ for some measurable subset } \omega_\Psi \subset \omega\,, \tag{6.13}$$

one can show that

$$\int_{\Psi} |\nu_\Psi \cdot \xi| \, \mathrm{d}\mathcal{H}^{d-1} \geq 2\mathcal{H}^{d-1}(\omega_\Psi) - C_\delta \mathcal{H}^{d-1}(\Psi) \tag{6.14}$$

for all $\xi \in \mathbb{S}^{d-1}$ with $|\xi_d| \geq 1 - \delta$, where C_δ is the constant in (6.11).

Step 4: Proof of $G_\infty = \emptyset$. Recall the definition of the graphs Γ_n in (6.2) and its σ^p_{sym}-limit Γ on the sets $U = \omega \times (-\frac{1}{2}, M)$ and $U' = \Omega$. As before, consider $\psi \in C^\infty(\Omega)$ with $\psi = 1$ in a neighborhood of Ω^+ and $\psi = 0$ on $\omega \times (-1, -\frac{1}{2})$. By employing (i) in Definition 4.1 for the sequence $v_n = \psi \chi_{\Omega_{h_n}} e_d$ and its limit $v = \psi \chi_{\Omega_h} e_d$, we get that $(\partial^* \Omega_h \cap \Omega) \setminus \Gamma \subset (G_\infty)^1$. Since $U' \cap \partial^* G_\infty \subset \Gamma$ by definition of σ^p_{sym}-convergence, we observe

$$\Gamma \supset \big(\partial^* G_\infty \cap \Omega\big) \cup \big(\partial^* \Omega_h \cap \Omega \cap (G_\infty)^0\big)\,. \tag{6.15}$$

We estimate the $\mathcal{H}^{d-1}$-measure of the two terms on the right separately. For the following, we fix $\delta > 0$, $\xi \in \mathbb{S}^{d-1}$ with $|\xi_d| \geq 1 - \delta$, and define $\Psi = \partial^* G_\infty \cap \Omega$.

The first term. Since G_∞ is contained in $U = \omega \times (-\frac{1}{2}, M)$ and $\Omega = \omega \times (-1, M+1)$, we observe $\Psi = \partial^* G_\infty \cap (\omega \times \mathbb{R})$. Choose $\omega_\Psi \subset \omega$ such that $\omega_\Psi \times \{0\}$ is the orthogonal projection of Ψ onto $\mathbb{R}^{d-1} \times \{0\}$. Note that Ψ and ω_Ψ satisfy (6.13) since G_∞ is a set of finite perimeter. Thus, by (6.14) and the area formula we get

$$\int_{\Pi^\xi} \mathcal{H}^0\big((\partial^* G_\infty \cap \Omega)^\xi_y\big) \, \mathrm{d}\mathcal{H}^{d-1}(y) = \int_{\Psi} |\nu_\Psi \cdot \xi| \, \mathrm{d}\mathcal{H}^{d-1} \geq 2\mathcal{H}^{d-1}(\omega_\Psi) - C_\delta \mathcal{H}^{d-1}(\partial^* G_\infty)\,. \tag{6.16}$$

The second term. By (6.12) applied for $\Lambda_1 = \partial^* \Omega_h \cap \Omega$ and $\omega_{\Lambda_1} = \omega$, we get

$$\int_{\Pi^\xi} \mathcal{H}^0\Big(\big(\partial^* \Omega_h \cap \Omega\big)^\xi_y\Big) \, \mathrm{d}\mathcal{H}^{d-1} = \int_{\Lambda_1} |\nu_{\Lambda_1} \cdot \xi| \, \mathrm{d}\mathcal{H}^{d-1} \geq \mathcal{H}^{d-1}(\omega) - C_\delta \mathcal{H}^{d-1}(\partial^* \Omega_h)\,. \tag{6.17}$$

In a similar fashion, letting $\Lambda_2 = (\partial^* \Omega_h \cap \Omega) \setminus (G_\infty)^0$ and denoting by $\omega_{\Lambda_2} \subset \omega$ its orthogonal projection onto $\mathbb{R}^{d-1} \times \{0\}$, we get

$$\int_{\Pi^\xi} \mathcal{H}^0\Big(\big((\partial^* \Omega_h \cap \Omega) \setminus (G_\infty)^0\big)^\xi_y\Big) \, \mathrm{d}\mathcal{H}^{d-1} = \int_{\Lambda_2} |\nu_{\Lambda_2} \cdot \xi| \, \mathrm{d}\mathcal{H}^{d-1} \leq \mathcal{H}^{d-1}(\omega_{\Lambda_2}) + C_\delta \mathcal{H}^{d-1}(\partial^* \Omega_h)\,. \tag{6.18}$$

As Λ_2 is contained in $(G_\infty)^1 \cup \partial^* G_\infty$, we get $\omega_{\Lambda_2} \subset \omega_\Psi$, see Figure 1. Therefore, by combining

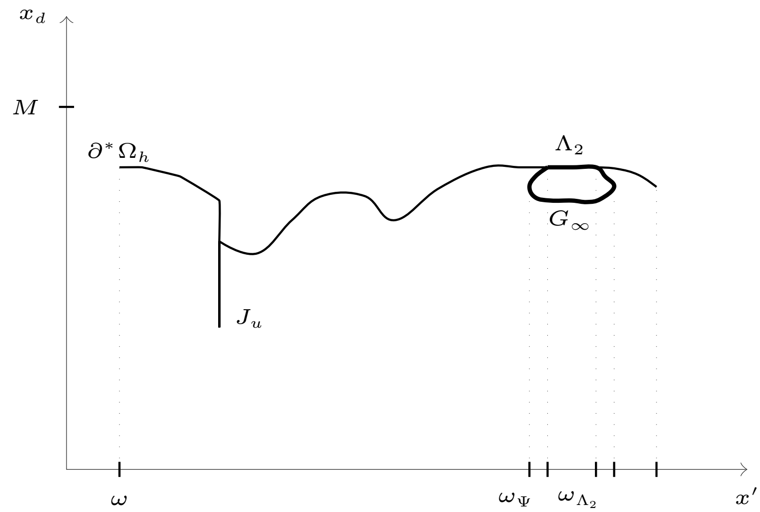

FIGURE 1. A picture of the situation in the argument by contradiction. We show that in fact $G_\infty = \emptyset$.

(6.17) and (6.18) we find

$$\begin{aligned}\int_{\Pi^\xi} \mathcal{H}^0\Big(\big(\partial^* \Omega_h \cap \Omega \cap (G_\infty)^0\big)^\xi_y\Big)\, \mathrm{d}\mathcal{H}^{d-1} &\geq \mathcal{H}^{d-1}(\omega) - \mathcal{H}^{d-1}(\omega_{\Lambda_2}) - 2C_\delta \mathcal{H}^{d-1}(\partial^* \Omega_h) \\ &\geq \mathcal{H}^{d-1}(\omega) - \mathcal{H}^{d-1}(\omega_\Psi) - 2C_\delta \mathcal{H}^{d-1}(\partial^* \Omega_h)\,. \quad (6.19)\end{aligned}$$

Now (6.15), (6.16), (6.19), and the fact that $\partial^* G_\infty \cap (G_\infty)^0 = \emptyset$ yield

$$\begin{aligned}\int_{\Pi^\xi} \mathcal{H}^0(\Gamma^\xi_y)\, \mathrm{d}\mathcal{H}^{d-1}(y) &\geq \int_{\Pi^\xi} \Big(\mathcal{H}^0\big((\partial^* G_\infty \cap \Omega)^\xi_y\big) + \mathcal{H}^0\Big(\big(\partial^* \Omega_h \cap \Omega \cap (G_\infty)^0\big)^\xi_y\Big)\Big)\, \mathrm{d}\mathcal{H}^{d-1}(y) \\ &\geq \mathcal{H}^{d-1}(\omega) + \mathcal{H}^{d-1}(\omega_\Psi) - 2C_\delta \mathcal{H}^{d-1}(\partial^* \Omega_h) - C_\delta \mathcal{H}^{d-1}(\partial^* G_\infty)\,. \quad (6.20)\end{aligned}$$

By the area formula, (6.7), and (6.12) we also have

$$\begin{aligned}\int_{\Pi^\xi} \mathcal{H}^0(\Gamma^\xi_y)\, \mathrm{d}\mathcal{H}^{d-1}(y) = \int_\Gamma |\nu_\Gamma \cdot \xi| \mathrm{d}\mathcal{H}^{d-1} &\leq \liminf_{n\to\infty} \int_{\Gamma_n} |\nu_{\Gamma_n} \cdot \xi|\, \mathrm{d}\mathcal{H}^{d-1} \\ &\leq \mathcal{H}^{d-1}(\omega) + C_\delta \sup\nolimits_{n\in\mathbb{N}} \mathcal{H}^{d-1}(\Gamma_n)\,.\end{aligned}$$

This along with (6.20) shows that $\mathcal{H}^{d-1}(\omega_\Psi) \leq \eta_\delta$ for some $\eta_\delta > 0$ with $\eta_\delta \to 0$ as $\delta \to 0$. (Recall that $C_\delta \to 0$, see (6.11).) As $\delta > 0$ is arbitrary, we obtain $\mathcal{H}^{d-1}(\omega_\Psi) = 0$. By recalling that $\omega_\Psi \times \{0\}$ is the orthogonal projection of $\partial^* G_\infty \cap (\omega \times \mathbb{R}) = \Psi$ onto $\mathbb{R}^{d-1} \times \{0\}$, we conclude that $G_\infty = \emptyset$.

This completes the proof of the lower inequality in Theorem 2.4. □

The upper inequality. To obtain the upper inequality, it clearly suffices to prove the following result.

Proposition 6.1. *Suppose that $f \geq 0$ is convex and satisfies* (2.1). *Consider (u,h) with $u = u\chi_{\Omega_h} \in GSBD^p(\Omega)$, $u = u_0$ in $\omega \times (-1,0)$, and $h \in BV(\omega;[0,M])$. Then, there exists a sequence $(u_n,h_n)_n$ with $h_n \in C^1(\omega) \cap BV(\omega;[0,M])$, $u_n|_{\Omega_{h_n}} \in W^{1,p}(\Omega_{h_n};\mathbb{R}^d)$, $u_n = 0$ in*

$\Omega \setminus \Omega_{h_n}$, *and* $u_n = u_0$ *in* $\omega\times(-1,0)$ *such that* $u_n \to u$ *in* $L^0(\Omega;\mathbb{R}^d)$, $h_n \to h$ *in* $L^1(\omega)$, *and*

$$\limsup_{n\to\infty} \int_{\Omega_{h_n}} f(e(u_n))\,\mathrm{d}x \le \int_{\Omega_h} f(e(u))\,\mathrm{d}x\,, \tag{6.21a}$$

$$\limsup\nolimits_{n\to\infty} \mathcal{H}^{d-1}(\partial\Omega_{h_n}\cap\Omega) \le \mathcal{H}^{d-1}(\partial^*\Omega_h\cap\Omega) + 2\mathcal{H}^{d-1}(J_u'\cap\Omega_h^1)\,. \tag{6.21b}$$

In particular, it is not restrictive to assume that $f \ge 0$. In fact, otherwise we consider $\tilde f := f + c_2 \ge 0$ changing the value of the elastic energy by the term $c_2\mathcal{L}^d(\Omega_h)$ which is continuous with respect to $L^1(\omega)$ convergence for h. Moreover, the integrals Ω_{h_n} and Ω_h can be replaced by $\Omega^+_{h_n}$ and Ω^+_h, respectively, since all functions coincide with u_0 on $\omega\times(-1,0)$.

Remark 6.2. The proof of the proposition will show that we can construct the sequence $(u_n)_n$ also in such a way that $u_n \in L^\infty(\Omega;\mathbb{R}^d)$ holds for all $n\in\mathbb{N}$. This, however, comes at the expense of the fact that the boundary data is only satisfied approximately, i.e., $u_n|_{\omega\times(-1,0)} \to u_0|_{\omega\times(-1,0)}$ in $W^{1,p}(\omega\times(-1,0);\mathbb{R}^d)$. This slightly different version will be instrumental in Subsection 6.3.

As a preparation, we first state some auxiliary results. We recall two lemmas from [18]. The first is stated in [18, Lemma 4.3].

Lemma 6.3. *Let* $h\in BV(\omega;[0,+\infty))$, *with* $\partial^*\Omega_h$ *essentially closed, i.e.,* $\mathcal{H}^{d-1}(\overline{\partial^*\Omega_h}\setminus\partial^*\Omega_h) = 0$. *Then, for any* $\varepsilon>0$, *there exists* $g\in C^\infty(\omega;[0,+\infty))$ *such that* $g\le h$ *a.e. in* ω, $\|g-h\|_{L^1(\omega)} \le \varepsilon$, *and*

$$\left|\int_\omega \sqrt{1+|\nabla g|^2}\,\mathrm{d}x' - \mathcal{H}^{d-1}(\partial^*\Omega_h\cap\Omega)\right| \le \varepsilon\,.$$

Lemma 6.4. *Let* $h\in BV(\omega;[0,M])$ *and let* $\Sigma\subset\mathbb{R}^d$ *with* $\mathcal{H}^{d-1}(\Sigma)<+\infty$ *be vertical in the sense that* $x=(x',x_d)\in\Sigma$ *implies* $(x',x_d+t)\in\Sigma$ *as long as* $(x',x_d+t)\in\Omega^1_h$. *Then, for each* $\varepsilon>0$ *there exists* $g\in C^\infty(\omega;[0,M])$ *such that*

$$\|g-h\|_{L^1(\omega)} \le \varepsilon\,, \tag{6.22a}$$

$$\mathcal{H}^{d-1}\big((\partial^*\Omega_h\cup\Sigma)\cap\Omega_g\big) \le \varepsilon\,, \tag{6.22b}$$

$$\Big|\int_\omega \sqrt{1+|\nabla g|^2}\,\mathrm{d}x' - \big(\mathcal{H}^{d-1}(\partial^*\Omega_h\cap\Omega) + 2\mathcal{H}^{d-1}(\Sigma)\big)\Big| \le \varepsilon\,. \tag{6.22c}$$

Proof. We refer to the first step in the proof of [18, Proposition 4.1], in particular [18, Equation (12)-(13)]. We point out that the case of possibly unbounded graphs has been treated there, i.e., $h\in BV(\omega;[0,+\infty))$. The proof shows that the upper bound on h is preserved and we indeed obtain $g\in C^\infty(\omega;[0,M])$ if $h\in BV(\omega;[0,M])$. $\square$

We now present an approximation technique based on [15]. To this end, we introduce some notation which will also be needed for the proof of Proposition 6.1. Let $k\in\mathbb{N}$, $k>1$. For any $z\in(2k^{-1})\mathbb{Z}^d$, consider the hypercubes

$$q^k_z := z + (-k^{-1},k^{-1})^d\,, \qquad Q^k_z := z + (-5k^{-1},5k^{-1})^d\,. \tag{6.23}$$

Given an open set $U\subset\mathbb{R}^d$, we also define the union of cubes well contained in U by

$$(U)^k := \mathrm{int}\Big(\bigcup\nolimits_{z:\ Q^k_z\subset U} \overline{q^k_z}\Big)\,. \tag{6.24}$$

(Here, $\mathrm{int}(\cdot)$ denotes the interior. This definition is unrelated to the notation E^s for the set of points with density $s\in[0,1]$.) In the following, $\psi\colon[0,\infty)\to[0,\infty)$ denotes the function $\psi(t) = t\wedge 1$.

Lemma 6.5. *Let $U \subset \mathbb{R}^d$ be open, bounded, $p > 1$, and $k \in \mathbb{N}$, $\theta \in (0,1)$ with k^{-1}, θ small enough. Let $\mathcal{F} \subset GSBD^p(U)$ be such that $\psi(|v|) + |e(v)|^p$ is equiintegrable for $v \in \mathcal{F}$. Suppose that for $v \in \mathcal{F}$ there exists a set of finite perimeter $V \subset U$ such that for each q_z^k, $z \in (2k^{-1})\mathbb{Z}^d$, intersecting $(U)^k \setminus V$, there holds that*

$$\mathcal{H}^{d-1}\big(Q_z^k \cap J_v\big) \le \theta k^{1-d} . \tag{6.25}$$

Then there exists a function $w_k \in W^{1,\infty}((U)^k \setminus V; \mathbb{R}^d)$ such that

$$\int_{(U)^k \setminus V} \psi(|w_k - v|)\,\mathrm{d}x \le R_k\,, \tag{6.26a}$$

$$\int_{(U)^k \setminus V} |e(w_k)|^p\,\mathrm{d}x \le \int_U |e(v)|^p\,\mathrm{d}x + R_k\,. \tag{6.26b}$$

where $(R_k)_k$ is a sequence independent of $v \in \mathcal{F}$ with $R_k \to 0$ as $k \to \infty$.

The lemma is essentially a consequence of the rough estimate proved in [15, Theorem 3.1]. For the convenience of the reader, we include a short proof in Appendix A.

Recall that $\Omega = \omega \times (-1, M+1)$ for given $M > 0$. Consider a pair (u,h) as in Proposition 6.1. We work with $u\chi_{\Omega_h} \in GSBD^p(\Omega)$ in the following, without specifying each time that $u = 0$ in the complement of Ω_h. Recall J'_u defined in (2.10), and, as before, set $\Sigma := J'_u \cap \Omega_h^1$. This implies $J_u \subset (\partial^*\Omega_h \cap \Omega) \cup \Sigma$. Since Σ is vertical, we can approximate $(\partial^*\Omega_h \cap \Omega) \cup \Sigma$ by the graph of a smooth function $g \in C^\infty(\omega; [0,M])$ in the sense of Lemma 6.4.

Our goal is to construct a regular approximation of u in (most of) Ω_g by means of Lemma 6.5. The main step is to identify suitable exceptional sets $(V_k)_k$ such that for the cubes outside of $(V_k)_k$ we can verify (6.25). In this context, we emphasize that it is crucial that each V_k is "vertical", see Step 1 of the proof of Proposition 6.1 below for details, i.e., it is constructed in such a way that the boundary of $(\Omega_g)^k \setminus V_k$ (recall (6.24)) can be written as the graph of a BV function. This will eventually allow us to approximate the boundary of $(\Omega_g)^k \setminus V_k$ from below by a smooth graph by means of Lemma 6.3. Before we start with the actual proof of Proposition 6.1, we introduce the notion of good and bad nodes for the construction of $(V_k)_k$, and collect some important properties.

Define the set of nodes

$$\mathcal{N}_k := \{z \in (2k^{-1})\mathbb{Z}^d \colon \overline{q_z^k} \subset \Omega_g\}\,. \tag{6.27}$$

Let us introduce the families of *good nodes* and *bad nodes* at level k. Let $\rho_1, \rho_2 > 0$ to be specified below. By $\mathcal{G}_k$ we denote the set of good nodes $z \in \mathcal{N}_k$, namely those satisfying

$$\mathcal{H}^{d-1}\big(\overline{q_z^k} \cap (\partial^*\Omega_h \cup \Sigma)\big) \le \rho_1 k^{1-d} \tag{6.28}$$

or having the property that there exists a set of finite perimeter $F_z^k \subset q_z^k$, such that

$$q_z^k \cap \Omega_h^0 \subset (F_z^k)^1, \qquad \mathcal{H}^{d-1}\big(\partial^* F_z^k\big) \le \rho_2 k^{1-d}, \qquad \mathcal{H}^{d-1}\big(\overline{q_z^k} \cap \Sigma \cap (F_z^k)^0\big) \le \rho_2 k^{1-d}\,. \tag{6.29}$$

We define the set of bad nodes by $\mathcal{B}_k = \mathcal{N}_k \setminus \mathcal{G}_k$. Moreover, let

$$\mathcal{G}_k^* := \{z \in \mathcal{G}_k \colon \text{(6.28) does not hold}\}\,. \tag{6.30}$$

For an illustration of the cubes in $\mathcal{G}_k$ we refer to Figure 2.

We partition the set of good nodes $\mathcal{G}_k$ into

$$\mathcal{G}_k^1 = \big\{z \in \mathcal{G}_k \colon \mathcal{L}^d(q_z^k \cap \Omega_h^0) \le \mathcal{L}^d(q_z^k \cap \Omega_h^1)\big\}, \qquad \mathcal{G}_k^2 = \mathcal{G}_k \setminus \mathcal{G}_k^1. \tag{6.31}$$

We introduce the terminology "$q_{z'}^k$ is above q_z^k" meaning that $q_{z'}^k$ and q_z^k have the same vertical projection onto $\mathbb{R}^{d-1} \times \{0\}$ and $z'_d > z_d$.

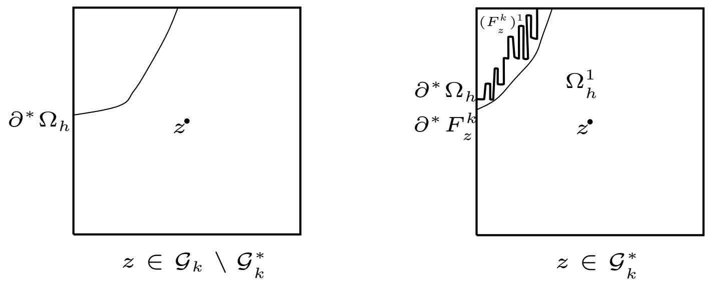

FIGURE 2. A simplified representation of nodes in $\mathcal{G}_k$, for $d = 2$ and with $\Sigma = \emptyset$. The set $\mathcal{G}_k \setminus \mathcal{G}_k^*$ corresponds to the cubes containing only a small portion of $\partial^* \Omega_h \cup \Sigma$, see first picture. For the cubes $\mathcal{G}_k^*$, the portion of $\partial^* \Omega_h$ is contained in a set F_z^k with small boundary, see second picture. Intuitively, this along with the fact that (6.28) does not hold means that $\partial^* \Omega_h$ is highly oscillatory in such cubes.

We remark that bad nodes have been defined differently in [18], namely as the cubes having an edge which intersects $\partial^* \Omega_h \cap \Sigma$. This definition is considerably easier than our definition. It may, however, fail in some pathological situations since, in this case, the union of cubes with bad nodes as centers does not necessarily form a "vertical set".

Lemma 6.6 (Properties of good and bad nodes)**.** *Given Ω_h and Σ, define Ω_g as in Lemma 6.4 for $\varepsilon > 0$ sufficiently small. We can choose $0 < \rho_1 < \rho_2$ satisfying $\rho_1, \rho_2 \le \frac{1}{2} 5^{-d} \theta$ such that the following properties hold for the good and bad nodes defined in* (6.27)–(6.31)*:*

(i) *if $q_{z'}^k$ is above q_z^k and $z \in \mathcal{B}_k \cup \mathcal{G}_k^2$, then $z' \in \mathcal{B}_k \cup \mathcal{G}_k^2$.*

(ii) *if $z, z' \in \mathcal{G}_k$ with $\mathcal{H}^{d-1}(\partial q_z^k \cap \partial q_{z'}^k) > 0$, then $z, z' \in \mathcal{G}_k^1$ or $z, z' \in \mathcal{G}_k^2$.*

(iii) $\#\mathcal{B}_k + \#\mathcal{G}_k^* \le 2\rho_1^{-1} k^{d-1} \varepsilon$.

(iv) $\sum_{z \in \mathcal{G}_k^2} \mathcal{L}^d(\Omega_h \cap q_z^k) \le \varepsilon$.

We suggest to omit the proof of the lemma on first reading and to proceed directly with the proof of Proposition 6.1.

Proof. By $c_\pi \ge 1$ we denote the maximum of the constants appearing in the isoperimetric inequality and the relative isoperimetric inequality on a cube in dimension d. We will show the statement for ε and $0 < \rho_2 < 1$ sufficiently small satisfying $\rho_2 \le \frac{1}{2} 5^{-d} \theta$, and for $\rho_1 = ((3d+1)c_\pi)^{-1} \rho_2$.

Preparations. First, we observe that for ρ_2 sufficiently small we have that $\mathcal{G}_k^* \subset \mathcal{G}_k^1$. Indeed, since for $z \in \mathcal{G}_k^*$ property (6.29) holds, the isoperimetric inequality implies

$$\mathcal{L}^d(q_z^k \cap \Omega_h^0) \le \mathcal{L}^d(F_z^k) \le c_\pi \big(\mathcal{H}^{d-1}(\partial^* F_z^k)\big)^{d/(d-1)} \le c_\pi \rho_2^{d/(d-1)} k^{-d}. \tag{6.32}$$

Then, for ρ_2 sufficiently small we get $\mathcal{L}^d(q_z^k \cap \Omega_h^0) < \frac{1}{2} \mathcal{L}^d(q_z^k)$, and thus $z \in \mathcal{G}_k^1$, see (6.31).

As a further preparation, we show that for each $z \in \mathcal{G}_k^1$ there exists a set of finite perimeter H_z^k with $\Omega_h^0 \cap q_z^k \subset H_z^k \subset q_z^k$ such that

$$\mathcal{L}^d(H_z^k) \le c_\pi \rho_2^{d/(d-1)} k^{-d}, \quad \mathcal{H}^{d-1}(\partial^* H_z^k) \le \rho_2 k^{1-d}, \quad \mathcal{H}^{d-1}\big(\overline{q_z^k} \cap \Sigma \cap (H_z^k)^0\big) \le \rho_2 k^{1-d}\,. \tag{6.33}$$

Indeed, if (6.29) holds, this follows directly from (6.29) and (6.32) for $H_z^k := F_z^k$.

Now suppose that $z \in \mathcal{G}_k^1$ satisfies (6.28). In this case, we define $H_z^k := \Omega_h^0 \cap q_z^k$. To control the volume, we use the relative isoperimetric inequality on q_z^k to find by (6.28)

$$\mathcal{L}^d(H_z^k) = \mathcal{L}^d(\Omega_h^0 \cap q_z^k) \wedge \mathcal{L}^d(\Omega_h^1 \cap q_z^k) \le c_\pi \big(\mathcal{H}^{d-1}(\partial^* \Omega_h \cap q_z^k)\big)^{d/(d-1)} \le c_\pi \rho_1^{d/(d-1)} k^{-d}\,, \tag{6.34}$$

i.e., the first part of (6.33) holds since $\rho_1 = ((3d+1)c_\pi)^{-1}\rho_2$. To obtain the second estimate in (6.33), the essential step is to control $\mathcal{H}^{d-1}(\partial q_z^k \cap \Omega_h^0)$. For simplicity, we only estimate $\mathcal{H}^{d-1}(\partial_d q_z^k \cap \Omega_h^0)$ where $\partial_d q_z^k$ denotes the two faces of ∂q_z^k whose normal vector is parallel to e_d. The other faces can be treated in a similar fashion. Write $z = (z', z_d)$ and define $\omega_z = z' + (-k^{-1}, k^{-1})^{d-1}$. By $\omega_* \subset \omega_z$ we denote the largest measurable set such that the cylindrical set $(\omega_* \times \mathbb{R}) \cap q_z^k$ is contained in Ω_h^0. Then by the area formula (cf. e.g. [53, (12.4) in Section 12]) and by recalling notation (3.1) we get

$$\begin{aligned}
\mathcal{H}^{d-1}(\partial_d q_z^k \cap \Omega_h^0) &\le 2\mathcal{H}^{d-1}(\omega_*) + 2\int_{(\omega_z \setminus \omega_*)\times\{0\}} \mathcal{H}^0\big((\partial^*\Omega_h)_y^{e_d}\big)\,\mathrm{d}\mathcal{H}^{d-1}(y) \\
&\le 2\mathcal{H}^{d-1}(\omega_*) + 2\int_{\partial^*\Omega_h \cap q_z^k} |\nu_{\Omega_h} \cdot e_d|\,\mathrm{d}\mathcal{H}^{d-1} \\
&\le 2\mathcal{H}^{d-1}(\omega_*) + 2\mathcal{H}^{d-1}(\partial^*\Omega_h \cap q_z^k)\,.
\end{aligned} \tag{6.35}$$

As $(\omega_* \times \mathbb{R}) \cap q_z^k \subset \Omega_h^0 \cap q_z^k$ and $\mathcal{L}^d(\Omega_h^0 \cap q_z^k) \le c_\pi \rho_1^{d/(d-1)} k^{-d}$ by (6.34), we deduce $2k^{-1}\mathcal{H}^{d-1}(\omega_*) \le c_\pi \rho_1^{d/(d-1)} k^{-d}$. This along with (6.28) and (6.35) yields

$$\mathcal{H}^{d-1}(\partial_d q_z^k \cap \Omega_h^0) \le c_\pi \rho_1^{d/(d-1)} k^{1-d} + 2\rho_1 k^{1-d} \le 3c_\pi \rho_1 k^{1-d}\,.$$

By repeating this argument for the other faces and by recalling $\mathcal{H}^{d-1}(\partial^*\Omega_h \cap q_z^k) \le \rho_1 k^{1-d}$, we conclude that $H_z^k = \Omega_h^0 \cap q_z^k$ satisfies

$$\mathcal{H}^{d-1}(\partial^* H_z^k) = \mathcal{H}^{d-1}(\partial^*\Omega_h \cap q_z^k) + \mathcal{H}^{d-1}(\partial q_z^k \cap \Omega_h^0) \le \rho_1 k^{1-d} + d \cdot 3c_\pi \rho_1 k^{1-d} \le \rho_2 k^{1-d}\,,$$

where the last step follows from $\rho_1 = ((3d+1)c_\pi)^{-1}\rho_2$. This concludes the the second part of (6.33). The third part follows from (6.28) and $\rho_1 \le \rho_2$. We are now in a position to prove the statement.

Proof of (i). We need to show that for $z' \in \mathcal{G}_k^1$ there holds $z \in \mathcal{G}_k^1$ for all $z \in \mathcal{N}_k$ such that $q_{z'}^k$ is above q_z^k. Fix such cubes q_z^k and $q_{z'}^k$.

Consider the set $H_{z'}^k$ with $\Omega_h^0 \cap q_{z'}^k \subset H_{z'}^k \subset q_{z'}^k$ introduced in (6.33), and define $F_z^k := H_{z'}^k - z' + z$. Since Ω_h is a generalized graph, we get $(F_z^k)^1 \supset \Omega_h^0 \cap q_z^k$. Moreover, since $\Sigma = J_u' \cap \Omega_h^1$ is vertical in Ω_h, see (2.10), and $(H_{z'}^k)^0 \subset \Omega_h^1 \cap q_{z'}^k$, we have

$$\Sigma \cap (F_z^k)^0 = \Sigma \cap \Omega_h^1 \cap (F_z^k)^0 \subset (\Sigma \cap \Omega_h^1 \cap (H_{z'}^k)^0) + z - z' = (\Sigma \cap (H_{z'}^k)^0) + z - z'\,.$$

By (6.33) we thus get $\mathcal{H}^{d-1}(\overline{q_z^k} \cap \Sigma \cap (F_z^k)^0) \le \mathcal{H}^{d-1}(\overline{q_{z'}^k} \cap \Sigma \cap (H_{z'}^k)^0) \le \rho_2 k^{1-d}$. Then the third property in (6.29) is satisfied for z. Again by (6.33) we note that also the first two properties of (6.29) hold, and thus $z \in \mathcal{G}_k$. Using once more that Ω_h is a generalized graph, we get $\mathcal{L}^d(\Omega_h \cap q_z^k) \ge \mathcal{L}^d(\Omega_h \cap q_{z'}^k)$. Then $z' \in \mathcal{G}_k^1$ implies $z \in \mathcal{G}_k^1$, see (6.31). This shows (i).

Proof of (ii). Suppose by contradiction that there exist $z \in \mathcal{G}_k^1$ and $z' \in \mathcal{G}_k^2$ satisfying $\mathcal{H}^{d-1}(\partial q_z^k \cap \partial q_{z'}^k) > 0$. Define the set $F := H_z^k \cup (\Omega_h^0 \cap q_{z'}^k)$ with H_z^k from (6.33), and observe that F is contained in the cuboid $q_*^k = \mathrm{int}(\overline{q_z^k} \cup \overline{q_{z'}^k})$. Since $H_z^k \supset \Omega_h^0 \cap q_z^k$, we find

$$\mathcal{H}^{d-1}(q_*^k \cap \partial^* F) \le \mathcal{H}^{d-1}(\partial^* H_z^k) + \mathcal{H}^{d-1}(\partial^* \Omega_h \cap q_{z'}^k)\,.$$

As $\mathcal{G}^2_k \cap \mathcal{G}^*_k = \emptyset$, cf. (6.32), for $z' \in \mathcal{G}^2_k$ estimate (6.28) holds true. This along with (6.33) yields

$$\mathcal{H}^{d-1}(q^k_* \cap \partial^* F) \le \rho_2 k^{1-d} + \rho_1 k^{1-d} \le 2\rho_2 k^{1-d}\,.$$

Then, the relative isoperimetric inequality on q^k_* yields

$$\mathcal{L}^d(q^k_* \cap F) \wedge \mathcal{L}^d(q^k_* \setminus F) \le C_* \big(\mathcal{H}^{d-1}(q^k_* \cap \partial^* F)\big)^{d/(d-1)} \le C_*(2\rho_2)^{d/(d-1)} k^{-d} \tag{6.36}$$

for some universal $C_* > 0$. On the other hand, there holds $\mathcal{L}^d(q^k_* \cap F) \ge \mathcal{L}^d(\Omega^0_h \cap q^k_{z'}) \ge \frac{1}{2}(2k^{-1})^d$ and $\mathcal{L}^d(q^k_* \setminus F) \ge \mathcal{L}^d(q^k_z \setminus H^k_z) \ge (2k^{-1})^d - c_\pi \rho_2^{d/(d-1)} k^{-d}$ by (6.33). However, for ρ_2 sufficiently small, this contradicts (6.36). This concludes the proof of (ii).

Proof of (iii). Note that $\mathcal{H}^{d-1}$-a.e. point in $\mathbb{R}^d$ is contained in at most two different closed cubes $\overline{q^k_z}$, $\overline{q^k_{z'}}$. Therefore, since the cubes with centers in $\mathcal{G}^*_k$ and $\mathcal{B}_k$ do not satisfy (6.28), we get

$$\#\mathcal{B}_k + \#\mathcal{G}^*_k \le \rho_1^{-1} k^{d-1} \sum_{z \in \mathcal{B}_k \cup \mathcal{G}^*_k} \mathcal{H}^{d-1}\big(\overline{q^k_z} \cap (\partial^* \Omega_h \cup \Sigma)\big) \le 2\rho_1^{-1} k^{d-1} \mathcal{H}^{d-1}\big((\partial^* \Omega_h \cup \Sigma) \cap \Omega_g\big)\,,$$

where the last step follows from (6.27). This along with (6.22b) shows (iii).

Proof of (iv). Recall that each $z \in \mathcal{G}^2_k$ satisfies (6.28), cf. (6.30) and before (6.32). The relative isoperimetric inequality, (6.27), and (6.31) yield

$$\begin{aligned}\sum\nolimits_{z \in \mathcal{G}^2_k} \mathcal{L}^d(\Omega_h \cap q^k_z) &= \sum\nolimits_{z \in \mathcal{G}^2_k} \mathcal{L}^d(\Omega^0_h \cap q^k_z) \wedge \mathcal{L}^d(\Omega^1_h \cap q^k_z) \le c_\pi \sum\nolimits_{z \in \mathcal{G}^2_k} \big(\mathcal{H}^{d-1}(\partial^* \Omega_h \cap q^k_z)\big)^{\frac{d}{d-1}} \\ &\le c_\pi \Big(\sum\nolimits_{z \in \mathcal{G}^2_k} \mathcal{H}^{d-1}(\partial^* \Omega_h \cap q^k_z)\Big)^{d/(d-1)} \le c_\pi \big(\mathcal{H}^{d-1}(\partial^* \Omega_h \cap \Omega_g)\big)^{d/(d-1)}\,.\end{aligned}$$

By (6.22b) we conclude for ε small enough that $\sum_{z \in \mathcal{G}^2_k} \mathcal{L}^d(\Omega_h \cap q^k_z) \le c_\pi \varepsilon^{d/(d-1)} \le \varepsilon$. □

Proof of Proposition 6.1. Consider a pair (u,h) and set $\Sigma := J'_u \cap \Omega^1_h$ with J'_u as in (2.10). Given $\varepsilon > 0$, we approximate $(\partial^* \Omega_h \cap \Omega) \cup \Sigma$ by the graph of a smooth function $g \in C^\infty(\omega; [0,M])$ in the sense of Lemma 6.4. Define the good and bad nodes as in (6.27)–(6.31) for $0 < \rho_1, \rho_2 \le \frac{1}{2} 5^{-d} \theta$ such that the properties in Lemma 6.6 hold. We will first define approximating regular graphs (Step 1) and regular functions (Step 2) for fixed $\varepsilon > 0$. Finally, we let $\varepsilon \to 0$ and obtain the result by a diagonal argument (Step 3). In the whole proof, $C > 0$ will denote a constant depending only on d, p, ρ_1, and ρ_2.

Step 1: Definition of regular graphs. Recall (6.24). For each $k \in \mathbb{N}$, we define the set

$$V_k := \bigcup\nolimits_{z \in \mathcal{G}^2_k \cup \mathcal{B}_k} Q^k_z \cap (\Omega_g)^k\,. \tag{6.37}$$

We observe that

$$\partial V_k \cap (\Omega_g)^k \subset \bigcup\nolimits_{z \in \mathcal{B}_k} \partial Q^k_z\,. \tag{6.38}$$

In fact, consider $z \in \mathcal{B}_k \cup \mathcal{G}^2_k$ such that $Q^k_z \cap V_k \neq \emptyset$ and one face of ∂Q^k_z intersects $\partial V_k \cap (\Omega_g)^k$. In view of (6.37), there exists an adjacent cube $q^k_{z'}$ satisfying $\mathcal{H}^{d-1}(\partial q^k_z \cap \partial q^k_{z'}) > 0$ and $z' \in \mathcal{G}^1_k$ since otherwise $\partial Q^k_z \cap \partial V_k \cap (\Omega_g)^k = \emptyset$. As $z' \in \mathcal{G}^1_k$, Lemma 6.6(ii) implies $z \notin \mathcal{G}^2_k$ and therefore $z \in \mathcal{B}_k$. This shows (6.38). A similar argument yields

$$V_k = \Big(\bigcup\nolimits_{z \in \mathcal{B}_k} Q^k_z \cup \bigcup\nolimits_{z \in \mathcal{G}^2_k} q^k_z\Big) \cap (\Omega_g)^k \tag{6.39}$$

up to a negligible set. Indeed, since V_k is a union of cubes of sidelength $2k^{-1}$ centered in nodes in $\mathcal{N}_k$, it suffices to prove that for a fixed $z \in \mathcal{N}_k \cap V_k$ there holds (a) $z \in \mathcal{G}^2_k$ or that (b) there exists $z' \in \mathcal{B}_k$ such that $z \in Q^k_{z'}$. Arguing by contradiction, if $z \in \mathcal{N}_k \cap V_k$ and neither (a) nor (b) hold, we deduce that $z \in \mathcal{G}^1_k$ and $Q^k_z \cap \mathcal{B}_k = \emptyset$. Then all $z' \in \mathcal{N}_k \cap Q^k_z$ lie in $\mathcal{G}_k$. More precisely, by $z \in \mathcal{G}_k^1$ and Lemma 6.6(ii) we get that all $z' \in \mathcal{N}_k \cap Q_z^k$ lie in $\mathcal{G}_k^1$. Then $Q_z^k \cap (\mathcal{G}_k^2 \cup \mathcal{B}_k) = \emptyset$, so that $q_z^k \cap V_k = \emptyset$ by (6.37). This contradicts $z \in V_k$.

Let us now estimate the surface and volume of V_k. By (6.38) and Lemma 6.6(iii) we get

$$\mathcal{H}^{d-1}(\partial V_k \cap (\Omega_g)^k) \le \sum\nolimits_{z \in \mathcal{B}_k} \mathcal{H}^{d-1}(\partial Q_z^k) \le C k^{1-d} \# \mathcal{B}_k \le C\varepsilon\,, \tag{6.40}$$

where C depends on ρ_1. In a similar fashion, by (6.39) and Lemma 6.6(iii),(iv) we obtain

$$\mathcal{L}^d(V_k \cap \Omega_h) \le C k^{-d} \# \mathcal{B}_k + \sum\nolimits_{z \in \mathcal{G}_k^2} \mathcal{L}^d(q_z^k \cap \Omega_h) \le C k^{-1}\varepsilon + \varepsilon \le C\varepsilon\,. \tag{6.41}$$

Note that V_k is vertical in the sense that $(x', x_d) \in V_k$ implies $(x', x_d + t) \in V_k$ for $t \ge 0$ as long as $(x', x_d + t) \in (\Omega_g)^k$. This follows from Lemma 6.6(i) and (6.37).

Our goal is to choose a regular graph lying below Ω_g and V_k. To this end, we need to slightly lift and dilate the involved sets. Recall definition (6.24), and define $\omega_k \subset \omega \subset \mathbb{R}^{d-1}$ such that $(\omega \times \mathbb{R})^k = \omega_k \times \mathbb{R}$. Since ω is uniformly star-shaped with respect to the origin, see (2.9), there exists a universal constant $\tau_\omega > 0$ such that

$$\omega_k \supset (1 - \tau k^{-1})\, \omega \quad \text{ for } \tau \ge \tau_\omega\,. \tag{6.42}$$

Define $\tau_g := 1 + \sqrt{d} \max_\omega |\nabla g|$. For k sufficiently large, it is elementary to check that

$$\Omega_g \cap (\omega_k \times (0, \infty)) \subset \big((\Omega_g)^k + 6\,\tau_g k^{-1} e_d\big)\,. \tag{6.43}$$

We now "lift" the set $(\Omega_g)^k \setminus V_k$ upwards: define the functions

$$g_k'(x') := \sup\big\{x_d < g(x') \colon (x', x_d - 6\,\tau_g/k) \in (\Omega_g)^k \setminus V_k\big\} \qquad \text{for } x' \in \omega_k\,. \tag{6.44}$$

We observe that $g_k' \in BV(\omega_k; [0, M])$. Define $(\Omega)^k$ as in (6.24) and, similar to (6.1), we let $\Omega_{g_k'} = \{x \in \omega_k \times (-1, M+1) \colon -1 < x_d < g_k'(x')\}$. Since V_k is vertical, we note that $\partial \Omega_{g_k'} \cap (\Omega)^k$ is made of two parts: one part is contained in the smooth graph of g and the rest in the boundary of $V_k + 6\tau_g k^{-1} e_d$. In particular, by (6.43) we get

$$\partial \Omega_{g_k'} \cap (\Omega)^k \subset (\partial \Omega_g \cap \Omega) \cup \big(\partial \Omega_{g_k'} \cap (\Omega)^k \cap \Omega_g\big) \subset (\partial \Omega_g \cap \Omega) \cup \big((\partial V_k \cap (\Omega_g)^k) + 6\,\tau_g k^{-1}\ e_d\big)\,.$$

Then, by (6.22c) and (6.40), we deduce

$$\mathcal{H}^{d-1}(\partial \Omega_{g_k'} \cap (\Omega)^k) \le \mathcal{H}^{d-1}(\partial^* \Omega_h \cap \Omega) + 2\,\mathcal{H}^{d-1}(\Sigma) + C\,\varepsilon\,. \tag{6.45}$$

Since by (6.43) and (6.44) there holds $(\Omega_g \setminus \Omega_{g_k'}) \cap (\Omega)^k \subset V_k + 6\,\tau_g k^{-1} e_d$, (6.22a), (6.41), and the fact that V_k is vertical imply

$$\begin{aligned} \mathcal{L}^d\big((\Omega_g \setminus \Omega_{g_k'}) \cap (\Omega)^k\big) &\le \mathcal{L}^d\big(\Omega_g \cap (V_k + 6\,\tau_g k^{-1} e_d)\big) \le \mathcal{L}^d(\Omega_g \cap V_k) \\ &\le \mathcal{L}^d(\Omega_h \cap V_k) + \mathcal{L}^d(\Omega_g \setminus \Omega_h) \le C\varepsilon\,. \end{aligned} \tag{6.46}$$

As g_k' is only defined on ω_k, we further need a dilation: letting $\tau_* := \tau_\omega \vee (6\tau_g + 6)$ and recalling (6.42) we define $g_k'' \in BV(\omega; [0, M])$ by

$$g_k''(x') = g_k'((1 - \tau_* k^{-1})\, x') \quad \text{for} \quad x' \in \omega\,. \tag{6.47}$$

(The particular choice of τ_* will become clear in (6.54) below.) By (6.46) we get

$$\mathcal{L}^d(\Omega_g \triangle \Omega_{g_k''}) \le C\varepsilon + C_{g,\omega} k^{-1}, \qquad \big|\mathcal{H}^{d-1}(\partial \Omega_{g_k''} \cap \Omega) - \mathcal{H}^{d-1}(\partial \Omega_{g_k'} \cap (\Omega)^k)\big| \le C_{g,\omega} k^{-1}\,, \tag{6.48}$$

where the constant $C_{g,\omega}$ depends also on g and ω. We also notice that $\mathcal{H}^{d-1}\big(\overline{\partial^* \Omega_{g_k''}} \setminus \partial^* \Omega_{g_k''}\big) = 0$. Then by Lemma 6.3 we find a function $h_k \in C^\infty(\omega; [0, M])$ with $h_k \le g_k''$ on ω such that

$$\|g_k'' - h_k\|_{L^1(\omega)} \le \varepsilon\,, \qquad \big|\mathcal{H}^{d-1}(\partial \Omega_{h_k} \cap \Omega) - \mathcal{H}^{d-1}(\partial^* \Omega_{g_k''} \cap \Omega)\big| \le \varepsilon\,. \tag{6.49}$$

By (6.22a), (6.45), (6.48), and (6.49) we finally get

$$\mathcal{L}^d(\Omega_h \triangle \Omega_{h_k}) \le C\varepsilon + C_{g,\omega} k^{-1}\,, \tag{6.50a}$$

$$\mathcal{H}^{d-1}(\partial\Omega_{h_k} \cap \Omega) \le \mathcal{H}^{d-1}(\partial^*\Omega_h \cap \Omega) + 2\,\mathcal{H}^{d-1}(\Sigma) + C\varepsilon + C_{g,\omega} k^{-1}\,. \tag{6.50b}$$

Step 2: Definition of regular functions. Recall (6.29)–(6.30), and observe that Lemma 6.6(iii) implies

$$\mathcal{L}^d(F^k) \le \sum\nolimits_{z\in\mathcal{G}_k^*} \mathcal{L}^d(q_z^k) \le Ck^{-d}\#\mathcal{G}_k^* \le C\varepsilon\, k^{-1}\,, \quad \text{ where } \ F^k := \bigcup\nolimits_{z\in\mathcal{G}_k^*} (F_z^k)^1\,.$$

We define the functions $v_k \in GSBD^p(\Omega)$ by

$$v_k := u(1-\chi_{F^k})\,\chi_{\Omega_g}\,. \tag{6.51}$$

Since $u = 0$ in $\Omega \setminus \Omega_h$ and $v_k = 0$ in $\Omega \setminus \Omega_g$, we get by (6.22a) and (6.51)

$$\limsup_{k\to\infty} \mathcal{L}^d(\{v_k \neq u\}) \le \limsup_{k\to\infty} \mathcal{L}^d\big(F^k \cup (\Omega_h \setminus \Omega_g)\big) \le \varepsilon\,. \tag{6.52}$$

We also obtain

$$\mathcal{H}^{d-1}(Q_z^k \cap J_{v_k}) \le \theta k^{1-d} \tag{6.53}$$

for each q_z^k intersecting $(\Omega_g)^k \setminus V_k$. To see this, note that the definitions of $\mathcal{N}_k$ in (6.27) and of V_k in (6.37) imply that for each q_z^k with $q_z^k \cap ((\Omega_g)^k \setminus V_k) \neq \emptyset$, each $z' \in \mathcal{N}_k$ with $q_{z'}^k \cap Q_z^k \neq \emptyset$ satisfies $z' \in \mathcal{G}_k$. In view of $\rho_1 < \rho_2 \le \frac{1}{2}5^{-d}\theta$ (see Lemma 6.6), the property then follows from (6.28), (6.29), $J_u \cap \Omega_g \subset \partial^*\Omega_h \cup \Sigma$, and the fact that Q_z^k consists of 5^d different cubes $q_{z'}^k$.

Notice that $|v_k| \le |u|$ and $|e(v_k)| \le |e(u)|$ pointwise a.e., i.e., the functions $\psi(|v_k|) + |e(v_k)|^p$ are equiintegrable, where $\psi(t) = t \wedge 1$. In view of (6.53), we can apply Lemma 6.5 on $U = \Omega_g$ for the function $v_k \in GSBD^p(\Omega_g)$ and the sets V_k, to get functions $w_k \in W^{1,\infty}((\Omega_g)^k \setminus V_k; \mathbb{R}^d)$ such that (6.26a) and (6.26b) hold for a sequence $R_k \to 0$.

Recall the definitions of g_k' and g_k'' in (6.44) and (6.47), as well as the definition of the smooth mapping h_k satisfying $h_k \le g_k''$. This yields

$$x = (x', x_d) \in \Omega_{h_k} \quad \Rightarrow \quad \big((1-\tau_*/k)\,x',\, x_d - 6\tau_g/k\big) \in (\Omega_g)^k \setminus V_k\,.$$

Recall $\tau_* = \tau_\omega \vee (6\tau_g + 6)$ and observe that $-(1-\tau_*/k) - 6\tau_g/k \ge -1 + 6/k$. Also note that $(\Omega_g)^k \supset (\Omega)^k \cap (\omega \times (-1+6/k, 0))$, cf. (6.24). This along with the verticality of V_k shows that

$$x = (x', x_d) \in \Omega_{h_k} \quad \Rightarrow \quad \big((1-\tau_*/k)\,x', (1-\tau_*/k)\,x_d - 6\tau_g/k\big) \in (\Omega_g)^k \setminus V_k\,. \tag{6.54}$$

We define the function $\hat{w}_k\colon \Omega \to \mathbb{R}^d$ by

$$\hat{w}_k(x) := \begin{cases} w_k\big((1-\tau_*/k)\,x', (1-\tau_*/k)\,x_d - 6\,\tau_g/k\big) & \text{if } -1 < x_d < h_k(x')\,, \\ 0 & \text{otherwise.} \end{cases}$$

In view of (6.54), the mapping is well defined and satisfies $\hat{w}_k|_{\Omega_{h_k}} \in W^{1,\infty}(\Omega_{h_k}; \mathbb{R}^d)$. By (6.22a) (6.26a), (6.50a), (6.52), and $\psi \le 1$ we get

$$\limsup_{k\to\infty} \|\psi(|\hat{w}_k - u|)\|_{L^1(\Omega)} \le \limsup_{k\to\infty} \big(\|\psi(|\hat{w}_k - v_k|)\|_{L^1(\Omega)} + \mathcal{L}^d(\{v_k \neq u\})\big) \le C\varepsilon\,. \tag{6.55}$$

In a similar fashion, employing (6.26b) in place of (6.26a) and the fact that $\|e(\hat{w}_k)\|_{L^p(\Omega)} \le (1 + C_M k^{-1})\|e(w_k)\|_{L^p((\Omega_g)^k\setminus V_k)}$ for some C_M depending on M and τ_*, we obtain

$$\begin{aligned} \limsup_{k\to\infty} \int_\Omega |e(\hat{w}_k)|^p \,\mathrm{d}x &\le \limsup_{k\to\infty} \int_{(\Omega_g)^k\setminus V_k} |e(w_k)|^p \,\mathrm{d}x \\ &\le \limsup_{k\to\infty} \int_{\Omega_g} |e(v_k)|^p \,\mathrm{d}x \le \int_{\Omega_h} |e(u)|^p \,\mathrm{d}x\,, \end{aligned} \tag{6.56}$$

where the last step follows from (6.51).

Step 3: Conclusion. Performing the construction above for $\varepsilon = 1/n$, $n \in \mathbb{N}$, and choosing for each $n \in \mathbb{N}$ an index $k = k(n) \in \mathbb{N}$ sufficiently large, we obtain a sequence $(\hat{w}_n, h_n)$ such that by (6.50a) and (6.55) we get

$$\hat{w}_n \to u = u\chi_{\Omega_h} \text{ in } L^0(\Omega;\mathbb{R}^d) \quad \text{and} \quad h_n \to h \text{ in } L^1(\omega)\,. \tag{6.57}$$

By (6.50b) and the definition $\Sigma = J'_u \cap \Omega^1_h$ we obtain (6.21b). By $GSBD^p$ compactness (see Theorem 3.5) applied on $\hat{w}_n = \hat{w}_n\chi_{\Omega_{h_n}} \in GSBD^p(\Omega)$ along with $\hat{w}_n \to u$ in $L^0(\Omega;\mathbb{R}^d)$ we get

$$\int_{\Omega_h} |e(u)|^p \,\mathrm{d}x \le \liminf_{n\to\infty} \int_{\Omega_{h_n}} |e(\hat{w}_n)|^p \,\mathrm{d}x.$$

This along with (6.56) and the strict convexity of the norm $\|\cdot\|_{L^p(\Omega)}$ gives

$$e(\hat{w}_n) \to e(u) \quad \text{in } L^p(\Omega;\mathbb{M}^{d\times d}_{\rm sym})\,. \tag{6.58}$$

In view of (2.1), this shows the statement apart from the fact that the configurations $\hat{w}_n$ do possibly not satisfy the boundary data. (I.e., we have now proved the version described in Remark 6.2 since $\hat{w}_n \in L^\infty(\Omega;\mathbb{R}^d)$.) It remains to adjust the boundary values.

To this end, choose a continuous extension operator from $W^{1,p}(\omega\times(-1,0);\mathbb{R}^d)$ to $W^{1,p}(\Omega;\mathbb{R}^d)$ and denote by $(w_n)_n$ the extensions of $(\hat{w}_n - u_0)|_{\omega\times(-1,0)}$ to Ω. Clearly, $w_n \to 0$ strongly in $W^{1,p}(\Omega;\mathbb{R}^d)$ since $(\hat{w}_n - u_0)|_{\omega\times(-1,0)} \to 0$ in $W^{1,p}(\omega\times(-1,0);\mathbb{R}^d)$. We now define the sequence $(u_n)_n$ by $u_n := (\hat{w}_n - w_n)\chi_{\Omega_{h_n}}$. By (6.57) we immediately deduce $u_n \to u$ in $L^0(\Omega;\mathbb{R}^d)$. Moreover, $u_n|_{\Omega_{h_n}} \in W^{1,p}(\Omega_{h_n};\mathbb{R}^d)$, $u_n = 0$ in $\Omega \setminus \Omega_{h_n}$, $u_n = u_0$ a.e. in $\omega \times (-1,0)$ and (6.58) still holds with u_n in place of $\hat{w}_n$. Due to (2.1), this shows (6.21a) and concludes the proof. □

Remark 6.7 (Volume constraint)**.** Given a volume constraint $\mathcal{L}^d(\Omega^+_h) = m$ with $0 < m < M\mathcal{H}^{d-1}(\omega)$, one can construct the sequence (u_n, h_n) in Proposition 6.1 such that also h_n satisfies the volume constraint, cf. [18, Remark 4.2]. Indeed, if $\|h\|_\infty < M$, we consider $h^*_n(x') = r_n^{-1}h_n(x')$ and $u^*_n(x', x_d) = u_n(x', r_n x_d)$, where $r_n := m^{-1}\int_\omega h_n \,\mathrm{d}x$. Then $\int_\omega h^*_n \,\mathrm{d}x = m$. Note that we can assume $\|h_n\|_\infty \le \|h\|_\infty$ (apply Proposition 6.1 with $\|h\|_\infty$ in place of M). Since $r_n \to 1$, we then find $h_n\colon \omega \to [0, M]$ for n sufficiently large, and (6.21) still holds.

If $\|h\|_{L^\infty(\omega)} = M$ instead, we need to perform a preliminary approximation: given $\delta > 0$, define $\hat{h}^{\delta,M} = h \wedge (M-\delta)$ and $h_\delta(x') = r_\delta^{-1}\hat{h}^{\delta,M}(x')$, where $r_\delta = m^{-1}\int_\omega \hat{h}^{\delta,M} \,\mathrm{d}x$. Since Ω_h is a subgraph and $m < M\mathcal{H}^{d-1}(\omega)$, it is easy to check that $r_\delta > (M-\delta)/M$ and therefore $\|h_\delta\|_\infty < M$. Moreover, by construction we have $\int_\omega h_\delta \,\mathrm{d}x = m$. We define $u_\delta(x', x_d) = u(x', r_\delta x_d)\chi_{\Omega_{h_\delta}}$. We now apply the above approximation on fixed (u_δ, h_δ), then consider a sequence $\delta \to 0$, and use a diagonal argument.

Remark 6.8 (Surface tension)**.** We remark that, similar to [9, 18, 31], we could also derive a relaxation result for more general models where the surface tension σ_S for the substrate can be different from the the surface tension σ_C of the crystal. This corresponds to surface energies of the form

$$\sigma_S\,\mathcal{H}^{d-1}(\{h=0\}) + \sigma_C\mathcal{H}^{d-1}\big(\partial\Omega_h \cap (\omega\times(0,+\infty))\big)\,.$$

In the relaxed setting, the surface energy is then given by

$$(\sigma_S \wedge \sigma_C)\,\mathcal{H}^{d-1}(\{h=0\}) + \sigma_C\Big(\mathcal{H}^{d-1}\big(\partial^*\Omega_h \cap (\omega\times(0,+\infty))\big) + 2\,\mathcal{H}^{d-1}(J'_u \cap \Omega^1_h)\Big)\,.$$

We do not prove this fact here for simplicity, but refer to [18, Subsection 2.4, Remark 4.4] for details how the proof needs to be adapted to deal with such a situation.

6.2. **Compactness and existence of minimizers.** In this short subsection we give the proof of the compactness result stated in Theorem 2.5. As discussed in Subsection 2.2, this immediately implies the existence of minimizers for problem (2.11).

Proof of Theorem 2.5. Consider $(u_n, h_n)_n$ with $\sup_n G(u_n, h_n) < +\infty$. First, by (2.8) and a standard compactness argument we find $h \in BV(\omega; [0, M])$ such that $h_n \to h$ in $L^1(\omega)$, up to a subsequence (not relabeled). Moreover, by (2.1), (2.8), and the fact that $J_{u_n} \subset \partial\Omega_{h_n} \cap \Omega$ we can apply Theorem 3.5 to obtain some $u \in GSBD^p_\infty(\Omega)$ such that $u_n \to u$ weakly in $GSBD^p_\infty$. We also observe that $u = u\chi_{\Omega_h}$ and $u = u_0$ on $\omega \times (-1, 0)$ by (3.7)(i), $u_n = u_n\chi_{\Omega_{h_n}}$, and $u_n = u_0$ on $\omega \times (-1, 0)$ for all $n \in \mathbb{N}$. It remains to show that $u \in GSBD^p(\Omega)$, i.e., $\{u = \infty\} = \emptyset$.

To this end, we take $U = \omega \times (-\frac{1}{2}, M)$ and $U' = \Omega = \omega \times (-1, M+1)$, and apply Theorem 4.2 on the sequence $\Gamma_n = \partial\Omega_{h_n} \cap \Omega$ to find that Γ_n σ^p_{sym}-converges (up to a subsequence) to a pair (Γ, G_∞). Consider $v_n = \psi u_n$, where $\psi \in C^\infty(\Omega)$ with $\psi = 1$ in a neighborhood of $\omega \times (0, M+1)$ and $\psi = 0$ on $\omega \times (-1, -\frac{1}{2})$. Clearly, v_n converges weakly in $GSBD^p_\infty(\Omega)$ to $v := \psi u$. As $J_{v_n} \subset \Gamma_n$ and $v_n = 0$ on $U' \setminus U$ for all $n \in \mathbb{N}$, it is not restrictive to assume that $\{v = \infty\} \subset G_\infty$, see Corollary 4.5. As by definition of v, we have $\{u = \infty\} = \{v = \infty\}$, we deduce $\{u = \infty\} \subset G_\infty$. It now suffices to recall $G_\infty = \emptyset$, see (6.3), to conclude $\{u = \infty\} = \emptyset$. $\square$

6.3. **Phase field approximation of $\overline{G}$.** This final subsection is devoted to the phase-field approximation of the functional $\overline{G}$. Recall the functionals introduced in (2.12).

Proof of Theorem 2.6. Fix a decreasing sequence $(\varepsilon_n)_n$ of positive numbers converging to zero. We first prove the liminf and then the limsup inequality.

Proof of (i). Let $(u_n, v_n)_n$ with $\sup_n G_{\varepsilon_n}(u_n, v_n) < +\infty$. Then, v_n is nonincreasing in x_d, and therefore

$$\widetilde{v}_n(x) := 0 \vee (v_n(x) - \delta_n x_d) \wedge 1 \qquad \text{for } x \in \Omega = \omega \times (-1, M+1)$$

is strictly decreasing on $\{0 < \widetilde{v}_n < 1\}$, where $(\delta_n)_n$ is a decreasing sequence of positive numbers converging to zero. For a suitable choice of $(\delta_n)_n$, depending on $(\varepsilon_n)_n$ and W, we obtain $\|v_n - \widetilde{v}_n\|_{L^1(\Omega)} \to 0$ and

$$G_{\varepsilon_n}(u_n, v_n) = G_{\varepsilon_n}(u_n, \widetilde{v}_n) + O(1/n)\,. \tag{6.59}$$

By using the implicit function theorem and the coarea formula for $\widetilde{v}_n$, we can see, exactly as in the proof of [18, Theorem 5.1], that for a.e. $s \in (0, 1)$ and $n \in \mathbb{N}$ the superlevel set $\{\widetilde{v}_n > s\}$ is the subgraph of a function $h^s_n \in H^1(\omega; [0, M])$. (Every h^s_n takes values in $[0, M]$ since $\widetilde{v}_n = 0$ in $\omega \times (M, M+1)$.) By the coarea formula for $\widetilde{v}_n$, $\partial^*\{\widetilde{v}_n > s\} \cap \Omega = \partial^*\Omega_{h^s_n} \cap \Omega$, and Young's inequality we obtain

$$\int_0^1 \sqrt{2W(s)}\,\mathcal{H}^{d-1}(\partial^*\Omega_{h^s_n} \cap \Omega)\,\mathrm{d}s \le \int_\Omega \sqrt{2W(\widetilde{v}_n)}\,|\nabla \widetilde{v}_n|\,\mathrm{d}x \le \int_\Omega \Big(\frac{\varepsilon_n}{2}|\nabla \widetilde{v}_n|^2 + \frac{1}{\varepsilon_n}W(\widetilde{v}_n)\Big)\,\mathrm{d}x\,.$$

Then, by Fatou's lemma we get

$$\int_0^1 \sqrt{2W(s)}\Big(\liminf_{n\to\infty}\int_\omega \sqrt{1 + |\nabla h^s_n(x')|^2}\mathrm{d}x'\Big)\mathrm{d}s \le \liminf_{n\to\infty}\int_\Omega \Big(\frac{\varepsilon_n}{2}|\nabla \widetilde{v}_n|^2 + \frac{1}{\varepsilon_n}W(\widetilde{v}_n)\Big)\,\mathrm{d}x < +\infty \tag{6.60}$$

and thus $\liminf_{n\to\infty}\int_\omega \sqrt{1 + |\nabla h^s_n(x')|^2}\,\mathrm{d}x'$ is finite for a.e. $s \in (0, 1)$. By a diagonal argument, we can find a subsequence (still denoted by $(\varepsilon_n)_n$) and $(s_k)_k \subset (0, 1)$ with $\lim_{k\to\infty} s_k = 0$ such that for every $k \in \mathbb{N}$ there holds

$$\lim_{n\to\infty}\int_\omega \sqrt{1 + |\nabla h^{s_k}_n(x')|^2}\,\mathrm{d}x' = \liminf_{n\to\infty}\int_\omega \sqrt{1 + |\nabla h^{s_k}_n(x')|^2}\,\mathrm{d}x' < +\infty\,. \tag{6.61}$$

Up to a further (not relabeled) subsequence, we may thus assume that $h_n^{s_k}$ converges in $L^1(\omega)$ to some function h^{s_k} for every k. Since $\sup_n G_{\varepsilon_n}(u_n,\widetilde{v}_n)<+\infty$ and thus $W(\widetilde{v}_n)\to 0$ a.e. in Ω, we obtain $\widetilde{v}_n\to 0$ for a.e. x with $x_d>h^{s_k}(x')$ and $\widetilde{v}_n\to 1$ for a.e. x with $x_d<h^{s_k}(x')$. (Recall $W(t)=0\Leftrightarrow t\in\{0,1\}$.) This shows that the functions h^{s_k} are independent of k, and will be denoted simply by $h\in BV(\omega;[0,M])$.

Let us denote by $u_n^k\in GSBD^p(\Omega)$ the function given by

$$u_n^k(x)=\begin{cases}u_n(x) & \text{if } x_d<h_n^{s_k}(x')\,,\\ 0 & \text{else}\,.\end{cases} \tag{6.62}$$

Then $(u_n^k)_n$ satisfies the hypothesis of Theorem 3.5 for every $k\in\mathbb{N}$. Indeed, $J_{u_n^k}\subset\partial^*\Omega_{h_n^{s_k}}$ and $\mathcal{H}^{d-1}(\partial^*\Omega_{h_n^{s_k}})$ is uniformly bounded in n by (6.61). Moreover, $(e(u_n^k))_n$ is uniformly bounded in $L^p(\Omega;\mathbb{M}^{d\times d}_{\rm sym})$ by (2.1) and the fact that

$$G_{\varepsilon_n}(u_n,\widetilde{v}_n)\geq(\eta_{\varepsilon_n}+s_k^2)\int_\Omega f(e(u_n^k))\,\mathrm{d}x\,.$$

Therefore, Theorem 3.5 implies that, up to a subsequence, u_n^k converges weakly in $GSBD^p_\infty(\Omega)$ to a function u^k. Furthermore, we infer, arguing exactly as in the proof of Theorem 2.5 above, that actually $u^k\in GSBD^p(\Omega)$, i.e., the exceptional set $\{u^k=\infty\}$ is empty. By (3.7)(i) this yields $u_n^k\to u^k$ in $L^0(\Omega;\mathbb{R}^d)$. By a diagonal argument we get (up to a further subsequence) that $u_n^k\to u^k$ pointwise a.e. as $n\to\infty$ for all $k\in\mathbb{N}$.

Recalling now the definition of u_n^k in (6.62) and the fact that $\lim_{n\to\infty}\|h_n^{s_k}-h\|_{L^1(\omega)}=0$ for all $k\in\mathbb{N}$, we deduce that the functions u^k are independent of k. This function will simply be denoted by $u\in GSBD^p(\Omega)$ in the following. Note that $u=u\chi_{\Omega_h}$ and that $u=u_0$ on $\omega\times(-1,0)$ since $u_n=u_0$ on $\omega\times(-1,0)$ for all $n\in\mathbb{N}$.

For the proof of (2.13), we can now follow exactly the lines of the lower bound in [18, Theorem 5.1]. We sketch the main arguments for convenience of the reader. We first observe that

$$\int_\Omega\widetilde{v}_n\,f(e(u_n))\,\mathrm{d}x=\int_\Omega\Big(2\int_0^{\widetilde{v}_n(x)}s\,\mathrm{d}s\Big)\,f(e(u_n)(x))\,\mathrm{d}x\geq\int_0^1 2s\Big(\int_{\{\widetilde{v}_n>s\}}f(e(u_n))\,\mathrm{d}x\Big)\,\mathrm{d}s\,.$$

This along with (6.60) and Fatou's lemma yields

$$\int_0^1\liminf_{n\to\infty}\Big(2s\int_{\{\widetilde{v}_n>s\}}f(e(u_n))\,\mathrm{d}x+c_W\sqrt{2W(s)}\int_\omega\sqrt{1+|\nabla h_n^s|^2}\,\mathrm{d}x'\Big)\,\mathrm{d}s\leq\liminf_{n\to\infty}G_{\varepsilon_n}(u_n,\widetilde{v}_n)\,. \tag{6.63}$$

Thus, the integrand

$$I_n^s:=2s\int_{\{\widetilde{v}_n>s\}}f(e(u_n))\,\mathrm{d}x+c_W\sqrt{2W(s)}\int_\omega\sqrt{1+|\nabla h_n^s|^2}\,\mathrm{d}x'$$

is finite for a.e. $s\in(0,1)$. We then take s such that $h_n^s\in H^1(\omega)$ for all n, and consider a subsequence $(n_m)_m$ such that $\lim_{m\to\infty}I_{n_m}^s=\liminf_{n\to\infty}I_n^s$. Exactly as in (6.62), we let $u_{n_m}^s$ be the function given by u_{n_m} if $x_d<h_{n_m}^s(x')$ and by zero otherwise. Repeating the compactness argument below (6.62), we get $u_{n_m}^s\to u$ a.e. in Ω and $h_{n_m}^s\to h$ in $L^1(\omega)$ as $m\to\infty$. We observe that this can be done for a.e. $s\in(0,1)$, for a subsequence depending on s.

By $\int_{\{\widetilde{v}_{n_m}>s\}}f(e(u_{n_m}))\,\mathrm{d}x=\int_\Omega f(e(u_{n_m}^s))\,\mathrm{d}x$ and the (lower inequality in the) relaxation result Theorem 2.4 (up to different constants in front of the elastic energy and surface energy) we obtain

$$2s\int_{\Omega_h^+} f(e(u))\,\mathrm{d}x + c_W\sqrt{2W(s)}\,\big(\mathcal{H}^{d-1}(\partial^*\Omega_h\cap\Omega)+2\,\mathcal{H}^{d-1}(J_u'\cap\Omega_h^1)\big) \le \lim_{n_m\to\infty} I^s_{n_m} = \liminf_{n\to\infty} I^s_n$$

for a.e. $s\in(0,1)$. We obtain (2.13) by integrating the above inequality and by using (6.59) and (6.63). Indeed, the integral on the left-hand side gives exactly $\overline{G}(u,h)$ as $c_W = (\int_0^1\sqrt{2W(s)}\,\mathrm{d}s)^{-1}$.

Proof of (ii). Let (u,h) with $\overline{G}(u,h) < +\infty$. By the construction in the upper inequality for Theorem 2.4, see Proposition 6.1 and Remark 6.2, we find $h_n\in C^1(\omega;[0,M])$ with $h_n\to h$ in $L^1(\omega)$ and $u_n\in L^\infty(\Omega;\mathbb{R}^d)$ with $u_n|_{\Omega_{h_n}}\in W^{1,p}(\Omega_{h_n};\mathbb{R}^d)$ and $u_n\to u$ a.e. in Ω such that

$$\overline{G}(u,h) = \lim_{n\to\infty} H(u_n,h_n) \quad \text{for } H(u_n,h_n) := \int_{\Omega^+_{h_n}} f(e(u_n))\,\mathrm{d}x + \mathcal{H}^{d-1}(\partial\Omega_{h_n}\cap\Omega) \tag{6.64}$$

as well as

$$(u_n-u_0)|_{\omega\times(-1,0)}\to 0 \quad \text{in } W^{1,p}(\omega\times(-1,0);\mathbb{R}^d)\,. \tag{6.65}$$

For each (u_n,h_n), we can use the construction in [18] to find sequences $(u_n^k)_k\subset W^{1,p}(\Omega;\mathbb{R}^d)$ and $(v_n^k)_k\subset H^1(\Omega;[0,1])$ with $u_n^k = u_n$ on $\omega\times(-1,0)$, $u_n^k\to u_n$ in $L^1(\Omega;\mathbb{R}^d)$, and $v_n^k\to\chi_{\Omega_{h_n}}$ in $L^1(\Omega)$ such that (cf. (6.64))

$$\limsup_{k\to\infty}\int_\Omega\bigg(\big((v_n^k)^2+\eta_{\varepsilon_k}\big)f(e(u_n^k)) + c_W\Big(\frac{W(v_n^k)}{\varepsilon_k}+\frac{\varepsilon_k}{2}|\nabla v_n^k|^2\Big)\bigg)\,\mathrm{d}x \le H(u_n,h_n)\,. \tag{6.66}$$

In particular, we refer to [18, Equation (28)] and mention that the functions $(v_n^k)_k$ can be constructed such that $v_n^k=1$ on $\omega\times(-1,0)$ and $v_n^k=0$ in $\omega\times(M,M+1)$. We also point out that for this construction the assumption $\eta_\varepsilon\varepsilon^{1-p}\to 0$ as $\varepsilon\to0$ is needed.

By (6.64), (6.66), and a standard diagonal extraction argument we find sequences $(\hat{u}^k)_k\subset(u_n^k)_{n,k}$ and $(v^k)_k\subset(v_n^k)_{n,k}$ such that $\hat{u}^k\to u$ a.e. in Ω, $v^k\to\chi_{\Omega_h}$ in $L^1(\Omega)$, and

$$\limsup_{k\to\infty}\int_\Omega\bigg(\big((v^k)^2+\eta_{\varepsilon_k}\big)f(e(\hat{u}^k)) + c_W\Big(\frac{W(v^k)}{\varepsilon_k}+\frac{\varepsilon_k}{2}|\nabla v^k|^2\Big)\bigg)\,\mathrm{d}x \le \overline{G}(u,h)\,. \tag{6.67}$$

By using (6.65) and the fact that $u_n^k = u_n$ for all $k,n\in\mathbb{N}$, we can modify $(\hat{u}^k)_k$ as described at the end of the proof of Proposition 6.1 (see below (6.58)): we find a sequence $(u^k)_k$ which satisfies $u^k=u_0$ on $\omega\times(-1,0)$, converges to u a.e. in Ω, and (6.67) still holds, i.e., $\limsup_{k\to\infty} G_{\varepsilon_k}(u^k,v^k)\le\overline{G}(u,h)$. This concludes the proof. □

A. Auxiliary results

In this appendix, we prove two technical approximation results employed in Sections 5 and 6, based on tools from [15].

Proof of Lemma 5.6. Let (v,H) be given as in the statement of the lemma. Clearly, it suffices to prove the following statement: for every $\eta>0$, there exists $(v^\eta,H^\eta)\in L^p(\Omega;\mathbb{R}^d)\times\mathfrak{M}(\Omega)$ with the regularity and the properties required in the statement of the lemma (in particular, $v^\eta = u_0$ in a neighborhood $V^\eta\subset\Omega$ of $\partial_D\Omega$), such that, for a universal constant C, one has $d(v^\eta,v)\le C\eta$ (cf. (3.13) for d), $\mathcal{L}^d(H\triangle H^\eta)\le C\eta$, and

$$\overline{F}'_{\mathrm{Dir}}(v^\eta,H^\eta)\le\overline{F}'_{\mathrm{Dir}}(v,H)+C\eta\,.$$

We start by recalling the main steps of the construction in [15, Theorem 5.5] and we refer to [15] for details (see also [16, Section 4, first part]). Based on this, we then explain how to construct (v^η,H^η) simultaneously, highlighting particularly the steps needed for constructing H^η.

Let $\varepsilon > 0$, to be chosen small with respect to η. By using the assumptions on $\partial\Omega$ given before (2.4), a preliminary step is to find cubes $(Q_j)_{j=1}^J$ with pairwise disjoint closures and hypersurfaces $(\Gamma_j)_{j=1}^J$ with the following properties: each Q_j is centered at $x_j \in \partial_N\Omega$ with sidelength ϱ_j, $\mathrm{dist}(Q_j, \partial_D\Omega) > d_\varepsilon > 0$ with $\lim_{\varepsilon\to 0} d_\varepsilon = 0$, and

$$\mathcal{H}^{d-1}(\partial_N\Omega \setminus \widehat{Q}) + \mathcal{L}^d(\widehat{Q}) \le \varepsilon, \qquad \text{for } \widehat{Q} := \bigcup\nolimits_{j=1}^J \overline{Q}_j\,. \tag{A.1}$$

Moreover, each Γ_j is a C^1-hypersurface with $x_j \in \Gamma_j \subset \overline{Q}_j$,

$$\mathcal{H}^{d-1}\big((\partial_N\Omega\triangle\Gamma_j) \cap \overline{Q_j}\big) \le \varepsilon(2\varrho_j)^{d-1} \le \frac{\varepsilon}{1-\varepsilon}\mathcal{H}^{d-1}(\partial_N\Omega \cap \overline{Q_j})\,,$$

and Γ_j is a C^1-graph with respect to $\nu_{\partial\Omega}(x_j)$ with Lipschitz constant less than $\varepsilon/2$. (We can say that $\partial_N\Omega \cap Q_j$ is "almost" the intersection of Q_j with the hyperplane passing through x_j with normal $\nu_{\partial\Omega}(x_j)$.) We can also guarantee that

$$\mathcal{H}^{d-1}\big((\partial^* H \cup J_u) \cap \Omega \cap \widehat{Q}\big) \le \varepsilon, \qquad \mathcal{H}^{d-1}\big((\partial^* H \cup J_u) \cap \partial Q_j\big) = 0 \tag{A.2}$$

for all $j = 1, \dots, J$. To each Q_j, we associate the following rectangles:

$$R_j := \Big\{x_j + \sum\nolimits_{i=1}^{d-1} y_i\, b_{j,i} + y_d\,\nu_j \colon y_i \in (-\varrho_j, \varrho_j),\, y_d \in (-3\varepsilon\varrho_j - t, -\varepsilon\varrho_j)\Big\}\,,$$
$$R'_j := \Big\{x_j + \sum\nolimits_{i=1}^{d-1} y_i\, b_{j,i} + y_d\,\nu_j \colon y_i \in (-\varrho_j, \varrho_j),\, y_d \in (-\varepsilon\varrho_j, \varepsilon\varrho_j + t)\Big\}\,,$$

and $\widehat{R}_j := R_j \cup R'_j$, where $\nu_j = -\nu_{\partial\Omega}(x_j)$ denotes the generalized outer normal, $(b_{j,i})_{i=1}^{d-1}$ is an orthonormal basis of $(\nu_j)^\perp$, and $t > 0$ is small with respect to η. We remark that $\Gamma_j \subset R'_j$ and that R_j is a small strip adjacent to R'_j, which is included in $\Omega \cap Q_j$. (We use here the notation $_j$ in place of $_{h,N}$ adopted in [15, Theorem 5.5].)

After this preliminary part, the approximating function u^η was constructed in [15, Theorem 5.5] starting from a given function u through the following three steps:

(i) definition of an extension $\widetilde{u} \in GSBD^p(\Omega + B_t(0))$ which is obtained by a reflection argument *à la* Nitsche [50] inside $\widehat{R}_j$, equal to u in $\Omega\setminus\bigcup_j \widehat{R}_j$, and equal to u_0 elsewhere. This can be done such that, for t and ε small, there holds (see below [15, (5.13)])

$$\int_{(\Omega+B_t(0))\setminus\Omega} |e(u_0)|^p \,\mathrm{d}x + \int_{\widehat{R}} |e(\widetilde{u})|^p \,\mathrm{d}x + \int_R |e(u)|^p \,\mathrm{d}x + \mathcal{H}^{d-1}\big(J_{\widetilde{u}} \cap \widehat{R}\big) \le \eta\,, \tag{A.3}$$

where $R := \bigcup_{j=1}^J R_j$ and $\widehat{R} := \bigcup_{j=1}^J \widehat{R}_j \cap (\Omega + B_t(0))$.

(ii) application of Theorem 3.4 on the function $\widetilde{u}^\delta := \widetilde{u} \circ (O_{\delta,x_0})^{-1} + u_0 - u_0 \circ (O_{\delta,x_0})^{-1}$ (for some δ sufficiently small) to get approximating functions $\widetilde{u}^\delta_n$ with the required regularity which are equal to $u_0 * \psi_n$ in a neighborhood of $\partial_D\Omega$ in Ω, where ψ_n is a suitable mollifier. Here, assumption (2.4) is crucial.

(iii) correcting the boundary values by defining u^η as $u^\eta := \widetilde{u}^\delta_n + u_0 - u_0 * \psi_n$, for δ and $1/n$ small enough.

After having recalled the main steps of the construction in [15, Theorem 5.5], let us now construct v^η and H^η at the same time, following the lines of the steps (i), (ii), and (iii) above. The main novelty is the analog of step (i) for the approximating sets, while the approximating functions are constructed in a very similar way. For this reason, we do not recall more details from [15, Theorem 5.5].

Step (i). Step (i) for v^η is the same done before for u^η, starting from v in place of u. Hereby, we get a function $\widetilde{v} \in GSBD^p(\Omega + B_t(0))$.

For the construction of H^η, we introduce a set $\widetilde{H} \subset \Omega + B_t(0)$ as follows: in R'_j, we define a set H'_j by a simple reflection of the set $H \cap R_j$ with respect to the common hyperface between R_j and R'_j. Then, we let $\widetilde{H} := H \cup \bigcup_{j=1}^J (H'_j \cap (\Omega + B_t(0)))$. Since H has finite perimeter, also $\widetilde{H}$ has finite perimeter. By (A.2) we get $\mathcal{H}^{d-1}(\partial^* \widetilde{H} \cap \widehat{R}) \le \eta/3$ for ε small, where as before $\widehat{R} := \bigcup_{j=1}^J \widehat{R}_j \cap (\Omega + B_t(0))$. We choose δ, ε, and t so small that

$$\mathcal{H}^{d-1}\Big(O_{\delta,x_0}\Big(\bigcup\nolimits_{j=1}^J \partial R'_j \setminus \partial R_j\Big) \cap \Omega\Big) \le \frac{\eta}{3}\,. \tag{A.4}$$

We let $H^\eta := O_{\delta,x_0}(\widetilde{H})$. Then, we get $\mathcal{L}^d(H^\eta \triangle H) \le \eta$ for ε, t, and δ small enough. By (A.1), (A.4), and $\mathcal{H}^{d-1}(\partial^* \widetilde{H} \cap \widehat{R}) \le \eta/3$ we also have (again take suitable ε, δ)

$$\int_{\partial^* H^\eta} \varphi(\nu_{H^\eta})\,\mathrm{d}\mathcal{H}^{d-1} \le \int_{\partial^* H \cap (\Omega \cup \partial_D \Omega)} \varphi(\nu_H)\,\mathrm{d}\mathcal{H}^{d-1} + \eta\,. \tag{A.5}$$

Moreover, in view of (2.4) and $\mathrm{dist}(Q_j, \partial_D \Omega) > d_\varepsilon > 0$ for all j, H^η does not intersect a suitable neighborhood of $\partial_D \Omega$. Define $\widetilde{v}^\delta := \widetilde{v} \circ (O_{\delta,x_0})^{-1} + u_0 - u_0 \circ (O_{\delta,x_0})^{-1}$ and observe that the function $\widetilde{v}^\delta \chi_{(H^\eta)^0}$ coincides with u_0 in a suitable neighborhood of $\partial_D \Omega$. By (A.5), by the properties recalled for $\widetilde{u}$, see (A.3), and the fact that $v = v\chi_{H^0}$, it is elementary to check that

$$\overline{F}'_{\mathrm{Dir}}(\widetilde{v}^\delta \chi_{(H^\eta)^0}, H^\eta) \le \overline{F}'_{\mathrm{Dir}}(v\chi_{H^0}, H) + C\eta = \overline{F}'_{\mathrm{Dir}}(v, H) + C\eta\,. \tag{A.6}$$

Notice that here it is important to take the same δ both for $\widetilde{v}^\delta$ and H^η, that is to "dilate" the function and the set at the same time.

Step 2. We apply Theorem 3.4 to $\widetilde{v}^\delta \chi_{(H^\eta)^0}$, to get approximating functions $\widetilde{v}^\delta_n$ with the required regularity. For n sufficiently large, we obtain $d(\widetilde{v}^\delta_n \chi_{(H^\eta)^0}, \widetilde{v}^\delta \chi_{(H^\eta)^0}) \le \eta$ and

$$|\overline{F}'_{\mathrm{Dir}}(\widetilde{v}^\delta_n \chi_{(H^\eta)^0}, H^\eta) - \overline{F}'_{\mathrm{Dir}}(\widetilde{v}^\delta \chi_{(H^\eta)^0}, H^\eta)| \le \eta\,.$$

Step 3. Similar to item (ii) above, we obtain $\widetilde{v}^\delta_n = u_0 * \psi_n$ in a neighborhood of $\partial_D \Omega$. Therefore, it is enough to define v^η as $v^\eta := \widetilde{v}^\delta_n + u_0 - u_0 * \psi_n$. Then by (A.6) and Step 2 we obtain $d(v^\eta, v) \le C\eta$ and $\overline{F}'_{\mathrm{Dir}}(v^\eta, H^\eta) \le \overline{F}'_{\mathrm{Dir}}(v, H) + C\eta$ for n sufficiently large. □

We now proceed with the proof of Lemma 6.5 which relies strongly on [15, Theorem 3.1]. Another main ingredient is the following Korn-Poincaré inequality in $GSBD^p$, see [13, Proposition 3].

Proposition A.1. *Let $Q = (-r, r)^d$, $Q' = (-r/2, r/2)^d$, $u \in GSBD^p(Q)$, $p \in [1, \infty)$. Then there exist a Borel set $\omega \subset Q'$ and an affine function $a\colon \mathbb{R}^d \to \mathbb{R}^d$ with $e(a) = 0$ such that $\mathcal{L}^d(\omega) \le cr\mathcal{H}^{d-1}(J_u)$ and*

$$\int_{Q' \setminus \omega} (|u - a|^p)^{1^*}\,\mathrm{d}x \le cr^{(p-1)1^*} \left(\int_Q |e(u)|^p\,\mathrm{d}x \right)^{1^*}. \tag{A.7}$$

If additionally $p > 1$, then there exists $q > 0$ (depending on p and d) such that, for a given mollifier $\varphi_r \in C_c^\infty(B_{r/4})$, $\varphi_r(x) = r^{-d}\varphi_1(x/r)$, the function $w = u\chi_{Q' \setminus \omega} + a\chi_\omega$ obeys

$$\int_{Q''} |e(w * \varphi_r) - e(u) * \varphi_r|^p\,\mathrm{d}x \le c \left(\frac{\mathcal{H}^{d-1}(J_u)}{r^{d-1}} \right)^q \int_Q |e(u)|^p\,\mathrm{d}x\,, \tag{A.8}$$

where $Q'' = (-r/4, r/4)^d$. The constant in (A.7) *depends only on p and d, the one in* (A.8) *also on φ_1.*

Proof of Lemma 6.5. We recall the definition of the hypercubes

$$q_z^k := z + (-k^{-1}, k^{-1})^d\,, \qquad \tilde{q}_z^k := z + (-2k^{-1}, 2k^{-1})^d\,, \qquad Q_z^k := z + (-5k^{-1}, 5k^{-1})^d\,,$$

where in addition to the notation in (6.23), we have also defined the hypercubes $\tilde{q}_z^k$. In contrast to [15, Theorem 3.1], the cubes Q_z^k have sidelength $10k^{-1}$ instead of $8k^{-1}$. This, however, does not affect the estimates. We point out that at some points in [15, Theorem 3.1] cubes of the form $z + (-8k^{-1}, 8k^{-1})^d$ are used. By a slight alternation of the argument, however, it suffices to take cubes Q_z^k. In particular it is enough to show the inequality [15, (3.19)] for a cube Q_j (of sidelength $10k^{-1}$) in place of $\widetilde{Q}_j$ (of sidelength $16k^{-1}$), which may be done by employing rigidity properties of affine functions. Let us fix a smooth radial function φ with compact support on the unit ball $B_1(0) \subset \mathbb{R}^d$, and define $\varphi_k(x) := k^d \varphi(kx)$. We choose $\theta < (16c)^{-1}$, where c is the constant in Proposition A.1 (cf. also [15, Lemma 2.12]). Recall (6.24) and set

$$\mathcal{N}_k' := \{z \in (2k^{-1})\mathbb{Z}^d \colon q_z^k \cap (U)^k \setminus V \neq \emptyset\}\,.$$

We apply Proposition A.1 for $r = 4k^{-1}$, for any $z \in \mathcal{N}_k'$ by taking v as the reference function and $z + (-4k^{-1}, 4k^{-1})^d$ as Q therein. (In the following, we may then use the bigger cube Q_z^k in the estimates from above.) Then, there exist $\omega_z \subset \tilde{q}_z^k$ and $a_z \colon \mathbb{R}^d \to \mathbb{R}^d$ affine with $e(a_z) = 0$ such that by (6.25), (A.7), and Hölder's inequality there holds

$$\mathcal{L}^d(\omega_z) \le 4ck^{-1}\mathcal{H}^{d-1}(J_v \cap Q_z^k) \le 4c\theta k^{-d}\,, \tag{A.9a}$$

$$\|v - a_z\|_{L^p(\tilde{q}_z^k \setminus \omega_z)} \le 4ck^{-1}\|e(v)\|_{L^p(Q_z^k)}\,. \tag{A.9b}$$

Moreover, by (6.25) and (A.8) there holds

$$\int_{q_z^k} |e(\hat{v}_z * \varphi_k) - e(v) * \varphi_k|^p \,\mathrm{d}x \le c\left(\mathcal{H}^{d-1}(J_v \cap Q_z^k)\, k^{d-1}\right)^q \int_{Q_z^k} |e(v)|^p \,\mathrm{d}x \le c\theta^q \int_{Q_z^k} |e(v)|^p \,\mathrm{d}x$$

for $\hat{v}_z := v\chi_{\tilde{q}_z^k \setminus \omega_z} + a_z \chi_{\omega_z}$ and a suitable $q > 0$ depending on p and d. Let us set

$$\omega^k := \bigcup\nolimits_{z \in \mathcal{N}_k'} \omega_z\,.$$

We order (arbitrarily) the nodes $z \in \mathcal{N}_k'$, and denote the set by $(z_j)_{j \in J}$. We define

$$\widetilde{w}_k := \begin{cases} v & \text{in } \left(\bigcup_{z \in \mathcal{N}_k'} Q_z^k\right) \setminus \omega^k\,, \\ a_{z_j} & \text{in } \omega_{z_j} \setminus \bigcup_{i<j} \omega_{z_i}\,, \end{cases} \tag{A.10}$$

and

$$w_k := \widetilde{w}_k * \varphi_k \quad \text{in } (U)^k \setminus V\,. \tag{A.11}$$

We have that w_k is smooth since $(U)^k \setminus V + \operatorname{supp} \varphi_k \subset \bigcup_{z \in \mathcal{N}_k'} \tilde{q}_z^k \subset U$ (recall (6.24)) and $v|_{\tilde{q}_z^k \setminus \omega^k} \in L^p(\tilde{q}_z^k \setminus \omega^k; \mathbb{R}^d)$ for any $z \in \mathcal{N}_k'$, by (A.9b).

We define the sets $G_1^k := \{z \in \mathcal{N}_k' \colon \mathcal{H}^{d-1}(J_v \cap Q_z^k) \le k^{1/2-d}\}$ and $G_2^k := \mathcal{N}_k' \setminus G_2^k$. By $\widetilde{G}_1^k$ and $\widetilde{G}_2^k$, respectively, we denote their "neighbors", see [15, (3.11)] for the exact definition. We let

$$\widetilde{\Omega}_{g,2}^k := \bigcup\nolimits_{z \in \widetilde{G}_2^k} Q_z^k\,.$$

There holds (cf. [15, (3.8), (3.9), (3.12)])

$$\lim_{k \to \infty} \left(\mathcal{L}^d(\omega^k) + \mathcal{L}^d(\widetilde{\Omega}_{g,2}^k)\right) = 0\,. \tag{A.12}$$

At this point, we notice that the set E_k in [15, (3.8)] reduces to ω^k since in our situation all nodes are "good" (see (6.25) and [15, (3.2)]) and therefore $\widetilde{\Omega}_b^k$ therein is empty.

The proof of (3.1a), (3.1d), (3.1b) in [15, Theorem 3.1] may be followed exactly, with the modifications described just above and the suitable slight change of notation. More precisely, by [15, equation below (3.22)] we obtain

$$\|w_k - v\|_{L^p(((U)^k\setminus V)\setminus\omega^k)} \le Ck^{-1}\|e(v)\|_{L^p(U)}\,, \tag{A.13}$$

for a constant $C>0$ depending only on d and p, and [15, equation before (3.26)] gives

$$\int_{\omega^k}\psi(|w_k - v|)\,\mathrm{d}x \le C\Big(\int_{\omega^k\cup\widetilde{\Omega}^k_{g,2}}\big(1+\psi(|v|)\big)\,\mathrm{d}x + k^{-1/2}\int_U\big(1+\psi(|v|)\big)\,\mathrm{d}x + k^{-p}\int_U |e(v)|^p\,\mathrm{d}x\Big)\,, \tag{A.14}$$

where $\psi(t) = t\wedge 1$. Combining (A.13)-(A.14), using (A.12), and recalling that ψ is sublinear, we obtain (6.26a). Note that the sequence $R_k \to 0$ can be chosen independently of $v\in\mathcal{F}$ since $\psi(|v|) + |e(v)|^p$ is equiintegrable for $v\in\mathcal{F}$.

Moreover, recalling (A.10)-(A.11), we sum [15, (3.34)] for $z = z_j \in \widetilde{G}^k_2$ and [15, (3.35)] for $z = z_j \in \widetilde{G}^k_1$ to obtain

$$\int_{(U)^k\setminus V}|e(w_k)|^p\,\mathrm{d}x \le \int_U |e(v)|^p\,\mathrm{d}x + Ck^{-q'/2}\int_U|e(v)|^p\,\mathrm{d}x + C\int_{\widetilde{\Omega}^k_{g,2}}|e(v)|^p\,\mathrm{d}x$$

for some $q'>0$. This along with (A.12) and the equiintegrability of $|e(v)|^p$ shows (6.26b). □

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