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A Relaxation Result in the Vectorial Setting and Power Law Approximation for Supremal Functionals.

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Abstract We provide relaxation for not lower semicontinuous supremal functionals defined on vectorial Lipschitz functions, where the Borel level convex density depends only on the gradient. The connection with indicator functionals is also enlightened, thus extending previous lower semicontinuity results in that framework. Finally we discuss the power law approximation of supremal functionals, with non-negative, coercive densities having explicit dependence also on the spatial variable, and satisfying minimal measurability assumptions.

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1 Introduction

Recently, a great attention has been devoted to supremal functionals, and to their connections with partial differential equatons such as ∞-harmonic, ∞-biharmonic equations or Hamilton-Jacobi ones, also in light of the many applications to optimal transport, continuum mechanics, see for instance [1],[2], [3], [4], [5], [6], [7], [8],[9] among a wider literature. Many of the above questions can be formulated in terms of suitable minimization problems involving supremal functionals, and the direct methods have been proven to be a powerful tool to provide solutions. A crucial property to ensure the existence of minimizers is the lower semicontinuity of such functionals. This in turn reflects in necessary and sufficient conditions on their densities. Such analysis started in the scalar case in [10] and [11], and later extended in [12],[13], [14] and so far a complete characterization is given: a supremal functional F is weakly* sequentially lower semicontinuous if and only if its density f is lower semicontinuous and level convex.

When the problem is truly vectorial, lower semicontinuity and level convexity of the supremand f are just sufficient conditions but no longer necessary. The notion, which has been proven to be necessary and sufficient for weak* sequential lower semicontinuity of F in the space of Lipschitz functions, is *strong Morrey quasiconvexity*, introduced by [12]. This notion is quite difficult to be verified in practice and stronger notions (but weaker than level convexity) have been introduced in order to ensure lower semicontinuity to supremal functionals or to approximate them through integral functionals (see [15], [12], [16]).

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Clearly, if such conditions fail to be satisfied by the supremand f, one has to look for the best weak* lower semicontinuous functional $\Gamma_{w^*}(F)$, which approximates F in the sense of admitting the same minimal values (see Theorem 3.1). The results available in literature are very satisfactory and quite exhaustive in the scalar case when F satisfies a coercivity assumption: in this case, $\Gamma_{w^*}(F)$ admits a sequential characterization. In [10] and [14] a complete representation formula for the relaxed functional $\Gamma_{w^*}(F)$ is given when $f = f(x, \xi)$ is a globally continuous function; in [17] the authors discuss the finslerian case and represent the relaxed functional as a difference quotient; in [14] it is shown that $\Gamma_{w*}(F)$ is level convex when f is a Carathéodory function, even in the second variable. In [18] the last assumption is dropped and the level convexity of the relaxed functional is proved for a class of discontinuos supremand, not even coercive. Despite of all these scalar results, very little is known in the vectorial setting, up to some sufficient conditions and in particular cases (see [19], [15], [12], [16]).

The first aim of this paper consists in providing a relaxation result for a class of supremal functionals in the vectorial case. Indeed in Theorem 2.1 we compute the lower semicontinuous envelope of F with respect to the weak* topology when the supremand $f = f(\xi)$ is level convex, only Borel measurable, and whose sublevel sets have not empty interior. We remark that, in general, level convex functions are not lower semicontinuous (see [20]). Our first main result (which is new also in the scalar case) states that the lower semicontinuous envelope of F is a supremal functional whose density f^{ls} is the lower semicontinuous envelope of f.

In order to prove Theorem 2.1, a key tool is the description of the level sets of the envelopes of the densities f, that is accomplished in Section 3. Indeed, after providing in Proposition 3.3 a characterization of the level convex envelope of functions defined in generical vector spaces, we specialize the result, giving a complete representation formula of the sublevel sets of level convex and lower semicontinuous envelope of f, in terms of closures and convexifications of the sublevel sets of f (see Proposition 3.9). For computational counterpart in the continuous and bounded case we refer to [21] while in the nonlocal setting formulas analogous to (18) can be found in [22].
We also underline that, despite of the results currently available in the literature, in the set of hypothesis of Theorem 2.1 we drop any coercivity assumptions on f thanks to arguments as in [18, Theorem 3.4]. On the other hand, the proof of representation formula (3) is given under homogeneity assumptions on the density f since it relies on a particular case of [23, Theorem 2.1] (see Theorem 4.1 below). Indeed a central role plays the connection with homogeneous indicator functionals of convex sets with nonempty interior, as already emphasized in similar context by [12], later exploited in [24], [5] and very recently in [22], [25] in the nonlocal framework. In turn, Theorem 2.1 allows us to generalize some relaxation results for indicator functionals or, equivalently, improves the understanding of the asymptotics for vectorial differential inclusions (cf. Corollary 4.2 below). The interest in this type of functionals is motivated by the many applications: we refer to [26] and the references therein for the scalar case, to [27], [28] for multidimensional control problems, to [29] for homogenization, to [30], [31], [32] for the analysis of thin structures, and to [33], and the bibliography contained therein for the applications in continuum mechanics.

Motivated by the connection with PDEs and norm approximation, in Theorem 2.2 we prove an L^p-approximation result which applies to a more general class of densities depending also on the spatial variables. In this way, we generalize [34, Theorem 3.2], since, under the same growth conditions, we just require measurability for f. Note that our power law approximation result is new in literature since the other L^p-approximation results suppose that f is lower semicontinuous with respect to the gradient variable. Indeed Theorem 3.2 in [34] requires that the density is a Carathéodory function satisfying a growth condition with respect to the second variable (uniformly with respect to spatial variable) of the type (5); anagously Theorem 3.1 in [16] applies when the density is lower semicontinuous w.r.t the second variable.

The paper is organized as follows: in Section 2 we establish notation and state the main results, Section 3 is devoted to preliminaries that will be exploited in the sequel and contains some results of broader scope on explicit representation of envelopes of functions and their effective domains, thus generalizing the results in [20, Section 2], (cf. [26] for their counterparts in the convex setting). Theorem 2.1 is proven in Section 4, together with an integral representation result for the relaxation of unbounded integral functionals (see Corollary 4.2). Finally in Section 5 we provide the proof of Theorem 2.2, and discuss particular cases and special representations in Remark 5.2.

2 Notation and Statement of Main Results

The following notation is adopted in the paper.

- (X, τ) denotes a topological (possibly vector) space whose generic elements will be denoted by x;
- for every $Y \subset X$, by $\mathrm{cl}_\tau(Y)$ we mean the closure of Y in X with respect to the topology τ. When X is an Euclidean space and τ is the natural topology, we adopt just the symbol $\bar{Y}$;
- for every set $S \subset X$ we denote by $\mathrm{conv}S$ its convex hull, namely the smallest convex set containing S. It is easily seen that $\mathrm{cl}_\tau(\mathrm{conv}S) = \mathrm{conv}(\mathrm{cl}_\tau(S))$;
- for every function $W : X \to [-\infty, +\infty]$, $\mathrm{dom}W$ denotes its effective domain, i.e.

$$\mathrm{dom}W := \{x \in X : W(x) < +\infty\},$$

  and for every $\lambda \in \mathbb{R}$, $L_\lambda(W)$,

$$L_\lambda(W) := \{x \in X : W(x) \leq \lambda\}$$

  is the sublevel set of W corresponding to λ;
- for every $N \in \mathbb{N}$, $\mathcal{B}_N$ and $\mathcal{L}^N$ denote the Borel measure in $\mathbb{R}^N$, and the Lebesgue one, respectively;
- w^* denotes the weak* topology on $W^{1,\infty}(\Omega, \mathbb{R}^d)$, unless differently stated.
- w^*_{seq} denotes the weak* sequential topology on $W^{1,\infty}(\Omega, \mathbb{R}^d)$.

We are now in position to state our main result concerning the supremal representation of the relaxation of an L^∞ functional.

Theorem 2.1 *Let Ω be a bounded open set of $\mathbb{R}^N$ with Lipschitz boundary and let $f : \mathbb{R}^{d\times N} \to [-\infty, +\infty]$ be a Borel function such that*

(H) for every $\lambda > \inf_{\mathbb{R}^{d\times N}} f$ the sublevel set, defined as

$$L_\lambda(f) := \{\xi \in \mathbb{R}^{d\times N} : f(\xi) \leq \lambda\}, \tag{1}$$

is convex and has nonempty interior.

Let $F : W^{1,\infty}(\Omega, \mathbb{R}^d) \to [-\infty, +\infty]$ be the supremal functional defined by

$$F(u) := \operatorname*{ess\,sup}_{x\in\Omega} f(\nabla u(x)). \tag{2}$$

Then it holds

$$\Gamma_{w*}(F)(u) = \Gamma_{w^*_{seq}}(F)(u) = \operatorname*{ess\,sup}_{x\in\Omega} f^{ls}(\nabla u(x)) \quad \text{for every } u \in W^{1,\infty}(\Omega, \mathbb{R}^d), \tag{3}$$

where f^{ls} denotes the lower semicontinuous envelope of f.

Note that assumption (H) is satisfied by a wide class of discontinuous functions. For instance, it is satisfied by Borel level convex functions f having an absolute minimum point $\bar{\xi}$ such that $f(\bar{\xi}) = \lim_{\xi\to\bar{\xi}} f(\xi)$ (see Remark 4.1). Moreover in Corollary 4.1 we show that if f satisfies (H) then f^{ls} is the greatest strong Morrey quasiconvex function less than or equal to f. To this end we recall that a Borel measurable function $f : \mathbb{R}^{d\times N} \to \mathbb{R}$ is said to be *strong Morrey quasiconvex* if for any $\varepsilon > 0$, for any $\xi \in \mathbb{R}^{d\times N}$, and for any $K > 0$, there exists a $\delta = \delta(\varepsilon, K, \xi) > 0$ such that if $\varphi \in W^{1,\infty}(Q; \mathbb{R}^d)$ satisfies $\|\nabla\varphi\|_{L^\infty(Q)} \leq K$, and $\max_{x\in\partial Q} |\varphi(x)| \leq \delta$, then

$$f(\xi) \leq \operatorname*{ess\,sup}_{x\in Q} f(\xi + \nabla\varphi(x)) + \varepsilon, \tag{4}$$

where Q denotes the cube $]0, 1[^N$.

Note that the class of Borel level convex functions is strictly contained in the class of Borel functions f (called *weak Morrey quasiconvex*) satisfying

$$f(\xi) \leq \operatorname*{ess\,sup}_{x\in Q} f(\xi + \nabla\varphi(x)), \quad \forall \varphi \in W^{1,\infty}_0(Q; \mathbb{R}^d).$$

Differently from strong Morrey quasiconvex functions which are lower semicontinuous (see [12, Proposition 2.5]), weak Morrey quasiconvex functions do not necessarily satisfy this property. On the other hand, the representation result by means of f^{ls} does not hold if we weaken the level convexity assumption on f, by requiring that f is only weak Morrey quasiconvex. Indeed, [34, Example 2.7] exhibits a weak Morrey quasiconvex function $f = f^{ls}$ that cannot represent the relaxed functional since it is not strong Morrey quasiconvex.

Our second main result deals with the following L^p-approximation.

Theorem 2.2 *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set with Lipschitz boundary. Let $f : \Omega \times \mathbb{R}^{d\times N} \to [0, +\infty[$ be a $\mathcal{L}^N \otimes \mathcal{B}_{d\times N}$-measurable function satisfying the following growth condition: there exist $\beta \geq \alpha > 0$ such that*

$$\frac{1}{\alpha}|\xi| - \alpha \leq f(x, \xi) \leq \beta(1 + |\xi|) \qquad \text{for a. e. } x \in \Omega \text{ and for every } \xi \in \mathbb{R}^{d\times N}. \tag{5}$$

For every $p \geq 1$ let $F_p : C(\bar{\Omega}, \mathbb{R}^d) \to [0, +\infty[$ be the functional given by

$$F_p(u) := \begin{cases} \left(\int_\Omega f^p(x, \nabla u(x))dx\right)^{1/p}, & \text{if } u \in W^{1,p}(\Omega, \mathbb{R}^d), \\ +\infty, & \text{otherwise.} \end{cases} \tag{6}$$

Then there exists a $\mathcal{L}^N \otimes \mathcal{B}_{d\times N}$-measurable function $f_\infty : \Omega \times \mathbb{R}^{d\times N} \to [0, +\infty[$, satisfying the same growth condition in (5), and such that $(F_p)_{p\geq 1}$ $\Gamma(L^\infty)$-converges, as $p \to \infty$, to the functional $\bar{F} : C(\bar{\Omega}, \mathbb{R}^d) \to [-\infty, +\infty]$ defined as

$$\bar{F}(u) := \begin{cases} \operatorname*{ess\,sup}_{x\in\Omega} f_\infty(x, \nabla u(x)), & \text{if } u \in W^{1,\infty}(\Omega, \mathbb{R}^d), \\ +\infty, & \text{otherwise.} \end{cases} \tag{7}$$

Moreover for a.e. $x \in \Omega$ $f_\infty(x, \cdot)$ is a strong Morrey quasiconvex function satisfying

$$f_\infty(x, \cdot) \geq Q_\infty f(x, \cdot) := \sup_{n\geq 1}(Qf^n)^{1/n}(x, \cdot), \tag{8}$$

where $Qf^n(x, \cdot) := Q(f^n)(x, \cdot)$ stands for the quasiconvex envelope of $f^n(x, \cdot)$ (cf. (22)).

We prove that for any diverging subsequence $(p_n)_n$ the $\Gamma(L^\infty)$-$\lim_{n\to\infty} F_{p_n}$ coincides with the supremal functional $\bar{F}$ whose supremand f_∞, defined in (61), admits the asymptotic formula (62).

In particular, under the assumptions of Theorem 2.1, the latter result guarantees that the relaxed functional $W^{1,\infty}(\Omega; \mathbb{R}^d) \ni u \to \Gamma_{w*}(F)(u) = \operatorname{ess\,sup}_{x\in\Omega} f^{lslc}(\nabla u(x))$ can be obtained as the Γ-limit with respect to the uniform convergence of the sequence of the integral functionals $(F_p(u))_{p\geq 1}$ defined by (6). In Remark 5.2 we will discuss several special cases of assumptions on f. More precisely, we prove that if the supremand $f(x, \cdot)$ is upper semicontinuous for a.e. $x \in \Omega$, then $f_\infty(x, \cdot) = Q_\infty f(x, \cdot)$. The same conclusion holds when $f \equiv f(\xi)$. In addition, if $f(x, \cdot)$ is upper semicontinuous and level convex for a.e. $x \in \Omega$, then (7) can be specialized, since

$$f_\infty(x, \cdot) = Q_\infty f(x, \cdot) = f^{ls}(x, \cdot) \quad \text{for a.e. } x \in \Omega.$$

The same conclusion holds when $f \equiv f(\xi)$ is level convex.
Moreover if $N = 1$ or $d = 1$, if $f(x, \cdot)$ is upper semicontinuous or $f \equiv f(\xi)$ then we get that

$$f_\infty(x, \cdot) = Q_\infty f(x, \cdot) = f^{lslc}(x, \cdot) \quad \text{for a.e. } x \in \Omega.$$

3 Preliminary Results

The aim of this section is twofold, from one hand we recall existing results which will be useful in the body of paper, and from the other, we provide some characterizations of level convex functions defined in general topological vector space (X, τ). In particular some of these results are new to our knowledge and of indipendent interest In Subsection 3.2, we recall the definition and the main properties of Γ-convergence. These topics, together with classical relaxation results for integral functionals in the Sobolev setting (see Subsection 3.3) enable us to deal with the L^p- approximation of Section 5. Finally in Subsection 3.4 we specialize the properties of the level convex and lower semicontinuous envelope f^{lslc} when $f : \mathbb{R}^{d\times N} \to [-\infty, +\infty]$.

3.1 Relaxation and Level Convex Envelopes

In this subsection we provide several relations among envelopes of functions in (X, τ) that will be used in the sequel, thus generalizing some of the results contained in [20, Section 2].

Definition 3.1 A function $F : (X, \tau) \to [-\infty, +\infty]$ is *level convex* if

$$F(tx_1 + (1-t)x_2) \le \max\{F(x_1), F(x_2)\} \quad \forall t \in [0,1],\ \forall x_1, x_2 \in X,$$

that is, for every $\lambda \in \mathbb{R}$ the sublevel set $L_\lambda(F)$ (see (1)) is convex.

Definition 3.2 Let $F : (X, \tau) \to [-\infty, +\infty]$ be a function.

1. The *lower semicontinuous envelope* (or *relaxed function*) of F is defined as

$$\Gamma_\tau(F) := \sup\{G : (X,\tau) \to [-\infty, +\infty]\text{: } G \ \tau\text{-lsc and } G \le F \text{ on } X\}.$$

2. The *level convex envelope* of F is defined as

$$F^{lc} := \sup\{G : (X,\tau) \to [-\infty, +\infty]\text{: } G \text{ level convex and } G \le F \text{ on } X\}.$$

Note that $\Gamma_\tau(F)$ (resp. F^{lc}) is the greatest τ-lower semicontinuous (shortly τ-l.s.c) (resp. level convex) function which is less than or equal to F. By [35, Proposition 3.5(a)] we have that

$$\{\xi \in X : \Gamma_\tau(F)(x) \le \lambda\} = \bigcap_{\varepsilon>0} \mathrm{cl}_\tau(L_{\lambda+\varepsilon}(F)). \tag{9}$$

Moreover, by definition, it easily follows that

$$\inf_X F = \inf_X \Gamma_\tau(F) = \inf_X F^{lc} = \inf_X \Gamma_\tau(F^{lc}). \tag{10}$$

Finally, if $F : (X, \tau) \to [-\infty, +\infty])$, we consider the envelope

$$F^{lslc} := \sup\{G : (X,\tau) \to [-\infty, +\infty]\text{: } G \ \text{ level convex and } \tau\text{-}l.s.c. \text{ and } G \le F \text{ on } X\},$$

that is the greatest lower semicontinuous and level convex function less than or equal to F. We recall that there exists a wide literature devoted to the study of a conjugation for level convex functions (see for example [36], [37] and [38] among the others).

Proposition 3.1 *Let $F : X \to [-\infty, +\infty]$. Then*

$$\Gamma_\tau(F^{lc}) = F^{lslc} \le (\Gamma_\tau(F))^{lc}. \tag{11}$$

In particular if F is level convex then $\Gamma_\tau(F)$ is level convex and

$$\Gamma_\tau(F) = F^{lslc}. \tag{12}$$

Proof Since F^{lslc} is τ-l.s.c. and level convex, we have that

$$F^{lslc} \le \min\{\Gamma_\tau(F^{lc}), (\Gamma_\tau(F))^{lc}\} \le F. \tag{13}$$

In order to conclude the proof of (11), observe that for every $\lambda \ge \inf F$ and for every $\varepsilon > 0$ the set $\{x \in X : F^{lc}(x) \le \lambda + \varepsilon\}$ is convex. Then its τ-closure is still convex. Thanks to (9), we can deduce that $\{x \in X : \Gamma_\tau(F^{lc})(x) \le \lambda\}$ is convex for every $\lambda \ge \inf F = \inf \Gamma_\tau(F^{lc})$. Thus, $\Gamma_\tau(F^{lc})$ is level convex and lower semicontinuous; consequently, exploiting (13), we get (11). In the particular case when F is level convex, (11) implies (12). □

Observe that inequality (11) can be strict (cf. Remark 3.5 below).

The following corollary of Proposition 3.1 holds:

Corollary 3.1 *Let X be a separable Banach space, and X' its dual. Let $F : X' \to [-\infty, +\infty]$ be level convex and let*

$$\Gamma_{w*}(F) = \sup\{G : X' \to [-\infty, +\infty] : G \text{ weak* lower semicontinuous}, G \leq F\},$$

where $w*$ denotes the weak topology in X'. Then $\Gamma_{w*}(F)$ is level convex and*

$$\Gamma_{w*}(F) = \Gamma_{w*_{seq}}(F). \tag{14}$$

Proof It is sufficient to observe that $\Gamma_{w*}(F) = F^{lslc}$ (where the symbol ls refers to the topology $w*$ in X') and to apply [18, Proposition 2.16], which, in turn, relies on Banach-Dieudonné' s Theorem on the sequential characterization of weak* closure of convex. □

Proposition 3.2 *For every $F : (X, \tau) \to [-\infty, +\infty]$ and for every continuous strictly increasing function $\Phi : [-\infty, +\infty] \to [a, b]$, it results*

$$\Gamma_\tau(\Phi(F)) = \Phi(\Gamma_\tau(F)), \tag{15}$$

$$(\Phi(F))^{lc} = \Phi(F^{lc}) \tag{16}$$

and

$$(\Phi(F))^{lslc} = \Phi(F^{lslc}).$$

Proof (15) follows by [35, Proposition 6.16]. In order to show (16), note that $\Phi(F^{lc}) \leq \Phi(F)$ implies

$$\Phi(F^{lc}) \leq (\Phi(F))^{lc}. \tag{17}$$

since the composition of an increasing function and a level convex one is still level convex. Moreover

$$\Phi^{-1}((\Phi(F))^{lc}) \leq \Phi^{-1}(\Phi(F)) = F.$$

Hence,

$$\Phi^{-1}((\Phi(F))^{lc}) \leq F^{lc}.$$

Thus,

$$(\Phi(F))^{lc} \leq \Phi(F^{lc}),$$

which, together with (17), gives (16). Finally, Proposition 3.1, (16) and (15) entail

$$(\Phi(F))^{lslc} = \Gamma_\tau((\Phi(F))^{lc}) = \Gamma_\tau(\Phi(F^{lc})) = \Phi(\Gamma_\tau(F^{lc})) = \Phi(F^{lslc}). \quad \square$$

Remark 3.1 By (15) it follows that

$$\Gamma_\tau(F) = \Phi^{-1}(\Gamma_\tau(\Phi(F))).$$

In particular, if $\Omega \subset \mathbb{R}^{d\times N}$ is a bounded open set, $g : \mathbb{R}^{d\times N} \to [-\infty, +\infty]$ is a Borel function and $G : W^{1,\infty}(\Omega; \mathbb{R}^d) \to [-\infty, +\infty]$ is the supremal functional defined as

$$G(u) := \operatorname*{ess\,sup}_\Omega g(\nabla u),$$

in order to detect $\Gamma_\tau(G)$, it suffices to detect $\Gamma_\tau(\arctan G)$. Since it holds that

$$(\arctan G)(u) = \operatorname*{ess\,sup}_\Omega \arctan g(\nabla u),$$

without loss of generality, we can assume that g is bounded.

We conclude this subsection by proving a general representation result for the functional F^{lc}.

Proposition 3.3 *Let* $F : (X, \tau) \to [-\infty, +\infty]$. *Then*

$$F^{lc}(x) = \inf\{\lambda \in \mathbb{R} : x \in \mathrm{conv}L_\lambda(F)\}. \tag{18}$$

Proof Let

$$\iota : x \in X \to \inf\{\lambda \in \mathbb{R} : x \in \mathrm{conv}L_\lambda(F)\}.$$

Clearly

$$\iota(x) \le F(x) \text{ for every } x \in X. \tag{19}$$

Moreover i is level convex. Indeed, fixed $x_1, x_2 \in X$, for every $\varepsilon > 0$ there exists λ_1 and λ_2 such that

$$\lambda_1 < \iota(x_1) + \varepsilon, \text{ and } \lambda_2 < \iota(x_2) + \varepsilon,$$

and

$$x_1 \in \mathrm{conv}L_{\lambda_1}(F), \text{ and } x_2 \in \mathrm{conv}L_{\lambda_2}(F).$$

Thus

$$tx_1 + (1-t)x_2 \in \mathrm{conv}L_{\max\{\lambda_1,\lambda_2\}}(F), \quad t \in [0,1],$$

and so

$$\iota(tx_1 + (1-t)x_2) \le \max\{\lambda_1, \lambda_2\} < \max\{\iota(x_1), \iota(x_2)\} + 2\varepsilon.$$

The arbitrariness of ε guarantees the level convexity of ι, which together with (19), guarantees

$$\iota(x) \le F^{lc}(x) \text{ for every } x \in X. \tag{20}$$

In order to prove the opposite inequality we have that for every level convex $G \le F$,

$$L_\lambda(G) \supseteq L_\lambda(F) \text{ for every } \lambda,$$

which gives

$$\mathrm{conv}L_\lambda(F) \subseteq \mathrm{conv}L_\lambda(G) = L_\lambda(G) \text{ for every } \lambda \in \mathbb{R}.$$

Since for every $x \in X$ it results

$$G(x) = \inf\{\lambda \in \mathbb{R} : x \in L_\lambda(G)\} \le \inf\{\lambda \in \mathbb{R} : x \in \mathrm{conv}L_\lambda(F)\} = \iota(x),$$

by choosing $G = F^{lc}$ we get $\iota \ge F^{lc}$. The latter inequality and (20) conclude the proof. □

3.2 Γ-Convergence

Now we recall the notion of Γ-convergence for family of functionals defined in the topological space (X, τ), (for more details on the theory we refer to [35]). To this end, we denote by $\mathcal{U}(x)$ the set of all open neighbourhoods of x in X .

Definition 3.3 Let $F_n : X \to \mathbb{R}$ be a sequence of functions. The $\Gamma(\tau)$-*lower limit* and the $\Gamma(\tau)$-*upper limit* of the sequence (F_n) are the functions from X into $[-\infty, +\infty]$ defined by

$$\Gamma(\tau)\text{-}\liminf_{n\to\infty} F_n(x) := \sup_{U \in \mathcal{U}(x)} \liminf_{n\to\infty} \inf_{y\in U} F_n(y),$$

$$\Gamma(\tau)\text{-}\limsup_{n\to\infty} F_n(x) := \sup_{U \in \mathcal{U}(x)} \limsup_{n\to\infty} \inf_{y\in U} F_n(y)$$

If there exists a function $F : X \to [-\infty, +\infty]$ such that $F = \Gamma(\tau)\text{-}\liminf_{n\to\infty} F_n = \Gamma(\tau)\text{-}\limsup_{n\to\infty} F_n$, then we write

$$F = \Gamma(\tau)\text{-}\lim_{n\to\infty} F_n$$

and we say that the sequence (F_n) $\Gamma(\tau)$-converges to F or that F is the $\Gamma(\tau)$-limit of $(F_n)_n$.

Definition 3.4 Given a family of functionals $G_p : X \to [-\infty, +\infty]$, we say that $(G_p)_p$ $\Gamma(\tau)$-converges to the functional G, as $p \to \infty$, if for every $(p_n) \to +\infty$ the sequence (G_{p_n}) $\Gamma(\tau)$-converges to G.

The introduction of this variational convergence by De Giorgi and Franzoni (see [35] and the bibliography therein) is motivated by the next theorem. Indeed, under the assumption of equicoercivity for the sequence (F_n), it holds the important property of convergence of the minimum values.

Theorem 3.1 *Suppose that the sequence (F_n) is equi-coercive in X, i.e. for every $t \in \mathbb{R}$ there exists a closed compact subset K_t of X such that $\{F_n \leq t\} \subset K_t$ for every $n \in \mathbb{N}$. If (F_n) $\Gamma(\tau)$-converges to a function F in X, then*

$$\min_{x\in X} F(x) = \lim_{n\to\infty} \inf_{x\in X} F_n(x). \tag{21}$$

Moreover if x_n is such that $F_n(x_n) \leq \inf_X F_n + \varepsilon_n$, where $\varepsilon_n \to 0$ and $x_{n_k} \to x$ for some subsequence $(x_{n_k})_k$ of (x_n) then $F(x) = \min_X F$.

For a proof, see [35, Theorem 7.8 and Corollary 7.17].

In the following proposition we summarize some properties of the Γ-convergence useful in the sequel (see [35, Proposition 6.8, Proposition 6.11, Proposition 5.7, Remark 5.5, Proposition 6.26]).

Proposition 3.4 *Let $F_n : X \to [-\infty, +\infty]$ be a sequence of functions. Then*

1. *Let $\hat{F} := \Gamma(\tau)\text{-}\lim_{n\to\infty} F_n$, then $\hat{F}$ is τ-lower semicontinuous on X;*
2. *if (F_n) is a not increasing sequence which pointwise converges to F then $\Gamma(\tau)\text{-}\lim_{n\to\infty} F_n = \Gamma_\tau(F)$. In particular if $F_n = F$ for every $n \in \mathbb{N}$ then $\Gamma(\tau)\text{-}\lim_{n\to\infty} F = \Gamma_\tau(F)$;*
3. *if $\Gamma_\tau(F_n)$ is the τ-l.s.c. envelope of F_n, then the sequence (F_n) $\Gamma(\tau)$-converges to F if and only if the sequence of the relaxed functions $(\Gamma_\tau(F_n))$ $\Gamma(\tau)$-converges to F, and*
$$\Gamma(\tau)\text{-}\lim_{n\to\infty} F_n = \Gamma(\tau)\text{-}\lim_{n\to\infty} \Gamma_\tau(F_n);$$
4. *if (F_n) is an increasing sequence of τ-lower semicontinuous functions which pointwise converges to F then $\Gamma(\tau)\text{-}\lim_{n\to\infty} F_n = F$;*
5. *for every $c \in \mathbb{R}$, $\Gamma_\tau(\max\{F, c\}) = \max\{\Gamma_\tau(F), c\}$.*

Next we recall the sequential characterization of $\Gamma(\tau)$-liminf, $\Gamma(\tau)$-limsup and $\Gamma(\tau)$-limit when the topological space (X, τ) satisfies the first axiom of countability (for a proof see [35, Proposition 8.1]).

Proposition 3.5 *Let $F_n : X \to [-\infty, +\infty]$ be a sequence of functions. Then the function*

$$F(x) = \Gamma(\tau)\text{-}\lim_{n\to\infty} F_n(x)$$

is characterized by the following inequalities:

- *(Γ-liminf inequality) for every $x \in X$ and for every sequence (x_n) converging to x in X it is*
$$F(x) \leq \liminf_{n\to\infty} F_n(x_n);$$
- *(Γ-limsup inequality) for every $x \in X$ there exists a sequence (x_n) (called a* recovering sequence*) converging to x in X such that*
$$F(x) = \lim_{n\to\infty} F_n(x_n).$$

Finally we note that the level convexity is stable under both pointwise and Γ-convergence (for a proof see [18, Proposition 2.9]).

Proposition 3.6 *Let (X, τ) be a topological vector space and let $F_n : X \to [-\infty, +\infty]$ be a sequence of level convex functions. Then*

1. *the function $F^{\#}(x) = \limsup_{n\to\infty} F_n(x)$ is level convex;*
2. *the function $\Gamma(\tau)\text{-}\limsup_{n\to\infty} F_n$ is level convex.*

3.3 Lower Semicontinuity and Relaxation Results in the Integral Setting

In the sequel we collect some definitions and results that will be crucial for the proof of Theorem 2.2. We refer the reader to [39] and [40] for a detailed treatment of this subject.

Definition 3.5 Let $g : \mathbb{R}^{d\times N} \to \mathbb{R}$ be a Borel function and let $Q :=]0,1[^N$. Then g is said quasiconvex (in the sense of Morrey) if

$$g(\xi) \leq \int_Q g(\xi + \nabla u(y))\,dy$$

for every $u \in W_0^{1,\infty}(Q;\mathbb{R}^d)$ and $\xi \in \mathbb{R}^{d\times N}$.

By [40, Theorem 5.3(4)] it follows that any quasiconvex function is continuous. The quasiconvexity is a sufficient (and necessary) condition for the lower semicontinuity of an integral functional on $W^{1,p}(\Omega;\mathbb{R}^d)$ with respect to the weak topology of $W^{1,p}(\Omega;\mathbb{R}^d)$. More precisely, let $1 \leq p < +\infty$ and let $g : \mathbb{R}^{d\times N} \to \mathbb{R}$ be a quasiconvex function, such that

$$0 \leq g(\xi) \leq \beta(1 + |\xi|^p) \qquad \text{for every } \xi \in \mathbb{R}^{d\times N}.$$

Let $G : W^{1,p}(\Omega;\mathbb{R}^d)$ be the integral functional defined by

$$G(u) := \int_\Omega g(\nabla u(y))dy.$$

Then G is sequentially weakly lower semicontinuous on $W^{1,p}(\Omega;\mathbb{R}^d)$ (see [40, Theorem 8.4]).

If the Borel function $g : \mathbb{R}^{d\times N} \to \mathbb{R}$ fails to be quasiconvex, one can introduce its quasiconvex envelope, namely

$$Qg := \sup\{h : \mathbb{R}^{d\times N} \to \mathbb{R} : h \text{ quasiconvex and } h \leq g\}. \tag{22}$$

Remark 3.2 It is worth to observe that, being Qg quasiconvex, then Qg is a continuous function (see [41, Lemma 5.42] and [40, Theorem 5.3]).

The following representation formula holds:

Theorem 3.2 [40, Theorem 6.9] *Let $g : \mathbb{R}^{d\times N} \to \mathbb{R}$ be a Borel and locally bounded function. Assume that there exists a quasiconvex function $h : \mathbb{R}^{d\times N} \to \mathbb{R}$ such that $g \geq h$. Then for every $\xi \in \mathbb{R}^{d\times N}$*

$$Qg(\xi) = \inf\Big\{\int_Q g(\xi + \nabla u(y))\,dy \;:\; u \in W_0^{1,\infty}(Q;\mathbb{R}^d)\Big\}.$$

The following result which holds under very general assumptions, i.e. when $g = g(x,\xi)$ is only $\mathcal{L}^N \otimes \mathcal{B}_{d\times N}$-measurable function, will be crucial in the proof of Theorem 2.2.

Theorem 3.3 [39, Theorem 4.4.1] *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set, let $1 \leq p < +\infty$ and let $g : \Omega \times \mathbb{R}^{d\times N} \to [0,+\infty[$ be a $\mathcal{L}^N \otimes \mathcal{B}_{d\times N}$-measurable function, satisfying*

$$0 \leq g(x,\xi) \leq \beta(1 + |\xi|^p) \qquad \text{for a. e. } x \in \Omega \text{ and for every } \xi \in \mathbb{R}^{d\times N}. \tag{23}$$

Then there exists a Caratheodory function $\tilde{g} : \Omega \times \mathbb{R}^{d\times N} \to [0,+\infty[$, quasiconvex in the second variable, satisfying the same growth condition (23) such that

$$\Gamma_{w_{seq}}(G)(u) = \int_\Omega \widetilde{g}(x,\nabla u(x))dx \text{ for every } u \in W^{1,p}(\Omega;\mathbb{R}^d),$$

where $\Gamma_{w_{seq}}(G)$ denotes the sequential lower semicontinuous envelope of G with respect to the weak topology in $W^{1,p}(\Omega,\mathbb{R}^d)$.
Moroever

$$Qg(x,\xi) \leq \tilde{g}(x,\xi) \text{ for a.e. } x \in \Omega, \text{ and for every } \xi \in \mathbb{R}^{d\times N}.$$

Remark 3.3 In general the above inequality can be strict on a set $\Omega' \times \mathbb{R}^{d\times N}$ of positive measure (cf. [39, Example 4.4.6]).
On the other hand, [39, Remark 4.4.5] guarantees that

1. if $g = g(\xi)$ then $Qg = \widetilde{g}$;
2. if $g(x,\cdot)$ is upper semicontinuous for a.e. $x \in \Omega$ then $Qg(x,\xi) = \widetilde{g}(x,\xi)$ for a.e. $x \in \Omega$, and for every $\xi \in \mathbb{R}^{d\times N}$.

3.4 Envelopes of Real Functions

In this subsection we detail the results of Subsection 3.1 in the special case when $X = \mathbb{R}^{d\times N}$ and τ is the natural topology.

Definition 3.6 Let $f : \mathbb{R}^{d\times N} \to [-\infty,+\infty]$ be a function. Set

$$\mathcal{F}_{lc}(f) := \{g : \mathbb{R}^{d\times N} \to [-\infty,+\infty] : \ g \leq f,\ g \text{ level convex}\},$$

$$\mathcal{F}_{ls}(f) := \{g : \mathbb{R}^{d\times N} \to [-\infty,+\infty] : \ g \leq f,\ g\ \tau\text{-lower semicontinuous}\},$$

and

$$\mathcal{F}_{lslc}(f) := \{g : \mathbb{R}^{d\times N} \to [-\infty,+\infty] : \ g \leq f,\ g\ \tau\text{-lower semicontinuous and level convex}\}.$$

Consequently define $f^{lc}, f^{ls}, f^{lslc} : \mathbb{R}^{d\times N} \to \text{cl}(\mathbb{R})$, as

$$f^{lc}(\xi) := \sup\{g(\xi) : \ g \in \mathcal{F}_{lc}(f)\},$$

$$f^{ls}(\xi) := \sup\{g(\xi) : \ g \in \mathcal{F}_{ls}(f)\},$$

and

$$f^{lslc}(\xi) := \sup\{g(\xi) : \ g \in \mathcal{F}_{lslc}(f)\}.$$

An explicit formula to compute f^{lc} in given by Proposition 3.3, applied to $F = f$ and to $X = \mathbb{R}^{d\times N}$.

Remark 3.4 If $f : \mathbb{R}^{d\times N} \to [-\infty,+\infty]$, by (11) we have that

$$f^{lslc} = (f^{lc})^{ls} \leq (f^{ls})^{lc}. \tag{24}$$

Then if f is level convex, we get that

$$f^{lslc} = f^{ls}. \tag{25}$$

In particular, thanks to [12, Theorems 3.4 and 2.7], we get that f^{ls} is a strong Morrey quasiconvex function less than or equal to f.

Remark 3.5 In general $(f^{lc})^{ls} \lneq (f^{ls})^{lc}$, since the level convex envelope of a lower semicontinuous function might not be lower semicontinuous. To this end, it suffices to consider the function

$$\chi_{\mathbb{R}^2\setminus C} = \begin{cases} 0, & \text{if } x \in C, \\ 1, & \text{otherwise,} \end{cases}$$

where $C := \{(x_1,0) : x_1 \in \mathbb{R}\} \cup \{(0,1)\}$. Indeed $\chi_{\mathbb{R}^2\setminus C}$ is lower semicontinuous but not level convex. On the other hand, $(\chi_{\mathbb{R}^2\setminus C})^{lc} = \chi_{\mathbb{R}^2\setminus D}$, where $D = \{(x_1,x_2) : x_1 \in \mathbb{R}, 0 \leq x_2 < 1\} \cup \{(0,1)\}$, which is not closed. Clearly $(\chi_{\mathbb{R}^2\setminus C})^{lslc} = \chi_{\mathbb{R}^2\setminus \text{cl}(D)} < \chi_{\mathbb{R}^2\setminus D}$.

Now we are in position to show a result characterizing the effective domain of f^{lc}, based on Carathéodory's Theorem (see [26, Theorem 1.2.1]).

Proposition 3.7 *For every* $f : \mathbb{R}^{d\times N} \to [-\infty,+\infty]$, *it results*

$$\text{dom}(f^{lc}) = \text{conv}(\text{dom} f)$$

Proof The result is achieved by proving a double inequality. If $\xi \in \text{dom}(f^{lc})$, then there exists $\lambda \in \mathbb{R}$ such that $\xi \in \text{conv} L_\lambda(f)$, thus there exist $\xi_1, \ldots, \xi_{d\times N+1} \in L_\lambda(f)$ and $t_i \in [0,1], i = 1, \ldots d \times N + 1$ such that $\sum_{i=1}^{d\times N+1} t_i = 1$ and $\xi = \sum_{i=1}^{d\times N+1} t_i \xi_i$. Clearly $\xi_1, \ldots \xi_{d\times N+1} \in \text{dom} f$, Hence, $\xi \in \text{conv}(\text{dom} f)$. Thus, it remains to prove the opposite inequality: if $\xi \in \text{conv}(\text{dom} f)$, again thanks to Carathéodory's

Theorem there exist almost $d\times N+1$ points $\xi_1,\xi_2,\cdots,\xi_{d\times N+1}\in \mathrm{dom}f$ and $t_1,\cdots,t_{d\times N+1}\in[0,1]$ such that $\sum_{i=1}^{d\times N+1} t_i=1$ and $\xi=\sum_{i=1}^{n+1} t_i\xi_i$. Hence, there exists $\lambda\in\mathbb{R}$ such that $f(\xi_i)\le\lambda$ for every $i\in\{1,\cdots,n+1\}$. Consequently Proposition 3.3 entails that $f^{lc}(\xi)\le\lambda$, i.e. $\xi\in\mathrm{dom}(f^{lc})$ and this concludes the proof. □

Proposition 3.8 *Let $f:\mathbb{R}^{d\times N}\to[-\infty,+\infty]$, then, for every $\lambda\in\mathbb{R}$*

1. $\mathrm{conv}L_\lambda(f)\subseteq L_\lambda(f^{lc})$*;*
2. *if f is coercive (i.e. $\lim_{|\xi|\to\infty} f(\xi)=+\infty$), then*

$$L_\lambda(f^{lc})\subseteq \mathrm{cl}(L_\lambda(f^{lc}))\subseteq \mathrm{conv}L_\lambda(f^{ls}). \tag{26}$$

In particular

$$L_\lambda(f^{lc})\subseteq \mathrm{conv}(\bigcap_{\varepsilon>0}\mathrm{cl}(L_{\lambda+\varepsilon}(f))); \tag{27}$$

3. *if f is lower semicontinuous and coercive then* $\mathrm{conv}L_\lambda(f)=L_\lambda(f^{lc})$*.*

Proof 1. It follows by the convexity of $L_\lambda(f^{lc})$ and by the fact that $L_\lambda(f)\subseteq L_\lambda(f^{lc})$.

2. Assume that $f^{lc}(\xi)\le\lambda$. By Proposition 3.3 there exists a sequence (λ_n) converging to $f^{lc}(\xi)$ such that $\xi\in co(L_{\lambda_n}(f))$. In particular, thanks to the Carathéodory's Theorem, for every $n\in\mathbb{N}$ there exist $\xi_n^1,\xi_n^2,\cdots,\xi_n^{d\times N+1}\in L_{\lambda_n}(f)$ and $t_n^i\in[0,1]$, $i\in\{1,\cdots,d\times N+1\}$ such that $\xi=\sum_{i=1}^{d\times N+1} t_n^i\xi_n^i$ and $\sum_{i=1}^{d\times N+1} t_n^i=1$. Since $L_{\lambda_n}(f)$ is bounded by coercivity, without loss of generality, we can assume, up to the extraction of not relabelled subsequences, that for every $i\in\{1,\cdots,d\times N+1\}$ there exist $\lim_{n\to\infty}\xi_n^i=\xi^i$ and $\lim_{n\to\infty} t_n^i=\bar{t}^i$. In follows that $\xi=\sum_{i=1}^{d\times N+1}\bar{t}^i\xi^i$ and $\sum_{i=1}^{d\times N+1}\bar{t}^i=1$. By definition of f^{ls} it follows that

$$f^{ls}(\xi^i)\le\liminf_{n\to\infty} f(\xi_n^i)\le\lim_{n\to\infty}\lambda_n=f^{lc}(\xi)\le\lambda.$$

Therefore $\xi\in\mathrm{conv}L_\lambda(f^{ls})$ and (26) follows. By (26) and (9), we obtain (27).

3. It follows by 1. and 2. □

Remark 3.6 Let $f:\mathbb{R}^{d\times N}\to\mathbb{R}$ be defined by

$$f(\xi)=\begin{cases}|\xi| & \text{if } \xi\neq 0,\\ 1 & \text{if } \xi=0.\end{cases}$$

Then $L_0(f)=\emptyset$, so $\mathrm{conv}L_0(f)=\emptyset$, while $f^{ls}(\xi)=f^{lc}(\xi)=|\xi|$, and so $L_0(f^{lc})=\{0\}$. Thus, we cannot expect equality in 1. Moreover this example proves also that in general $L_\lambda(f^{lc})\not\subseteq\mathrm{cl}(\mathrm{conv}L_\lambda(f))$ and $L_\lambda(f^{ls})\neq\mathrm{cl}(L_\lambda(f))$.

The following result specializes (9) when $X=\mathbb{R}^{d\times N}$, thus providing a useful description of the sublevel sets of f^{lslc}.

Proposition 3.9 *Let $f:\mathbb{R}^{d\times N}\to[-\infty,+\infty]$. Then for every $\lambda\in\mathbb{R}$ it holds*

$$L_\lambda(f^{lslc})=\bigcap_{\varepsilon>0}\mathrm{cl}(\mathrm{conv}(L_{\lambda+\varepsilon}(f))).$$

Proof First of all, we notice that, thanks to (24) and (9), we have that

$$L_\lambda(f^{lslc})=L_\lambda((f^{lc})^{ls})=\bigcap_{\delta>0}\mathrm{cl}(L_{\lambda+\delta}(f^{lc})), \tag{28}$$

in particular, Proposition 3.8(1) implies

$$\bigcap_{\varepsilon>0}\mathrm{cl}(\mathrm{conv}(L_{\lambda+\varepsilon}(f)))\subseteq L_\lambda(f^{lslc}).$$

The proof of the opposite inclusion will be developed in several steps.

Step 1. First we consider the case when f is coercive. Under this extra assumption, by applying (26), we have that for every $\lambda \in \mathbb{R}$ and for every $\delta > 0$

$$\mathrm{cl}(L_{\lambda+\delta}(f^{lc})) \subseteq \mathrm{conv}\left(\bigcap_{\varepsilon>0} \mathrm{cl}(L_{\lambda+\delta+\varepsilon}(f))\right). \tag{29}$$

By putting together (28) and (29), it follows

$$\begin{aligned} L_\lambda(f^{lslc}) &\subseteq \bigcap_{\delta>0} \mathrm{conv}(\bigcap_{\varepsilon>0} \mathrm{cl}(L_{\lambda+\delta+\varepsilon}(f))) \subseteq \bigcap_{\delta>0}\bigcap_{\varepsilon>0} \mathrm{conv}(\mathrm{cl}(L_{\lambda+\delta+\varepsilon}(f))) \\ &= \bigcap_{r>0} \mathrm{conv}(\mathrm{cl}(L_{\lambda+r}(f))) = \bigcap_{r>0} \mathrm{cl}(\mathrm{conv}(L_{\lambda+r}(f))) \end{aligned}$$

and this identity concludes the proof in the coercive case.

Step 2. In the second step we consider the general case when $f : \mathbb{R}^{d\times N} \to [0,+\infty]$. We define $f_n(\xi) := \max\{f(\xi), \frac{1}{n}|\xi|\}$. Since $f \le f_n$ then $f^{lc} \le (f_n)^{lc} := f_n^{lc}$ that implies

$$f^{lc} \le (f^{lc})_n \le f_n^{lc} \le f_n$$

for every $n \in \mathbb{N}$. In particular

$$f^{lc} \le \inf_n f_n^{lc} \le \inf f_n = f.$$

By Proposition 3.6(1), since (f_n^{lc}) is monotone, the function $g(\xi) := \inf_n f_n^{lc}(\xi)$ is level convex. Then

$$f^{lc} = \inf_n f_n^{lc}.$$

Since f_n is coercive, by applying (26), we have that for every $n \in \mathbb{N}$, for every $\lambda \ge 0$ and $\varepsilon > 0$

$$L_{\lambda+\varepsilon}(f_n^{lc}) \subseteq \mathrm{conv}L_{\lambda+\varepsilon}(f_n^{ls}) \subseteq \mathrm{conv}L_{\lambda+\varepsilon}(f^{ls}).$$

Now, for fixed $\lambda \ge 0$ and $\varepsilon > 0$, if $\xi \in L_\lambda(f^{lc})$ then for $n = n(\xi)$ big enough we get that $\xi \in L_{\lambda+\varepsilon}(f_n^{lc})$. Thus,

$$L_\lambda(f^{lc}) \subseteq \mathrm{conv}L_{\lambda+\varepsilon}(f^{ls})$$

that implies

$$\mathrm{cl}(L_\lambda(f^{lc})) \subseteq \bigcap_{\varepsilon>0} \mathrm{conv}L_{\lambda+\varepsilon}(f^{ls}). \tag{30}$$

Thanks to (28), (30) and Proposition 3.8 (1), we have that

$$\begin{aligned} L_\lambda(f^{lslc}) &= \bigcap_{\delta>0} \mathrm{cl}(L_{\lambda+\delta}(f^{lc})) \subseteq \bigcap_{\delta>0}\bigcap_{\varepsilon>0} \mathrm{conv}L_{\lambda+\delta+\varepsilon}(f^{ls}) \\ &= \bigcap_{\varepsilon>0} \mathrm{conv}L_{\lambda+\varepsilon}(f^{ls}) = \bigcap_{\varepsilon>0} \mathrm{conv}\big(\bigcap_{\delta>0} \mathrm{cl}(L_{\lambda+\varepsilon+\delta}(f))\big) \subseteq \bigcap_{\varepsilon>0}\bigcap_{\delta>0} \mathrm{conv}\left(\mathrm{cl}(L_{\lambda+\varepsilon+\delta})\,(f)\right) \\ &= \bigcap_{r>0} \mathrm{conv}\left(\mathrm{cl}(L_{\lambda+r}(f))\right) = \bigcap_{r>0} \mathrm{cl}(\mathrm{conv}L_{\lambda+r}(f)) \end{aligned}$$

and this identity concludes the proof.

Step 3. Now we consider the case when $f : \mathbb{R}^{d\times N} \to \bar{\mathbb{R}}$ is such that $\inf f > -\infty$. Then, it is sufficient to apply the previous step to the non negative function $g := f - \inf f$ and use the fact that $g^{lc} = f^{lc} - \inf f$ and $g^{lslc} = f^{lslc} - \inf f$.

Step 4. Finally, when $f : \mathbb{R}^{d\times N} \to \bar{\mathbb{R}}$ is such that $\inf f = -\infty$ we can consider the approximation $\varphi_n := \max\{f, -n\} \ge f$. Then for every $n \in \mathbb{N}$ and $\lambda \ge -n$, thanks to the previous step, it holds

$$\bigcap_{\varepsilon>0} \mathrm{cl}(\mathrm{conv}(L_{\lambda+\varepsilon}(\varphi_n))) = L_\lambda(\varphi_n^{lslc}). \tag{31}$$

Denote by φ_n^{lc} the function $(\varphi_n)^{lc}$. Then $f^{lc} \leq \varphi_n^{lc} \leq \varphi_n$ for every $n \in \mathbb{N}$. In particular

$$f^{lc} \leq \inf_n \varphi_n^{lc} \leq \inf \varphi_n = f.$$

Applying again Proposition 3.6(1), in light of the monotonicity of (φ_n^{lc}), it turns out that $g := \inf_n \varphi_n^{lc}$ is level convex. Then

$$f^{lc} = \inf_n \varphi_n^{lc}$$

and, by Proposition 3.4 (2)-(3) we have that

$$f^{lslc} = \Gamma\text{-} \lim_{n\to\infty} \varphi_n^{lc} = \Gamma\text{-} \lim_{n\to\infty} \varphi_n^{lslc}.$$

Then for fixed $\lambda \in \mathbb{R}$, and $\xi \in L_\lambda(f^{lslc})$ there exists a sequence (ξ_n) converging to ξ such that for every $\varepsilon > 0$ one can find $n_0 = n_0(\varepsilon)$ such that

$$\varphi_n^{lslc}(\xi_n) \leq f^{lslc}(\xi) + \varepsilon \leq \lambda + \varepsilon \quad \forall n \geq n_0$$

that is

$$(\xi_n)_{n\geq n_0} \subseteq \bigcup_{n\geq n_0} L_{\lambda+\varepsilon}(\varphi_n^{lslc})$$

that implies

$$\xi \in \mathrm{cl}(\bigcup_{n\geq n_0} L_{\lambda+\varepsilon}(\varphi_n^{lslc})).$$

By applying (31), we get that, for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$\xi \in \mathrm{cl}\left(\bigcup_{n\geq n_0} \bigcap_{\delta>0} \mathrm{cl}(\mathrm{conv}(L_{\lambda+\varepsilon+\delta}(\varphi_n)))\right).$$

Since $f \leq \varphi_n$ for every $n \in \mathbb{N}$ it follows that for every $\varepsilon > 0$

$$\xi \in \mathrm{cl}\left(\bigcup_{n\geq n_0} \bigcap_{\delta>0} \mathrm{cl}(\mathrm{conv}(L_{\lambda+\varepsilon+\delta}(f)))\right) = \bigcap_{\delta>0} \mathrm{cl}\left(\mathrm{conv}(L_{\lambda+\varepsilon+\delta}(f))\right),$$

that implies

$$L_\lambda(f^{lslc}) \subseteq \bigcap_{\varepsilon>0} \bigcap_{\delta>0} \mathrm{cl}(\mathrm{conv}(L_{\lambda+\varepsilon+\delta}(f))) = \bigcap_{\varepsilon>0} \mathrm{cl}(\mathrm{conv}(L_{\lambda+\varepsilon}(f))). \qquad \square$$

4 Relaxation Results

This section is mainly devoted to the proof of Theorem 2.1. First of all, we give an equivalent formulation of assumption (H).

Remark 4.1 Assumption (H) is equivalent to require the following property:

(H'): f is level convex and there exist two sequences $(\xi_n) \subseteq \mathbb{R}^{d\times N}$ and $(\lambda_n) \searrow \inf_{\mathbb{R}^{d\times N}} f$ such that

$$f(\xi_n) \leq \lambda_n \text{ and } \limsup_{\xi\to 0} f(\xi_n + \xi) \leq \lambda_n \quad \forall n \in \mathbb{N}. \tag{32}$$

Indeed, assume that (H) holds. In order to show that f is level convex, it remains to check that when $\inf_{\mathbb{R}^{d\times N}} f = \min_{\mathbb{R}^{d\times N}} f =: \bar{\lambda} \in \mathbb{R}$ the sublevel set $L_{\bar{\lambda}}(f)$ is convex. This holds since the sublevel set corrisponding to the minimum value $\bar{\lambda}$ satisfies

$$L_{\bar{\lambda}}(f) = \bigcap_{\lambda > \bar{\lambda}} L_\lambda(f)$$

and $L_\lambda(f)$ is convex for every $\lambda > \bar{\lambda}$ by hypothesis. In order to prove (32) it suffices to take (λ_n) such that $(\lambda_n) \to \inf_{\mathbb{R}^{d\times N}} f$ and choose ξ_n in the interior of E_{λ_n}.

Viceversa, assume that (H') holds, thus $L_\lambda(f)$ is convex for any $\lambda \in \mathbb{R}$ such that $\lambda \geq \inf_{\mathbb{R}^{d\times N}} f$. In order to show that $L_\lambda(f)$ has nonempty interior for any $\lambda > \inf_{\mathbb{R}^{d\times N}} f$, let us choose n big enough such that $\lambda_n < \lambda$. Let $0 < \epsilon < \lambda - \lambda_n$.Thanks to (32) the set $L_{\lambda_n+\epsilon}(f)$ has nonempty interior and since $L_{\lambda_n+\epsilon}(f) \subseteq L_\lambda(f)$, the same holds for $L_\lambda(f)$.

The proof of Theorem 2.1 relies on the following result, which is a consequence of [23, Theorem 2.1] and exploits arguments as in [18, Theorem 3.4].

Theorem 4.1 *Let $I_C : \mathbb{R}^{d\times N} \to [0, +\infty]$ be the indicator function of a nonempty open bounded convex set C, such that $\underline{0} \in C$, i.e.*

$$I_C(\xi) := \begin{cases} 0, & \text{if } \xi \in C, \\ +\infty, & \text{if } \xi \notin C. \end{cases} \tag{33}$$

Let $\mathcal{I}, \mathrm{cl}(\mathcal{I}) : W^{1,\infty}(\Omega;\mathbb{R}^d) \to [0, +\infty]$ be the functionals defined by

$$\mathcal{I}(u) := \int_\Omega I_C(\nabla u)dx, \tag{34}$$

and

$$\overline{\mathcal{I}}(u) := \int_\Omega I_{\mathrm{cl}(C)}(\nabla u)dx. \tag{35}$$

Then

$$\Gamma_{L^1}(\mathcal{I})(u) = \overline{\mathcal{I}}(u) \quad \text{for every } u \in W^{1,\infty}(\Omega;\mathbb{R}^d).$$

Remark 4.2 Note that when Ω is a bounded open subset with Lipschitz boundary

$$C \text{ bounded} \Longrightarrow \Gamma_{w^*}(\mathcal{I}) = \Gamma_{w^*_{seq}}(\mathcal{I}) = \Gamma_{L^\infty}(\mathcal{I}) = \Gamma_{L^1}(\mathcal{I}).$$

Since $\mathcal{I}(u)$ is finite if and only if $\nabla u(x) \in C$ for a.e. $x \in \Omega$, the first equality is a consequence of Banach-Alaoglu-Bourbaki's Theorem. The second one follows by Rellich-Kondrachov Theorem. For what concerns the last one, it is trivially observed that $\Gamma_{L^\infty}(\mathcal{I}) \geq \Gamma_{L^1}(\mathcal{I})$. In order to show the converse inequality, we note that if $(u_n) \subseteq W^{1,\infty}(\Omega;\mathbb{R}^d)$ converges to u in L^1 and $\liminf_{n\to\infty} I(u_n) = \lim \mathcal{I}(u_n) < +\infty$ then $(\nabla u_n(x))_n \in C$ for a.e. $x \in \Omega$. Since (up to a subsequence) the sequence $(u_n)_n$ pointwise converge to u, by Morrey's inequality and by Rellich-Kondrachov's Theorem, we get that the sequence $(u_n)_n$ uniformly converges to u.

Now, inspired by the arguments in [18, Theorem 3.4], we prove our result dealing with the relaxation of the functional F in (2).

Proof of Theorem 2.1 Taking into account (25), by [11, Remark 4.4] the functional $\overline{F} : W^{1,\infty}(\Omega, \mathbb{R}^d) \to \mathbb{R}$ defined by

$$\overline{F}(u) := \operatorname*{ess\,sup}_{x\in\Omega} f^{ls}(\nabla u(x))$$

is w^*- lower semicontinuous. Therefore we have that

$$\operatorname*{ess\,sup}_{x\in\Omega} f^{ls}(\nabla u(x)) \leq \Gamma_{w*}(F)(u) = \Gamma_{w^*_{seq}}(F)(u) \tag{36}$$

for every $u \in W^{1,\infty}(\Omega;\mathbb{R}^d)$, where in the latter equality we have exploited Corollary 3.1.

The proof of the inequality $\Gamma_{w*}(F) \leq \overline{F}$ will be developed in several steps.

Step 1. First we assume that f satisfies the further hypotheses that

$$f(\xi) \geq \alpha|\xi| \tag{37}$$

for $\alpha > 0$ and that there exists $\bar{\xi}$ such that $f(\bar{\xi}) = \min_{\mathbb{R}^{d\times N}} f$. Up to a translation argument there is no loss of generality in assuming $\overline{\xi} = \underline{0}$ and $\min_{\mathbb{R}^{d\times N}} f = 0$.
Let $\overline{u} \in W^{1,\infty}(\Omega;\mathbb{R}^d)$ and set

$$\lambda := \operatorname*{ess\,sup}_{x\in\Omega} f^{ls}(\nabla\overline{u}(x)). \tag{38}$$

We determine a sequence $(u_{\varepsilon_n}) \subset W^{1,\infty}(\Omega;\mathbb{R}^d)$ such that

$$u_{\varepsilon_n} \overset{*}{\rightharpoonup} \overline{u} \text{ in } W^{1,\infty}(\Omega;\mathbb{R}^d)$$

and

$$\lim_{n\to\infty} \operatorname*{ess\,sup}_{x\in\Omega} f(\nabla u_{\varepsilon_n}(x)) \leq \lambda.$$

With this aim for fixed $\varepsilon > 0$ let

$$C_\varepsilon := \{\xi \in \mathbb{R}^{d\times N} : f(\xi) \leq \lambda + \varepsilon\},$$

denote by I_{C_ε} the indicator function of C_ε, i.e.,

$$I_{C_\varepsilon}(\xi) := \begin{cases} 0, & \text{if } \xi \in C_\varepsilon, \\ +\infty, & \text{otherwise.} \end{cases}$$

Clearly $\underline{0} \in C_\varepsilon$. Since $\lambda + \varepsilon > \inf f$ we get that C_ε is convex, and has nonempty interior. Moreover the coercivity of f guarantees that C_ε is bounded. Set

$$C^\infty := \{\xi \in \mathbb{R}^{N\times d} : f^{ls}(\xi) \leq \lambda\}.$$

By (38), $\nabla\overline{u}(x) \in C^\infty$ for a.e. $x \in \Omega$ and, by Proposition 3.9, it holds

$$C^\infty = \bigcap_{\varepsilon>0} \mathrm{cl}(C_\varepsilon).$$

Then $\nabla\overline{u}(x) \in \mathrm{cl}(C_\varepsilon)$ for a.e. $x \in \Omega$ and for every $\varepsilon > 0$.
For fixed $\varepsilon > 0$ denote by $\mathcal{G}_\varepsilon$ and $\widetilde{\mathcal{G}}_\varepsilon$ the unbounded integral functionals defined in $W^{1,\infty}(\Omega;\mathbb{R}^d)$ with values in $[0,+\infty]$, as

$$\mathcal{G}_\varepsilon(u) := \int_\Omega I_{C_\varepsilon}(\nabla u(x))dx,$$

and

$$\widetilde{\mathcal{G}}_\varepsilon(u) := \int_\Omega I_{\mathrm{int}(\mathrm{C}_\varepsilon)}(\nabla u(x))dx.$$

Let $\Gamma_{L^1}(\mathcal{G}_\varepsilon)$ and $\Gamma_{L^1}(\widetilde{\mathcal{G}}_\varepsilon)$ be their lower semicontinuous envelopes with respect to the L^1-topology. Since $\mathrm{int}(C_\varepsilon) \neq \emptyset$, by [26, Proposition 1.1.5] we have that $\mathrm{cl}(\mathrm{int}(\mathrm{C}_\varepsilon)) = \mathrm{cl}(\mathrm{C}_\varepsilon)$. Therefore, by Theorem 4.1 and Remark 4.2, we get

$$\Gamma_{w^*}(\widetilde{\mathcal{G}}_\varepsilon)(u) = \int_\Omega I_{\mathrm{cl}(C_\varepsilon)}(\nabla u(x))dx, \tag{39}$$

for every $u \in W^{1,\infty}(\Omega;\mathbb{R}^d)$.
On the other hand, since

$$\int_\Omega I_{\mathrm{cl}(C_\varepsilon)}(\nabla u(x))dx \leq \mathcal{G}_\varepsilon(u) \leq \widetilde{\mathcal{G}}_\varepsilon(u),$$

we have $\Gamma_{w^*}(\widetilde{\mathcal{G}}_\varepsilon) = \Gamma_{w^*}(\mathcal{G}_\varepsilon)$.
We notice that the latter equality and the representation formula (39) imply that for every $u \in W^{1,\infty}(\Omega;\mathbb{R}^d)$

$$\Gamma_{w^*}(\mathcal{G}_\varepsilon)(u) = 0 \Longleftrightarrow \nabla u(x) \in \mathrm{cl}(C_\varepsilon) \text{ for a.e. } x \in \Omega.$$

In particular, if $u \in W^{1,\infty}(\Omega;\mathbb{R}^d)$ is such that $\nabla u(x) \in \mathrm{cl}(C_\varepsilon)$ for a.e. $x \in \Omega$ then there exists a sequence (v_k^ε) converging weakly* to u in $W^{1,\infty}(\Omega;\mathbb{R}^d)$ such that

$$0 = \int_\Omega I_{\mathrm{cl}(C_\varepsilon)}(\nabla u(x))dx = \lim_k \int_\Omega I_{C_\varepsilon}(\nabla v_k^\varepsilon(x))dx.$$

Thus, by the regularity of Ω, the previous identity implies that there exists $\overline{k}$ (depending on ε) such that for every $k \geq \overline{k}$

$$\begin{cases} \nabla v_k^\varepsilon(x) \in C_\varepsilon \text{ for a.e. } x \in \Omega, \\ \\ \|u - v_k^\varepsilon\|_{L^\infty} \leq \varepsilon, \end{cases}$$

which equivalently means that for every $k \geq \overline{k}$

$$\begin{cases} f(\nabla v_k^\varepsilon(x)) \leq \lambda + \varepsilon \text{ for a.e. } x \in \Omega, \\ \\ \|u - v_k^\varepsilon\|_{L^\infty} \leq \varepsilon. \end{cases} \tag{40}$$

Now for every $n \in \mathbb{N}$ let $\varepsilon_n > 0$ be such that $\varepsilon_n \to 0$. Since $\nabla \bar{u}(x) \in \mathrm{cl}(C_{\varepsilon_n})$ for a.e. $x \in \Omega$ and for every $n \in \mathbb{N}$, by applying (40) with ε_n, we can find two sequences (k_n) strictly increasing and such that $k_n \geq n$, and $(v_{k_n}^{\varepsilon_n}) \subseteq W^{1,\infty}(\Omega;\mathbb{R}^d)$ satisfying

$$\begin{cases} f(\nabla v_{k_n}^{\varepsilon_n}(x)) \leq \lambda + \varepsilon_n \text{ for a.e. } x \in \Omega, \\ \\ \|\overline{u} - v_{k_n}^{\varepsilon_n}\|_{L^\infty} \leq \varepsilon_n. \end{cases} \tag{41}$$

Thus, we can conclude that for every $n \in \mathbb{N}$ and $\varepsilon_n > 0$ there exists $v_{k_n}^{\varepsilon_n}$ such that $\|\overline{u} - v_{k_n}^{\varepsilon_n}\|_{L^\infty} \leq \varepsilon_n$ and

$$\operatorname*{ess\,sup}_{x\in\Omega} f(\nabla v_{k_n}^{\varepsilon_n}) \leq \lambda + \varepsilon_n.$$

Thanks to the coercivity assumption (37), it results that $(v_{k_n}^{\varepsilon_n})$ weakly$*$ converges to $\bar{u}$ in $W^{1,\infty}(\Omega;\mathbb{R}^d)$. As consequence, it results that

$$\Gamma_{w^*}(F)(u) \leq \lim_{\varepsilon_n\to 0} \operatorname*{ess\,sup}_{x\in\Omega} f(\nabla v_{k_n}^{\varepsilon_n}) \leq \lambda.$$

Thus, it suffices to define $u_{\varepsilon_n} := v_{k_n}^{\varepsilon_n}$, to conclude the proof.

Step 2. Next we remove the coercivity assumption on f, just assuming that f admits minimum and $f(\underline{0}) = \min_{\mathbb{R}^{d\times N}} f = 0$.
For every $n \in \mathbb{N}$ and every $\xi \in \mathbb{R}^{d\times N}$, define f_n the level convex function given by

$$f_n(\xi) := \max\{f(\xi), \frac{1}{n}|\xi|\}.$$

Clearly f_n satisfies all the assumptions in Step 1. Thus, defining $f_n^{ls} := (f_n)^{ls}$, and denoting by F_n the functional defined as $W^{1,\infty}(\Omega,\mathbb{R}^d) \ni u \to F_n(u) := \operatorname{ess\,sup}_{x\in\Omega} f_n(\nabla u(x))$, we deduce that

$$\Gamma_{w*}(F_n)(u) = \operatorname*{ess\,sup}_{x\in\Omega} f_n^{ls}(\nabla u), \tag{42}$$

for every $u \in W^{1,\infty}(\Omega,\mathbb{R}^d)$
Moreover F_n decreasingly converges to F since $F_n(u) = \max\{F(u), \frac{1}{n}\|\nabla u\|_{L^\infty}\}$ (see [18, Remark 3.7]). Thus, by virtue of Proposition 3.4 (2)-(3) we can conclude that

$$\Gamma_{w*}(F)(u) = \Gamma(w^*)\text{-}\lim_{n\to+\infty} F_n(u) = \Gamma(w^*)\text{ - }\lim_{n\to+\infty} \Gamma_{w*}(F_n)(u), \tag{43}$$

for every $u \in W^{1,\infty}(\Omega,\mathbb{R}^d)$. Since $f_n(\xi) \leq f(\xi) + \frac{1}{n}|\xi|$ for every $\xi \in \mathbb{R}^{d\times N}$, then

$$f_n^{ls}(\xi) - \frac{1}{n}|\xi| \leq f(\xi)$$

for every $\xi \in \mathbb{R}^{d\times N}$. The continuity of $\frac{1}{n}|\cdot|$ entails

$$f_n^{ls}(\xi) - \frac{1}{n}|\xi| \le f^{ls}(\xi). \tag{44}$$

that yields to

$$\begin{aligned}\operatorname*{ess\,sup}_{x\in\Omega} f_n^{ls}(\nabla u(x)) &\le \operatorname*{ess\,sup}_{x\in\Omega}(f^{ls}(\nabla u(x)) + \frac{1}{n}|\nabla u(x)|)\\ &\le \operatorname*{ess\,sup}_{x\in\in\Omega} f^{ls}(\nabla u(x)) + \frac{1}{n}\|\nabla u\|_{L^\infty},\end{aligned}$$

for every $u \in W^{1,\infty}(\Omega,\mathbb{R}^d)$. Thanks to (42), we obtain that

$$\Gamma_{w*}(F_n)(u) \le \operatorname*{ess\,sup}_{x\in\in\Omega} f^{ls}(\nabla u(x)) + \frac{1}{n}\|\nabla u\|_{L^\infty}.$$

By the latter inequality, by (43) and by (36) we get that

$$\begin{aligned}\Gamma_{w*}(F)(u) &= \Gamma(w*)\text{-}\lim_{n\to+\infty}\Gamma_{w*}(F_n)(u)\\ &\le \lim_n(\operatorname*{ess\,sup}_{x\in\Omega} f^{ls}(\nabla u(x)) + \frac{1}{n}\|\nabla u\|_{L^\infty})\\ &= \operatorname*{ess\,sup}_{x\in\Omega} f^{ls}(\nabla u(x) = \overline{F}(u).\end{aligned}$$

Step 3. Now we remove the assumption that f admits a minimum. We assume that f admits a real infimum. The existence of the real infimum of f guarantees that F also admits a real infimum and they coincide. By (10) it results that

$$\inf_{W^{1,\infty}(\Omega,\mathbb{R}^d)} F(u) = \inf_{W^{1,\infty}(\Omega,\mathbb{R}^d)} \Gamma_{w*}(F)(u) = \inf_{\mathbb{R}^{d\times N}} f.$$

Thanks to Remark 4.1 there exist two sequences $(\xi_n) \subseteq \mathbb{R}^{d\times N}$ and $(\lambda_n)_n \searrow \inf_{\mathbb{R}^{d\times N}} f$ such that

$$f(\xi_n) \le \lambda_n \text{ and } \limsup_{\xi\to 0} f(\xi_n+\xi) \le \lambda_n \quad \forall n\in\mathbb{N}.$$

Then $(u_n) \subseteq W^{1,\infty}(\Omega;\mathbb{R}^d)$ given by $u_n(x) := \xi_n\cdot x$ is an infimizing sequence since

$$\lim_{n\to+\infty} F(u_n) = \lim_{n\to+\infty} f(\xi_n) = \inf_{\mathbb{R}^{d\times N}} f = \inf_{W^{1,\infty}(\Omega,\mathbb{R}^d)} F. \tag{45}$$

Consider, for every $n\in\mathbb{N}$ and for every $u \in W^{1,\infty}(\Omega;\mathbb{R}^d)$ the functional

$$G_n(u) := \max\{F(u+u_n),\lambda_n\} - \lambda_n = \max\{F(u+u_n)-\lambda_n, 0\} = \operatorname*{ess\,sup}_{x\in\Omega} g_n(\nabla u(x)),$$

where g_n is the function defined as

$$\mathbb{R}^{d\times N} \ni \xi \to g_n(\xi) := \max\{f(\xi+\xi_n),\lambda_n\} - \lambda_n = \max\{f(\xi+\xi_n)-\lambda_n, 0\} \ge 0.$$

Then $g_n(0) = 0 = \min_{\mathbb{R}^{d\times N}} g_n$. Then G_n verifies all the assumptions in Step 2, g_n being in particular level convex. Thus, applying the previous step and Proposition 3.4(5) we obtain that

$$\operatorname*{ess\,sup}_{x\in\Omega} g_n^{ls}(\nabla u(x)) = \Gamma_{w*}(G_n)(u) = \max\{\Gamma_{w*}(F)(u+u_n),\lambda_n\} - \lambda_n \tag{46}$$

On the other hand, by (15), it results,

$$g_n^{ls} = \max\{f(\cdot+\xi_n)^{ls},\lambda_n\} - \lambda_n.$$

In particular, for every $\xi \in \mathbb{R}^{d\times N}$,

$$g_n^{ls}(\xi) = \max\{(f(\cdot + \xi_n)^{ls})(\xi), \lambda_n\} - \lambda_n = \max\{f^{ls}(\xi + \xi_n), \lambda_n\} - \lambda_n.$$

From the latter equality, and the first identity in (46), we deduce that

$$\Gamma_{w*}(G_n)(u) = \max\{\operatorname*{ess\,sup}_{x\in\Omega} f^{ls}(\nabla u + \nabla u_n), \lambda_n\} - \lambda_n.$$

By the last equality in (46) and a translation argument,

$$\max\{\Gamma_{w*}(F)(u), \lambda_n\} = \Gamma_{w*}(G_n)(u - u_n) + \lambda_n = \max\{\operatorname*{ess\,sup}_{x\in\Omega} f^{ls}(\nabla u(x)), \lambda_n\}.$$

Taking the limit as $n \to +\infty$ and exploiting (10) and (45), we have

$$\Gamma_{w*}(F)(u) = \lim_{n\to+\infty} \max\{\Gamma_{w*}(F)(u), \lambda_n\} = \lim_{n\to+\infty} \max\{\operatorname*{ess\,sup}_{x\in\Omega} f^{ls}(\nabla u(x)), \lambda_n\} = \operatorname*{ess\,sup}_{x\in\Omega} f^{ls}(\nabla u(x)).$$

Step 4. Now we treat the last case, where $\inf_{\mathbb{R}^d} f = -\infty$. Defining, for every $m \in \mathbb{R}^+$ the function $f_m := \sup\{f, -m\}$ we have that f_m admits a real infimum and falls into the case described in Step 3. Thus, if for every $u \in W^{1,\infty}(\Omega;\mathbb{R}^d)$ we define $F_m(u) := \operatorname{ess\,sup}_{x\in\Omega} f_m(\nabla u(x))$, then it results that (once again, exploiting the level convexity of f, and applying [18, Proposition 2.6])

$$\max\{\Gamma_{w*}(F)(u), -m\} = \Gamma_{w*}(F_m)(u) := \operatorname*{ess\,sup}_{x\in\Omega} f_m^{ls}(\nabla u(x)) = \max\{\operatorname*{ess\,sup}_{x\in\Omega} f^{ls}(\nabla u), -m\}.$$

The proof is concluded by sending $m \to +\infty$. □

Remark 4.3 In the same spirit of Remark 4.2, the assumptions on Ω guarantee that if $f = f(\xi)$ is coercive, then the relaxed functional $\Gamma_{w*}(F)$ coincides on $W^{1,\infty}(\Omega;\mathbb{R}^d)$ with the lower semicontinuous envelopes of F with respect to the L^∞ and L^1 convergences, i.e. $\Gamma_{w*}(F) = \Gamma_{w^*_{seq}}(F) = \Gamma_{L^\infty}(F) = \Gamma_{L^1}(F)$ by the classical embedding theorems. On the other hand, Theorem 2.1, shows that even without coercivity assumptions on f, it holds

$$\Gamma_{w*}(F) = \Gamma_{w^*_{seq}}(F). \tag{47}$$

This fact is not surprising since the level convexity of F entails the validity of Corollary 3.1.

Thanks to Theorem 2.1, we can deduce that f^{ls} is the strong Morrey quasiconvex "envelope" of f, i.e. the greatest strong Morrey quasiconvex minorant of f, provided that f satisfies (H).

Corollary 4.1 *Let $f : \mathbb{R}^{d\times N} \to [-\infty, +\infty]$ be a level convex Borel function satisfying (H). Then f^{ls} is the greatest strong Morrey quasiconvex function less than or equal to f.*

Proof Thanks to Remark 3.4, it is sufficient to show that $h \le f^{ls}$ for every strong Morrey quasiconvex function such that $h \le f$. Let $h : \mathbb{R}^{d\times N} \to [-\infty, +\infty]$ be a strong Morrey quasiconvex function such that $h \le f$. Then, if $Q =]0,1[^N$, the associated supremal functional $W^{1,\infty}(Q;\mathbb{R}^d) \ni u \to S_h(u) := \operatorname{ess\,sup}_{x\in Q} h(\nabla u)$ satisfies $S_h \le F$ on $W^{1,\infty}(Q;\mathbb{R}^d)$ and, by [12, Theorem 2.6], is a w^*_{seq}-lower semicontinuous functional. Then (47) and Theorem 2.1 imply that

$$S_h(u) \le \Gamma_{w*}(F)(u) \text{ for every } u \in W^{1,\infty}(Q;\mathbb{R}^d)$$

and evaluating this latter expression on affine functions $u(x) := \xi \cdot x$, with $\xi \in \mathbb{R}^{d\times N}$, we get $h \le f^{ls}$. □

Theorem 2.1 allows us to extend the relaxation results for indicator functionals provided by Theorem 4.1 to the case where the convex set is unbounded, and not necessarily open, and with no requirement that $\underline{0} \in \operatorname{int} C$.

Corollary 4.2 *Let Ω be a bounded open set of $\mathbb{R}^N$ with Lipschitz boundary. Let $C \subseteq \mathbb{R}^{d\times N}$ be a convex Borel set with nonempty interior. Let $\mathcal{I}, \overline{\mathcal{I}} : W^{1,\infty}(\Omega;\mathbb{R}^d) \to [0,+\infty]$ be the functionals defined by (34) and (35). Then*

$$\overline{\mathcal{I}}(u) = \Gamma_{w*}(\mathcal{I})(u) = \Gamma_{w^*_{seq}}(\mathcal{I})(u) \quad \forall u \in W^{1,\infty}(\Omega, \mathbb{R}^d). \tag{48}$$

Proof We start observing that, due to convexity of C, Corollary 3.1 guarantees the second equality in (48). First we note that, thanks to Ioffe's Theorem (see, for example, Theorem 2.3.1. in [39]) for every $u \in W^{1,\infty}(\Omega;\mathbb{R}^d)$

$$\overline{\mathcal{I}}(u) \le \Gamma_{w*}(\mathcal{I})(u).$$

In order to conclude the proof, it is sufficient to show that for every $u \in W^{1,\infty}(\Omega;\mathbb{R}^d)$

$$\Gamma_{w^*}(\mathcal{I})(u) \le \overline{\mathcal{I}}(u).$$

Without loss of generality, assume that $u \in W^{1,\infty}(\Omega,\mathbb{R}^d)$ is such that $\int_\Omega I_{\mathrm{cl}(C)}(\nabla u(x))dx = 0$, i.e. $\nabla u(x) \in \mathrm{cl}(C)$ for a.e. $x \in \Omega$. Thus, arguing as above, we have that $\Gamma_{w^*}(G)(u) = \operatorname{ess\,sup}_{x\in\Omega} I_{\mathrm{cl}(C)}(\nabla u(x)) = 0$. In particular, by Definition 3.3 and Proposition 3.4(2.), for every weak* neighborhood U of u it results that

$$\inf_{v\in U} \operatorname*{ess\,sup}_{x\in\Omega} I_C(\nabla v(x)) = 0,$$

which in turns guarantees that there exists $\bar{v} \in U$ such that

$$I_C(\nabla \bar{v}(x)) = 0 \text{ for a.e. } x \in \Omega.$$

Consequently for every U,

$$\inf_{v\in U} \int_\Omega I_C(\nabla v(x))dx = 0,$$

i.e.

$$\Gamma_{w^*}\mathcal{I}(u) = 0 = \overline{\mathcal{I}}(u). \qquad \square$$

5 The L^p-Approximation

In this section we prove Theorem 2.2, in details, we study Γ-convergence, as $p \to +\infty$, of the functionals $F_p : C(\bar{\Omega},\mathbb{R}^d) \to [0,+\infty]$ given by

$$F_p(u) := \begin{cases} \left(\int_\Omega f^p(x,\nabla u(x))dx\right)^{1/p}, & \text{if } u \in W^{1,p}(\Omega,\mathbb{R}^d), \\ +\infty, & \text{otherwise} \end{cases} \tag{49}$$

where $f : \Omega \times \mathbb{R}^{d\times N}$ is $\mathcal{L}^N \otimes \mathcal{B}_{d\times N}$ function satisfying the growth condition (5). We show that, as $p \to \infty$, $(F_p)_{p\ge1}$ Γ-converges with respect to the uniform convergence to the functional $\bar{F} : C(\bar{\Omega},\mathbb{R}^d) \to [0,+\infty]$ given by (7).

With this aim, we first prove the following result, containing an L^p- approximation for f^{lslc}, that will be useful in the proof of some particular cases of Theorem 2.2. It generalizes [15, Proposition 2.9] where f is assumed to be level convex and lower semicontinuous.

Proposition 5.1 *Let $f : \mathbb{R}^{d\times N} \to \mathbb{R}$ be a Borel function satisfying*

$$f(\xi) \ge \alpha|\xi| \tag{50}$$

for a fixed $\alpha > 0$ and for every $\xi \in \mathbb{R}^{d\times N}$. For every $p \ge 1$, let $(f^p)^{**}$ be the lower semicontinuous and convex envelope of f^p. Then

$$\lim_{p\to\infty}((f^p)^{**})^{1/p}(\xi) = f^{lslc}(\xi). \tag{51}$$

Moreover if f is level convex, then

$$\lim_{p\to\infty}((f^p)^{**})^{1/p}(\xi) = \lim_{p\to\infty}(Qf^p)^{1/p}(\xi) = f^{ls}(\xi) \tag{52}$$

where $Qf^p := Q(f^p)$ is the quasiconvex envelope of f^p in (22).

Proof Clearly the family $((f^p)^{**})^{1/p})_p$ is not decreasing and for every $\xi \in \mathbb{R}^{d\times N}$ and $p \in [1,+\infty[$ we have that

$$((f^p)^{**})^{1/p}(\xi) \le f(\xi).$$

Since $((f^p)^{**})^{1/p}$ is lower semicontinuous and level convex, it results that

$$((f^p)^{**})^{1/p}(\xi) \leq f^{lslc}(\xi) \tag{53}$$

for every $\xi \in \mathbb{R}^{d\times N}$, and $p \in [1, +\infty[$. Thus, the first inequality in (51) follows as $p \to +\infty$. Moreover, by [15, Proposition 2.9] applied to f^{lslc}, we have that

$$f^{lslc}(\xi) = \lim_{p\to\infty}(((f^{lslc})^p)^{**})^{1/p}(\xi) \leq \lim_{p\to\infty}((f^p)^{**})^{1/p}(\xi) \tag{54}$$

for every $\xi \in \mathbb{R}^{d\times N}$. Now we assume that f is level convex. By (51) and (25) we get that

$$f^{ls}(\xi) = \lim_{p\to\infty}((f^p)^{**})^{1/p}(\xi)$$

We note that for every fixed $p \geq 1$ the function $(f^p)^{**}$ is quasiconvex (see Definition 3.5). Then $(f^p)^{**} \leq Qf^p \leq f^p$ that yields to $((f^p)^{**})^{1/p} \leq (Qf^p)^{1/p} \leq f$. By the continuity of Qf^p (see Remark 3.2), it follows that for every $p \geq 1$

$$((f^p)^{**})^{1/p} \leq (Qf^p)^{1/p} \leq f^{ls}. \tag{55}$$

By applying Hölder's inequality, it is easy to show that the family $(\left(Qf^p\right)^{1/p})_p$ is not decreasing. So, by (55), it results

$$f^{ls}(\xi) = \lim_{p\to\infty}((f^p)^{**})^{1/p}(\xi) \leq \lim_{p\to\infty}(Qf^p)^{1/p}(\xi) \leq f^{ls}(\xi),$$

for every $\xi \in \mathbb{R}^{d\times N}$, which proves formula (52).

□

Remark 5.1 Let $f : \mathbb{R}^{d\times N} \to \mathbb{R}$ be a Borel function. For every $\xi \in \mathbb{R}^{d\times N}$ we denote

$$Q_\infty f(\xi) := \lim_{p\to\infty}\left(Qf^p\right)^{1/p}(\xi) = \sup_{p\geq 1}\left(Qf^p\right)^{1/p}(\xi). \tag{56}$$

Observe that (56) coincides with (8) since f does not depend on x.
Note that, if $N = 1$ or $d = 1$, then $Qf^p = (f^p)^{**}$ for every $p \geq 1$. Therefore, if f satisfies (50), by Proposition 5.1, we get that $Q_\infty f = f^{lslc}$.

In [15] it has been introduced the class of functions $f : \mathbb{R}^{d\times N} \to [0, +\infty[$ satisfying $f = \lim_{p\to\infty}(Q(f^p))^{1/p}$. They have been referred as curl-∞ quasiconvex. If f is continuous, in [34] it has been remarked that any curl-∞ quasiconvex function is strong Morrey quasiconvex (see (4)), while it is currently an open question whether the converse is true for coercive functions. The proposition below establishes, without further assumptions, that the supremum of strong Morrey quasiconvex functions is itself strong Morrey quasiconvex. In particular, if $f : \mathbb{R}^{d\times N} \to [0, +\infty[$ is a Borel function then, by the very definition (56), it results that $Q_\infty f$ is strong Morrey quasiconvex.

Proposition 5.2 *Let I be a family of indices and let $(f_\eta)_{\eta\in I}$, be a family of strong Morrey quasiconvex functions ($f_\eta : \mathbb{R}^{d\times N} \to \mathbb{R}$ for any $\eta \in I$). Then the function $\hat{f} := \sup_\eta f_\eta$ is strong Morrey quasiconvex. In particular, if $f : \mathbb{R}^{d\times N} \to [0, +\infty[$ is a Borel function then the sequence $(\left(Qf^p\right)^{1/p})$ converges to the strong Morrey quasiconvex function $Q_\infty f$.*

Proof Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary as above. For every $\eta \in I$ the functional

$$W^{1,\infty}(\Omega, \mathbb{R}^d) \ni u \to F_\eta(u) := \operatorname*{ess\,sup}_\Omega f_\eta(\nabla u)$$

is sequentially weakly* lower semicontinuous on $W^{1,\infty}(\Omega, \mathbb{R}^d)$, see [12, Theorem 2.6]. This implies that the functional $W^{1,\infty}(\Omega, \mathbb{R}^d) \ni u \to \hat{F}(u) := \sup_\eta F_\eta(u)$ is also sequentially weakly* lower semicontinuous. Since

$$\hat{F}(u) = \sup_\eta \operatorname*{ess\,sup}_\Omega f_\eta(\nabla u) = \operatorname*{ess\,sup}_\Omega \sup_\eta f_\eta(\nabla u) = \operatorname*{ess\,sup}_\Omega \hat{f}(\nabla u),$$

then, thanks to the necessary condition for sequentially weak* lower semicontinuity of supremal functionals in [12, Theorem 2.7], we can conclude that $\hat{f}$ is strong Morrey quasiconvex.

In particular, in order to show that $Q_\infty f$ is strong Morrey quasiconvex, it is sufficient to recall that, by [12, Proposition 2.4], for any $p \geq 1$ the function Qf^p is strong Morrey quasiconvex. □

Proof of Theorem 2.2 It will be achieved in several steps, some of them follow along the lines of [18, Proof of Theorem 3.4]. First we prove that for every $p > N$, the relaxed functional $\Gamma_{L^\infty}(F_p)$ admits an integral representation. In the step 2. we introduce the function f_∞ appearing in (7) and obtain the comparison in (8).Then step 3. is devoted to show that the sequence $\Gamma_{L^\infty}(F_n)$ $\Gamma(L^\infty)$-converges, as $n \to \infty$, to the functional $\bar{F}$ while in step. 4 we prove that for every $(p_n) \to +\infty$ the sequence $\Gamma_{L^\infty}(F_{p_n})$ $\Gamma(\tau)$-converges to $\bar{F}$ as $n \to \infty$, which concludes the proof, in light of Definition 3.4.

Step 1. For every $p \geq 1$ let $\Gamma_{L^\infty}(F_p) : C(\bar{\Omega}, \mathbb{R}^d) \to [-\infty, +\infty]$ be the lower semicontinuous envelope of the functional F_p in (6) with respect to the uniform convergence. Let $G_p : W^{1,p}(\Omega, \mathbb{R}^d) \to [-\infty, +\infty]$ be the functional given by

$$G_p(u) := \left(\int_\Omega f^p(x, \nabla u(x)) dx \right)^{1/p}.$$

Then, taking into account (5), by Theorem 3.3, there exists a Carathéodory function $\widetilde{f^p}$, quasiconvex in the second variable, such that

$$\max\left\{ (\frac{1}{\alpha}|\xi| - \alpha)^p, 0 \right\} \leq Qf^p(x, \xi) \leq \widetilde{f^p}(x, \xi) \leq \beta^p(1 + |\xi|)^p, \tag{57}$$

for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^{d\times N}$, and

$$\Gamma_{w_{seq}}(G_p)(u) := \left(\int_\Omega \widetilde{f^p}(x, \nabla u(x)) dx \right)^{1/p}$$

for every $u \in W^{1,p}(\Omega, \mathbb{R}^d)$. Now we show that for every $p > N$, $\Gamma_{L^\infty}(F_p)$ coincides with the functional $\phi_p : C(\bar{\Omega}, \mathbb{R}^d) \to [-\infty, +\infty]$ given by

$$\phi_p(u) := \begin{cases} \left(\int_\Omega \widetilde{f^p}(x, \nabla u(x)) dx \right)^{1/p}, & \text{if } u \in W^{1,p}(\Omega, \mathbb{R}^d), \\ +\infty & \text{otherwise.} \end{cases}$$

In order to show that $\phi_p \leq \Gamma_{L^\infty}(F_p)$ we notice that for every $p > 1$ the functional ϕ_p is lower semicontinuous on $C(\overline{\Omega}, \mathbb{R}^d)$ with respect to the uniform convergence. In fact, let $(u_n) \subseteq C(\bar{\Omega}, \mathbb{R}^d)$ be such that $u_n \to u$ uniformly and $\liminf_{n\to\infty} \phi_p(u_n) < +\infty$. Without relabelling, take a subsequence such that $\lim_{n\to\infty} \phi_p(u_n) = \liminf_{n\to\infty} \phi_p(u_n)$. Thanks to the coercivity assumption (5), we have that the sequence (u_n) is bounded in $W^{1,p}(\Omega, \mathbb{R}^d)$. Therefore, up to a not relabelled subsequence, (u_n) weakly converges to u in $W^{1,p}(\Omega, \mathbb{R}^d)$. Then

$$\phi_p(u) = \Gamma_{w_{seq}}(G_p)(u) \leq \liminf_{n\to\infty} \Gamma_{w_{seq}}(G_p)(u_n) = \liminf_{n\to\infty} \phi_p(u_n).$$

Since $\phi_p \leq F_p$ on $C(\bar{\Omega}, \mathbb{R}^d)$ and ϕ_p is lower semicontinuous with respect to the uniform convergence, we obtain that

$$\phi_p(u) \leq \Gamma_{L^\infty}(F_p)(u) \quad \forall\, u \in C(\bar{\Omega}, \mathbb{R}^d). \tag{58}$$

On the other hand, for every $p > N$ the functional $\Gamma_{L^\infty}(F_p)$ is sequentially lower semicontinuous on $W^{1,p}(\Omega, \mathbb{R}^d)$ with respect to the weak convergence of $W^{1,p}(\Omega, \mathbb{R}^d)$. In fact, if $(u_n) \subseteq W^{1,p}(\Omega, \mathbb{R}^d)$ is such that $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega, \mathbb{R}^d)$ then, thanks to Rellich-Kondrachov Theorem, we have that $u_n \in C(\bar{\Omega}, \mathbb{R}^d)$ and $u_n \to u$ uniformly. In particular it follows that $\Gamma_{L^\infty}(F_p)(u) \leq \liminf_{n\to\infty} \Gamma_{L^\infty}(F_p)(u_n)$.

Since

$$\Gamma_{L^\infty}(F_p) \leq F_p = G_p \quad \text{on } W^{1,p}(\Omega, \mathbb{R}^d),$$

we get that for every $p > N$

$$\Gamma_{L^\infty}(F_p)(u) \leq \Gamma_{w_{seq}}(G_p)(u) = \phi_p(u) \quad \forall\, u \in W^{1,p}(\Omega, \mathbb{R}^d). \tag{59}$$

Inequalities (58) and (59) imply that for every $p > N$

$$\Gamma_{L^\infty}(F_p)(u) = \phi_p(u) = \left(\int_\Omega \widetilde{f^p}(x, \nabla u(x))dx\right)^{1/p} \quad \forall\, u \in W^{1,p}(\Omega, \mathbb{R}^d).$$

If we show that $\Gamma_{L^\infty}(F_p)(u) < +\infty$ if and only if $u \in W^{1,p}(\Omega, \mathbb{R}^d)$ then we can conclude that $\Gamma_{L^\infty}(F_p) = \phi_p$ on $C(\bar{\Omega}, \mathbb{R}^d)$ for every $p > N$. In fact if $u \in C(\bar{\Omega}, \mathbb{R}^d)$ is such that $\Gamma_{L^\infty}(F_p)(u) < +\infty$ then there exists a sequence $(u_n) \subseteq C(\bar{\Omega}, \mathbb{R}^d)$ such that $u_n \to u$ uniformly and $\lim_{n\to\infty} F_p(u_n) = \Gamma_{L^\infty}(F_p)(u) < +\infty$. Thanks to the coercivity assumption (5), we have that the sequence (u_n) is bounded in $W^{1,p}(\Omega, \mathbb{R}^d)$ and, up to a subsequence, weakly converges to u in $W^{1,p}(\Omega, \mathbb{R}^d)$ when $p > 1$. In particular $u \in W^{1,p}(\Omega, \mathbb{R}^d)$. The viceversa is trivial.

Step 2. If $p < q$ then, by applying Hölder's inequality, we have that $F_p \leq (\mathcal{L}^N(\Omega))^{1/p-1/q} F_q$. In particular

$$\Gamma_{L^\infty}(F_p) \leq (\mathcal{L}^N(\Omega))^{1/p-1/q} \Gamma_{L^\infty}(F_q).$$

Since for every $p \geq 1$, $\widetilde{f^p}$ is a Carathédory function, we deduce that

$$(\widetilde{f^p})^{1/p}(x, \xi) \leq (\widetilde{f^q})^{1/q}(x, \xi) \tag{60}$$

for a.e. $x \in \Omega$ and $\xi \in \mathbb{R}^{d\times N}$. Then, set

$$f_\infty(x, \xi) := \sup_{n\geq 1} (\widetilde{f^n})^{1/n}(x, \xi), \tag{61}$$

it results that f_∞ is $\mathcal{L}^N \otimes \mathcal{B}_{d\times N}$-measurable function, being the countable supremum of Carathéodory functions, and for a.e. $x \in \Omega$ and $\xi \in \mathbb{R}^{d\times N}$

$$f_\infty(x, \xi) = \lim_{n\to\infty} \left(\widetilde{f^n}\right)^{1/n}(x, \xi). \tag{62}$$

Hence, taking into account (57), f_∞ satisfies the growth condition

$$\max\left\{\frac{1}{\alpha}|\xi| - \alpha, 0\right\} \leq f_\infty(x, \xi) \leq \beta(1 + |\xi|). \tag{63}$$

Moreover, thanks to Proposition 5.2, for a.e. fixed $x \in \Omega$ the function $f_\infty(x, \cdot)$ is strong Morrey quasiconvex. Finally, by (57), it results that

$$Qf^n(x, \xi) \leq \widetilde{f^n}(x, \xi),$$

for every $n \in \mathbb{N}$, for a.e. $x \in \Omega$ and $\xi \in \mathbb{R}^{d\times N}$. This implies that $Q_\infty f(x, \xi) \leq f_\infty(x, \xi)$ for a.e. $x \in \Omega$ and $\xi \in \mathbb{R}^{d\times N}$.
Moreover, it is easy to show that if $(p_n)_n$ is a divergent sequence, that is $p_n \to +\infty$, the $\mathcal{L}^N \otimes \mathcal{B}_{d\times N}$-measurable function $h_\infty : \Omega \times \mathbb{R}^{d\times N} \to [0, \infty]$ defined by

$$h_\infty(x, \xi) := \sup_{n\geq 1} (\widetilde{f^{p_n}})^{1/p_n}(x, \xi),$$

satisfies

$$\operatorname*{ess\,sup}_\Omega h_\infty(x, Du(x)) = \operatorname*{ess\,sup}_\Omega f_\infty(x, Du(x)) \quad \forall u \in W^{1,\infty}(\Omega, \mathbb{R}^d). \tag{64}$$

Indeed, for every fixed $u \in W^{1,\infty}(\Omega, \mathbb{R}^d)$ and for every fixed $\epsilon > 0$ there exists $\bar{n} \in \mathbb{N}$ such that

$$\operatorname*{ess\,sup}_{\Omega} h_\infty(x, Du(x)) = \sup_{n\geq 1} \operatorname*{ess\,sup}_{\Omega} \left(\widetilde{f^{p_n}}\right)^{1/p_n}(x, Du(x)) \leq \operatorname*{ess\,sup}_{\Omega} (\widetilde{f^{p_{\bar{n}}}})^{1/p_{\bar{n}}}(x, Du(x)) + \epsilon. \tag{65}$$

Then, by (60) there exists a measurable set $\Omega' \subseteq \Omega$ such that $\mathcal{L}^N(\Omega \setminus \Omega') = 0$ and

$$(\widetilde{f^{p_{\bar{n}}}})^{1/p_{\bar{n}}}(x, \xi) \leq (\widetilde{f^n})^{1/n}(x, \xi),$$

for every $n \geq p_{\bar{n}}$, for every $x \in \Omega'$ and $\xi \in \mathbb{R}^{d\times N}$. In particular, (65) and (61) imply

$$\operatorname*{ess\,sup}_{\Omega} h_\infty(x, Du(x)) \leq \operatorname*{ess\,sup}_{\Omega} (\widetilde{f^n}(x, Du(x)))^{\frac{1}{n}} + \epsilon \leq \operatorname*{ess\,sup}_{\Omega} f_\infty(x, Du(x)) + \epsilon.$$

By sending ϵ to 0 we get that

$$\operatorname*{ess\,sup}_{\Omega} h_\infty(x, Du(x)) \leq \operatorname*{ess\,sup}_{\Omega} f_\infty(x, Du(x)).$$

The proof of the converse inequality is analoguous.

Step 3. We consider the sequence $(p_n)_n = (n)_n$ and we show that

$$\Gamma(L^\infty)\text{-}\lim_{n\to\infty} F_n = \bar{F}. \tag{66}$$

Note that, since the family $(\mathcal{L}^N(\Omega)^{-1/n} F_n)_{n\geq 1}$ is increasing, by Proposition 3.4(3)-(4), it follows that

$$\Gamma(L^\infty)\text{-}\lim_{n\to\infty} F_n = \Gamma(L^\infty)\text{-}\lim_{n\to\infty} \Gamma_{L^\infty}(F_n) = \lim_{n\to\infty} \Gamma_{L^\infty}(F_n).$$

Now we show that

$$\lim_{n\to\infty} \Gamma_{L^\infty}(F_n) \geq \bar{F}(u) \quad \forall u \in C(\bar{\Omega}, \mathbb{R}^d). \tag{67}$$

Without loss of generality we consider the case when $u \in C(\bar{\Omega}, \mathbb{R}^d)$ is such that $\sup_{n\geq 1} \Gamma_{L^\infty}(F_n)(u) < +\infty$. Thanks to the coercivity assumption (5), we have that $\sup_{n\geq 1} ||u||_{W^{1,p_n}(\Omega;\mathbb{R}^d)} =: M < +\infty$. It follows that $u \in W^{1,\infty}(\Omega; \mathbb{R}^d)$ and, by (63), it holds

$$\bar{F}(u) = \operatorname*{ess\,sup}_{x\in\Omega} f_\infty(x, \nabla u(x)) \leq \beta(1 + M) < +\infty.$$

Therefore, for every fixed $\varepsilon > 0$, there exists a measurable set $B_\varepsilon \subset \Omega$ such that $\mathcal{L}^N(B_\varepsilon) > 0$ and

$$\operatorname*{ess\,sup}_{x\in\Omega} f_\infty(x, \nabla u(x)) \leq f_\infty(x, \nabla u(x)) + \varepsilon$$

for every $x \in B_\varepsilon$. This implies

$$\operatorname*{ess\,sup}_{x\in\Omega} f_\infty(x, \nabla u(x))\mathcal{L}^N(B_\varepsilon) \leq \int_{B_\varepsilon} f_\infty(x, \nabla u(x))dx + \varepsilon\mathcal{L}^N(B_\varepsilon).$$

By Beppo Levi's Theorem, and Hölder's inequality we obtain

$$\begin{aligned}\operatorname*{ess\,sup}_{x\in\Omega} f_\infty(x, \nabla u(x))\mathcal{L}^N(B_\varepsilon) &\leq \lim_{n\to\infty} \int_{B_\varepsilon} (\widetilde{f^n})^{1/n}(x, \nabla u(x)dx + \varepsilon\mathcal{L}^N(B_\varepsilon) \\ &\leq \lim_{n\to\infty} \Big(\int_{B_\varepsilon} (\widetilde{f^n})(x, \nabla u(x))dx\Big)^{1/n} \big(\mathcal{L}^N(B_\varepsilon)\big)^{1-1/n} + \varepsilon\mathcal{L}^N(B_\varepsilon).\end{aligned}$$

It follows that

$$\operatorname*{ess\,sup}_{x\in\Omega} f_\infty(x, \nabla u(x)) \leq \lim_{n\to\infty} \Gamma_{L^\infty}(F_n)(u)\big(\mathcal{L}^N(B_\varepsilon)\big)^{-1/n} + \varepsilon \leq \lim_{n\to\infty} \Gamma_{L^\infty}(F_n)(u) + \varepsilon\,. \tag{68}$$

By passing to the limit when $\varepsilon \to 0$, we get (67).

In order to show the converse inequality

$$\lim_{n\to\infty} \Gamma_{L^\infty}(F_n)(u) \le \bar{F}(u) \quad \forall u \in C(\bar{\Omega},\mathbb{R}^d),$$

without loss of generality, we consider the case when $u \in W^{1,\infty}(\Omega;\mathbb{R}^d)$. Then

$$\Gamma_{L^\infty}(F_n)(u) = \Big(\int_\Omega \widetilde{f^n}(x,\nabla u(x))dx\Big)^{1/n} \le \big(\mathcal{L}^N(\Omega)\big)^{1/n} \operatorname*{ess\,sup}_{x\in\Omega} f_\infty(x,\nabla u(x)).$$

In particular, it follows

$$\lim_{n\to\infty} \Gamma_{L^\infty}(F_n)(u) \le \lim_{n\to\infty} \big(\mathcal{L}^N(\Omega)\big)^{1/n} \bar{F}(u) = \bar{F}(u),$$

for every $u \in W^{1,\infty}(\Omega,\mathbb{R}^d)$. The last inequality, together with (67), implies (66).

Step 4. Now we show that if $(p_n)_n$ diverges, i.e. $p_n \to +\infty$, then

$$\Gamma(L^\infty)\text{-}\lim_{n\to\infty} F_{p_n}(u) = \bar{F}(u) \quad \forall u \in C(\bar{\Omega},\mathbb{R}^d).$$

First of all, we show the Γ-liminf inequality, that is

$$\Gamma(L^\infty)\text{-}\lim_{n\to\infty} F_{p_n}(u) \ge \bar{F}(u) \quad \forall u \in C(\bar{\Omega},\mathbb{R}^d).$$

Let $u,(u_{p_n})_n \subseteq C(\bar{\Omega},\mathbb{R}^d)$ be such that $u_{p_n} \to u$ uniformly in Ω. Fix $q \in \mathbb{N}$. Then, there exists $\bar{n} \in \mathbb{N}$ such that $p_n \ge q$ for every $n \ge \bar{n}$. Hence,

$$\Gamma_{L^\infty}(F_q)(u_{p_n}) \le \big(\mathcal{L}^N(\Omega)\big)^{1/q-1/p_n} \Gamma_{L^\infty}(F_{p_n})(u_{p_n})$$

for every $n \ge \bar{n}$ and, by passing to the liminf when $n \to \infty$, we get that for every $q \ge 1$

$$\Gamma_{L^\infty}(F_q)(u) \le \big(\mathcal{L}^N(\Omega)\big)^{1/q} \liminf_{n\to\infty} \Gamma_{L^\infty}(F_{p_n})(u_{p_n}).$$

By passing to the limit when $q \to \infty$, taking into account (66), we get

$$\bar{F}(u) \le \liminf_{n\to\infty} \Gamma_{L^\infty}(F_{p_n})(u_{p_n}) \le \liminf_{n\to\infty} F_{p_n}(u_{p_n}). \tag{69}$$

Taking into account (64), we have that

$$\Gamma_{L^\infty}(F_{p_n})(u) \le \big(\mathcal{L}^N(\Omega)\big)^{1/n}\Big(\int_\Omega \widetilde{f^{p_n}}(x,\nabla u(x))dx\Big)^{1/p_n} \le \big(\mathcal{L}^N(\Omega)\big)^{1/p_n} \operatorname*{ess\,sup}_{x\in\Omega} f_\infty(x,\nabla u(x)),$$

and we get that

$$\limsup_{n\to\infty} \Gamma_{L^\infty}(F_{p_n})(u) \le \operatorname*{ess\,sup}_{x\in\Omega} f_\infty(x,\nabla u(x)).$$

The last inequality, (69) and Proposition 3.4, (3) imply the Γ-convergence:

$$\Gamma(L^\infty)\text{-}\lim_{n\to\infty} F_{p_n}(u) = \bar{F}(u) \quad \forall u \in C(\bar{\Omega},\mathbb{R}^d). \qquad \square$$

Remark 5.2 We note the following facts.

1. If $f(x,\cdot)$ is continuous for a.e. $x \in \Omega$, Theorem 2.2 gives the same representation result for the Γ-limit shown in [34].
2. If the supremand $f(x,\cdot)$ is upper semicontinuous for a.e. $x \in \Omega$, then $f_\infty(x,\cdot) = Q_\infty f(x,\cdot)$ by Remark 3.3 and (61). The same conclusion holds when $f \equiv f(\xi)$.
   In addiction, if $f(x,\cdot)$ is upper semicontinuous and level convex for a.e. $x \in \Omega$, then, in view of (52) evaluated along the the sequence $(p_n) = (n)$, (7) can be specialized, since

   $$f_\infty(x,\cdot) = Q_\infty f(x,\cdot) = f^{ls}(x,\cdot) \quad \text{for a.e. } x \in \Omega.$$

   The same conclusion holds when $f \equiv f(\xi)$ is level convex.

3. If $N=1$ or $d=1$, then $Q_\infty f(x,\cdot)=f^{lslc}(x,\cdot)$ for a.e. $x\in\Omega$, (see Remark 5.1). Consequently, by the above arguments, if $f(x,\cdot)$ is upper semicontinuous or $f\equiv f(\xi)$ then we get that
$$f_\infty(x,\cdot)=Q_\infty f(x,\cdot)=f^{lslc}(x,\cdot)\quad \text{for a.e. } x\in\Omega.$$
4. In the case when $f_\infty(x,\cdot)=f^{lslc}(x,\cdot)$, the proof of the Γ-liminf inequality can be simplified. Indeed f^{lslc} satisfies the assumptions of [16, Theorem 3.1] and $f^{lslc}\le f$, then for every $u\in W^{1,\infty}(\Omega;\mathbb{R}^d)$
$$\begin{aligned}\operatorname*{ess\,sup}_{x\in\Omega} f^{lslc}(x,\nabla u(x)) &\le \Gamma(L^\infty)\text{-}\lim_{p\to\infty}\left(\int_\Omega (f^{lslc}(x,\nabla u(x)))^p dx\right)^{1/p}\\ &\le \Gamma(L^\infty)\text{-}\lim_{p\to\infty}\left(\int_\Omega (f^p(x,\nabla u(x)))dx\right)^{1/p}.\end{aligned}$$
It is also worth to note that the above inequality holds without imposing any growth from above on f.
5. We observe that if $f\equiv f(\xi)$, under the weaker assumption that f is a Borel function locally bounded and satisfying (up to a constant) (50), we can show that the family of functionals $\mathcal{F}_p : C(\bar{\Omega},\mathbb{R}^d)\to[0,+\infty]$ given by
$$\mathcal{F}_p(u):=\begin{cases}\left(\int_\Omega f^p(\nabla u(x))dx\right)^{1/p}, & \text{if } u\in W^{1,\infty}(\Omega,\mathbb{R}^d),\\ +\infty, & \text{otherwise}\end{cases}$$
$\Gamma(L^\infty)$-converges to the functional $\mathcal{F}: C(\bar{\Omega},\mathbb{R}^d)\to[0,+\infty]$ given by
$$\mathcal{F}(u):=\begin{cases}\operatorname*{ess\,sup}_{\Omega} Q_\infty f(\nabla u(x)), & \text{if } u\in W^{1,\infty}(\Omega,\mathbb{R}^d),\\ +\infty, & \text{otherwise.}\end{cases}$$
Indeed, in this case, it is sufficient to apply the relaxation result for integral functionals on Sobolev space with respect to the uniform convergence (see [40, Theorem 9.1]) to get that
$$\Gamma_{L^\infty}(\mathcal{F}_p)(u)=\begin{cases}\left(\int_\Omega Qf^p(\nabla u(x))dx\right)^{1/p}, & \text{if } u\in W^{1,\infty}(\Omega,\mathbb{R}^d),\\ +\infty, & \text{otherwise.}\end{cases}$$
Then the proof develops along the lines of the one of Theorem 2.2 and takes into account the identity $f_\infty=Q_\infty f$.
6. For the sake of completeness, with the same notations of Theorem 2.2, if N or $d=1$, one can assume Ω to be also convex and f to be only Borel measurable to obtain a representation formula for $\Gamma_{L^1}(G_p)$, see [42, Theorem 3.10]. In particular, one obtains, that
$$\Gamma_{L^1}(G_p)(u)=\left(\int_\Omega (f^p)^{**}(\nabla u(x))dx\right)^{1/p}\qquad \forall\, u\in W^{1,p}(\Omega,\mathbb{R}^d).$$
Then, assuming also that f satisfies (50), (7) is obtained in the same way as before, relying on the equality $\Gamma_{L^\infty}(F_p)=\phi_p=\Gamma_{L^1}(G_p)$ in $W^{1,p}(\Omega,\mathbb{R}^d)$.

6 Conclusions

Two main results have been obtained: the first deals with supremal representation in the vectorial case of the relaxed envelope of a level convex supremal functional in the homogeneous setting, (see Theorem 2.1); the second one concerns the variational approximation of a supremal functional through a sequence of power-law integral functionals, in the inhomogeneous setting, under a further growth condition (Theorem 2.2). The proof of relaxation formula stated in Theorem 2.1 is given under homogeneity assumption on the density f since it relies on a particular case of [23, Theorem 2.1] (see Theorem 4.1 below). Indeed a

central role plays the connection with homogeneous indicator functionals of convex sets with nonempty interior, as already emphasized in similar context by [12], later exploited in [5], and very recently in [22], [31] in the nonlocal framework. It is worth to underline that, despite of the results currently available in the literature, in the set of hypotheses of Theorem 2.1 we drop any coercivity assumptions on f, thanks to arguments as in [18, Theorem 3.4]. In turn, Theorem 2.1 allows us to generalize some relaxation results for indicator functionals whose density have unbounded convex effective domain or, equivalently, improves the understanding of the asymptotics for vectorial differential inclusions (cf. Corollary 4.2).
Finally we note that the Γ-convergence result stated in Theorem 2.2 is new in literature since we require only the necessary measurability hypothesis on f and a linear growth condition. Indeed Theorem 3.1 in [16] applies when f is lower semicontinuous and level convex w.r.t the gradient variable and satisfies a linear growth condition from below; the next results remove the level convexity assumption on f (see Theorem 3.9 in [41] in the scalar case and Theorem 3.2 in [34] in the vectorial case) but require that f is a Carathéodory function (satisfying a suitable growth condition with respect to the gradient variable of the type (5).

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