STABILITY OF CMC HYPERSURFACES IN THE COMPLEX HYPERBOLIC SPACE

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Abstract. We show that in the complex hyperbolic space there are no compact stable hypersurfaces satisfying a certain quadratic bound on mean curvature and characteristic curvature. The proof follows from a stability inequality that is obtained embedding the complex hyperbolic space into the space Hermitian matrices endowed with a Minkowski product.

1. Introduction

We study stability of compact constant mean curvature (CMC) hypersurfaces in the complex hyperbolic space $\mathbf{C}H^n$. By “stable” we mean a CMC hypersurface for which the second variation of the area is non-negative under deformations preserving the enclosed volume.

Barbosa, do Carmo, and Eschenburg proved that geodesics spheres in $\mathbf{C}H^n$ are stable, see [2, Theorem 1.4]. It is conjectured that isoperimetric sets in the complex hyperbolic space are geodesics spheres. An incorrect attempt to classify compact embedded CMC hypersurfaces in $\mathbf{C}H^2$ (without using stability) was done in [7],[8] trying to use Alexandrov's moving plane method. If the CMC hypersurface is assumed to be Hopf, then it must be a geodesic sphere, see [13]. Without the Hopf assumption, however, there seems to be no result on the classification of CMC and of stable CMC hypersurfaces.

In this paper we prove some negative results about stable CMC hypersurfaces in $\mathbf{C}H^n$ continuing the research line initiated in [3] in the case of the complex projective space. For a hypersurface $\Sigma \subset \mathbf{C}H^n$, we denote by $\kappa = h(JN, JN)$ its characteristic curvature, where J is the complex structure of $\mathbf{C}H^n$, N is the unit normal, and h is the second fundamental form of Σ. The trace of h (the non-normalized mean curvature) is denoted by $H = \mathrm{tr}(h)$.

Our main result is the following non-existence theorem for compact stable CMC hypersurfaces in $\mathbf{C}H^n$.

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Theorem 1.1. Let $n \geq 2$. There exists no compact oriented stable CMC hypersurface $\Sigma \subset \mathbf{C}H^n$ satisfying

$$p(\kappa; n, H) := (2n+1)\kappa^2 - 2H\kappa + 4(n^2-1) - H^2 \geq 0. \tag{1.1}$$

In particular, no such a hypersurface exists if $H^2 \leq 2(n-1)(2n+1)$.

We shall show that stability forces inequality (1.1) to be an equality and forces Σ to be a sphere. On the other hand, spheres satisfy the strict inequality $p(\kappa; n, H) < 0$, the equality case corresponding to the limit case of a sphere with infinite radius.

The proof of Theorem 1.1 is based on the following stability inequality. Let $Y_1, \dots, Y_{2n-2}$ be any local orthonormal frame for the complex tangent space $\mathbf{C}T\Sigma = T\Sigma \cap J(T\Sigma)$. We define the complex tangent vector-field

$$h_N = \sum_{j=1}^{2n-2} h(JN, Y_j)Y_j \in \mathbf{C}T\Sigma. \tag{1.2}$$

The characteristic curvature κ is principal (i.e., Σ is a Hopf hypersurface) precisely when $h_N = 0$ identically on Σ.

Theorem 1.2. If $\Sigma \subset \mathbf{C}H^n$, $n \geq 2$, is a compact oriented stable CMC hypersurface then

$$\int_\Sigma \left\{ H^2 - (n-1)\big(2n + |h|^2\big) - \frac{(H+\kappa)^2 + |h_N|^2}{2(n+1)} \right\} d\mu \geq 0, \tag{1.3}$$

where μ is the Riemannian hypersurface measure.

Our first step towards the proof of Theorem 1.2 is the isometric embedding of $\mathbf{C}H^n$ into the space of Hermitian matrices H^{n+1}, endowed with a suitable inner product of the Minkowski-type. The isometric embedding into the Euclidean space was used in the case of the complex projective space in [3] and it is an effective procedure to study the topology of minimal surfaces, see [1] and the references therein. The embedding for $\mathbf{C}H^n$ was introduced for the first time by Garay and Romero in [9]. Our presentation is different from [9] in some respects and in Section 2 we provide some details on this embedding.

In Section 4 we compute the tangential Laplacian of the position matrix $A \in \Sigma$, of the unit normal N to Σ, and of the curvature of the embedding $\mathbf{C}H^n \subset H^{n+1}$ in the direction N. This is the technical and computational part of the paper.

The matrix-valued function $u = \Delta_\Sigma A$ is the key tool in our proof of the stability inequality (1.3), see Section 5. We introduce the quadratic form $Q_\Sigma : H^{n+1} \to \mathbb{R}$,

$$Q_\Sigma(V) = -\int_\Sigma \langle u, V_M \rangle_H \langle \mathscr{L} u, V_M \rangle_H d\mu,$$

where $\langle\cdot,\cdot\rangle_H$ is the standard inner product on H^{n+1}, $u = \Delta_\Sigma A$ is the Laplacian of the position matrix,

$$\mathscr{L} = \Delta_\Sigma + |h|^2 + \mathrm{Ric}(N) = \Delta_\Sigma + |h|^2 - 2(n+1) \tag{1.4}$$

is the Jacobi operator of Σ, $M = \mathrm{diag}\{-1, 1, \dots, 1\}$ is the Minkowski matrix, and $V_M = MV$ is the product of M with $V \in H^{n+1}$.

If Σ is stable, then the trace of Q_Σ is non-negative, and this fact is expressed by the inequality (1.3). Finally, in Section 6 we show how to deduce Theorem 1.1 from Theorem 1.2.

2. Algebraic preliminaries

This section has a preliminary character and can be skipped by the reader familiar with the complex hyperbolic space and with its embedding into the space of Hermitian matrices.

Let $\mathbf{C}^{n+1}$ be endowed with the Minkowski inner product $\langle z, w\rangle_M := -z_0\bar{w}_0 + z_1\bar{w}_1 + \ldots + z_n\bar{w}_n$. On $\mathbf{C}^{n+1}\setminus\{0\}$ we define the equivalence relation $z \sim w$ if there exists $\alpha \in \mathbf{C}$ such that $z = \alpha w$. The complex projective space $\mathbf{C}P^n$ is the set of all equivalence classes $[z]$, $z \neq 0$, and the complex hyperbolic space is $\mathbf{C}H^n = \big\{[z] \in \mathbf{C}P^n : \langle z, z\rangle_M < 0\big\}$.

The real hyperboloid $S = \big\{z \in \mathbf{C}^{n+1} : \langle z, z\rangle_M = -1\big\}$ is known as anti-de Sitter space, and the complex hyperbolic space can be equivalently defined as $\mathbf{C}H^n = S/\sim$. The complex tangent space $\mathbf{C}T_zS = \big\{w \in \mathbf{C}^{n+1} : \langle z, w\rangle_M = 0\big\}$ passes to the quotient and we can define the tangent space $T_{[z]}\mathbf{C}H^n = \mathbf{C}T_zS$. The inner product

$$\langle w, \zeta\rangle_R := \mathrm{Re}\langle w, \zeta\rangle_M, \quad w, \zeta, \in T_{[z]}\mathbf{C}H^n, \tag{2.1}$$

is positive and $\mathbf{C}H^n$, equipped with this product, is the Riemannian complex hyperbolic space.

We identify $\mathbf{C}H^n$ with a subset of $H^{n+1} = \big\{A \in \mathrm{gl}(n+1, \mathbf{C}) : \overline{A} = A^t\big\}$, the space of $(n+1)$-dimensional Hermitian matrices. We fix on H^{n+1} the inner product

$$\langle A, B\rangle := -\frac{1}{2}\mathrm{tr}(A_M B_M), \qquad A, B \in H^{n+1}. \tag{2.2}$$

Here and throughout the paper, we will use the notation $A_M := MA$ where MA is a matrix-matrix product and $M := \mathrm{diag}\{-1, 1, \dots, 1\}$ is the Minkowski matrix.

The transposed operator with respect to the Minkowski product of a $\mathbf{C}$-linear operator $A \in \mathrm{End}(\mathbf{C}^{n+1})$ is the $\mathbf{C}$-linear operator $A^* \in \mathrm{End}(\mathbf{C}^{n+1})$ defined by $\langle A(z), w\rangle_M = \langle z, A^*(w)\rangle_M$ for $z, w \in \mathbf{C}^{n+1}$. To $A \in \mathrm{End}(\mathbf{C}^{n+1})$ we associate the matrix $(A_{jk})_{j,k=0,\dots,n} \in \mathrm{gl}(n+1, \mathbf{C})$ given by $A_{jk} = \langle A(\mathrm{e}_k), \mathrm{e}_j\rangle_M$, where $\mathrm{e}_0, \mathrm{e}_1, \dots, \mathrm{e}_n$ is the canonical basis of $\mathbf{C}^{n+1}$. Thus, we have $A = A^*$ if and only if $A_{jk} = \bar{A}_{kj}$. The linear action $z \mapsto A(z)$ is given by the matrix-vector product

$$A(z) = A_M z^t, \tag{2.3}$$

where $z^t \in \mathbf{C}^{n+1}$ is a column vector.

Now we define the mapping $\mathscr{F} : S \to H^{n+1}$, $\mathscr{F}(z) = A$, where A is the matrix associated with the Minkowski-orthogonal projection from $\mathbf{C}^{n+1}$ onto the complex line $[z]$, namely

$$A(w) = -\langle w, z\rangle_M z, \qquad w \in \mathbf{C}^{n+1}. \tag{2.4}$$

We remark that for $z_0 = \mathrm{e}_0 = (1, 0, \dots, 0) \in \mathbf{C}^{n+1}$ we have $\mathscr{F}(z_0) = A_0 = \mathrm{diag}\{-1, 0, \dots, 0\}$. The mapping $\mathscr{F}$ passes to the quotient and so we have a map $\mathscr{F} : \mathbf{C}H^n \to H^{n+1}$. Now we determine the image of this map.

Let $U(1, n)$ be the group of the endomorphisms $Q \in \mathrm{End}(\mathbf{C}^{n+1})$ preserving the Minkowski product, i.e., $\langle Q(z), Q(w)\rangle_M = \langle z, w\rangle_M$ for all $z, w \in \mathbf{C}^{n+1}$. Each $Q \in U(1, n)$ satisfies $Q \circ Q^* = Q^* \circ Q = \mathrm{Id}$, where Id is the identity operator. Thus, as matrices, we have the identities

$$Q_M Q_M^* = Q_M^* Q_M = I. \tag{2.5}$$

The group $U(1, n)$ acts transitively on S, i.e., for any $z \in S$ there exists $Q \in U(1, n)$ such that $z = Q(z_0)$. Therefore, the projection onto $[z]$ is the $\mathbf{C}$-linear operator $A = Q \circ A_0 \circ Q^*$. Again by (2.3), the corresponding matrix equation is $A_M = Q_M (A_0)_M Q_M^*$, which is equivalent to $A = QA_0Q^*$.

From the above discussion, we deduce that the image of $\mathscr{F}$ in H^{n+1} is

$$\mathscr{F}(\mathbf{C}H^n) = \big\{A \in H^{n+1} : A = QA_0Q^* \text{ for some } Q \in U(1, n)\big\}.$$

With a slight abuse of notation, from now on we identify $\mathbf{C}H^n$ with $\mathscr{F}(\mathbf{C}H^n) \subset H^{n+1}$.

Lemma 2.1. The mapping $\mathscr{F} : \mathbf{C}H^n \to \mathscr{F}(\mathbf{C}H^n) \subset H^{n+1}$ is an isometry, when $\mathbf{C}H^n$ and $\mathscr{F}(\mathbf{C}H^n)$ are endowed with the metrics (2.1) and (2.2), respectively.

The proof is a direct check.

Each matrix $A \in \mathbf{C}H^n$ is characterized by the following three properties:

(1) $A_M A_M = A_M$;
(2) $\mathrm{tr}(A_M) = 1$;
(3) if $z \in \mathbf{C}^{n+1} \setminus \{0\}$ solves the equation $A_M z = z$, then $\langle z, z\rangle_M < 0$.

Indeed, the operator A satisfies the equation $A \circ A = A$ because it is a projection and this is equivalent to (1). By (2.5), we get

$$\mathrm{tr}(A_M) = \mathrm{tr}(Q_M (A_0)_M Q_M^*) = \mathrm{tr}(MA_0) = 1.$$

Property (3) ensures that A projects onto a negative line.

Let ∇ be the standard Levi-Civita connection of H^{n+1}. For any $X \in H^{n+1}$ we have the identity

$$\nabla_X A = X. \tag{2.6}$$

Differentiating the projection equation $A_M A_M = A_M$ in the direction X, we obtain $X_M = A_M X_M + X_M A_M$. This is the equation for the tangent space of the complex hyperbolic space. The normal space $T_A^\perp \mathbf{C}H^n$ of $\mathbf{C}H^n$ at $A \in \mathbf{C}H^n$ is the orthogonal complement of $T_A \mathbf{C}H^n$ in H^{n+1} with respect to the inner product (2.2).

Lemma 2.2. The tangent and normal spaces at $A \in \mathbf{C}H^n$ are given by

$$T_A \mathbf{C}H^n = \big\{X \in H^{n+1} : X_M = A_M X_M + X_M A_M\big\}, \tag{2.7}$$

$$T_A^\perp \mathbf{C}H^n = \big\{Z \in H^{n+1} \,:\, A_M Z_M = Z_M A_M\big\}\,. \tag{2.8}$$

The proof of this lemma is elementary and is left to the reader. Some more details on analogous formulas for the complex projective case can be found in [12].

Let $A, A_0 \in \mathbf{C}H^n$ be related by $A = QA_0Q^*$ for some $Q \in U(1,n)$. We define the mapping $T_Q : H^{n+1} \to H^{n+1}$ by setting

$$(T_Q X)_M := Q_M X_M Q_M^*.$$

This map preserves the scalar product (2.2) and, in particular, it maps isometrically the tangent space $T_{A_0}\mathbf{C}H^n$ onto $T_A\mathbf{C}H^n$.

Using these isometries, all the isometric invariant quantities will be computed only at the point $A_0 = \mathscr{F}(z_0)$. For $i, j \in \{0, 1, \dots, n\}$, let E_{ij} be the $(n+1) \times (n+1)$ matrix with entry 1 at the position (i, j) and with 0 elsewhere. Then, the matrices $X_1, \dots, X_n, \widehat{X}_1, \dots, \widehat{X}_n$, where

$$X_j = E_{j0} + E_{0j} \quad \text{and} \quad \widehat{X}_j = iE_{j0} - iE_{0j}, \qquad j = 1, \dots, n, \tag{2.9}$$

form an orthonormal basis for the tangent space of $\mathbf{C}H^n$ at the point A_0.

Using $\mathscr{F}$, we can transfer the complex structure of $\mathbf{C}^{n+1}$ to $\mathbf{C}H^n$. Indeed, at the point $A = \mathscr{F}(z) \in \mathbf{C}H^n$, with $z \in S \subset \mathbf{C}^{n+1}$, we can define the linear mapping $J_A : T_A\mathbf{C}H^n \to T_A\mathbf{C}H^n$ by setting $J_A d\mathscr{F}(z)[w] := d\mathscr{F}(z)[iw]$, for any $w \in \mathbf{C}T_zS$. The basis in (2.9) satisfies $\widehat{X}_j = JX_j$ at A_0.

We observe that the complex structure J commutes with the isometries T_Q, i.e., for any $A, B \in \mathbf{C}H^n$ such that $A = T_Q B$ for $Q \in U(1,n)$, and for any $X \in T_B\mathbf{C}H^n$, we have

$$J_A(T_Q(X)) = T_Q(J_B(X)). \tag{2.10}$$

In terms of matrix-multiplications, the complex structure is described in the following lemma.

Lemma 2.3. For any $A \in \mathbf{C}H^n$, the complex structure J_A is given by

$$J_A(X) = i(M - 2A)X_M, \qquad X \in T_A\mathbf{C}H^n. \tag{2.11}$$

Formula (2.11) can be checked first at the point A_0 and then at any point using the isometries T_Q and identity (2.10).

We conclude this introductory section with a list of algebraic identities that can be proved using the projection equation $A_M A_M = A_M$ and the equation $X_M = A_M X_M + X_M A_M$ for the tangent space. For any $A \in \mathbf{C}H^n$ and $X, Y \in T_A\mathbf{C}H^n$ we have:

$$A_M X_M Y_M = X_M Y_M A_M, \tag{2.12}$$

$$A_M X_M A_M = 0, \tag{2.13}$$

$$X_M(I - 2A_M) = -(I - 2A_M)X_M, \tag{2.14}$$

$$(I - 2A_M)^2 = I, \tag{2.15}$$

$$(I - 2A_M)X_M Y_M = X_M Y_M(I - 2A_M). \tag{2.16}$$

3. Geometry of the immersion $\mathscr{F}$

We define the tangential and normal connections of $\mathbf{C}H^n$ using the orthogonal decomposition $H^{n+1} = T_A\mathbf{C}H^n \oplus T_A^\perp\mathbf{C}H^n$. Denoting by $\pi_A^\top : H^{n+1} \to T_A\mathbf{C}H^n$ and $\pi_A^\perp : H^{n+1} \to T_A^\perp\mathbf{C}H^n$ the orthogonal projections, and by ∇ the standard Levi-Civita connection of H^{n+1} with respect to the metric (2.2), we set, for any $X, Y \in \Gamma(T\mathbf{C}H^n)$,

$$\nabla_X^\top Y(A) := \pi_A^\top(\nabla_X Y) \qquad \text{and} \qquad \nabla_X^\perp Y(A) := \pi_A^\perp(\nabla_X Y),$$

whenever $A \in \mathbf{C}H^n$.

In order to compute some algebraic formulas for these connections, we first define the bilinear map $\pi : H^{n+1} \times H^{n+1} \to H^{n+1}$,

$$\pi(X, Y) := XMY + YMX, \qquad X, Y \in H^{n+1}.$$

Formula (2.12) implies that π maps $T_A\mathbf{C}H^n \times T_A\mathbf{C}H^n$ into $T_A^\perp\mathbf{C}H^n$.

Lemma 3.1. For any $A \in \mathbf{C}H^n$ and $X \in H^{n+1}$, we have:

$$\pi_A^\top(X) = \pi(A, X) - 2AX_M A_M, \tag{3.17}$$

$$\pi_A^\perp(X) = X - \pi(A, X) + 2AX_M A_M. \tag{3.18}$$

Proof. For $A \in \mathbf{C}H^n$, we have $H^{n+1} \cong T_A H^{n+1} = T_A\mathbf{C}H^n \oplus T_A^\perp\mathbf{C}H^n$, and hence $X = \pi_A^\top(X) + \pi_A^\perp(X)$ for any $X \in H^{n+1}$. Thus, identity (3.18) follows from (3.17).

We prove the first identity. If $X \in T_A\mathbf{C}H^n$, then $\pi_A^\top(X) = X$ and using (2.12) we obtain

$$\pi_A^\top(X)_M = X_M = A_M X_M + X_M A_M = M(\pi(A, X) - 2AX_M A_M),$$

as wished. We show that the right-hand side of (3.17) satisfies the equation for the tangent space at A whenever $X \in H^{n+1}$. In fact, we have

$$(\pi(A,X) - 2AX_M A_M)_M A_M + A_M(\pi(A,X) - 2AX_M A_M)_M =$$
$$= X_M A_M - 2A_M X_M A_M + A_M X_M = (\pi(A,X) - 2AX_M A_M)_M.$$

This finishes the proof.

□

By Lemma 3.1, we obtain the following formulas for the tangent and normal connections:

$$\nabla^{\top}_X Y(A) = \pi(A, \nabla_X Y) - 2A(\nabla_X Y)_M A_M, \tag{3.19}$$
$$\nabla^{\perp}_X Y(A) = \nabla_X Y(A) - \pi(A, \nabla_X Y) + 2A(\nabla_X Y)_M A_M. \tag{3.20}$$

The second fundamental form of the immersion $\mathscr{F}$ of $\mathbf{C}H^n$ into H^{n+1} is the mapping $\sigma : \Gamma(T\mathbf{C}H^n) \times \Gamma(T\mathbf{C}H^n) \longrightarrow \Gamma(T^{\perp}\mathbf{C}H^n)$ defined for each $A \in \mathbf{C}H^n$ by

$$\sigma_A(X,Y) := \nabla^{\perp}_X Y(A).$$

The (non-normalized) mean curvature vector of $\mathscr{F}$ is the mapping $\mathscr{H} : \mathbf{C}H^n \rightarrow \Gamma(T^{\perp}\mathbf{C}H^n)$ defined as the trace of σ, i.e.,

$$\mathscr{H}(A) := \sum_{i=1}^{2n} \sigma_A(X_i, X_i), \tag{3.21}$$

where $X_1, \ldots, X_{2n}$ is any orthonormal basis of $T_A\mathbf{C}H^n$. For notational simplicity, we drop the dependence on $A \in \mathbf{C}H^n$ in σ and $\mathscr{H}$.

Proposition 3.2. Let $A \in \mathbf{C}H^n$. Then, for every $X, Y \in T_A\mathbf{C}H^n$ we have

$$\sigma(X,Y) = \pi(X,Y)(I - 2A_M). \tag{3.22}$$

Moreover, the mean curvature vector $\mathscr{H}$ of the immersion $\mathscr{F}$ at $A \in \mathbf{C}H^n$ is given by

$$\mathscr{H} = 4\left((n+1)A - M\right), \qquad A \in \mathbf{C}H^n. \tag{3.23}$$

Proof. We show that formula (3.22) defines a mapping such that:

$$\sigma(X,Y)_M A_M \ = \ A_M \sigma(X,Y)_M, \tag{3.24}$$
$$\nabla_X Y \ = \ \nabla^{\top}_X Y + \sigma(X,Y). \tag{3.25}$$

We first check identity (3.25). Using identities (2.12) and (2.13) we have

$$\begin{aligned}
(\nabla^{\top}_X Y)_M + \sigma(X,Y)_M &= A_M(\nabla_X Y)_M + (\nabla_X Y)_M A_M - 2A_M(\nabla_X Y)_M A_M \\
&\quad +(X_M Y_M + Y_M X_M)(I - 2A_M) \\
&= A_M(\nabla_X Y)_M + (\nabla_X Y)_M A_M + X_M Y_M + Y_M X_M \\
&\quad -2(A_M(\nabla_X Y)_M A_M + X_M Y_M A_M + Y_M X_M A_M) \\
&= A_M(\nabla_X Y)_M + (\nabla_X Y)_M A_M + X_M Y_M + Y_M X_M \\
&= (\nabla_X \pi(A,Y))_M \\
&= (\nabla_X Y)_M.
\end{aligned}$$

where we used Lemma 3.1.

We prove identity (3.24). By (2.12) and (2.16) we have

$$\begin{aligned}
\sigma(X,Y)_M A_M &= (X_M Y_M + Y_M X_M)(I - 2A_M)A_M \\
&= (X_M Y_M + Y_M X_M)(-A_M) \\
&= -A_M(X_M Y_M + Y_M X_M) \\
&= A_M(I - 2A_M)(X_M Y_M + Y_M X_M) \\
&= A_M(X_M Y_M + Y_M X_M)(I - 2A_M) \\
&= A_M \sigma(X,Y)_M.
\end{aligned}$$

We check formula (3.23) at the point A_0. Let $X_1, \dots, X_n, \widehat{X}_1, \dots, \widehat{X}_n$ be the orthonormal basis of $T_{A_0}\mathbf{C}H^n$ introduced in (2.9). The mean curvature vector of the immersion reads

$$\mathscr{H} = \sum_{j=1}^n \sigma(X_j, X_j) + \sum_{j=1}^n \sigma(\widehat{X}_j, \widehat{X}_j).$$

Using (3.22) we get

$$\sigma(X_j, X_j) = 2X_j M X_j (I - 2MA_0) = 2X_j M X_j M, \qquad j = 1, \dots, n,$$

and a direct calculation shows that

$$\sigma(X_j, X_j) = 2(A_0 - E_{jj}), \qquad j = 1, \dots, n. \tag{3.26}$$

Above and in the sequel, we denote by E_{ij} the $(n+1)\times(n+1)$-matrix with 1 at the position (i,j) with $i,j = 0,1,\dots,n$ and 0 otherwise. A similar computation shows that

$$\sigma(\widehat{X}_j, \widehat{X}_j) = \sigma(X_j, X_j), \quad j = 1, \dots, n. \tag{3.27}$$

From (3.26) and (3.27) we deduce that

$$\mathscr{H} = 4\sum_{j=1}^n (A_0 - E_{jj}) = 4\big((n+1)A_0 - M\big).$$

This concludes the proof of (3.23) when $A = A_0$. The general case follows using the isometries T_Q.

□

Let $Z \in \Gamma(T^\perp \mathbf{C}H^n)$ be a normal vector field and $A \in \mathbf{C}H^n$. The *Weingarten endomorphism* associated with the immersion $\mathscr{F}$ is the mapping $\Lambda_Z : T_A\mathbf{C}H^n \longrightarrow T_A\mathbf{C}H^n$ defined by

$$\Lambda_Z(X) := -\nabla_X^\top Z(A).$$

Proposition 3.3. Let $Z \in \Gamma(T^\perp \mathbf{C}H^n)$ and $A \in \mathbf{C}H^n$. For any $X \in T_A\mathbf{C}H^n$, we have

$$\Lambda_Z(X) = (XZ_M - ZX_M)(I - 2A_M). \tag{3.28}$$

Proof. Let $X \in T_A\mathbf{C}H^n$, $Z \in T_A^\perp \mathbf{C}H^n$ and, with a slight abuse of notation, denote in the same way any extension of Z to an element of $\Gamma(T^\perp \mathbf{C}H^n)$. We know that $Z \in T_A^\perp \mathbf{C}H^n$ if and only if $Z \in H^{n+1}$ and $Z_M A_M = A_M Z_M$. Thus, we get

$$X_M Z_M - Z_M X_M + A_M(\nabla_X Z)_M - (\nabla_X Z)_M A_M = 0. \tag{3.29}$$

Now, using formulas (3.19) and (3.29) we conclude that

$$\begin{aligned}
(\nabla_X^\top Z)_M &= 2A_M(\nabla_X Z)_M A_M - (\nabla_X Z)_M A_M - A_M(\nabla_X Z)_M \\
&= (\nabla_X Z)_M A_M - 2(\nabla_X Z)_M A_M - A_M(\nabla_X Z)_M + 2A_M(\nabla_X Z)_M A_M \\
&= ((\nabla_X Z)_M A_M - A_M(\nabla_X Z)_M)\,(I - 2A_M) \\
&= (X_M Z_M - Z_M X_M)(I - 2A_M),
\end{aligned}$$

as wished.

□

For future reference, we establish some identities linking the mappings Λ and σ. Let $X_1, \dots, X_{2n}$ be an orthonormal frame for $T\mathbf{C}H^n$. We use the alternative notation $N = X_{2n}$. Later, $N = X_{2n}$ will be the normal vector to Σ. We set

$$\begin{aligned}
\pi_{ij} := \pi(X_i, X_j) \quad &\text{and} \quad \pi_{i,N} := \pi(X_i, N), \\
\sigma_{ij} := \sigma(X_i, X_j) \quad &\text{and} \quad \sigma_{i,N} := \sigma(X_i, N),
\end{aligned}$$

and we finally let $\sigma_N := \sigma(N, N)$ and $\pi_N := \pi(N, N)$.

The second fundamental form $\sigma(X, Y)$ is defined when X and Y are tangent sections of $\mathbf{C}H^n$. However, the right-hand side of (3.22) is defined for any $X, Y \in H^{n+1}$. In the next lemma and in the next section, we will use (3.22) as the general definition of σ.

Lemma 3.4. Let $X_1, \dots, X_{2n-1}, N$ be an orthonormal frame for $T\mathbf{C}H^n$. Then for any $i, j = 1, \dots, 2n-1$ we have

$$\Lambda_{\sigma_{j,N}}(X_i) = \pi(\pi_{j,N}, X_i) = 2\pi_{j,N} M X_i - \sigma(\sigma_{ij}, N) - \sigma(X_j, \sigma_{i,N}). \tag{3.30}$$

Proof. We first prove the identity on the left hand-side of (3.30). By (3.22), (3.28), (2.14), and (2.15) we get

$$\begin{aligned}
\Lambda_{\sigma_{j,N}}(X_i) &= (X_i M \sigma_{j,N} - \sigma_{j,N} M X_i)(I - 2A_M) \\
&= \big(X_i M \pi_{j,N}(I - 2A_M) - \pi_{j,N}(I - 2A_M) M X_i\big)(I - 2A_M) \\
&= X_i M \pi_{j,N} + \pi_{j,N} M X_i = \pi(\pi_{j,N}, X_i).
\end{aligned} \tag{3.31}$$

Next, we check the identity on the right hand-side. Using (2.14) and (2.15) yields

$$\begin{aligned}
& 2\pi_{j,N} M X_i - [\pi(\sigma_{ij}, N) + \pi(X_j, \sigma_{i,N})](I - 2A_M) = \\
= \; & -(\pi_{ij}(I - 2A_M) M N + N M \pi_{ij}(I - 2A_M) + X_j M \pi_{i,N}(I - 2A_M) + \\
& \quad + \pi_{i,N}(I - 2A_M) M X_j)(I - 2A_M) + 2\pi_{j,N} M X_i \\
= \; & \pi_{ij} M N - N M \pi_{ij} - X_j M \pi_{i,N} + \pi_{i,N} M X_j + 2\pi_{j,N} M X_i \\
= \; & X_i M X_j M N + N M X_j M X_i + X_j M N M X_i + X_i M N M X_j \\
= \; & X_i M \pi_{j,N} + \pi_{j,N} M X_i \\
= \; & \pi(X_i, \pi_{j,N}).
\end{aligned}$$

□

Lemma 3.5. Let $X_1, \dots, X_{2n-1}, N$ be an orthonormal frame for $T\mathbf{C}H^n$. Then we have

$$\sum_{i=1}^{2n-1} \Lambda_{\sigma_{i,N}}(X_i) = -2(n-1)N. \tag{3.32}$$

Proof. It is enough to verify (3.32) when $A = A_0 \in \mathbf{C}H^n$. We can use the orthonormal basis in (2.9) with $\widehat{X}_n = N$. Applying formula (3.28) and the identities (2.14) and (2.15), we find that

$$\begin{aligned}
\Lambda_{\sigma_{i,N}}(X_i) &= \delta_{in}\widehat{X}_i - N, & i &= 1, \dots, n, \\
\Lambda_{\sigma_{\hat{j},N}}(\widehat{X}_j) &= -\delta_{jn}\widehat{X}_j - N = -N, & j &= 1, \dots, n-1,
\end{aligned}$$

where $\delta_{\ell k}$ denotes the Kronecker delta, i.e., $\delta_{\ell k} = 1$ if $\ell = k$ and $\delta_{\ell k} = 1$ otherwise. Summing up, we obtain (3.32).

□

4. Tangential Laplacian of A, N and σ_N

Let $\Sigma \subset \mathbf{C}H^n$ be a hypersurface. Throughout the paper, by "hypersurface" we mean an oriented smooth real hypersurface embedded in $\mathbf{C}H^n$. We denote by N the unit normal vector field giving the orientation of Σ. For any $A \in \mathbf{C}H^n$ the second fundamental form of Σ is the mapping $h : T_A\Sigma \times T_A\Sigma \to \mathbb{R}$, defined choosing the following sign convention $h(X,Y) := \left\langle \nabla^\top_X N, Y \right\rangle$. The (non-normalized) mean curvature of Σ is defined as $H := \mathrm{tr}(h)$. Finally, we denote by ∇^Σ the Levi-Civita connection of Σ (it is the one induced by $\nabla^\top$), and by Δ_Σ the Laplace-Beltrami operator of Σ.

In this section we compute the tangential Laplacian $\Delta_\Sigma A$ of the position matrix, the tangential Laplacian of the unit normal N and of σ_N. The computations are done at $A_0 \in \Sigma$ using a system of normal coordinates. Let $X_1, \dots, X_{2n-1}$ be a geodesic frame centered at A_0. Then we have

$$\nabla^\Sigma_{X_i} X_j(A_0) = 0, \qquad i,j = 1, \dots, 2n-1. \tag{4.33}$$

From now on, the Einstein summation will be sometimes used to sum over repeated indices.

Lemma 4.1. Let $\Sigma \subset \mathbf{C}H^n$ be a hypersurface. Then, the position matrix $A \in \mathbf{C}H^n$ satisfies the equation

$$\Delta_\Sigma A = \mathscr{H} - \sigma_N - \mathrm{tr}(h)N, \qquad A \in \Sigma, \tag{4.34}$$

where σ is the second fundamental form of $\mathscr{F}$.

Proof. Without loss of generality, we assume that $A_0 \in \Sigma$ and we check formula (4.34) using normal coordinates at A_0. Using the identity (2.6), we obtain

$$\Delta_\Sigma A|_{A=A_0} = \nabla_{X_j}\nabla_{X_j} A|_{A=A_0} = \nabla_{X_j} X_j(A_0) = \nabla^\top_{X_j} X_j(A_0) + \sigma_{A_0}(X_j, X_j).$$

In the last equality, we used the very definition of σ; see the beginning of Section 3. Again when $A = A_0$, by (4.33) we get

$$\nabla^\top_{X_j} X_j = \langle \nabla^\top_{X_j} X_j, N\rangle N = -\langle X_j, \nabla^\top_{X_j} N\rangle N = -\mathrm{tr}(h)N.$$

Since $X_1, \dots, X_{2n-1}, N$ is an orthonormal frame of $\mathbf{C}H^n$, from the definition (3.21) of $\mathscr{H}$ we get $\sigma(X_j, X_j) = \mathscr{H} - \sigma_N$, and this concludes the proof. □

Our next goal is to compute the Laplacian of the unit normal N along Σ. The proof of the following lemma is identical to the one of Lemma 4.3 in [3].

Lemma 4.2. Let $\Sigma \subset \mathbf{C}H^n$ be a hypersurface with constant mean curvature. At the center of a system of normal coordinates, the entries of the second fundamental form

$h_{ij} = h(X_i, X_j)$ satisfy the equations

$$\sum_{i=1}^{2n-1} X_i h_{ij} = 0, \qquad j = 1, \dots, 2n-1. \tag{4.35}$$

For any $A \in \Sigma \subset \mathbf{C}H^n$, the second fundamental form h of Σ can be identified with a linear operator on $T_A\Sigma$. Moreover, the restriction of σ to $T_A\Sigma$ can be viewed as a linear operator from $T_A\Sigma$ to $\mathrm{End}(T_A\Sigma, T_A^\perp \mathbf{C}H^n)$. In this way, the composition $\sigma h = \sigma \circ h$ becomes a linear operator from $T_A\Sigma$ to $\mathrm{End}(T_A\Sigma, T_A^\perp \mathbf{C}H^n)$. Namely, for any $X, Y \in T_A\Sigma$ we have $\sigma h(X)[Y] := \sigma\big(h(X), Y\big)$. We denote its trace by

$$\mathrm{tr}(\sigma h) := \sigma h(X_i)[X_i] = \sigma\left(h_{ij} X_j, X_i\right) = h_{ij}\sigma_{ij} \in T_A^\perp \mathbf{C}H^n,$$

where $\sigma_{ij} = \sigma(X_i, X_j)$ and $h_{ij} = h(X_i, X_j)$, for any orthonormal tangent frame $X_1, \dots, X_{2n-1}$ of Σ.

Theorem 4.3. Let $\Sigma \subset \mathbf{C}H^n$ be a hypersurface with constant mean curvature $H = \mathrm{tr}(h)$. Then, the unit normal satisfies the equation

$$\Delta_\Sigma N = 2\mathrm{tr}(\sigma h) - \big(|h|^2 - 2(n-1)\big)N - H\sigma_N. \tag{4.36}$$

The proof is based on the following:

Lemma 4.4. Let $\Sigma \subset \mathbf{C}H^n$ be a hypersurface. At the center of a system of normal coordinates, we have

$$\sum_{i=1}^{2n-1} \nabla_{X_i}\sigma_{i,N} = \mathrm{tr}(\sigma h) + 2(n-1)N - H\sigma_N. \tag{4.37}$$

Proof. Using (3.22), (3.28), and the definition of σ (see (3.25)) we obtain

$$\begin{aligned}
\nabla_{X_i}\sigma_{i,N} &= \nabla_{X_i}\pi_{i,N}(I - 2A_M) + \pi_{i,N}\nabla_{X_i}(I - 2A_M) \\
&= \sigma(\nabla_{X_i}X_i, N) + \sigma(X_i, \nabla_{X_i}N) - 2\pi_{i,N}MX_i \\
&= -h_{ii}\sigma_N + \sigma(\sigma_{ii}, N) + h_{ij}\sigma_{ij} + \sigma(X_i, \sigma_{i,N}) - 2\pi_{i,N}MX_i. \\
&= -h_{ii}\sigma_N + h_{ij}\sigma_{ij} + S,
\end{aligned}$$

where we set $S := \sigma(\sigma_{ii}, N) + \sigma(X_i, \sigma_{i,N}) - 2\pi_{i,N}MX_i$. Using formula (3.30) with $i = j$, we get $S = -\Lambda_{\sigma_{i,N}}(X_i)$. Hence, we proved that

$$\nabla_{X_i}\sigma_{i,N} = -h_{ii}\sigma_N + h_{ij}\sigma_{ij} - \Lambda_{\sigma_{i,N}}(X_i), \tag{4.38}$$

and formula (4.37) follows from (3.32). □

Proof of Theorem 4.3. Using $|N|^2 = 1$ and $\nabla^\perp_{X_i}N = \sigma_{i,N}$, we obtain

$$\Delta_\Sigma N = \nabla_{X_i}\nabla_{X_i}N = \nabla_{X_i}\left(\nabla^\perp_{X_i}N + \nabla^\top_{X_i}N\right) = \nabla_{X_i}\big(\sigma_{i,N} + h_{ij}X_j\big). \tag{4.39}$$

From (4.35) and (4.33), we deduce that

$$\nabla_{X_i}(h_{ij}X_j) = h_{ij}\nabla_{X_i}X_j = h_{ij}\left(\sigma_{ij} - h_{ij}N\right) = -|h|^2 N + h_{ij}\sigma_{ij}.$$

By (4.37) we have

$$\nabla_{X_i}\sigma_{i,N} = -H\sigma_N + h_{ij}\sigma_{ij} + 2(n-1)N,$$

and finally $\Delta_\Sigma N = -\left(|h|^2 - 2(n-1)\right)N + 2h_{ij}\sigma_{ij} - H\sigma_N$, which proves the lemma. □

Our final task is to compute the tangential Laplacian of σ_N. We fix an orthonormal frame $X_1, \ldots, X_{2n-1}$ for $T\Sigma$ such that $X_{n+j} = JX_j$ for any $j = 1, \ldots, n-1$ and we also assume that $X_n = -JN$, where N is the unit normal to Σ and J is the complex structure of $\mathbf{C}H^n$. The complex vector field $h_N \in \Gamma(\mathbf{C}T\Sigma)$ is defined in (1.2).

Lemma 4.5. Let $\Sigma \subset \mathbf{C}H^n$ be a hypersurface. Then we have

$$\sum_{i,j=1}^{2n-1} h_{ij}\Lambda_{\sigma_{j,N}}(X_i) = (\kappa - H)N - Jh_N, \tag{4.40}$$

where κ is the characteristic curvature of Σ.

Proof. The proof is a computation based on the formula $\Lambda_{\sigma_{j,N}}(X_i) = \pi(X_i, \pi_{j,N})$, see (3.31). In particular, the following identities hold:

$$\begin{aligned}
\Lambda_{\sigma_{j,N}}(X_i) &= \delta_{in}\widehat{X}_j - \delta_{ij}N,\\
\Lambda_{\sigma_{j,N}}(\widehat{X}_\ell) &= \delta_{\ell j}X_n,\\
\Lambda_{\sigma_{\widehat{\ell},N}}(X_i) &= -\delta_{in}X_l - \delta_{i\ell}X_n,\\
\Lambda_{\sigma_{\widehat{\ell},N}}(X_k) &= -\delta_{\ell k}N,
\end{aligned}$$

whenever $i, j = 1, \ldots, n$ and $\ell, k = 1, \ldots, n-1$. Using these identities, formula (4.40) easily follows.

□

The following lemma contains a technical calculation that will be used later.

Lemma 4.6. Let $\Sigma \subset \mathbf{C}H^n$ be a hypersurface. For any tangent orthonormal frame $X_1, \ldots, X_{2n-1}$, we set

$$\begin{aligned}
S_1 &:= \sigma(\sigma_{i,N}, \sigma_{i,N}) - 4\pi(\sigma_{i,N}, N)MX_i,\\
S_2 &:= -\pi_N M\Delta_\Sigma A.
\end{aligned}$$

Then, we have

$$S_1 + S_2 = -2\mathrm{tr}(h)N + \mathrm{tr}(\sigma) - 2(n-1)\sigma_N. \tag{4.41}$$

The proof will be given at the end of this section. In Section 4, we introduced the linear operator σh. In the same way, we now define the linear operator

$$\sigma h^2(X)[Y] := \sigma \circ h \circ h(X)[Y] = \sigma\big(h^2(X), Y\big), \qquad X, Y \in T_A\Sigma.$$

Its trace is

$$\mathrm{tr}(\sigma h^2) = \sigma h^2(X_j)[X_j] = h_{ij}h_{ik}\sigma_{jk} \in T_A^{\perp}\mathbf{C}H^n.$$

We are ready to prove the last result of this section.

Theorem 4.7. Let $\Sigma \subset \mathbf{C}H^n$ be a hypersurface. Then, the following holds

$$\Delta_\Sigma \sigma_N = -4\kappa N - 2|h|^2\sigma_N + 2\mathrm{tr}(\sigma h^2 + \sigma) + 4Jh_N. \tag{4.42}$$

Proof. We prove formula (4.42) using a system of normal coordinates centered at $A_0 \in \Sigma$. By (4.36) and (3.22) we have

$$\begin{aligned}
\Delta_\Sigma \sigma_N &= \nabla_{X_i}\nabla_{X_i}\sigma_N = \nabla_{X_i}\left(2\pi(\nabla_{X_i}N, N)(I - 2A_M) - 2\pi_N M X_i\right)\\
&= 2\big\{\sigma(\nabla_{X_i}\nabla_{X_i}N, N) + \sigma(\nabla_{X_i}N, \nabla_{X_i}N) - 4\pi(\nabla_{X_i}N, N)MX_i - \pi_N M\nabla_{X_i}X_i\big\}\\
&= 2\big\{\sigma(\Delta_\Sigma N, N) + 2\sigma\big(h_{ij}X_j + \sigma_{i,N}, h_{ik}X_k + \sigma_{i,N}\big)\\
&\qquad -4\pi\big(h_{ij}X_j + \sigma_{i,N}, N\big)MX_i - \pi_N M\left(-h_{ii}N + \sigma_{ii}\right)\big\}\\
&= -2\left(|h|^2 - 2(n-1)\right)\sigma_N + 4h_{ij}\left\{\left(\sigma(\sigma_{ij}, N) + \sigma(X_j, \sigma_{i,N})\right) - 2\pi_{j,N}MX_i\right\}\\
&\qquad + 2h_{ij}h_{ik}\sigma_{jk} + 2(\sigma(\sigma_{i,N}, \sigma_{i,N}) - 4\pi(\sigma_{i,N}, N)MX_i) - 2\pi_N M\Delta_\Sigma A,
\end{aligned}$$

where we used the identity $\sigma(\sigma_N, N) = 0$. By (3.30) and (4.41), we have

$$\begin{aligned}
\Delta_\Sigma \sigma_N &= -2\left(|h|^2 - 2(n-1)\right)\sigma_N + 2h_{ij}h_{ik}\sigma_{jk} - 4h_{ij}\Lambda_{\sigma_{j,N}}(X_i)\\
&\qquad - 4(n-1)\sigma_N + 2\mathrm{tr}(\sigma) - 4\mathrm{tr}(h)N\\
&= -4\mathrm{tr}(h)N + 2\mathrm{tr}(\sigma) - 2|h|^2\sigma_N + 2h_{ij}h_{ik}\sigma_{jk} - 4h_{ij}\Lambda_{\sigma_{j,N}}(X_i).
\end{aligned}$$

By formula (4.40), this ends the proof.

□

Proof of Lemma 4.6. We check the formula at $A_0 \in \Sigma$. By (2.15), (2.16), and (2.14), we obtain

$$\begin{aligned}
S_1 &= 2\pi_{X_i,N}(I - 2A_M)M\pi_{X_i,N}(I - 2A_M)^2 - 4\pi(\sigma_{X_i,N}, N)MX_i\\
&= 2\{\pi_{i,N}M\pi_{i,N}(I - 2A_M) - 2\left(\pi_{i,N}(I - 2A_M)MN + \; NM\pi_{i,N}(I - 2A_M)\right)MX_i\}\\
&= 2\{\left[\pi_{i,N}M\pi_{i,N} - 2\left(\pi_{i,N}MN - NM\pi_{i,N}\right)MX_i\right](I - 2A_M)\}.
\end{aligned}$$

A simple computation gives

$$\begin{aligned}
\pi_{X_i,N}M\pi_{X_i,N}(I - 2MA_0) &= -\delta_{in}(E_{in} + E_{ni}) + E_{nn} + E_{ii},\\
\pi_{\widehat{\jmath},N}M\pi_{\widehat{\jmath},N}(I - 2MA_0) &= E_{nn} + E_{jj}.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}&\big(-\pi_{i,N}MNMX_i+NM\pi_{X_i,N}MX_i\big)(I-2MA_0)=\delta_{in}E_{ni}-E_{ii}-(1-\delta_{in})A_0,\\&\big(-\pi_{\widehat{\jmath},N}MNM\widehat{X}_j+NM\pi_{\widehat{\jmath},N}M\widehat{X}_j\big)(I-2MA_0)=-E_{jj}-A_0.\end{aligned}$$

Therefore, at A_0 we have

$$\begin{aligned}S_1&=2\sum_{i=1}^{n}\left[-\delta_{in}(E_{in}+E_{ni})+E_{nn}+E_{ii}+2\left(\delta_{in}E_{ni}-E_{ii}-(1-\delta_{in})A_0\right)\right]+\\&\quad+2\sum_{j=1}^{n-1}\left[E_{nn}+E_{jj}-2\left(E_{jj}+A_0\right)\right]\\&=4nE_{nn}-4M+(12-8n)A_0.\end{aligned}$$

Moreover, using (3.23) and $\sigma_N=2(-E_{nn}+A_0)$, we get $4M=-\mathrm{tr}(\sigma)+4(n+1)A_0$ and $2E_{nn}=-\sigma_N+2A_0$. Thus, we have

$$S_1=-2n\sigma_N+\mathrm{tr}(\sigma)-8(n-1)A_0. \tag{4.43}$$

In order to compute S_2 at A_0, we note that $\pi_N MN=-2N$, and $\pi_N M\mathrm{tr}(\sigma)=-8nA_0+8E_{nn}$, and $\pi_N M\sigma_N=-4A_0+4E_{nn}$. Thus, we get

$$\begin{aligned}S_2&=-\pi_N M\left(-\mathrm{tr}(h)N+\mathrm{tr}(\sigma)-\sigma_N\right)\\&=-\left(2\mathrm{tr}(h)N-8nA_0+8E_{nn}+4A_0-4E_{nn}\right)\\&=-2\mathrm{tr}(h)N+8(n-1)A_0+2\sigma_N.\end{aligned}\tag{4.44}$$

The proof follows by adding the formulas for S_1 and S_2 in (4.43) and (4.44) . □

5. Second variation of the area and geometric stability inequality

In this section, we prove Theorem 1.2. Let $\Omega\subset\mathbf{C}H^n$ be a relatively compact domain with smooth boundary $\Sigma=\partial\Omega$ and let $u\in C^\infty(\Sigma)$ be a function with zero mean, i.e., $\int_\Sigma u\,d\mu=0$. Let $\Psi:\mathbf{C}H^n\times]-\varepsilon,\varepsilon[\to\mathbf{C}H^n$ be a differentiable map such that $\Psi_t:=\Psi(\cdot,t)$ is an embedding for each $t\in]-\varepsilon,\varepsilon[$, $\Psi_0=\mathrm{Id}_\Sigma$ and $\partial\Psi(x,0)/\partial t=u(x)N(x)$ for $x\in\Sigma$.

If Σ is a critical point of the area, then it has constant mean curvature. The second variation of the area subject to deformations preserving the enclosed volume takes the following form

$$A''(u):=\frac{d^2}{dt^2}\mu(\Sigma_t)\bigg|_{t=0}=-\int_\Sigma u\mathscr{L}u\,d\mu,$$

where $\mathscr{L}$ the Jacobi operator (1.4), see e.g. [5, pp. 169-171]. The Ricci curvature $\mathrm{Ric}(N)$ appearing in the Jacobi operator can be computed using the following formula for the Riemann curvature tensor

$$R(X,Y)Z=-\left\{\langle Z,Y\rangle X-\langle Z,X\rangle Y+\langle Z,JY\rangle JX-\langle Z,JX\rangle JY+2\langle X,JY\rangle JZ\right\}$$

whenever $X, Y, Z \in T_A\mathbf{C}H^n$, see [10, p. 285]. Notice that the holomorphic curvature of the complex hyperbolic space $\mathbf{C}H^n$ for our choice of metric (2.1) is $c = -4$. It turns out that the Ricci curvature is

$$\mathrm{Ric}(N) = -2(n+1). \tag{5.45}$$

Now we consider the matrix valued function $u = \Delta_\Sigma A$ and for any $V \in H^{n+1}$ we let $u_V = \langle u, V_M\rangle_H$, where

$$\langle A, B\rangle_H := \frac{1}{2}\mathrm{tr}(AB), \qquad A, B \in H^{n+1},$$

is the standard inner product on H^{n+1}. We define the quadratic form $Q_\Sigma : H^{n+1} \to \mathbb{R}$

$$Q_\Sigma(V) := A''(u_V) = -\int_\Sigma \langle u, V_M\rangle_H \langle \mathscr{L}u, V_M\rangle_H d\mu, \qquad V \in H^{n+1}.$$

If $\mathscr{V}$ is any orthonormal basis of H^{n+1} for the standard inner product, then we have

$$\mathrm{tr}(Q_\Sigma) = -\sum_{V\in\mathscr{V}} \int_\Sigma \langle u, V_M\rangle_H \langle \mathscr{L}u, V_M\rangle_H d\mu.$$

Let us compute the sum:

$$\begin{aligned}\sum_{V\in\mathscr{V}} \langle u, V_M\rangle_H \langle \mathscr{L}u, V_M\rangle_H &= \Big\langle u, \sum_{V\in\mathscr{V}} \langle \mathscr{L}u, V_M\rangle_H V_M \Big\rangle_H = \frac{1}{2}\Big\langle u, M \sum_{V\in\mathscr{V}} \mathrm{tr}(\mathscr{L}uMV)V \Big\rangle_H \\ &= \Big\langle u, M \sum_{V\in\mathscr{V}} \langle \mathscr{L}uM, V\rangle_H V \Big\rangle_H = \langle u, M\mathscr{L}uM\rangle_H \\ &= \frac{1}{2}\mathrm{tr}(uM\mathscr{L}uM) = \frac{1}{2}\mathrm{tr}(MuM\mathscr{L}u) = -\langle u, \mathscr{L}u\rangle_M.\end{aligned}$$

Eventually, also using (5.45), the trace of Q_Σ is given by

$$\mathrm{tr}(Q_\Sigma) = \int_\Sigma \langle u, \mathscr{L}u\rangle_M d\mu = \int_\Sigma \big\{ \langle u, \Delta_\Sigma u\rangle_M + \big(|h|^2 - 2(n+1)\big)\langle u, u\rangle_M \big\}\, d\mu. \tag{5.46}$$

The proof of Theorem 1.2 follows from formula (5.47) below and from the fact that we have $\mathrm{tr}(Q_\Sigma) \geq 0$ when Σ is stable. The functions u_V are admissible because they have zero mean on Σ, since Σ has no boundary.

Lemma 5.1. Let $\Sigma \subset \mathbf{C}H^n$ be a compact CMC hypersurface. The trace of the quadratic form Q_Σ is given by

$$\mathrm{tr}(Q_\Sigma) = 4\int_\Sigma \Big\{ 2(n+1)H^2 - 2(n^2-1)\big(2n + |h|^2\big) - (H+\kappa)^2 - |h_N|^2 \Big\} d\mu, \tag{5.47}$$

where $H = \mathrm{tr}(h)$, κ is the characteristic curvature of Σ, and h_N is defined in (1.2).

Proof. We compute the quantities appearing in (5.46). Using formula (4.34), we get

$$\begin{aligned}\langle u, u\rangle_M &= \langle -\mathrm{tr}(h)N + \mathrm{tr}(\sigma) - \sigma_N, -\mathrm{tr}(h)N + \mathrm{tr}(\sigma) - \sigma_N\rangle_M \\ &= H^2 + \langle \mathscr{H}, \mathscr{H}\rangle_M - 2\langle \mathscr{H}, \sigma_N\rangle_M + \langle \sigma_N, \sigma_N\rangle_M,\end{aligned}$$

where we used the orthogonality relations $\langle N, \sigma_N \rangle_M = \langle N, \mathscr{H} \rangle_M = 0$.

We compute $\langle \mathscr{H}, \mathscr{H} \rangle_M$, $\langle \mathscr{H}, \sigma_N \rangle_M$, and $\langle \sigma_N, \sigma_N \rangle_M$ at $A = A_0 \in \Sigma$ using the orthonormal frame (2.9) with $N = \widehat{X}_n$. By (3.27) and (3.26) we obtain

$$\mathscr{H} = 2\sum_{i=1}^{n} \sigma_{ii} = 2\sum_{i=1}^{n} (2A_0 - 2E_{ii}) = 4nA_0 - 4(M - A_0) = 4(n+1)A_0 - 4M$$

and thus

$$\begin{aligned} \langle \mathscr{H}, \mathscr{H} \rangle_M &= 16\left[(n+1)^2 \langle A_0, A_0 \rangle_M - 2(n+1)\langle A_0, M \rangle_M + \langle M, M \rangle_M\right] \\ &= 16\left[(n+1)^2\left(-\frac{1}{2}\right) - 2(n+1)\left(-\frac{1}{2}\right) - \frac{1}{2}(n+1)\right] \\ &= -8n(n+1). \end{aligned} \tag{5.48}$$

Moreover, we have

$$\langle \sigma_N, \sigma_N \rangle_M = -\frac{1}{2}\mathrm{tr}(M\sigma_N M\sigma_N) = -4, \tag{5.49}$$

and

$$\begin{aligned} \langle \mathscr{H}, \sigma_N \rangle_M &= \langle 4(n+1)A_0 - 4M, 2A_0 - 2E_{nn} \rangle_M \\ &= 8\left[(n+1)\langle A_0, A_0 \rangle_M - (n+1)\langle A_0, E_{nn} \rangle_M - \langle A_0, M \rangle_M + \langle M, E_{nn} \rangle_M\right] \\ &= 8\left[(n+1)\left(-\frac{1}{2}\right) - \left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right)\right] \\ &= -4(n+1). \end{aligned} \tag{5.50}$$

Therefore, by (5.48), (5.49) and (5.50) we get

$$\langle u, u \rangle_M = H^2 - 8n(n+1) + 8(n+1) - 4 = H^2 - 4(2n^2 - 1). \tag{5.51}$$

Secondly, we compute the term $\langle u, \Delta_\Sigma u \rangle_M$. We have

$$\langle u, \Delta_\Sigma u \rangle_M = -H\langle u, \Delta_\Sigma N \rangle_M + \langle u, \Delta_\Sigma \mathscr{H} \rangle_M - \langle u, \Delta_\Sigma \sigma_N \rangle_M.$$

Thus, by (4.36) we get

$$\begin{aligned} \langle u, \Delta_\Sigma N \rangle_M = H(|h|^2 - 2(n-1)) + 2\langle \mathscr{H}, \mathrm{tr}(\sigma h) \rangle_M - (2+H)\langle \mathscr{H}, \sigma_N \rangle_M \\ + H\langle \sigma_N, \sigma_N \rangle_M. \end{aligned}$$

At the point $A = A_0$ we obtain

$$\begin{aligned} \langle \mathscr{H}, \mathrm{tr}(\sigma h) \rangle_M &= \langle 4(n+1)A_0 - 4M, h_{ij}\sigma_{ij} \rangle_M = 4h_{ij}\langle (n+1)A_0 - M, \sigma_{ij} \rangle_M \\ &= 4h_{ij}\left[(n+1)\langle A_0, \sigma_{ij} \rangle_M - \langle M, \sigma_{ij} \rangle_M\right] \\ &= 4(n+1)\left(-\sum_{i,j=1}^{n} h_{ij}\delta_{ij} - \sum_{i,j=1}^{n-1} h_{\widehat{ij}}\delta_{ij}\right) \\ &= -4(n+1)H, \end{aligned} \tag{5.52}$$

where we used the following identities:

$$\langle A_0, \sigma_{ij}\rangle_M = \langle A_0, \sigma_{\widehat{\widehat{ij}}})\rangle_M = -\delta_{ij},$$
$$\langle A_0, \sigma_{i\widehat{j}}\rangle_M = \langle A_0, \sigma_{\widehat{i}j}\rangle_M = 0,$$

and $\langle M, \sigma_{ij}\rangle_M = 0$, for every $i, j = 1, ..., 2n-1$. Then, we have

$$\begin{aligned}
\langle \sigma_N, \mathrm{tr}(\sigma h)\rangle_M &= h_{ij}\langle \sigma_N, \sigma_{ij}\rangle_M \\
&= -2\sum_{i,j=1}^{n} h_{ij}(\delta_{ij} + \delta_{in}\delta_{jn}) - 2\sum_{i,j=1}^{n-1} h_{\widehat{\widehat{ij}}}(\delta_{ij} + \delta_{in}\delta_{jn}) \\
&= -4h_{nn} - 2\sum_{i=1, i\neq n}^{2n-1} h_{ii} = -2\kappa - 2H,
\end{aligned} \tag{5.53}$$

where $\kappa = h_{nn}$. Here, we also used the following identities

$$\langle \sigma_N, \sigma_{ij}\rangle_M = \langle \sigma_N, \sigma_{\widehat{\widehat{ij}}}\rangle_M = -2\delta_{ij} - 2\delta_{in}\delta_{jn},$$
$$\langle \sigma_N, \sigma_{i\widehat{j}}\rangle_M = \langle \sigma_N, \sigma_{\widehat{i}j}\rangle_M = 0,$$

holding for $i, j = 1, \dots, n$ and for $\widehat{i} = i + n$, $\widehat{j} = j + n$ ranging from $n+1$ to $2n-1$.
for every $i, j = 1, \dots, 2n-1$. Hence, we obtain

$$\langle u, \Delta_\Sigma N\rangle_M = H(|h|^2 - (6n+2)) + 4\kappa. \tag{5.54}$$

Thirdly, using formula (4.42) we have

$$\begin{aligned}
\langle u, \Delta_\Sigma \sigma_N\rangle_M &= 4\kappa H - 2(|h|^2+1)\langle \mathscr{H}, \sigma_N\rangle_M + 2\langle \mathscr{H}, \mathrm{tr}(\sigma h^2)\rangle_M \\
&\quad + 2\langle \mathscr{H}, \mathscr{H}\rangle_M + 2|h|^2\langle \sigma_N, \sigma_N\rangle_M - 2\langle \sigma_N, \mathrm{tr}(\sigma h^2)\rangle_M,
\end{aligned}$$

where we used $\langle u, Jh_N\rangle_M = 0$. Moreover, we have

$$\begin{aligned}
\langle \mathscr{H}, \mathrm{tr}(\sigma h^2)\rangle_M &= \big\langle 4(n+1)A_0 - 4M, h_{ij}h_{ik}\sigma_{jk}\big\rangle_M \\
&= 4h_{ij}h_{ik}\left[(n+1)\langle A_0, \sigma_{jk}\rangle_M - \langle M, \sigma_{jk}\rangle_M\right] \\
&= 4(n+1)h_{ij}h_{ik}\langle A_0, \sigma_{jk}\rangle_M \\
&= 4(n+1)\sum_{i=1}^{2n-1}\left(-\sum_{j,k=1}^{n} h_{ij}h_{ik}\delta_{jk} - \sum_{j,k=1}^{n-1} h_{\widehat{ij}}h_{\widehat{ik}}\delta_{jk}\right) \\
&= -4(n+1)|h|^2.
\end{aligned} \tag{5.55}$$

In addition, we compute

$$\left\langle \sigma_N, \mathrm{tr}(\sigma h^2)\right\rangle_M = \sum_{i,j,k=1}^{2n-1} h_{ij}h_{ik}\langle \sigma_N, \sigma_{jk}\rangle_M$$
$$= \sum_{i=1}^{2n-1}\left(-2\sum_{j,k=1}^{n} h_{ij}h_{ik}(\delta_{jk}+\delta_{jn}\delta_{kn}) - 2\sum_{j,k=1}^{n-1} h_{i\widehat{j}}h_{i\widehat{k}}(\delta_{jk}+\delta_{jn}\delta_{kn})\right)$$
$$= -2\sum_{i=1}^{2n-1} h_{in}^2 - 2|h|^2 = -2\left(|h|^2+|h_N|^2+\kappa^2\right). \tag{5.56}$$

Therefore, we get

$$\begin{aligned}\langle u, \Delta_\Sigma \sigma_N\rangle_M &= 4\kappa H + 8|h|^2(n+1) + 8|h|^2(n+1) - 16n(n+1) - 4|h|^2 \\ &\quad + 4(|h|^2+|h_N|^2+\kappa^2) + 8(n+1) \\ &= 4\kappa H - 4|h|^2 - 8(n+1)(2n-1) + 4|h_N|^2 + 4\kappa^2.\end{aligned} \tag{5.57}$$

In order to finish the proof of (5.47), we just have to sum up formulas (5.51), (5.54), and (5.57).

□

Remark 5.2. The geometric trace formula (5.47) is similar to the one discovered in [3, Theorem 6.2] in the case of $\mathbf{C}P^n$. The complex projective space can be isometrically embedded in the space H^{n+1}, endowed with its standard inner product. The quadratic form Q_Σ can be defined in a similar way and in $\mathbf{C}P^n$ its trace reads

$$\mathrm{tr}(Q_\Sigma) = 4\int_\Sigma \Big\{2(n+1)H^2 + 2(n^2-1)\big(2n-|h|^2\big) - (H+\kappa)^2 - |h_N|^2\Big\}d\mu.$$

The unique difference with (5.47) is the sign of the additive constant $4n(n^2-1)$ appearing in the integrand.

6. Proof of Theorem 1.1

In this section we prove Theorem 1.1. We follow an idea contained in the proof of Theorem 1.1 in [3]. Let $p(\kappa; H, n)$ be the polynomial in (1.1). Its discriminant is $8(n+1)\big[H^2-2(n-1)(2n+1)\big]$ and so $H^2 \le 2(n-1)(2n+1)$ implies that $p(\kappa; H, n) \ge 0$. So the last claim in Theorem 1.1 follows from the first one.

We prove that there is no stable CMC hypersurface satisfying $p(\kappa; H, n) \ge 0$. We argue by contradiction. Let Σ be a stable CMC hypersurface such that $p(\kappa; H, n) \ge 0$. We shall show that this implies that $p(\kappa; H, n) = 0$ and that Σ is a sphere. This will be a contradiction for the following reason.

Let $\Sigma_r \subset \mathbf{C}H^n$ be a geodesic sphere with radius $r > 0$. The sphere is a Hopf hypersurface and so $h_N = 0$ identically, and the characteristic curvature κ is constant.

Namely, we have $\kappa = 2\coth 2r$. The restriction of the second fundamental form of Σ_r to the complex tangent plane is umbilical, i.e., the principal curvatures in the complex tangent directions are equal and constant. In fact, they are given by $\lambda = \coth r$. As a reference for these facts, see [11].

Setting $t := \tanh r \in (0,1)$, we have $\lambda = \frac{1}{t}$ and $\kappa = \frac{1}{t} + t$. So, we get

$$\begin{aligned} |h|^2 &= \frac{2n-1}{t^2} + t^2 + 2, \\ H^2 &= \frac{(2n-1)^2}{t^2} + t^2 + 2(2n-1), \\ (H+\kappa)^2 &= 4\Big(\frac{n^2}{t^2} + t^2 + 2n\Big). \end{aligned} \tag{6.58}$$

Inserting these values into formula (5.47), we obtain

$$\operatorname{tr}(Q_{\Sigma_r}) = 8n(n-1)\frac{1-t^2}{t^2}\left(t^2+2n+1\right)\mu(\Sigma_r).$$

Thus, we have $\operatorname{tr}(Q_{\Sigma_r}) > 0$ for any $r > 0$.

We now prove our initial claim. Let $\widehat{h}$ be the restriction of the second fundamental form h of Σ to the complex tangent space $\mathbf{C}T\Sigma$ and let $\widehat{H} = \operatorname{tr}(\widehat{h})$. Then, the following identities hold at any point of Σ

$$H = \widehat{H} + \kappa \quad \text{and} \quad |h|^2 = |\widehat{h}|^2 + 2|h_N|^2 + \kappa^2$$

and we have the inequalities

$$|h|^2 \geq |\widehat{h}|^2 + \kappa^2 \quad \text{and} \quad |\widehat{h}|^2 \geq \frac{\widehat{H}^2}{2(n-1)} = \frac{(H-\kappa)^2}{2(n-1)}. \tag{6.59}$$

Starting from (5.47), by these inequalities and by $|h_N| \geq 0$ we obtain

$$\begin{aligned} \operatorname{tr}(Q_\Sigma) &\leq 4\int_\Sigma \left\{2(n+1)H^2 - 4n(n^2-1) - 2(n^2-1)\left[\frac{(H-\kappa)^2}{2(n-1)} + \kappa^2\right] - (H+\kappa)^2\right\} d\mu \\ &= -4n\int_\Sigma p(\kappa; H, n)d\mu, \end{aligned} \tag{6.60}$$

where $p(\kappa; H, n)$ is the polynomial in (1.1).

If Σ is stable then we have $\operatorname{tr}(Q_\Sigma) \geq 0$. Thus our assumption $p(\kappa; H, n) \geq 0$ implies that $p(\kappa; H, n) = 0$ and that we have equalities in (6.59), i.e.,

$$|h|^2 = |\widehat{h}|^2 + \kappa^2 \quad \text{and} \quad |\widehat{h}|^2 = \frac{\widehat{H}^2}{2(n-1)}. \tag{6.61}$$

In addition, we have $h_N = 0$, which implies that JN is an eigenvector of h, i.e., Σ is a Hopf hypersurface. By Maeda's theorem, it follows that the characteristic curvature κ is constant (in each connected component). This also simply follows from the fact that κ is a root of $p(\kappa; H, n) = 0$.

The equality in the right hand side of (6.61) implies that the principal curvatures along the complex tangent directions are equal and constant. Thus Σ is a hypersurface with precisely two different principal curvatures. By the Berndt's classification [4] of Hopf hypersurfaces with constant principal curvatures in $\mathbf{C}H^n$ $(n \geq 2)$, it turns out that Σ is an open subset of a hypersurface in the so-called Montiel's list (see [6], [11]). Finally, the compactness implies that Σ is a geodesic sphere. The same conclusion can be obtained using [13, Theorem E]. However, we showed that for geodesics spheres $p(\kappa; H, n) < 0$. This contradiction proves Theorem 1.1.

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