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THE CANHAM-HELFRICH MODEL FOR THE ELASTICITY OF BIOMEMBRANES AS A LIMIT OF MESOSCOPIC ENERGIES

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Abstract. In this paper we review some recent results concerning the variational deduction of a Canham-Helfrich model for biomembranes obtained starting from a mesoscopic model which implements the amphiphilic behavior of the lipid molecules and the head-tail connection. The 2-dimensional analysis is complete while in the 3-dimensional case we have partial results and open problems.

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1. Introduction

A prominent way to model biomembranes is given by shape energies of *Canham-Helfrich* type [2, 5]. These type of energies have the general form

$$E(S) = \int_S \kappa_1 (H - H_0)^2 - \kappa_2 K \, d\mathcal{H}^2 \tag{1}$$

where S denotes a smooth surface in $\mathbb{R}^3$, H and K are the mean curvature and the Gaussian curvature of S respectively, and the bending moduli κ_1, κ_2 and the spontaneous curvature H_0 are constant. Typically, $\kappa_1 > \kappa_2 > 0$ is a compatibility condition coming both from mathematical considerations and from experiments [12, 14]. The shape of the membrane is an absolute minimimizer of E among a suitable class of surfaces. We notice that, thanks to the Gauss-Bonnet's Theorem, when the spontaneous curvature is zero and the topology of S is fixed the minimization problem for the Canham-Helfrich functional reduces to the minimization problem for the very well studied *Willmore functional* [7, 11, 13]. The Canham-Helfrich energy functional had been introduced starting from physical experiments while much less is known about its deduction from simpler models. In this paper we review some recent results concerning a rigorous deduction of the Canham-Helfrich energy functional. We refer to the microscopic model proposed by Peletier and Röger in 2009 [10, App. A]. Here the authors implemented the amphiphilic behavior of the lipid molecules that constitute the cell membrane and the covalent bond between head and tail of any molecule. A mesoscopic model had been formally derived from the microscopic one [10, App. A] and in the same paper a complete analysis in the 2-dimensional case had been performed. Precisely, the authors proved that the limit, in the sense of Γ-convergence, of the mesoscopic energies introduced by them is the Euler elastica functional on suitable families of closed curves in the plane. The analysis in the 3-dimensional case is much harder and there are only partial results [8, 9]. In such a case deep tools from Geometric Measure Theory, like currents and curvature varifolds, are necessary.

The paper is organized as follows. In Section 2 we recall the mesoscopic model proposed by Peletier and Röger [10]. In Section 3 we review the notion of Γ-convergence, essential in order to understand the correct way to pass to the limit in a family of variational problems. Then, in Section 4 we describe the 2-dimensional analysis done by Peletier and Röger [10]. Finally, the last section is dedicated to the partial results obtained in the 3-dimensional case [8, 9].

2. The Peletier-Röger mesoscopic model

In 2009 Peletier and Röger [10] proposed a mesoscale model for biomembranes in the form of an energy for idealized and rescaled head and tail densities. Such a model originates from a probabilistic micro-scale description in which heads and tails are treated as separate particles. The energy functional introduced by Peletier and Röger has essentially two contributions: the first one penalizes the proximity of tail to polar (head or water) particles, and the second one implements the head-tail connection as an energetic penalization. Configurations of head and tail particles are described by two rescaled density functions

$$u \in BV(\mathbb{R}^n; \{0, \varepsilon^{-1}\}), \quad v \in L^1(\mathbb{R}^n; \{0, \varepsilon^{-1}\})$$

with $uv = 0$ a.e. in $\mathbb{R}^n$ and with prescribed total mass, namely

$$\int u(x)\, dx = \int v(x)\, dx = M_T.$$

Here $\varepsilon > 0$ is a small parameter. We call

$$K_\varepsilon \subset X := L^1(\mathbb{R}^n) \times L^1(\mathbb{R}^n)$$

the set of such a configurations. The energy functional is defined by

$$F_\varepsilon(u,v) := \begin{cases} \varepsilon \displaystyle\int |\nabla u| + \frac{1}{\varepsilon} d_1(u,v) & \text{if } (u,v) \in K_\varepsilon \\ +\infty & \text{otherwise in } X. \end{cases}$$

In this model u corresponds to the tail density while v is the density of heads. The term

$$\varepsilon \int |\nabla u|$$

is, up to the constant ε, the total variation of u and it measures the boundary size of the support of tails: this corresponds to the contribution which arises from the amphiphilic behavior of the polar particles. The second term which appear in the energy functional F_ε, that is

$$\frac{1}{\varepsilon} d_1(u,v)$$

takes into account the implicit implementation of the head-tail connection and it is given by the *Monge-Kantorovich distance between* u *and* v. Let us explain briefly what is d_1 and the relation with the optimal transport problem; for details we refer to [1, 4]. Consider two mass distributions $u, v \in L^1(\mathbb{R}^n)$ with compact support and with

$$\int u(x)\, dx = \int v(x)\, dx = 1.$$

We denote by $\mathcal{A}(u,v)$ the set of all Borel vector fields $T\colon \mathbb{R}^n \to \mathbb{R}^n$ *pushing* u *forward to* v, that is

$$\int \eta(T(x)) u(x)\, dx = \int \eta(y) v(y)\, dy, \quad \forall \eta \in C^0(\mathbb{R}^n).$$

The *Monge-Kantorovich distance between* u *and* v is therefore defined by

$$d_1(u,v) := \min_{T \in \mathcal{A}(u,v)} \int |x - T(x)| u(x)\, dx. \tag{2}$$

Moreover, it turns out that there exists the so called *Kantorovich potential* ϕ, that is a 1-Lipschitz map $\mathbb{R}^n \to \mathbb{R}$ characterized by

$$\phi(x) - \phi(T(x)) = |x - T(x)|, \quad \text{a.e. } x \in \operatorname{spt}(u)$$

whenever T solves the optimization problem (2). A key property of d_1 is the presence of *transport rays*. Let ϕ be a Kantorovich potential as above. A *transport ray* is a maximal line segment in $\mathbb{R}^n$ with endpoints $a, b \in \mathbb{R}^n$ such that ϕ has slope one on that segment, that is

$$\begin{aligned}
&a \in \operatorname{spt}(u), \quad b \in \operatorname{spt}(v), \quad a \neq b,\\
&\phi(a) - \phi(b) = |a - b|,\\
&|\phi(a + t(a - b)) - \phi(b)| < |a + t(a - b) - b|, \quad \forall t > 0,\\
&|\phi(b + t(b - a)) - \phi(a)| < |b + t(b - a) - a|, \quad \forall t > 0.
\end{aligned}$$

Two transport rays can only intersect in a common endpoint and if z lies in the interior of a ray with endpoints $a \in \operatorname{spt}(u), b \in \operatorname{spt}(v)$ then ϕ is differentiable in z and

$$\nabla\phi(z) = \frac{a - b}{|a - b|}.$$

Let us back to F_ε. In order to understand what happens we consider ring structures (for details on the computations see [10]). Let the supports of u and v be the ring structures of Fig.2: the support of u is a single ring between circles of radii r_2 and r_3, and the support of v is given by two rings flanking $\operatorname{spt}(u)$, namely between radii r_1 and r_2 and between radii r_3 and r_4. Expanding F_1 we find

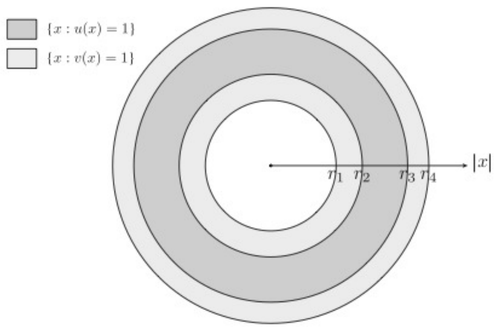

Figure 1. The densities u and v are disposed forming a ring structure (courtesy of [10]).

$$F_1 \sim 2M_T + M_T \left(\frac{r_4 - r_1}{2} - 2\right)^2 + \frac{M_T}{(r_4 + r_1)^2}.$$

The constant term $2M_T$ is simply the Lagrange multiplier due to the total mass constraint. We then see a preference for thickness

$$\frac{r_4 - r_1}{2} = 2.$$

Moreover, we also notice a penalization of the curvature of the structure do to the term

$$\frac{M_T}{(r_4+r_1)^2}.$$

After rescaling and renormalization we see that

$$F_\varepsilon \sim 2M_T + M_T\left(\frac{r_4-r_1}{2\varepsilon}-2\right)^2 + \frac{M_T\varepsilon^2}{(r_4+r_1)^2}.$$

Then, the ε-ring structure prefers the thickness 2ε and again we notice a penalization of the curvature. In order to capture such a penalization, the right energy to investigate is given by

$$G_\varepsilon(u,v) := \frac{F_\varepsilon(u,v)-2M_T}{\varepsilon^2}.$$

The main problem now is the following one: what happens when $\varepsilon \to 0$? The limit structure should be a surface S (the membrane) and the energy G_ε should converge, in a suitable way, to an energy functional defined on S which penalizes the curvatures of S.

3. An overwiew on Γ-convergence

In this section we review the notion of Γ-convergence which is the right way to pass to the limit in a family of variational problems. The theory of Γ-convergence dates back to De Giorgi (1975), for the general theory see [3]. We give the definition only for metric spaces even if it is possible to extend to topological spaces. Let (X,d) be a metric space. Let (F_h) be a sequence of functions $X \to \mathbb{R}\cup\{\pm\infty\}$. We say that (F_h) Γ-*converges*, as $h \to +\infty$, to $F\colon X \to \mathbb{R}\cup\{\pm\infty\}$, if for all $u \in X$ we have:

(a) (*liminf inequality*) For every $u \in X$ and for every sequence $u_h \to u$ it holds

$$F(u) \le \liminf_{h\to+\infty} F_h(u_h).$$

(b) (*existence of a recovery sequence*) For every $u \in X$ there exists a sequence $u_h \to u$ such that

$$F(u) \ge \limsup_{h\to+\infty} F_h(u_h).$$

It is easy to extend this definition of convergence to families depending on a real parameter. Given a family $(F_\varepsilon)_{\varepsilon>0}$ of functions $X \to \mathbb{R}\cup\{\pm\infty\}$, we say that it Γ-*converges*, as $\varepsilon \to 0$, to $F\colon X \to \mathbb{R}\cup\{\pm\infty\}$ if for every positive infinitesimal sequence (ε_h) the sequence (F_{ε_h}) Γ-converges to F. The most important consequence of the definition of Γ-convergence is the following result about the convergence of minimizers [3, Cor. 7.20].

Theorem 1. *Let $F_h \colon X \to \mathbb{R} \cup \{\pm\infty\}$ be a sequence of functions which Γ-converges to some $F \colon X \to \mathbb{R} \cup \{\pm\infty\}$. Assume that*

$$\inf_{v \in X} F_h(v) > -\infty$$

for every $h \in \mathbb{N}$. Let (ε_h) be a positive infinitesimal sequence, and for every $h \in \mathbb{N}$ let $u_h \in X$ be an ε_h-minimizer of F_h, i.e.

$$F_h(u_h) \le \inf_{v \in X} F_h(v) + \varepsilon_h.$$

Assume that $u_h \to u$ for some $u \in X$. Then u is a minimum point of F, and

$$F(u) = \lim_{h \to +\infty} F_h(u_h).$$

4. The 2D analysis

The 2-dimensional analysis had been investigated in 2009 by Peletier and Röger [10]. The mathematical analysis of the mesoscopic model in dimension 2 confirms that such a model shows the key properties of biomembranes, that is a preference for uniformly thin structures without ends and a resistance to bending of the structure. In [10] the authors proved a full Γ-convergence result for the family $\{G_\varepsilon\}_{\varepsilon>0}$ in two space dimensions. In that limit the densities concentrate on families of $W^{2,2}$-curves and a generalized Euler elastica energy is obtained for moderate-energy structures. To be precise first of all we recall the notion of system of $W^{2,2}$-curves. Let $\mathcal{C} = \{\gamma_i\}_{i=1,\dots,N}$ be a finite collection of maps $W^{2,2}_{\text{loc}}(\mathbb{R};\mathbb{R}^2)$. We say that $\mathcal{C}$ is a *$W^{2,2}$-system of closed curves* if $\gamma_i' \neq 0$ and γ_i are L_i-periodic for some $L_i > 0$, $i = 1, \dots, N$. We also let

$$\operatorname{spt}(\mathcal{C}) := \bigcup_{i=1}^N \gamma_i(\mathbb{R}), \quad |\mathcal{C}| := \sum_{i=1}^N \int_0^{L_i} |\gamma_i'(s)|\, ds.$$

Moreover, we define the corresponding Radon measure $\mu_{\mathcal{C}}$ on $\mathbb{R}^2$ to be the measure that satisfies

$$\int \varphi\, d\mu_{\mathcal{C}} = \sum_{i=1}^N \int_0^{L_i} \varphi(\gamma_i(s))|\gamma_i'(s)|\, ds, \quad \forall \varphi \in C^0_c(\mathbb{R}^2).$$

We finally say that $\mathcal{C}$ has *no transversal crossings* if for any $1 \le i, j \le N$, $s_i, s_j \in \mathbb{R}$

$$\gamma_i(s_i) = \gamma_j(s_j) \Longrightarrow \gamma_i'(s_i) \text{ and } \gamma_j'(s_j) \text{ are parallel.}$$

We remark that we can represent a given system of closed curves $\mathcal{C}$ as a finite collection $\{\gamma_i\}_{i=1,\dots,N}$ where for any $i = 1, \dots, N$ we have that γ_i is one-periodic, with 1 being the smallest possible period, and γ_i is parametrized proportional to arclength. We are therefore able to generalize the classical curve bending energy to $W^{2,2}$-systems of closed curves. Precisely, we let

$$\mathcal{W}(\mathcal{C}) := \frac{1}{2}\sum_{i=1}^{N} L_i^{-3}\int_0^1 \gamma_i''(s)^2\,ds.$$

We are ready to state the main theorem by Peletier and Röger [10, Thm. 4.1] which essentially says that the family $\{G_\varepsilon\}_{\varepsilon>0}$ Γ-converges to $\mathcal{W}$ with respect to the weak*-convergence of Radon measures on $\mathbb{R}^2$.

Theorem 2. *The following facts hold true.*

(a) *Let* $(u_\varepsilon, v_\varepsilon) \in L^1(\mathbb{R}^2)\times L^1(\mathbb{R}^2)$*,* $R>0$ *and a Radon measure* μ *on* $\mathbb{R}^2$ *be given with*

$$\mathrm{spt}(u_\varepsilon) \subset B_R(0), \quad \textit{for all } \varepsilon > 0$$
$$u_\varepsilon \mathcal{L}^2 \overset{*}{\rightharpoonup} \mu \quad \textit{as Radon measures on } \mathbb{R}^2$$

and

$$\liminf_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon, v_\varepsilon) < +\infty.$$

Then there is a $W^{2,2}$*-system of closed curves* $\mathcal{C} = \{\gamma_i\}_{i=1,\dots,N}$ *such that* $2\mu_{\mathcal{C}} = \mu$*,* $2|\mathcal{C}| = M_T$*,* $\mathrm{spt}(\mathcal{C})$ *is bounded,* $\mathcal{C}$ *has no transversal crossings, and*

$$\mathcal{W}(\mathcal{C}) \le \liminf_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon, v_\varepsilon).$$

(b) *Let* $\mathcal{C} = \{\gamma_i\}_{i=1,\dots,N}$ *be a* $W^{2,2}$*-system of closed curves such that* $2|\mathcal{C}| = M$*,* $\mathrm{spt}(\mathcal{C})$ *is bounded and with no transversal crossings. Then there exists* $(u_\varepsilon, v_\varepsilon) \in K_\varepsilon$ *such that* $\mathrm{spt}(u_\varepsilon) \subset B_R(0)$ *for all* $\varepsilon > 0$ *and for some* $R > 0$*, such that*

$$u_\varepsilon \mathcal{L}^2 \overset{*}{\rightharpoonup} 2\mu_{\mathcal{C}} \quad \textit{as Radon measures on } \mathbb{R}^2$$

and such that

$$\mathcal{W}(\mathcal{C}) \ge \limsup_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon, v_\varepsilon).$$

5. The 3D analysis

The analysis in the 3-dimensional case is much more complicated and there are only partial results. The main starting point of such an analysis is the following theorem [8, Thm. 2.1] that can be proved parametrizing $\mathrm{spt}(u)$ by means of transport rays. Let $(u, v) \in K_\varepsilon$ and let ϕ be the corresponding Kantorovich potential as in Section 2. We let $\theta := \nabla\phi$. Moreover, for an arbitrary 3×3 matrix A we let

$$Q(A) := \frac{1}{4}(\mathrm{tr}A)^2 - \frac{1}{6}\,\mathrm{tr}(\mathrm{cof}A)$$

with $\mathrm{cof}A$ denoting the cofactor matrix of A.

Theorem 3. *Let $(u,v) \in K_\varepsilon$ and assume that $J_u =: S$ is compact, orientable surfaces of class C^1 in $\mathbb{R}^3$. Then there exist non-negative measurable functions $M\colon S \to \mathbb{R}$ such that*

$$M_T = \int_S M \, d\mathcal{H}^2$$

such that θ and the inner unit normal field ν of $\operatorname{spt}(u)$ on S satisfy $\theta \cdot \nu > 0$ everywhere on $\{M > 0\}$, and such that

$$G_\varepsilon(u,v) \geq \frac{1}{\varepsilon^2}\int_S (M-1)^2 \, d\mathcal{H}^2 + \frac{1}{\varepsilon^2}\int_S \left(\frac{1}{\theta\cdot\nu} - 1\right) M^2 \, d\mathcal{H}^2 + \int_S \frac{M^4}{(\theta\cdot\nu)^3} Q(D\theta)\, d\mathcal{H}^2. \tag{3}$$

Estimate (3) suggests the form of the Γ-limit. Indeed, take $(u,v) \in K_\varepsilon$ such that $G_\varepsilon(u_\varepsilon, v_\varepsilon) \leq c$. Correspondingly we have $S_\varepsilon, M_\varepsilon, \theta_\varepsilon, \nu_\varepsilon$ satisfying Theorem 3. If we assume that some compactness for $\{S_\varepsilon\}$ hold true, say $S_\varepsilon \to S$ in some sense, thanks to

$$\frac{1}{\varepsilon^2}\int_{S_\varepsilon} (M_\varepsilon - 1)^2 \, d\mathcal{H}^2 \leq c$$

we expect that functions M_ε tend to be 1 as $\varepsilon \to 0$. As a consequence, since

$$\frac{1}{\varepsilon^2}\int_{S_\varepsilon} \left(\frac{1}{\theta_\varepsilon\cdot\nu_\varepsilon} - 1\right) M_\varepsilon^2 \, d\mathcal{H}^2 \leq c$$

we can conjecture that θ_ε tends to be orthogonal to S. Putting all together these informations, if the estimate (3) was optimal, the limit of $G_\varepsilon(u_\varepsilon, v_\varepsilon)$ should be

$$\int_S Q(D\nu)\, d\mathcal{H}^2 = \int_S \frac{1}{4}H^2 - \frac{1}{6}K \, d\mathcal{H}^2$$

which is a functional of Canham-Helfrich-type: choose, in (1), $\kappa_1 = \frac{1}{4}, \kappa_2 = \frac{1}{6}$ and $H_0 = 0$. This heuristic explanation can be formalized at least for the existence of a recovery sequence accordingly with the very definition of Γ-convergence. Indeed, the following theorem holds true [8, Thm. 2.5].

Theorem 4. *Fix a smooth compact orientable surface $S \subset \mathbb{R}^3$ without boundary such that $\mathcal{H}^2(S) = M_T$. Then there exists a family $(u_\varepsilon, v_\varepsilon)_{\varepsilon>0}$ in K_ε such that*

$$u_\varepsilon \mathcal{L}^3 \overset{*}{\rightharpoonup} \mathcal{H}^2 \llcorner S \quad \text{as Radon measures on } \mathbb{R}^2$$

and

$$G_\varepsilon(u_\varepsilon, v_\varepsilon) \to \int_S \frac{1}{4}H^2 - \frac{1}{6}K \, d\mathcal{H}^2.$$

The main problem is the compactness and the liminf inequality. The first difficulty stems in the fact that we are not able to prove rigorously that $M_\varepsilon \to 1$, so that in order to have some compactness and liminf inequality we need to simplify the setting. Precisely, fix $\Omega \subset \mathbb{R}^3$ open and let $\mathcal{M}$ be the set of tuples (S,θ), where S is a compact and orientable surface of class C^2 in $\mathbb{R}^3$ that is given by the boundary of an open set $A(S) \subset\subset \Omega$, and $\theta\colon S \to \mathbb{R}^3$ is a Lipschitz vector field such that

$$|\theta| \,=\, 1 \text{ and } \theta\cdot\nu \,>\, 0 \text{ on } S$$

where $\nu\colon S \to \mathbb{R}^3$ denotes the outer unit normal field on S. For any $p \in S$ denote by $L(p)\colon \mathbb{R}^3 \to \mathbb{R}^3$ the extension of $D\theta(p) : T_pS \to \mathbb{R}^3$ defined by the properties

$$L(p)\tau \,=\, D\theta(p)\tau \quad \text{for all } \tau \in T_pS, \qquad L(p)\theta(p) \,=\, 0$$

if $D\theta(p)$ exists and $L(p) = 0$ else. We next define $\mathcal{Q}_\varepsilon : \mathcal{M} \to [0,+\infty)$ as

$$\mathcal{Q}_\varepsilon(S,\theta) := \frac{1}{\varepsilon^2}\int_S \frac{1}{\theta\cdot\nu} - 1\, d\mathcal{H}^2 + \int_S Q(L(p))\, d\mathcal{H}^2(p). \tag{4}$$

The functional $\mathcal{Q}_\varepsilon$ is a simplification of the right-hand side of (3) and could represent a good functional to study in order to understand the general case. The analysis of $\mathcal{Q}_\varepsilon$ in terms of compactness and liminf inequality is contained in [9]. A bound $\mathcal{Q}_\varepsilon(S_\varepsilon,\theta_\varepsilon) \le c$ at a first sight produces only a bound on the area of S_ε but we have to produce curvatures in the limit. The idea is to look at the family of measures $\mathcal{H}^2 \llcorner S_\varepsilon$ and its weak*-limit μ in the sense of Radon measures on $\mathbb{R}^2$. Indeed, it is possibile to prove, but the proof is very complicated [9], that μ is supported on a sort of *weak surface* for which curvatures make sense, precisely an *integral curvature varifold*. We briefly recall the main definitions, and we refer to Hutchinson [6] for details. Let $G(2,3)$ denote the Grassmann manifold of all two-dimensional unoriented planes in $\mathbb{R}^3$. An *integral curvature varifold* V in $\mathbb{R}^3$ is a Radon measure on $\mathbb{R}^3 \times G(2,3)$ characterized by

$$V(\psi) \,=\, \int_S \psi(x,T_xS)\beta(x)\, d\mathcal{H}^2(x), \qquad \text{for all } \psi \in C^0_c(\mathbb{R}^3\times G(2,3))$$

where $S \subset \mathbb{R}^3$ is a 2-rectifiable set, $\beta\colon S \to \mathbb{N}$ is locally $\mathcal{H}^2$-integrable, and such that there exist V-measurable functions $A_{ijk} : \mathbb{R}^3 \times G(2,3) \to \mathbb{R}$, $1 \le i,j,k \le 3$ such that for any $\varphi \in C^1(\mathbb{R}^3 \times \mathbb{R}^{3\times 3})$ compactly supported with respect to the first variable

$$0 \,=\, \int \Big(P_{ij}\partial_j\varphi + A_{ijk}\partial^*_{jk}\varphi + A_{jij}\varphi \Big)\, dV(x,P), \quad i = 1,2,3 \tag{5}$$

where we identify $P \in G(2,3)$ and the associated orthogonal projection $\mathbb{R}^3 \to P$ with matrix representation (P_{ij}) and where ∂^* denotes the derivatives with respect to the P variable. We also let $\mu_V := \beta\mathcal{H}^2 \llcorner S$, which therefore is a Radon measure on $\mathbb{R}^3$. Formula (5) generalizes the integration by parts on smooth manifolds without boundary, and the idea is that starting from $A = (A_{ijk})$ it is possibile, as in the smooth differential geometry, to construct mean curvature vector and Gauss curvature. Precisely, we let

$$H_i := A_{jij}, \quad K := \sum_k \mathrm{tr}(\mathrm{cof}(A_{ijk})_{ij}).$$

As a consequence the mean curvature square is defined as the norm square of H_i, so that for an integral curvature varifolds the quantity H^2 and K are well defined. We are ready to go back to compactness and liminf inequality for the family $\{\mathcal{Q}_\varepsilon\}_{\varepsilon>0}$. The main result is the following compactness and lower bound statements [9, Thm. 2.2].

Theorem 5. *Let $(\varepsilon_j)_{j\in\mathbb{N}}$ be an infinitesimal sequence of positive numbers and $(S_j,\theta_j)_{j\in\mathbb{N}}$ be a sequence in $\mathcal{M}$ such that $\sup_j \mathcal{H}^2(S_j) < \infty$ and*

$$\bigcup_j S_j \subset \tilde{\Omega} \quad \textit{for some } \tilde{\Omega} \subset\subset \Omega$$

and that for a fixed $\Lambda > 0$

$$\mathcal{Q}_{\varepsilon_j}(S_j,\theta_j) \le \Lambda \quad \textit{for all } j \in \mathbb{N}.$$

Assume furthermore that in the sense of Radon measures on Ω

$$\mathcal{H}^2 \llcorner S_j \to \mu \quad \textit{as } j \to \infty.$$

Then $\mu = \mu_V$ where V is an integral curvature varifold and

$$\int \frac{1}{4}H^2 - \frac{1}{6}K\, dV \le \liminf_{j\to+\infty} \mathcal{Q}_{\varepsilon_j}(S_j,\theta_j).$$

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