REGULARITY OF MINIMAL SURFACES NEAR QUADRATIC CONES

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ABSTRACT. Hardt-Simon [7] proved that every area-minimizing hypercone $\mathbf{C}$ having only an isolated singularity fits into a foliation of $\mathbb{R}^{n+1}$ by smooth, area-minimizing hypersurfaces asymptotic to $\mathbf{C}$. In this paper we prove that if a stationary varifold M in the unit ball $B_1 \subset \mathbb{R}^{n+1}$ lies sufficiently close to a minimizing quadratic cone (for example, the Simons' cone $\mathbf{C}^{3,3}$), then $\mathrm{spt} M \cap B_{1/2}$ is a $C^{1,\alpha}$ perturbation of either the cone itself, or some leaf of its associated foliation. In particular, we show that singularities modeled on these cones determine the local structure not only of M, but of any nearby minimal surface. Our result also implies the Bernstein-type result [12], which characterizes area-minimizing hypersurfaces asymptotic to a quadratic cone as either the cone itself, or some leaf of the foliation.

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1. Introduction

In this paper we are interested in the following question:

Question 1.1. *Suppose M_i is a sequence of minimal n-dimensional surfaces, converging to some $\tilde{M}$ with multiplicity one in the unit ball in $\mathbb{R}^{n+1}$. How does the singular structure of $\tilde{M}$ determine the singular or regular structure of the M_i in $B_{1/2}$?*

This question and its variants underly significant amount of research in the field of minimal surfaces, and other variational problems. Question 1.1 arises when attempting to study the singularity structure or compactness properties of some class of surfaces. For example, often the M_i form some kind of "blow-up" sequence for a singularity, and the resulting $\tilde{M}$ is a singularity model. In the very important special case when M_i are dilations around a fixed point of a given minimal M, i.e. when

$$M_i = \lambda_i(M - x), \quad \lambda_i \to \infty,$$

then $\tilde{M}$ is dilation invariant, and any such $\tilde{M}$ arising in this fashion is called a tangent cone of M at x.

Question 1.1 is entirely answered when $\tilde{M}$ is smooth: in this case Allard's theorem [1] implies that for i large, the $M_i \cap B_{1/2}$ must be smooth also (in fact must be $C^{1,\alpha}$ perturbations of $\tilde{M}$).

For singular $\tilde{M}$, Question 1.1 has been answered under some structral assumptions. When $\tilde{M}$ has at most a "strongly isolated" singularity[1], *and* each M_i has at least one singularity of the same type, then profound work of [9] shows that the $M_i \cap B_{1/2}$ must be C^1 perturbations of $\tilde{M}$ for large i. The conditions on M_i are naturally satisfied when $\tilde{M}$ is the tangent cone of some minimal surface, and M_i are the dilations around a fixed point.

In certain particular cases one can use the topology of $\tilde{M}$ to deduce singular structure on M_i. For example, when $\tilde{M}$ is a union of half-planes, then [10] showed that $M_i \cap B_{1/2}$ are a $C^{1,\alpha}$ perturbation of the $\tilde{M}$. Similar results hold if $\tilde{M}$ has tetrahedral singularities, and the M_i have an associated "orientation structure" ([6]); or when $\tilde{M}$ is a union of two planes, and the M_i are 2-valued graphs ([3]).

Notice that in these theorems, either by assumption or by the nature of $\tilde{M}$, all the M_i have the same regular or singular structure as $\tilde{M}$.

Question 1.1 becomes more subtle when one does not assume anything about the singular nature of the M_i. In this case it is possible

[1]Strongly isolated here means that some tangent cone of M at the singularity is a multiplicity-one cone with an isolated singularity at 0.

for a sequence of smooth minimal (even area-minimizing!) surfaces to limit to a singular one. For example, [4] have constructed a foliation by smooth minimal surfaces of the complement of the Simons's cone

$$\mathbf{C}^{3,3} = \{(x, y) \in \mathbb{R}^8 : |x| = |y|\}.$$

More generally, [7] showed the same holds for any area-minimizing cone $\mathbf{C}^n \subset \mathbb{R}^{n+1}$, having an isolated singularity. In these circumstances the M_i need not have the same singularity structure as $\tilde{M}$.

More generally, it is conceivable that singularities of one type can limit to a singularity of a different type. Or, even worse, that multiple singularities of various types or dimensions could collapse into a singularity of some other type. One can build toy examples of these behaviors using geodesic nets, but to our knowledge no examples exist for surfaces of dimension > 1.

In this paper we answer Question 1.1 in the case when $\tilde{M}$ has singularities modeled on area-minimizing quadratic cones, i.e. so called minimizing (p, q)-singularities (see Definition 1.1.1). One of the main results of this paper is that any singularity in *any minimal surface* that is sufficiently nearby a minimizing (p, q)-singularity, must be of the same type, and in particular none of the pathologies of the previous paragraph can occur for these singularities.

Definition 1.1.1. Following [12], define the $\mathbf{C}^{p,q}$ quadratic minimal cone as the hypersurface

$$\mathbf{C}^{p,q} = \{(x, y) \in \mathbb{R}^{p+1} \times \mathbb{R}^{q+1} : q|x|^2 = p|y|^2\} \subset \mathbb{R}^{n+1},$$

where $p, q > 0$ are integers, and $p + q = n - 1$. Some people refer to the cones $\mathbf{C}^{p,q}$ as generalized Simons' cones.

Given a surface or varifold M, we say that $x \in \mathrm{spt}M$ is a (p, q)-singularity if some tangent cone of M at x is (up to rotation) equal to $\mathbf{C}^{p,q}$ with multiplicity one. We say x is a minimizing (p, q)-singularity if the associated cone $\mathbf{C}^{p,q}$ is area-minimizing.

Some remarks are in order.

Remark 1.2. An easy computation shows each $\mathbf{C}^{p,q}$ is minimal. However, there are no non-flat area-minimizing hypercones in $\mathbb{R}^{n+1}$, for $n < 7$, and of course by dimension-reducing there are no singular area-minimizing hypersurfaces in these dimensions either. When $n = 7$, the cones $\mathbf{C}^{3,3}$ and $\mathbf{C}^{2,4}$ are area-minimizing, and in fact up to rigid motions these are the only known area-minimizing hypercones in $\mathbb{R}^8$. When $n > 7$, then every $\mathbf{C}^{p,q}$ cone is area-minimizing. See, e.g. [8] and the references therein. Thus our results are motivated by, and most relavant to, the regularity theory for area-minimizing hypersurfaces.

Remark 1.3. By work of [2], if x is a (p,q)-singularity of an M which is stationary (or has L^p mean curvature, for $p > n$), then M is nearby a $C^{1,\alpha}$ perturbation of $\mathbf{C}^{p,q}$.

As a corollary to our main theorem, we obtain the following answer to 1.1.

Theorem 1.4. *Let M be a multiplicity-one, stationary integral n-varifold in B_1, which is regular away from 0, and has a minimizing (p,q)-singularity at 0. Let M_i be a sequence of stationary, integral varifold in B_1, so that $M_i \to M$ as varifolds. Then for each i sufficiently large, $M_i \llcorner B_{1/2}$ has either an isolated singularity of the same type (p,q), or is entirely regular.*

More precisely, we have the following. Let S_+, S_- be leaves of the foliation by minimal surfaces of $\mathbb{R}^{n+1} \setminus \mathbf{C}^{p,q}$ (as constructed by [7]) which lie in different connected components. Let $M_\pm$ be the smooth manifolds obtained by replacing a small neighborhood of 0 in $M_0 := \mathrm{spt} M$ with a scaled-down copy of $S_\pm$. Then, for i sufficiently large, $\mathrm{spt} M_i \cap B_{1/2}$ is a $C^{1,\alpha}$ perturbation of one of M_0, M_+, M_-.

We remark that, only from the information that $M_i \to M$, one cannot distinguish a priori whether each M_i is regular or singular. See Section 3 for a more detailed statement of our main regularity theorem, and for other corollaries.

The main novelty of Theorem 3.1 is that, unlike previous regularity results for minimal surfaces near isolated singularities (e.g. [2], [9]), we do not prescribe a priori the density of the M_i at any point, that is we do not impose them to be singular at the origin, nor at any other point. As a consequence, even if a minimal surface is close at scale 1 to a cone with an isolated singularity, the surface itself may be entirely smooth.

We can give a further characterization of the M_i when they are singular: in this case the M_i must be one of the examples of minimal surfaces as constructed by [5]. Finally, we can use our regularity theorem to reprove the rigidity result of [12], which characterize complete minimal surfaces asymptotic to quadratic cones.

It would be interesting to know whether our results carry over to other (area-minimizing) singularity models. Unfortunately, the only other known area-minimizing hypercones are of so-called isoparametric type, and for these other examples our techniques do not seem to work. See Section 4 of [12] for further discussions.

2. Notation and preliminaries

We work in $\mathbb{R}^{n+1}$. We denote by $\mathcal{H}^n$ the n-dimensional Hausdorff measure. Given a subset $A \subset \mathbb{R}^{n+1}$, we let $d_A(x) = \inf_{a\in A} |x - a|$ be the Euclidean distance to A, and given $r > 0$ write

$$B_r(A) := \{x \in \mathbb{R}^{n+1} : d_A(x) < r\}$$

for the (open) r-tubular neighborhood of A. Similarly $B_r(x)$ is the open r-ball centered at $x \in \mathbb{R}$. If $x = 0$, we may sometimes just write B_r. We write $\overline{B_r(A)}$ for the closed tubular neighborhood or r-ball. We set $\omega_n = \mathcal{H}^n(\mathbb{R}^n \cap B_1)$ to be the volume of the n-dimensional unit ball. We define the translation/dilation map $\eta_{x,r}(z) := (z - x)/r$.

We may occasionally use the notation of Cheeger: we denote with $\Psi(\epsilon_1, \ldots, \epsilon_k|c_n, \ldots, c_N)$ a non-negative function, which for any fixed $c_1, \ldots, c_N$, satisfies

$$\lim_{\epsilon_1,\ldots,\epsilon_k\to 0} \Psi = 0.$$

We shall always treat graphing functions as scalars. Given oriented hypersurfaces M, N, and an open subset $U \subset N$, if we write

$$M = \mathrm{graph}_N(u)$$

we mean that $M = \{x + u(x)\nu_N(x) : x \in U \subset N\}$, where ν_N is the unit normal of N.

Given an (oriented) hypersurface N, a function $u : N \to \mathbb{R}^n$, and a $\beta \in (0, 1)$, we define the Holder semi-norm

$$[u]_{\beta,r} = \sup_{x\neq y\in N\cap B_r\setminus B_{r/2}} \frac{|u(x) - u(y)|}{|x - y|},$$

and, given an integer $k \geq 0$, the Holder norm

$$|u|_{k,\beta,r} = [D^k u]_{\beta,r} + \sum_{i=0}^{k} r^{1-k} \sup_{x\in N\cap B_r\setminus B_{r/2}} |\nabla^i u|.$$

Typically when we use these norms, N will be conical or nearly conical.

2.1. **Varifolds and first variation.** We are concerned with integral n-varifolds that are stationary, or have L^p mean curvature. Recall that an n-varifold M is integral if it has the following structure: there is a countably n-rectifiable set $\tilde{M}$, and a $\mathcal{H}^n\llcorner\tilde{M}$-integrable, non-negative $\mathbb{Z}$-valued function θ, so that

$$M(\phi(x, V)) = \int_{\tilde{M}} \phi(x, T_x\tilde{M})\theta(x)\, d\mathcal{H}^n(x), \quad \forall\phi \in C^0_c(\mathbb{R}^{n+1}\times Gr(n, n+1)).$$

Here $Gr(n, n + 1)$ is the Grassmanian bundle, i.e. the space of unoriented n-planes in $\mathbb{R}^{n+1}$. If N is an n-manifold, then N induces a natural n-varifold in the obvious fashion, which we write as as $[N]$. Given a proper, C^1 map η, we write $\eta_\sharp M$ for the pushforward of M.

We write μ_M for the mass measure of M. The first variation of M in an open subset $U \subset \mathbb{R}^{n+1}$ is the linear functional

$$\delta M(X) = \int \mathrm{div}_M(X) d\mu_M, \quad X \in C^1_c(U, \mathbb{R}^{n+1}),$$

where $\mathrm{div}_M(X)$ is defined for μ_M-a.e. x as follows: if e_i is an orthonormal basis for the tangent space $T_x M$, then

$$\mathrm{div}_M(X) = \sum_i \langle e_i, D_{e_i} X \rangle.$$

A integral n-varifold M (or surface) is stationary in $U \subset \mathbb{R}^{n+1}$ if $\delta M(X) = 0$ for all X compactly supported in U. M is said to have generalized mean curvature H_M, and zero generalized boundary in U, if

$$\delta M(X) = -\int H \cdot X d\mu_M \quad \forall X \in C^1_c(U, \mathbb{R}^{n+1}),$$

where H is some μ_M-integrable vector field.

Let M have generalized mean curvature H_M in B_1, and zero generalized boundary, and suppose $||H_M||_{L^p(B_1;\mu_M)} \leq \Lambda < \infty$ for some $p > n$. Then M admits the area monotonicity (see [1])

$$\left(\frac{\mu_M(B_s(x))}{s^n} \right)^{1/p} \leq \frac{\Lambda}{p-n}(r^{1-n/p} - s^{1-n/p}) + \left(\frac{\mu_M(B_r(x))}{r^n} \right)^{1/p}. \tag{1}$$

for any $x \in B_1$, and $0 < s < r < 1 - |x|$. Of course if M is stationary, then $r^{-n}\mu_M(B_r(x))$ is increasing for all $r < 1 - |x|$.

Further, Allard's theorem [1] implies M as in the previous paragraph admits the following regularity: there is a $\delta(n, p)$ so that if for some n-plane V^n we have

$$\int_{B_1} d_V^2 \, d\mu_M + ||H_M||^2_{L^p(B_1;\mu_M)} \leq E \leq \delta^2,$$
$$\mu_M(B_1) \leq 3/2\omega_n \quad \text{and} \quad \mu_M(B_{1/10}) \geq (1/2)\omega_n(1/10)^n,$$

then there is a $C^{1,1-n/p}$ function $u : V \cap B_{1/2} \to V^\perp$, so that

$$\mathrm{spt} M \cap B_{1/2} = \mathrm{graph}(u) \cap B_{1/2}, \quad |u|_{C^{1,1-n/p}} \leq c(n) E^{1/2}.$$

2.2. **Jacobi fields.** Given a smooth, oriented minimal hypersurface N^n, let us write $\mathcal{M}_N$ for the mean curvature operator on graphs over N, i.e. so that given $u : U \subset N \to \mathbb{R}$, and $x \in U$, then $\mathcal{M}_N(u)(x)$ denotes the mean curvature of $\mathrm{graph}_U(u)$ at the point $x + u(x)\nu_N(x)$.

Equivalently, $-\mathcal{M}_N$ is the Euler-Lagrange operator for the area functional on graphs over N. $\mathcal{M}_N$ is a second-order, quasi-linear elliptic operator

$$\mathcal{M}_N(u) = a_N(x, u, \nabla u)^{ij} \nabla^2_{ij} u + b_N(x, u, \nabla u),$$

whose coefficients $a_N(x, z, p)$, $b_N(x, z, p)$ depend smoothly on x, z, p and the submanifold N.

Write $\mathcal{L}_N$ for the linearization of $\mathcal{M}_N$ at $u = 0$. $\mathcal{L}_N$ is called the Jacobi operator, and any solution w to $\mathcal{L}_N(w) = 0$ is called a Jacobi field. $\mathcal{L}_N$ is a linear, elliptic operator:

$$\mathcal{L}_N = \Delta_N + |A_N|^2.$$

Here A_N is the second fundamental form of $N \subset \mathbb{R}^{n+1}$, and Δ_N is the connection Laplacian. If ϕ_t is a family of compactly supported diffeomorphisms of $\mathbb{R}^{n+1}$, and $\partial_t \phi_t|_{t=0, x\in N} = f\nu_N$ on N, then

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathrm{vol}(\phi_t(N)) = -\int_N f\mathcal{L}_N f \, d\mathcal{H}^n.$$

N is called stable if $\mathcal{L}_N \leq 0$ when restricted to any compact subset of N.

When $N = \mathbf{C}$ is a cone, with smooth, compact cross section $\Sigma = \mathbf{C} \cap S^n$, then we can further decompose

$$\mathcal{L}_N = \partial_r^2 + (n-1)r^{-1}\partial_r + r^{-2}\mathcal{L}_\Sigma, \quad \mathcal{L}_\Sigma = \Delta_\Sigma + |A_\Sigma|^2,$$

where $r = |x|$ is the radial distance, $\omega = x/|x|$, and Δ_Σ, A_Σ are the connection Laplacian, second fundamental form (resp.) of $\Sigma \subset S^n$.

Since Σ is compact, there is $L^2(\Sigma)$-orthonormal basis of eigenfunctions ϕ_i of $\mathcal{L}_\Sigma$, with corresponding eigenvalues $\mu_1 < \mu_2 \leq \cdots \to \infty$:

$$\mathcal{L}_\Sigma \phi_i + \mu_i \phi_i = 0, \quad \int_\Sigma \phi_i \phi_j d\mathcal{H}^{n-1} = \delta_{ij}.$$

By the Rayleigh quotient $\mu_1 \leq -(n-1)$. On the other hand, when $\mathbf{C}$ is stable we have $\mu_1 \geq -((n-2)/2)^2$, and we have strict inequality when $\mathbf{C}$ is strictly stable (see [5]). If we define

$$\gamma_i^{\pm} = -((n-2)/2) \pm \sqrt{((n-2)/2)^2 + \mu_i},$$

then for any solution w to $\mathcal{L}_{\mathbf{C}}(w) = 0$, with $\mathbf{C}$ being strictly stable, we can expand in $L^2_{loc}(\mathbf{C})$

$$w(r\omega) = \sum_{i=1}^{\infty} (a_i^+ r^{\gamma_i^+} + a_i^- r^{\gamma_i^-})\phi_i(\omega),$$

where for each r the sum is $L^2(\Sigma)$ orthogonal.

2.3. **Hardt-Simon foliation.** Taking $\mathbf{C}$, Σ as above, then $\mathbf{C}$ divides $\mathbb{R}^{n+1}$ into two connected, open, disjoint regions E_+ and E_-. We can choose an oriented unit normal $\nu_{\mathbf{C}}$ for $\mathbf{C}$, so that $\nu_{\mathbf{C}}$ points into E_+.

When $\mathbf{C}$ is area-minimizing, in the sense of currents, then [7] have shown there are *smooth*, area-minimizing hypersurfaces $S_\pm \subset E_\pm$, which are asymptotic to $\mathbf{C}$. Moreover, the $S_\pm$ radial graphs, and hence the collection of dilations $\lambda S_\pm$ ($\lambda > 0$) forms a foliation of $E_\pm$ by smooth, area-minimizing hypersurfaces, sometimes called the Hardt-Simon foliation. Let us orient $S_\pm$ with unit normals $\nu_{S_\pm}$ compatible with $\mathbf{C}$, so that as $|x| \to \infty$, $\nu_{S_\pm} \to \nu_{\mathbf{C}}$.

When $\mathbf{C}$ is *strictly* minimizing, then $S_\pm$ decays to $\mathbf{C}$ like the larger homogeneity $r^{\gamma_1^+}$. In particular, after a normalization as necessary, there is a radius $R_0 \geq 1$ and $\alpha_0 > 0$ so that

$$S_\pm \setminus B_{R_0} = \mathrm{graph}_{\mathbf{C}}(v_\pm),$$

where $v_\pm : \mathbf{C} \setminus B_{R_0/2} \to \mathbb{R}$ is a smooth function satisfying

$$v_\pm(r\omega) = \pm r^{\gamma_1^+} f_\pm(r\omega), \quad \sum_{k=0}^{2} r^k |\nabla^k (f_\pm - \phi_1)| = O(r^{-\alpha_0}).$$

For shorthand we will set $\gamma = \gamma_1^+$. See [7] for details about strictly minimizing.

Given $\lambda \in \mathbb{R}$, define

$$S_\lambda = \begin{cases} \lambda S_+ & \lambda > 0 \\ \mathbf{C} & \lambda = 0 \\ |\lambda| S_- & \lambda < 0 \end{cases} \qquad v_\lambda(r\omega) = \begin{cases} \lambda v_+(r/\lambda) & \lambda > 0 \\ 0 & \lambda = 0 \\ |\lambda| v_-(r/|\lambda|) & \lambda < 0 \end{cases}$$

so that

$$S_\lambda \setminus B_{\lambda R_0} = \mathrm{graph}_{\mathbf{C}}(v_\lambda).$$

Let S_λ have the same orientation as $S_{\mathrm{sign}(\lambda)}$. Observe that

$$v_\lambda(r) = \mathrm{sign}(\lambda)|\lambda|^{1-\gamma} r^\gamma f_{\mathrm{sign}(\lambda)}(r/|\lambda|). \tag{2}$$

For shorthand, we will often write $\lambda^\alpha := \mathrm{sign}(\lambda)|\lambda|^\alpha$.

The following straightforward Lemma will be useful.

Lemma 2.4. *For $|\mu|, |\lambda| \leq 1$, and $r \geq R \geq R_0$, we have*

$$v_\mu(r) - v_\lambda(r) = (1 + O((\max\{|\mu|, |\lambda|\})^{\alpha_0} R^{-\alpha_0}))(\mu^{1-\gamma} - \lambda^{1-\gamma}) r^\gamma.$$

In particular, if $R_0(\mathbf{C})$ is sufficiently large, then

$$\frac{1}{4}(\mu^{1-\gamma} - \lambda^{1-\gamma})^2 r^{2\gamma} \leq (v_\mu(r) - v_\lambda(r))^2 \leq 4(\mu^{1-\gamma} - \lambda^{1-\gamma})^2 r^{2\gamma}$$

for all $r \geq \max\{|\mu|, |\lambda|\} R_0$.

Proof. If $\lambda \neq 0$, $|\lambda| \leq 1$, and $r \geq |\lambda| R$, then we have

$$\begin{aligned}\frac{d}{d\lambda} v_\lambda(r) &= (1-\gamma)|\lambda|^{-\gamma} r^\gamma f_{\mathrm{sign}(\lambda)}(r/|\lambda|) - |\lambda|^{-1-\gamma} r^{1+\gamma} f'_{\mathrm{sign}(\lambda)}(r/|\lambda|) \\ &= (1-\gamma)|\lambda|^{-\gamma} r^\gamma (1 + O(|\lambda|^{\alpha_0} R^{-\alpha_0})).\end{aligned}$$

If $\mu\lambda \geq 0$, then the required result follows from the above and the fundamental theorem of calculus.

If $\mu\lambda < 0$, $|\lambda| \geq |\mu| > 0$, then we have (recalling our shorthand $\mu^\beta = \mathrm{sign}(\mu)|\mu|^\beta$)

$$\begin{aligned}(v_\mu(r) - v_\lambda(r))^2 &= (|\mu|^{1-\gamma} + |\lambda|^{1-\gamma})^2 r^{2\gamma} (1 + O(|\lambda|^{\alpha_0} R^{-\alpha_0})) \\ &= (\mu^{1-\gamma} - \lambda^{1-\gamma})^2 r^{2\gamma} (1 + O(|\lambda|^{\alpha_0} R^{-\alpha_0})).\end{aligned}$$

□

2.5. **Minimizing quadratic cones.** Take $\mathbf{C} = \mathbf{C}^{p,q}$ an area-minimizing quadratic cone. There are two key properties of $\mathbf{C}$ which we shall need. These are proven in [12, Proposition 2.7].

(1) $\mathbf{C}$ is *strictly-minimizing*, so that the foliation decays like $r^{\gamma_1^+}$. In fact, since $\mathbf{C}$ is rotationally symmetric, we have that

$$v_\pm(r\omega) = \pm r^{\gamma_1^+} f_\pm(r\omega), \quad \sum_{k=0}^{2} r^k |\nabla^k (f_\pm - 1)| = O(r^{-\alpha_0}). \tag{3}$$

Recall that for shorthand we write $\gamma = \gamma_1^+$.

(2) $\mathbf{C}$ is *strongly integrable*, in the following sense: any solution of $\mathcal{L}_{\mathbf{C}}(w) = 0$ can be written

$$w(x = r\omega) = \sum_{j \geq 1} a_i^- r^{\gamma_i^-} \phi_i(\omega) + e r^{\gamma_1^+} + (b + Ax) \cdot \nu_{\mathbf{C}}(\omega) + \sum_{j \geq 4} a_i^+ r^{\gamma_i^+},$$

where $e \in \mathbb{R}$, $b \in \mathbb{R}^{n+1}$, and A is a skew-symmetric $(n+1) \times (n+1)$ matrix. In other words, $\gamma_2^+ = 0$, $\gamma_3^+ = 1$, and the eigenfunctions ϕ_2, ϕ_3 are generated by translations, rotations.

Every result in our paper holds for any area-minimizing hypercone satisfying the above two conditions. Rotational symmetry as in (3) simplifies our computations slightly, but has no bearing on our proof. We write all our results for quadratic cones because these are the only area-minimizing cones which we can verify as "strongly integrable."

3. Main theorems

For the duration of this paper, we fix $\mathbf{C} = \mathbf{C}^{p,q}$ to be an area-minimizing quadratic cone, S_λ the Hardt-Simon foliation, and we use the notation associated to $\mathbf{C}$, S_λ as introduced in Section 2.3.

Our main theorem is the following, from which Theorem 1.4 follows directly.

Theorem 3.1. *There are constants* $\delta_1(\mathbf{C})$*,* $\Lambda_1(\mathbf{C})$*,* $c_1(\mathbf{C})$*,* $\beta(\mathbf{C})$ *so that the following holds. Take* $|\lambda| \leq \Lambda_1$*, and let* M *be a stationary integral varifold in* B_1*, satisfying*

$$\int_{B_1} d^2_{S_\lambda} d\mu_M \leq E \leq \delta_1^2, \tag{4}$$

$$\mu_M(B_1) \leq (3/2)\mu_{\mathbf{C}}(B_1), \quad \text{and} \quad \mu_M(\overline{B_{1/10}}) \geq (1/2)\mu_{\mathbf{C}}(B_{1/10}).$$

Then there is an $a \in \mathbb{R}^{n+1}$*,* $q \in SO(n+1)$*,* $\lambda' \in \mathbb{R}$*, with*

$$|a| + |q - Id| + \left|\mathrm{sign}(\lambda')|\lambda'|^{1-\gamma} - \mathrm{sign}(\lambda)|\lambda|^{1-\gamma}\right| \leq c_1 E^{1/2}, \tag{5}$$

and a $C^{1,\beta}$ *function* $u : (a + q(S_{\lambda'})) \cap B_{1/2} \to \mathbb{R}$*, so that*

$$\mathrm{spt} M \cap B_{1/2} = \mathrm{graph}_{a+q(S_{\lambda'})}(u) \cap B_{1/2},$$

and u *satisfies the estimates*

(6)

$$r^{-1}|u|_{C^0(B_r(a))} + |\nabla u|_{C^0(B_r(a))} + r^\beta [\nabla u]_{\beta, B_r(a)} \leq c_1 r^\beta E^{1/2} \quad \forall r \leq 1/2.$$

In particular, $M \cap B_{1/2}$ *is either smooth, or has an isolated singularity modeled on* $\mathbf{C}$*.*

Remark 3.2. The precise form of the lower bound on $\mu_M(B_{1/10})$ in (4) is of no consequence, nor is the precise ball radius $1/10$. One could easily assume (for example) $\mu_M(B_{1/10}) \geq v > 0$, and obtain the same conclusions, except that the constants δ_1 and Λ_1 would depend on the choice of v also. The upper bound on $\mu_M(B_1)$ is more important: we require it to be strictly less than $2\mu_{\mathbf{C}}(B_1)$.

A further characterization is possible in the case when M as in Theorem 3.1 is singular. [5] have constructed a large class of examples of minimal surfaces in B_1, which are singular perturbations of a given minimal cone (see Section 9.1). In fact, in a sufficiently small neighborhood, these are the only minimal surfaces which are graphical over $\mathbf{C}$. It would be interesting to know whether examples like those in [5] exist as perturbations over a foliate S_λ.

Proposition 3.3. *Let* M*,* λ' *be as in Theorem 3.1. If* $\lambda' = 0$*, and* $E \leq \delta_2(\mathbf{C})$ *is sufficiently small, then* spt$M \cap B_{1/4}$ *coincides with one of the graphical solutions as constructed by* [5].

The most interesting consequence of Theorem 3.1 is that singularities modeled on (minimizing) Simons's cones propagate out their structure not only to a neighborhood of the original surface, but also of *nearby* surfaces. If these nearby surfaces are not minimal, but instead of L^p mean curvature, then essentially the same structure holds, but with slightly less regularity. In this sense the minimizing Simons's singularities can be thought of as "very strongly isolated."

Corollary 3.4. *Given any $p > n$, there are constants $\delta_3(\mathbf{C})$, $\epsilon_3(p, \mathbf{C})$, $\Lambda_3(\mathbf{C})$, $c_3(\mathbf{C})$ so that the following holds. Let M be an integral n-varifold in B_1 with generalized mean curvature H_M, zero generalized boundary, satisfying 4 with δ_3^2 in place of E, and*

$$\left(\int_{B_1} |H_M|^p d\mu_M\right)^{1/p} \leq \epsilon_3.$$

Then there are $a \in \mathbb{R}^{n+1}$, $\lambda' \in \mathbb{R}$ so that

- *either $\lambda' = 0$, in which case $\mathrm{spt} M \cap B_{1/2}$ is a $C^{1,\beta}$ perturbation of $\mathbf{C}$;*
- *or $\lambda' \neq 0$, and we can find for every $0 < r \leq 1/2$ a $q_r \in SO(n+1)$, so that $\mathrm{spt} M \cap B_r(a) \setminus B_{r/100}(a)$ is a $C^{1,\beta}$ graph over $a + q_r(S_{\lambda'})$.*

In particular, M is either entirely regular, or has an isolated singularity modeled on $\mathbf{C}$.

Example 3.5. This Corollary rules out many possible examples of singularity formation. For example, in an 8 dimensional manifold this rules out the possibility that $S^3 \times S^3$ singularites are collapsing into an $S^2 \times S^4$ singularity, or even worse, that multiple types of isolated singularities are collapsing into and single $S^3 \times S^3$ or $S^2 \times S^4$ singularity.

Remark 3.6. We cannot obtain directly that the q_r have a limit as $r \to 0$. If $\lambda' = 0$, then we can use [2] to deduce a posteriori that $M \cap B_{1/2}$ is a $C^{1,\alpha}$ perturbation of $\mathbf{C}$. If $\lambda' \neq 0$, then we do not need to worry about the limiting behavior of the q_r to deduce that $M \cap B_{1/2}$ is some $C^{1,\alpha}$ perturbation of $S_{\lambda'}$. However, because we have no control over q_r as $r \to 0$, we cannot obtain any effective estimates on the $C^{1,\alpha}$ map in question. It would be interesting to resolve this.

Another direct consequence of our regularity theorem is the following rigidity theorem for area-minimizing surfaces asymptotic to Simons's cones, which was originally proven by Simon-Solomon:

Corollary 3.7 ([12])**.** *Let M be an area-minimizing hypersurface in $\mathbb{R}^{n+1}$, and $\mathbf{C}$ be a strongly integrable cone, with associated foliation S_λ. Suppose there is a sequence of radii $R_i \to \infty$ so that*

$$R_i^{-1} M \to \mathbf{C}$$

in the flat disance. Then up to translation, rotation, and dilation, $M = \mathbf{C}, S_1,$ *or* S_{-1}.

Remark 3.8. We remark that our result is much stronger than the characterization of [12]. As illustrated by the examples of [5], being close to the Simons's cone at scale 1 is much weaker than being close on all of $\mathbb{R}^{n+1}$ – in particular, the latter precludes any modes growing faster than 1-homogeneous. One can think of [12] as a Bernstein-type theorem, while our theorem as an Allard-type regularity theorem.

4. Outline of proof

Our strategy is to prove the following excess-decay type theorem (Proposition 6.1): provided both λ and $\int_{B_1} d^2_{S_\lambda} d\mu_M$ are sufficiently small (plus some restrictions on the mass of M), then we have a decay estimate of the form

$$\theta^{-n-2} \int_{B_\theta} d^2_{a'+q'(S_{\lambda'})} d\mu_M \leq (1/2) \int_{B_1} d^2_{S_\lambda} d\mu_M. \tag{7}$$

That is, provided S_λ is sufficiently close to the cone $\mathbf{C}$, and we are sufficiently L^2 close to S_λ, then after a translation/rotation/dilation as necessary, our L^2 distance to the foliation improves at a smaller scale.

We can continue iterating (7) while S_λ is scale-invariantly close to $\mathbf{C}$, and obtain a decay of the form: there is an $a'' \in \mathbb{R}^{n+1}$, $q'' \in SO(n+1)$, and $\lambda'' \in \mathbb{R}$ so that

$$r^{-n-2} \int_{B_r} d^2_{a''+q''(S_{\lambda''})} d\mu_M \leq C r^{2\beta} \quad \forall c|\lambda''| \leq r \leq 1. \tag{8}$$

By two straightforward contradiction arguments (one for M close to $S_\lambda \cap B_1 \setminus B_{1/100}$ with λ small, and one for M close to $S_{\pm 1} \cap B_c$), we can use [1] with (8) to deduce $\mathrm{spt} M \cap B_{1/2}$ is graphical over $a'' + q''(S_{\lambda''})$.

We would like to prove (7) by contradiction,with an argument that loosely resembles the original "excess decay" proof due to De Giorgi and, as implemented in a fashion closer to our style, [2], [10], [11]. Briefly, we would like to suppose (7) fails for some sequence M_i and $\lambda_i \to 0$, $E_i = \int_{B_1} d^2_{S_{\lambda_i}} d\mu_{M_i} \to 0$. Then over larger and larger annuli $B_{1/2} \setminus B_{\tau_i}$ ($\tau_i \to 0$) we can write $\mathrm{spt} M = \mathrm{graph}_{S_{\lambda_i}}(u_i)$. If we rescale $v_i = E_i^{-1/2} u_i$, then the v_i have uniformly bounded $||v_i||_{L^2(B_1)}$, and after passing to a subsequence we get convergence

$$v_i \to w \quad \text{with} \quad \mathcal{L}_{\mathbf{C}}(w) = 0\,.$$

The idea now, in vague terms, is to use good decay properties for solutions to the linearized problem $\mathcal{L}_{\mathbf{C}}(w) = 0$, to prove good decay for solutions to the non-linear problem $\mathcal{M}_{\mathbf{C}}(u) = 0$, that is we'd like to arrange so that $w = O(r^{1+\epsilon})$, and then use this to deduce L^2 decay of the u_i as in (7).

To ensure this argument works we need to

(i) ensure the decaying norm for the non-linear problem is comparable to the linear one (a.k.a. non-concentration of L^2 norm at singularities), that is for any ρ small

$$E_i^{-1} \int_{B_\rho} d^2_{S_{\lambda_i}} d\mu_{M_i} \to ||w||^2_{L^2(\mathbf{C} \cap B_\rho)} ,$$

(ii) prove good decay for the linear problem (a.k.a. killing bad homogeneities through integrability), that is $w = O(r^{1+\epsilon})$.

The latter issue is where the concept of *integrability* arises. A minimal cone $\mathbf{C}$ is called integrable if every 1-homogeneous Jacobi field arises from a 1-parameter family of minimal cones. The idea of [1], [2] is that, under suitable density assumptions in the argument above, one can typically show that $w = O(r)$, but one needs it to grow like $r^{1+\epsilon}$. For a cone with an isolated singularity, the homogeneities are discrete, and so provided $\mathbf{C}$ is integrable, one can rewrite the minimal surface as a graph over a slightly adjusted cone, chosen to cancel the r term in the Fourier expansion of w.

The main novelty in our approach is in treating the foliation S_λ as a direction of integrability. In other words, we are relaxing the original notion of integrability, as a movement through cones, to allow one to push off the cone into families of entirely smooth hypersurfaces, and in particular we are allowing for a notion of integrability in which the singularity behavior changes. In order to handle this we require new decay and non-concentration estimates for minimal surfaces near an arbitrary foliate S_λ without any structural assumption on M. This is the content of Theorem 5.1.

More precisely, the key observation is that the foliation is generated by a positive Jacobi field of the form

$$v(r) = r^\gamma, \quad 0 > \gamma > -(n-2)/2,$$

and this Jacobi field has itself good L^2 decay:

$$\int_{B_\rho \cap \mathbf{C}} v^2 d\mathcal{H}^n \le c\rho^2 \int_{B_1 \cap \mathbf{C}} v^2 d\mathcal{H}^n. \tag{9}$$

To deal with point (i), we use the maximum principle to "trap" M between two foliates, and thereby show that M cannot diverge from a given S_λ any faster than the foliation itself. This allows us to prove that $\int_{B_\rho} d^2_{S_\lambda} d\mu_M$ has a decay similar to (9), and hence no L^2 norm can accumulate near the non-graphical region (away from 0 we of course have strong L^2 convergence since the v_i converge smoothly there).

To deal with point (ii), we can prove that the v_i, and hence the resulting Jacobi field w, grow at least as fast as $v(r) = r^\gamma$ as r increases. Using the strongly integrable nature of $\mathbf{C}$, we can then deduce that w looks like

$$w(x = r\omega) = er^\gamma + (b + Ax) \cdot \nu_{\mathbf{C}} + O(r^{1+\epsilon}),$$

for $f \in \mathbb{R}$, $b \in \mathbb{R}^{n+1}$, A skew-symmetric. In other words, w has the growth we require except for terms generated by moving into the foliation, translation, and rotation. By replacing the S_{λ_i} with a new sequence of foliates $a_i + q_i(S_{\lambda'_i})$, and repeating the above contradiction argument with this new sequence, we can arrange so that these three lower homogeneities disappear, and thereby deduce $w = O(r^{1+\epsilon})$.

5. Non-concentration of L^2-Excess

Our main theorem of this section is the following. It is spiritually similar to Theorem 2.1 in [11], except we are proving non-concentration with respect to an arbitrary foliate S_λ instead of just $\mathbf{C}$, and we additionally obtain a pointwise decay estimate on the graphing function. Recall the shorthand $\gamma = \gamma_1^+$.

Theorem 5.1. *For every $0 < \tau < 1/4$, $\beta > 0$, there is $\Lambda_4(\mathbf{C}, \tau)$, $\epsilon_4(\mathbf{C}, \beta, \tau)$, $c_4(\mathbf{C})$ so that the following holds: if $|\lambda| \leq \Lambda_4$ and M is a stationary integral n-varifold in B_1 satisfying*

(10)
$$\int_{B_1} d_{S_\lambda}^2 d\mu_M \leq \epsilon_4^2, \quad \mu_M(B_1) \leq (7/4)\mu_{\mathbf{C}}(B_1), \quad \mu_M(\overline{B_{1/10}}) \geq \frac{1}{2}\mu_{\mathbf{C}}(B_{1/10}),$$

then there is a smooth function $u : S_\lambda \cap B_{1/2} \setminus B_{\tau/2} \to \mathbb{R}$ so that

$$\text{spt} M \cap B_{1/2} \setminus B_\tau = \text{graph}_{S_\lambda}(u) \cap B_{1/2} \setminus B_\tau, \quad \sum_{k=0}^{3} r^{k-1} |\nabla^k u| \leq \beta. \tag{11}$$

For every $\tau \leq \rho \leq 1/4$, we have:

$$\int_{B_\rho \setminus B_\tau} u^2 d\mu_{\mathbf{C}} + \int_{B_\rho} d_{S_\lambda}^2 d\mu_M \leq c_4 \rho^{n+2\gamma} \int_{B_1} d_{S_\lambda}^2 d\mu_M \tag{12}$$
$$\leq c_4 \rho^2 \int_{B_1} d_{S_\lambda}^2 d\mu_M$$

Moreover, u has the following L^∞ decay bound:

$$\sup_{S_\lambda \cap \partial B_r} u^2 \leq c_4 r^{2\gamma} \int_{B_1} d_{S_\lambda}^2 \qquad \forall r \in (\tau, 1/4). \tag{13}$$

Proof. Let $R_0(\mathbf{C})$ be as in Lemma 2.4. Taking $\Lambda_4(\mathbf{C}, \tau)$ sufficiently small, we can assume $\Lambda_4 R_0 < \tau/100$, and

$$\sum_{k=0}^{2} r^{k-1}|\nabla^k v_\lambda| \leq \beta \quad \forall r \in (\tau/100, 1). \tag{14}$$

By a straightforward contradiction argument, allowing $|\lambda| \leq \Lambda_4$ to vary, we get that (taking $\epsilon_4(\mathbf{C}, \Lambda_4, \tau, \beta)$ small):

$$\text{spt}M \cap B_{3/4} \setminus B_{\tau/10} = \text{graph}_{S_\lambda}(u), \quad \sum_{k=0}^{3} r^{k-1}|\nabla^k u| \leq \beta. \tag{15}$$

Indeed, otherwise there is a sequence of stationary integral n-varifolds M_i, and numbers $\epsilon_i \to 0$, $\lambda_i \in [-\Lambda_4, \Lambda_4]$, for which (10) holds but (15) fails. We can without loss assume $\lambda_i \to \lambda$, for some $|\lambda| \leq \Lambda_4$. By compactness of stationary varifolds with bounded mass, we can pass to a subsequence (also denoted i), and get varifold convergence $M_i \to M$ for some stationary integral n-varifold M. The resulting M satisfies

$$\int_{B_1} d_{S_\lambda}^2 d\mu_M = 0, \quad \mu_M(B_1) \leq (7/4)\mu_{\mathbf{C}}(B_1), \quad \mu_M(B_{2/10}) \geq (1/2)\mu_{\mathbf{C}}(B_{1/10}). \tag{16}$$

The constancy theorem implies that $M = k[S_\lambda]$, for some integer k. The lower bound of (16) implies $k \geq 1$. Ensuring $\Lambda_4(\mathbf{C})$ is sufficiently small, the upper bound in (16) implies $k \leq 1$. So in fact $M_i \to [S_\lambda]$, and hence by Allard's theorem convergence is smooth on compact subsets of $B_1 \setminus \{0\}$. This proves our assertion.

It will be more convenient in this proof to work with graphs over $\mathbf{C}$. By a similar contradiction argument as above, we have (again taking $\epsilon_4(\mathbf{C}, \tau, \beta)$, $\Lambda_4(\mathbf{C}, \tau, \beta)$ sufficiently small):

$$\text{spt}M \cap B_{3/4} \setminus B_{\tau/10} = \text{graph}_{\mathbf{C}}(h), \quad \sum_{k=0}^{3} r^{k-1}|\nabla^k h| \leq \beta. \tag{17}$$

Ensuring $\beta(\mathbf{C}, \tau)$ is sufficiently small, u is effectively equivalent to $h - v_\lambda$. Precisely, if $r\omega \in \mathbf{C}$, and $x = r\omega + v_\lambda(r)\nu_{\mathbf{C}}(r\omega) \in S_\lambda$, then an elementary computation shows that

$$|u(x)\nu_{S_\lambda}(x) - (h(r\omega) - v_\lambda(r))\nu_{\mathbf{C}}(r\omega)| \leq \Psi(\beta|\mathbf{C}, \tau)|h(r\omega) - v_\lambda(r)|.$$

Choosing a possibly small $\beta(\mathbf{C}, \tau)$, this implies

$$\sup_{S_\lambda \cap \partial B_r} |u| \leq 2 \sup_{\mathbf{C} \cap B_{2r} \setminus B_{r/2}} |h - v_\lambda| \qquad \forall r \in (\tau/2, 1/4)\,,$$

and

$$\int_{B_r \setminus B_{\tau/2}} u^2 d\mu_{S_\lambda} \leq 2 \int_{B_{2r} \setminus B_{\tau/4}} |h - v_\lambda|^2 d\mu_{\mathbf{C}}$$
$$\int_{B_r \setminus B_{\tau/2}} |h - v_\lambda|^2 d\mu_{\mathbf{C}} \leq 2 \int_{B_{2r} \setminus B_{\tau/4}} d^2_{S_\lambda} d\mu_M$$
$$\int_{B_r \setminus B_{\tau/2}} d^2_{S_\lambda} d\mu_M \leq 2 \int_{B_{2r} \setminus B_{\tau/4}} u^2 d\mu_{S_\lambda}.$$

So we can prove the required estimates for $h - v_\lambda$ instead of u.

For $\rho \in (\tau/10, 3/4)$, define

$$\lambda^+_\rho = \inf\{\mu : v_\mu(\rho\omega) \geq h(\rho\omega) \quad \forall \omega \in \Sigma\}$$
$$\lambda^-_\rho = \sup\{\mu : v_\mu(\rho\omega) \leq h(\rho\omega) \quad \forall \omega \in \Sigma\}.$$

From (2), (17), we have $|\lambda^\pm_\rho| = \Psi(\beta|\mathbf{C}, \tau)$, and so ensuring $\beta(\mathbf{C}, \tau)$ is sufficiently small, $\lambda^\pm_\rho R_0 < \tau/10$ for all admissible ρ. By the maximum principle (e.g. [13]), we have that $\mathrm{spt} M \cap B_\rho$ is trapped between $S_{\lambda^-_\rho}$ and $S_{\lambda^+_\rho}$. This implies that λ^+_ρ is increasing in ρ, while λ^-_ρ is decreasing in ρ, and

$$v_{\lambda^-\rho}(r) \leq h(r\omega) \leq v_{\lambda^+_\rho}(r) \quad \forall r \in (\tau/2, \rho) \tag{18}$$

Both h and v_λ solve the minimal surface equation over $\mathbf{C}$, and on any compact subset of $\mathbf{C} \cap B_1 \setminus \{0\}$ have uniformly bounded derivatives. Therefore the difference $h - v_\lambda$ solves a linear, second order, uniformly elliptic operator. So by standard iteration techniques at scale r, we get

(19)
$$\sup_\omega |h(r\omega) - v_\lambda(r)|^2 \leq c(\mathbf{C}) r^{-n} \int_{B_{2r} \setminus B_{r/2}} |h - v_\lambda|^2 d\mu_{\mathbf{C}} \quad \forall \tau/5 < r < 1/4.$$

Since λ^+_ρ is increasing in ρ, λ^-_ρ is decreasing in ρ, and $\lambda^-_\rho \leq \lambda^+_\rho$, we get that

$$\max\{((\lambda^+_\rho)^{1-\gamma} - \lambda^{1-\gamma})^2, ((\lambda^-_\rho)^{1-\gamma} - \lambda^{1-\gamma})^2\} \quad \text{is increasing in } \rho. \tag{20}$$

For any $\tau/5 < r < \rho < 1/4$, we have by Lemma 2.4, (18), (20), and (19):

$$
\begin{aligned}
|h(r\omega) - v_\lambda(r)|^2 &\leq 2\max\{(v_{\lambda_\rho^+}(r) - v_\lambda(r))^2, (v_{\lambda_\rho^-}(r) - v_\lambda(r))^2\} \\
&\leq cr^{2\gamma} \max\{((\lambda_\rho^+)^{1-\gamma} - \lambda^{1-\gamma})^2, ((\lambda_\rho^-)^{1-\gamma} - \lambda^{1-\gamma})^2\} \\
&\leq cr^{2\gamma} \max\{((\lambda_{1/4}^+)^{1-\gamma} - \lambda^{1-\gamma})^2, ((\lambda_{1/4}^-)^{1-\gamma} - \lambda^{1-\gamma})^2\} \\
&\leq cr^{2\gamma} \max\{(v_{\lambda_{1/4}^+}(1/4) - v_\lambda(1/4))^2, (v_{\lambda_{1/4}^-}(1/4) - v_\lambda(1/4))^2\} \\
&= cr^{2\gamma} \max\{(\sup_\omega h(\omega/4) - v_\lambda(1/4))^2, (\inf_\omega h(\omega/4) - v_\lambda(1/4))^2\} \\
&= cr^{2\gamma} \sup_\omega |h(\omega/4) - v_\lambda(1/4)|^2 \\
&\leq cr^{2\gamma} \int_{B_{1/2}\setminus B_{1/8}} |h - v_\lambda|^2 d\mu_{\mathbf{C}}, \qquad (21)
\end{aligned}
$$

where $c = c(\mathbf{C})$. This implies the estimate (13).

Integrating this relation in $r \in (\tau/2, \rho)$, gives:

$$
\begin{aligned}
\int_{B_\rho \setminus B_{\tau/2}} |h - v_\lambda|^2 d\mu_{\mathbf{C}} &\leq \frac{c(\mathbf{C})}{n+2\gamma} \rho^{n+2\gamma} \int_{B_1} d_{S_\lambda}^2 d\mu_M \\
&\leq c(\mathbf{C})\rho^2 \int_{B_1} d_{S_\lambda}^2 d\mu_M,
\end{aligned}
$$

since $2\gamma + n \geq -(n-2) + n \geq 2$. By the bounds (14), (17), we have

$$
\int_{B_\rho \setminus B_\tau} d_{S_\lambda}^2 d\mu_M \leq c\rho^2 \int_{B_1} d_{S_\lambda}^2 d\mu_M. \qquad (22)
$$

We focus now on proving the final part of (12), i.e. the excess in the ball B_τ. Since S is graphical over $\mathbf{C}$ near ∂B_{R_0}, we have

$$
d_H(S_\lambda \cap B_{\max\{|\lambda|,|\mu|\}R_0}, S_\mu \cap B_{\max\{|\lambda|,|\mu|\}R_0}) \leq c(\mathbf{C})|\lambda - \mu|.
$$

Let $\lambda_\tau = \max\{|\lambda_\tau^+|, |\lambda_\tau^-|\}$. Then, recalling how $\text{spt}M \cap B_\tau$ is trapped between $S_{\lambda_\tau^+}$ and $S_{\lambda_\tau^-}$, we get

$$
\begin{aligned}
&\int_{B_{\lambda_\tau R_0}} d_{S_\lambda}^2 d\mu_M \\
&\leq \max\{d_H(S_{\lambda_\tau^+} \cap B_{\lambda_\tau R_0}, S_\lambda \cap B_{\lambda_\tau R_0})^2, d_H(S_{\lambda_\tau^-} \cap B_{\lambda_\tau R_0}, S_\lambda \cap B_{\lambda_\tau R_0})^2\}\mu_M(B_{\lambda_\tau R_0}) \\
&\leq c(\mathbf{C})\lambda_\tau^n \max\{(\lambda_\tau^+ - \lambda)^2, (\lambda_\tau^- - \lambda)^2\} \\
&\leq c(\mathbf{C})\tau^{n+2\gamma} \max\{((\lambda_\tau^+)^{1-\gamma} - \lambda^{1-\gamma})^2, ((\lambda_\tau^-)^{1-\gamma} - \lambda^{1-\gamma})^2\}. \qquad (23)
\end{aligned}
$$

The last line follows because there is a constant $c(n)$ so that whenever $|\mu|, |\lambda| \leq 1$, we have (recall $\gamma < 0$)

$$(\mu^{1-\gamma} - \lambda^{1-\gamma})^2 \geq \frac{1}{c(n)} \max\{|\mu|, |\lambda|\}^{-2\gamma}(\mu - \lambda)^2.$$

Choose I so that $2^I \lambda_\tau R_0 \leq \tau < 2^{I+1}\lambda_\tau R_0$. We compute:

$$\begin{aligned}
&\int_{B_\tau \setminus B_{\lambda_\tau R_0}} d^2_{S_\lambda} d\mu_M \\
&\leq \sum_{i=0}^{I} \int_{B_{2^{i+1}\lambda_\tau R_0} \setminus B_{2^i \lambda_\tau R_0}} d^2_{S_\lambda} d\mu_M \\
&\leq \sum_{i=0}^{I} \sup_{2^i\lambda_\tau R_0 \leq r \leq 2^{i+1}\lambda_\tau R_0} \max\{(v_{\lambda_\tau^+}(r) - v_\lambda(r))^2, (v_{\lambda_\tau^-}(r) - v_\lambda(r))^2\} \mu_M(B_{2^{i+1}\lambda_\tau R_0}) \\
&\leq \sum_{i=0}^{I} c(\mathbf{C})(2^i\lambda_\tau R_0)^{2\gamma} \max\{((\lambda_\tau^+)^{1-\gamma} - \lambda^{1-\gamma})^2, ((\lambda_\tau^-)^{1-\gamma} - \lambda^{1-\gamma})^2\}(2^i\lambda_\tau R_0)^n \\
&\leq c(\mathbf{C})\tau^{n+2\gamma} \max\{((\lambda_\tau^+)^{1-\gamma} - \lambda^{1-\gamma})^2, ((\lambda_\tau^-)^{1-\gamma} - \lambda^{1-\gamma})^2\}.
\end{aligned} \tag{24}$$

Combining (23), (24), with the computations of (21), we obtain

$$\begin{aligned}
\int_{B_\tau} d^2_{S_\lambda} d\mu_M &\leq c(\mathbf{C})\tau^{n+2\gamma} \max\{((\lambda_\tau^+)^{1-\gamma} - \lambda^{1-\gamma})^2, ((\lambda_\tau^-)^{1-\gamma} - \lambda^{1-\gamma})^2\} \\
&\leq c(\mathbf{C})\tau^2 \int_{B_1} d^2_{S_\lambda} d\mu_M.
\end{aligned}$$

Together with (22), this gives the required estimate (12). □

The following corollary will also be useful.

Corollary 5.2. *There is a $\Lambda_5(\mathbf{C})$ so that if $|\lambda|, |\lambda'| \leq \Lambda_5$, and M satisfies the hypotheses of Theorem 5.1, then*

$$\int_{B_1} d^2_{S_{\lambda'}} d\mu_M \leq c(\mathbf{C}) \int_{B_1} d^2_{S_\lambda} d\mu_M + c(\mathbf{C})(\lambda^{1-\gamma} - (\lambda')^{1-\gamma})^2.$$

Proof. The computations of Theorem 5.1 show that, provided $|\lambda|, |\lambda'| \leq \Lambda_4(\mathbf{C})$, we have

$$\begin{aligned}
\int_{B_{1/4}} d^2_{S_{\lambda'}} d\mu_M &\leq c(\mathbf{C}) \max\{((\lambda_{1/4}^+)^{1-\gamma} - (\lambda')^{1-\gamma})^2, ((\lambda_{1/4}^-)^{1-\gamma} - (\lambda')^{1-\gamma})^2\} \\
&\leq c(\mathbf{C}) \int_{B_1} d^2_{S_\lambda} d\mu_M + c(\mathbf{C})(\lambda^{1-\gamma} - (\lambda')^{1-\gamma})^2.
\end{aligned}$$

It remains only to control the annular region $B_1 \setminus B_{1/4}$.

Choose $\epsilon(\Lambda_4, \mathbf{C})$ sufficiently small so that if $|\lambda| \leq \Lambda_4$, then the nearest point projection from $B_\epsilon(\mathbf{C}) \cap B_1 \setminus B_{1/4}$ onto S_λ is smooth and lies in $S_\lambda \cap B_2 \setminus B_{1/8}$. Ensure $\Lambda_5(\epsilon, \Lambda_4, \mathbf{C}) \leq \Lambda_4$ is sufficiently small, so that $|\lambda| \leq \Lambda_5$ implies $S_\lambda \cap B_2 \setminus B_{1/8} \subset B_{\epsilon/2}(\mathbf{C})$.

Given $x \in B_1 \setminus B_{1/4}$, first assume that $x \notin B_\epsilon(\mathbf{C})$. In this case

$$\epsilon/2 \leq d(x, S_\lambda) \leq 2,$$

and hence we have

$$\int_{(B_1 \setminus B_{1/4}) \setminus B_\epsilon(\mathbf{C})} d^2_{S_{\lambda'}} d\mu_M \leq 4\mu_M(B_1) \leq (16/\epsilon^2) c(\mathbf{C}) \int_{(B_1 \setminus B_{1/4}) \setminus B_\epsilon(\mathbf{C})} d^2_{S_\lambda} d\mu_M.$$

Now assume $x \in B_\epsilon(\mathbf{C})$. Let x' be the nearest point projection to $\mathbf{C}$, and let $u(x') = x - x'$. Since $|v_\lambda| + |\nabla v_\lambda| \leq c(\mathbf{C})|\Lambda_5|^{-\gamma}$ on $B_2 \setminus B_{1/8}$, we have

$$d(x, S_\lambda) = (1 + \psi(\Lambda_5|\mathbf{C}))|x - v_\lambda(x')|.$$

In particular, ensuring $\Lambda_5(\mathbf{C})$ is sufficiently small and using Lemma 2.4, we get

$$\begin{aligned} d(x, S_{\lambda'}) &\leq 2|x - v_{\lambda'}(x')| \\ &\leq 2|x - v_\lambda(x)| + 2|v_{\lambda'}(x) - v_\lambda(x)| \\ &\leq 4d(x, S_\lambda) + c(\mathbf{C})|(\lambda')^{1-\gamma} - \lambda^{1-\gamma}|. \end{aligned}$$

Integrating $d\mu_M$ over $B_1 \cap B_\epsilon(\mathbf{C}) \setminus B_{1/4}$ gives the required result. □

6. L^2-Excess decay

In this section we work towards the following decay theorem.

Proposition 6.1 (Decay Lemma)**.** *Given any $\theta \leq 1/8$, there are positive constants $\delta_6(\mathbf{C}, \theta)$, $\Lambda_6(\mathbf{C}, \theta)$, $c_6(\mathbf{C})$, $\alpha(\mathbf{C})$, so that the following holds: If $|\lambda| \leq \Lambda_6$, and M is a stationary integral n-varifold in B_1, satisfying*

(25)
$$\int_{B_1} d^2_{S_\lambda} d\mu_M \leq E \leq \delta_6^2, \quad \mu_M(B_1) \leq (7/4)\mu_{\mathbf{C}}(B_1), \quad \mu_M(\overline{B_{1/10}}) \geq (1/2)\mu_{\mathbf{C}}(B_{1/10}),$$

then we can find $a \in \mathbb{R}^{n+1}$, $q \in SO(n+1)$, $\lambda' \in \mathbb{R}$, with

$$|a| + |q - Id| + |(\lambda')^{1-\gamma} - \lambda^{1-\gamma}| \leq c_6 E^{1/2}, \tag{26}$$

so that

(27)
$$\theta^{-n-2} \int_{B_\theta(a)} d^2_{a+q(S_{\lambda'})} d\mu_M \leq c_6 \theta^{2\alpha} E, \quad \mu_M(\overline{B_{\theta/10}(a)}) \geq (1/2)\mu_{\mathbf{C}}(B_{1/10}).$$

We first define a general notion of blow-up sequence and show how any blow-up sequence gives rise to a Jacobi field, i.e. a solution of the linearized problem $\mathcal{L}_{\mathbf{C}}(w) = 0$.

Definition 6.1.1. Consider the sequences $a_i \in \mathbb{R}^{n+1}$, $\lambda_i \in \mathbb{R}$, $q_i \in SO(n+1)$, M_i stationary integral varifolds in $B_1 \subset \mathbb{R}^{n+1}$, and $E_i \in \mathbb{R}$. We say the collection $(M_i, E_i, a_i, \lambda_i, q_i)$ is a blow-up sequence if:

(1) $a_i \to 0$, $\lambda_i \to 0$, $q_i \to Id$, $E_i \to 0$
(2) $\mu_{M_i}(B_1) \leq (7/4)\mu_{\mathbf{C}}(B_1)$, $\mu_{M_i}(\overline{B_{1/10}}) \geq (1/2)\mu_{\mathbf{C}}(B_{1/10})$
(3) $\limsup_i E_i^{-1} \int_{B_1} d^2_{a_i+q_i(S_{\lambda_i})} d\mu_{M_i} < \infty$

Proposition 6.2. *Let $(M_i, \beta_i, a_i, \lambda_i, q_i)$ be a blow-up sequence. From Theorem 5.1, there is a sequence of radii $\tau_i \to 0$, so that*

$$M \cap B_{1/2} \setminus B_{\tau_i} = \mathrm{graph}_{a_i+q_i(S_{\lambda_i})}(u_i), \quad \sum_{k=0}^{3} r^{k-1}|\nabla^k u_i| \to 0,$$

and

$$(a_i + q_i(S_{\lambda_i})) \setminus B_{\tau_i} = \mathrm{graph}_{\mathbf{C}}(\phi_i), \quad \sum_{k=0}^{3} r^{k-1}|\nabla^k \phi_i| \to 0.$$

Write $\Phi_i(x) = x + \phi_i(x)\nu_{\mathbf{C}}$ for the graphing function associated to ϕ_i.

There is a subsequence, also denoted i, and a solution $w : \mathbf{C} \cap B_{1/4} \to \mathbb{R}$ to $\mathcal{L}_{\mathbf{C}}(w) = 0$, satisfying the following:

(1) *smooth convergence $E_i^{-1/2} u_i \circ \Phi_i \to w$ on compact subsets of $\mathbf{C} \cap B_{1/4} \setminus \{0\}$;*
(2) *L^∞ decay: for all $r < 1/4$:*

$$w(r\omega)^2 \leq c(\mathbf{C}) r^{2\gamma} \left(\limsup_i E_i^{-1} \int_{B_1} d^2_{a_i+q_i(S_{\lambda_i})} d\mu_{M_i} \right);$$

(3) *strong L^2 convergence:*

$$E_i^{-1} \int_{B_r} d^2_{a_i+q_i(S_{\lambda_i})} d\mu_{M_i} \to \int_{B_r} w^2 d\mu_{\mathbf{C}} \quad \forall r \leq 1/4.$$

Remark 6.3. For shorthand, we will often say $E_i^{-1/2} u_i$ converges smoothly to w to indicate convergence as in Proposition 6.2, conclusion 1.

Proof. Part 1 is a fairly standard argument (see e.g. [10]), and parts 2, 3 follow directly from Theorem 5.1. We outline the argument of part 1. Since $a_i + q_i(S_{\lambda_i})$ converges smoothly to $\mathbf{C}$ on compact subsets of $B_1 \setminus \{0\}$, the coefficients of $\mathcal{M}_{a_i+q_i(S_{\lambda_i})}$, $\mathcal{L}_{a_i+q_i(S_{\lambda_i})}$ converge locally smoothly to those of $\mathcal{M}_{\mathbf{C}}$, $\mathcal{L}_{\mathbf{C}}$. Using this and standard elliptic estimates, we have for any compact $K \subset\subset (a_i + q_i(S_{\lambda_i})) \cap B_{1/2} \setminus \{0\}$, and $l = 0, 1, 2. \ldots,$ uniform estimates of the form

$$\sup_K |\nabla^l u_i| \leq c(K,l) \left(\int_{(a_i+q_i(S_{\lambda_i}))\cap B_{1/2}} u_i^2 d\mathcal{H}^n \right)^{1/2} \tag{28}$$

$$\leq c(K,l) E_i^{1/2} \left(\limsup_i E_i^{-1} \int_{B_r} d^2_{a_i+q_i(S_{\lambda_i})} d\mu_{M_i} \right) \tag{29}$$

Of course by assumption $\Phi_i \to 0$ in $C^\infty_{loc}(\mathbf{C} \cap B_1 \setminus \{0\})$. After passing to a subsequence, we deduce $C^\infty_{loc}(\mathbf{C} \cap B_{1/4} \setminus \{0\})$ convergence of the functions $E_i^{-1/2} u_i \circ \Phi_i$ to some $w \in C^\infty(\mathbf{C} \cap B_{1/4})$.

Now we can write

$$0 = \mathcal{M}_{a_i+q_i(S_{\lambda_i})}(u_i) = \mathcal{L}_i(u_i) + \mathcal{E}_i(u_i),$$

where $\mathcal{L}_i \equiv \mathcal{L}_{a_i+q_i(S_{\lambda_i})}$ converges smoothly away from 0 to the operator $\mathcal{L}_{\mathbf{C}}$, and where

$$\sup_K |\mathcal{E}_{S_{\lambda_i}}(u_i)| = C(K)|u|^2_{C^2(K)} = o(1)E_i^{1/2}.$$

It follows easily that w solves the Jacobi operator $\mathcal{L}_{\mathbf{C}}(w) = 0$. □

Proof of Proposition 6.1. Choose $\alpha(\mathbf{C})$ so that $\gamma_3^+ = 1 < 1 + \alpha \leq \gamma_4^+$. Fix $\theta \leq 1/8$. We first prove the decay estimate. Suppose, towards a contradiction, there are sequences of numbers $\delta_i \to 0$, $\lambda_i \to 0$, $E_i \to 0$, and stationary integral varifolds M_i in B_1, which satisfy:

$$\int_{B_1} d^2_{S_{\lambda_i}} d\mu_{M_i} \leq E_i \leq \delta_i, \quad \mu_{M_i}(B_1) \leq (7/4)\mu_{\mathbf{C}}(B_1), \quad \mu_{M_i}(\overline{B_{1/10}}) \geq (1/2)\mu_{\mathbf{C}}(B_{1/10}),$$

but for which

$$\theta^{-n-2} \int_{B_\theta(a)} d^2_{a'+q'(S_{\lambda'})} d\mu_{M_i} \geq c_6 \theta^{2\alpha} E_i$$

for any $a \in \mathbb{R}^{n+1}$, $q \in SO(n+1)$, $\lambda' \in \mathbb{R}$ satisfying

$$|a| + |q - Id| + |\lambda' - \lambda_i|^{1-\gamma} \leq c_6 E_i^{1/2}.$$

Here $c_6(\mathbf{C})$ will be fixed shortly.

For any sequence τ_i, β_i tending to 0 sufficiently slowly, by Theorem 5.1 we can write

$$\mathrm{spt} M_i \cap B_{1/2} \setminus B_{\tau_i} = \mathrm{graph}_{S_{\lambda_i}}(u_i), \quad \sum_{k=0}^{3} r^{k-1}|\nabla^k u_i| \leq \beta_i.$$

By definition, $(M_i, E_i, 0, \lambda_i, Id)$ is a blow-up sequence, and so by Proposition 6.2 there is a solution $w : \mathbf{C} \cap B_{1/4} \to \mathbb{R}$ to $Lw = 0$ satisfying:

$$\int_{B_{1/4}} w^2 d\mu_{\mathbf{C}} \leq 1, \quad |w(r\omega)| \leq c(\mathbf{C}) r^\gamma, \tag{30}$$

so that, after passing to a subsequence (also denoted i),

$$E_i^{-1/2} u_i \to w$$

smoothly on compact subsets of $\mathbf{C} \cap B_{1/4} \setminus \{0\}$, and

$$E_i^{-1} \int_{B_\rho} d^2_{S_{\lambda_i}} d\mu_{M_i} \to \int_{B_\rho} w^2 d\mu_{\mathbf{C}} \quad \forall \rho \leq 1/4.$$

Using the pointwise bound (30) combined with [12, (2.10) Lemma] to kill the modes γ_i^-, $i \in \mathbb{N}$, and the strongly integrable nature of $\mathbf{C}$, there are $e \in \mathbb{R}$, $b \in \mathbb{R}^{n+1}$, and A a skew-symmetric $(n+1) \times (n+1)$ matrix, so that we can expand w in $L^2(\mathbf{C} \cap B_{1/4})$ as

$$w(r\omega = x) = er^\gamma + \nu_{\mathbf{C}}(r\omega) \cdot (b + Ax) + \sum_{j : \gamma_j^+ \geq 1+\alpha} r^{\gamma_j^+} z_j(\omega)$$

where the sum is $L^2(\Sigma)$-orthogonal for each fixed r. In particular, using the L^2 bound (30) and an appropriate choice of $r \in (1/8, 1/4)$, we get

$$|e| + |b| + |A| \leq c_7(\mathbf{C}),$$

and by Fubini we have

$$\sum_{j : \gamma_j^+ \geq 1+\alpha} \frac{(1/4)^{2\gamma_j^+ + n}}{2\gamma_j^+ + n} \int_\Sigma z_j^2 \leq 1. \tag{31}$$

We first show that, by replacing λ_i with appropriate λ_i', we can arrange so that $e = 0$. Let us define

$$\lambda_i' = (eE_i^{1/2} + \lambda_i^{1-\gamma})^{1/(1-\gamma)}.$$

Trivially $\lambda_i' \to 0$. Corollary 5.2 implies that $(M_i, E_i, 0, \lambda_i', Id)$ is a blow-up sequence also.

Our choice implies that

$$v_{\lambda_i'} - v_{\lambda_i} = (1 + o(1))((\lambda_i')^{1-\gamma} - \lambda_i^{1-\gamma}) r^\gamma = (1 + o(1)) e E_i^{1/2} r^\gamma,$$

where we write $o(1)$ to signify any function which tends to 0 as $i \to \infty$. We can write

$$S_{\lambda_i'} \cap B_1 \setminus B_{\tau_i} = \mathrm{graph}_{S_{\lambda_i}}(v_i),$$

where, setting $x = r\omega + v_{\lambda_i}(r\omega)\nu_{\mathbf{C}}(r\omega)$:

$$|v_i(x) - (v_{\lambda_i'} - v_{\lambda_i})(x)| \leq o(1)|(v_{\lambda_i'} - v_{\lambda_i})(x)|.$$

In other words,

$$v_i(x) = (1 + o(1))eE_i^{1/2}r^\gamma.$$

Write

$$\mathrm{spt}M_i \cap B_{1/2} \setminus B_{\tau_i} = \mathrm{graph}_{S_{\lambda_i'}}(\tilde{u}_i).$$

If we set $y = x + v_i(x)\nu_{S_{\lambda_i}}(x)$ (for x as above), then

$$\tilde{u}_i(y) = (1 + o(1))(u_i(x) - v_i(x)) = (1 + o(1))u_i(x) - (1 + o(1))eE_i^{1/2}r^\gamma.$$

Applying Proposition 6.2 to the blow-up sequence $(M_i, E_i, 0, \lambda_i', Id)$, we deduce that, after passing to a further subsequence, $E_i^{-1/2}\tilde{u}_i$ converges smoothly on compact subsets to

$$w - er^\gamma \equiv \nu_{\mathbf{C}}(r\omega) \cdot (b + Ax) + \sum_{j:\gamma_j^+ \geq 1+\alpha} r^{\gamma_j^+} z_j(\omega),$$

and we have strong L^2 convergence

$$E_i^{-1} \int_{B_\rho} d^2_{S_{\lambda_i'}} d\mu_{M_i} \to \int_{B_\rho} (w - er^\gamma)^2 d\mu_{\mathbf{C}} \quad \forall \rho \leq 1/4.$$

We now show how to pick a_i, q_i to arrange so that $b = 0$, $A = 0$. This more standard, and essentially follows the usual "integrability through rotations" argument.

Choose $a_i = bE_i^{1/2}$ and $A_i = AE_i^{1/2}$, and let $q_i = \exp(A_i)$. It's easy to check that $|a_i| + |q_i - Id| \leq c(\mathbf{C})E_i^{1/2}$. Since

$$d_H(S_{\lambda_i'} \cap B_2, (a_i + q_i(S_{\lambda_i'}) \cap B_2) \leq c(\mathbf{C})E_i^{1/2}$$

it follows that $(M_i, E_i, a_i, \lambda_i', q_i)$ is a blow-up sequence also.

We can write

$$(a_i + q_i(S_{\lambda_i'})) \cap B_1 \setminus B_{\tau_i} = \mathrm{graph}_{S_{\lambda_i'}}(v_i)$$

(for $\tau_i \to 0$ sufficiently slowly) where

$$v_i(x) = (1 + o(1))\nu_{S_{\lambda_i'}}(x) \cdot (a_i + A_i(x)).$$

So now if $\tilde{u}_i$ is the graphing function of M_i over $a_i + q_i(S_{\lambda_i'})$, and u_i is the graphing function of M_i over $S_{\lambda_i'}$, then we have

$$|\tilde{u}_i(x + v_i(x)\nu_{S_{\lambda_i}}(x)) - (u_i(x) - v_i(x))| \leq o(1)|u_i(x) - v_i(x)|.$$

This implies that

$$\begin{aligned}\tilde{u}_i(y) &= (1+o(1))(u_i(x) - v_i(x)) \\ &= (1+o(1))u_i(x) - (1+o(1))\nu_{S_{\lambda_i'}}(x)\cdot(b+A(x))E_i^{1/2}.\end{aligned}$$

Applying Proposition 6.2 to this new blow-up sequence, we deduce that (after passing to a further subsequence) $E_i^{1/2}\tilde{u}_i$ converges smoothly on compact subsets to

$$w - er^\gamma - \nu_{\mathbf{C}}(r\omega)\cdot(b+Ax) \equiv \sum_{j:\gamma_j^+\geq 1+\alpha} r^{\gamma_j^+} z_j(\omega),$$

and we have strong L^2 convergence

$$E_i^{-1}\int_{B_\rho} d^2_{a_i+q_i(S_{\lambda_i'})} d\mu_{M_i} \to \int_{B_\rho} (w - er^\gamma - \nu_{\mathbf{C}}\cdot(b+Ax))^2 d\mu_{\mathbf{C}} \quad \forall \rho \leq 1/4.$$

We've demonstrated that by judiciously choosing our a_i, q_i, λ_i', we can arrange so that

$$w = \sum_{j:\gamma_j^+\geq 1+\alpha} r^{\gamma_j^+} z_j(\omega),$$

where z_j continue to satisfy the bound (31). Using (31), and the fact that $4\theta \leq 1$, we compute:

$$\begin{aligned}\int_{B_{2\theta}} w^2 d\mu_{\mathbf{C}} &= \sum_{j:\gamma_j^+\geq 1+\alpha} \frac{(2\theta)^{2\gamma_j^+ + n}}{2\gamma_j^+ + n}\int_\Sigma z_j^2(\omega)d\omega \\ &\leq \max_{j:\gamma_j^+\geq 1+\alpha} (8\theta)^{2\gamma_j^+ + n} \\ &\leq (8\theta)^{n+2+2\alpha}.\end{aligned}$$

So by the strong L^2 convergence, for sufficiently large i we must have

$$\int_{B_{2\theta}} d^2_{a_i+q_i(S_{\lambda_i'})} d\mu_{M_i} \leq 16^{n+2+2\alpha}\theta^{n+2+2\alpha}E_i,$$

To recenter, we simply observe that for i sufficiently large, we have

$$B_\theta(a_i) \subset B_\theta,$$

and hence we get

$$\theta^{-n-2}\int_{B_\theta(a_i)} d^2_{a_i+q_i(S_{\lambda_i'})} d\mu_{M_i} \leq 32^{n+4}\theta^{2\alpha}E_i,$$

which is a contradiction for sufficiently large i.

Finally let us establish the lower volume bound (27). This is also a straightforward proof by contradiction. Suppose otherwise: there is a sequence $\delta_i \to 0$, $\lambda_i \to 0$, and M_i satisfying the hypotheses (25) , and the decay of (26), (27), but for which

$$\mu_M(\overline{B_{\theta/10}(a_i)}) < (1/2)\mu_{\mathbf{C}}(B_{\theta/10})$$

for all i.

By compactness of stationary varifolds, we can pass to a subsequence (also denoted i) so that $M_i \to M$ in B_1, for some integral stationary n-varifold M in B_1. Since

$$\int_{B_1} d^2_{\mathbf{C}} d\mu_{M_i} \to 0,$$

by the constancy theorem $M = k[\mathbf{C}]$, for some integer k. Since $\mu_M(\overline{B_{1/10}}) \geq (1/2)\mu_{\mathbf{C}}(B_{1/10})$ for all i, we must have $k \geq 1$. Since $\mu_M(B_1) \leq (7/4)\mu_{\mathbf{C}}(B_1)$, we must have $k \leq 1$. So in fact $k = 1$, and we deduce that M_i varifold converge to $[\mathbf{C}]$.

Since $|a_i| \to 0$, for any $0 < \epsilon < 1$ and sufficiently large i we have

$$\mu_{M_i}(\overline{B_{\theta/10}(a_i)}) \geq \mu_{M_i}(B_{(1-\epsilon)\theta/10}) \to \mu_{\mathbf{C}}(B_{(1-\epsilon)\theta/10}) = (1-\epsilon)^n \mu_{\mathbf{C}}(B_{\theta/10}).$$

Choosing $\epsilon(n)$ sufficiently small, so that $(1-\epsilon)^n \geq 3/4$, and we obtain a contradiction for large i. This completes the proof of Proposition 6.1 □

7. Regularity

The key idea to obtain regularity is to iterate Proposition 6.1 at decreasing scales, until λ scale-invariantly becomes too big. This is the radius at which we start to "see" the foliation as separate from the cone, and this is the radius at which we stop. If no such radius exists, we keep iterating until radius 0, to deduce regularity over the cone. Note that, from only the information we start with, we have no way of predetermining how large this radius is.

Proposition 7.1. *There are $\beta(\mathbf{C})$, $\delta_7(\mathbf{C})$, $c_7(\mathbf{C})$ so that the following holds. Take $|\lambda| \leq \Lambda_6$, and let M be a stationary integral varifold in B_1 satisfying*

(32)
$$\int_{B_1} d^2_{S_\lambda} d\mu_M \leq E \leq \delta_7^2, \quad \mu_M(B_1) \leq (3/2)\mu_{\mathbf{C}}(B_1), \quad \mu_M(\overline{B_{1/10}}) \geq (1/2)\mu_{\mathbf{C}}(B_{1/10}).$$

Then there is a $a \in R^{n+1}$, $q \in SO(n+1)$, $\lambda' \in \mathbb{R}$, with

$$|a| + |q - Id| + |(\lambda')^{1-\gamma} - \lambda^{1-\gamma}| \leq c_7 E^{1/2}, \tag{33}$$

so that, for all $1 \geq r > c_7|\lambda'|$*, we have the decay:*

$$r^{-n-2}\int_{B_r(a)} d^2_{a+q(S_{\lambda'})} d\mu_M \leq c_7 r^{2\beta} E, \tag{34}$$

and the volume bounds

$$\mu_M(B_r(a)) \leq (7/4)\mu_{\mathbf{C}}(B_r), \quad \mu_M(\overline{B_{r/10}(a)}) \geq (1/c_7)\mu_{\mathbf{C}}(B_{r/10}). \tag{35}$$

Remark 7.2. In fact one can take β to be anything in the interval $(0, \alpha)$, except of course in this case the various constants δ_7, c_7 will depend on the choice of β also.

Proof. Choose $\theta(\mathbf{C}) \leq 1/4$ sufficiently small so that $c_6\theta^{2\alpha} \leq 1/4$. Set $r_i = \theta^i$. We claim that we can find an integer $I \leq \infty$, and sequences $a_i \in \mathbb{R}^{n+1}$, $q_i \in SO(n+1)$, $\lambda_i \in \mathbb{R}$, $(i = 0, 1, \ldots, I)$, so that for all $i < I$ we have:

(36)

$$a_0 = 0, \quad q_0 = Id, \quad \lambda_0 = \lambda,$$

(37)

$$r_i^{-1}|a_{i+1} - a_i| + |q_{i+1} - q_i| + r_i^{-1}|(\lambda_{i+1})^{1-\gamma} - (\lambda_i)^{1-\gamma}| \leq c_6 2^{-i} E^{1/2},$$

(38)

$$|\lambda_i| \leq \Lambda_6 r_i,$$

and the decay:

$$r_i^{-n-2}\int_{B_{r_i}(a_i)} d^2_{a_i+q_i(S_{\lambda_i})} d\mu_M \leq 4^{-i} E, \tag{39}$$

and the volume bounds:

$$\mu_M(B_{r_i}(a_i)) \leq (7/4)\mu_{\mathbf{C}}(B_{r_i}), \quad \mu_M(\overline{B_{r_i/10}(a_i)}) \geq (1/2)\mu_{\mathbf{C}}(B_{r_i}). \tag{40}$$

Moreover, if $I < \infty$, then

$$|\lambda_I| > \Lambda_6 r_I.$$

Let us prove this by induction. Let us first show how the upper volume bound of (40) follows from (37). If we have $a_0, \ldots, a_i$, satisfying (37), then

$$|a_i| \leq \sum_{j=0}^{i-1} |a_{j+1} - a_j| \leq c_6 E^{1/2} \sum_{j=0}^{i-1} r_i \leq 2c_6\delta_7.$$

Therefore, provided $\delta_7(\mathbf{C})$ is sufficiently small, we have by volume monotonicity

$$\begin{aligned}\mu_M(B_{r_i}(a_i)) &\leq (1-2c_6\delta_7)^{-n}\mu_M(B_1)r_i^n\\ &\leq (1-2c_6\delta_7)^{-n}(3/2)\mu_{\mathbf{C}}(B_1)r_i^n\\ &\leq (7/4)\mu_{\mathbf{C}}(B_{r_i}).\end{aligned}$$

Let us also ensure $\delta_7(\mathbf{C})$ is sufficiently small so that $2c_6\delta_7 < \theta$.

It remains to show the existence of the a_i, q_i, λ_i. Since $|\lambda_0| = |\lambda| \leq \Lambda_6$, we can apply Proposition 6.1 to M to obtain a_1, q_1, λ_1, which satisfy the required estimates. If $|\lambda_1| > \Lambda_6 r_1$, we set $I = 1$ and stop. This proves the base case of our induction.

Suppose, by inductive hypothesis, we have found a_i, q_i, λ_i satisfying (37), (38), (39), (40), and for which $|\lambda_i| \leq \Lambda_6 r_i$. By inductive hypotheses, we can apply Proposition 6.1 to the varifold $M_i = (q_i^{-1})_\sharp(\eta_{a_i,r_i})_\sharp M$ and foliate $S_{r_i^{-1}\lambda_i}$ to obtain $\tilde{a}_{i+1}, \tilde{q}_{i+1}, \tilde{\lambda}_{i+1}$ satisfying (26), (27). If we let

$$a_{i+1} = q_i(r_i\tilde{a}_{i+1}) + a_i, \quad q_{i+1} = q_i \circ \tilde{q}_{i+1}, \quad \lambda_{i+1} = r_i\tilde{\lambda}_{i+1},$$

then it follows by scaling that this $a_{i+1}, q_{i+1}, \lambda_{i+1}$ satisfies the requirement estimates. If $|\lambda_{i+1}| > \Lambda_6 r_{i+1}$, we stop and set $I = i+1$. Otherwise, continue. By mathematical induction this proves the existence of the sequence.

If $I < \infty$, then let $a = a_I$, $q = q_I$, and $\lambda' = \lambda_I$. Otherwise, observe that (37) imply that a_i, q_i form a Cauchy sequence, and hence we take $a = \lim_i a_i$, $q = \lim_i q_i$, $\lambda' = 0$.

From (37), we have for every $i < I$:

$$r_i^{-1}|a - a_i| + |q - q_i| + r_i^{-1}|(\lambda')^{1-\gamma} - (\lambda_i)^{1-\gamma}| \leq 2c_6 2^{-i}E^{1/2}.$$

In particular, taking $i = 0$ gives (33).

Given any $r_I \leq r < 1$, choose integer $i \leq I$ so that $r_{i+1} \leq r < r_i$. Then, ensuring that $c_6\delta_7 << \theta$, and using Corollary 5.2, we have

$$\begin{aligned}&r^{-n-2}\int_{B_r(a)} d^2_{a+q(S_{\lambda'})}d\mu_M\\ &\leq cr_i^{-2}|a_{i+1} - a_i|^2 + c|q_{i+1} - q_i|^2 + cr_i^{-2}((\lambda')^{1-\gamma} - (\lambda_i)^{1-\gamma})^2\\ &\quad + cr_i^{-n-2}\int_{B_{r_i}(a_i)} d^2_{a_i+q_i(S_{\lambda_i})}d\mu_M\\ &\leq c(\mathbf{C},\theta)4^{-i}E\\ &\leq c(\mathbf{C},\theta)r^{2\beta}E. \qquad (41)\end{aligned}$$

where $\beta = \log(1/2)/\log(\theta) > 0$. Finally, observe that (37), (38) imply

$$\Lambda_6 r_I < |\lambda'| \leq \Lambda_6 r_{I-1} + (c_6\delta_7)^{1/(1-\gamma)} r_{I-1} \leq c(\mathbf{C}) r_I.$$

Therefore, up to changing enlarging our constant c, we can take $r \geq \lambda'$ in (41). The volume bounds follow directly from (40), (37), and monotonicity. This finishes the proof of Proposition 7.1. □

The proof of Theorem 3.1 is now essentially a straightforward application of Allard's theorem.

Proof of Theorem 3.1. Ensure $\delta_1 \leq \delta_7$, and take $\Lambda_1 = \Lambda_6$. Apply Proposition 7.1 to obtain a, q, λ'. For every $c_7|\lambda'| \leq r \leq 1$, a straightforward contradiction argument as in the proof of Theorem 5.1, implies that

$$\mathrm{spt} M \cap B_{r/2}(a) \setminus B_{r/100}(a) = \mathrm{graph}_{a+q(S_{\lambda'})}(u), \quad r^{-1}|u| + |\nabla u| + [\nabla u]_{\beta,r} \leq c(\mathbf{C}) r^\beta E^{1/2},$$

for some $u : (a + q(S_{\lambda'})) \cap B_{r/2}(a) \setminus B_{r/200}(a) \to \mathbb{R}$. If $\lambda' = 0$ we are done.

Suppose $\lambda' \neq 0$. After scaling up by $|\lambda'|$, it suffices to show that there is a $u : (a + q_{\mathrm{sign}(\lambda')}) \cap B_{c_7/2}(a) \to \mathbb{R}$ so that

$$\mathrm{spt} M \cap B_{c_7/2}(a) = \mathrm{graph}_{a+q(S_{\mathrm{sign}(\lambda')})}(u), \quad |u| + |\nabla u| + [\nabla u]_\beta \leq c(\mathbf{C}) E^{1/2},$$

provided

$$c_7^{-n-2} \int_{B_{c_7}(a)} d^2_{a+q(S_{\mathrm{sign}(\lambda')})} d\mu_M \leq c_7 E$$

and

$$\mu_M(B_{c_7}(a)) \leq (7/4)\mu_{\mathbf{C}}(B_{c_7}), \quad \mu_M(\overline{B_{c_7/10}(a)}) \geq (1/c_7)\mu_{\mathbf{C}}(B_{c_7/10}).$$

However this is also an easy contradiction argument, taking $E \leq \delta_1$ to zero. □

8. Mean curvature

When M has mean curvature, then we cannot use the maximum principle to conclude as in Proposition 6.1. However, provided the mean curvature is sufficiently small and L^2 excess sufficiently large, then we can still get decay to one scale (Proposition 8.1), by a straightforward contradiction argument, which suffices to characterize the singular nature, if not the precise local structure.

Iterating this gives scale-invariant smallness of the excess (Proposition 8.2), rather than decay. At each scale we can deduce closeness to some rotate/translate of the cone $\mathbf{C}$ (and hence foliate S_λ), but we cannot deduce that the rotations form a Cauchy sequence as we progress in scale. If at some scale λ becomes big, we stop, and we can deduce that $M \cap B_{1/2}$ is a $C^{1,\alpha}$ deformation of S_1 (*without* effective estimates). If we continue all the way to scale 0, then we deduce that $M \cap B_{1/2}$ has an isolated singularity modeled on $\mathbf{C}$, and a posteriori by [2] we can deduce $M \cap B_{1/2}$ is a $C^{1,\alpha}$ graph over $\mathbf{C}$.

Proposition 8.1. *For any $\theta \in (0, 1/4)$, $E_0 > 0$, there is a $\epsilon_8(\mathbf{C}, \theta, E_0)$ so that the following holds. Assume that M is an integral n-varifold in B_1, satisfying $||\delta M||(B_1) \leq \epsilon_8$, and for some $E_0 \leq E$, $|\lambda| \leq \Lambda_6$ we have*

(42)
$$\int_{B_1} d^2_{S_\lambda} d\mu_M \leq E \leq \delta_6^2, \quad \mu_M(B_1) \leq (7/4)\mu_{\mathbf{C}}(B_1), \quad \mu_M(\overline{B_{1/10}}) \geq (1/2)\mu_{\mathbf{C}}(B_{1/10}).$$

Then there is an $a \in \mathbb{R}^{n+1}$, $q \in SO(n+1)$, $\lambda' \in \mathbb{R}$, satisfying

$$|a| + |q - Id| + |(\lambda')^{1-\gamma} - \lambda^{1-\gamma}| \leq c_6 E^{1/2}, \tag{43}$$

so that

(44)
$$\theta^{-n-2} \int_{B_\theta(a)} d^2_{a+q(S_{\lambda'})} d\mu_M \leq c_6 \theta^{2\alpha} E, \quad \mu_M(\overline{B_{\theta/10}(a)}) \geq (1/2)\mu_{\mathbf{C}}(B_{\theta/10}).$$

Here $\delta_6(\mathbf{C}, \theta)$, $c_6(\mathbf{C})$ are the constants from Theorem 6.1.

Proof. Suppose, towards a contradiction, there is a sequence of integral n-varifolds M_i, and numbers $\epsilon_i \to 0$, $E_i \in [E_0, \delta_6^2]$, so that M_i satisfy (42) and $||\delta M_i||(B_1) \leq \epsilon_i$, but for which (44) fails for all a, q, λ' satisfying (43).

We can pass to a subsequence, also denoted i, and obtain varifold converge $M_i \to M$, and convergence $E_i \to E \in [E_0, \delta_6^2]$. The resulting n-varifold M is stationary in B_1, continues to satisfy (42), but fails (44) for all a, q, λ' satisfying (43). However, M satisfies the hypotheses of Proposition 6.1, contradicting the conclusions of Proposition 6.1. This proves Proposition 8.1. □

Proposition 8.2. *There are constants $\delta_9(\mathbf{C})$, $c_9(\mathbf{C})$ which yield the following. Given $E \in (0, \delta_9^2]$ and $p > n$, we can find an $\epsilon_9(\mathbf{C}, E, p)$ so that: If $|\lambda| \leq \Lambda_6$, and M is an integral n-varifold in B_1 with generalized mean-curvature H_M, with zero generalized boundary, and which satisfies*

$$\int_{B_1} d^2_{S_\lambda} d\mu_M \leq E, \quad \mu_M(B_1) \leq (3/2)\mu_{\mathbf{C}}(B_1), \quad \mu_M(\overline{B_{1/10}}) \geq (1/2)\mu_{\mathbf{C}}(B_{1/10}),$$

and

$$\left(\int_{B_1} |H_M|^p d\mu_M \right)^{1/p} \leq \epsilon_9.$$

Then there is an $a \in \mathbb{R}^{n+1}$, $\lambda' \in \mathbb{R}$, *and for each* $1 \geq r \geq c_9|\lambda'|$ *there is a* $q_r \in SO(n+1)$, *which satisfy*

$$|a| + |\log(r)|^{-1}|q_r - Id| + |(\lambda')^{1-\gamma} - \lambda^{1-\gamma}| \leq c_9 E^{1/2},$$

so that for all $1 \geq r \geq c_9|\lambda'|$ *we have the smallness:*

$$r^{-n-2} \int_{B_r(a)} d^2_{a+q_r(S_{\lambda'})} d\mu_M \leq c_9 E,$$

and volume bounds

$$\mu_M(B_r(a)) \leq (7/4)\mu_{\mathbf{C}}(B_r), \quad \mu_M(\overline{B_{r/10}(a)}) \geq (1/c_9)\mu_{\mathbf{C}}(B_{r/10}).$$

Proof. The proof is almost verbatim to Proposition 7.1, except we use Proposition 8.1 in place of Proposition 6.1. Notice that Proposition 8.1 requires a lower bound on E, and so we cannot deduce decay of the L^2 excess, only smallness. Choose $\theta(\mathbf{C})$ as in Proposition 8.1, and set $r_i = \theta^i$. We claim that we can find an integer $I \leq \infty$, and sequences $a_i \in \mathbb{R}^{n+1}$, $q_i \in SO(n+1)$, $\lambda_i \in \mathbb{R}$ $(i = 0, \ldots, I)$ so that for all $i < I$ we have:

(45) $a_0 = 0, \quad q_0 = Id, \quad \lambda_0 = 0,$

(46) $r_i^{-1}|a_{i+1} - a_i| + |q_{i+1} - q_i| + r_i^{-1}|(\lambda_{i+1})^{1-\gamma} - (\lambda_i)^{1-\gamma}| \leq c_6 E^{1/2},$

(47) $|\lambda_i| \leq \Lambda_6 r_i,$

and the smallness:

(48)
$$r_i^{-n-2} \int_{B_{r_i}(a_i)} d^2_{a_i+q_i(S_{\lambda_i})} d\mu_M \leq E, \quad r_i^{1-n/p} \left(\int_{B_{r_i}(a_i)} |H_M|^p d\mu_M \right)^{1/p} \leq \epsilon_9,$$

and the volume bounds

(49) $\mu_M(B_{r_i}(a_i)) \leq (7/4)\mu_{\mathbf{C}}(B_{r_i}), \quad \mu_M(\overline{B_{r_i/10}(a_i)}) \geq (1/2)\mu_{\mathbf{C}}(B_{r_i}).$

We proceed by induction. Ensure $\delta_9 \leq \delta_7(\mathbf{C})$. As in the proof of Proposition 7.1, given $a_0, \ldots, a_i$ satisfying (46), we have $|a_i| \leq 2c_6\delta_9$, and hence by volume monotonicity (1)

$$\mu_M(B_{r_i}(a_i)) \leq (1 + c(p, \mathbf{C})\epsilon_9)(1 - 2c_6\delta_9)^{-n}\mu_M(B_1)r_i^n \leq (7/4)\mu_{\mathbf{C}}(B_{r_i})$$

provided $\delta_9(\mathbf{C})$ and $\epsilon_9(\mathbf{C}, p)$ are sufficiently small. This proves the upper volume bounds (49). The mean curvature bound in (48) follows trivially from $p > n$.

To obtain the a_i, q_i, λ_i, we first observe that, given any a_i, q_i above, if we set $M_i = (q_i^{-1})_\sharp(\eta_{a_i,r_i})_\sharp M$, then

$$\begin{aligned} ||\delta M_i||(B_1) &= r_i^{-n+1}\int_{B_{r_i}(a_i)} |H_M| d\mu_M \\ &\leq r_i^{-n+1}\left(\int_{B_{r_i}(a_i)} |H_M|^p d\mu_M\right)^{1/p} \mu_M(B_{r_i}(a_i))^{1-1/p} \\ &\leq c(\mathbf{C})\epsilon_9. \end{aligned}$$

In particular, ensuring $\epsilon_9(\mathbf{C}, p)$ is sufficiently small, we can ensure that $||\delta M_i||(B_1) \leq \epsilon_8(\mathbf{C}, p)$. We can therefore proceed like the proof of Proposition 7.1, using Proposition 8.1 in place of Proposition 6.1. □

Proof of Corollary 3.4. Ensuring $\delta_3 < \delta_9$, $\epsilon_3 < \epsilon_9(\delta, p)$, then we can obtain a, λ' and q_r as in Proposition 8.1. A straightforward argument by contradiction, like in the proof of Theorem 3.1, shows that for every $1 \geq r \geq c_9|\lambda'|$,

$$\mathrm{spt} M \cap B_{r/2}(a) \setminus B_{r/100}(a) = \mathrm{graph}_{a+q_r(S_{\lambda'})}(u), \quad r^{-1}|u| + |\nabla u| + [\nabla u]_{\beta,r} \leq c(\mathbf{C})\delta,$$

(provided δ_3, ϵ_3 are sufficiently small, depending on $\mathbf{C}$, p). If $\lambda' = 0$, this shows that any tangent cone at a is rotation of $\mathbf{C}$. Therefore, shrinking δ_3, ϵ_3 as necessary, [2] implies $\mathrm{spt} M \cap B_{1/2}$ is a $C^{1,\beta}$ perturbation of $\mathbf{C}$.

If $\lambda' \neq 0$, then arguing as in the proof of Theorem 3.1, we get that

$$\mathrm{spt} M \cap B_{c_9|\lambda'|/2}(a) = \mathrm{graph}_{a+q_{c_9|\lambda'|}(S_{\lambda'})}(u), \quad r^{-1}|u| + |\nabla u| + [\nabla u]_\beta \leq c(\mathbf{C})\delta.$$

In particular, $\mathrm{spt} M \cap B_{1/2}$ is regular. □

9. Corollaries, related results

In this section we give an alternate proof of [12, Theorem 0.3] (i.e., Corollary 3.7 of this paper) using our main regularity Theorem 3.1. We also prove, using results of [5], a uniqueness result for minimal graphs over $\mathbf{C}$ (Proposition 3.3).

Proof of Corollary 3.7. Let

$$E_i = R_i^{-n-2}\int_{B_{R_i}} d_{\mathbf{C}}^2 d\mu_M$$

By hypothesis, $E_i \to 0$. For all i sufficiently large, we can apply Theorem 3.1 to deduce the existence of $a_i \in \mathbb{R}^{n+1}, q_i \in SO(n+1), \lambda_i \in \mathbb{R}$, and $u_i : (a_i + q_i(S_{\lambda_i})) \cap B_{R_i}(a_i) \to \mathbb{R}$, so that for all $c_7|\lambda_i| \leq r \leq R_i$,

$$\mathrm{spt} M \cap B_r(a_i) = \mathrm{graph}_{a_i+q_i(S_{\lambda_i})}(u_i), \quad r^{-1}|u_i| + |\nabla u_i| \leq c(\mathbf{C})E_i^{1/2}. \tag{50}$$

Suppose, towards a contradiction, that $a_i \to \infty$. Let $U_i(x) = x + u_i(x)\nu_{a_i+q_i(S_{\lambda_i})}(x)$ be the graphing function associated to u_i. From (50) we have that $|U_i(x) - x| \leq o(1)|x - a_i|$.

Fix any $\rho > 0$, then by the previous paragraph we have

$$U_i^{-1}(\mathrm{spt}M \cap B_\rho) \subset (a_i + q_i(S_{\lambda_i})) \cap B_{2|a_i|}(a_i) \setminus B_{|a_i|/2}(a_i)$$

for i sufficiently large. Now the curvature of $(a_i + q_i(S_{\lambda_i})) \cap B_{2|a_i|}(a_i) \setminus B_{|a_i|/2}(a_i)$ tends to zero uniformly as $i \to \infty$, and $|\nabla u_i| = o(1)$, and so we must have that $\mathrm{spt}M \cap B_\rho$ is contained in a plane. Taking $\rho \to \infty$, we deduce $\mathrm{spt}M$ is planar, and hence $\mathbf{C}$ is planar also. This is a contradiction, and so a_i must be bounded.

By (50), we have

$$d(0, \mathrm{spt}M) = d(0, a_i + q_i(S_{\lambda_i})) + o(1)|a_i| \geq \frac{1}{c(\mathbf{C})}|\lambda_i| - c(\mathbf{C})|a_i|$$

for i large. Since a_i is bounded, we must have that λ_i is bounded also. We can pass to a subsequence, also denoted i, so that $a_i \to 0$, $\lambda_i \to \lambda$, and $q_i \to q \in SO(n+1)$. We get smooth convergence on compact subsets of $\mathbb{R}^{n+1} \setminus \{a\}$

$$a_i + q_i(S_{\lambda_i}) \to a + q(S_\lambda),$$

and C^1_{loc} convergence $u_i \to 0$. We deduce that $\mathrm{spt}M = a + q(S_\lambda)$, which is the desired conclusion. □

9.1. **Uniqueness of graphs over C.** [5] prove the following theorem constructing a plethora of examples of minimal surfaces in B_1, which are perturbations of a given minimal cone. To state their theorem properly we need a little notation. Given $J \in \mathbb{N}$, define the projection mapping $\Pi_J : L^2(\Sigma) \to L^2(\Sigma)$

$$\Pi_J(g)(r\omega) = \sum_{j \geq J+1} < g, \phi_j >_{L^2(\Sigma)} r^{\gamma_j^+} \phi_j(\omega).$$

For shorthand write $\mathbf{C}_1 = \mathbf{C} \cap \overline{B_1}$.

Theorem 9.2 ([5])**.** *Take $m > 1$, $\alpha \in (0,1)$, and $J \in \mathbb{N}$ so that $\gamma_J^+ \leq m < \gamma_{J+1}^+$. There are $\epsilon(\mathbf{C}, m, \alpha)$, $\Lambda(\mathbf{C}, m, \alpha)$ so that given any $g \in C^{2,\alpha}$ satisfying $|g|_{2,\alpha} \leq \epsilon$, and any $\lambda \in (0, \Lambda)$, then there is a solution $u_\lambda \in C^{2,\alpha}(\mathbf{C}_1)$ to the problem*

$$M_{\mathbf{C}}(u) = 0 \text{ on } \mathbf{C}_1, \quad \Pi_J(u_\lambda) = \lambda \Pi_J(g) \text{ on } \Sigma,$$

satisfying

$$r^{-m}|u|_{2,\alpha,r} \leq c(\mathbf{C}, m, \alpha)\lambda|g|_{2,\alpha} \quad \forall 0 < r \leq 1.$$

Loosely speaking, [5] are solving a boundary-value-type problem, where one is allowed to specify the decay rate at $r = 0$, and the Fourier modes at $r = 1$ which decay *faster* than the prescribed rate. Though they do not comment on it, implicit in their work is the uniqueness statement of Proposition 3.3. The basic idea is that solutions to the minimal surface operator can be written as fixed points to a contraction mapping, provided the solutions decay like $r^{1+\epsilon}$, and have boundary data sufficiently small. We illustrate this below.

Proof of Proposition 3.3. We use the notation of Theorem 3.1. Fix an $\alpha \in (0, 1)$, and given $w \in C^{2,\alpha}(\mathbf{C}_1)$, define the norm

$$|w|_B = \sup_{0<r\leq 1} r^{-\beta}|u|_{2,\alpha,r}. \tag{51}$$

Pick J so that $\gamma_J^+ = 1$, and recall that $1 + \beta \in (\gamma_J^+, \gamma_{J+1}^+)$.

By assumption, we have $\mathrm{spt}M \cap B_{1/2} = \mathrm{graph}_{U\subset a+q(\mathbf{C})}(u)$, where u satisfies the estimates (6). Since we are concerned with $M \cap B_{1/4}$, ensuring $\delta_2(\mathbf{C})$ is small, there is no loss in assuming (after scaling, translating, rotating) that $\mathrm{spt}M = \mathrm{graph}_{\mathbf{C}_1}(u)$. By standard interior elliptic estimates and decay (6), we can assume that u is smooth, and satisfies

$$|u|_B \leq c(\mathbf{C}, \alpha)\delta_2.$$

We can write the mean curvature operator as

$$\mathcal{M}_{\mathbf{C}}(u) = \mathcal{L}_{\mathbf{C}}(u) + \mathcal{E}(u),$$

where the non-linear error part $\mathcal{E}(u)$ satisfies certain, relatively standard scale-invariant structure conditions (see [5]). Given $g \in C^{2,\alpha}(\Sigma)$, [5] show there are numbers $\epsilon_{10} < \delta_{10}$, depending only on $\mathbf{C}, \alpha$, so that provided $|g|_{2,\alpha} < \epsilon_{10}$, then for every w in the convex space

$$\mathcal{B} := \{w \in C^{2,\alpha}(\mathbf{C}_1) : \quad \Pi_J(w_{r=1}) = \Pi_J(g), \quad |w|_B \leq \delta_{10}\},$$

there is a unique solution $v =: \mathcal{U}(w) \in \mathcal{B}$ to the linear problem

$$\mathcal{L}_{\mathbf{C}}(v) = -\mathcal{E}(w) \text{ on } \mathbf{C}_1, \quad \Pi_J(v_{r=1}) = \Pi_J(g) \text{ on } \Sigma, \quad \sup_{0<r\leq 1} r^{-\beta} \left(\int_\Sigma v(r\omega)^2 d\omega \right)^{1/2} < \infty,$$

and *moreover*, that $\mathcal{U}$ is a contraction mapping on $\mathcal{B}$.

To prove Proposition 3.3 it therefore suffices to show that, with $g = u|_{r=1}$, then $u \in \mathcal{B}$ and $|g|_{2,\alpha} \leq \epsilon_{10}$. However, both of these follow from (51) by ensuring $\delta_2(\mathbf{C}, \alpha)$ is sufficiently small. $\square$

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